Thermodynamics of the Quark-Gluon-Plasma

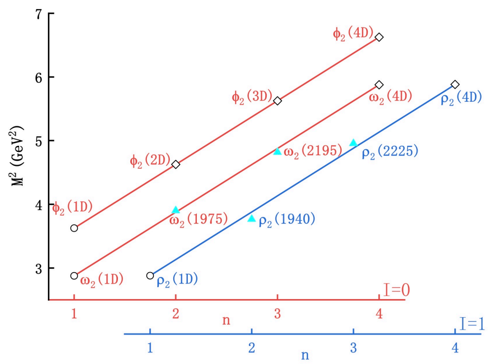
Intermezzo: Hagedorn temperature

Context: Masses of resonances of hadrons – which have the same quantum numbers – are related linearly in m_H^2, namely

$$\alpha' m_J^2 - \alpha_0 = J$$

Inspired by the phenomenological success of Regge theory, Hagedorn observed that such a scaling meant self-similarity in the spectral function of the internal dynamics of hadronic physics (later QCD). The spectral function, $\rho(M)$, is to be here understood as the mass distribution of hadronic states. He coined his famous phrase:

"Resonances are made of resonances, are made of resonances, are made of resonances....."

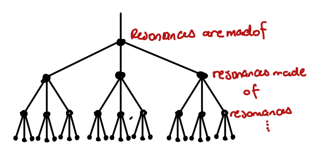


Regge trajectories

Boot-strap model: Assumes a self-similar master equation for $\rho(m, V_0)$ where he assumes a dependence in the volume:

$$\rho(m, V_0) = \delta(m - m_0) + \sum_N \left[\frac{V_0}{(2\pi)^3}\right]^{N-1} \int \prod_{i=1}^{N} \left[dm_i\, \rho(m_i) d^3p_i\right] \delta^{(4)}(\Sigma p_i - p)$$

mass of lowest resonance (π)



Assuming $N \to \infty$, which assumes this self-similarity keeps being valid forever (assumption!) we can solve this eq.

$$\rho(m, V_0) \sim m^{-3} e^{m/T_H} \qquad \text{with} \qquad T_H: \quad (m_0 r_0)^3 \left(\frac{2}{3\pi}\right)\left(\frac{T_H}{m_0}\right) K_2\left(\frac{m_0}{T_H}\right) = 2\ln 2 - 1.$$

The interesting thing is to check out the thermodynamics of the system. Namely calculate the

$$F = -T\ln Z(T, V) = \frac{VT}{(2\pi)^3}\int_m^\infty dm\, \rho(m) \int d^3p \exp\left[\frac{\sqrt{p^2+m^2}}{T}\right] = \frac{VT^2}{2\pi^2}\int_{m_0}^\infty dm\, \rho(m)\, m^2 K_2\left(\frac{m}{T}\right)$$

where F is the free energy, and Z is the partition function. Using $\rho = \rho_0 m^{-a} e^{m/T_H}$.

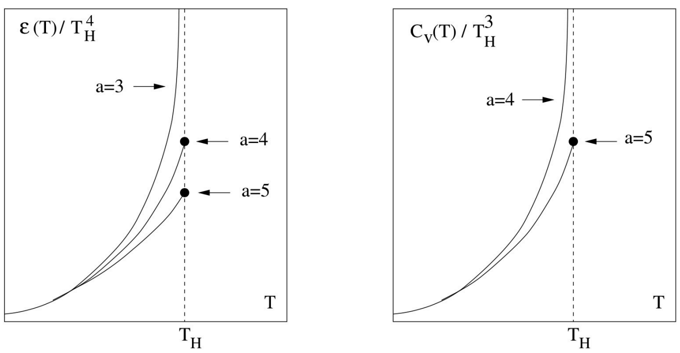
$$F = \frac{VT^2}{2\pi^2}\int_{m_0}^\infty dm\, \rho(m)\, m^2 K_2\left(\frac{m}{T}\right) \xrightarrow{m/T \gg 1} \frac{VT^2}{2\pi^2}\int_{m_0}^\infty dm\, m^{3/2-a} \exp\left\{m\left(T_H^{-1} - T\right)\right\}$$

$\Rightarrow$ Free energy diverges for $T > T_H$!

