Effects of eccentricity on the precession of orbital periapsides due to General Relativity in the weak field limit

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(Dated: June 8, 2019)

Mercury's close proximity to the Sun and relatively high eccentricity make it the only object in the solar system to exhibit significant general relativistic (GR) perihelion precession. Due to not having more observational data, we must resort to simulating how eccentricity changes the relativistic precession of an orbiting body's periapsis (the point in a body's orbit that is closest to the object in which it is orbiting). Since the Sun is only massive enough to display weak GR effects, and Mercury is far enough away, the gravitational interaction is modeled via perturbing (i.e. adding a small variation to) Newton's inverse-square law of gravity/modified Newtonian dynamics (MOND). To fully describe the physics of the system, I investigate how the rate of precession due to General Relativity varies as a function of eccentricity. As a corollary, using it's parameters, Mercury's rate of precession is found to match the known value, thus confirming General Relativity is the current best explanation for Mercury's anomalous perihelion precession.

I. INTRODUCTION

The effects of General Relativity (GR) are *mostly* unobservable within our solar system. One of the exceptions being the precession of Mercury's perihelion. This is due to Mercury's close proximity to the Sun, and is accentuated by its higher-than-average eccentricity (around 0.2). But, what if Mercury's eccentricity was larger? What about smaller? How would GR affect Mercury if it got so close it grazed the Sun? What would happen if Mercury was nearly unbounded? All of these questions are encompassed by the single question "How do GR effects depend on the eccentricity of Mercury?".

II. MODIFIED GRAVITY

Newtonian gravity is simple enough that it is a center piece of all introductory physics courses. This placement and difficulty is primarily due to the fact that the only mathematical theory encapsulated in this formalism is basic vectors in a Euclidean space, or rather arrows in our 3D world. Of course the form of gravity a la Newton is

$$\mathbf{g} = -\frac{Gm}{r^3}\mathbf{r} = -\frac{Gm}{r^2}\hat{r}, \tag{1}$$

where of course, G_N is the Newtonian gravitational constant, m is the mass of the *central mass*, or rather the object that is causing the gravity, and r is the magnitude of the separation $\mathbf{r}$ between the central mass and a point in space. This form works extremely well for everyday situations like objects interacting with the Earth, or *most* planets interacting with the Sun. It is in the latter case, namely Mercury interacting with the Sun, where we find Newtonian gravity is not suitable to describe objects that are interacting via a stronger gravity. For such situations we must extend our model to the standard treatment of gravity, General Relativity (GR).

General Relativity is, to put it *absurdly* lightly, a step up in difficulty from Newtonian gravity. Albert Einstein's famous and elegant theory needs much more difficult mathematical machinery to describe the gravitational interactions between objects. Without going into too much detail, GR requires the use of a field of math called *Differential Geometry* (DG) which describes geometric properties, like distance, in/on curved spaces/surfaces. The full general relativistic description of gravity gives the Einstein Field Equations (EFE)[1]

$$G_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu} \tag{2}$$

where G is the Einstein tensor that describes how space and time curve, and T is the stress-energy tensor that describes how much energy there is in a region. There are few analytic solutions to this equation, and therefore few instances of exact descriptions of "true" gravity. Therefore, we must usually resort to approximate numerical solutions in order to model the systems that require the full form of GR. It turns out, even approximating solutions to the EFE are really hard. Though, there are situations that lend themselves to being modeled via a mix of Newtonian gravity and GR.

GR is definitely needed and to be fully used in the case of masses strongly interacting via gravitation, e.g. near black holes and neutron stars, but what about when the objects aren't massive enough, or are far enough away

