02/10/2013

za $a, b \in \mathbb{R}$ velja natančno ena od možnosti $a>b, a<b$ ali $a=b$

: $a<b$ in $b<c \Rightarrow a<c$

: $a<b \Rightarrow a+c<b+c$

: $a<b$ in $c>0 \Rightarrow ac<bc$

: $a \neq 0 \Rightarrow a^2>0$

: $1>0$

: $c<0$ in $a<b \Rightarrow ac>bc$

: $a<b \Rightarrow (-a)>(-b)$ and if $a<0 \Rightarrow (-a)>0$

: $ab>0 \Rightarrow$ both (+) or both (−)

: $a<c$ and $b<d \Rightarrow (a+b)<(c+d)$

CELA IN RACIONALNA ŠTEVILA

1) 1 is a positive integer whose existance is guaranted by axiom 4.

2) $2=1+1$, $3=2+1$, ...

Positive integers $= \{1, 2, 3, \ldots\}$

Def: A set of real number is called an INDUCTIVE SET if it has the following properties: - number 1 is in the set
- for every x in this set we have that $(x+1)$ is also in the set

example: - $\mathbb{R}$ is an inductive set

- $\mathbb{R}^+$ is also an inductive set

Def: A real number is called POSITIVE INTEGER if it belongs to an inductive set!

We denote by $\mathbb{P}$ the set of all positive integer. $\mathbb{P}$ itself is an inductive set.
Since $\mathbb{P}$ belongs to any inductive set, then $\mathbb{P}$ can be viewed as the <u>smalest inductive set</u>!

$N = \{-1, -2, -3, \ldots\}$ - negative integer

$\mathbb{Z} = P \cup \{0\} \cup N$ - set of all integers

$\mathbb{Q}$ - racional numbers = quotient of integers $\frac{a}{b}$; $b \neq 0 \wedge a, b \in \mathbb{Z}$

$\mathbb{R} \setminus \mathbb{Q}$ - iracional numbers → Is $\mathbb{R} \setminus \mathbb{Q} \neq \emptyset$? Yes, because $\sqrt{2} \notin \mathbb{Q}$!

Proff: use contradiction $\Rightarrow \sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{a}{b}$; $a, b \in \mathbb{Z}$; $b \neq 0$
with no loss of generality we assume that a/b is already simplyfied to the lowest type (POKRAJŠAN) $\Rightarrow a, b$ cannot both be even numbers, otherwise you can simplyfy by 2...

$\sqrt{2} = \frac{a}{b} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ even number $\Rightarrow a$ is even number, $a=2k \Rightarrow 4k^2=2b^2 \Rightarrow b^2=2k^2 \Rightarrow$ <u>b evenn</u>

Def.: Upper bound of a set, maximum element

We consider S is not empty set $S \neq \emptyset$ and assume that B is such that $x \leq B \ \forall x \in S$. Than B is upper bound for S.

If in addition we know that $B \in S$ than $B = \max S \rightarrow$ maximum element of B (~~SUPREMUM~~)

A set with no upper bound is called to be UNBOUNDET SET.

example: - positive numbers $P = \{1, 2, \dots\}$ is an unboundet set
- $[0,1] \Rightarrow \max [0,1] = 1$
- $[0,1) \Rightarrow 1$ is an upper bound but is not maximum element because $1 \notin [0,1)$!

Def.: A number B is called LEAST UPPER BOUND (SUPREMUM) for set S if B satisfies the following properties: - B is an upper bound for S
- no number less than B is an upper bound for S

remark: $m = \max S$ and : a set that has supremum need not to have a maximum!

Theorem: Two diferent numbers cannot be least upper bound for the same set.

proff: Suppose that B and C are both supremum of S, but since B is supremum than (by def.) $C \geq B$ ... $B \geq C \Rightarrow B = C$

Axiom 10: (completness axiom)

! Every non empty set S of real numbers which is bounded above has a supremum ($\exists B \in \mathbb{R}$; $B = \sup S$) !

In anology we introduce the terms LOWER BOUND of a set S b ; $b \leq x$; $\forall x \in S$

The greatest lower bound of S is called INFINIMUM $= \inf S$

Def.: Some number L is called the greatest ~~number~~ lower bound (INFINIMUM) for S if L is lower bound for S and there is no greater number than L to be lower bound

- $(0,1]$ – 0 is lower bound but is not a minimum
- $[0,1]$ – $0 = \inf [0,1] = \text{minimum } [0,1]$

Theorem (1.7): Every nonempty set S that is bounded bellow has a greatest lower bound! ($\exists L \in \mathbb{R}$; $L = \inf S$)

ARCHIMEDION PROP. of $\mathbb{R}$: The set P of positive integer is unbounded from abowe!

Proff: (by contradiction) we asume that P is bounded above $\Rightarrow$ (by Axiom 10) $\Rightarrow$ we know that P has supremum b ($b = \sup P$). The number $(b-1)$ cannot be an upper bound for P. Hence there is as least a positive integer...

Theorem: $\forall x \in \mathbb{R}$ $\exists$ a pos. integer n such that $n > x$ $\forall x \in \mathbb{R}$ $\exists n > 0$ - pos. integ. ; $n > x$

proff: If not some x would be an upper bound for P!

Theorem: If $x > 0$ and y is an arbitrary number, than $\exists n$ positive integer such that $nx > y$

proff: Apply the theorem before with x defined by y/x

Theorem: If $a, x, y \in \mathbb{R}$ are such that $a \leq x \leq a+\frac{y}{n}, \forall n > 1$ than $x = a$

Pf: If $x > a$, by Arch. prop. exists an int. n s.t. $n(x-a) > y$ ...

Properties of supremum and infinum.

Theorem: Let h be a given (small) number and let S be a set of $\mathbb{R}$ number.

(*)

1.) if S has a supremum, than for some $x \in S$ we have $x > (\sup S - h)$

2.) if S has an infinum, than exists some $x \in S$ we have $x < (\inf S + h)$

Proff: If we had $x < (\sup S - h)$; $\forall x \in S$ than $(\sup S - h)$ would be an upper bound, but beeing $(\sup S - h) < (\sup S)$ we arrive at contradiction! (ther is only one sup S)

Theorem: (ADDITIVE PROPERTY): Given nonempty sets A and B, let denote the set $C = \{a+b; a \in A \wedge b \in B\}$, then: a) if each A and B has a supremum, than also C has a supremum, $\boxed{\sup C = \sup A + \sup B}$!

b) if each A and B has an infimum than also C has a infimum: $\boxed{\inf C = \inf A + \inf B}$

Proff: Take $c \in C$; $c = a+b$; $a \in A \wedge b \in B \Rightarrow c \leq \sup A + \sup B$ for any $a \in A \wedge b \in B$

$\Rightarrow \sup C \leq \sup A + \sup B$; by a previous theorem (*) we have that $h = \frac{1}{n}$;

$a > \sup A - \frac{1}{n}$ and $b > \sup B - \frac{1}{n}$.

$\sup A + \sup B < \underbrace{a+b}_{C} + \frac{2}{n} \leq \sup C + \frac{2}{n} \Rightarrow \sup C \leq \sup A + \sup B \leq \sup C + \frac{2}{n}; \forall n \geq 1$

$\Rightarrow$ by archimedian prop. $\Rightarrow \sup C = \sup A + \sup B$ ✓

Theorem: Given two nonempty sets S an T of $\mathbb{R}$; $s < t$, $\forall s \in S \wedge \forall t \in T$, than S has supremum and T has infimum and $\boxed{\sup S \leq \inf T}$!

Pf: any $\forall t \in T$ is an upper bound for S, therefore S has a supremum that $\sup S \leq t$; $\forall t \in T$. Hence $\sup S$ is a lower bound for T, so T has an infinum which $\inf T \geq \sup S$

THE PRINCIPLE OF MATHEMATHICAL INDUCTION

Method of the proff by induction: Let $A(n)$ is an ~~susta~~ certation involving an integer n. We conclude that $A(n)$ is true for any integer n if: $\boxed{1.)}$ Proff that $A(n_1)$ is true! (usualy is $n_1=1$ but is not necesary)

$\boxed{2.)}$ Proff that if k is any integer and $k>n_1$ and asume that $A(k)$ is true and prof that $A(k+1)$ is also true!

example: $A(n): 1^2+2^2+\dots+(n-1)^2<\frac{n^3}{3}$; $\sum_{i=1}^{n}(i-1)^2<\frac{n^3}{3}$

1.) $n_1=1 \Rightarrow 0^2<\frac{1^3}{3} \Rightarrow 0<\frac{1}{3}$ ✓

2.) Assume that $A(n)$ is true for $n=k$: $A(k) \overset{?}{\Rightarrow} A(k+1)$: $1^2+2^2+\dots+(k-1)^2<\frac{k^3}{3}=$
$1^2+2^2+\dots+(k-1)^2<\frac{k^3}{3} \Rightarrow 1^2+2^2+\dots+(k-1)^2+k^2<\frac{(k+1)^3}{3} \Rightarrow \frac{k^3}{3}+k^2<\frac{(k+1)^3}{3} \Rightarrow$
$\Rightarrow \frac{(k+1)^3}{3}=\frac{1}{3}(k^3+3k^2+3k+1)=\frac{k^3}{3}+k^2+k+\frac{1}{3} \Rightarrow \cancel{\frac{k^3}{3}}+\cancel{k^2}<\cancel{\frac{k^3}{3}}+\cancel{k^2}+k+\frac{1}{3} \Rightarrow 0<k+\frac{1}{3}$ ✓

ABSOLUTE VALUES AND TRIANGLE INEQUALITY

- If $x\in\mathbb{R}$, then $\boxed{|x|=\begin{cases}x, & \text{is } x>0\\ -x, & \text{is } x<0\end{cases}}$; grapf is →

Theorem: If $a>0$, than $|x|\le a$, if and only if $-a\le x\le a$.

Pf: assome that $|x|\le a$, than we have $-a\le-|x|$ that either $x=|x|$ or $x=-|x|$, and hence we have $-a\le-|x|\le x\le|a|\le a$
assome that $-a\le x\le a$ then if $x\ge 0$ then we have $|x|=x\le a$ ... (beri knjigo)

- Δ NEENAKOST: For any $\mathbb{R}$ number x, y, we have that $\boxed{|x+y|\le|x|+|y|}$

Pf: We know that $-|x|\le x\le|x|$ and $-|y|\le y\le|y|$ then we sum up $\Rightarrow$
$\Rightarrow -|x|-|y|\le x+y\le|x|+|y|$! By previous theorem we have $|x+y|\le|x|+|y|$ ✓

Theorem: Sum of finaly many $\mathbb{R}$ numbers $a_1, a_2, \dots a_k$, we have that $\boxed{\left|\sum_{l=1}^{k}a_l\right|\le\sum_{l=1}^{k}|a_l|}$

Pf: • $k=2$ is true by Δ inequality
• if true for $(k) \Rightarrow$ true for $(k+1)$ $\quad \left|\sum_{1}^{n+1}a_i\right|=\left|\sum_{1}^{n}a_i+a_{n+1}\right|\le\left|\sum_{1}^{n}a_i\right|+|a_{n+1}|$ ✓

Theorem: CAUCHY-SCHWARTZ INEQUALITY $\quad (a_1b_1+a_2b_2+\dots+a_nb_n)^2\le(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2)$

If $a_1,\dots a_n$ and $b_1,\dots b_n$ are any $\mathbb{R}$ number than $\boxed{\left(\sum_{k=1}^{n}a_kb_k\right)^2\le\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}b_k^2\right)}$

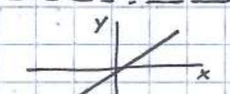
Equality holds only if there is $x\in\mathbb{R}$; $a_kx+b_k=0$; $\forall k$ od $1 \dots n$

We assome that $\sum(a_kx-b_k)^2\ge 0^{(0)}$: Write (0) as $Ax^2+2Bx+C\ge 0$ where $A=\sum a_k^2$; $B=-\sum a_kb_k$; $C=\sum b_k^2$
$B^2\le AC$; if $A=0$ then $a_i=0$ and so $B=0$ so ... so we assome that $A\ne 0$
$Ax^2+2Bx+C=A\left(x^2+\frac{2B}{A}+\frac{C}{A}\right)=A\left(x+\frac{B}{A}\right)^2+\frac{AC-B^2}{A} \Rightarrow$ minimum če $x=-\frac{B}{A}$;
tedaj je dosežena enakost, za vse druge x (poglej knjigo)

Def.: Given two sets X and Y, a function is a correspondance which PRIREDITI each element of X one and only one element of Y.

$$\boxed{\begin{array}{l} f: X \to Y \;;\; X, Y \subseteq \mathbb{R} \\ x \to f(x) \end{array}}$$

npr.: the identity function
$f: \mathbb{R} \to \mathbb{R}$
$x \to f(x) = x$



Def.: ! We say that two ordered pairs (a,b) and (c,d) are equal $(a,b)=(c,d)$, if and only if $a=c$ and $b=d$. P.S. $(a,b) \neq (b,a)$!

Alternative Def.: A function f is a set of ordered pairs (x,y), no two of which have the same first number.

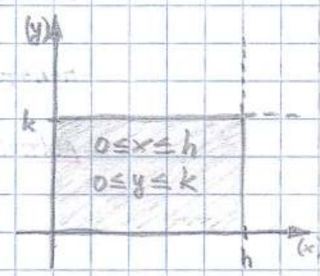
Def.: We have two functions; $f(x)$ and $g(x)$:

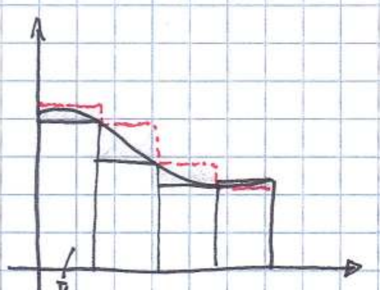
- $(f+g)(x) = f(x) + g(x)$
- $(f \cdot g)(x) = f(x) \cdot g(x)$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$; $g(x) \neq 0$

CONCEPT OF AREA OF A SET FUNCTION

$a: C$ (Colection of sets) $\to \mathbb{R}$
$S \to a(S)$

We denote by m the collection of mesurable set.
Elementary set as rectangle: $\{(x,y);\ 0 \le x \le h,\ 0 \le y \le k\}$





step region

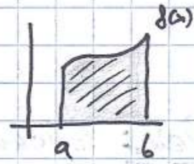
AXIOMATIC definition of the area:

We assume that a class of m-mesurable set, exists and also a function a, whose domain is m, whith the following properties:

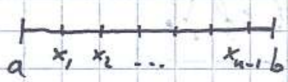
- non-negative
  $\forall S \in m \Rightarrow a(S) \ge 0$
- aditive pr.
  if $S, T \in m$ then $S \cup T$, $S \cap T \in m$
  $$a(S \cup T) = a(S) + a(T) - a(S \cap T)$$
- difference pr.
  if $S, T \in m$ with $S \subseteq T$
  $a(T \setminus S) = a(T) - a(S)$
- invariance under transport. pr.
  Is S and T are comsport. sets (same shape, same size)
  $a(S) = a(T)$
- choice of scale
  For $\forall R \in m$ (R-rectangle), If the edge of R are h and k, than $a(R) = hk$
- exhorstion properties

Let Q be a set which is enclosed by two step regions S, T : $S \subseteq Q \subseteq T$ $(*)$. If there

Ordered set of f: $f:[a,b]\to\mathbb{R}$, is the collection of points (x,y) $(a\le x\le b)$ and $(0\le y\le f(x))$!



