# MLT: Week 8

## GENERATIVE MODELS - NAIVE BAYES

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# Naive Bayes Algorithm

# Naive Bayes Algorithm

Given a dataset $\{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ where $\mathrm{x}_i \in \{0,1\}^d$,
let $\{y_1, \ldots, y_n\}$ be the labels, where $y_i \in \{0,1\}$.

$$\mathrm{X} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathrm{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

The Naive Bayes Algorithm is characterized by two main things:

1. Class Conditional Independence Assumption
2. Bayes Rule

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Based on the first characteristic, we get $2d+1$ parameters where $d$ is the number of dimensions of the features set.
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$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

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Using MLE, the estimates of the parameters are given by,

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\widehat{p}_j^{\,y} = \frac{\sum_{i=1}^{n} \mathbb{I}(f_j^i = 1, y_i = y)}{\sum_{i=1}^{n} \mathbb{I}(y_i = y)} \quad \forall j \in \{1, 2, \ldots, d\} \quad \forall y \in \{0, 1\}$$

Estimating the GMM parameters for our dataset would result in parameters whose values are zero. Therefore, we need to apply **Laplace Smoothing**.
After applying it on our dataset, we get,

$$X = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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In this new dataset, parameters don't end up as zero probability estimates.

Calculating $\widehat{p}$ using MLE for our dataset, we get,

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{8}(1+1+0+0+1+0+1+0) = \frac{4}{8} = 0.5$$

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In this new dataset, parameters don't end up as zero probability estimates.
Calculating $\widehat{p}$ using MLE for our dataset, we get,

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{8}(1+1+0+0+1+0+1+0) = \frac{4}{8} = 0.5$$

$$\therefore \widehat{p} = 0.5$$

Calculating $\widehat{p}_j^y$ using MLE for our dataset, we get,

$$\widehat{p}_j^y = \frac{\sum_{i=1}^{n} \mathbb{I}(f_j^i = 1, y_i = y)}{\sum_{i=1}^{n} \mathbb{I}(y_i = y)}$$

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$$\widehat{p}_2^1 = \frac{1+1}{4} = 0.50 \qquad \widehat{p}_2^0 = \frac{1+1}{4} = 0.50$$

$$\widehat{p}_3^1 = \frac{1}{4} = 0.25 \qquad \widehat{p}_3^0 = \frac{1+1}{4} = 0.50$$

## Prediction using Naive Bayes

If

$$\left(\prod_{i=1}^{d}\left(\widehat{p}_i^1\right)^{f_i}\left(1-\widehat{p}_i^1\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{d}\left(\widehat{p}_i^0\right)^{f_i}\left(1-\widehat{p}_i^0\right)^{1-f_i}\right)(1-\widehat{p})$$

we predict $y = 1$, otherwise $y = 0$.

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we predict $y=1$, otherwise $y=0$.

Predicting $y_1$ for our dataset,

$$y_1 = \mathbb{I}\left(\left(\prod_{i=1}^{3}\left(\widehat{p}_i^{1}\right)^{f_i}\left(1-\widehat{p}_i^{1}\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{3}\left(\widehat{p}_i^{0}\right)^{f_i}\left(1-\widehat{p}_i^{0}\right)^{1-f_i}\right)(1-\widehat{p})\right)$$

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Predicting $y_1$ for our dataset,

$$\begin{aligned} y_1 &= \mathbb{I}\left(\left(\prod_{i=1}^{3}\left(\widehat{p}_i^{\,1}\right)^{f_i}\left(1-\widehat{p}_i^{\,1}\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{3}\left(\widehat{p}_i^{\,0}\right)^{f_i}\left(1-\widehat{p}_i^{\,0}\right)^{1-f_i}\right)(1-\widehat{p})\right) \\ &= \mathbb{I}\Big(\big((0.75)^1(0.75)^0(0.50)^0(0.50)^1(0.25)^0(0.75)^1\big)0.5 \geq \big((0.25)^1(0.75)^0(0.50)^0(0.50)^1(0.50)^0(0.50)^1\big)0.5\Big) \end{aligned}$$

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we predict $y=1$, otherwise $y=0$.