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that they experience GR effects only slightly? This is called the *weak field limit* (WFL). The set of systems that obey Newtonian gravity fall into this regime, but also the ones that behave slightly non-Newtonian-ly. In the WFL, such systems can be modeled with a combination of Newtonian gravity and GR via corrections to the Newtonian form of gravity (Eq. 1). This combination of theories takes the form of adding corrections to the Newtonian form of gravity as

$$\mathbf{g} = -\frac{c^2}{2}\frac{r_s}{r^2}\left(1 + 3\frac{r_L^2}{r^2}\right)\hat{r}, \tag{3}$$

where r_s is the Schwarzchild radius of a mass defined by

$$r_s = \frac{2G_N}{c^2}m, \tag{4}$$

where m is the mass of the object, and

$$r_L^2 = \frac{(\mathbf{r} \times \dot{\mathbf{r}})^2}{c^2}. \tag{5}$$

Even though the WFL can be modeled via modified Newtonian dynamics (MOND), Newtonian gravity is still easier to implement and understand. So the natural question arises, at what point are the GR effects significant enough to invoke, at least, MOND? In figure (1), we can see that there is a small but significant difference between Newtonian gravity and the MOND gravity as the distance between masses approaches zero. At a certain point though, the difference is indeed negligible, and Newtonian gravity can be used without worry.

With this form of gravity, any investigation of a system in the WFL can be carried out as it would with Newtonian gravity. Though in this work, the precession of periapsides is done computationally, there is an analytic form of the angular precession due to GR.

GR directly tells us that the radial equation for an orbit is defined as[1]

$$r(\phi) = \frac{(1-e^2)a}{1 + e\cos[(1 - \delta\phi_0/2\pi)\phi]} \tag{6}$$

From equation (6), the angular precession of the orbit is exactly

$$\delta\phi_0 = \frac{6\pi M}{a(1-e^2)} \tag{7}$$

where M is mass "causing" the gravity, and a, e are the semi-major-axis and eccentricity respectively.

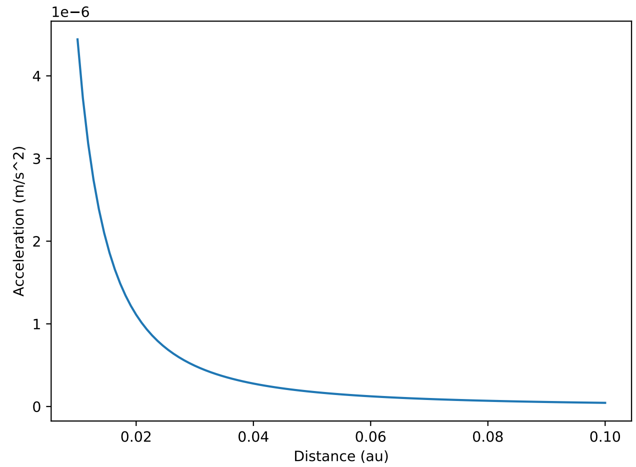

FIG. 1: The absolute difference in gravity between the Newtonian model and the GR model. The mass used was one solar mass, and the velocity was Mercury's average orbital velocity.

III. COMPUTATIONAL MODEL

A. Algorithm

Initialization & first step: To start the simulation of these two celestial bodies, one of which is fixed in space at the origin, the program takes as it's initialization parameters the masses of the two orbiting bodies, the initial position of the orbiting body at its periapsis, the initial velocity of the orbiting body at its periapsis, the number of orbits the body will undergo, and whether or not the gravity used will be the modified gravity (Eq. 3) or the standard Newtonian gravity (Eq. 1). From these parameters, the gravity due to the fixed mass is found at the initial position of the orbiting mass, and the first step in the orbiting mass's orbit is found through an Euler-Cromer method as

$$\mathbf{v}_1 = \mathbf{v}_0 + \mathbf{g}_0\Delta t \tag{8a}$$
$$\mathbf{r}_1 = \mathbf{r}_0 + \mathbf{v}_1\Delta t \tag{8b}$$

where g_0 is the gravity at the initial position r_0 acting on the orbiting mass with velocity v_0, and Δt is the time step. We make note of the velocity dependence of gravity here because in Newtonian gravity, the strength of the gravitational acceleration only depends on the distance from the mass, whereas in GR, the strength of gravity is dependant upon the distance and velocity of the mass on which gravity is acting.

Time step: Here we define our time step as above as

$$\Delta t = \frac{2v_0}{\alpha a_0}, \tag{9}$$

where α is the factor by which we decrease the time step, and in this work $\alpha = 2000$ (While this seems drastic in the sense that this produces on average a time step of about 15 minutes/sub-hour precision, if any smaller value was chosen, the end results *seemed* completely unphysical). We choose this time step so the Euler-Cromer evolution of the position is indeed a valid one. Since the time step is defined as such, it is actually determined by the initial properties of the system, and not in fact a parameter that is given by the user. After the first step is taken, the model transitions to evolving under a Verlet method.

