HW5: Entropy of a Damped Driven Pendulum

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1. Introduction

Entropy is often described as the amount of disorder in a system, with higher entropy corresponding with more disorder. In statistical mechanics we often talk about entropy along with heat and use it to explain why heat spreads out. Entropy is part of a thermodynamic law stating that in the universe, entropy only can remain constant or increase. But we can also explain entropy in terms of the amount of possible micro-states that correspond to a single macro-state. With this definition we can look at kinematic systems in terms of their position and define entropy as the different possible configurations of that system and the likely hood that each configuration takes place. In this paper we do exactly that by examining how the entropy of a damped driven pendulum changes between a range of driving forces. We examine specifically a range of driving forces for the pendulum that begin in the non-chaotic regime and then transition into the chaotic regime. In this paper we explain our modelling of a damped driven pendulum, along with our numerical method in calculating entropy. We then discuss our finding for the physical motion of chaotic and non-chaotic pendulums and the relation to the calculation of entropy.

2. Method & Model

Runge-Kutta Method: To model a damped driven pendulum we utilized the second order Runge-Kutta method, also known as RK2. The method uses what is called a "midpoint" method where it approximates the values of the pendulum's angle θ and angular velocity ω by finding the first derivative at the original position, then using that information to take half a step in time, then again estimating the derivative of values at that half step which it then uses to approximate its full step.

System & Simulation: As in previous work (HW 3), the pendulum was simulated with the same parameters (e.g. the driving frequency, initial angle, initial angular velocity, total time) and iterative method (RK2 described above) as before. In this work, the time step in this simulation was chosen as $\Delta t = 2\pi/800f = 3\pi/800$, where $f = 2/3\,\mathrm{Hz}$ is the driving frequency, in order to produce data that was numerically accurate and aligned with the driving period. This produces a sequence of angles that represent the position of the pendulum at integer multiples of the driving period.

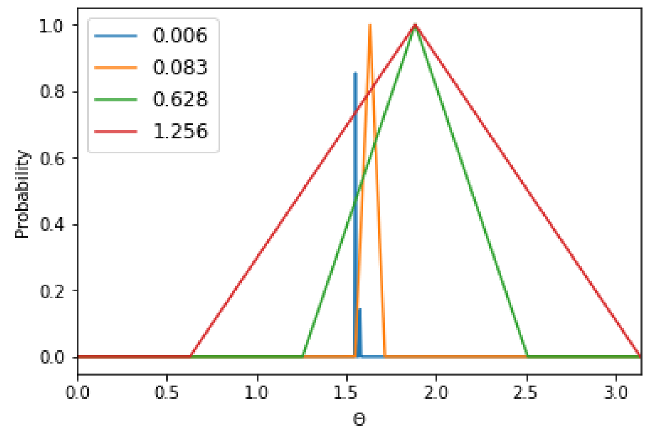

Fig. 1. The probability distribution vs θ of a pendulum with a driving force of 1.4 N for different values of bin size to split the range of possible θ values. The orange plot corresponding to $\Delta\theta = 0.083$ is the bin size used for calculations.

Binning and Probability: We then partition our domain of $\mathcal{D} = [-\pi, \pi]$ into bins of size $\Delta\theta = 2\pi/75$. This bin size was chosen because it is small enough to yield meaningful data (i.e. a bin size of 2π or π wouldn't give any significant information), while also not being so small that multiple bins are being populated due to slight numerical variations in the calculations. This variance is shown in Figure 1. In this figure, the two widest triangles correspond to large bin sizes that capture the angles well, but provide almost no information as to where the angles are actually lying, while the most narrow plot (which is actually two spikes next to each other) conveys exactly where the angles lie, but does not reach unit probability because another bin captures some numerical variation. Then count the number of angles that fall in each bin/partition and scale each population by the total number of angles measured to yield the probability of measuring the pendulum to be at that position at a driving period.