Predicting $y_1$ for our dataset,

$$\begin{aligned}
y_1 &= \mathbb{I}\left(\left(\prod_{i=1}^{3}\left(\widehat{p}_i^{1}\right)^{f_i}\left(1-\widehat{p}_i^{1}\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{3}\left(\widehat{p}_i^{0}\right)^{f_i}\left(1-\widehat{p}_i^{0}\right)^{1-f_i}\right)(1-\widehat{p})\right) \\
&= \mathbb{I}\Big(\big((0.75)^1(0.75)^0(0.50)^0(0.50)^1(0.25)^0(0.75)^1\big)0.5 \geq \big((0.25)^1(0.75)^0(0.50)^0(0.50)^1(0.50)^0(0.50)^1\big)0.5\Big) \\
&= \mathbb{I}(0.141 \geq 0.031)
\end{aligned}$$

**Prediction using Naive Bayes**

If

$$\left(\prod_{i=1}^{d}\left(\widehat{p}_i^{1}\right)^{f_i}\left(1-\widehat{p}_i^{1}\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{d}\left(\widehat{p}_i^{0}\right)^{f_i}\left(1-\widehat{p}_i^{0}\right)^{1-f_i}\right)(1-\widehat{p})$$

we predict $y=1$, otherwise $y=0$.

Predicting $y_1$ for our dataset,

$$\begin{aligned}
y_1 &= \mathbb{I}\left(\left(\prod_{i=1}^{3}\left(\widehat{p}_i^{1}\right)^{f_i}\left(1-\widehat{p}_i^{1}\right)^{1-f_i}\right)\widehat{p} \geq \left(\prod_{i=1}^{3}\left(\widehat{p}_i^{0}\right)^{f_i}\left(1-\widehat{p}_i^{0}\right)^{1-f_i}\right)(1-\widehat{p})\right) \\
&= \mathbb{I}\Big(\big((0.75)^1(0.75)^0(0.50)^0(0.50)^1(0.25)^0(0.75)^1\big)0.5 \geq \big((0.25)^1(0.75)^0(0.50)^0(0.50)^1(0.50)^0(0.50)^1\big)0.5\Big) \\
&= \mathbb{I}(0.141 \geq 0.031) \\
y_1 &= 1
\end{aligned}$$

Prediction table for all datapoints:

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| Label | $P(\widehat{y}=1\|\widehat{x})$ | $P(\widehat{y}=0\|\widehat{x})$ | Prediction | Actual |
|---|---|---|---|---|
| $y_1$ | $0.141$ | $0.031$ | 1 | 1 |
| $y_2$ | $0.141$ | $0.031$ | 1 | 1 |
| $y_3$ | $0.047$ | $0.094$ | 0 | 0 |
| $y_4$ | $0.016$ | $0.094$ | 0 | 0 |
| $y_5$ | $0.047$ | $0.031$ | 1 | 1 |
| $y_6$ | $0.047$ | $0.031$ | 1 | 0 |
| $y_7$ | $0.047$ | $0.094$ | 0 | 1 |
| $y_8$ | $0.047$ | $0.094$ | 0 | 0 |

# Gaussian Naive Bayes Algorithm

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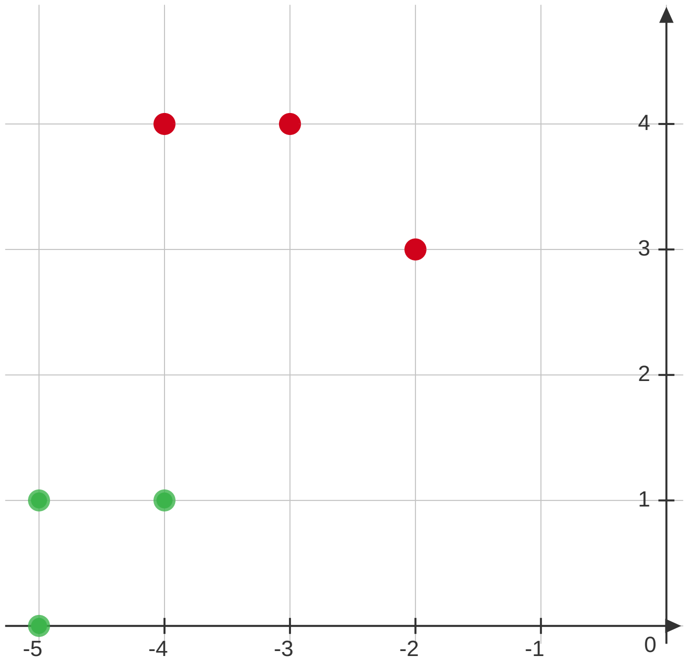
Given a dataset $\{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ where $\mathrm{x}_i \in \mathbb{R}^d$,
let $\{y_1, \ldots, y_n\}$ be the labels, where $y_i \in \{0, 1\}$.