We can also find the energy and the heat capacity

$$\epsilon(T) = \frac{VT^{7/2}}{2\pi^2} \int_{m_0}^{\infty} dm \; m^{5/2-a} \; \exp\left\{m\left(T_H^{-1} - T\right)\right\}$$

$$C_v(T) = \frac{VT^{3/2}}{2\pi^2} \int_{m_0}^{\infty} dm \; m^{7/2-a} \; \exp\left\{m\left(T_H^{-1} - T\right)\right\}$$



Energy density $\epsilon(T)$ and specific heat $C_v(T)$ of a Hagedorn-type resonance gas

Thermodynamics of the Quark-Gluon-Plasma: Kinetic Medium

As we saw in the first lecture, at high energy transfers, the coupling of quarks and gluons, α_s, becomes vanishingly small. It will be then interesting to explore the properties of a collection of free quarks and gluons.

In such a context, one can characterize the system by counting the distribution of states using the so-called distribution of modes,

$$dN = h^d f(x,p)\, d^dx\, d^dp \qquad\qquad f(x,p) \equiv \frac{dN}{d^3x\, d^3p}$$

In this context, f counts the distribution of particle states per differential phase-space volume, $\Delta^3x\,\Delta^3p$. The integration of different parts of the phase space give out different quantities, for example $\longrightarrow \int d^3x\, f(x,p) = dN/d^3p \propto$ yield

$\longrightarrow \int d^3p\, f(x,p) \equiv \frac{dN}{d^3x} \equiv n$: the „number density".

The distribution function is a Lorentz invariant quantity, thanks to the simultaneous changes of d^3x and d^3p —themselves not invariant— which cancel out on the full distribution.

Moments of the distribution: Given a distribution $f = f(\vec{x})$ for d-dimensional collection of variables $\vec{x}$, belonging to a domain Ω, one can define the nth moment of such a distribution as follows:

$$K^{i_1 \ldots i_n} = \int_\Omega d^dx\; x^{i_1} \ldots x^{i_n} f(\vec{x})$$

This process is -in principle- invertible if one would know every single moment of the distribution

$\Rightarrow$ Important moments: We can define two sets of physically relevant quantities

1) The Energy-stress tensor $\quad T^{\mu\nu}_{(x)} = \int \frac{d^dp}{(2\pi)^d} \frac{p^\mu p^\nu}{p^0} f(x,p)$

$\hookrightarrow$ Ideal fluid $\to$ no friction/viscosity, no diffusion $\quad \Rightarrow \quad T^{\mu\nu} = (\underbrace{\epsilon}_{\text{energy density}} + \underbrace{p}_{\text{pressure (isotropic)}}) u^\mu u^\nu - \eta^{\mu\nu} p$

Velocity field $u^\mu = u^\mu(x)$ of the fluid.

From this, notice $\quad \epsilon \equiv u_\mu T^{\mu\nu} u_\nu \quad$ and defining the projector $\Delta^{\mu\nu} \equiv \eta^{\mu\nu} - u^\mu u^\nu$

$$\Rightarrow \quad \Delta_{\mu\nu} T^{\mu\nu} = \eta_{\mu\nu} T^{\mu\nu} - u_\mu T^{\mu\nu} u_\nu = (\epsilon + p)\cancel{x}^{\,1} - d\,p - \epsilon \quad \Longrightarrow \quad p = -\frac{1}{(d-1)}\, \Delta : T$$

• Keep in mind $\quad \eta_{\mu\nu} T^{\mu\nu} = \mathrm{tr}[T] = \epsilon - (d-1)p \quad$ is called the trace anomaly (as it is not usually 0)