Evolution: Whereas some models use forward biased methods, like the Euler method [2], here we use the time-reversible Verlet method

$$y_{i+1} = 2y_i - y_{i-1} + \left.\frac{d^2y}{dt^2}\right|_{y_i,t_i} \Delta t^2 \tag{10}$$

to ensure energy conservation. Conservation of energy is of the utmost importance in long-term gravitational simulations because over such long time periods, the error/energy-leak accumulates and makes the end result completely inaccurate. Now we evolve our system using this time step and integration method.

The main evolution of the system is defined by

$$\mathbf{r}_{i+1} = 2\mathbf{r}_i - \mathbf{r}_{i-1} + \mathbf{g}(\mathbf{r}_i, \mathbf{v}_i, t_i)\Delta t^2 \tag{11a}$$

$$\mathbf{v}_{i+1} = \frac{\mathbf{r}_{i+1} - \mathbf{r}_{i-1}}{2\Delta t} \tag{11b}$$

where the position $\mathbf{r}$ evolves under our Verlet integration, and the velocity $\mathbf{v}$ is determined via a simple difference quotient. Through this updating algorithm, the system evolves throughout it's orbit for one orbital period.

Orbit: The system steps through time under (Eq. 11) for one orbital period. This orbital period τ is found through the classical Keplerian formula[3]

$$\tau^2 = \frac{4\pi^2\mu}{G_N m_1 m_2} a^3 \tag{12}$$

where G_N is the Newtonian gravitational constant, m_1, m_2 are the masses of the objects, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system, and a is the semi-major-axis (SMA). Here the SMA is defined in terms of other parameters of the system as

$$a = \frac{c}{1 - \epsilon^2} \tag{13}$$

where c is the *principal orbital parameter* defined by

$$c = \frac{l^2}{G_N m_1 m_2 \mu}, \tag{14}$$

where l is the magnitude of the angular momentum defined by

$$l = m_2 ||\mathbf{r}_0 \times \mathbf{v}_0|| \tag{15}$$

and ϵ is the eccentricity of the orbit defined by

$$\epsilon = \frac{c}{r_0} - 1 \tag{16}$$

After one orbit has been simulated, the periapsis of that orbit is found by simply selecting the position that is closest to the fixed body. Once the location of the periapsis is known, and since we're interested in how the periapsis precesses, the angle of the periapsis is found with respect to the initial position (i.e. the x-axis because we started there) in arcseconds. We choose to work in units of arcseconds because we know that the perihelion precession of Mercury due to GR is about 43 arcseconds per century and so the average precession angle should be on that scale.

Now that we have an algorithm to evolve the system for one orbital period, we let the system evolve for as many periods as we want, and measure the positions, velocities, periapsides, the angles of the periapsides, and the eccentricities as we go.

After several orbits have been simulated, we preform a linear regression (with numpy's polyfit function) on the angles of the periapsides throughout the orbits to find the average rate of precession of the orbiting body. (All that work for one number!)

B. Numerical analysis

Of course no computational model is complete without an analysis of the numerical accuracy. While there is no *easy* analytic solution to this problem, we can test the sub-processes that govern the overall results.

As above, several parameters are not calculated through recursive formulations but rather through their closed

	$r_{S,\odot}$	$g_{\oplus}$	$\epsilon_{\text{Mercury}}$	τ_{Mercury}
Predicted	2.953 km	9.807 m/s^2	0.205	88 Earth days
Calculated	2.949 km	9.813 m/s^2	0.206	88.1 Earth days
Difference	0.004 km	0.006 m/s^2	0.001	0.1 Earth days
% Difference	0.16	0.06	0.49	0.11

TABLE I: The calculated and measured values of the Schwarzchild radius of the Sun, the gravity at the surface of the Earth, the eccentricity of Mercury, and the orbital period of Mercury, along with their absolute and percent differences (i.e. percent difference = |predicted − calculated|/predicted).

form mathematical definitions. Even though these properties have a closed form definition, computational calculations still carry error.