$$\mathrm{X} = \begin{bmatrix} -3 & -4 & -2 & -4 & -5 & -5 \\ 4 & 4 & 3 & 1 & 1 & 0 \end{bmatrix} \quad \mathrm{y} \ = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Gaussian Naive Bayes Algorithm

Given a dataset $\{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$ where $\mathrm{x}_i \in \mathbb{R}^d$, let $\{y_1, \ldots, y_n\}$ be the labels, where $y_i \in \{0, 1\}$.

$$\mathrm{X} = \begin{bmatrix} -3 & -4 & -2 & -4 & -5 & -5 \\ 4 & 4 & 3 & 1 & 1 & 0 \end{bmatrix} \quad \mathrm{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Let $P(x|y=1) \sim \mathcal{N}(\mu_1, \Sigma_1)$ and $P(x|y=0) \sim \mathcal{N}(\mu_0, \Sigma_0)$.

The parameters to be estimated are $\widehat{p}$, $\widehat{\mu}_0$, $\widehat{\mu}_1$, and $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$.

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Using Maximum Likelihood Estimation, we get the following results:

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Using Maximum Likelihood Estimation, we get the following results:

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

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The parameters to be estimated are $\widehat{p}$, $\widehat{\mu}_0$, $\widehat{\mu}_1$, and $\widehat{\Sigma}_0$ and $\widehat{\Sigma}_1$.

Using Maximum Likelihood Estimation, we get the following results:

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\widehat{\mu}_1 = \frac{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = 1)x_i}{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = 1)}$$

$$\widehat{\mu}_0 = \frac{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = 0)x_i}{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = 0)}$$

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

Where $\widehat{p}$ is the proportion of data points labeled $1$, $\widehat{\mu}_1$ is the sample mean of data points labeled $1$, $\widehat{\mu}_0$ is the sample mean of data points labeled $0$, and $\widehat{\Sigma}_1$ and $\widehat{\Sigma}_0$ are the covariance matrices for classes $1$ and $0$ respectively.

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

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Calculating $\widehat{p}$ using MLE for our dataset, we get,

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{6}(1+1+1+0+0+0) = \frac{3}{6} = 0.5$$

$$\therefore \widehat{p} = 0.5$$

Calculating $\widehat{\mu}_k$ using MLE for our dataset, we get,

$$\widehat{\mu}_k = \frac{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = k) x_i}{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

Calculating $\widehat{\mu}_k$ using MLE for our dataset, we get,

$$\widehat{\mu}_k = \frac{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = k)x_i}{\displaystyle\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

$$\widehat{\mu}_1 = \frac{\begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}}{3} = \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}$$

$$\widehat{\mu}_0 = \frac{\begin{bmatrix} -4 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \end{bmatrix}}{3} = \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}$$

Calculating $\widehat{\Sigma}_k$ using MLE for our dataset, we get,

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

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$$\widehat{\Sigma}_1 = \frac{\left(\begin{bmatrix}-3\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-3\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-4\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-4\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-2\\3\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-2\\3\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T}{3}$$

Calculating $\widehat{\Sigma}_k$ using MLE for our dataset, we get,

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

$$\widehat{\Sigma}_1 = \frac{\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)^T + \left(\begin{bmatrix} -4 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)\left(\begin{bmatrix} -4 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)^T + \left(\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} -3 \\ 3.666 \end{bmatrix}\right)^T}{3}$$

$$\widehat{\Sigma}_1 = \begin{bmatrix} 0.666 & 0.333 \\ -0.333 & 0.222 \end{bmatrix}$$