2) The particle current $\quad j^\mu_{(x)} = \int \frac{d^dp}{(2\pi)^d} \cdot \frac{p^\mu}{p^0} f(x,p)$

We can deconstruct the structure of the current in the ideal case as $\quad j^\mu = n u^\mu \quad \Longrightarrow \quad \boxed{u \cdot j \equiv n}$

$\longrightarrow$ notice that if we are at the rest-frame of the fluid, $u^\mu = (1,0,0,0)$, then we can get

$$j^0 \equiv n, \text{ the particle density}$$
$$\epsilon = T^{00}, \text{ the energy density.}$$

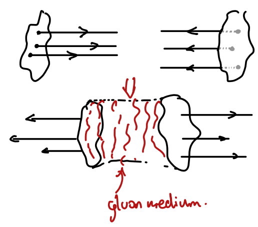
Equilibrium distributions

In a thermally equilibrated system with temperature T, the average occupation number is given by

$$f(x,p) = \begin{cases} z\, e^{-\beta E(p)} & : \text{classical particles} \\ \dfrac{1}{z^{-1} e^{+\beta E(p)} \pm 1} & : \text{quantum DoF.} \quad \begin{matrix} + \to \text{fermions} \\ - \to \text{bosons} \end{matrix} \end{cases}$$

Additionally, $\beta \equiv 1/T$ while the z factor is called fugacity and it is given by $z \equiv e^{\beta\mu}$. The chemical potential μ_i is the necessary energy needed to add a particle of species i to the system. Under general conditions $\mu = 0$ for massless, neutral particles. For charged particles, $\mu \neq 0$ controls the particle-antiparticle asymmetry, that is the total charge of the system.

⊛ At very high collision energies, all baryon number carriers (valence quarks) carry on with full speed, and the deposited energy around $y=0$ is depleted from any quarks. The „midrapidity" region becomes roughly neutral $\Rightarrow \mu_q \approx 0$.
At lower energies, Baryons are more effectively stopped, producing a region with a non-vanishing (proton/neutron) density $\Rightarrow \mu_B \sim \mu_q > 0$



Degeneracy (ν_i): Number of equivalent degrees of freedom in a system. Example: In an isotropic, unmagnetized system, spin states are energetically equivalent. In the QGP

Gluons: $\nu_g = \nu_{\text{polarizations}} \cdot \nu_{\text{color states}}$
$= 2 \cdot (\underset{\uparrow}{N_c^2} - 1) = 2 \cdot 8$
colours = 3

Quarks: $\nu_q = \nu_{\text{quarks}} + \nu_{\text{anti-quarks}} \equiv 2\,\nu_{\text{quarks}}$
$= 2 \cdot \nu_{\text{spin}} \cdot N_c \cdot \underset{\uparrow}{N_f} \equiv 12 \cdot N_f$
# of equivalent flavours.

All quarks are non-degenerate, since they all possess different masses. However in the large-T limit one may count some of those masses as equal since $m/T \to 0$. In such a case $N_f = 2\ (u,d)$ or $N_f = 3\ (u,d,s)$ depending on the temperature.

Gas of free quarks and gluons

First let's go to the fluid restframe, $u^\mu = (1,0,0,0)$, for simplicity. We can easily compute the relevant quantities. These are the energy densities $(\epsilon_q, \epsilon_{\bar q}, \epsilon_g)$, the number densities, $(n_q, n_{\bar q}, n_g)$, and finally the pressures $(P_q, P_{\bar q}, P_g)$.

