Systems of Linear Equations - Matrices

The use of Elementary Row Operations is required when solving a system of equations using matrices.

Elementary Row Operations

I. Interchange two rows.

II. Multiply one row by a nonzero number.

III. Add a multiple of one row to a different row.

$$\left[\begin{array}{ccc|c} 3 & 6 & -2 & -8 \\ 2 & 0 & 5 & 13 \\ 1 & 3 & -7 & 12 \end{array}\right] \quad R_{13} \qquad \left[\begin{array}{ccc|c} 1 & 3 & -7 & 12 \\ 2 & 0 & 5 & 13 \\ 3 & 6 & -2 & -8 \end{array}\right] \quad 2R_3 \qquad \left[\begin{array}{ccc|c} 1 & 3 & -7 & 12 \\ 2 & 0 & 5 & 13 \\ 6 & 12 & -4 & -16 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -7 & 12 \\ 2 & 0 & 5 & 13 \\ 6 & 12 & -4 & -16 \end{array}\right] \quad R_2 - 2R_1 \qquad \left[\begin{array}{ccc|c} 1 & 3 & -7 & 12 \\ 0 & -6 & 19 & -11 \\ 6 & 12 & -4 & -16 \end{array}\right]$$

Systems of Linear Equations - Matrices

Row Echelon Form

A matrix is in **row echelon form** when it satisfies the following conditions.

The first non-zero element in each row, called the **leading entry**, is 1.

Each leading entry is in a column to the right of the leading entry in the previous row.

Rows with all zero elements, if any, are below rows having a non-zero element.

$$\left[\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 8 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & -7 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4 & 7 & 5 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Systems of Linear Equations

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** when it satisfies the following conditions.

The matrix is in row echelon form (i.e., it satisfies the three conditions listed for row echelon form).

The leading element is the only non-zero entry in its column.

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example: Solve the following systems of equations by Gauss-elimination method (Row echelon form)

Augmented matrix [A|B]

$$x - y + z = -4$$

$$2x - 3y + 4z = -15$$

$$5x + y - 2z = 12$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 2 & -3 & 4 & -15 \\ 5 & 1 & -2 & 12 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & -1 & 2 & -7 \\ 5 & 1 & -2 & 12 \end{array}\right] \quad R_3 - 5R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & -1 & 2 & -7 \\ 0 & 6 & -7 & 32 \end{array}\right] \quad R_2(-1)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -2 & 7 \\ 0 & 6 & -7 & 32 \end{array}\right] \quad R_3 - 6R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 5 & -10 \end{array}\right] \quad R_3/5$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

which is row echelon form

$$x - y + z = -4$$

$$y - 2z = 7$$

$$z = -2$$

$$y - 2(-2) = 7$$

$$y = 3$$

$$x - (3) + (-2) = -4$$

$$x = 1$$

Solution:

$$(x, y, z)$$

$$(1, 3, -2)$$

Example: Solve the following systems of equations by Gauss-Jordan method. (Reduced row echelon form)

$$x - y + z = -4$$

$$2x - 3y + 4z = -15$$

$$5x + y - 2z = 12$$

Continue to Reduced Row Echelon Form

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -4 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_2 + 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

which is reduced row echelon form

$$x = 1$$
$$y = 3$$
$$z = -2$$

Solution:

$$(x, y, z)$$

$$(1, 3, -2)$$

Consistent or Inconsistent System?

$$\left.\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right.$$

$x = 1$
$y = 5$ $\quad$ *Consistent system: Unique solution*
$z = 2$

$$\left.\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right.$$

$x = 4$
$y = 7$ $\quad$ *Inconsistent system: No solution*
$0 = 2$ (not true

$$\left.\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right.$$

$x + 5z = 2$ $\quad$ *Consistent system: Infinite solutions*
$y + 3z = 2$
Let $z = k$; *any real number*

$(x = 2 - 5\mathrm{k}, y = 2 - 3\mathrm{k}, z = k;$ *any arbitrary number*$)$

Example : Solve the following systems of equations.

$$x + 2y - z = 3$$
$$2x - y + 2z = 6$$
$$x - 3y + 3z = 4$$

Augmented matrix [A|B]

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right] \quad R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 1 & -3 & 3 & 4 \end{array}\right] \quad R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -5 & 4 & 0 \\ 0 & -5 & 4 & 1 \end{array}\right] \quad (-\tfrac{1}{5})R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -4/5 & 0 \\ 0 & -5 & 4 & 1 \end{array}\right] \quad R_3 + 5R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -4/5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad 0 = 1; \text{ Not possible}$$

System is inconsistent system and it has no solution