ELECTRO-DYNAMICS

[redacted]

knjiga: Griffiths Introduction to El-Dy.

VECTOR ANALYSIS – ponovitev matematike

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ ... $df = (\nabla f)\cdot d\vec{r}$

GRAD $\nabla(fg) = f\nabla g + g\nabla f$

DIV $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\nabla f$

CURL $\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\nabla f)$

$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$

$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$

Sedaj pravila za 2nd derivative...

$\vec{\nabla}\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$ ... LAPLACE OPERATOR

$\vec{\nabla}\times(\nabla f) = 0$ ... curl od gradienta je vedno nič!

Manjka za $\nabla(\vec{\nabla}\cdot\vec{A})$
gradient divergence!

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ ... divergenca curla je vedno nič!

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

1D fondamental th. $\int_a^b \left(\frac{df}{dx}\right)dx = f(b) - f(a)$ → 3D fundamental theorem: $\int_A^B \nabla f\cdot d\vec{r} = f(B) - f(A)$

Theorem: $\int_A^B \nabla f\cdot d\vec{r}$ is PATH INDEPENDENT $\Rightarrow \oint \nabla f\cdot d\vec{r} = 0$ (integral on closed path)

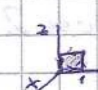
GAUSS: $\boxed{\int_V (\vec{\nabla}\cdot\vec{A})\,dV = \oint_S \vec{A}\cdot d\vec{a}}$ ... surface encloses the volume... $d\vec{a}$ = surface element

(PS. TO SPREGA / IMO STOKES THEOREM) $\Rightarrow \int(\vec{\nabla}\times\vec{A})\,d\vec{a}$ – depends only on the boundery, not on surface

posledica: $\boxed{\oint(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = 0}$ ... for any closed surface! (pri stokes theoremu moraš zaprt surface ...
... izbereš poljubno pot ⇒ lahko izbereš pot z dolžino 0 (= točka))

⇑

STOKES: $\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_\Gamma \vec{A}\cdot d\vec{r}}$ ... path Γ encloses the surface S! (pot Γ v ⊕ smeri!)

VAJA: verify the "stokes" for $\vec{v} = (0, 2xz + 3y^2, 4yz^2)$ for unit square surface...  NOTE x=0 on surface

HELMHOLZ

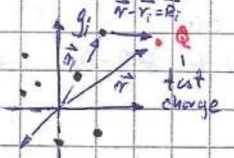
Supose that we know divergence and curl of some field!

$\vec{\nabla}\cdot\vec{F} = D$ and $\vec{\nabla}\times\vec{F} = \vec{C}$ ... Then $\vec{F}$ is uniquely determined only if proper boundery condition is suplied. (as npr. $\vec{F}\to 0$ at ∞)

Theorem: If $\vec{\nabla}\times\vec{F} = 0$ for some field $\Rightarrow \int_A^B \vec{F}\cdot d\vec{r}$ is PATH INDEPENDENT ... $\Rightarrow \oint \vec{F}\cdot d\vec{r} = 0$
or $\vec{F}$ is the gradient of some scalar potential ... $\vec{F} = \nabla V$ ... all 4 conditions are equivalent!

Theorem: If $\vec{\nabla}\cdot\vec{F} = 0$ everywhere $\Rightarrow \int_S \vec{F}\cdot d\vec{a}$ is SURFACE INDEPENDENT $\Rightarrow \oint_S \vec{F}\cdot d\vec{a} = 0 \Rightarrow \vec{F} = \vec{\nabla}\times\vec{A}$ VECTOR POTENTIAL
$\vec{F}$ can be written as curl of some vector

Any vector field can be written as: $\boxed{\vec{F} = -\nabla V + \vec{\nabla}\times\vec{A}}$



$$\vec{F} = Q\left[\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{R_i^2}\cdot\hat{R}_i\right] = Q\vec{E}$$

test charge — electric field

NOTE... for countinous charge distributions the sum → integral $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r})\,dV}{R^2}\hat{R}_i$

permittivity of free-space ... $\varepsilon_0 = 8{,}85 \cdot 10^{-12}\ As/Nm^2$

$\vec{F} = m\vec{a}$ (mass of Q) ; $\vec{F} = \frac{1}{4\pi\varepsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{R_i^2}\hat{R}_i = Q\cdot\vec{E}$; $\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{R_i^2}\cdot\hat{R}_i$

Asumption: charges are known and AT REST ... and in vacuum! → ELECTROSTATICS in VACUUM

By Helmholz ... we know $\vec{E}$ if we know $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\times\vec{E}$ (... and Boundary Conditions)

Imagine we have only one particle ... (we want to find divergence) ... particle is in origin!

– what is the flux of $\vec{E}$ through a sphere centred at origin

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ ... define FLUX: $\boxed{\Phi(\vec{E}) = \int \vec{E}\cdot d\vec{a}}$ element poovšine

... in spheric coords: $d\vec{a} = r^2 \sin\Theta\, d\Theta\, d\phi \cdot \hat{r}$

$$\Phi(\vec{E}) = \int \vec{E}\cdot d\vec{a} = \frac{1}{4\pi\varepsilon_0}\cdot q\int \frac{1}{r^2}\hat{r}\left(r^2\sin\Theta\, d\Theta\, d\phi\, \hat{r}\right) = \frac{1}{4\pi\varepsilon_0}\cdot q\cdot 4\pi = \boxed{q/\varepsilon_0}$$

True for any charge distrib. and for any ... (not only sphere)

By divergence theorem we know:

$$\Phi(\vec{E}) = \int(\vec{\nabla}\cdot\vec{E})\,dV = \frac{1}{\varepsilon_0}\int \rho\, dV \implies \boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

(ρ … charge density distribution) GAUSS LAW

The equation $\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \iff \Phi(\vec{E}) = q_{internal}/\varepsilon_0}$ (equation are equivalent)

EXAMPLE:

USE SYMETRY

Rod of length $2L$ ... line charge density λ ... point above the centre ($\perp$) at distance z!

$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{R_i^2}\hat{R}_i \to \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r})}{R^2}\hat{R}\, dl$ — Ta del se izniči s simetričnim delom!

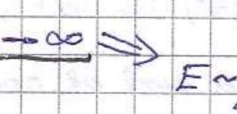
We can see ... $d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx}{R^2}\sin\Theta\,\hat{x} + \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,dx}{R^2}\cos\Theta\,\hat{z}$ ... $\cos\Theta = \frac{z}{R}$; $R^2 = z^2 + x^2$

$$\vec{E} = 2\cdot\frac{1}{4\pi\varepsilon_0}\int_0^L \frac{\lambda z}{(z^2+x^2)^{3/2}}\,dx = \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]_0^L\cdot\hat{z} \implies \boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{z}}$$

Let's discuss 2 extrems ... $\underline{z \gg L} \implies E \sim \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$ — $2\lambda L$ = charge = Q → just columb low

... $\underline{L\to\infty} \implies E \sim \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$ ... fals like z not like z^2!

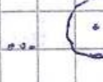
Infinite wire ... let's use Gauss low to find same result ... we see that we have cylindrical symetry!



the basis contribute nothing ... $\vec{E}$ is const. on the surface of cylinder ...

$\Phi(\vec{E}) = \int \vec{E}\cdot d\vec{a} = \underbrace{2\pi z}_{\text{obseg}}\underbrace{l}_{\text{dolžina cilindra}} E = \overbrace{\lambda l}^{\text{NABOJ}}/\varepsilon_0 \to E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{z}$ ... same result!

