NUMERICAL METHODS 2

Mini-project Task 1 - Report

By: Noman Noor

(Student ID: 302343)

Introduction

❖ Goal of the task:

This task was about creating a function for calculation of 2-Dimensional Integrals of the form $\int_c^d \int_a^b \boldsymbol{f(x,y)dx\,dy}$ using composite Newton $\frac{3}{8}$ rule *(i.e., composite Simpson's $\frac{3}{8}$ quadrature, which is a Newton-Cotes quadrature due to equidistant x points)* with respect to x, and the composite Trapezoidal rule with respect to y.

(Please note that the term 'rule' and 'quadrature' in our context may be used interchangeably as they refer to the same concept.)

❖ My approach to the task:

Keeping in mind the formulae for the composite Simpson's $\frac{3}{8}$ quadrature and composite Trapezoidal quadrature (provided in the appendix), the function is implemented to take out the common scalar multiplier of each term's coefficients and a vector containing the coefficients of all the terms for composite Simpson's 3/8 and composite Trapezoidal rule. Let their labels be c_S,v_S and c_T,v_T for each rule respectively. A matrix fxy containing values of f(x,y) for each (x,y) combination is computed. Then, due to the nature of Matrix Multiplication, the following expression:

$$c_S c_T sum(sum(v_T * fxy * {v_S}^T))$$

is the same as the result of applying the needed quadratures one after another. This is because when inspected closely, we see that the underlying elements of the expression given above are the same as the expansion of the following formula:

$$(Q_1 \times Q_2)(f) := \sum_{k=0}^{n} A_k \sum_{j=0}^{m} B_j f(x_k, y_j)$$

Please note that further clarification and explanations have been provided in the source code in the form of comments.

❖ Notation Scheme/Table:

Symbol	Denotes
[a,b]	Interval of Integration when using Simpson's 3/8 rule.
nx	Value of n for the Simpson's Formula. Number of points excluding x_0.
[c,d]	Interval of Integration when using Trapezoidal rule.
ny	Value of n for the Simpson's Formula. Number of points excluding x_0.

❖ Usage:

$$\int_c^d \int_a^b \boldsymbol{f(x,y)dx\,dy} \approx \mathbf{Int2D(func, a, b, nx, c, d, ny)}$$

Note that the testing script “TestInt2D.m” may be called by simply typing its name in the console. Both the testing script and the function itself can only be called when the file is present in the current working directory.

Numerical Tests

❖ For $I = \int_1^2 \int_1^3 x^3 y\, dx\ dy$:

Finding actual value of the integral I:

Let’s first compute the integral in a way that avoids numerical errors. So, we can use Wolfram Alpha or the like to get analytical answers (without computations). We find:

$\int x^3 y dx = \frac{yx^4}{4} + C$

so, when we apply the bounds [1 to 3], we get

$\frac{(81-1)}{4} y = 20y.$

Now, integrating again, we get:

$I = 20 \int_1^2 y dy = 20 \left[\frac{y^2}{2}\right]_1^2 = 20\left(\frac{3}{2}\right) = 30$

Analyzing the approximations by the Int2D Function:

Now we know the actual (accurate) value is supposed to be 30. The following table contains the values (for each nx, ny pair) computed in the “TestInt2D.m” script.

nx	ny	Int2D Approximation	Actual Value (I)	Error
3	1	30	30	0
3	7	30		0
9	1	30		0
6	10	30		0

We know (composite) Simpson’s 3/8 rule is exact for polynomials of degree lesser than or equal to 3, and similarly (composite) Trapezoidal rule is exact for polynomials of degree 1 no matter what values of nx or ny are, therefore, when applied to x^3 and y in that order we should get the exact result (no error), no matter what nx and ny values are. This is confirmed by the above table.

Also, notice (refer to the formulas in the appendix), when nx=3, the first rule becomes simple Simpson’s 3/8 (*not* composite). Similarly, when ny=1, the second rule becomes simple Trapezoidal rule (*not* composite). The first row shows the results when both become non-composite, meanwhile the second and third row show the results when either Simpson’s 3/8 or trapezoidal rule respectively become non-composite. The accuracy of these results shows that the function can use and works for the non-composite forms of the rules if the correct values of nx and/or ny are provided for the required case.