First let's take a look at gluons...

$$n_g = \nu_g \int \frac{d^3p}{(2\pi)^3} \cdot n_B(p) = \nu_g \cdot \int \frac{dp\, p^2}{(2\pi)^3} \underbrace{\int d\Omega}_{4\pi}\, n_B(p) = \frac{\nu_g}{2\pi^2} \int_0^\infty dp \frac{p^2}{e^{\beta p} - 1} = \frac{\nu_g T^3}{2\pi^2} \int_0^\infty \frac{dx\, x^2}{e^x - 1}$$

Using a bit of algebra, one can find that $n_g = \dfrac{\nu_g\, \zeta(3) T^3}{\pi^2} \xrightarrow[T \equiv 0.3\,\text{GeV},\ \nu_g = 16]{\zeta(3) \sim 1.202} \sim 6.9\ \text{fm}^{-3}$

Now let's look at the energy

$$\epsilon_g = \nu_g \int \frac{d^3p}{(2\pi)^3} p_0 f_{eq}(x,p) = \frac{\nu_g T^4}{2\pi^2} \int_0^\infty \frac{dx\, x^3}{e^x - 1} = \nu_g \frac{\pi^2 T^4}{30} \xrightarrow[T \equiv 0.3\,\text{GeV}]{\nu_g = 16} 5{,}57 \frac{\text{GeV}}{\text{fm}^3}$$

Finally the pressure.

$$P_g \equiv -\frac{1}{3}(\Delta : T) = -\frac{1}{3}\left[\overset{-\delta_{ij}}{\cancel{\eta_{\mu\nu} - u_\mu u_\nu}}\right] T^{\mu\nu} = \frac{\nu_g}{3}\int \frac{d^3p}{(2\pi)^3} \cdot \frac{p^{\cancel{2}}}{\cancel{p^0}} n_B(p) = \frac{\nu_g}{3}\int \frac{d^3p}{(2\pi)} \overset{\sim p_0 \text{ if } m=0}{|p|}\, n_F(p) = \frac{1}{3}\epsilon_g$$

For which the trace of $T^{\mu\nu}$ vanishes : $\eta_{\mu\nu} T^{\mu\nu} = 0$

Now let's do the quarks. For simplicity we will only consider the light quarks and use the $m/T \to 0$ limit. We will nevertheless keep the chemical potential $\mu_q \neq 0$. However, it is important to note that charge conservation requires that $\mu_q + \mu_{\bar{q}} = 0$. We will now have to differentiate for (anti) quarks. Thankfully, we need to compute only for quarks and then use $\mu_q \to \mu_{\bar{q}} = -\mu_q$

Quark density: $$n_q = \frac{\nu_q}{2\pi^2} \cdot \int_0^\infty dp \frac{p^2}{e^{\beta(p-\mu)} + 1} = \frac{\nu_q T^3}{2\pi^2}\left[-2\, Li_3(-e^{\beta\mu})\right] \sim \frac{\nu_q T^3}{2\pi^2}\left[\frac{3}{2}\zeta(3) + \frac{\pi^2}{6}\frac{\mu}{T} + \log(2)\frac{\mu^2}{T^2} + \frac{\mu^3}{6T^3} + O\left(\frac{\mu^4}{T^4}\right)\right]$$

Quark energy $$\epsilon_q = \frac{\nu_q}{2\pi^2}\int_0^\infty dp \frac{p^3}{e^{\beta(p-\mu)} + 1} = \frac{\nu_q T^4}{2\pi^2}\left[-6\, Li_4(-e^{\beta\mu})\right] \sim \frac{\nu_q T^4}{2\pi^2}\left[\frac{7\pi^4}{120} + \frac{9}{2}\zeta(3)\frac{\mu}{T} + \frac{\pi^2}{4}\frac{\mu^2}{T} + \log(2)\frac{\mu^3}{T^3} + O(\mu^4/T^4)\right]$$

Quark pressure $$P_q = \frac{1}{3}\epsilon_q$$

Notice 1) The relationship between ϵ and p is called the EQUATION OF STATE (EoS)

2) that the inclusion of a mass would create a deviation from the $\epsilon = 3P$ EoS

3) Odd terms of μ cancel if one sums $q + \bar{q}$ contributions!