PARTITIONS and STEP FUNCTIONS

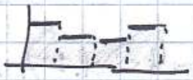


$a\le x_1<x_2<\dots x_{n-1}<b$

$P=\{\underset{a}{\overset{\shortparallel}{x_0}}, x_1, \dots x_{n-1}, \underset{b}{\overset{\shortparallel}{x_n}}\}$ - PARTITION

Def: A function $s:[a,b]\to\mathbb{R}$ is a STEP FUNCTION (peacwise constant function), if there exists a partition $P=\{x_0\dots x_n\}$, such that $s=$ const on each open subinterval (x_i, x_{i+1}); $0\le i\le(n-1)$

For any $i=1,\dots n$ $\exists$ (real number) s_i, such that $s(x)=s_i$ in (x_{i-1}, x_i)



PS: Ex: The sum and the product of two step functions is still a step function!

Def: If s is a step function, then the integral of s in $[a,b]$, $\int_a^b s(x)dx=\sum_{i=1}^{n} s_i(x_i, x_{i+1})$ (D)

The definition D does not depent on the partition that we use to express s.

PROPERTIES: (of the integral of a step function)

- ADDITIVE: $\int_a^b [s(x)+t(x)]\,dx = \int_a^b s(x)\,dx + \int_a^b t(x)\,dx$
- HOMOGENIOUS: $\forall c\in\mathbb{R}$: $\int_a^b c\cdot s(x)\,dx = c\cdot\int_a^b s(x)\,dx$
- LINEARITY: $\forall c_1, c_2\in\mathbb{R}$: $\int_a^b [c_1 s(x)+c_2 t(x)]\,dx = c_1\int_a^b s(x)\,dx + c_2\int_a^b t(x)\,dx$
- MONOTONICITY: If $s(x)\le t(x) \Rightarrow \int_a^b s(x)dx \le \int_a^b t(x)\,dx$
- ADITIVITY w.r.t. the interval of integ.

$$\int_c^b s(x)dx + \int_a^c s(x)\,dx = \int_a^b s(x)\,dx$$

- INVARIANT UNDER TRANSLATION

$$\int_c^b s(x)\,dx) = \int_{a+c}^{b+c} s(x-c)\,dx$$

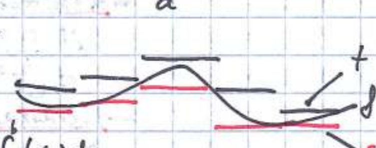
- EXPANSION of interval of integration

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)dx = \int_a^b s(x)\,dx$$

- $\int_a^b s(x)\,dx = -\int_b^a s(x)\,dx$; $a\le b$ and $\int_a^a s(x)dx=0$

The integral of more general function:

to aproximate f by two step functions s and t; $s(x)\le f(x)\le t(x)$. $\int_a^b s(x)dx \le \int_a^b t(x)dx$



Consider $f:[a,b]\to\mathbb{R}$ bounded $\left(\equiv \exists M>0; |f(x)|\le M, \forall x\in[a,b]\right)$ → def. for BOUNDED F.

Definition of integral for bounded functions: Let $f:[a,b]\to\mathbb{R}$ bounded, and s,t two step functions s.t. (**) $s(x)\le f(x)\le f(x)$; $\forall x\in[a,b]$. If there is one and only one number $I\in\mathbb{R}$ s.t.: $\int_a^b s(x)dx \le I \le \int_a^b t(x)dx$; $\forall s,t$ step functions satisfying (**)

The UPPER and LOWER INTEGRALS:

Consider s, t – two step function: $s,t:[a,b] \to \mathbb{R}$, s.t. $s(x) \le \underset{\text{f - bounded}}{f(x)} \le t(x)$; $\forall x \in [a,b]$

$\boxed{S = \{\int_a^b s(x)dx;\ s \le f\}}$ and $\boxed{T = \{\int_a^b t(x)dx;\ t \ge f\}}$. Since f is bounded $S, T \ne \emptyset$

$\exists(\sup S)$ and $(\inf T)$ from previous result $\Longrightarrow \int_a^b s(x)dx \le \sup S \le \inf T \le \int_a^b t(x)dx$

Therefore f is INTEGRABALE if and only if $\boxed{\sup S = \inf T.}$

$\sup S$ = lower integral of f, and $\inf T$ = upper integral of f!

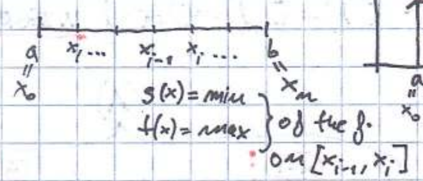
Theorem: Let f be a bounded function: $f:[a,b] \to \mathbb{R}$, than f has lower integral $\underline{I}(f)$ and an upper integral $\overline{I}(f)$ satisfying the inequalaties:

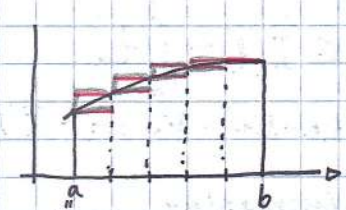
$$\int_a^b s(x)\,dx \le \underline{I}(f) \le \overline{I}(f) \le \int_a^b t(x);\ \forall s,t\text{-step function s.t. } s \le f \le t$$

Def: A bounded function $f:[a,b] \to \mathbb{R}$ is said to be INTEGRABLE if and only if $\boxed{\underline{I}(f) = \overline{I}(f)}$ and we set $\boxed{\int_a^b f(x)\,dx = \underline{I}(f) = \overline{I}(f)}$!

np.r.:

$\underline{I}(f) \le \overline{I}(f)$





$\underline{I}(f) = \sup \int_a^b s(x)dx$ over all the possible partitions over $[a,b]$

$\overline{I}(f) = \inf \int_a^b (t)(x)dx$ — // —

IF $\underline{I}(f) = \overline{I}(f)$ THAN f is INTEGRABLE!

Fix $x \in \mathbb{R}$. $\forall \varepsilon > 0\ \exists q \in \mathbb{Q}$ s.t. $|x - q| < \varepsilon$ ($\mathbb{Q}$ is DENSE in $\mathbb{R}$) [if $x \in \mathbb{R} \setminus \mathbb{Q}$; otherwise is trivial]

NPR.:

The DIRAC function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \Rightarrow f \text{ is bounded}$$

For the Dirac function we have that $\underline{I}(f) = 0$ and $\overline{I}(f) = 1$.

Exists function f which is bounded but not integrable!

Theorem: Let $f:[a,b] \to \mathbb{R}^+ \cup \{0\}$ integrable and let Q be the ordinate set of f over $[a,b]$. Then Q is mesurable and its area is equivalent to $\int_a^b f(x)\,dx$

proof:

Consider S and T two step regions, which enclose Q: $S \subseteq Q \subseteq T$ and let s and t two step functions, s.t. $s(x) \le f(x) \le t(x)$: $\forall x \in [a,b]$.

$a(S) = \int_a^b s(x)dx$ and $a(T) = \int_a^b t(x)dx$.

Since f is integrable we have $I(f) = \int_a^b f(x)dx$ is the only number satisfying $\int_a^b s(x)dx \le I(f) \le \int_a^b t(x)dx$; $\forall s,t$-step function; $s \le t$

Moreover this is the only number that satisfy $a(S) \le I(f) \le a(T)$; $\forall S,T$-step region

By the exhaustion property we have $a(Q) = I(f)$

Theorem: $f:[a,b] \to \mathbb{R}$ non negative, integrable

Then the graph of f ($\{(x,y),\ x \in [a,b],\ y = f(x)\}$) has area zero!

pf: $Q' = \{(x,y);\ a \le x \le b;\ 0 \le y < f(x)\}$, by previous theorem we have that $a(Q) = a(Q')$

$a(\text{graph}) = a(Q - Q') \overset{\text{DIFF. PR.}}{\le} a(Q) - a(Q') = 0$

Theorem 1: $f:[a,b] \to \mathbb{R}$ is bounded and monotone function $\Rightarrow f$ INTEGRABLE !

P.S. – Monotone increasing function: $x < y \Rightarrow f(x) \leq f(y)$
$\Rightarrow f(x) < f(y) \to$ strictly increasing f.
– Monotone decreasing function: $x < y \Rightarrow f(x) \geq f(y)$
$\Rightarrow (f(x) > f(y) \to$ strictly decreasing f.

pf: Let's proof for monotone increasing function.
Let n – positive integer and partition $P = \{a, x_1, \dots x_{n-1}, b\}$ so that $[x_{k-1}, x_k]$ are s.t. $x_k - x_{k-1} = (b-a)/n$

Define special step function $s_n(x) = f(x_{k-1})$ and $t_n(x) = f(x_k)$; $\forall x$; $x_{k-1} \leq x \leq x_k$

We know by hypotesis (f-monotone) $\Rightarrow s_n(x) \leq f(x) \leq t_n(x)$; $\forall n$

Consider $\int_a^b t_n - \int_a^b s_n = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) = \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = \frac{(b-a)(f(b)-f(a))}{n}$

The lower and upper integral of f:

$$\int_a^b s_n \leq \underline{I}(f) \leq \int_a^b t_n \text{ and } \int_a^b s_n \leq \overline{I}(f) \leq \int_a^b t_n$$

$$\overline{I}(f) - \underline{I}(f) \leq \int_a^b t_n - \int_a^b s_n \leq \frac{1}{n}(b-a)(f(b)-f(a)), \forall n \Rightarrow \overline{I}(f) - \underline{I}(f) = 0 \Rightarrow f \text{ INTEGRABLE}$$

Theorem 2: $f:[a,b] \to \mathbb{R}$ is bounded and increasing
we define $x_k = a + k\left(\frac{b-a}{n}\right)$ for $k = 0, 1, 2, \dots, n$

If I is any number that satisfies inequality $\overbrace{\underbrace{\frac{b-a}{n} \cdot \sum_{k=0}^{n-1} f(x_k)}_{\int s_n} \leq I \leq \underbrace{\frac{b-a}{n} \cdot \sum_{k=1}^{n} f(x_k)}_{\int t_n}}^{(A)}$; $n \geq 1$! !

than $\boxed{I = \int_a^b f(x)\,dx}$!

pf: Let s_n and t_n be two special approximating function than by (A):
$\int_a^b s_n \leq I \leq \int_a^b t_n$; $\forall n$, But by defenition of integral we also have $\int_a^b s_n \leq \int_a^b f \leq \int_a^b t_n$; $\forall n$! We have that $0 \leq \left| I - \int_a^b f(x)dx \right| \leq \left| \int_a^b t_n - \int_a^b s_n \right| \leq$
$\leq (b-a)\frac{f(b)-f(a)}{n}$; $\forall n \Rightarrow I = \int_a^b f(x)dx$!

Theorem 3: If p – positive integer and $b > 0$; $b \in \mathbb{R}$, we have $\boxed{\int_0^b x^p dx = \frac{1}{p+1} b^{p+1}}$

pf: We consider $\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$ (proof by induction); $\forall n \geq 1$, $\forall p \geq 1$ (*)

Multyply (*) by $\frac{b^{p+1}}{n^{p+1}}$ we obtain $\frac{b}{n}\sum_{k=1}^{n-1}\left(\frac{kb}{n}\right)^p < \frac{b^{p+1}}{p+1} \leq \frac{b}{n}\sum_{k=1}^{k}\left(\frac{kb}{n}\right)^p$

If $f(x) = x^p$ and $x_k = \frac{kb}{n}$; $k = 0, 1, \dots, n$ than $\frac{b}{n}\sum_{k=0}^{n-1} f(x_k) < \frac{b^{p+1}}{p+1} < \frac{b}{n}\sum_{k=1}^{n} f(x_k)$

so by Theorem 2 with $f(x) = x^p$ and $a = 0$ and $I = b^{p+1}/(p+1)$ we have that $\int_0^b x^p dx = \frac{x^{p+1}}{p+1}$!

>

BASIC PROPERTIES OF INTEGRALS

- LINEARITY with respect to (=w.r.t.) the integrand

$$\int_a^b (c_1 f + c_2 g) = c_1 \int_a^b f + c_2 \int_a^b g \; ; \forall c_1, c_2 \in R$$

- ADDITIVITY w.r.t. the interval of integration

$$\int_a^b f + \int_b^c f = \int_a^c f$$

- INVARIANT under translation

$$\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx$$

- EXPANSION or CONTRACTION

$$\int_a^b f(x)\,dx = \frac{1}{k} \int_{ka}^{kb} f\left(\frac{x}{k}\right) dx$$

- COMPARISON

$$f(x) \leq g(x) \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Remark: in theorem 3 we saw that $\int_0^b x^p dx = b^{p+1}/(p+1)$

$\oplus$

(LINEARITY w.r.t. the integrand)

$\Downarrow$

$\int_0^b (\text{polinoms})\,dx$

TLE SEM MANJKAL ZRAD ZAVODA ZA ZAPOSLOVANJE (09/10/2013)

ZVEZNOST FUNKCIJ

Def: Funkcija $f:[a,b] \to \mathbb{R}$ je zvezna v $x=p$ če:

(i) funkcija f je definirana v $x = p \in [a,b]$

(ii) $\lim_{x\to p} f(x) = f(p)$

torej: $\forall \varepsilon > 0, \exists \delta > 0$ tako da $\forall x, |x-p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

Theorem: Funkciji $f, g: [a,b] \to \mathbb{R}$ sta obe v $x=p$ zvezni in $\lim_{x\to p} f(x) = A$ in $\lim_{x\to p} g(x) = B$:

1.) $\lim [f+g] = \lim f + \lim g = A + B$

2) $\lim [-f(x)] = -A$

3) $\lim [f \cdot g] = A \cdot B$

4) $\lim \left[\frac{f}{g}\right] = \frac{A}{B}$; $B \neq 0$

Theorem: SQUIZING PRINCIPLE

Naj bodo $f(x) \leq g(x) \leq h(x)$ $\forall x, x \neq p$

in naj $\lim_{x\to p} f(x) = \lim_{x\to p} h(x) = A \Longrightarrow \lim_{x\to p} g(x) = A$

Theorem: Naj bosta $f(x)$ in $g(x)$ zvezni na $[a,b]$, potem so

1) $f \pm g$

2) $f \cdot g$

3) $\frac{f}{g}$; $g \neq 0$ tudi zvezne funkcije na $[a,b]$!