All other valid values of nx and ny use the composite versions of the rules, e.g., ny in the second row of the table, nx in the third row, and both in the fourth row. From the correctness/accuracy of the solutions from these rows, it is seen that the function also works for composite variants of both the rules. This shall be further tested when testing the next function.

❖ For $I = \int_1^0 \int_0^1 (e^x - e^y) dx\, dy$

The actual (accurate) calculation of actual value of I is done similarly to the one above and shall be provided in the Appendix section. We have, actual value of the integral $I = 0$.

nx	ny	Int2D Approximation	Actual Value (I)	Error
3	10	0.0011733	0	0.0011733
18	30	0.00015889		0.00015889
600	500	0.00000057276		0.00000057276
6000	5000	0.0000000057276		0.0000000057276

As can be seen, though we aren't able to get a perfectly accurate value, we keep getting closer and closer as the values of nx and ny increase. So, we could get even better approximations (closer to actual value) as we increase nx and ny. However, it must be noted that while the accuracy increases with nx and ny, the computational time and resource usage also increases.

Comparison of "Int2D" with MatLab Implementation "integral2"

Consider the function $\int_1^3 \int_1^2 (y^3 x) dx dy$, notice that it's equivalent to the first function we considered in "Numerical Tests" section (i.e., $\int_1^2 \int_1^3 \mathrm{x}^3 \mathrm{y}\, \mathrm{dx}\ \ \mathrm{dy}$). So, the accurate/actual value of the integral is I=30. First, we note that Simpson's 3/8 rule is exact for x (because polynomial degree <= 3) but Trapezoidal rule is not exact for y^3 (because polynomial degree >= 1). So, we can just let nx remain 3 (i.e., non-composite Simpson's 3/8 rule).

Then, for changing values of ny, we have:

nx	ny	Int2D Approximation	MatLab integral2 approximation	Actual Value (I)
3	10	42	30	30
3	50	30.0048		
3	100	30.0012		
3	200	30.0003		
3	300	30.0001		
3	490	30		

So, we conclude, our function can be as accurate as we want depending on the values of nx and ny. Of course, computational resources serve as limiting factor. Clearly then, our function is just as viable (or maybe even better) for MatLab's default integral2 for non-mission-critical use-cases, as long as we have proper nx and ny pairs and computational resources. If not, it's recommended to use integral2 in MatLab instead.

Conclusion

We conclude that the function Int2D is implemented according to the task description and we have shown the characteristic traits have been shown to be the same (exactness and increasing accuracy for increasing nx,ny). This can be further verified from the source code (with comments explaining most aspects).

Appendix

- **Actual (accurate) solution to $\boldsymbol{I} = \int_1^0 \int_0^1 (e^x - e^y)\,dx\,dy$:**
  - We find, $\int_0^1 e^x - e^y dx = [-e^y x + e^x + C]_0^1 = -e^y + e - 1$
  - $\Rightarrow \boldsymbol{I} = \int_1^0 (-e^y + e - 1)dy = [-e^y + ey - y + C]_1^0 = \mathbf{0}$
- **Task description:**
  - Numerical calculation of the integral $\iint_D f(x,y)dxdy$, where $D = [a,b] \times [c,d]$. Use the composite Newton („$\frac{3}{8}$") rule with respect to x and the composite trapezoid rule with respect to y.
- **Simpson's 3/8 rule formula:**

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$
$$= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right],$$

  where $b - a = 3h$.

- **Composite Simpson's 3/8 rule formula:** Let n be a multiple of 3,

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + 2f(x_3) + 3f(x_4) + 3f(x_5) + 2f(x_6) + \cdots + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n)\right]$$
$$= \frac{3h}{8}\left[f(x_0) + 3\sum_{i \neq 3k}^{n-1} f(x_i) + 2\sum_{j=1}^{n/3-1} f(x_{3j}) + f(x_n)\right] \quad \text{for } k \in \mathbb{N}_0.$$

- **Trapezoidal rule:**

$$\int_a^b f(x)\,dx \approx (b-a) \cdot \tfrac{1}{2}(f(a) + f(b)).$$

- **Composite Trapezoidal rule:**

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + \cdots + 2f(x_{N-1}) + f(x_N)\right).$$

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N} \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x_k.$$