

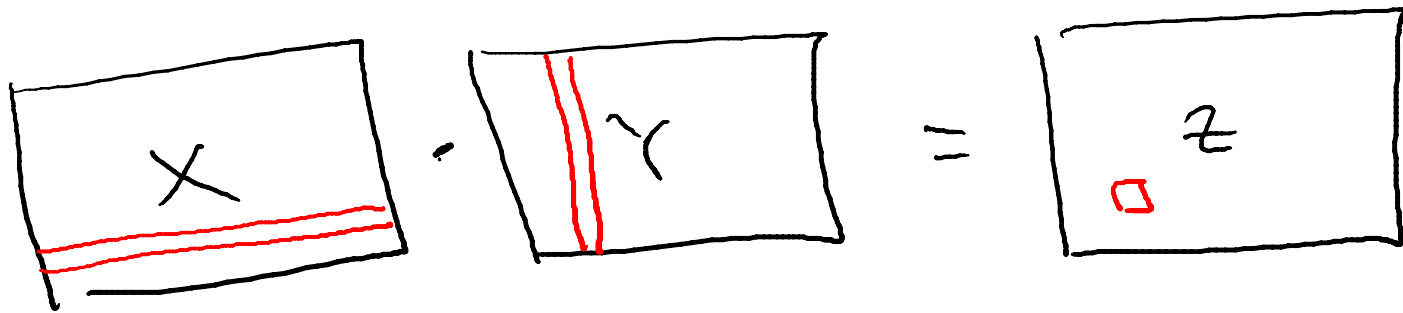


Design and Analysis
of Algorithms I

Divide and Conquer

Matrix Multiplication

Matrix Multiplication



(all $n \times n$ matrices)


Where $z_{ij} = (\text{i}^{\text{th}} \text{ row of } X) \cdot (\text{j}^{\text{th}} \text{ column of } Y)$

$$= \sum_{k=1}^n X_{ik} \cdot Y_{kj}$$

Note : input size
 $= \theta(n^2)$

Example (n=2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$z_{ij} = \sum_{k=1}^n X_{ik} \cdot Y_{kj}$$

$$\theta(n)$$

What is the asymptotic running time of the straightforward iterative algorithm for matrix multiplication?

☐ $\theta(n \log n)$

☐ $\theta(n^2)$

 ☒ $\theta(n^3)$

☐ $\theta(n^4)$

The Divide and Conquer Paradigm

1. DIVIDE into smaller subproblems
2. CONQUER subproblems recursively.
3. COMBINE solutions of subproblems into one for the original problem.

Applying Divide and Conquer

Idea :

Write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

[where A through H are all $n/2$ by $n/2$ matrices]

Then : (you check)

$$X \cdot Y = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}$$

Recursive Algorithm #1

Step 1 : recursively compute the 8 necessary products.

Step 2 : do the necessary additions ($\theta(n^2)$ *time*)

Fact : runtime is $\theta(n^3)$ [follows from the master method]

Strassen's Algorithm (1969)

Step 1 : recursively compute only 7 (cleverly chosen) products

Step 2 : do the necessary (clever) additions + subtractions
(still $\theta(n^2)$ time)

Fact : better than cubic time!

[see Master Method lecture]

$$7(1/4 n + n^2) + (12-1)*n^2 = O(n^2)$$

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

The Details

$$Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

The Seven Products : $P_1 = A(F-H)$, $P_2 = (A+B)H$,
 $P_3 = (C+D)E$, $P_4 = D(G-E)$, $P_5 = (A+D)(E+H)$,
 $P_6 = (B-D)(G+H)$, $P_7 = (A-C)(E+F)$

Claim : $X \cdot Y = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix} = \begin{pmatrix} P_5+P_4-P_2+P_6 & P_1+P_2 \\ P_3+P_4 & P_1+P_5-P_3-P_7 \end{pmatrix}$

Proof: $AE + \cancel{AH} + \cancel{DE} + \cancel{DH} + \cancel{DG} - \cancel{DE} - \cancel{AH} - \cancel{BH}$
 $+ \cancel{BG} + \cancel{BH} - \cancel{DG} - \cancel{DH} = AE + BG$

Q.E.D

Question : where did this come from ? (remains open!)