ZVEZNOST NEDOLOČENEGA INTEGRALA

$f: [a,b] \to \mathbb{R}$ naj bo INTEGRABILNA in OMEJENA in naj bo $A(x) = \int_a^x f(t)\,dt$. Potem je ~~tudi~~ $A(x)$ ZVEZNA za $\forall x \in [a,b]$

Example:

$$\sin(x) = \int_0^x \cos t \, dt \quad \text{and} \quad \cos(x) = 1 - \int_0^x \sin t \, dt$$

Example:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \, ?$$

Pf: $0 < \cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$; $\forall x \in (0, \frac{\pi}{2})$ and $\forall x \in (-\frac{\pi}{2}, 0)$

$\downarrow_{x \to 0}$ $\qquad$ $\downarrow_{x \to 0}$

$1 \longrightarrow 1 \Rightarrow$ squizing principle $\Rightarrow \lim_{x \to 0} \frac{\sin x}{x} = 1$

KOMPOZITUM FUNKCIJ

Def: $x \to \underbrace{v(x)}_{y} \to u(y) := u \circ v(x) = u(v(x)) \to$ NOT KOMUTATIVE!

Ex: $v(x) = x^2$; $u(y) = \sin y$

$f(x) = (u \circ v)(x) = \sin(x^2)$

$g(y)$

Example: PAZI DEFINICIJSKO OBMOČJE KOMPOZITUMA

$u(x) = \sqrt{x}$; $x \geq 0$; $v(x) = 1 - x^2$

$(u \circ v)(x) = \sqrt{1 - x^2}$; $1 - x^2 \geq 0 \Rightarrow -1 \leq x \leq 1$

Theor: Consider v countineous at p an u countineous at q ; $q = v(p)$

So $f = u \circ v$ is countineous at p!

Pf: (i) $\forall$ neigherbur $N_1[u(q)]$ $\exists$ $N_2(q)$ s.t. $u(y) \in N_2[u(q)]$ if $y \in N_2(q)$

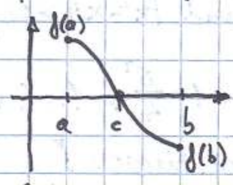
(ii) $\forall$ — " — $N_2[q]$ $\exists$ $N_3(p)$ s.t. $v(x) \in N_2(q)$ if $x \in N_3(p)$

If $y = v(x)$ and combine (i) and (ii)

$\forall$ neigh. $N_1[u \circ v(x)]$ $\exists$ $N_3(p)$ s.t. $u[v(x)] \in N_1(u(v(x)))$ $\forall x \in N_3(p)$

Theorem: Let $f: [a,b] \to \mathbb{R}$ contimeous and assume that $f(a)$ and $f(b)$ have opposite sign, then $\exists c \in [a,b)$ s.t. $f(c) = 0$! BOLZANO THEOREM

Pf.:



Theorem: (SIGN PRESERVING PROPERTY OF COUNTINOUS FUNCTION)

Let $f: [a,b] \to \mathbb{R}$ contimeous and supose that $f(c) \neq 0$; $c \in (a,b)$ then exists $\delta > 0$ s.t. $\forall x \in (c - \delta, c + \delta)$, $f(x)$ has the same sign of $f(c)$!

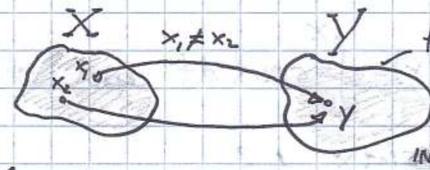
Pf: Chose $\varepsilon = f(c)/2$; $\exists \delta = \delta(\varepsilon)$ s.t. $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$; $\forall x \in (c - \delta, c + \delta)$

$0 < \frac{f(c)}{2} < f(x) < \frac{3}{2} f(c) \Rightarrow f(x) > 0$, $\forall x \in (c - \delta, c + \delta)$

Theorem: INTERMEDIATE VALUE TH. Let $f: [a,b] \to \mathbb{R}$ countineous. Take $x_1, x_2 \in [a,b]$ s.t. $f(x_1) \neq f(x_2)$, then f takes on any values between $f(x_1)$ and $f(x_2)$!

Pf: $f(x_1) < f(x_2)$ Let k s.t. $f(x_1) < k < f(x_2)$! Define $g(x) = f(x) - k$, countineous. So we have $g(x_1) < 0$ and $g(x_2) > 0$. By Boltzano theorem exists $c \in (x_1, x_2)$ s.t. $g(c) = 0$!

DEFINITION OF INVERSE FUNCTION



to ne sme bit! (en original, ena sliko)
$\forall y \in f(Y)$ $\exists$ one and only one $x \in X$ s.t. $f(x) = y$!

INJECTIVITY:
$\forall y \in f(X)\ \exists!\ x \in X;\ f(x) = y$

SURJECTIVITY:
$f(X) = Y$

BIJECTIVITY = INJECTIVITY + SURJECTIV.

Ex: Strictly monotone functions are inversible!

$$f(x) = 2x + 1 \Rightarrow g(y) = \frac{1}{2}y - 1 \rightarrow \text{INVERSE}$$

Theorem: $f:[a,b] \rightarrow \mathbb{R}$, strictly increasing and countinuous
$f(a) = c$ and $f(b) = d$. Let g the inverse, $g:[c,d] \rightarrow \mathbb{R}$
then: 1) g is strictly increasing
2) g is countinuous on $[c,d]$

Theorem: $f: S \subseteq \mathbb{R} \rightarrow \mathbb{R}$, contineous (tudi če ni zvezna ta definicija je vedno velja)

Def.: → f has an absolute maximum on S if $\exists c \in S$, s.t. $f(x) \leq f(c)$, $\forall x \in S$
$f(c)$ is the absolute maximum!
→ f has an absolute minimum os S if $\exists c \in S$, s.t. $f(x) \geq f(c)$, $\forall x \in S$
$f(c)$ is the absolute minimum!

Theorem: BOUNDEDNES THEOREM FOR CONTINUOS FUNCTIONS
Let $f:[a,b] \rightarrow \mathbb{R}$ contineous, then f is bounded in $[a,b]$!
$\exists M > 0$, s.t. $|f(x)| < M$, $\forall x \in [a,b]$

PS. Obratno ni res, iz tega da je fun. omejena ne sledi da je zvezna!

Theorem: Let $f:[a,b] \rightarrow \mathbb{R}$ continuos, $\exists c, d \in [a,b]$ s.t. $f(c) = \sup\limits_{[a,b]} f$ and $f(d) = \inf\limits_{[a,b]} f$

CONTINUITY and UNIFORM CONTINUITY $f: S \rightarrow \mathbb{R}$

continuity: Fix $x_0 \in S$; $\forall \varepsilon > 0$ $\exists \delta = \underline{\delta(x_0, \varepsilon)} > 0$, s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x$ s.t. $|x - x_0| < \delta$

Uniform. c.: Fix $x_0 \in S$; $\forall \varepsilon > 0$ $\exists \delta = \underline{\delta(\varepsilon)} > 0$, s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x$ s.t. $|x - x_0| < \delta$

Example: $f(x) = x^2$ on $(0, \infty)$ is continuies but not uniform continues!
We want to profe $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$ $\exists x_0 \in S$ s.t. $|x - x_0| < \delta$ and $|x^2 - x_0^2| > 0$
Take $\varepsilon = 1$, consider $\delta > 0$ and $x_0 = 1/\delta$.
$x = x_0 + \frac{\delta}{2}$, then $x - x_0 = \frac{\delta}{2} < \delta \ldots$ $x^2 - x_0^2 = \left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2 = 1 + \frac{\delta^2}{4} > 1 = \varepsilon$ ✓

Theorem: Let $f:[a,b] \rightarrow \mathbb{R}$ be continuos, then f is uniformly continuos!
OMEJEN INTERVAL, NE SME BIT NEOMEJEN ALI ODPRT...
And then f is also INTEGRABLE!

Theorem: MEAN-VALUE THEOREM FOR INTEGRALS
$f:[a,b] \rightarrow \mathbb{R}$ continuos, then for some $c \in [a,b]$ we have $\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$

Pf: Let m and M be the maximum and minimum of f in $[a,b]$; $m \leq f(x) \leq M$, $\forall x \in [a,b]$
$m(b-a) \leq \int_a^b f(t)\,dt \leq M(b-a) \Rightarrow m \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq M$; $\exists c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$!
BY THE INTERMEDIET VALUE TH.

Theorem: WEIGHTED MEAN-VALUE TH. FOR INTEGRALS
$f:[a,b] \rightarrow \mathbb{R}$ and $g:[a,b] \rightarrow \mathbb{R}$ both continuos. If g never changes sign ($g(x) \geq 0$ or $g(x) \leq 0$ $\forall x \in [a,b]$)
than we have that

$$\int_a^b f(x) \cdot \underbrace{g(x)}_{\text{WEIGHT}}\,dx = f(c)\int_a^b g(x)\,dx$$

Pf: npr. $g(x) \geq 0 \Rightarrow m \cdot g(x) \leq f(x) \cdot g(x) \leq M \cdot g(x)$. If we integrate $\int$

$$m \cdot \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \cdot \int_a^b g(x)\,dx \Rightarrow m \leq \frac{1}{\int_a^b g(x)\,dx} \cdot \int_a^b f(x)g(x)\,dx \leq M,$$

by the intermediate value theorem $\exists c \in (a,b)$, s.t. $f(c) = \dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$. ✓

14/10/2013 # DIFFERENCIAL CALCULUS

Def.: $f:(a,b)\to\mathbb{R}$, the derivative at x of f is $\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$, if the limit exists!

Example: $f(x)=x^m$; m positive integer $\Rightarrow \lim_{h\to 0}\frac{(x+h)^m-x^m}{h}$ + we know $a^m-b^m=(a-b)\sum_{k=0}^{m-1}a^k b^{m-1-k}$ ...

we can choose $a=x+h$ and $b=x \Rightarrow (x+h)^m-x^m = h\cdot\sum_{k=0}^{m-1}\underbrace{(x+h)^k x^{m-1-k}}_{\to x^{m-1}} \Rightarrow (x^m)' = m\cdot x^{m-1}$

(similary for $f(x)=\sin x, \cos x$ ... nisem prepisav izpeljav)

Example: $f(x)=x^{1/m}$; m-positive integer $\Rightarrow \frac{f(x+h)-f(x)}{h}=\frac{(x+h)^{1/m}-(x)^{1/m}}{h}$; let $u=(x+h)^{1/m}$; $v=x^{1/m} \Rightarrow u^m=x+h$; $v^m=x$

$h=u^m-v^m \Rightarrow f'(x)=\frac{u-v}{u^m-v^m}=\frac{1}{(u^{m-1}+u^{m-2}v+\ldots+uv^{m-2}+v^{m-1})}\to\frac{1}{m}x^{1/m-1}$ $\to$ ($m-1$ terms that go to $x^{1-1/m}$)

Theorem: $\lim_{h\to 0}f(x+h)=f(x) \Rightarrow f(x+h)=f(x)+h\cdot\left[\frac{f(x+h)-f(x)}{h}\right]\xrightarrow{h\to 0}$ (If f is differenciable) $\Rightarrow$ (f is countinus at x!)

DIFFERENCIABLE at x $\Rightarrow$ COUNTINUS at x $\to$ v obratni smeri ni res, glej npr. $f(x)=|x|$ $\to$ 

Theorem: Let $f,g:(a,b)\to\mathbb{R}$; f,g have derivative at $x\in(a,b)$, than also: $f\pm g$, $f\cdot g$ and f/g (providet that g is different from zero in a neigh. of x) have derivative ...

i) $(f\pm g)'=f'\pm g'$

ii) $(f\cdot g)'=f'\cdot g+f\cdot g'$

iii) $(f/g)'=\frac{f'\cdot g-f\cdot g'}{g^2}$

Prof: i) $\frac{1}{h}\cdot\left[f(x+h)+g(x+h)-f(x)-g(x)\right]=\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}=f'+g'$

ii) $\frac{1}{h}\left[f(x+h)g(x+h)-f(x)g(x)\right]+\frac{f(x)g(x+h)}{h}-\frac{f(x)g(x+h)}{h}=\ldots f'\cdot g+f\cdot g'$

The derivative is linear: $\Rightarrow (af+bg)'=af'+bg'$; $a,b\in\mathbb{R}$! $\Rightarrow$ (derivative of polynoms)

- $R(x)=\frac{p(x)}{q(x)}$ - RATIONAL FUNCTION; p,q are polynomial $\Rightarrow \boxed{R'(x)=\frac{p'(x)q(x)-p(x)q'(x)}{q^2(x)}}$!

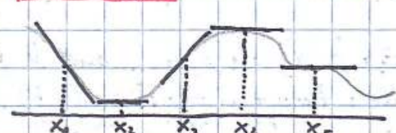
- rational power; $\boxed{f(x)=x^{m/n}}$; m,n are positive integer

Look at $f(x)=x^{1/n}$; $f'(x)=\frac{1}{n}x^{\frac{1}{n}-1}$ than proof by induction $f(x)=x^{2/n}=x^{1/n}\cdot x^{1/n}\to f'(x)=\frac{2}{n}x^{\frac{2}{n}-1}$

than by induction over $m \Rightarrow \boxed{f'(x)=\frac{m}{n}x^{\frac{m}{n}-1}}$!

