

Warsaw University of Technology

FACULTY OF  
MATHEMATICS AND INFORMATION SCIENCE



# Bachelor's diploma thesis

in the field of study Computer Science and Information Systems

Procedural world generation and the challenges of rendering in  
non-Euclidean spaces

**Aleksy Bałaziński**

student record book number 313173

**Karol Denst**

student record book number 305962

thesis supervisor

Paweł Kotowski, Ph.D.

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## Abstract

Procedural world generation and the challenges of rendering in non-Euclidean spaces

Procedural generation, as a method used in creating video games, has experienced a significant increase in popularity in recent years. It has been employed extensively in acclaimed titles such as *Minecraft* and *No Man's Sky* to create virtually infinite worlds that the player is free to interact with. On the other hand, a recent game *Hyperbolica* popularized a novel idea of making the virtual worlds even more interesting by setting them in non-Euclidean spaces. The objective of this thesis is to create a small video game incorporating both of the aforementioned concepts.

**Keywords:** keyword1, keyword2, ...



## Streszczenie

Proceduralne generowanie świata i wyzwania związane z renderowaniem w przestrzeniach nieeuklidesowych

Proceduralne generowanie świata jest techniką zdobywającą rosnącą popularność przy tworzeniu gier komputerowych. Zostało ono wykorzystane z powodzeniem m. in. w takich produkcjach jak *Minecraft* czy *No Man's Sky*. Z drugiej strony za przyczyną gier takich jak *Hyperbolica*, szersze zainteresowanie zyskała kwestia osadzania gier komputerowych w przestrzeniach nieeuklidesowych. Za cel niniejszej pracy obrano stworzenie aplikacji graficznej łączącej obydwie wspomniane elementy.

**Słowa kluczowe:** slowo1, slowo2, ...



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## 1. Introduction

What is the thesis about? What is the content of it? What is the Author's contribution to it?

WARNING! In a diploma thesis which is a team project: Description of the work division in the team, including the scope of each co-author's contribution to the practical part (Team Programming Project) and the descriptive part of the diploma thesis.

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## 2. Theoretical foundations

### 2.1. Non-Euclidean geometry

The *Euclidean* geometry is based on a set of five postulates originally given by Euclid. The postulates read as follows [5]:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.
4. All right angles are congruent.
5. Given any straight line and a point not on it, there exists one and only one straight line that passes through that point and never intersects the first line, no matter how far they are extended. [4]

The fifth postulate, also called the *parallel postulate*, has been for hundreds of years a subject of debate if it can be proven from the former four postulates. It was discovered, however, that the negation of the parallel postulate doesn't lead to a contradiction[1]. The postulate can be negated in one of two ways.

- Given any straight line and a point not on it, there exist **at least two straight lines** that pass through that point and never intersect the first line, no matter how far they are extended.
- Given any straight line and a point not on it, there exists **no straight line** that passes through that point and never intersects the first line; in other words, there are no parallel lines, since any two lines must intersect.

When the parallel postulate is replaced with the first statement, we obtain a new geometry, called the *hyperbolic geometry*. Similarly, replacing it with the second statement yields the *spherical geometry*. These geometries are collectively referred to as *non-Euclidean geometries*.

### 2.1.1. Analytic description

To describe the non-Euclidean geometries analytically, we will follow the approach given by [3]. This approach allows us to view the points of a 3-dimensional non-Euclidean space as a subset of the 4-dimensional *embedding space*. Since imagining the fourth dimension is not something particularly easy, we will be decreasing the dimensionality whenever we give examples.