E je na plašču cilindra konstanten!

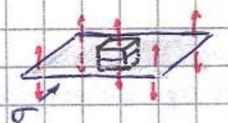
EXAMPLE... spherical charge...  ρ → we use spherical symetry ... find $\vec{E}$ ... (for inside and outside sphere)

We could find E by definition ... or better by Gauss law...

For inside sphere (shell) ... $\Phi_1(\vec{E}) = 0 \Rightarrow \vec{E} = 0$ ... there is no charge inside

For outside sphere (shell) ... $\Phi_2(\vec{E}) = \int \vec{E}\cdot d\vec{a} = E\cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\cdot \underbrace{q}_{\text{charge on sphere}} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cdot\hat{r}$

EXAMPLE... INFINITE SURFACE ... find $\vec{E}$ ... charge density σ...



→ By symety we know the $\vec{E}$ is perpendicular ... as surface we can take a box with surface A

$$\int \vec{E}\cdot d\vec{a} = \underbrace{2AE}_{\text{zgornja in spodnja stranica škatle ... zato imaš 2!}} = \underbrace{\sigma A}_{\text{(zajet naboj)}}/\varepsilon_0 \Rightarrow \boxed{E = \sigma/2\varepsilon_0}$$

Now we know the divergance ... what about the curl? (for ELECTRO-STATICS) → use stokes-theorem

We have single charge in the origin ... → Look at $\int_a^b \vec{E}\cdot d\vec{l}$ = work per unit charge!

Spherical coords: $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}q\left(\frac{-1}{r}\right)\Big|_a^b = \frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

→ FOR CLOSED PATH ($\vec{r}_a = \vec{r}_b$) this is ZERO!

So... $\oint \vec{E}\cdot d\vec{l} = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ ... true also for many static charges! - GENERAL RESULT!

So this is CONSERVATIVE FIELD ⇒ CAN BE WRITTEN AS GRADIENT of SCALAR FUN.

$\boxed{\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\nabla V}$ ... V is SCALAR FUNCTION! → POTENTIAL

So from now on (for ELECTRO-STATICS) our attention is focused only on V!

$\vec{E} = -\nabla V \Rightarrow V = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{l}$ → depends only on $\vec{r}$ (and 0 = origin) not on path

If you use different origin $0 \to 0' \Rightarrow V'(\vec{r}) = \int_{0'}^{\vec{r}} \vec{E}\cdot d\vec{l} = \overbrace{\int_{0'}^{0} \vec{E}\cdot d\vec{l}}^{\text{const.}} + \int_0^{\vec{r}} \vec{E}\cdot d\vec{l} = V + \text{const.}$

Constant is not important because we are intrested in gradient ... GAUGE FREEDOM

$[V] = [\overbrace{Nm}^{=J}/C] = [V]$ ... VOLT!

POISSON EQUATION

Now ... $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\nabla V) = \boxed{-\nabla^2 V = \rho/\varepsilon_0}$ ... we used $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ and $\vec{E} = -\nabla V$

We need the oposite ... how to find V if we know ρ...

Find an equation givind expliciettly V as a function of ρ!

Consider simple-case of one charge q in the origin of reference sytem!

$V(\vec{r}) = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{l}$ ... origin → ∞ $\quad V(\vec{r}) = -\int_\infty^{\vec{r}} \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_\infty^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$

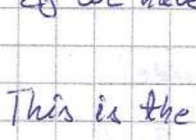
If charge is not in the origin ... $v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ → this R is distance ... $R = \|\vec{r} - \vec{r}_q\|$

If we have many charges... $\boxed{v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{R_i} \rightarrow v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')dV}{R}}$

This is the way to calculate potential v if we know ρ... LOOK PAGE 87 (Δ scheme)

What about ENERGY:

ENERGY



How much work is done on charge Q if we move from $a \rightarrow b$?

$$\boxed{W_{a\to b}} = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l}$$ ... minus ker delamo proti $\vec{E}$ ... ta minus kompenzira tistega pri gradient

$$= \boxed{Q\left[v(b) - v(a)\right]}$$ ... we used $v(\vec{r}) = -\int_0^{r}\vec{E}\cdot d\vec{r}$

$\Longrightarrow$ potential difference $\boxed{\Delta v = W/Q}$

How much energy we need to assamble given charge distribution?

1^{ST} charge ... no work because there is no potential yet!
2^{ND} charge ... we do work because the potential of the 1^{ST} charge ... and so on...
LOOK at the BOOK page 91! (energy of point charge distribution)

$$\ldots \boxed{W = \frac{1}{2}\sum_{i=1}^{N} q_i\, v(\vec{r}_i) \quad \text{where } v(\vec{r}_i) = \sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}}$$ ... potential due to all charges except one

FOR COUNTINOUS DISTRIBUTION: $\boxed{W = \frac{1}{2}\int_V \rho(\vec{r})\, v(\vec{r})\, dV}$ ... we can write this in terms of electric field

use: $\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$

PER-PARTES

$$\ldots W = \frac{1}{2}\varepsilon_0\int_V(\vec{\nabla}\cdot\vec{E})\, v\, dV = \frac{1}{2}\varepsilon_0\left[\int\vec{\nabla}\cdot[\vec{E}v]\,dV - \int\vec{E}\cdot(\nabla v)\,dV\right] =$$

$$= \frac{1}{2}\varepsilon_0\left[\oint(\vec{E}v)\cdot d\vec{a} + \int_V E^2 dV\right]$$ ... The volume V must contain the charge... but we can make the volume infinite ($V\to\infty$)

we changed to surface integral

By $V \to 0$ ... (najbrž mišljeno ∞)

$$W = \frac{1}{2}\varepsilon_0\left[\oint(\vec{E}v)\cdot d\vec{a} + \int_V E^2 dV\right] = \boxed{\frac{1}{2}\varepsilon_0\int_V E^2 dV}$$ V = all space

goes to 0 as $\frac{1}{r^2}$, as $\frac{1}{r}$, increase as r^2 ... together as $\frac{1}{r}$!

$\Longrightarrow$ goes like $\frac{1}{r}$ as $V\to\infty$ ($\equiv r\to\infty$) $\Rightarrow$ INTEGRAL $\to$ 0

MIHA RAZLOŽI ENERGIJO CHARGE DISTRIBUTION

1^{ST} CHARGE ... we move it to it's possition ... work = 0 (ker ni še nobenega $\vec{E}$-polja) $W_1 = 0$

2^{ND} CHARGE ... we do some work against $\vec{E}$-field of 1^{ST} charge ... $W_2 = \frac{1}{4\pi\varepsilon_0}q_2\left(\frac{q_1}{R_{12}}\right)$; R_{12} - razdalja q_1 do q_2

3^{RD} CHARGE ... work against 1^{ST} and 2^{ND} CHARGE ... $W_3 = q_3\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}}\right)$

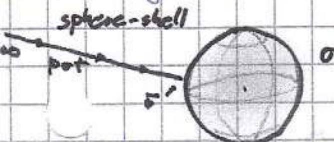
⋮

N-th CHARGE ... $W_N = q_N\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N-1} q_i/R_{iN}$

For given distribution we have to sum all this works... $W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j>i}^{N} q_i q_j/R_{ij} =$

$$= \frac{1}{\ldots}\sum^{N}\sum^{N}(q_i q_j)/\ldots = \frac{1}{\ldots}\sum^{N} q_i\sum^{N}\frac{1}{\ldots}\frac{q_j}{\ldots}$$

EXCESISE from CHARGE DISTRIBUTION ... How much work do we do to make sphere-shell charge distribution?