GEOMETRIC INTERPRETATION OF THE DERIVATIVE AS A SLOPE

$\boxed{f'(x)=\operatorname{tg}\alpha}$, or the SLOPE of the tangent line on f at x; $y=mx+q \Rightarrow m=f'(x)$



$f'(x_1)<0 \Rightarrow f$ is decreasing at x_1

$f'(x_2)=0 \Rightarrow f$ has minimum at x_2 (global)

$f'(x_3)>0 \Rightarrow f$ is increasing at x_3

$f'(x_4)=0 \Rightarrow f$ has maximum x_4

$f'(x_5)=0 \Rightarrow f$ has sadle in x_5

Theorem: (CHAIN RULE) Let f be a compositum of two functions u and v ($f=u\circ v$). Assume that both derivatives $v'(x)$ and $u'(y)$ exists, where $y=v(x)$. Than also the derivative $f'(x)$ exists and we have $\boxed{f'(x)=u'(y)\cdot v'(x)=u'[v(x)]\,v'(x)}$!

Pf: $\frac{f(x+h)-f(x)}{h}=\frac{u(v(x+h))-u(v(x))}{h}\ (*)=\frac{u(v(x+h))-u(v(x))}{v(x+h)-v(x)}\cdot\frac{v(x+h)-v(x)}{h}$; $k=v(x+h)-v(x)\neq 0$

$=\frac{u(v(x)+k)-u(v(x))}{k}\cdot\frac{v(x+h)-v(x)}{h}\xrightarrow{h\to 0}u'(v(x))\cdot v'(x)$

If $v(x+h)-v(x)=0$ we introduce $g(t)=\frac{u(y+t)-u(y)}{t}-u'(y)$; $t\neq 0 \Rightarrow u(y+t)-u(y)=t[g(t)+u'(y)]$ (**)

If we asign $g(0)=0$, than (**) is well defined at $t=0 \Rightarrow g$ is countinus!

Than you replace t in (**) with $k=v(x+h)-v(x)$, we also substitute r.h.s in (**) in $\frac{f(x+h)-f(x)}{h}=\frac{u(y+k)-u(y)}{k}$, we get $\frac{f(x+h)-f(x)}{h}=\frac{k}{h}[g(k)+u'(y)]$. If we let $h\to 0$ than $\frac{k}{h}\to v'(x)$ and $g(k)\to 0$, hence we have $\frac{f(x+h)-f(x)}{h}\to u'(v(x))\cdot v'(x)$!

APLICATION OF DIFFERENTIATION

Def.: $f: S \to \mathbb{R}$ has an ABSOLUTE MAXIMUM on a set S if $\exists c \in S$ s.t. $f(x) \leq f(c)$ $\forall x \in S$!

has an ABSOLUTE MINIMUM on a set S if $\exists c \in S$ s.t. $f(x) \geq f(c)$ $\forall x \in S$!

Def.: $f: S \to \mathbb{R}$ has a RELATIVE MAXIMUM at a point $c \in S$, if $\exists I$ interval s.t. $c \in I$ and $f(x) \leq f(c)$ for $\forall x \in S \cap I$! (podobno za RELATIVE MINIMUM)

Def.: A number which is either a relative max. or min. of a function f is called an EXTREME VALUES of f!

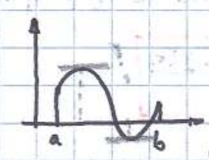
Theorem: $f: (a,b) \to \mathbb{R}$, if f has relative max. or relative min. at an interior point $c \in (a,b)$, than $\boxed{f'(c) = 0}$ (if f is differenciable at $x = c$)!

Pf: $Q(x) = \dfrac{f(x) - f(c)}{x - c}$; $x \neq c$; f is differenciable $\Rightarrow Q(x) \xrightarrow[x \to c]{} Q(c)$ (because we know $f'(c)$ exists)

We argue by contradiction, assume that $Q(c) > 0$. Since Q is continuous at c, than by sign preserving theorem $Q(x) > 0$ in a neigh. I of c. If $f(x) > f(c)$, $x > c$ or $f(x) < f(c)$, $x < c \Rightarrow$ ni extrem

Theorem: (ROLLE'S THEOREM)
$f: [a,b] \to \mathbb{R}$; f continuous in $[a,b]$ and has derivative at any point of open interval (a,b). Assume $f(a) = f(b)$ than $\exists c \in (a,b)$ s.t. $f'(c) = 0$!

Pf:



We argue by contradiction! Assume that $f'(x) \neq 0$ $\forall x \in (a,b)$. By the extrem value theorem for continuous funct. $\exists m$ (min.) and $\exists M$ (max.) on $[a,b]$. By the previous theorem m and M cannot be achived in (a,b). Hence both extremes must be achived at a and b → $f(a) = f(b) \Rightarrow f$ is const. $\Rightarrow f' = 0$, $\forall x \in (a,b) \Rightarrow$ contrad.
→ $f(a) \neq f(b) \Rightarrow$ contradiction with hypotesis.

Theorem: (MEAN-VALUE THEOREM FOR DERIVATIVES)
$f: [a,b] \to \mathbb{R}$, continuous $(:= f \in C'(a,b))$ Then $\exists c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$!

(Uniform. continuouty) $|f'(x)| \leq M$, $\forall x \in (a,b) \Rightarrow$ (f is uniform. contouns.)

Pf: $h(x) = f(x)(b-a) - x[f(b) - f(a)] \Rightarrow h(a) = h(b) = \dots$ and h is continuous on $[a,b]$.
$h(x)$ satisfies the hyp. of the Rolle's theorem $\Rightarrow \exists c \in (a,b)$ s.t. $h'(c) = 0$
$h'(x) = f'(x)(b-a) - [f(b) - f(a)]$. If $x = c \Rightarrow f(b) - f(a) = f'(c)(b-a)$! ✓

DEFINITION OF LIPSCHITZ FUNCTION

Def.: $f: I \to \mathbb{R}$, I interval. We say that $f(x)$ is LIPSCHITZ FUNCTION if $\exists L > 0$ s.t. $|f(x) - f(y)| \leq L|x - y|$, $\forall x, y \in I$! Remark: f is lipschitz $f. \Rightarrow f$ is continuos (v obratni smeri ni nes...!)

Pf: (Mean value th. for der.) + $|f'(x)| \leq M$ $\forall x \in (a,b) \Rightarrow |f(b) - f(a)| = |f'(c)| \cdot |b-a| \leq M \cdot |b-a| \Rightarrow f$ is lipschitz
$\{\text{Lipschitz fun.}\} \subseteq \{\text{uniform. cont. fun.}\} \subseteq \{\text{count. func.}\}$

Theorem: (CAUCHY MEAN-VALUE FORMULA)
Let $f, g: [a,b] \to \mathbb{R}$, countinuos $f, g \in C'(a,b)$. Then $\exists c \in (a,b)$ s.t. $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$!

Pf: $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)] \Rightarrow h(a) = h(b)$, h is cont. in $[a,b]$ and $h \in C'(a,b)$.
By Rolle's theorem $\exists c \in (a,b)$ s.t. $h'(c) = 0$. $h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)]$ if $x = c \Rightarrow \boxed{OK!}$

P.S.: The extreme values for f may occour at: – the end-points a, b
$f: [a,b] \to \mathbb{R}$ count.
$f \in C'(a,b)$ – at those interior points c s.t. $f'(c) = 0$

Theorem: $f:[a,b]\to\mathbb{R}$, f cont., $f\in C'(a,b)$ then:

a) If $f'(x)>0$, $\forall x\in(a,b)$ then f is strictly increasing on (a,b)

b) If $f'(x)<0$, $\forall x\in(a,b)$ then f is strictly decreasing on (a,b)

c) If $f'(x)=0 \Rightarrow f$ is constant.

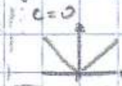
Pf: a) we must proofe that if $x<y \Rightarrow f(x)<f(y)$. We know that $\exists c\in(x,y)$ s.t. $f(y)-f(x)=\underbrace{f'(c)}_{>0}\underbrace{(y-x)}_{>0}$
$\Rightarrow f(y)-f(x)>0$ ✓

b) (analogus)

c) Take any two x and y in (a,b); $x\neq y$. $\exists c\in(x,y)$ s.t. $f'(c)=0 \Rightarrow f(y)-f(x)=\overbrace{f'(c)}^{=0}\overbrace{(x-y)}^{\neq 0} \Rightarrow f(y)-f(x)=0$

Theorem: $f:[a,b]\to\mathbb{R}$, f cont. and assume that $f'(x)$ exists at any point of (a,b) except maybe at a point c.

npr. $f(x)=|x|$, $c=0$



a) If $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c$ than $f(c)$ is a relative maximum.

b) If $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c$ than $f(c)$ is a relative minimum.

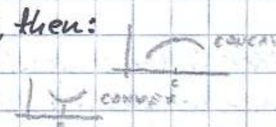
Pf: By the previous theorem ⊕ $(f(x)>f(c)\ \forall x>c$ and $f(x)<f(c)\ \forall x<c) \Rightarrow f(c)$ is rel. max.

Theorem (2ND DERIVATIVE TEST)

Let $c\in(a,b)$ s.t. $f'(c)=0$. Assume that $f''(x)$ exists on (a,b), then:

a) If $f''<0$ on $(a,b) \Rightarrow f$ has relative maximum at c!

b) If $f''>0$ on $(a,b) \Rightarrow f$ has relative minimum at c!



Pf: a) $f''<0$ on $(a,b) \Rightarrow f'$ is strictly decreasing on (a,b) and we know that $f'(c)=0 \Rightarrow$
$f'(x)>f'(c)=0$ for $x<c$ and $f'(x)<f'(c)=0$ for $x>c \Rightarrow f(c)$ is relative maximum.

Theorem: $f:[a,b]\to\mathbb{R}$ cont.; $f\in C'(a,b)$
If f' is increasing on (a,b) then f is convex on $[a,b]$

CONVEX: $\forall x,y\in(a,b)$ $\forall\lambda\in[0,1]$
$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$

Pf: Apply the mean-value theorem twice. $\exists c$ ... (POGLEJ KNJIGO, TA JE ZMEDENA) → str. 189 (→ 209 v PDF)

THE RELATION BETWEEN THE INTEGRATION AND DIFFERENTIATION

Theorem: FIRST FUNDAMENTAL THEOREM OF CALCULUS

Let $f:[a,b]\to\mathbb{R}$ s.t. f is integrable on $[a,x]$ where $a\leq x\leq b$.
We define $A(x)=\int_a^x f(t)dt$. Than the derivative of A exists in $x\in(a,b)$, where f is continous and for such x we have $A'(x)=f(x)$

$A(x)=\int_a^x f(t)dt$
$\Downarrow$
$A'(x)=f(x)$

Pf: Let x be a point of continuity of f and we consider $\frac{1}{h}[A(x+h)-A(x)] \xrightarrow[h\to 0]{} f(x)$.
We have $A(x+h)-A(x)=\int_a^{x+h}f(t)dt-\int_a^x f(t)dt=\int_x^{x+h}f(t)dt$; now $\pm f(x)-f(x)$: $\Rightarrow$
$\int_x^{x+h}f(x)dt+\int_x^{x+h}f(x)-f(t)dt = f(x)h+\int_x^{x+h}f(t)-f(x)dt$. So we have $\frac{A(x+h)-A(x)}{h}=f(x)+\frac{1}{h}\int_x^{x+h}f(t)-f(x)dt$.
Define $G(h)=\frac{1}{h}\int_x^{x+h}f(t)-f(x)dt$. We want to show that $G(h)\to 0$ as $h\to 0$! Since f is continous in x, than given $\frac{\varepsilon}{2}$, there exists $\delta>0$ s.t. $|f(t)-f(x)|<\frac{\varepsilon}{2}$ when $|t-x|<\delta$. If we take h s.t. $0<h<\delta$ we have that $\left|\int_x^{x+h}f(t)-f(x)dt\right|\leq\int_x^{x+h}|f(t)-f(x)|dt<\int_x^{x+h}\frac{\varepsilon}{2}dt=\frac{1}{2}\varepsilon h<h\varepsilon \Rightarrow |G(h)|<\varepsilon$ when $0<|h|<\delta$
$\Rightarrow G$ is countinous at zero and $G(h)\to 0$ as $h\to 0$!

Theorem: ZERO DERIVATIVE THEOREM

If $f'(x)=0$ $\forall x\in I$, I-open interval, than f is constant in I!

$f'(x)=0$, $\forall x\in I \Rightarrow f(x)=$ const.

Pf: Consequence of former theorem.

Def: A function P is called a PRIMITIVE (or an ANTIDERIVATIVE) of a function f on an open interval I, if the derivative of P is $f \Rightarrow P'(x) = f(x) \ \forall x \in I$!

The primative is NOT UNIQUE, because $f(x) + C, \forall C \in \mathbb{R}$ is also a primative!

Any two primatives P and Q of the same function f, can differ only by a constant: $(P(x) - Q(x))' = P'(x) - Q'(x) = f(x) - f(x) = 0 \Rightarrow P(x) - Q(x) = \text{const.} \ \forall x \in I$

Theorem: SECOND FUNDAMENTAL THEOREM OF CALCULUS

Assume that f is continuous on an open interval I and let P be any primitive of f on I. Than for any x and c in I, we have $P(x) = P(c) + \int_c^x f(t)\,dt$ (T)

Pf: Let $A(x) = \int_c^x f(t)\,dt$, since f is continuous on I than by the first theorem of calculus we have $A'(x) = f(x) \ \forall x \in I \Rightarrow A$ is primitive of f, but two primitives can differ only by a constant $\Rightarrow A(x) - P(x) = K$ for some $K \in \mathbb{R}$. But when we have $x = c \Rightarrow P(c) = A(c) + K = 0 + K = K$
$\Rightarrow A(x) - P(x) = -P(c) \Rightarrow$ ok ✓ mpr. $\int_0^b \cos x\,dx = \sin x \Big|_0^b = \sin b - \sin(0)$

INTEGRATION BY SUBSTITUTION

example: $f(x) = \cos x$; $(\sin' x = \cos x)$; $\int \cos(g(x)) \cdot g'(x)\,dx = \sin(g(x)) + C$, $C \in \mathbb{R}$

Consider Q the composition of two functions P and g. $Q(x) = P[g(x)]$. If for instance we know $P'(x) = f(x)$, than by chain rule $Q'(x) = P'[g(x)]\,g'(x) = f[g(x)]\,g'(x)$.
$\int f[g(x)]g'dx = \int P'[g(x)]\,g'(x)\,dx = Q(x) + c$!