The elliptic space can be modeled as a unit 3-sphere, *embedded* in a 4-dimensional Euclidean space. By saying that a space is embedded in another, we mean that the embedded space inherits the distance from the embedding space. In this case, the spherical distance, given by  $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$  is derived from the Euclidean distance. This is similar to how we may model the 2-dimensional elliptic space as a sphere, where lines are identified with great circles. The inner product of two vectors  $u$  and  $v$  in the Euclidean space is given by

$$\langle u, v \rangle_E = u_x v_x + u_y v_y + u_z v_z + u_w v_w.$$

Thus, we can define that a point  $p$  belongs to the elliptic geometry if

$$\langle p, p \rangle_E = 1.$$

The model that we use for the hyperbolic geometry is the so-called *hyperboloid model*. In this model, points  $p$  of the hyperbolic space satisfy the equation

$$p_x^2 + p_y^2 + p_z^2 - p_w^2 = -1,$$

with  $p_w > 0$ . The set of these points creates the upper sheet of a hyperboloid, which could be visualized as shown in Figure 2.1 if the embedding space was Euclidean. However, the

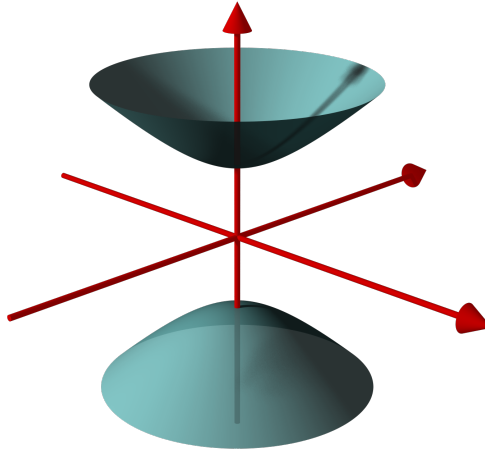


Figure 2.1: 2-dimensional hyperboloid embedded in Euclidean space

hyperbolic space is not embedded in the Euclidean space, but in the *Minkowski space* instead. In

the Minkowski space, the inner product of vectors  $u, v$  is given by the Lorentzian inner product:

$$\langle u, v \rangle_L = u_x v_x + u_y v_y + u_z v_z - u_w v_w.$$

Thus, the points  $p$  belonging to the hyperbolic geometry satisfy the equation

$$\langle p, p \rangle_L = -1.$$

It could be interpreted that they are located on a sphere with a radius of imaginary length  $\sqrt{-1}$  (and hence are equidistant from the origin).

To build a unified framework for discussing both types of geometries, we introduce the notion of *sign of curvature*,  $\mathcal{L}$ , that attains the value  $+1$  for spherical, and  $-1$  for hyperbolic space. We also define  $\langle u, v \rangle = u_x v_x + u_y v_y + u_z v_z + \mathcal{L} u_w v_w$ .

### 2.1.2. Transformations

We will now define transformations that can be used in non-Euclidean geometries.

#### Reflection

A vector  $v_R$  obtained from reflecting vector  $v$  on vector  $m$ , see Figure 2.2, can be defined as

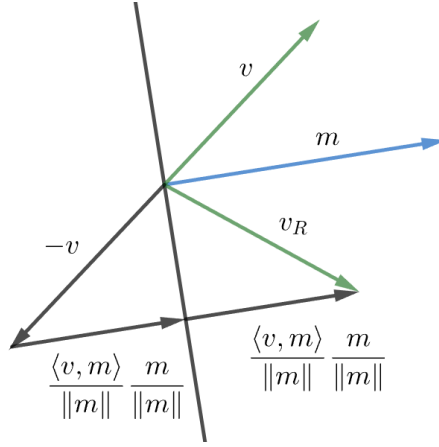


Figure 2.2: Reflection of  $v$  on vector  $m$

$$v_R = 2 \frac{\langle v, m \rangle}{\|m\|} \frac{m}{\|m\|} - v = 2 \frac{\langle v, m \rangle}{\langle m, m \rangle} m - v.$$

It can be verified that this definition satisfies the intuitive conditions of a reflection:

- The reflected vector  $v_R$  lies in the plane spanned by  $v$  and  $m$ ,
- The transformation is an isometry, i.e.  $\|u - v\| = \|u_R - v_R\|$ .