Calculating $\widehat{\Sigma}_k$ using MLE for our dataset, we get,

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

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Calculating $\widehat{\Sigma}_k$ using MLE for our dataset, we get,

$$\widehat{\Sigma}_k = \frac{\sum_{i=1}^{n} \mathbb{I}(y_i = k)\Big((x_i - \widehat{\mu}_k)(x_i - \widehat{\mu}_k)^T\Big)}{\sum_{i=1}^{n} \mathbb{I}(y_i = k)}$$

$$\widehat{\Sigma}_1 = \frac{\left(\begin{bmatrix}-3\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-3\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-4\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-4\\4\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-2\\3\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)\left(\begin{bmatrix}-2\\3\end{bmatrix} - \begin{bmatrix}-3\\3.666\end{bmatrix}\right)^T}{3}$$

$$\widehat{\Sigma}_1 = \begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}$$

$$\widehat{\Sigma}_0 = \frac{\left(\begin{bmatrix}-4\\1\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)\left(\begin{bmatrix}-4\\1\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-5\\1\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)\left(\begin{bmatrix}-5\\1\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)^T + \left(\begin{bmatrix}-5\\0\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)\left(\begin{bmatrix}-5\\0\end{bmatrix} - \begin{bmatrix}-4.66\\0.666\end{bmatrix}\right)^T}{3}$$

$$\widehat{\Sigma}_0 = \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}$$

# Prediction using Gaussian Naive Bayes

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Predict $y_i = 1$ if:

$$f(x_i; \widehat{\mu}_1, \widehat{\Sigma}_1)\widehat{p} \geq f(x_i; \widehat{\mu}_0, \widehat{\Sigma}_0)(1 - \widehat{p})$$

**Prediction using Gaussian Naive Bayes**

Predict $y_i = 1$ if:

$$f(x_i; \widehat{\mu}_1, \widehat{\Sigma}_1)\widehat{p} \geq f(x_i; \widehat{\mu}_0, \widehat{\Sigma}_0)(1 - \widehat{p})$$

$$e^{-(x_i - \widehat{\mu}_1)^T \widehat{\Sigma}_1^{-1} (x_i - \widehat{\mu}_1)} \widehat{p} \geq e^{-(x_i - \widehat{\mu}_0)^T \widehat{\Sigma}_0^{-1} (x_i - \widehat{\mu}_0)} (1 - \widehat{p})$$

## Prediction using Gaussian Naive Bayes

Predict $y_i = 1$ if:

$$f(x_i; \widehat{\mu}_1, \widehat{\Sigma}_1)\widehat{p} \geq f(x_i; \widehat{\mu}_0, \widehat{\Sigma}_0)(1 - \widehat{p})$$

$$e^{-(x_i - \widehat{\mu}_1)^T \widehat{\Sigma}_1^{-1} (x_i - \widehat{\mu}_1)} \widehat{p} \geq e^{-(x_i - \widehat{\mu}_0)^T \widehat{\Sigma}_0^{-1} (x_i - \widehat{\mu}_0)} (1 - \widehat{p})$$

$$-(x_i - \widehat{\mu}_1)^T \widehat{\Sigma}_1^{-1} (x_i - \widehat{\mu}_1) + \log(\widehat{p}) \geq -(x_i - \widehat{\mu}_0)^T \widehat{\Sigma}_0^{-1} (x_i - \widehat{\mu}_0) + \log(1 - \widehat{p})$$

## Prediction using Gaussian Naive Bayes

Predict $y_i = 1$ if:

$$f(x_i; \widehat{\mu}_1, \widehat{\Sigma}_1)\widehat{p} \geq f(x_i; \widehat{\mu}_0, \widehat{\Sigma}_0)(1-\widehat{p})$$

$$e^{-(x_i-\widehat{\mu}_1)^T\widehat{\Sigma}_1^{-1}(x_i-\widehat{\mu}_1)}\widehat{p} \geq e^{-(x_i-\widehat{\mu}_0)^T\widehat{\Sigma}_0^{-1}(x_i-\widehat{\mu}_0)}(1-\widehat{p})$$

$$-(x_i-\widehat{\mu}_1)^T\widehat{\Sigma}_1^{-1}(x_i-\widehat{\mu}_1) + \log(\widehat{p}) \geq -(x_i-\widehat{\mu}_0)^T\widehat{\Sigma}_0^{-1}(x_i-\widehat{\mu}_0) + \log(1-\widehat{p})$$

$$x_i^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_i - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_i + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0$$