$$W = \frac{1}{2}\int \sigma(\vec{r}')\, v(\vec{r}')\, da = ? \;\ldots\; da = r^2 \sin\Theta\, d\Theta\, d\phi \;;\; \sigma = \text{const.}$$

integral over all shell

From previous results ... $v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dV}{r} = \frac{1}{4\pi\varepsilon_0}\cdot\sigma\cdot 4\pi R^2 \cdot \frac{1}{R} = \frac{\sigma R}{\varepsilon_0}$

Na tabli smo iskali potencial tako:

$$v(\vec{r}) = -\int_\infty^R \vec{E}\cdot d\vec{l} = -\int_\infty^R \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\,dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r}\right]_\infty^R = \frac{\overbrace{q}^{4\pi R^2\sigma}}{4\pi\varepsilon_0 R} = \frac{\sigma R}{\varepsilon_0}$$

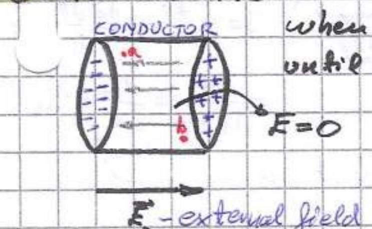
$$\ldots\; W = \frac{1}{2}\int \frac{\sigma^2}{\varepsilon_0} r \cdot r^2 \sin\Theta\, d\Theta\, d\phi = \frac{1}{2}\frac{\sigma^2 R^3}{\varepsilon_0}\cdot 2\pi\cdot 2 = \frac{2\pi\sigma^2 R^3}{\varepsilon_0} \;\ldots\; \sigma = q/4\pi R^2 \;\ldots\; W = \frac{q^2}{8\pi\,\varepsilon_0 R}$$

// => OK!

Now by $W = \frac{1}{2}\varepsilon_0 \int_{\text{all space}} E^2\, dV = \frac{1}{2}\varepsilon_0 \int_0^\infty \frac{q^2}{16\pi^2\varepsilon_0^2}\frac{1}{r^4}\cdot 4\pi r^2\, dr = \frac{q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0 R}$

razdelimo na $\infty \to R$ in $R \to 0$ (od $R \to 0$ je $E = 0$)

CONDUCTORS

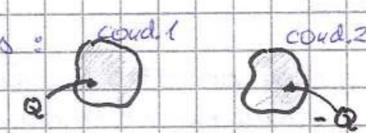


when we apply external $\vec{E}$-field the electrons begin to move until equlibrium is established ... INSIDE CONDUCTOR THERE IS NO $\vec{E}$-FIELD!

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = 0 \;\ldots\; \Delta v = v_b - v_a = 0$$

net-charge (there is charge ... but net charge = 0)

Two conductors:



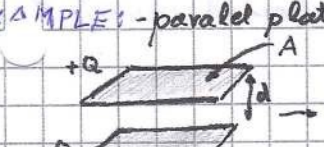
define pot. diff. $\Delta v = v_{(+)} - v_{(-)} = -\int_{(-)}^{(+)} \vec{E}\cdot d\vec{l}$ $\quad v \propto Q$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{r^2}\hat{r}\, dV \propto Q$$ ... field is proportional to Q

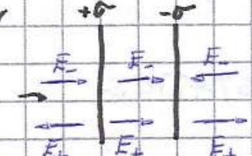
We define quantaty $\boxed{C = Q/\Delta v}$ ... CAPACITANCE ... $[C/V = F]$ ... Farad!

Capacitance is ability of matterial to store charge ... stored charge = energy (previous results)

EXAMPLE: - paralel plate capacitor



we do like for infinite plate

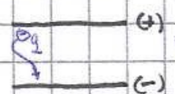


Outside the capacitor the sum of $E_+ + E_-$ is zero ... no $\vec{E}$-field! Inside the fields are in the same direction!

$$E_+ = \frac{\sigma}{2\varepsilon_0} = E_- \implies \text{INSIDE } E = \frac{\sigma}{\varepsilon_0} \implies \Delta v = -\int_{(-)}^{(+)} \frac{\sigma}{\varepsilon_0}\, dl = \frac{\sigma}{\varepsilon_0}\cdot d$$

So the potential difference $\Delta v = \frac{\sigma}{\varepsilon_0}\cdot d = \frac{Q}{\varepsilon_0 A}\cdot d \implies$ CAPACITANCE $\boxed{C = \frac{Q}{\Delta v} = \varepsilon_0 \frac{A}{d}}$

How much energy is storred in the capacitance?

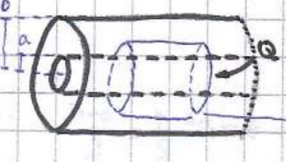


we move charge from one plate to another.

At a given step the charge is q: $dW = dq\, v = dq\,\frac{1}{C}q \implies W = \int_0^Q \frac{1}{C}q\, dq = \frac{1}{2}\frac{Q^2}{C}$

$$\boxed{W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\Delta v^2}$$

OTHER EXAMPLE : Find the capacitance per unit length of two co-axial metal cylindrical tubes of radius a and b!



Also here the $\vec{E}$-field out-side the cable is ZERO!

Gaussian surface : $\int \vec{E}\cdot d\vec{a} = E\cdot 2\pi r L = Q/\varepsilon_0 \Rightarrow \vec{E} = \frac{Q}{2\pi\varepsilon_0 L}\cdot\frac{1}{r}\hat{r}$

$$V(b)-V(a) = -\int_a^b \vec{E}\cdot d\vec{l} = -\frac{Q}{2\pi\varepsilon_0 L}\int_a^b \frac{1}{r}dr = -\frac{Q}{2\pi\varepsilon_0 L}\cdot\ln(b/a)$$ we get (-) because inner cylinder has higher pot!

So capacitance: $\boxed{C = \frac{Q}{|V(b)-V(a)|} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}}$ ... per unit length you remove L!

End of electro-statics... now we let charges to move... (moving charges = CURRENT)

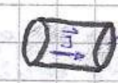
EXPERIMENT $\rightarrow$ $\boxed{\vec{F} = Q[\vec{E} + \vec{v}\times\vec{B}]}$ LORENZ FORCE

work of magnetic force... $dW_{mag} = \vec{F}_{mag}\cdot d\vec{l} = \left(\frac{d\vec{l}}{dt}\times\vec{B}\right)\cdot d\vec{l} = 0$ ... work of magnetic force is zero!

CURRENT $\boxed{I = \int_{surface} \vec{J}\cdot d\vec{a}}$ $\vec{J}$ = current density ... $\boxed{\vec{J} = \rho\vec{v}}$

CHARGE DENSITY

CHARGES VELOCITY (hitrost)



da

$J = \rho\frac{dl}{dt} = \rho\frac{da}{da}\frac{dl}{dt} = \frac{dQ}{da\,dt} = \frac{dI}{da} = J$ ✓ OK!

In electro-statics to "know" the enviroment was to know charge distribution ρ... in electrodynamics to "know" the enviroment is to know $\vec{J}$!