Recall that a bifurcation diagram tells us the the exact angles the pendulum takes at each driving period, as well as how the period doubles, as a function of driving force. We can represent the number of driving periods the system takes to get back to where it started as $T_n = 2^n$, where $n = 0, 1, 2, \dots$ is the "chaos level". T_n can then be re-thought of as the number of different angles (i.e. branches in the bifurcation diagram for a specific driving force) that would be measured over 2^n driving periods. Therefore T_n is exactly the number of angles with non-zero probability of measuring for a given driving force. We can also think of T_n as the number of states the system can be in for a given driving force. With this in mind, investigating the entropy as a function of driving force is a natural course.

Entropy: For a given driving force F_D, the probability distribution $P_{F_D}(\theta)$ can be used to find the entropy associated with that driving force. For a sequence of states, or in our case angle-bins, $\{\theta_i\}$ that has as its probability sequence $\{P_{F_D,i}\}$, where $P_{F_D}(\theta_i) \equiv F_D, i$, the entropy S_{F_D} takes the form

$$S_{F_D} = -\sum_i P_{F_D,i} \log \left(P_{F_D,i}\right). \qquad [1]$$

Therefore entropy is proportional to the *number* of bins/states that have non-zero probability, which is exactly T_n. Therefore, since T_n is constant in different regimes and discontinuously increases at certain driving forces (i.e. when the period doubles), the entropy should also be constant in those regimes and discontinuously increase at the period-doubling driving forces.

3. Results

How the probability distribution relates to the chaos level is shown in Figure 2 where in different regimes, or for different chaos levels, the number of populated bins increases and in fact doubles with the period. We can see where this corresponds to the splitting points in the bifurcation diagram as seen in Figure 3. Therefore we can see the corresponding sharp increases in entropy for the increasing driving force

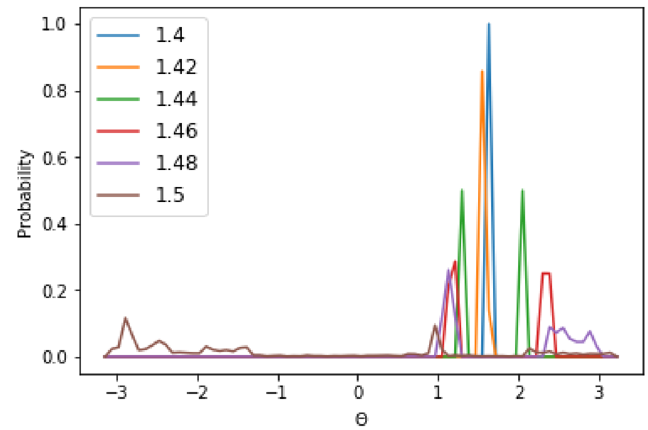

Fig. 2. Probability distributions vs θ for pendulums with different driving forces.

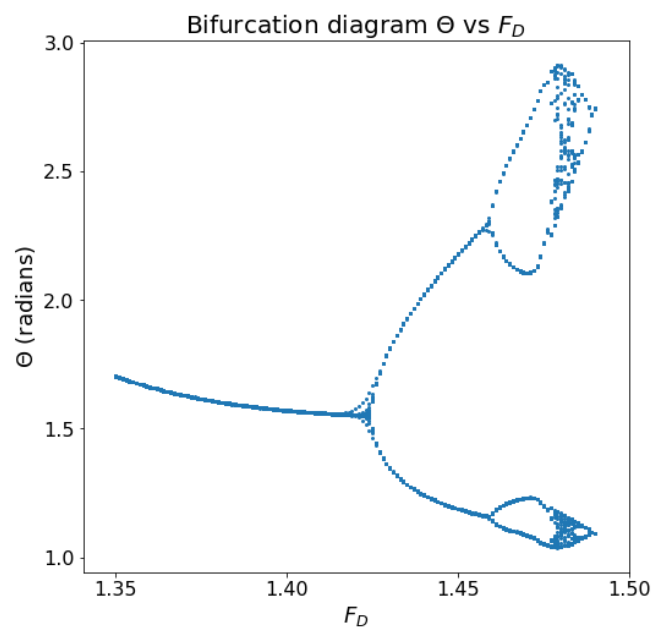

Fig. 3. Bifurcation diagram for θ values vs different driving forces.

in Figure 4 whenever we come across some force that splits the periodic nature of our pendulum.