In the aplications mpr.: $\int \cos(\underbrace{x^3}_{u}) \cdot \underbrace{3x^2 dx}_{du} \to \int \cos(u)\,du$; $u = x^3$

The method of integration by substitution: $\int f[g(x)] \cdot g'(x)\,dx$ (1)
By substitution $u = g(x)$; $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$!
From (1) we get that (1) $= \int f(u)\,du$!

Ex: $\int_2^3 \frac{(x+1)}{\sqrt{x^2+2x+3}}\,dx$ $\quad u = x^2 + 2x + 3$, $du = (2x+2)dx = 2(x+1)dx$

$= \frac{1}{2} \int_{11}^{18} \frac{1}{\sqrt{u}}\,du = \sqrt{u}\Big|_{11}^{18} = \sqrt{18} - \sqrt{11}$

Theorem: SUBSTITUTION THEOREM FOR INTEGRALS

Assume that g has continuous derivative g' on I open interval. Let J be the set of values taken by g on I and assume that f is continuous on J. Than $\forall x, c \in I$ we have $\int_c^x f[g(t)]\,g'(t)\,dt = \int_{g(c)}^{g(x)} f(u)\,du$

Pf: Let $a = g(c)$; $P(x) = \int_a^x f(u)\,du$; $x \in J$ and $Q(x) = \int_c^x f[g(t)]\,g'(t)\,dt$; $x \in I$.
By the fundamental theorem for calculus for P and Q we have $P'(x) = f(x)$ and $Q'(x) = f[g(x)]\,g'(x)$. Consider $R(x) = P[g(x)]$, by the chain rule $R'(x) = P'[g(x)]\,g'(x) =$
$= f[g(x)]\,g'(x) = Q'(x)$. Applying the 2^{nd} fund. th. of calculus twice we have
$\int_{g(c)}^{g(x)} f(u)\,du = \int_{g(c)}^{g(x)} P'(u)\,du = P[g(x)] - P[g(c)] = R(x) - R(c)$ and $\int_c^x f[g(t)]\,g'(t)\,dt = \int_c^x Q'(t)\,dt = \int_c^x R'(t)\,dt = R(x) - R(c)$ ✓

=

INTEGRATION "PER PARTES"

$h(x) = f(x) \cdot g(x) \Rightarrow h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$
Take the integral: $\int f(x)\,g'(x)\,dx + \int f'(x)\,g(x)\,dx = \int (f(x)\,g(x))'dx = f(x)\,g(x) + C$; $C \in \mathbb{R}$

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx + C; \ C \in \mathbb{R}}$$

Ex:

$\int x \cos x\,dx = x \sin x - \int \sin x \ldots$

Ex: $-u'' = f$; $f \in C$; $u \in C^2$
↳ 2^{nd} order dif. equation

$\mathbb{R}^{n=1}$ (*) $\begin{cases} -u'' = f & \text{in } (a,b) \\ u(a) = u(b) = 0 \end{cases} \to$ $n>1$ $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$

INTRODUCE WEEK FORMULATION of (*) $\Rightarrow -\int_a^b u''\varphi = \int_a^b f\varphi \Rightarrow \int_a^b u''\varphi = -\int_a^b u'\varphi' + (u'\underbrace{\varphi}_{0})(b) - (u'\underbrace{\varphi}_{0})(a)$
φ - test func. $\varphi \in C_0^\infty$ (PER PARTES)

$x \in (a,b)$; $\varphi(x) \neq 0 \subsetneq (a,b)$
$\varphi(a) = \varphi(b) = 0$

Theorem: SECOND MEAN-VALUE THEOREM FOR INTEGRALS

$g:[a,b] \to \mathbb{R}$ continuous

$f:[a,b] \to \mathbb{R}$ continuous, has derivative which is continuous and never change sign

Then for any $c \in [a,b]$ we have $\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$

Pf: Let $G(x)=\int_a^x g(t)dt$, since g is continuous $G'(x)=g(x)$. By integration "per partes" we have

$\int_a^b f(x)g(x)dx = \int_a^b f(x)G'(x)dx = f(b)G(b) - \int_a^b f'(x)G(x)dx$ (ps. $G(a)=0$ by definition) (*)

By the weighted mean-value theorem $\int_a^b f'(x)G(x)dx = G(c)\int_a^b f'(x)dx = G(c)[f(b)-f(a)]$ for $c\in[a,b]$

Hence (*) becomes $\int_a^b f(x)g(x)dx = f(b)G(b) - G(c)[f(b)-f(a)] = f(\ )G(c) + f(b)[G(b)-G(c)]$

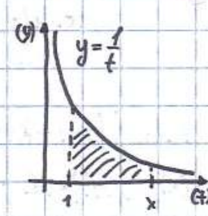
Def: If $x>0$; $x\in\mathbb{R}$ we define the NATURAL LOGARITHM $\boxed{\ln(x)=\int_1^x \frac{1}{t}dt}$!

Theorem: The $\ln(x)$ has the following properties:

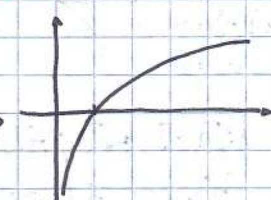
1.) $\ln(1)=0$

2.) $\frac{d}{dx}(\ln(x))=\frac{1}{x}$; $\forall x>0$

3.) $\ln(a\cdot b)=\ln(a)+\ln(b)$; $\forall a,b>0$



$\Rightarrow$



Pf: 1.) Trivial

2.) From 1st fundamental th. of calculus...

3.) $\ln(a\cdot b)=\int_1^{a\cdot b}\frac{1}{t}dt=\int_1^a \frac{1}{t}dt+\int_a^{ab}\frac{1}{t}dt=\ln(a)+\int_a^{ab}\frac{1}{t}dt \stackrel{\circledcirc}{} $ Let $t=au \Rightarrow u=\frac{1}{a}t$, $du=\frac{1}{a}dt \Rightarrow$

$\circledcirc\ \ln(a)+\int_1^b \frac{1}{au}a\,du=\ln(a)+\ln(b)$ ok✓

Theorem: $\forall b\in\mathbb{R}$ there is exactly one positive real number a whose logarithm is b $\ln(a)=b$

$\Rightarrow$ logarithm is an injective function...

Def: If $b>1$, $b\neq 1$ and if $x>0$, the logarithm of x to the base b is the number

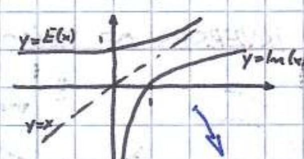
$\boxed{\log_b(x)=\ln(x)/\ln(b)}$!

We define $\log|x|=\int_1^{|x|}\frac{1}{t}dt \Rightarrow \boxed{\int\frac{du}{u}=\log|u|+C}$

P.S.: $\int\frac{1}{f(x)}f'(x)dx=\ln(f(x))+C$

npr. $\int tg\,x=\int\frac{\sin x}{\cos x}dx=-\ln(\cos x)+C$

npr. $\int^y \log x\,dx=\int^y \underbrace{1}_{f'}\cdot\underbrace{\log x}_{g}\,dx = y\log y - y + C$; $C\in\mathbb{R}$!



THE EXPONENTIAL FUNCTION

Def: $\forall x\in\mathbb{R}$, we define $E(x)$ to be the number y whose logarithm is $x \Rightarrow y=E(x)$, $\ln(y)=x$

Theor: $E(x)$ has the following properties:

1.) $E(0)=1$

2.) $E(1)=e$

3.) $E'(x)=E(x)$ $\forall x$

4.) $E(a+b)=E(a)\cdot E(b)$

Pf: 1.) $\ln(1)=0$ and $\ln(e)=1$

4) $x=E(a)$, $y=E(b)$, $c=\ln(x\cdot y)$

By the definition $\ln(x)=a$, $\ln(y)=b$, $E(c)=xy$

The properties of logarithm $\Rightarrow c=\ln(xy)=\ln(x)+\ln(y)=a+b$ • $c=a+b$ Then $E(c)=E(a+b)$

$E(c)=x\cdot y=E(a)\cdot E(b) \Rightarrow$ ok✓

3) $\frac{E(x+h)-E(x)}{h}=\frac{E(x)\cdot E(h)-E(x)}{h}=E(x)\left[\frac{E(h)-1}{h}\right]$; set $k=E(h)-1 \Rightarrow E(h)=k+1$; $\ln(k+1)=\ln(E(h))=h$

$\frac{E(h)-1}{h}=\frac{k}{\ln(k+1)}$. As $h\to 0$ does $k=E(h)-1\to 0$; $\frac{\ln(k+1)}{k}=\frac{\ln(k+1)-\ln(1)}{k}\xrightarrow{k\to 0}\ln'(1)=1$

EXPONENTIAL EXPRESSED AS A POWER OF e

$E(r)=e^r$; $r\in\mathbb{Q}$ By the properties $E(a+b)=E(a)\cdot E(b)$, we have $E(na)=E(a)^n$ in particular if $a=1 \Rightarrow E(n)=e^n$. For $a=\frac{1}{n}$ we get $E(1)=E(\frac{1}{n})^n$. Since $E(\frac{1}{n})>0$ then $E(\frac{1}{n})=e^{1/n}$ (**). By $E(na)=E(a)^n$ with $a=1/m \Rightarrow E(n\cdot\frac{1}{m})=E(\frac{1}{m})^n=e^{n/m}$. $E(-r)=\frac{1}{E(r)}=e^{-r}$

We define $e^x=E(x)$ $\forall x\in\mathbb{R}$!

Once we have defined e^x, $x\in\mathbb{R}$ we can also define $a^x=e^{x\ln(a)}$ where $a>0$

The following properties hold:

- $\ln(a^x) = \ln(e^{x \ln a}) = x \ln(a) \ln(\overset{1}{e}) = x \ln(a)$
- $(ab)^x = e^{x \ln(a \cdot b)} = e^{x \ln a + x \ln b} = e^{x \ln a} \cdot e^{x \ln b} = a^x b^x$
- $a^x \cdot a^y = a^{x+y}$ Pf. DN
- $(a^x)^y = a^{x \cdot y}$ Pf. DN

THE HYPERBOLIC FUNCTIONS

$\boxed{\sinh(x) = \dfrac{e^x - e^{-x}}{2}}; x \in \mathbb{R}$ HYPERBOLIC SINUS

$\boxed{\cosh(x) = \dfrac{e^x + e^{-x}}{2}}; x \in \mathbb{R}$ HYPERBOLIC COSINUS

$\boxed{\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}$

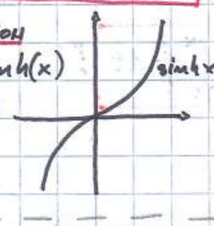
ODD FUNCTION

$\sinh(-x) = -\sinh(x)$

$\sin(-x) = -\sin(x)$

$\sinh(0) = 0$

$\sin(0) = 0$



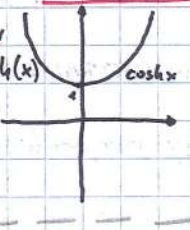
EVEN FUNCTION

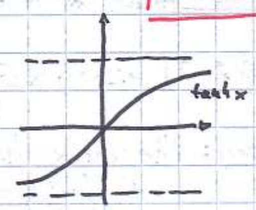
$\cosh(-x) = \cosh(x)$

$\cos(-x) = \cos(x)$

$\cosh(0) = 1$

$\cos(0) = 1$





VME: $f:[a,b] \to \mathbb{R}$; $\exists L > 0$ s.t. $|f(x) - f(y)| \le L|x-y|, \forall x, y \in [a,b]$ – LIPSCHITZ FUNCTION

$\forall \varepsilon > 0 \ \exists \delta > 0 \ |f(x) - f(y)| < \varepsilon$; $L\delta < \varepsilon \Rightarrow \delta = \frac{\varepsilon}{L}$; LIPSCHITZ is stronger than CONTINUOUS!

VME: $f:[1,2] \to \mathbb{R}$ $f(x) = \ln(x) + e^x + x^2 + \sqrt{x}$ is Lipschitz

$f'(x) = \dots$ – continuous $\Rightarrow \exists c: |f(x) - f(y)| = |f'(c)||x-y| \le M|x-y|$

DERIVATIVES OF INVERSE FUNCTION

Theorem: Assume f is strictly increasing and continuous function in the interval $[a,b]$ and let g be the inverse of f. Assume $f'(x)$ exists and $f'(x) \neq 0$, then $g'(x)$ exists and

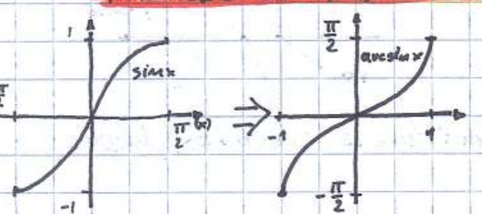
$\boxed{g'(y) = 1/f'(x),\ y = f(x)}$ $\qquad \dfrac{dx}{dy} = 1 \Big\backslash \left(\dfrac{dy}{dx}\right)$

Pf: Consider a point x where $f'(x) \neq 0$ and let $y = f(x)$. Consider $\frac{g(y+k) - g(y)}{k}$; $h = g(y+k) - g(\overset{x}{y}) = g(y+k) - x$

$x + h = g(y+k)$, $f(x+h) = y + k \Rightarrow k = f(x+h) - y = f(x+h) - f(x)$; $h, k \neq 0$ because f is increasing

$\frac{g(y+k) - g(y)}{k} = \frac{h}{f(x+h) - f(x)} = 1 \Big/ \frac{f(x+h) - f(x)}{h} \xrightarrow[h \to 0]{} 1/f'(x)$ Ps. If $k \to 0 \Rightarrow h \to 0$ because g is continuous!