Mathematicaly : CHARGE is CONSERVED

$$\int_S \vec{J}\cdot d\vec{a} = -\frac{d}{dt}\int_V \rho\, dV$$ če teče naboj skozi površino S volumna V se mora naboj v volumnu za enako spremeniti!

$$\int_V (\vec{\nabla}\cdot\vec{J})dV = -\frac{d}{dt}\int_V \rho\, dV = -\int_V \frac{\partial\rho}{\partial t}dV \Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}}$$ CONTIUNTY EQUATION

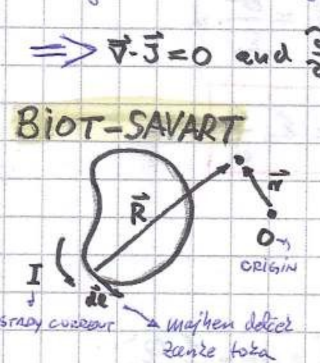
PS. $\frac{d}{dt}$ se spremeni v $\frac{\partial}{\partial t}$ ... ker tu smo ρ že integrirali po vsem volumnu ⇒ torej ni več odvisno od položaja... ko nesemo odvod noter pa je ρ še vedno odvisen od pozicije!

Now... assume stady current $\rightarrow$ MAGNETOSTATICS

Stady current ... flowing from $-\infty$ to $+\infty$ flowing forever → no charge accumulation

$\Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ and $\frac{\partial\rho}{\partial t} = 0$! By experiments we can determine $\vec{B}$!

BiOT-SAVART



CAN GO OUT BECAUSE IS STADY CURRENT (in this case)

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{I}\times\hat{R}}{R^2}dl' = \frac{\mu_0}{4\pi}I\int\frac{d\vec{l}'\times\hat{R}}{R^2}}$$ BiOT-SAVART LAW

I gre ven ker je "stady current"

μ_0 – PERMEABILITY OF FREE SPACE

$\mu_0 = 4\pi\cdot 10^{-7}\ N/A^2$

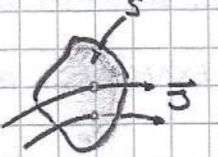
Before was for linear current... what abaut a volume current?

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\times\hat{R}}{R^2}dV}$$ ... for volume current

Now like for $\vec{E}$-field we want to find divergence and curl of $\vec{B}$!

mathematics... $\Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ → NO SOURCES (NO MONOPOL)

$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}}$ → Differential form of AMPERE LAW



INTEGRAL FORM

CURRENT THROUGH THE SURFACE

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int_S \vec{J}\cdot d\vec{a} \rightarrow \text{Stokes: } \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I}$$

↔ equivalent to GAUSS LAW in electrostatics

The fact that $\vec{\nabla}\cdot\vec{B}=0$ will allow us to write $\vec{B}$ as vector potential $\vec{B}=\vec{\nabla}\times\vec{A}$!

In electro-static case we described $\vec{E}$... from $\vec{E}$ we found the force... with force and Newton law we can find trajectory of particle... Now is the same, but we have two fields $\vec{E}$ and $\vec{B}$! By knowing $\vec{E}$ and $\vec{B}$ we can find trajectory!

P.S. Differential form $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \longrightarrow$ Integral form $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$! I is the current enclosed by the surface! $\left(\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 \int_S \vec{J}\cdot d\vec{a} = \text{definition of current}\right)$

So for now ... ELECTRO-STATICS and MAGNETO-STATICS ① expression for FORCE

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \rightarrow \text{Gauss} \qquad \boxed{\vec{\nabla}\cdot\vec{B} = 0} \qquad \boxed{\vec{F} = Q\left[\vec{E} + \vec{v}\times\vec{B}\right]}$$

$$\boxed{\vec{\nabla}\times\vec{E} = 0} \qquad \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}} \rightarrow \text{Ampere}$$

... we can say ρ and $\vec{J}$ are sources!

In this REDUCED form (of Maxwell equations) the $\vec{E}$ and $\vec{B}$ are completly decupled (INDEPENDENT)

In electro-statics $\vec{\nabla}\times\vec{E}=0$ enabled us to define potential $v \rightarrow \vec{E}=-\nabla v$! (SCALAR POTENTIAL)

Similary $\vec{\nabla}\cdot\vec{B}=0$ meant that we can write $\vec{B}=\vec{\nabla}\times\vec{A}$ ($\vec{A}$ is VECTOR POTENTIAL)

is $\vec{B}=\vec{\nabla}\times\vec{A}$ ... $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ ✓ ... divergenca curla je vedno 0!

... $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$ ... equivalent equation to $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$!

How to simplyfy ... remember that el-potential v is not unigly defined, we can add a constant! Similar we can add a gradient of scalar potential to $\vec{A}$!

If $\vec{A'} = \vec{A} + \nabla\lambda \Rightarrow \vec{\nabla}\times\vec{A'} = \vec{\nabla}\times\vec{A}$ (because $\vec{\nabla}\times(\nabla\lambda) = 0$...allways)... we can add gradient!

Now... we would like to have first term in $\nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A$ ... $\nabla(\vec{\nabla}\cdot\vec{A}) = 0$! can do this?

Let's see... assume $\vec{A} = \vec{A}_0 + \nabla\lambda$ ... we want $\vec{\nabla}\cdot\vec{A} = 0$ ... P.S. $\lambda = \lambda(\vec{r})$ – scalar potential

... $\vec{\nabla}\cdot(\vec{A}_0 + \nabla\lambda) = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda = 0$ ... REMEMBER $\nabla^2 v = -\rho/\varepsilon_0$ ... POISSON EQUATION

$$\nabla^2 v = -\rho/\varepsilon_0 \Rightarrow v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r})}{R}\,d\tau$$

... z $d\tau$ je označen element volumna!

Our equation is mathematicaly equivalent to poisson equation!

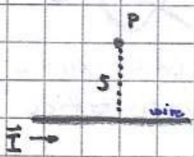
$$\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0 \Rightarrow \lambda(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}_0}{R}\,d\tau$$

... so we can allways find such $\lambda(\vec{r})$ that we can make vector potential $\vec{A} \rightarrow \vec{\nabla}\cdot\vec{A}=0$

... we see that we can chose $\vec{A}$ such that $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ COMPARE TO $v(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho}{R}d\tau$

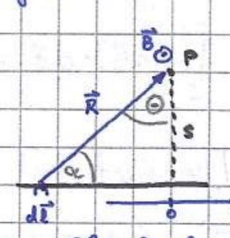
... solution to this equation is similar like in electro-statics... $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{R}d\tau}$

EXAMPLE... application of Biot-Savart... find $\vec{B}$-field generated by straight wire carrying stady current I at a distance s!



by def.: $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{R}}{R^2}$ ... $dl' \times \hat{R} = dl' \cdot 1 \cdot \sin\alpha = dl' \cos\Theta$; $r\cos\Theta = s$, $r\sin\Theta = l$

$$\Rightarrow l = s\,tg\,\Theta \rightarrow dl = \frac{s}{\cos^2\Theta}d\Theta$$



$R^2 = s^2 + x^2$

$d\vec{l} \times \vec{R}$ ... ven iz lista

So we can write...

$$B = \frac{\mu_0}{4\pi} I \int_{\Theta_1}^{\Theta_2} \underbrace{\frac{\cos^2\Theta}{s^2}}_{= 1/R^2} \cdot \underbrace{\cos\Theta}_{= \sin\alpha} \cdot \underbrace{\frac{s}{\cos^2\Theta}d\Theta}_{dl} = \frac{\mu_0 I}{4\pi s}\int_{\Theta_1}^{\Theta_2}\cos\Theta\, d\Theta = \frac{\mu_0 I}{4\pi s}(\sin\Theta_2 - \sin\Theta_1)$$

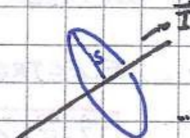
Infinite wire ... $\Rightarrow \Theta_1 = -\pi/2$, $\Theta_2 = \pi/2 \Rightarrow \boxed{B = \mu_0 I / 2\pi s}$!