With closer examination of Figure 4, we can see some interesting characteristics. First, we can see that the entropy starts at exactly zero which physically makes sense. With a non-chaotic pendulum, by sampling the angular position at the same time, every time, we should get that the pendulum is in the same position. By populating only one of our bins, the system is perfectly in order which would correspond to zero entropy. We can also see that there are long ranges where the entropy does not

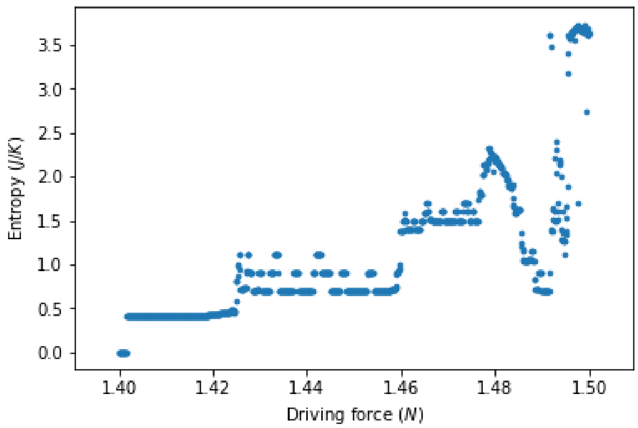

Fig. 4. Entropy vs driving force for a damped driven pendulum.

change or seems to stay at a specific value for entropy. We can once again relate this to our bifurcation diagram. After some period doubling, we do not change from being single, double, or quadruple periodic for a range of driving forces. So while the position that we are measuring the pendulum changes, the number of specific places we are seeing the pendulum does not change. Not changing in the amount of bins being populated by our sampling would correspond to the overall entropy not changing. One of the next interesting features is the drop in entropy that occurs around the driving force of 1.49 N. While this may initially seem to violate the 2^{nd} law of thermodynamics, it is important to note that the law corresponds to the fact that entropy does not decrease with *time* not necessarily with different system configurations. Therefore it is physically okay for the entropy to decrease at this driving force and by looking at Figure 3, we can explain this feature. As the force gets closer to 1.5 N the distinct number of places we see the pendulum with our sampling decreases and seems to converge to 2 specific values of θ again like we saw for the range of 1.43 to 1.46 N. The entropy value around 1.49 N does drop to what appears to be that same value for the entropy in that range of values confirming this observation. We then see a sharp increase in entropy after 1.49 N as the pendulum tends toward more chaotic behavior.

We do acknowledge that with Figure 4 that we see some unexpected behavior. For the range of driving forces between roughly 1.43 to 1.46 N, we see unexpected jumping up and down of the entropy to seemingly specific values. Over this range of driving forces, we would expect the entropy to stay constant as the period does not double. We conclude that there must be some systematic error therefore where the small range of theta values that we are measuring for the driving period are being split into multiple bins when they really correspond to roughly one theta value. This would also explain the specific levels that the entropy is evaluating to as the splitting between bins would be systematic and not random. We see this small variation in entropy elsewhere in the graph but it does not change the overall characteristics and therefore does not change the claims we make about how the entropy changes with the differing driving forces.

4. Conclusion

Entropy is not something you would usually think of for a pendulum, but through the introduction chaos from damping and driving forces, it is a natural consequence. Thus the entropy of uncommon systems, like the pendulum, is worth studying because it quantitatively describes how chaotic a system is. And we do that here.

Via a second order Runge-Kutta method, we simulated a damped, driven pendulum over a range of representative driving forces to analyze the effects of chaos on entropy. More specifically, for several driving forces we found the angles of the associated Poincare sections and the probability distributions of those angles to analyze the entropy as a function of driving force.

Using the idea of a “chaos level” as the integer number of doublings the period has undergone, we can predict that the entropy will discontinuously increase with the chaos level and is constant otherwise. We also find the chaos level corresponds to the number of angles that could be measured at a driving period (bifurcation diagram), or equivalently the number of peaks in the probability distribution. Using any of these tools, we can *qualitatively* determine how entropic the system is for a range of driving forces.