Calculus 1: Summary

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Limits

Limit exists if and only if the approach from left and right is the same:

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

Properties of Limits

$$\lim_{x\to a}\big(f(x)+g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \lim_{x\to a}\big(f(x)\times g(x)\big) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\big(f(x)\cdot g(x)\big) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \qquad \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n \qquad \lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

Calculating for Limits

Direct Substitution

If f is polynomial or rational function or is continuous at a, then $\lim_{x\to a} f(x) = f(a)$.

Limit via Similar Function

If $f(x) = g(x)$ except at $x = a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

If $f(x) \le g(x)$ for x is near a (possibly except at $x = a$), then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

Squeeze Theorem or Sandwich Theorem

If $f(x) \le g(x) \le h(x)$ when x is near a (possibly except at $x = a$), then

$$\left(\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L\right) \quad \to \quad \lim_{x\to a} g(x) = L$$

Applications

Asymptotes

Vertical: $x = a$ if $\lim_{x\to a^\pm} f(x) = \pm\infty$

Horizontal: $y = a$ if $\lim_{x\to \infty^\pm} f(x) = a$

Tip: For rational function, you only need to consider the largest degree term as it is the most significant.

Continuous

If f is continuous at a, then $\lim_{x\to a} f(x) = f(a)$.

If f and g is continuous, then $f+g,\ f-g,\ f\cdot g,\ \frac{f}{g}$ is also continuous.

If g is continuous at a and f is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

Intermediate Value Theorem

If f is continuous at $[a,b]$, then there exists c such that $f(c)$ is between $f(a)$ and $f(b)$.

Derivatives

Properties of Derivatives

$$\frac{d}{dx}f(x) = f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$\frac{d}{dx}c = 0$	$\frac{d}{dx}ax = a$
$\frac{d}{dx}x^n = n\cdot x^{n-1}$	$\frac{d}{dx}a^x = \ln(a)\cdot a^x$
$\frac{d}{dx}\ln(x) = \frac{1}{x}$	$\frac{d}{dx}\log_a x = \frac{1}{x\cdot\ln(a)}$
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$(f+g)' = f'+g'$	$(f-g)' = f'-g'$
$(fg)' = fg'+f'g$	$\left(\frac{f}{g}\right)' = \frac{f'g-g'f}{g^2}$
$\left(f(g(x))\right)' = f'(g(x))\cdot g'(x)$	$\frac{d}{dx} = \frac{du}{dx}\frac{d}{du}$

Intermediate Form and L'Hospital's Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

If it is an intermediate form of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

This can only be used on the above intermediate form, but some form can be transformed:

$$f(x)g(x) = \frac{f(x)}{\left(g(x)\right)^{-1}} = \frac{g(x)}{\left(f(x)\right)^{-1}}$$

$$f(x)-g(x) = g(x)\left(\frac{f(x)}{g(x)}-1\right)$$

$$y = [f(x)]^{g(x)} \quad\rightarrow\quad \ln y = g(x)\ln(f(x))$$

Minimum and Maximum

Global Maximum and Minimum is where the function is the greatest or the least in all of domain.

Local Maximum and Minimum is where the function is the greatest or the least near some points.

The concept is intuitive, although domain endpoints cannot be local maximum and minimum.

Critical Number

Point where $f'(c) = 0$ or $f'(c)$ does not exist.

$$\text{(Local Minimum or Maximum)} \subseteq \text{(Critical Numbers)}$$

Closed Interval Theorem for Global Maximum or Minimum

Global Maximum or Minimum of f in range $[a, b]$, is the maximum or minimum of

- Value of f at a and b.
- Value of f at all critical points in $(a,\ b)$.

Derivative Test for Local Extreme Values

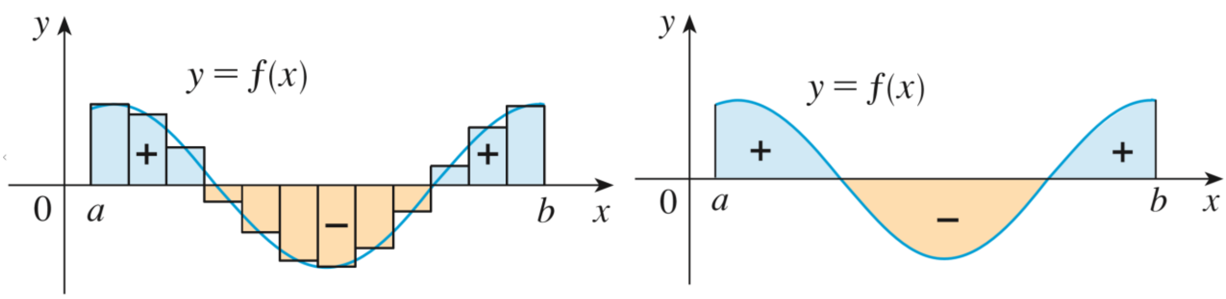
Let c be a critical number of continuous function f.

- If $f'(c)$ changes from positive to negative, then c is the local maximum.
- If $f'(c)$ changes from negative to positive, then c is the local minimum.
- If $f'(c) = 0$ and $f''(c) < 0$, then c is the local maximum.
- If $f'(c) = 0$ and $f''(c) > 0$, then c is the local minimum.