Now... can we make this simpler? ... write Ampere law:

Like in el. stat. we found Gauss surface... find path!

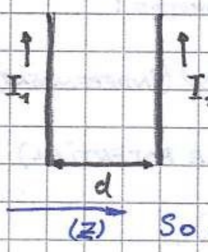
$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \rightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enclosed}$$ ... now we use this...

We select path... circle around the current... centred at wire and $\perp$!



... and so on...

What is the force bettween two paralel wires?



$\vec{B}$-field of 1. wire at 2. wire ... $B = \mu_0 I_1 / 2\pi d$

$dq = \lambda dl \rightarrow \lambda\vec{v} = \vec{I}$

Remember... $\vec{F} = q\,\vec{v}\times\vec{B} \Rightarrow d\vec{F} = dq\,\vec{v}\times\vec{B} \Rightarrow \vec{F} = \int(\vec{v}\times\vec{B})dq = \int(\vec{v}\times\vec{B})\lambda\,dl =$

$$= \int \vec{I}\times\vec{B}\,dl = \boxed{I\int d\vec{l}\times\vec{B}}$$

This is I_2 in our case!

So now... $F = I_2 \overbrace{\frac{\mu_0 I_1}{2\pi d}}^{B}\int dl$ ... but $\int dl = \infty$ INFINITE WIRE

FORCE PER UNIT LENGTH : $\boxed{F/L = \mu_0 I_1 I_2 / 2\pi d}$ ... force can be attractive or repulsive... depends on current directions!

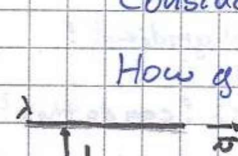
$$dF/dl = \frac{\mu_0 I_1 I_2}{2\pi d}\cdot\hat{z}$$

EXAMPLE...

we have Lorenz forze... $\vec{F} = Q[\vec{E} + \vec{v}\times\vec{B}]$ ... usualy elec. force is dominant... but we can increase v!

Consider 2 long paralel wires (charge density λ) moving with $\vec{v}$!

How great would v have to be in order for B-force to be comparable with E-force?



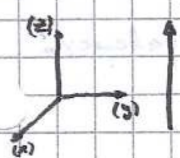
wires are moving $\Rightarrow$ current $I = \lambda v$

mag. force per unit length... $f_{mag} = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$

elec. field generated by 1 wire... $E = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{d} \Rightarrow$ el. force per unit length... $f_{el} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{d}$

So... $f_{mag} \simeq f_{el} \Rightarrow \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d} \simeq \frac{1}{2\pi\varepsilon_0}\frac{\lambda^2}{d} \Rightarrow \boxed{v = 1/\sqrt{\varepsilon_0\mu_0} = c}$ ... speed of light!

Consider... charged particle moving in uniform $\vec{B}$-field! ... motion = ?



We need... $\vec{F} = m\vec{a} = Q(\vec{v}\times\vec{B})$

Decompose... $\vec{F} = Q\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ v_x & v_y & v_z\\ 0 & 0 & B_z\end{vmatrix} = (v_y B_z\hat{x} - v_x B_z\hat{y})Q = QB_z(v_y, -v_x, 0)$

So we get... $m\dot{v}_x = QB_z v_y$; $m\dot{v}_y = QB_z v_x$; $m\dot{v}_z = 0$ ... 3 equations...

So we have... $m\ddot{z} = 0 \Rightarrow z = v_z t + z_0$; v_z, z_0 - constants $\Rightarrow$ constant motion in z-direction!

denote $\omega = QB_z/m$ ... $\begin{cases}\ddot{x} = \omega\dot{y} \\ \ddot{y} = -\omega\dot{x}\end{cases}$ ← odvajaj po t še 1x

$\Rightarrow \dddot{x} = \omega\ddot{y} = \omega(-\omega\dot{x}) = -\omega^2\dot{x}$ ... mamo... $\dddot{x} = -\omega^2\dot{x}$

We solve... let $w = \dot{x} \Rightarrow \ddot{w} = -\omega^2 w \Rightarrow w(t) = A\cos(\omega t + \phi) \Rightarrow \dot{x}(t) = A\cos(\omega t + \phi)$

we need initial conditions... $\ddot{x}(0) = -A\sin\phi = \underline{\omega\dot{y}(0)} = 0$ ← we selected $\Rightarrow$ NO v_z component ($v_z = 0$)

$\dot{x}(0) = A\cos\phi$ ⊕ $\phi = 0 \Rightarrow \dot{x}(0) = A$ ... nato vse integriramo!

Na kraju dobimo... $x(t) = \frac{A}{\omega}\sin(\omega t) + x_0$ and $y(t) = \frac{A}{\omega}\cos(\omega t) + y_0$

Lahko vidimo, da je krožnica v x-y ravnini... $x^2(t) + y^2(t) = A^2/\omega^2 = m^2A^2/Q^2B_z^2$

The radius is... $\boxed{r = \underbrace{m\dot{x}(0)}_{\text{MOMENTUM}}/QB_z}$ CYCLOTRON FORMULA ... we can determine momentum of a particle!

In order to make current in a circuit we have to apply force...

$\boxed{\vec{J} = \sigma\vec{f}}$ ... $\vec{f}$ - force per unit charge

↳ σ - conductivity ... depends on material

$\vec{f}$ can be any kind of force ... chemical, grav, el-magnetic ...

We are intrested in electro-magnetic forces... $\underline{\vec{f} = \vec{E} + \vec{v}\times\vec{B}}$

But we have seen before that mag. force is small compared to ele. force.

So we can simplify the force to $\vec{f} = \vec{E} \Rightarrow \boxed{\vec{J} = \sigma\vec{E}}$ OHMS LAW

equivalent to $\Delta V = RI$

Consider:

... we are in magnetostatics ... $\Rightarrow \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = 0$ ← SIGNATURE OF MAGNETOSTATICS

... so also ... $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \rho = 0$ (by Gauss law)

} NO NET CHARGE IN CONDUCTOR

P.S. (Electrostatics $\vec{E} = 0 \Rightarrow \rho = 0$ ... Magnetostatics $\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \rho = 0$)

we know... $\nabla^2 v = \rho/\varepsilon_0 \Rightarrow \nabla^2 v = 0$ → LAPLACE EQUATION

$\vec{J}\cdot\hat{n} = 0 \Rightarrow \vec{E}\cdot\hat{n} = 0 \Rightarrow \frac{\partial v}{\partial n} = 0$ ... derivative of pot. with respect to normal direction

BY DEFENITION

... now we can calculate the current... $I = \int \vec{J} \cdot d\vec{a} = JA = \sigma E A = \frac{\sigma A}{L} \cdot V_0$

... so we have ... $V_0 = R I$ where $R = L/\sigma A$ ... so we have demonstrated equivalency!

PS. we can say that $\vec{J} = \sigma \vec{E}$ is LOCAL EQ. and $\Delta V = RI$ is GLOBAL EQ. !

Now... closed circuit... work per unit-charge in one loop

$$\vec{f} = \vec{E} \rightarrow \text{BUT!} \rightarrow \mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 0$$ => So electrostatic force can not be responsable for current in circuit!

So we need...
(need BATERY)

$$\vec{f} = \vec{E} + \vec{f}_s \qquad \mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l}$$ $\mathcal{E}$-electro-motive-force

f_s CHEMICAL FORCE

Faraday 1831 ... playing with circuits... found a way to produce flowing current with-out a battery!