We should also note that given a point  $p$  in the geometry, i.e. satisfying  $\langle p, p \rangle = \mathcal{L}$ , its reflection,  $p'$ , is also in the geometry.

### Translation

Just like in Euclidean space, we can define translation in terms of an even number of reflections. More specifically, the translation will be defined by specifying two points: *geometry origin*,  $g = (0, 0, 0, 1)$  and *translation target*,  $q$ , which is the point that the geometry origin is translated to. Now we can define that the translation is the composition of two reflections: one on the vector  $m_1 = g$  and the second one on the vector  $m_2 = g + q$ , which is halfway between  $g$  and  $q$ . Applying the first reflection to an arbitrary point  $p$  gives a point

$$p' = 2 \frac{\langle p, g \rangle}{\langle g, g \rangle} g - p = 2p_w g - p,$$

and the second reflection applied to  $p'$  yields a point

$$p'' = 2 \frac{\langle p', g + q \rangle}{\langle g + q, g + q \rangle} (g + q) - p' = 2p_w q + p - \frac{p_w + \mathcal{L}\langle p, q \rangle}{1 + q_w} (g + q). \quad (2.1)$$

The effect of applying translation to an arbitrary point  $a$  is shown in Figure 2.3.

It can be verified that the geometry origin  $g$  is indeed translated to point  $g'' = q$ . We can

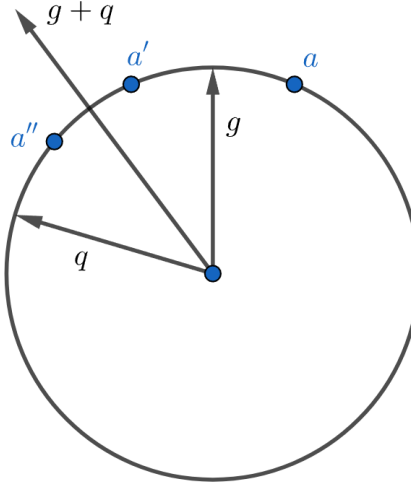


Figure 2.3: Translation of a point  $a$

evaluate the formula for the basis vectors  $i = (1, 0, 0, 0)$ ,  $j = (0, 1, 0, 0)$ ,  $k = (0, 0, 1, 0)$ , and  $l = (0, 0, 0, 1)$  obtaining the translation matrix

$$T(q) = \begin{bmatrix} 1 - \mathcal{L} \frac{q_x^2}{1+q_w} & -\mathcal{L} \frac{q_x q_y}{1+q_w} & -\mathcal{L} \frac{q_x q_z}{1+q_w} & -\mathcal{L} q_x \\ -\mathcal{L} \frac{q_y q_x}{1+q_w} & 1 - \mathcal{L} \frac{q_y^2}{1+q_w} & -\mathcal{L} \frac{q_y q_z}{1+q_w} & -\mathcal{L} q_y \\ -\mathcal{L} \frac{q_z q_x}{1+q_w} & -\mathcal{L} \frac{q_z q_y}{1+q_w} & 1 - \mathcal{L} \frac{q_z^2}{1+q_w} & -\mathcal{L} q_z \\ q_x & q_y & q_z & q_w \end{bmatrix}$$

It can be seen that the translation is an isometry since the row vectors of the matrix are orthonormal.

## Rotation

It can be shown that a rotation about an axis through the origin is the same as the Euclidean rotation about the same axis [2].

## Camera transformation

The camera transformation allows us to describe the scene from the viewer's perspective. The transformation is defined in terms of the *eye position*  $e$ , and three orthonormal vectors in the tangent space of the eye:

1. the right direction  $i'$ ,
2. the up direction  $j'$ , and
3. the negative view direction  $k'$ .

An example in Figure 2.4 shows the tangent space of the eye, with the  $e$  vector marked green,  $-k'$  marked blue, and  $i'$  marked orange.