INVERSES OF TRIGONOMETRIC FUNCTIONS

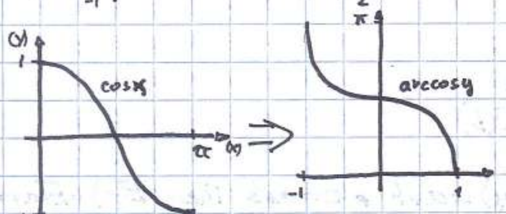


$\arcsin [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y \to \arcsin y$

$g'(y) = \dfrac{1}{f'(x)} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1 - \sin^2 x}} = \boxed{\dfrac{1}{\sqrt{1 - y^2}}}$

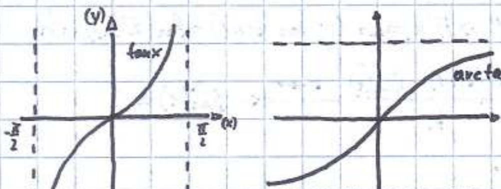
for $-1 < y < 1$



$\arccos [-1,1] \to [0, \pi]$

$y \to \arccos y$

$\boxed{[\arccos(y)]' = -\dfrac{1}{\sqrt{1 - y^2}}}$ for $-1 < y < 1$



$\arctan \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$\boxed{[\arctan(y)]' = \dfrac{1}{1 + y^2}}$

INTEGRATION BY PARTIAL FRACTION

Consider f/g ; f and g are polynomial:

1.) If $\deg f < \deg g$ than (f/g) is proper rational function.

2.) If $\deg f \geq \deg g$ than (f/g) is improper rational function.

$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$ where $\deg R < \deg g$ and $Q(x)$ is polynomial.

Case 1. The denominator is product of distinct linear factors $\Rightarrow g(x) = (x-x_1)\cdot(x-x_2)\cdot\ldots\cdot(x-x_n)$

$g(x) = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \ldots + \frac{A_n}{x-x_n}$ mpr: $\int \frac{2x^2+5x+1}{x^3+x^2-2x}dx = \frac{1}{2}\int\frac{dx}{x} + 2\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+2} = \ldots$

Casa 2. The denuminator has repeated linear factors!

mpr. : $\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx \longrightarrow \frac{A_1}{x-1} + \frac{A_2}{x+1} + \frac{A_3}{(x+1)^2} \Rightarrow$ you get A_1, A_2 and A_3. ; $A_1 = \frac{3}{2}; A_2 = \frac{1}{2}; A_3 = 1$

Can also be solved by differentiation of both sides and you get $2x+2 = 2A_1(x+1) + A_2(x+1) + A_3$!

We get $\int \frac{x^2+2x+3}{(x-1)(x+1)^2}dx = \frac{3}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} + \int\frac{dx}{(x+1)^2} = \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \frac{1}{x+1} + C ; C\in\mathbb{R}$

Case 3.

The denominator cointanes irreducable quadratic factors, none of which is repeated!

mpr.: $\int\frac{3x^2+2x-2}{x^3-1}dx \longrightarrow x^3-1 = (x-1)\underbrace{(x^2+x+1)}$ → irreducable quadratic factor

$\frac{3x^2+2x-2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \Rightarrow$ clear the fraction $\Rightarrow \left.\begin{array}{l} A+B=3 \\ A+C-B=2 \\ A-C=-2 \end{array}\right\} \Rightarrow A=1, B=2, C=3 \Rightarrow$

$\int\frac{3x^2+2x-2}{x^3-1}dx = \int\frac{dx}{x-1} + \int\frac{2x+3}{x^2+x+1}dx = \ln|x-1| + \ln|x^2+x+1| + 2\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$

$\int\frac{2x+3}{x^2+x+1}dx = \int\frac{2x+1}{x^2+x+1}dx + \int\frac{2}{x^2+x+1}dx = \boxed{\ln|x^2+x+1|} + 2\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} \rightarrow \begin{array}{l} u = x+1/2 \\ du = dx \\ \alpha^2 = 3/4 \end{array} \rightarrow \int\frac{du}{u^2+\alpha^2} = \frac{1}{\alpha}\operatorname{arctg}\frac{u}{\alpha} = \frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{x+1/2}{\sqrt{3/4}}\right) = \boxed{\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)}$

Case 4. The denominator contains irreducable quadratic factors, some of which are repeated!

mpr.: $\int\frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2}dx = \int\frac{A\,dx}{x-1} + \int\frac{Bx+C}{x^2+2}dx + \int\frac{Dx+E}{(x^2+2)^2}dx \Rightarrow A=\frac{1}{3}, B=\frac{2}{3}, C=-\frac{1}{3}, D=-1, E=0$

$\int\frac{x^4-x^3+2x^2-x+2}{(x-1)(x^2+2)^2}dx = \frac{1}{3}\int\frac{dx}{x-1} + \frac{1}{3}\int\frac{2x\,dx}{x^2+2} - \frac{1}{3}\int\frac{dx}{x^2+2} - \frac{1}{2}\int\frac{2x\,dx}{(x^2+2)^2} =$

$= \frac{1}{3}\ln|x-1| + \frac{1}{3}|x^2+2| - \frac{\sqrt{2}}{6}\operatorname{arctg}\frac{x}{\sqrt{2}} - \frac{1}{2}\frac{1}{x^2+2} + C ; C\in\mathbb{R}$

THE TAYLOR POLYNOMIAL GENERATED BY A FUNCTION → T_n

Theorem:

Let $f(x)$ be a function with derivatives of order n at the point $x=0$! Than exists one and only one polynomial $P(x)$ of degree $\leq n$, which satisfies the $(n+1)$ conditions!

$P(0) = f(0), P'(0) = f'(0), \ldots P^{(n)}(0) = f^{(n)}(0)$.

The polynomial is given by formula: $P(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k$; $x=0 \rightarrow x=a \Rightarrow P(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$

mpr.: $f(x) = e^x$ in $x=0 \Rightarrow P(x) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}$

$f(x) = e^x$ in $x=1 \Rightarrow P(x) = \sum_{k=0}^{n}\frac{e}{k!}(x-1)^k$

Theorem: The Taylor polynomial T_n has the following properties:

a) LINEARITY PROPERTY

$T_n(c_1 f + c_2 g) = c_1 T_n f + c_2 T_n g$

b) DIFFERENTIAL PROPERTY

$(T_n f)' = T_{n-1}(f')$

Theorem: Let P_n be a polynomial of degree $n \geq 1$. Let $f(x)$ and $g(x)$ be two functions (with derivatives of order n in 0) and assume that:
$f(x) = P_n + g(x) \cdot x^n$, where $g(x) \to 0$ as $x \to 0$. Than P_n is the Taylor pol. of $f(x)$ at $x=0$!

Pf: Let $h(x) = f(x) - P(x) = x^n g(x)$. Differentiating the product $x^n g(x)$ we have that h and it's first n derivatives in $x=0$ are zero. $\Rightarrow P_n$ is the Taylor polynomial!

TAYLOR POLYNOMIAL WITH REMINDER

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k + \overbrace{E(x)}$$

Theorem: Assume that $f(x)$ has continous second derivative $f''(x)$ in some neighboorhod of a point a. Than for any x in this neighboorhod we have $f(x) = f(a) + f'(a)(x-a) + E_1(x)$
Where $E_1(x) = \int_a^x (x-t) f''(t)\,dt$!

$$|E_1(x)| = \left|\int_a^x (x-t) f''(t)\,dt\right| \leq \int_a^x |(x-t)| \cdot |f''(t)|\,dt \overset{f''\text{-continous} \Rightarrow f''\text{-bounded}}{\leq} M \cdot \int_a^x |x-t|\,dt \sim M \cdot (x-a)^2$$

Pf: $E_1(x) = f(x) - f(a) - f'(a)(x-a) = \int_a^x f'(t)\,dt - f'(a)\int_a^x dt = \int_a^x [f'(t) - f'(a)]\,dt \to \left\{\begin{matrix} u = f'(t) - f'(a) \\ v = t - x \end{matrix}\right\} \begin{matrix} \frac{du}{dt} = f''(t) \\ dv = dt \end{matrix} \Rightarrow$

by integration per-partes $E(x) = \int_a^x u\,du = uv\Big|_a^x - \int_a^x (t-x) f''(t)\,dt = \int_a^x (x-t) f''(t)\,dt$ since $\begin{matrix} u=0 \text{ if } t=a \\ v=0 \text{ if } t=x \end{matrix}$!

Theorem: Assume that $f(x)$ has countinous derivatives of order $(n+1)$ in some neighboorhod N of $x=a$, than for $\forall x \in N$: $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + E_n(x)$ where $E_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$.

Theorem: If $(n+1)$-derivative of $f(x)$ satisfies the inequality $m \leq f^{(n+1)}(t) \leq M$; $\forall t \in I(a)$-INTERVAL $a \in I$ than: $\forall x \in I: \frac{m(x-a)^{n+1}}{(n+1)!} \leq E_n(x) \leq \frac{M(x-a)^{n+1}}{(n+1)!}$; $x > a$ or ...

THE SEQUENCE

Definition: A function f whose domain of definition is the set of all positive integer is called an INFINITE SEQUENCE. The function value $f(n)$ is called the n-th term of sequence!

Def: A sequence $\{f(n)\}$ is said to have a limit L, if for any $\epsilon > 0$, $\exists N = N(\epsilon)$ s.t. $|f(n) - L| < \epsilon$, $\forall n > N$!
$(f(n) \to L) \Rightarrow$ CONVERGENT SEQUENCE!
A sequence that does not converge is called DIVERGENT SEQUENCE!

Def: A sequence $\{f(n)\}$ is said to be INCREASING if $f(n) \leq f(n+1)$ for $\forall n \in \mathbb{N}$ - positive integer!
A sequence $\{f(n)\}$ is said to be DECREASING if $f(n) \geq f(n+1)$ for $\forall n \in \mathbb{N}$!
A sequence $\{f(n)\}$ is said to be BOUNDED if $\exists M > 0$ s.t. $|f(n)| < M$ for $\forall n \in \mathbb{N}$!
If such M does not exists the $\{f(n)\}$ is UNBOUNDED!

Def: A CAUCHY SEQUENCE — CONVERGENT $\Leftrightarrow$ CAUCHY
$\forall \epsilon > 0$ $\exists N$ s.t. $\forall n, m > N \Rightarrow |f(n) - f(m)| < \epsilon$. If $\{f(n)\}$ is CONVERGENT, than $\{f(n)\}$ is CAUCHY seq.!

Theorem: A monotonic sequence converges if and only if sequence is bounded!

Pf: - If sequence is unbounded it cannot converge! So let L be the least upper bound for $f(n) \Rightarrow$ it exists $\Rightarrow f(n)$ is bounded. Than $f(n) \leq L$ $\forall n$. For any $\epsilon > 0$ $\exists N$ s.t. $|L - \epsilon| < f(N)$. If $n \geq N$ we have $f(N) \leq f(n)$ (for increasing $f(n)$). Hence $L - \epsilon < f(n) < L$; $\forall n \geq N \longrightarrow L - \epsilon < f(n) < L + \epsilon \Rightarrow |f(n) - L| < \epsilon$, $\forall n \geq N$!
For decreasing $f(n)$ the proof is similar...

P.S.: Usualy we denote $f(n)$ with a_n: $f(n) = a_n$!

INFINITE SERIES

Sequence $a_1, a_2, \dots, a_n, \dots$ Let $s_n = a_1 + a_2 + \dots + a_n$, the sum of first n-terms of a_n!
If there is S-real number s.t. $\lim_{n\to\infty} s_n = S$, than we say that $\sum_{n=1}^{\infty} a_n$ is CONVERGENT.
If S does not exist we say that series is DIVERGENT!

Example: The harmonic series: $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent! $s_n = \sum_{k=1}^{n} 1/k$, s_n is not a cauchy sequence $\Rightarrow$ DIVERGENT
$$|s_{2n} - s_n| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}$$

Example: $\sum_{k=1}^{\infty} 2^{-k}$ converges...

Theorem: Let $\sum a_n$ and $\sum b_n$ be convergent series and let α and β be real numbers!
The series $\sum (\alpha a_n + \beta b_n)$ is convergent and $\sum (\alpha a_n + \beta b_n) = \alpha \sum a_n + \beta \sum b_n$!

Pf: $\sum_{k=1}^{n} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{n} a_n + \beta \sum_{k=1}^{n} b_n \to$ finite sum (we know it's true), than let $n \to \infty$!

Theorem: If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges than $\sum_{k=1}^{\infty} (a_k + b_k)$ diverges!

TELESCOPIC SERIES

$$\sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1}$$ – is it also true if we let $n \to \infty$?

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be two sequences s.t. $a_n = b_n - b_{n+1}$ $\forall n$!
Than the series $\{a_n\}$ converges if and only if the sequence $\{b_n\}$ converges: $\sum_{n=1}^{\infty} a_n = b_1 - L$; $L = \lim_{n\to\infty} b_n$

Pf: $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_k - b_{k+1}) = b_1 - b_{n+1}$; $b_{n+1} = L$ if $n \to \infty$! If we know $b_n \to L \Rightarrow s_n = b_1 - b_{n+1} \to b_1 - L$
If we know $s_n \to S \Rightarrow s_n = b_1 - b_{n+1} \Rightarrow b_{n+1} = b_1 - s_n \Rightarrow b_n = b_1 - S$!

Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \to \frac{1}{n(n+1)} = \frac{(n+1)-n}{n(n+1)} \Longrightarrow \frac{1}{n} - \frac{1}{n+1} = b_n - b_{n+1} \dots \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = b_1 - \lim_{n\to\infty} b_n = 1 - 0 = 1$ ✓

THE GEOMETRIC SERIES

Theorem: If $|x| < 1$, the geometric series $\sum_{k=0}^{\infty} x^k$ is convergent and it converges to $\frac{1}{1-x}$! If $|x| \geq 1$ it diverges!

TEST FOR CONVERGENCE

If the series $\sum a_k$ converges than its n-th term tends to zero! $\lim_{k\to\infty} a_k = 0$

Pf: $s_n = a_1 + \dots + a_n \Rightarrow a_n = \overset{L}{s_n} - \overset{L}{s_{n-1}} \to 0$ P.S.: THIS IS NECESARY CONDITION BOT NOT SUFFICIENT! (npr. $a_k = \frac{1}{k}$ is div.)

COMPARISON TEST FOR SERIES (of non-negative terms)

P.S.: Assume that $a_n \geq 0$ $\forall n$. Than $\sum_{n=1}^{\infty} a_n$ converges if and only if $s_n = \sum_{k=1}^{n} a_k$ is bounded above!

Theorem: Assume that $a_n \geq 0$ and $b_n \geq 0$. and $\exists c > 0$ s.t. $a_k \leq c \cdot b_k$, $\forall k \geq 1$.
Than if $\sum b_k$ converges than also $\sum a_k$ converges!