# Prediction using Gaussian Naive Bayes

Predict $y_i = 1$ if:

$$f(x_i; \widehat{\mu}_1, \widehat{\Sigma}_1)\widehat{p} \geq f(x_i; \widehat{\mu}_0, \widehat{\Sigma}_0)(1-\widehat{p})$$

$$e^{-(x_i-\widehat{\mu}_1)^T \widehat{\Sigma}_1^{-1} (x_i-\widehat{\mu}_1)}\widehat{p} \geq e^{-(x_i-\widehat{\mu}_0)^T \widehat{\Sigma}_0^{-1} (x_i-\widehat{\mu}_0)}(1-\widehat{p})$$

$$-(x_i-\widehat{\mu}_1)^T \widehat{\Sigma}_1^{-1} (x_i-\widehat{\mu}_1) + \log(\widehat{p}) \geq -(x_i-\widehat{\mu}_0)^T \widehat{\Sigma}_0^{-1} (x_i-\widehat{\mu}_0) + \log(1-\widehat{p})$$

$$x_i^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_i - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_i + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0$$

Hence, we can say that the decision function is of the form $x^TQx - 2b^Tx + c \geq 0$ where $Q = \widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}$, $b = \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}$, and $c = (\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1) + log(\frac{1-\widehat{p}}{\widehat{p}})$.

Predicting $y_1$ for our dataset,

$$y_1 = \mathbb{I}\left(x_1^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_1 - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_1 + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0\right)$$

Predicting $y_1$ for our dataset,

$$y_1 = \mathbb{I}\left(x_1^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_1 - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_1 + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0\right)$$

$$= \mathbb{I}(\begin{bmatrix} -3 \\ 4 \end{bmatrix}^T \left(\begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix} - 2\left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T \begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$+\left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 3.666 \end{bmatrix} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T \begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}\right) + log\left(\frac{1-0.5}{0.5}\right) \geq 0)$$

Predicting $y_1$ for our dataset,

$$y_1 = \mathbb{I}\left(x_1^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_1 - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_1 + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0\right)$$

$$= \mathbb{I}(\begin{bmatrix} -3 \\ 4 \end{bmatrix}^T\left(\begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix} - 2\left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T\begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T\begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$+\left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T\begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\begin{bmatrix} -3 \\ 3.666 \end{bmatrix} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T\begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1}\begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}\right) + log\left(\frac{1-0.5}{0.5}\right) \geq 0)$$

$$= \mathbb{I}(1.0 \geq 0)$$

Predicting $y_1$ for our dataset,

$$y_1 = \mathbb{I}\left(x_1^T\left(\widehat{\Sigma}_1^{-1} - \widehat{\Sigma}_0^{-1}\right)x_1 - 2\left(\widehat{\mu}_1^T\widehat{\Sigma}_1^{-1} - \widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\right)x_1 + \left(\widehat{\mu}_0^T\widehat{\Sigma}_0^{-1}\widehat{\mu}_0 - \widehat{\mu}_1^T\widehat{\Sigma}_1^{-1}\widehat{\mu}_1\right) + log\left(\frac{1-\widehat{p}}{\widehat{p}}\right) \geq 0\right)$$

$$= \mathbb{I}(\begin{bmatrix} -3 \\ 4 \end{bmatrix}^T \left(\begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix} - 2\left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T \begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1}\right)\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$+ \left(\begin{bmatrix} -3 \\ 3.666 \end{bmatrix}^T \begin{bmatrix} 0.222 & 0.111 \\ 0.111 & 0.222 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 3.666 \end{bmatrix} - \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}^T \begin{bmatrix} 0.666 & -0.333 \\ -0.333 & 0.222 \end{bmatrix}^{-1} \begin{bmatrix} -4.66 \\ 0.666 \end{bmatrix}\right) + log\left(\frac{1-0.5}{0.5}\right) \geq 0)$$

$$= \mathbb{I}(1.0 \geq 0)$$

$$y_1 = 1$$

Prediction table for all datapoints:

Prediction table for all datapoints:

| Label | $P(\widehat{y}=1\|\widehat{x})$ | $P(\widehat{y}=0\|\widehat{x})$ | Prediction | Actual |
|---|---|---|---|---|
| $y_1$ | $1.0$ | $0$ | 1 | 1 |
| $y_2$ | $1.0$ | $0$ | 1 | 1 |
| $y_3$ | $1.0$ | $0$ | 1 | 1 |
| $y_4$ | $0$ | $1.0$ | 0 | 0 |
| $y_5$ | $0$ | $1.0$ | 0 | 0 |
| $y_6$ | $0.00019$ | $0.99981$ | 0 | 0 |

# Gaussian Naive Bayes Algorithm