The transformation can be described by the matrix

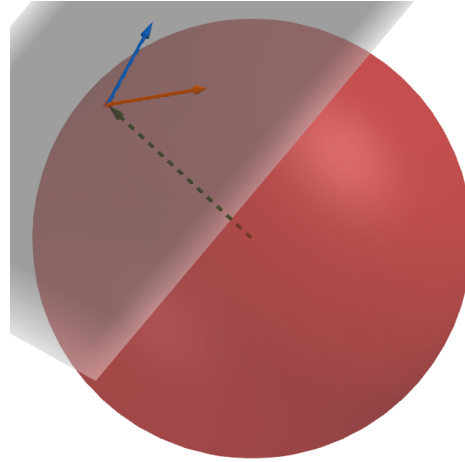


Figure 2.4: Tangent space of the camera

$$V = \begin{bmatrix} i'_x & j'_x & k'_x & \mathcal{L}e_x \\ i'_y & j'_y & k'_y & \mathcal{L}e_y \\ i'_z & j'_z & k'_z & \mathcal{L}e_z \\ \mathcal{L}i'_w & \mathcal{L}j'_w & \mathcal{L}k'_w & e_w \end{bmatrix}$$

As the result of the transformation, the eye position is mapped to the geometry origin  $g$ . Furthermore, the vectors  $i'$ ,  $j'$ , and  $k'$  are mapped to  $i$ ,  $j$ , and  $k$ , respectively.

### Perspective transformation

The perspective transformation is described using a projection matrix  $P$ . The projection matrix we use in spherical geometry is identical to the one used in the *Unity* implementation of [3] (see <https://github.com/mmagdics/noneuclideanunity>). It is parameterized by the *near plane distance*  $n$ , *far plane distance*  $f$ , *aspect ratio* ASP, and *field of view* FOV:

$$P = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -n & 0 \end{bmatrix},$$

where  $s_x = 2n/(r - l)$ ,  $s_y = 2n/(t - b)$ , and  $r, l, t, b$  are defined in terms of  $u = f \tan(\text{FOV})$ :

$$r = u \cdot \text{ASP}, l = -u \cdot \text{ASP}, t = u, b = -u.$$

For hyperbolic and Euclidean geometries, the standard projection matrix is used.

### Porting objects

The positions of objects in the scene are specified in a 3-dimensional Euclidean space. They are then "transported" or *ported* to a non-Euclidean space of choice. One possible mapping that could be used for this purpose is called the exponential map. For a given point  $p$  in the 3-dimensional Euclidean space with coordinates  $(x, y, z)$  the mapping to elliptic geometry is given by

$$\mathcal{P}_E(p) = (p/d \sin(d), \cos(d)), \quad (2.2)$$

and for hyperbolic space, it is given by

$$\mathcal{P}_E(p) = (p/d \sinh(d), \cosh(d)).$$

The effect of porting a 1-dimensional point  $p$  onto a 1-dimensional elliptic space can be seen in Figure 2.5.

#### 2.1.3. Practical considerations

There are two ways we could implement placing objects in the scene:

1. Port the object to non-Euclidean geometry and then use the translation given by Equation 2.1,
2. Translate the object using ordinary Euclidean translation and then port it to non-Euclidean geometry.

We will now shortly discuss both options.

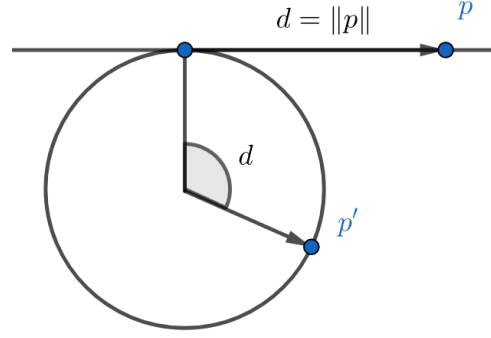


Figure 2.5: Exponential map

### Port, then translate

The first option is undesirable, as it may significantly change the relative positions of objects in the scene. To see why, let's consider two copies of a 2-dimensional rectangle that we will first port to the spherical geometry, and then translate using the non-Euclidean translation. The rectangle with vertices  $a = (-0.5, -0.7)$ ,  $b = (0.5, -0.7)$ ,  $c = (0.5, 0.5)$ ,  $d = (-0.5, 0.5)$  is ported to spherical geometry using Equation 2.2. As a result, we obtain points on a unit sphere:

$$\mathcal{P}(a) = (-0.441, -0.617, 0.652)$$

$$\mathcal{P}(b) = (0.441, -0.617, 0.652)$$

$$\mathcal{P}(c) = (0.459, 0.459, 0.760)$$

$$\mathcal{P}(d) = (-0.459, 0.459, 0.760)$$

If we were to translate the first copy of the rectangle to point  $t_1 = (0.5, 0.5)$  and the second copy to  $t_2 = (1.5, 1.7)$  in Euclidean geometry, the two copies should meet at the point  $(1, 1)$ .

When we perform the translation to point  $t_1$  (the corresponding translation target is obtained by porting  $t_1$  using Equation 2.2, i.e. the translation target is  $q_1 = \mathcal{P}(t_1)$ ), we get the following vertices:

$$T_{q_1} \mathcal{P}(a) = (-0.014, -0.190, 0.982)$$

$$T_{q_1} \mathcal{P}(b) = (0.761, -0.296, 0.577)$$

$$T_{q_1} \mathcal{P}(c) = (0.698, 0.698, 0.156)$$

$$T_{q_1} \mathcal{P}(d) = (-0.110, 0.809, 0.578)$$



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The translation to  $t_2$  (with  $q_2 = \mathcal{P}(t_2)$ ) gives

$$T_{q_2}\mathcal{P}(a) = (0.709, 0.686, 0.160)$$

$$T_{q_2}\mathcal{P}(b) = (0.957, -0.031, -0.287)$$

$$T_{q_2}\mathcal{P}(c) = (0.141, 0.099, -0.985)$$

$$T_{q_2}\mathcal{P}(d) = (-0.117, 0.847, -0.519)$$

Even though we would expect the third vertex of the first copy of the rectangle to be identical to the first vertex of the second copy, there is a difference between the two. This effect can be seen in Figure 2.6. The effect is even more visible in the implementation. Figure 2.7 shows how

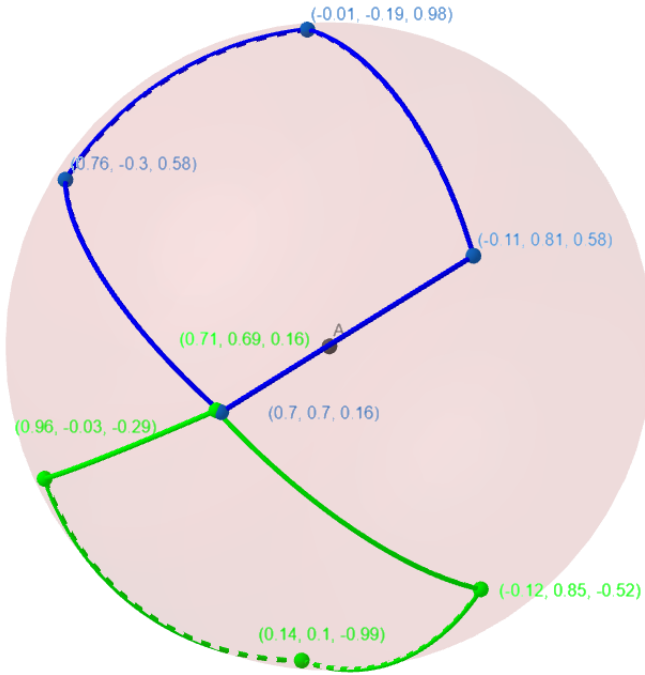


Figure 2.6: Rectangles ported onto a sphere and then translated

the wheels of the car get misaligned from their wheel arches.