Example: $\sum_{n=1}^{\infty} 1/n!$ is convergent! pf... $k! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k = \overbrace{2 \cdot 3 \cdot \dots \cdot (k-1) \cdot k}^{(k-1) \text{ factors each of them } \geq 2} \geq 2^{k-1} \Rightarrow \frac{1}{k!} \leq \frac{1}{2^{k-1}} = 2 \cdot \left(\frac{1}{2}\right)^k \to$
$\to$ this is general term of geometric series for $x = 1/2 \Rightarrow$ geom. series converges $\Rightarrow \sum 1/k!$ also converges!

Theorem: LIMIT COMPARISON TEST

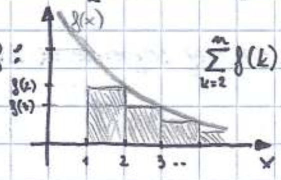
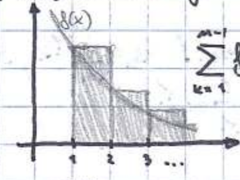
Assume that $a_n > 0$ and $b_n > 0$, $\forall n \geq 1$ and assume that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $(*)$
than $\sum a_n$ converges if and only if $\sum b_n$ converges!

Pf: By $(*)$ we have that $\exists N$ s.t. $\forall n \geq N$ is $1/2 < a_n/b_n < 3/2 \Rightarrow b_n < 2a_n$ and $a_n < \frac{3}{2} b_n$,
than by the comparison test we have the proof!

P.S.: The limit comparison test stil holds if we have $\lim_{n\to\infty} \frac{a_n}{b_n} = c$; $0 < c < \infty$

THE INTEGRAL TEST

Theorem: Let $f(x)$ be a positive decreasing function, defined for all $x \geq 1$. For each $n \geq 1$ we define $s_n = \sum_{k=1}^{n} f(k)$ and $t_n = \int_1^n f(x)\,dx$. Than both sequences $\{s_n\}$ and $\{t_n\}$ converge or both diverge!

Pf:  $\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\,dx$  $\sum_{k=1}^{n-1} f(k) \geq \int_1^n f(x)\,dx$

we have that $s_n - f(1) = \sum_{k=2}^{n} f(k) \leq \int_1^n f(x)dx = t_n \leq \sum^{n-1} f(k)$. Both sequences $\{s_n\}$ and $\{t_n\}$ are monotone increasing, either they are both bounded or both unbounded ... ✓

Example: $\sum_{n=1}^{\infty} 1/n^s$ converges if and only if $s > 1$. Pf: Take $f(x) = x^{-s}$ and we have $t_n = \int_1^n \frac{1}{x^s}dx = \begin{cases} \frac{n^{1-s}-1}{1-s}; & \text{if } s \neq 1 \\ \log n; & \text{if } s = 1 \end{cases}$

If $s > 1 \Rightarrow n^{1-s} \to 0$ as $n \to +\infty \Rightarrow t_n$ converges $\Rightarrow$ series converges!

If $s \leq 1$ then t_n diverges $\Rightarrow$ series diverges!

ROOT TEST AND RATIO TEST FOR SERIES

Theorem: (ROOT TEST) Let $\sum a_n$ be a series with non-negative terms, s.t. $\sqrt[n]{a_n} \to R$ as $n \to \infty$

a) if $R < 1$ than the series converges ($\sum a_n < \infty$)!

b) if $R > 1$ than the series diverges ($\sum a_n = \infty$)! P.S.: if $R=1$ the test is inconclusive!

$\sum \frac{1}{n}$ in $\sum \frac{1}{n^2}$ imate obe $R=1$!

Pf: a) assume $R<1$ and take x s.t. $R < x < 1$, than $0 \leq a_n^{1/n} \leq x$ for all $n \geq N$.
Since $\sum x^n < \infty$ for $0<x<1$, than by the comparison test we have $\sum a_n < \infty$!
b) assume $R>1$, implies that $a_n > 1$ for infinitly many values of $n \Rightarrow a_n \not\to 0 \Rightarrow \sum a_n$ diverges!

Theorem: (RATIO TEST) Let $\sum a_n$ be a series of non-negative terms s.t. $\lim_{n\to\infty} (a_{n+1}/a_n) \to L < \infty$ than:

a) if $L<1$ than the series converges ($\sum a_n < \infty$)!

b) if $L>0$ than the series diverges ($\sum a_n = \infty$)! P.S.: if $L=1$ the test is inconclusive!

obe $\frac{1}{n}$ in $\frac{1}{n^2}$ imata $R=1$ ena konvergira druga ne!

Pf: a) Assume that $L<1$ and chose x s.t. $L<x<1$, we have $\exists N>0$ s.t. $\frac{a_{n+1}}{a_n} < x$; $\forall n > N$.
So we have that $(a_{n+1}/x^{n+1}) < (a_n/x^n) \Rightarrow \{a_n/x_n\}$ is a decreasing sequence $\Rightarrow$
$\Rightarrow (a_n/x^n) < (a_N/x^N)$; $\forall n > N \Rightarrow a_n \leq c \cdot x^n$; $c = a_N/x^N$; $\forall n>N \Rightarrow$ by the comparison test $\sum x^n < \infty \Rightarrow \sum a_n < \infty$.
b) $a_n > a_{n+1} \Rightarrow a_n \not\to 0$ as $n \to \infty \Rightarrow \sum a_n = \infty$

ALTERNATING SERIES

Def: Alternatig series have the form: $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n-1} a_n + \ldots$; $a_i > 0$

Example: $\to \log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots + (-1)^{n-1} x^n/n + \ldots$ converges for $-1 < x \leq 1$
$\to \log(2) = 1 - 1/2 + 1/3 - 1/4 + \ldots$ (alternating harmonic series, P.S. harmonic series diverges)
$\to \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots + (-1)^{n-1}/(2n-1) + \ldots$

Theorem: LEIBNIZ'S RULE

If $\{a_n\}$ is a monotonic decreasing sequence with limit $=0$, than the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges! If S denotes its sum and s_n its n-th partial sum, we also have the inequalities: $0 < (-1)^n (S - s_n) < a_{n+1}$ for each $n \geq 1$!

Pf: $\overset{n\text{-even}}{\longrightarrow}$ $\overset{n\text{-odd}}{\longleftarrow}$
$s_2 \; s_4 \ldots s_{2n} \; \underset{a_{2n+1}}{\longleftrightarrow} \; s_{2n-1} \ldots s_3 \; s_1$ The partial sums s_{2n} (even numbers of terms) form an increasing sequence because $s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} > 0$. Similary the partial sums s_{2n-1} form a decreasing sequence. Both are bounded below by s_2 and above by s_1. Therefore each sequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$, being monotonic and bounded converges to a limit, say $s_{2n} \to S'$ and $s_{2n-1} \to S''$.
But $S' = S''$ because $S' - S'' = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty}(s_{2n} - s_{2n-1}) = \lim_{n\to\infty}(-a_n) = 0$. If we denote this common limit by S, it is clear that the series conveges and has sum S.
To derive inequalities we look: $s_{2n} \nearrow$ and $s_{2n-1} \searrow$, we have $s_{2n} < s_{2n+2} \leq S$ and $S \leq s_{2n+1} < s_{2n-1}$ for all $n \geq 1$. $\Rightarrow 0 < S - s_{2n} \leq s_{2n+1} - s_{2n} = a_{2n+1}$ and $0 < s_{2n-1} - S \leq s_{2n-1} - s_{2n} = a_{2n}$ which,

CONDITIONAL AND ABSOLUTE CONVERGENCE

Theorem: Assume that $\sum |a_n|$ converges. Than $\sum a_n$ also converges, and we have $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$

Pf: (glej knjigo str. 406)

Def: A series $\sum a_n$ is called ABSOLUTELY CONVERGENT if $\sum |a_n|$ converges.
It is called CONDITIONALLY CONVERGENT if $\sum a_n$ converges but $\sum |a_n|$ diverges!

P.S.: If $\sum a_n$ and $\sum b_n$ are absolutely convegent, than so are the series $\sum (\alpha a_n + \beta b_n)$!

IMPROPER INTEGRALS

We know the concept of an integral $\int_a^b f(x)dx$ under the restriction that the function f is defined and bounded on a finite interval $[a,b]$. The scope of integration may be extended by relaxing these restrictions.

Def: $\int_a^b f(x)dx$ as $b \to \infty$ $(:= \int_a^\infty f(x)dx)$ → IMPROPER INTEGRAL OF THE FIRST KIND!
If we keep the interval $[a,b]$ finite and allow $f(x)$ to become unbounded at one or more points → IMPROPER INTEGRAL OF THE SECOND KIND!

If the proper integral $\int_a^b f(x)dx$ exists for every $b \geq a$, we define the function I as follows: $I(b) = \int_a^b f(x)dx$; $\forall b \geq a$. The function I defined in this way is called an INFINITE INTEGRAL, or an IMPROPER INTEGRAL OF THE FIRST KIND, and it is denoted by the symbol $\int_a^\infty f(x)dx$. The integral is said to CONVERGE if the limit: $\lim_{b\to+\infty} I(b) = \lim_{b\to+\infty} \int_a^b f(x)dx$ exists and is finite. Otherwise the integral $\int_a^\infty f(x)dx$ is said to DIVERGE! If the limit exists and equals A, the number A is called the value of the integral: $A = \int_a^\infty f(x)dx$!

Example: The improper integral $\int_1^\infty x^{-s}dx$ converges if $s>1$ and diverges if $s \leq 1$!

$$I(b) = \int_1^b x^{-s}dx = \begin{cases} \frac{b^{1-s}-1}{1-s} & \text{if } s \neq 1 \\ \log b & \text{if } s = 1 \end{cases} \Rightarrow I(b) \text{ tends to limit if and only if } s>1 \Rightarrow \int_1^\infty x^{-s}dx = \frac{1}{s-1} \text{ if } s>1!$$

Example: The integral $\int_0^\infty \sin(x)dx$ diverges because $I(b) = \int_0^b \sin(x)dx = 1 - \cos(b)$ → does not tend to a limit as $b \to \infty$!

The definition for infinite integrals of the form $\int_{-\infty}^b f(x)dx$ is analogous! Moreover, if $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ are both convergent for some c, we say that the integral $\int_{-\infty}^\infty f(x)dx$ is convergent, and its value is defined by the sum $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$! It is easy to show that the choice of c is unimportant!

Ex: The integral $\int_{-\infty}^\infty e^{-a|x|}dx$ converges if $a>0$. For $b>0$ we have $\int_0^b e^{-a|x|}dx = \int_0^b e^{-ax}dx = \frac{1}{(-a)}[e^{-ab}-1] \to$ $\to 1/a$ if $b \to \infty$! Hence $\int_0^\infty e^{-a|x|}dx$ converges and has the value $1/a$.
Also, if $b>0$, we have $\int_{-b}^0 e^{-a|x|}dx = \int_{-b}^0 e^{ax}dx = -\int_b^0 e^{-at}dt = \int_0^b e^{-at}dt$. Hence $\int_{-\infty}^0 e^{-a|x|}dx$ also converges and has the value $1/a$. ⇒ we have $\int_{-\infty}^\infty e^{-a|x|}dx = 2/a$.

Ex: $\lim_{x\to 0} \frac{\sin x}{x} = 1$? $x_n \to 0$ $\left(\frac{1}{\frac{1}{x_n}}\right)$: by mean value th. $\sin(x_n) = \overset{0}{\sin(0)} + \overset{\sin'}{\cos}(\xi_n)\cdot(x_n - 0) \Rightarrow$
$\Rightarrow \frac{\sin x_n}{x_n} = \frac{\cos(\xi_n)\cdot \cancel{x_n}}{\cancel{x_n}} = \cos(\xi_n)$ but $\sin(x)$ is continuous ⇒ as $x_n \to 0 \Rightarrow \xi_n \to 0 \Rightarrow \lim_{x_n\to 0} \frac{\sin x_n}{x_n} = \cos(0) = 1$ ✓

P.S.: $\boxed{\lim \frac{N(x)}{D(x)} = \lim \frac{N'(x)}{D'(x)}}$ - L'Hopital rule

Theorem: L'HOPITAL RULE 0/0 (or ∞/∞)

$f, g: (a,b) \to \mathbb{R}$, both continuous and have derivatives in (a,b). $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$.
$\lim_{x\to a+} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"}$ Assume that $g'(x) \neq 0$, $\forall x \in (a,b)$. If the limit $\lim_{x\to a+}(f'(x)/g'(x))$ exists and has the value L, than $\lim_{x\to a+} \frac{f(x)}{g(x)}$ also exists and has the same value L!

PS.: If $\lim_{x\to a+} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ than $= \lim_{x\to a+} \frac{1/g(x)}{1/f(x)} = \frac{0}{0}$ ⇒ we can use L'Hopital rule!

Theorem: Assume that the proper integral $\int_a^b f(x)dx$ exists $\forall b>a$ and suppose that $f(x)\geq 0;\ \forall x\geq a$ than $\int_a^\infty f(x)dx$ converges if and only if $\exists M>0$ s.t. $\int_a^b f(x)dx \leq M \quad \forall b\geq a$!

Theorem: Assume that the proper integral $\int_a^b f(x)dx$ exists for $\forall b>a$ and $0\leq f(x)\leq g(x);\ \forall x\geq a$ and $\int_a^\infty g(x)dx$ converge, than $0\leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$!

Theorem: LIMIT COMPARISON TEST

Assume $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ both exist for $\forall b\geq a$, and $f(x)\geq 0, g(x)\geq 0$ for $\forall x\geq a$. Than if the limit $f(x)/g(x) \xrightarrow[x\to\infty]{} C\neq 0$, than both integrals $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$ converge or both diverge!

IMPROPER INTEGRAL OF THE SECOND KIND

Def: $f:(a,b]\mapsto \mathbb{R}$ and assume that the integral $\int_x^b f(t)dt$ exists $\forall x$ s.t. $a<x\leq b$. Than $I(x)=\int_x^b f(t)dt$ is called IMPROPER INTEGRAL OF THE 2nd KIND and is denoted by $\int_{a+}^b f(t)dt$. The integral converges if $\lim_{x\to a+} I(x)$ exists and is finite!

Example: $f(t)=t^{-s};\ t>0$. If $b>0$ and $x>0$ than $I(x)=\int_x^b t^{-s}dt = \begin{cases}\frac{b^{1-s}-x^{1-s}}{1-s} & ; s\neq 1\\ \log b-\log x & ; s=1\end{cases}$ $\Rightarrow$
$\Rightarrow$ limite is finite $\left(\lim_{x\to 0+} I(x)\right)$ if and only if $s<1$!