FLUX

$$\mathcal{E} = -\frac{d}{dt} \phi(\vec{B}) = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{a}$$ ... S-surface that has edge on circuit

P.S. we can move circuit or change $\vec{B}$ in time ... if we move by $\vec{F} = \vec{v} \times \vec{B}$ we know the reason why charge move... but why if change $\vec{B}$ in time the charge move ?

P.S. FLUX : $\phi(\vec{B}) = \int_S \vec{B} \cdot d\vec{a}$

If $\vec{B}$ changes ... must be some new E-field ... $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \phi(\vec{B})$

THIS IS NON CONSERVATIVE ELECTRIC FIELD!

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \phi(\vec{B}) \rightarrow \text{STOKE'S} \rightarrow \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} \Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ FARADAY'S LAW

P.S. Note the minus sign in equation... this minus is not there just because of the convention (like $\vec{E} = -\nabla V$)... but is important (found by experiment)

The fact that there is minus there is called LENZ'S LAW!

I $\quad I = -\frac{1}{R} \frac{d\phi}{dt}$

Changing $\vec{B}$-field → $\vec{E}$-field → current I → new $\vec{B}$-field !

$\vec{B}_I = \frac{\mu_0}{2\pi} I \int \frac{d\vec{l} \times \hat{R}}{R^2} \propto I \Rightarrow \phi(B_I) \propto I$ ... $\vec{B}$-field created by current I (which is created by original $\vec{B}$)

So we get ... $\boxed{\phi(B_I) \propto -\frac{d}{dt} \phi(B)}$ ... NASPROTUJE SPREMEMBI!

Flux (generated) is proportional to current ... $\boxed{\phi(\vec{B}_I) = L I}$ L... [Vs/A = H... Henry]

INDUCTANCE

INDUCIRANA NAPETOST

So we can write ... $\boxed{\mathcal{E} = -L \frac{dI}{dt}}$ P.S. $\mathcal{E}$ je napetostna razlika [V]

How much work is needed to start a current (per unit charge) ... ?

$(W)_{\text{UNIT CHARGE TO START THE CURRENT}} = -\mathcal{E} \Rightarrow \frac{dW}{dt} = -\mathcal{E} \frac{dq}{dt} = -\mathcal{E} I$ ... work per unit time to start the current...

$$\Rightarrow \frac{dW}{dt} = -\mathcal{E} L = L I \frac{dI}{dt} = \frac{1}{2} L \frac{d}{dt} I^2 \Rightarrow \boxed{W = \frac{1}{2} L I^2}$$

we can say that this work (energy) is stored in current distribution!

... let's try to write this in diferent way ...

GENERATED FLUX $\rightarrow \phi(\vec{B}) = \int_S \vec{B}\cdot d\vec{a} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_{\partial S} \vec{A}\cdot d\vec{l} = LI$

... so we can write ... $\boxed{W = \frac{1}{2} I \oint \vec{A}\cdot d\vec{l} = \frac{1}{2}\oint (\vec{I}\cdot\vec{A})\, dl}$ ... FOR LINE CURRENT

FOR VOLUME CURRENT:

$$I\,dl = \frac{dq}{dt}dl = \frac{\rho\, d\tau}{dt}dl = \rho\frac{dl}{dt}d\tau = \overbrace{\rho v}^{J}\, d\tau = J\, d\tau \Rightarrow \boxed{W = \frac{1}{2}\int_V (\vec{J}\cdot\vec{A})\, d\tau}$$ ... FOR VOLUME CURRENT

From this equation we can more clearly see that energy is stored in current distr.!

We can use Ampere law ... $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \rightarrow W = \frac{1}{2\mu_0}\int_V (\vec{A}\cdot\vec{\nabla}\times\vec{B})\, d\tau$

Remember: $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$

$$\Rightarrow W = \frac{1}{2\mu_0}\left[\int_V \vec{B}\cdot\underbrace{\vec{\nabla}\times\vec{A}}_{\text{"}\vec{B}\text{"}}\, d\tau - \int_V \underbrace{\vec{\nabla}\cdot(\vec{A}\times\vec{B})}_{\text{DIV. THEOREM}}\, d\tau\right] =$$

$$= \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$$ ... like in electro-statics we can expand volume...

From beginning the volume V (integral. volume) must enclose $\vec{J}$... but we can make volume $V\rightarrow\infty$ ... The surface integral will vanish!

$$W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S (\underbrace{\vec{A}}_{\sim\frac{1}{r}}\times\underbrace{\vec{B}}_{\sim\frac{1}{r^2}})\cdot \underbrace{d\vec{a}}_{\sim r^2}\right] \Rightarrow \text{all together} \sim\frac{1}{r}$$

So we can write: $\boxed{W = \frac{1}{2\mu_0}\int_V B^2 d\tau}$ $\longleftrightarrow$ el-stat: $W = \frac{1}{2}\varepsilon_0\int_V E^2 d\tau$

all space

All 3 eq. are equivalent!

$$W = \frac{1}{2}LI^2 \longleftrightarrow W = \frac{1}{2}\int_V (\vec{J}\cdot\vec{A})\, d\tau \longleftrightarrow W = \frac{1}{2\mu_0}\int_V B^2 d\tau$$

Here we are neglecting energy lost as Joul Heat ... else we can get energy back!

Lets sumarize Maxwell equations until now:

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$	GAUSS	$\vec{\nabla}\cdot\vec{B} = 0$	
$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	FARADAY	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$	AMPERE

we expet that changing $\vec{E}$-field would produce $\vec{B}$-field...

Now we will try to fix this lack of symetry ... NEW PAGE!

Now ... $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0$ ... divergence of curl is allways zero ... $= -\frac{\partial}{\partial t}\,\vec{\nabla}\cdot\vec{B} = 0$ ✓ OK.

But ... $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 \ldots = \mu_0\,\underbrace{\vec{\nabla}\cdot\vec{J}}_{\neq 0}$ (not true in general ... only true in magneto-statics)

PROBLEM!

Maxwell modified Ampere's law ... in mathematical way ... to solve problem ...

Additional term is small ... that's why wasn't noticed by experiment!

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\varepsilon_0\,\vec{\nabla}\cdot\vec{E}\right) = -\varepsilon_0\,\vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)$$

So we get ... $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}}$ → MAXWELL TERM (DISPLACEMENT CURRENT)

MAXWELL EQUATIONS (IN VACUUM)

$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

SOURCES

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

We have to add only the force equation (in vacuum)

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

Now continuity equation can be derived ... $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ (conservation of charge)

EXAMPLE ... derive the conservation of charge equation ...

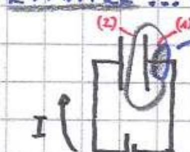
Charge flowing out of closed surface ... $\underbrace{\oint_S \vec{J}\cdot d\vec{a}}_{=I} \overset{\text{DIV. THEOREM}}{=} \int_V (\vec{\nabla}\cdot\vec{J})\,d\tau$ ... use $I = \frac{dq}{dt} = \frac{d}{dt}\underbrace{\int_V \rho\,d\tau}_{=q}$

So we get ... $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ → CHARGE IS CONSERVED

From Maxwell ... uporabimo $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$ in $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ ...

$$\vec{\nabla}\cdot\left(\frac{\partial\vec{E}}{\partial t}\right) = \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t} \;\ldots\; \frac{\partial\vec{E}}{\partial t} = \frac{1}{\mu_0\varepsilon_0}\left[\vec{\nabla}\times\vec{B} - \mu_0\vec{J}\right] \ldots \text{dobiš } \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \;!$$

EXAMPLE ... consider condenser ...