In table (I) we compare the values calculated here against the either measured or more accurately computed values. We can see all properties are within 1% of their "accurate" counter-parts. While we can quite easily quantify the error in these parameters, we can also at least qualitatively describe the error in the evolution of the system as well.

To test the orbital procedure, we test the trajectory of Earth around the Sun because we know it should be mostly circular. In figure (2), we see how the trajectory of the Earth behaves for different time steps. The different trajectories in the figure are not actually the time steps exactly, but rather the scaling parameter α as in equation (9). In each of these simulations, Earth started at it's average orbital distance (1 AU) on the x-axis, with its average orbital velocity orthogonal to the position ($v = 2\pi(1\text{AU})/(1\text{year})$). We can see that even for relatively small scaling factors ($\alpha = 20$ compared to the factor used for the results was $\alpha = 2000$), the trajectory of Earth converges to its known circular orbit. Another property we can test to get at least a qualitative idea of the error in our procedure is how the rate of precession varies with the length of the simulation.

In figure (3) we can see that the rate of precession is on the scale of hundreds of arcseconds per century for a relatively short simulation time of around one Earth year. The precession rate quickly decreases and seems to converge as the length of the simulation is increased past 20 Earth years. We stop this test at a length of 100 Earth years because, in this simulation, the parameters used were that of the Sun-Mercury system, and the known value of Mercury's perihelion shift is about 43 arcseconds per century.

Now that we're (hopefully) at least somewhat confident the algorithm produces physically accurate results, let's analyze the Sun-Mercury system.

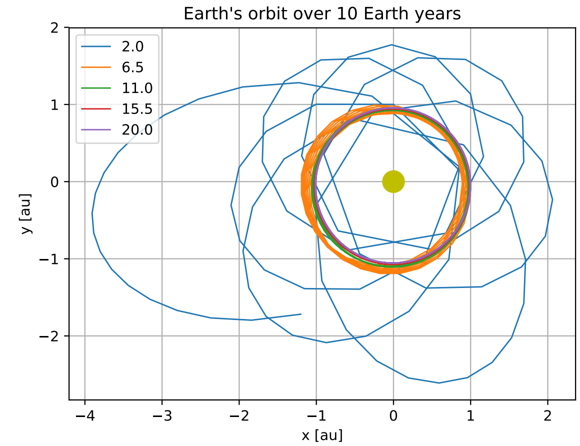

FIG. 2: The trajectory of Earth over 10 Earth years for several different scaling parameters α, and therefore different time steps.

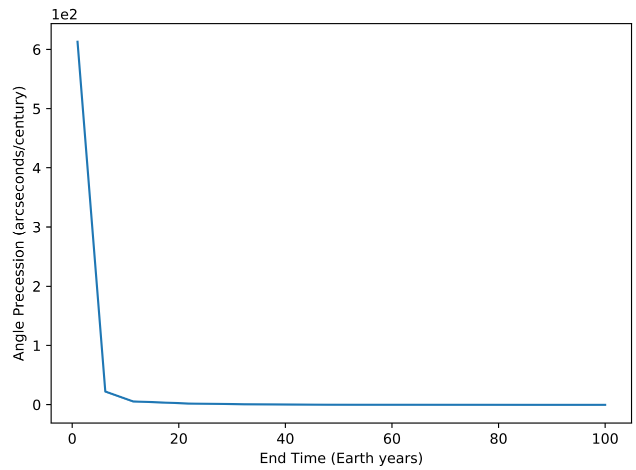

FIG. 3: The precession rate as a function of the length of the simulation. The parameters used in this simulation were that of the Sun-Mercury system.

IV. RESULTS

Starting off, all the following results were done for the Sun-Mercury system, and with Mercury starting at it's perihelion (about 0.3 AU) on the x-axis with its velocity at that point, a time-step scaling parameter of $\alpha = 2000$, and a simulation time of 100 Earth years. We start at perihelion because then we know that the velocity is orthogonal to the position, and it's a good starting point nonetheless. Before we actually look at some results, let's see the actual strength of the GR effects at Mercury. In figure (4), we can see the effects of GR are *quite* minuscule, on the order of nanometers, compared to the closer regimes as shown in figure (1).