A general - free - QGP (up to $O(\mu^3)$)

$$n_{QGP} = n_g + n_q + n_{\bar{q}} = \frac{\nu_g \zeta(3) T^3}{\pi^2} + \frac{\nu_q T^3}{2\pi^2}\left[\frac{3}{2}\zeta(3) \cancel{+ \frac{\pi^2}{6}\frac{\mu}{T}} + \log(2)\frac{\mu^2}{T^2}\right] + \frac{\nu_{\bar{q}} T^3}{2\pi^2}\left[\frac{3}{2}\zeta(3) \cancel{- \frac{\pi^2}{6}\frac{\mu}{T}} + \log(2)\frac{\mu^2}{T^2}\right]$$

$$= T^3\left\{\frac{\nu_g \zeta(3)}{\pi^2} + \frac{\overbrace{(\nu_q + \nu_{\bar{q}})}^{2\nu_q}}{2\pi^2}\left[\frac{3}{2}\zeta(3) + \frac{\mu^2}{T^2}\log(2)\right]\right\}$$

$\nu_q = \nu_{\bar{q}}$ but we separate them because $\mu_q = -\mu_{\bar{q}}$!!

$$\epsilon_{QGP} \equiv \epsilon_g + \epsilon_q + \epsilon_{\bar{q}} = \nu_g \frac{\pi^2 T^4}{30} + \frac{\nu_q T^4}{2\pi^2}\left[\frac{7\pi^4}{120} \cancel{+ \frac{9}{2}\zeta(3)\frac{\mu}{T}} + \frac{\pi^2}{4}\frac{\mu^2}{T}\right] + \frac{\nu_q T^4}{2\pi^2}\left[\frac{7\pi^4}{120} \cancel{- \frac{9}{2}\zeta(3)\frac{\mu}{T}} + \frac{\pi^2}{4}\frac{\mu^2}{T}\right]$$

$$\mathcal{E}_{QGP} = T^4 \left\{ V_g \frac{\pi^2}{30} + \frac{(V_q + V_{\bar q})}{2\pi^2}\left[\frac{7\pi^4}{120} + \frac{\pi^2 \mu^2}{4\,T}\right]\right\} \overset{\mu=0}{=} \frac{\pi^2 T^4}{30} \underbrace{\left\{ V_g + \frac{7}{8}\cdot(V_q + V_{\bar q})\right\}}_{\equiv V_{QGP}}$$

$$P_{QGP} = P_g + P_q + P_{\bar q} = \frac{1}{3}\left[\mathcal{E}_g + \mathcal{E}_q + \mathcal{E}_{\bar q}\right] = \frac{1}{3}\mathcal{E}_{QGP}$$

On the other hand let's take a look at the net charge

$$n_{q-\bar q} = n_q - n_{\bar q} = \frac{V_q T^3}{2\pi^2}\left[\cancel{\frac{3}{2}\zeta(3)} + \frac{\pi^2}{6}\frac{\mu}{T} + \cancel{\log(2)\frac{\mu^2}{T^2}} + \frac{\mu^3}{6T^3}\right] - \frac{V_{\bar q} T^3}{2\pi^2}\left[\cancel{\frac{3}{2}\zeta(3)} - \frac{\pi^2}{6}\frac{\mu}{T} + \cancel{\log(2)\frac{\mu^2}{T^2}} - \frac{\mu^3}{6T^3}\right]$$