SEQUENCE AND SERIES OF FUNCTIONS

$f_n(x)\to f(x)$ as $n\to\infty$

Example: $f_n\ [0,1]\to\mathbb{R};\ x\mapsto x^n;\ 0\leq x\leq 1 \Rightarrow$ 

$\Rightarrow$ in the limit as $n\to\infty$ $f(x)=\begin{cases}0 & ; 0\leq x<1\\ 1 & ; x=1\end{cases}$
$\Rightarrow f(x)$ is not countinous in $x=1$, even if our sequence $f_n(x)$ are all countinous functions in $x=1$!

Example: $f_n:[0,1]\mapsto\mathbb{R};\ x\mapsto nx(1-x^2)^n;\ 0\leq x\leq 1 \Rightarrow f_n(x)\to 0$ pointwise as $n\to+\infty$; now $\int_0^1 f_n(x)dx=$
$=n\int_0^1 x(1-x^2)^n dx = -\frac{n}{2}\frac{(1-x^2)^{n+1}}{n+1}\Big|_0^1 = \frac{n}{2(n+1)} \mapsto 1/2$ as $n\to+\infty$!

Def: $f_n, f:[a,b]\mapsto\mathbb{R};\ f_n\mapsto f$ UNIFORMLY if $\forall\varepsilon>0\ \exists N=N(\varepsilon)$ s.t. $|f_n(x)-f(x)|<\varepsilon;\ \forall x\in[a,b];\ \forall n\geq N$!

Def: $f_n, f:[a,b]\mapsto\mathbb{R};\ f_n\mapsto f$ POINTWISE if $\forall\varepsilon>0\ \exists N=N(\varepsilon,x)$ s.t. $|f_n(x)-f(x)|<\varepsilon;\ \forall n\geq N(\varepsilon,x)$!

$\to$ za zadosti velik n leži $f_n(x)$ v pasu med $f(x)-\varepsilon$ in $f(x)+\varepsilon$

Theorem: Assume that $f_n\mapsto f$ uniformly on $[a,b]$. If any f_n is countinous in $p\in[a,b]$, than $f(x)$ is also countinous in $p\in[a,b]$!

Pf: $|x-p|<\delta;\ |f(x)-f(p)| \leq |f(x)-f_N(x)| + |f_N(x)-f_N(p)| + |f_N(p)-f(p)|$ ($\to \pm f_N(x)$ and $\pm f_N(p)$)
- $\forall\varepsilon>0\ \exists N=N(\varepsilon)$ s.t. $|f(x)-f_N(x)|\leq\varepsilon/3;\ \forall x\in(p-\delta,p+\delta)$
- f_N continous $\Rightarrow \forall\varepsilon>0\ \exists\delta$ s.t. $|x-p|<\delta \Rightarrow |f_N(x)-f_N(p)|<\varepsilon/3$
- $|f_N(p)-f(p)|$ is small because $f_n\to f$ uniformly! $\Rightarrow \forall\varepsilon>0\ \exists N=N(\varepsilon)$ s.t. $|f_N(p)-f(p)|<\varepsilon/3$ ... ✓

Consequence: Assume $f_n(x)=\sum_{k=1}^n u_k(x)$ and $f_n\mapsto f$ pointwise, than $f(x)=\lim_{n\to\infty} f_n(x)=\sum_{k=1}^\infty u_k(x)$.
If $f_n\mapsto f$ uniformly, we say $\sum u_k\mapsto f$ uniformly.

Theorem: If $\sum u_k\mapsto f$ uniformly and if u_k is continous at $p \Rightarrow f$ is also continous!

Theorem: $f_n\mapsto f$ uniformly on $[a,b]$, f_n continous. Define $g_n(x)=\int_a^x f_n(x)dx$ and $g(x)=\int_a^x f(x)dx \Rightarrow g_n\to g$ uniformly!

Pf: $x\in[a,b];\ |g_n(x)-g(x)| = \left|\int_a^x (f_n(t)-f(t))dt\right| \leq \int_a^x |f_n(t)-f(t)|dt$ (*)
because $f_n\to f$ uniformly $\Rightarrow \forall\varepsilon>0\ \exists N=N(\varepsilon)$ s.t. $|f_n(t)-f(t)|<\varepsilon\ \ \forall n>N\ \forall x\in[a,b] \Rightarrow$
$\Rightarrow$ (*) $\leq \int_a^b \varepsilon\, dt = \varepsilon(b-a)$ ✓

Theorem: $\sum_{k=1}^{n} u_k \to f$ uniformly on $[a,b]$. If $x\in[a,b]$ define $g_n(x)=\sum_{k=1}^{n}\int_0^1 u_k(t)\,dt$, $g(x)=\sum_{k=1}^{\infty}\int_0^1 u_k(t)\,dt$ than $g_n \to g$ as $n\to\infty$ uniformly.

Pf: Apply the previous theorem to $f_n(t)=\sum_{k=1}^{n} u_k(t)$ ...

A SUFFICIENT TEST FOR UNIFORM CONVERGENCE

Theorem: THE WEIESTRASS M-TEST

$\sum_{k=1}^{n} u_k \to f$ pointwise convergent. If there exists a convergent series of positive constant $\sum M_k$; $0\le |u_k| \le M_k$ $\forall m\ge 1$ $\forall x \Rightarrow \sum u_k \to f$ UNIFORMLY!

Pf: By the comparison test $\sum u_n(x)$ converges absolutly $\forall x\in[a,b]$. For $x\in[a,b]$
$|f(x)-\sum_{k=1}^{n} u_k(x)| = |\sum_{k=n+1}^{\infty} u_k(x)| \le \sum_{k=n+1}^{\infty} |u_k(x)| \le \sum_{k=n+1}^{\infty} M_k$. Since $\sum M_k$ converges $\Rightarrow$
$\Rightarrow \forall \varepsilon>0\ \exists N$ s.t. $\forall n\ge N$ $\sum_{k=n+1}^{\infty} M_k < \varepsilon$. ✓

COMPLEX NUMBERS

Def: $a,b\in\mathbb{R}$; (a,b)
1) $(a,b)=(c,d) \Rightarrow$ if and only if $a=c$ and $b=d$!
2) $(a,b)+(c,d)=(a+c,b+d)$
3) $(a,b)\cdot(c,d)=(ac-bd,ad+bc)$

Properties: - commutative law: $x+y=y+x$; $xy=yx$
- associative law: $x+(y+z)=(x+y)+z$; $(xy)z=x(yz)$
- distributive low: $x(y+z)=xy+xz$

Axiom 4: Existance of the identity element: $(0,0)$ identity element for $\oplus$
$(1,0)$ identity element for $\odot$

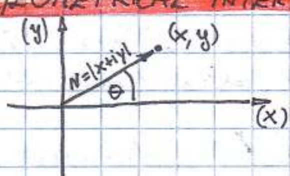
Axiom 5: Existance of negatives: $(a,b)+(-a,-b)=(0,0)$

Axiom 6: Each $z=a+ib$ has a reciprocal element with respect to the ident. ele. $(1,0)$
$(a+ib)(c+id)=(1+i\cdot 0) \Rightarrow c=a/(a^2+b^2)$, $d=-b/(a^2+b^2)$!

i – IMMAGINARY UNIT, (EX: $x^2+1=0 \Rightarrow x_{1,2}=\pm i$)

GEOMETRICAL INTERPRETATION OF COMPLEX NUMBER

$z=a+ib$



$x=r.\cos\theta$
$y=r.\sin\theta$
$r=|x+iy|=\sqrt{x^2+y^2}$
$\theta=\arg(x+iy)$ – argument of $z=x+iy$; $\theta\in[0,2\pi)$

P.S: We also know $|z_1+z_2|\le|z_1|+|z_2|$ (Δ neenakost) and $|z_1\cdot z_2|\le|z_1|\cdot|z_2|$; $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$; $z_2\neq 0$

COMPLEX CONJUGATE of $z=x+iy$ is $\boxed{\bar z = x-iy}$!
- $\overline{z_1+z_2}=\bar z_1+\bar z_2$
- $\overline{z_1 z_2}=\bar z_1\cdot\bar z_2$
- $\overline{z_1/z_2}=\bar z_1/\bar z_2$

COMPLEX EXPONENTIAL

e^z; $z\in\mathbb{C} \Rightarrow e^z=e^{x+iy}=e^x\cdot e^{iy} \Rightarrow$ we have to define $e^{iy}=A(y)+iB(y)$ (*)
Now we differentiate two times: $ie^{iy}=A'(y)+iB'(y)$ and $-e^{iy}=i^2e^{iy}=A''(y)+iB''(y)$
We get that $A''(y)=-A(y)$ and $B''(y)=-B(y)$. If we put $y=0$ in (*) and $[(*)]'$
we get $A(0)=1$, $B(0)=0$ and $A'(0)=0$, $B'(0)=1$!

$$\left.\begin{array}{ll} A''(y)=-A(y) & B''(y)=-B(y) \\ A(0)=1 & B(0)=0 \\ A'(0)=0 & B'(0)=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} A(y)=\cos(y) \\ B(y)=\sin(y)\end{array}\right\} \Rightarrow \boxed{e^{iy}=\cos(y)+i\sin(y)}$$

Def: $\boxed{e^z=e^{x+iy}=e^x(\cos y+i\sin y)}$

Theorem: If $a, b \in \mathbb{C} \Rightarrow e^a e^b = e^{a+b}$ Pf: ...

Theorem: $z \in \mathbb{C}; z \neq 0$ can be expressed in the form $z = r \cdot e^{i\theta}$; $r = |z|$, $\theta = \arg(z) + 2\pi n$

COMPLEX VALUED FUNCTIONS

$f: \mathbb{R} \longrightarrow \mathbb{C} \Rightarrow f(x) = u(x) + i v(x) \Rightarrow f'(x) = u'(x) + i v'(x)$ and $\int f(x)dx = \int u(x)dx + i\int v(x)dx$

$f: \mathbb{R} \to \mathbb{C}$ $t \in \mathbb{C}$ fixed
$x \to e^{tx}$ $t = \alpha + i\beta \Rightarrow e^{tx} = e^{(\alpha+i\beta)x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$
$\boxed{f'(x) = te^{tx}}$

$z = \cos\theta + i\sin\theta \Rightarrow z^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos(n\theta) + i\sin(2\theta)$
In general: $z^n = \cos(n\theta) + i\sin(n\theta)$
So we get the general formula for $z = r(\cos\theta + i\sin\theta)$: $z^n = r^n(\cos n\theta + i\sin n\theta)$
PS.: $|z^n| = |z|^n$ and $\arg(z^n) = n\cdot\arg(z) + 2\pi k$

Example: Solve the equation $z^n = w$; $z, w \in \mathbb{C}$. Assume that $w \neq 0$, so we can write $w = r(\cos\varphi + i\sin\varphi)$ and $z = \rho(\cos\theta + i\sin\theta)$. $\Rightarrow z^n = \rho^n(\cos n\theta + i\sin n\theta) = w$
$\Rightarrow \rho^n = r$ and $n\theta = \varphi + 2\pi k \Rightarrow \rho = \sqrt[n]{r}$ and $\theta = \frac{\varphi}{n} + \frac{2\pi k}{n}$! We get n solutions! (COMPLEX SOLUTIONS)
$z^n = \rho^n(\cos n\theta + i\sin n\theta)$; $\rho = \sqrt[n]{r}$ and $\theta = \frac{\varphi}{n} + \frac{2k\pi}{n}$; $k = 0, 1, \ldots, (n-1)$!

LINEAR ALGEBRA

Def: Let V be a non-empty set! We call V a LINEAR SPACE if it:

- Axiom 1: $x, y \in V \Rightarrow (x+y) \in V$ } CLOSURE AXIOMS
- Axiom 2: $x \in V, a \in \mathbb{R}\ (\mathbb{C}) \Rightarrow a\cdot x \in V$ }
- Axiom 3: $x + y = y + x$; $\forall x, y \in V$
- Axiom 4: $(x+y) + z = x + (y+z)$; $\forall x, y, z \in V$
- Axiom 5: Existance of element 0; $0 \in V$; $x + 0 = x$
- Axiom 6: Existance of negative: $\forall x \in V\ \exists (-1)x$ that satisfies $x + (-1)x = 0$
- Axiom 7: $\forall x \in V; \forall a, b \in \mathbb{R}$: $a(bx) = (ab)x$
- Axiom 8: $\forall x, y \in V; \forall a \in \mathbb{R}$: $a(x+y) = ax + ay$
- Axiom 9: $\forall x \in V; \forall a, b \in \mathbb{R}$: $(a+b)x = ax + bx$
- Axiom 10: Existance of element 1: $\exists 1$ s.t. $\forall x \in V: 1\cdot x = x$

Theorem: Let S be a non-empty subset of linear space V. Than S is a SUBSPACE if and only if it satisfies the CLOSURE AXIOMS (axiom 1 and axiom 2)!

Def: Let S be a non-empty sub-set of V linear space! An element $x \in V$ of the form $x = \sum_{i=1}^{k} a_i x_i$ (FINITE LINEAR COMBINATON); $x_1, x_2, \ldots x_k \in S$; $a_i \in \mathbb{R}$; $i = 1, 2, \ldots, k$.
The set of all finite linear combinations of element of S satisfies the closure axioms and hence it is a sub-space of V which is denoted with $\boxed{L(S) - \text{SUB-SPACE SPANNED BY } S}$!

Example: The set of polynomials $p(t)$ of degree $\leq n$ is spanned by the set of $(n+1)$ polynomials $S = \{1, t, t^2, \ldots, t^n\}$: $p(t) = \sum a_i t^i$; $a_i \in \mathbb{R}$

Def: A set S is called DEPENDENT if $\exists x_1, \ldots, x_k \in S$ and corresponding $c_1, \ldots, c_k \in \mathbb{R}$ (c_i not all zero!) s.t. $\sum_{i=1}^{k} c_i x_i = 0$!
A set is called INDEPENDENT when it is not-dependent! $\Rightarrow$ if $\sum_{i=1}^{k} c_i x_i = 0 \Rightarrow c_1 = c_2 = \ldots = c_k = 0$!

Theorem: Let S be an independent set consisting in k elements in a linear space V and let $L(S)$ be the sub-space spanned by S. Than every $(k+1)$ elements in $L(S)$ is a dependent set!