Integrals

$$\int_a^b f(x)\,dx = \lim_{dx \to 0} \left(\sum_{i=1}^{n} f(x_i^*) \cdot dx \right)$$

Integral is a way to find the area under the curve. It is the limit of the approximation of the area by dividing it into tiny strips. The area may be negative if the curve is below the x-axis.



Properties of Integrals

$\int_a^a f(x)\,dx = 0$	$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
$\int_a^b c\,dx = c\cdot(b-a)$	$\int_a^b c\cdot f(x)\,dx = c\cdot\int_a^b f(x)\,dx$
$\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$	$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Calculating for Integrals

Squeeze Theorem

If $g(x) \le f(x) \le h(x)$ for $a \le x \le b$, then

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$$

Antiderivative

Useful for finding integral of most simple functions, such as $\sin x$, x^n, $\ln x$.

$\frac{d\,F(x)}{dx} = f(x)$	$F(x)$ is "antiderivative" of $f(x)$.
$\int_a^b f(x)\,dx = F(b) - F(a)$	Integral is the subtraction of antiderivative function.
$\frac{d}{dx}\int_a^x f(t)\,dt = f(x),\qquad \int \frac{d}{dx} f(x)\,dx = f$	Derivative and Integral is inverse of each other.

Substitution

Useful for finding integral of composite functions.

$$\int_a^b f\big(g(x)\big)\,dx$$

Let $u = g(x)$

$$\frac{du}{dx} = u' \quad\rightarrow\quad dx = \frac{du}{u'}$$

$$\int_a^b f\big(g(x)\big)\,dx = \int_{g(a)}^{g(b)} f(u)\cdot\frac{1}{u'}\,du$$

Symmetries

If f is even (y-axis symmetry), then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

If f is odd ($(0,0)$ symmetry), then $\int_{-a}^{a} f(x)\,dx = 0$.

Integration by parts

Useful for finding integral of multiplication products.

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

$$\int \frac{d}{dx}[f(x)\,g(x)] = \int f(x)\cdot g'(x)\,dx + \int f'(x)\cdot g(x)\,dx$$

$$\int f(x)\cdot g'(x)\,dx = f(x)g(x) - \int f'(x)\cdot g(x)\,dx$$

Sequences and Series

A sequence is an ordered set of values:

$$a_1, a_2, a_3, a_4, a_5, \dots$$

A series is the sum of the first n terms of the sequence:

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n$$

A sequence/series "converges" if limit approach infinity exists (otherwise it "diverges"):

$$\lim_{n\to\infty} a_n = L \qquad \lim_{n\to\infty} S_n = L \qquad (L \neq \pm\infty)$$

A series will diverge if its sequence doesn't converge to 0: $\lim_{n\to\infty} a_n \neq 0 \;\rightarrow\; \lim_{n\to\infty} S_n = \pm\infty$

To calculate the limit of a sequence or series, it may be useful to extend it into real numbers:

Let $f(x)$ be a function where $f(n) = a_n$ for $n \in \mathbb{I}^+$ (Don't forget this step!)

$$\lim_{x\to\infty} f(x) = L \quad \rightarrow \quad \lim_{n\to\infty} a_n = L$$

Squeeze Theorem for Sequences

If $a_n \leq b_n \leq c_n, and \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

The absolute function is useful for this: $-|a_n| \leq a_n \leq |a_n|$.

Arithmetic Sequence and Series

A sequence with a constant increase/decrease. Such as $3, 6, 9, 12, 15, \dots$

$$a_n = a_1 + (n-1)\cdot d \qquad a_n = a_m + (n-m)\cdot d$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Arithmetic Sequence and Series **always diverges** (if $d \neq 0$).

Geometric Sequence and Series

A sequence with a constant ratio. Such as $2, 4, 8, 16, 32, 64, \ldots$

$$a_n = a_1 \cdot r^{n-1} \qquad a_n = a_m \cdot r^{n-m}$$

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

Geometric Sequence and Series **converges if $|r| < 1$**.

Proof of Geometric Series Formula:

$$S_n = a_1 r^0 + a_1 r^1 + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}$$

$$(1 - r) \cdot S_n = (1 - r) \cdot (a_1 r^0 + a_1 r^1 + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1})$$

$$(1 - r) \cdot S_n = (a_1 r^0 + a_1 r^1 + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1}) - (a_1 r^1 + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + \cdots + a_1 r^n)$$

$$(1 - r) \cdot S_n = a_1 r^0 - a_1 r^n \qquad \rightarrow \qquad S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

Faulhaber's formula

$S_k(n) = 1^k + 2^k + 3^k + 4^k + 5^k + 6^k + \cdots + n^k$

$S_2(n) = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \cdots + n^2$ (when $k = 2$)

Calculating Formula:

$$(n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3 = 3n^2 + 3n + 1$$

n	$(n+1)^3 - n^3$	$3n^2 + 3n + 1$
1	$2^3 - 1^3$	$3(1^2) + 3(1) + 1$
2	$3^3 - 2^3$	$3(2^2) + 3(2) + 1$
3	$4^3 - 3^3$	$3(3^2) + 3(3) + 1$
…	…	…
n	$(n+1)^3 - n^3$	$3(n^2) + 3(n) + 1$

Add it all together:

$$(n + 1)^3 - 1 = 3\big(S_2(n)\big) + 3\big(S_1(n)\big) + 3(n)$$

Solve for $S_2(n)$.

Harmonic Series

$S_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n}$

Harmonic Series **always diverges**.