### Translate, then port

The second option isn't unfortunately free of distortions either. For example, consider two identical squares of side length 0.5. The first one with the bottom-left corner at the point  $(0,0)$  and the second one with the corresponding corner at  $(0.5,0.5)$ . After porting to spherical geometry using the Equation 2.2, we get squares with side lengths (listed counter-clockwise starting at the bottom edge):

$$0.500, 0.479, 0.479, 0.500$$



Figure 2.7: Non-Euclidean translation causing a misalignment of objects

for the first square and with side lengths

$$0.480, 0.425, 0.425, 0.480$$

for the second square. The side lengths of the square have been calculated as the lengths of geodesics<sup>1</sup> between the square's vertices. As we can see, the side lengths of the ported square are no longer equal to each other, and the distortion increases as the square is farther away from the origin. This effect can be seen in Figure 2.8.

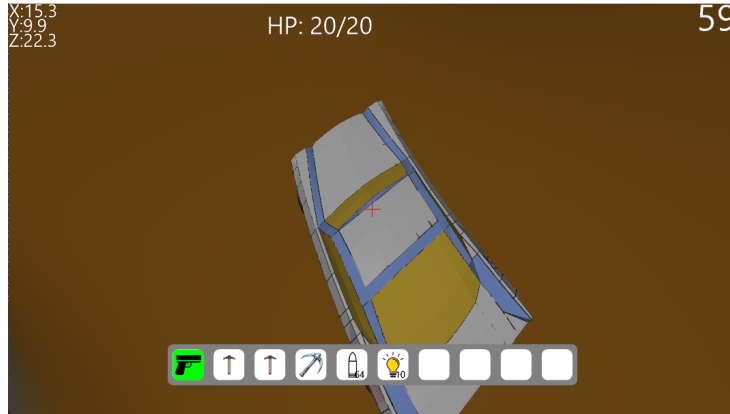


Figure 2.8: Distortions caused by porting to spherical space

### Spherical space

To minimize the distortions in spherical space we employed the following method. Due to the periodic nature of the porting given by Equation 2.2, the scene has to be confined inside a 3-dimensional ball of radius  $2\pi$ . Since the distortions increase as an object is farther from the origin, we decided to split the scene into two physical regions – balls of radius  $\pi$ . Points  $p$  in the

<sup>1</sup>This is the "great-circle distance" equal to  $2 \arcsin(c/2)$ , where  $c$  is the chord length.

## 2.1. NON-EUCLIDEAN GEOMETRY

first ball (centered at the origin) are mapped to spherical geometry using the mapping

$$\mathcal{P}_{E,1}(p) = (p/\|p\| \sin(\|p\|), \cos(\|p\|)) \quad (2.3)$$

and points  $p$  in the second ball (which is centered at a point  $c$ ) using the formula

$$\mathcal{P}_{E,2}(p) = (p'/\|p'\| \sin(\|p'\|), -\cos(\|p'\|)), \quad (2.4)$$

where  $p' = p - c$ . In the 2-dimensional case, the regions become disks with radii of length  $\pi$ , and the effect of using Equation 2.3 and Equation 2.4 can be visualized as "wrapping" the first disk on the upper half of a unit sphere, and "wrapping" the second one on the lower half of the sphere.

### Hyperbolic space

Dealing with distortions in hyperbolic space requires a more drastic approach because the scene we wished to port was potentially infinite. The main goal was to keep the distortions as small as possible near the camera and still show the visual aspects indicative of setting the scene in non-Euclidean geometry. To achieve this, the camera position is fixed at some point close to the origin, e.g.  $(0, 1, 0)$ . The movement of the camera is then simulated by moving all of the objects in the scene in the direction opposite to the camera's movement direction.

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## List of symbols and abbreviations

nzw.    nadzwyczajny

\*       star operator

~       tilde

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