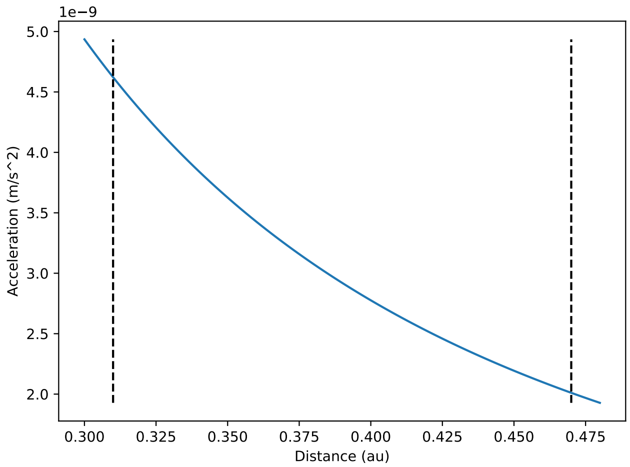

FIG. 4: The strength of the gravitational acceleration from the Sun due to GR in the regime of Mercury's orbital distance. The vertical dashed, black lines correspond to the distance between Mercury and the Sun at Mercury's perihelion and its aphelion.

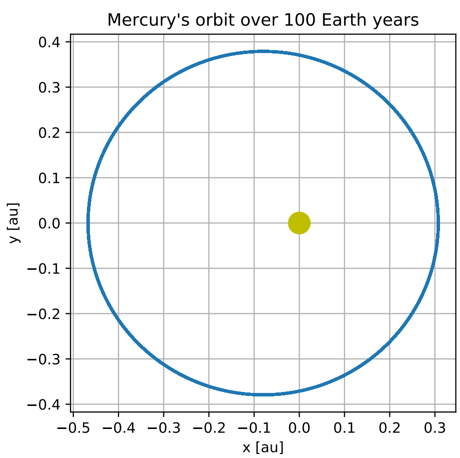

FIG. 5: The trajectory of Mercury under the influence of MOND over the course of 100 Earth years.

Since the effects of GR are so small at Mercury, the orbit should be almost negligibly close to the Newtonian orbit. That is, we shouldn't be able to see any difference over any reasonable time scale. In figure (5), that's exactly what we see.

Though if we take a closer look, specifically at the angle of the perihelion at each orbit, we can see GR's almost hidden effects. In figure (6), we can see that the perihelion actually oscillates in it's angle, while still having a trend to decrease linearly. It's here that the importance of the (ridiculously) small time step (of scale factor $\alpha = 2000$) is needed. If the time step were any bigger the angular precession would not be on the right scale as the known

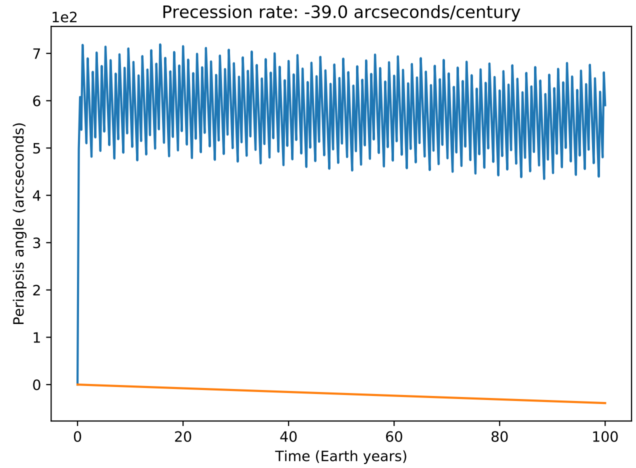

FIG. 6: The angular precession of the perihelion of Mercury over the course of 100 Earth years.