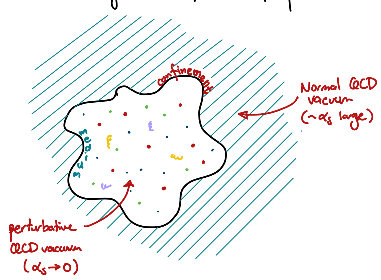
$$= \frac{V_q T^3}{2\pi^2}\left[\frac{\pi^2}{6}\frac{\mu}{T} + \frac{\mu^3}{6T^3}\right] + \frac{V_{\bar q} T^3}{2\pi^2}\left[\frac{\pi^2}{6}\frac{\mu}{T} + \frac{\mu^3}{6T^3}\right]$$

$$= \frac{V_q T^3}{\pi^2}\left\{\frac{\pi^2}{6}\frac{\mu}{T} + \frac{\mu^3}{6T^3}\right\}$$

$$= \frac{V_q T^2 \mu}{6\pi^2}\left\{\pi^2 + \frac{\mu^2}{T^2}\right\} \xrightarrow[N_f = 3,\ T \equiv 0.3\,\text{GeV},\ \mu = 0{,}1\,\text{GeV}]{V_g = 16,\ V_q = V_{\bar q} = 6 \cdot N_f} 1{,}2\ \frac{N_f}{fm^3}$$

The MIT Bag Model

Literally the simplest of pheno models. Build in confinement and asymptotic freedom in one simple model.

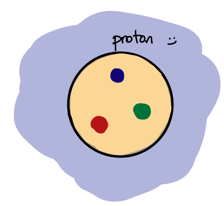


↳ Confined medium (or hadron) as a bag of free partons

- $\mathcal{E}_{perturbative} := \mathcal{E}_{in} > \mathcal{E}_{out} =: \mathcal{E}_{vacuum}$.

  $\mathcal{E}_{in} - \mathcal{E}_{out} \equiv B$.

- "Compressing" all complex dynamics to one number



- But what is B? Let's use the proton!

  ↳ homogeneous in B, spherically symmetric. ✓

$E_p = N_q \cdot E_{kin} + V \cdot B$ using $V = \frac{4\pi}{3}R^3$ and $E_{kin} = \frac{\gamma \hbar c}{R}$ [Lowest energy solution of the Dirac equation for an infinite potential!]

Stability? $\Rightarrow \frac{dE_p}{dR} = \frac{dV}{dR}\cdot B + N_q \cdot \frac{d\langle E_{kin}\rangle}{dR} \equiv 0$

$\gamma \cong 2{,}404$ ↳ numerical value of the solution

$$\Rightarrow \quad B = -\frac{\left[\frac{d\langle E_{kin}\rangle}{dR}\right] N_q}{\left[dV/dR\right]} = \frac{+\gamma/R^2}{4\pi R^2} N_q \longrightarrow B^{1/4} = \left(\frac{\gamma N_q}{4\pi}\right)^{1/4}\frac{1}{R}$$

Setting $N_q \equiv 3$ and $R \sim 0{,}8\ fm$ we get $B^{1/4} \sim 220$ MeV

$\Rightarrow \epsilon_{QGP} \equiv \epsilon_{QGP}^{pert} + B$ Why? First law of thermodynamics $E = TS - pV$

$P_{QGP} \equiv p_{QGP}^{pert} - B$

$\Rightarrow \epsilon = Ts - p$ → Addition of B keeps

$\Rightarrow \epsilon + p = Ts$ the same entropy!

We would like to compare this to a gas of hadrons. Why? We would like to estimate the critical temperature by looking at the intercept. This is, by no means supposed to be an accurate account of QCD dynamics. It is supposed to help us with the intuition building.

⟹ Extreme simplification ⇒ Hadron gas is taken to be a gas of -massless- pions. Why? Because it is easier...