Ampere law ... $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$ (brez Maxwell člena)

If we choose the surface that has wire inside ... NO PROBLEM ... (SURFACE (1))

But we can choose any surface ... even the one that goes in between the plates of condenser (SURFACE (2)) ⇒ NO CURRENT ⇒ AMPERE LAW ITS NOT WORKING!

Now lets try if MAXWELL TERM (displacement current) solves the problem!

Ampere → $\int(\vec{\nabla}\times\vec{B})\,d\vec{a} = \int \mu_0\vec{J}\cdot d\vec{a} = \mu_0 I_{enc} \overset{\text{STOKES}}{=} \oint \vec{B}\cdot d\vec{l}$

So Maxwell → $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\int_S \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$ → we have to integrate over surface!

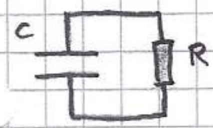
Surface (1): $\mu_0 I_{enc} = \mu_0 I \;\ldots\; \int\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = 0$

Surface (2): $\mu_0 I_{enc} = 0$ ... what is $\vec{E}$-field inside capacitor ... $E = \frac{\sigma}{\varepsilon_0} \Rightarrow \frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial\sigma}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{\partial q}{\partial t}$ $\left(\sigma = \frac{q}{A}\right)$

So we have ... $\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{\partial q}{\partial t} = \frac{1}{\varepsilon_0 A} I \;\longrightarrow\; \mu_0\varepsilon_0\int_S \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} = \mu_0\varepsilon_0\frac{1}{\varepsilon_0 A} I A = \mu_0 I$ OK! ✓

P.S. Upoštevali smo, da je $\vec{E}$-polje različno od 0 samo znotraj kondenzatorja!

EXAMPLE... charged capacator and we connect a resistor...



$$V = RI$$

moraš paziti kako definiraš tok... tok odteka iz C... zato je minus

$$VC = Q \rightarrow C\frac{dV}{dt} = \frac{dQ}{dt} = -I$$

začetni pogoji — karakteristični čas

Damo skupaj in... $V = RI = -RC\frac{dV}{dt} \Rightarrow \frac{\dot{V}}{V} = -\frac{1}{RC} \Rightarrow V(t) = V_0 \exp(-t/RC)$

What is the power disipated by resistor R...

$$P_0 = V_0^2/R$$

$$P = RI^2 = R\frac{V^2(t)}{R^2} = \frac{V_0^2}{R}\exp(-2t/RC) = P_0\exp(-2t/RC)$$

Total energy... $W = \int_0^\infty P(t)\,dt = P_0\int_0^\infty e^{-2t/RC}dt = -P_0\frac{RC}{2}\cdot\left[e^{-2t/RC}\right|_0^\infty = \frac{1}{2}P_0 RC = \frac{1}{2}CV_0^2$ ✓ok.

... we get exactly the energy that is initialy stored in capacitor!

EXAMPLE... inductance and resistance in a circuit...

$\mathcal{E}_L$ ... electromotive force by inductance... $\mathcal{E}_L = -L\frac{dI}{dt}$

... sum of voltages over the circuit must be zero ... $\mathcal{E}_0 + RI + \mathcal{E}_L = 0$

... so we get... $\mathcal{E}_0 + RI - L\frac{dI}{dt} = 0 \Rightarrow \dot{I} = \frac{R}{L}I + \frac{\mathcal{E}_0}{L} \rightarrow (Ie^{-\frac{R}{L}t})^{\bullet} = \frac{\mathcal{E}_0}{L}e^{-\frac{R}{L}t} \rightarrow$

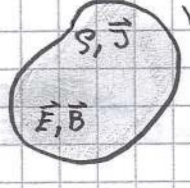
$$\rightarrow Ie^{-\frac{R}{L}t} = \frac{\mathcal{E}_0}{L}\int e^{-\frac{R}{L}t}dt = -\frac{\mathcal{E}_0}{R}e^{-\frac{R}{L}t} + \frac{\mathcal{E}_0}{L}\cdot\text{Const.}$$ ... sm pozabu en minus ones!

Na koncu bi moral dobit... $I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$

Now... we saw charge conservation law included in Maxwell equations...

WHAT ABOUT ENERGY CONSERVATION?

Consider volume ... there is also some $\vec{E}$ and $\vec{B}$-field inside acting on $\rho, \vec{J}$



We look at work done on some small charge ... $(\vec{F}\cdot d\vec{l})_{dq} = (dW)_{dq}$

$$(dW)_{dq} = dq\,(\vec{E} + \cancel{\vec{v}\times\vec{B}})\cdot d\vec{l} = dq\,\vec{E}\cdot\overbrace{\vec{v}\,dt}^{=d\vec{l}}$$

NO WORK IS DONE BY B-FIELD

So we have... $(dW/dt)_{dq} = dq\,\vec{E}\cdot\vec{v}$ → Elementary work per unit time on dq! WE USED $\vec{J} = \rho\vec{v}$

Total work/time on charges... $dW/dt = \int_V dq\,\vec{E}\cdot\vec{v} = \int_V \rho\,\vec{E}\cdot\vec{v}\,d\tau = \int_V \underbrace{\vec{E}\cdot\vec{J}}\,d\tau$

Now we work on this!

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \overbrace{\varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}}^{=\frac{\varepsilon_0}{2}\frac{\partial}{\partial t}E^2}$$

USE... $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$ and $\vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) - \vec{\nabla}(\vec{E}\times\vec{B})$

So we have... $\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\left(-\frac{1}{2}\frac{\partial}{\partial t}B^2\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B}) - \frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}E^2$

$$\vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left[\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right] - \frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

DIV. THEOREM

POYINTING

Energy per unit time and unit area ... $\boxed{\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}}$ POYINTING VECTOR

We are going to focus on this ... propagation of energy ...

WAVE EQUATION ... empty space => $\rho = 0$; $\vec{J} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \nabla(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0}) - \nabla^2 \vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

So we get ... $\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$

For $\vec{B}$-field ... $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \nabla(\underbrace{\vec{\nabla} \cdot \vec{B}}_{=0}) - \nabla^2 B = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$

And again ... $\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$

P.S. E-M valovanje je edino, ki ne rabi medija ...

$v = 1/\sqrt{\varepsilon_0 \mu_0} = c$!

Lets look in 1-dimension ... $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$

most general solution ...

$$f(z,t) = g(z + vt) + h(z - vt)$$

But we can review main facts only with sinosodial waves ... because any function can be written as sum of sin. by Fourier series ...

$f(z,t) = A \cos[k(z - vt) + \delta]$ ... A - amplitude; k - wave number $(= 2\pi/\lambda)$; δ - phase

... period $T = 2\pi/kv$; frequency $\nu = 1/T = \frac{kv}{2\pi} = v/\lambda$

Exponential form ...

$$f(z,t) = \mathrm{Re}\left[A \exp(i[kz - \omega t + \delta])\right]$$

we can incorporate the phase in A →

$$\tilde{f}(z,t) = \tilde{A} \exp(i[kx - \omega t])$$

$\tilde{A}$ is complex! $\tilde{A} = A e^{i\delta}$

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

Now ... MONOCHROMATIC PLANE WAVES

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz - \omega t)}$$
$$\tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz - \omega t)}$$

→ By Maxwell $\left.\begin{array}{l}\vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0\end{array}\right\} \rightarrow (\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$ → TRANSVERSE WAVE

→ By Maxwell $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \left\{\begin{array}{l} -k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x \\ k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y \end{array}\right.$

To so splošne rešitve, toda Maxwell nam da še nekaj dodatnih pogojev.

$$\Rightarrow \tilde{\vec{B}}_0 = \frac{k}{\omega} \hat{z} \times \tilde{\vec{E}}_0$$

$\vec{B}, \vec{E}$ are $\perp$

Now general ... $\vec{k}$ - wave vector ; $|\vec{k}| = 2\pi/\lambda$ direction of $\vec{k}$ is along dirr. of propagation!