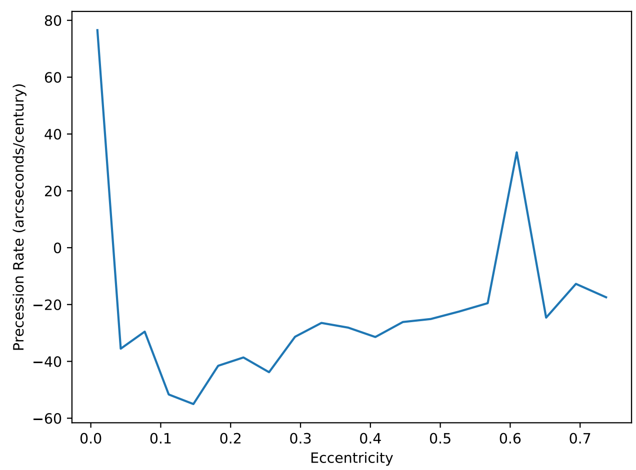

FIG. 7: The rate precession of Mercury if it had different eccentricities.

43 arcseconds per century.

That being said, I also don't know why the angle jumps up and oscillates with such high amplitude. The main takeaway here though is that the overall behavior of the perihelion is that it does linearly shift with time. In this case the overall rate of precession is around the known value (in magnitude) of 43 arcseconds per century, but, getting back to the main idea, what how would this change with different eccentricity?

In figures (7) and (8), we see how the rate of precession changes as the eccentricity of the orbit changes. Well, the overall behavior is there. As we increase eccentricity, the rate of precession decreases.

In this analysis, only eccentricities $\epsilon < 0.8$ were considered because the implementation evolved the simulation in time, and since an increase in eccentricity corresponds

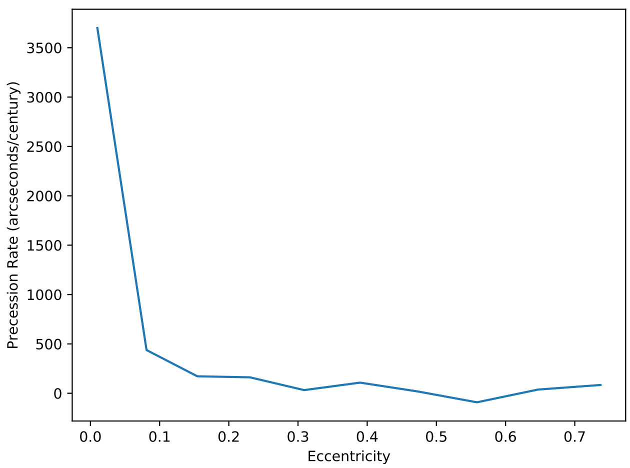

FIG. 8: The rate precession of Mercury if it had different eccentricities. Each simulation here was done with a simulation time of 20 Earth years.

to a non-linear increase in orbital period, the computation times associated with eccentricities $\epsilon > 0.8$ were unreasonably large.

V. CONCLUSIONS

While GR may be mostly unobservable in our solar system, there is at least one candidate for the experimental verification of such a beautiful theory. The perihelion shift of Mercury was one of the first confirmations of Einstein's radical idea, and has since offered itself for us to study and gather insight into GR. Though since Mercury, with it's high eccentricity, is the one of the few data points to measure, we must turn to computational simulations to study the effects of eccentricity on the general relativistic precession of orbital periapsides. From this analysis, it was found the rate of precession due to GR decreases as the eccentricity of the orbit increases.

VI. FUTURE WORK

In the future I would find a way to increase accuracy while increasing the size of the time step because the amount of time it took to produce some of this data seemed to be too long. Another numerical idea to think about would be how to minimize error and such because these simulations are over such long time periods, and thus there are thousands of times the error associated with a single step is compounded. In terms of physical aspects, the power loss due to the GR precession causing the system to emit gravitational waves could be investigated. From there, the amount of time it would take for the bodies to collide could be found. If this were done though, the consideration of whether or not the application of the modified gravity would stil be valid because once the bodies are close enough, the system transitions into the strong-field regime. Once this is done, the full general relativistic calculations would need to be done.

[1] M. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, 1972).

[2] C. Körber, I. Hammer, J.-L. Wynen, J. Heuer, C. Müller, and C. Hanhart, Physics Education **53**, 055007 (2018).

[3] J. Taylor, *Classical Mechanics* (University Science Books, 2005).