Pions are bosons with 3 different internal states (π^+, π^0, π^-) ⇒ $\nu_\pi = 3$

$$n_{\pi G} = \frac{\nu_\pi \zeta(3) T^3}{\pi^2} \qquad \epsilon_{\pi G} = \nu_\pi \frac{\pi^2 T^4}{30} \qquad p_{\pi G} = \frac{\epsilon_{\pi G}}{3}$$

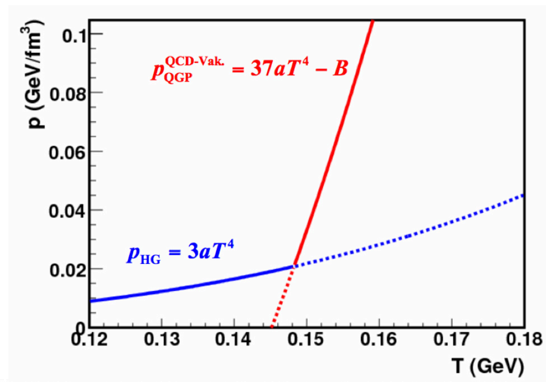
⊛ Gibbs phase transition criterion: $P_{phase\,1}(T_c) = P_{phase\,2}(T_c)$

Then by choosing $\mu = 0$ we get

$$\nu_{QGP} \frac{\pi^2 T_c^4}{90} - B \equiv \nu_\pi \frac{\pi^2 T_c^4}{90}$$

Which can be solved for T_c to find

$$\Rightarrow T_c = \left[\frac{90 B}{\pi^2 \Delta\nu}\right]^{1/4} \sim 150\,\text{MeV}$$

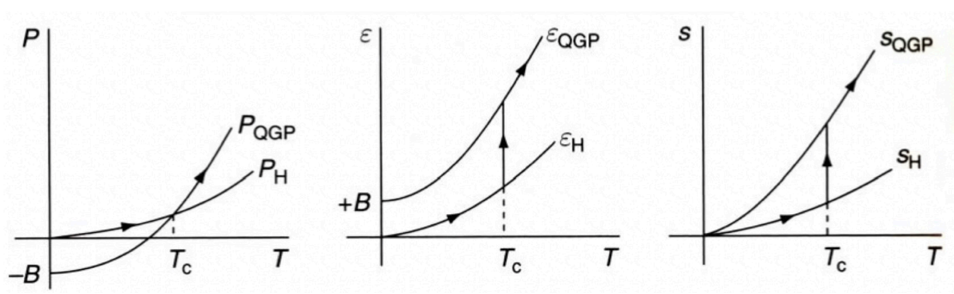


Here, $\Delta\nu = \nu_{QGP} - \nu_{\pi G} \sim 34$ stands for the change in the degrees of freedom. Now take a look at the energy density @ the transition boundary

$$\epsilon_{QGP}(T_c) - \epsilon_{\pi G}(T_c) = \Delta\nu \frac{\pi^2 T_c^4}{30} + B = 4B$$

⇒ Can be interpreted as latent heat.

⇒ First order phase transition

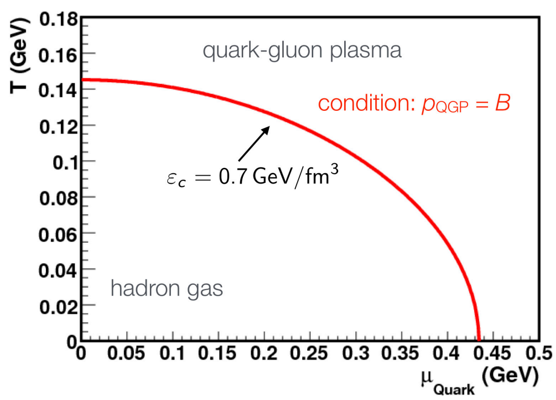


In the limit $T \to 0$ $\mu \neq 0$ the same process can be re-done to find the transition chemical potential, $\mu_c(B)$
This is left as exercise for the tutorial, but the result is roughly

$$\mu_c(T=0) = (2\pi^2 B)^{1/4} \approx 0{,}45 \text{ GeV} \qquad \text{with} \qquad n_c \approx \frac{2}{3\pi^2}\mu_c^3 \approx 5\, n_{nucleus}.$$

↳ Maybe attainable @ neutron stars!

So, the full picture now looks like this!



But this is not correct! Full QCD studies have found a smooth crossover @ $\mu \sim 0$