Calculus: Integration References

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Trigonometric Function References

Pythagorean Identity:

$$\sin(\theta)^2 + \cos(\theta)^2 = 1$$

Divide by $\sin(\theta)^2$ *or* $\cos(\theta)^2$:

$$\tan(\theta)^2 + 1 = \sec(\theta)^2, \qquad 1 + \cot(\theta)^2 = \csc(\theta)^2$$

Sum of Angle Formulas:

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Power Reduction Derivation:

$$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$$

$$= \cos(\theta)^2 - (1 - \cos(\theta))^2 = (1 - \sin(\theta))^2 - \sin(\theta)^2$$

$$\cos(\theta)^2 = \frac{1 + \cos(2\theta)}{2}, \qquad \sin(\theta)^2 = \frac{1 - \cos(2\theta)}{2}$$

Product to Sum Derivation:

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$$

Cofunction Identities:

$$\text{Identity: } f\left(\tfrac{\pi}{2} - \theta\right) = g(\theta) \leftrightarrow g\left(\tfrac{\pi}{2} - \theta\right) - f(\theta)$$

$$sin \leftrightarrow cos, \qquad tan \leftrightarrow cot, \qquad sec \leftrightarrow csc$$

$$\frac{d}{dx}\sin(x) = \cos(x), \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x), \qquad \frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\tan(x) = \sec(x)^2, \qquad \frac{d}{dx}\cot(x) = -\csc(x)^2$$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}, \qquad \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1 + x^2}$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)|$$

$$\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)|$$

$$\int \tan(x)\,dx = \ln|\sec(x)|$$

$$\int \cot(x)\,dx = \ln|\sin(x)|$$

Substitution

Let $u = g(x), du = g'(x)dx$

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du$$

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Trigonometric Functions

$$\int \sin(x)^m \cos(x)^n \, dx$$

If m is odd:

$$= \int \sin(x)^{m-1} \cos(x)^n \, d(-\cos(x))$$

$$\sin(x)^{m-1} = \left(1 - \cos(x)^2\right)^{\frac{m-1}{2}}$$

If n is odd:

$$= \int \sin(x)^m \cos(x)^{n-1} \, d(\sin(x))$$

$$\cos(x)^{n-1} = \left(1 - \sin(x)^2\right)^{\frac{n-1}{2}}$$

If both are even:
Use Power Reduction

$$\int \tan(x)^m \sec(x)^n \, dx$$

If m is odd:

$$= \int \tan(x)^{m-1} \sec(x)^{n-1} \, d(\sec(x))$$

$$\tan(x)^{m-1} = \left(\sec(x)^2 - 1\right)^{\frac{m-1}{2}}$$

If n is even:

$$= \int \tan(x)^m \sec(x)^{n-2} \, d(\tan(x))$$

$$\sec(x)^{n-2} = \left(\tan(x)^2 + 1\right)^{\frac{n-2}{2}}$$

If m is even and n is odd:
Use $\tan(x)^{2k} = \left(\sec(x)^2 - 1\right)^k$

Trigonometric Substitutions

Let $x = c\tan(\theta)$

$$\sqrt{x^2 + c^2} = c\sec(\theta)$$

Let $x = c\sec(\theta)$

$$\sqrt{x^2 - c^2} = c\tan(\theta)$$

Let $x = c\sin(\theta)$

$$\sqrt{c^2 - x^2} = c\cos(\theta)$$

Partial Fraction Decomposition

Let $R(x) = \frac{P(x)}{Q(x)}$ be proper rational function.

Factorize $Q(x)$ into linear factors or quadratic factors:

$$Q(x) = \left[\ldots\left(a_i x + b_i\right)^{r_i}\right]\left[\ldots\left(a_i x^2 + b_i x + c_i\right)^{r_i}\right]$$

For each factor, decompose into:

$$(ax+b)^r \to \frac{C_1}{(ax+b)^1} + \frac{C_2}{(ax+b)^2} + \frac{C_3}{(ax+b)^3} + \cdots + \frac{C_r}{(ax+b)^r}$$

$$(ax^2+bx+c)^r \to \frac{C_1x + D_1}{(ax^2+bx+c)^1} + \frac{C_2x + D_2}{(ax^2+bx+c)^2} + \frac{C_3x + D_3}{(ax^2+bx+c)^3} + \cdots + \frac{C_rx + D_r}{(ax^2+bx+c)^r}$$

Then, $Q(x)$ is equal to the sum of all decomposed factors.