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$
$$\tilde{\vec{B}}(\vec{r},t) = \frac{1}{c} \tilde{\vec{E}}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \frac{1}{c} \hat{k} \times \tilde{\vec{E}}$$

Back to Poyinting ...

GOSTOTA ENERGIJE E in B POLJA

energy of $\vec{E}$ and $\vec{B}$ field per unit volume ... $\boxed{u_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)}$

For a plane wave ... $B^2 = \frac{1}{c^2} E^2 = \mu_0 \varepsilon_0 E^2$

So ... $u_{EM} = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$

fast charges in time

$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) \rightarrow$ plane wave $\rightarrow$ $\vec{S} = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\,\hat{z}$

But when we measure this things ... you get only averige (because the small period)

AVERAGE

$\langle u_{EM} \rangle = \frac{1}{2}\varepsilon_0 E_0^2$

$\langle \vec{S} \rangle = \frac{1}{2} c\varepsilon_0 E_0^2 \hat{z}$ ... $\boxed{I = |\langle \vec{S} \rangle| = \frac{1}{2} c\varepsilon_0 E_0^2}$ INTENSITY

EXAMPLE ... monochromatic wave

Example of a function for mono-chrom wave ... $f(t) = \begin{cases} \cos\omega_0 t & ; -\tau < t < \tau \\ 0 & ; \text{else} \end{cases}$

To see spectre we must take Fourier transform ...

$$f(t) = \int_0^\infty a(\omega)\cos(\omega t)\,dt \longrightarrow a(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt \rightarrow (\text{SPECTRE} \ldots \|a(\omega)\|^2)$$

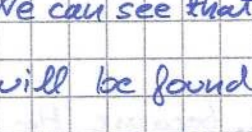
$$\ldots\ a(\omega) = \frac{1}{\pi}\int_{-\tau}^{\tau}\cos(\omega_0 t)\cos(\omega t)\,dt = \frac{1}{\pi}\int_0^{\tau}\left[\cos([\omega_0+\omega]t) + \cos([\omega_0-\omega]t)\right]dt = \ldots$$

$$\ldots = a(\omega) = \frac{\tau}{\pi}\left[\frac{\sin(\omega_0+\omega)\tau}{(\omega_0+\omega)\tau} + \frac{\sin(\omega_0-\omega)\tau}{(\omega_0-\omega)\tau}\right]$$

Now lets say we try to have mono-chrom w.
For this we need big τ!
Look what happens when ... $\tau \gg T = \frac{2\pi}{\omega_0}$

wave period

When $\tau \gg T = 2\pi/\omega_0$ (big τ) → The 1ST term with $(\omega_0+\omega)$ goes to zero! But the 2ND term depends on how close is ω to ω_0 ... When $\omega_0 \approx \omega$ the 2ND term → 1!



Lets try to find the width of this spectral distribution! ...

Let $\omega_0 + \Delta\omega = \omega'$; ω' such that $\sin[(\omega_0 - \omega')\tau] = 0$ ...

so we get ... $(\omega_0 - \omega')\tau = \Delta\omega\,\tau = \pi \implies \boxed{\Delta\omega\,\Delta\tau = 2\pi}$ UNCERTAINTY PRINCIPLE

We can see that frequency and time domain are coupled! Similar principle will be found in Quantum Mechanics ... there the uncertainties of position and velocoty are coupled ⇒ we can not determine exact initial conditions!

Given the sources we can determine the fields: $\rho(\vec{r},t), \vec{J}(\vec{r},t) \rightarrow \vec{E}(\vec{r},t), \vec{B}(\vec{r},t)$

Remember → Electrostatics ⊕ Magnetostatics

$$\vec{E} = -\nabla V \qquad \vec{B} = \vec{\nabla}\times\vec{A} \quad \Big\} \rightarrow \text{in terms of potentials!}$$

Solutions...

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r})}{R}\,d\tau \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J(\vec{r})}{R}\,d\tau$$

Now we try to find the potentials for Electrodynamics...

$$\left.\begin{array}{l}\vec{B} = \vec{\nabla}\times\vec{A} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}\end{array}\right\} \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = -\frac{\partial}{\partial t}\nabla\times\frac{\partial\vec{A}}{\partial t} \Rightarrow \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$$

PS... we start from $\vec{B} = \vec{\nabla}\times\vec{A}$ because $\vec{\nabla}\cdot\vec{B} = 0$ also in dynamics...

So... Electrodynamics... $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$; $\vec{B} = \vec{\nabla}\times\vec{A}$ $\rightarrow \begin{cases}\vec{A}\rightarrow\vec{A}+\nabla\lambda \text{ ... does not change anything} \\ V \rightarrow V - \frac{\partial\lambda}{\partial t}\end{cases}$ ... lets try ...

Let... $\begin{cases}\vec{A}' = \vec{A}+\nabla\lambda \\ V' = V - \frac{\partial\lambda}{\partial t}\end{cases}$ ... $\vec{E} + \frac{\partial\vec{A}}{\partial t} = $ ... pride isto ōc ustonis ... GAUGE FREEDOM

$$\vec{E}' = -\nabla V' - \frac{\partial\vec{A}'}{\partial t} = -\nabla V + \cancel{\frac{\partial}{\partial t}\nabla\lambda} - \frac{\partial\vec{A}}{\partial t} - \cancel{\frac{\partial}{\partial t}\nabla\lambda} = \vec{E} \quad \checkmark \text{ OK}$$

→ ... MAXWELL EQUATIONS IN TERMS OF POTENTIALS

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \vec{\nabla}\cdot\left(-\nabla V - \frac{\partial\vec{A}}{\partial t}\right) = -\nabla^2 V - \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = \rho/\varepsilon_0 \quad (1)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial\vec{A}}{\partial t}\right)$$

... so we get... $\left(\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\vec{J}$ (2)

Now use GAUGE FREEDOM... (we can fix something ... we use LORENZ GAUGE)

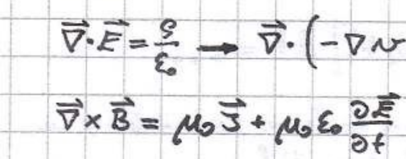
LORENZ GAUGE : $\vec{\nabla}\cdot\vec{A} = -\mu_0\varepsilon_0\frac{\partial V}{\partial t}$ (we fix the divergance of vector potential)

(1) & (2) ⊕ LORENZ $\begin{cases}\boxed{\nabla^2 V - \mu_0\varepsilon_0\frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0} \\ \boxed{\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}}\end{cases}$ → We introduce new operator...

$$\square^2 = \nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2} \quad \text{D'ALAMBERTIAN}$$

And we can writte in compact form... $\boxed{\square^2\vec{A} = -\mu_0\vec{J} \; ; \; \square^2 V = -\rho/\varepsilon_0}$

Now we want to find the solutions ... to find potentials...



... there is some delay between cause and effect... because the perturbation moves with finite speed (speed of light)

$$t_r = t - R/c \rightarrow \text{RETARDED TIME}$$

We will try to se if the potentials with retarded time instead of time solve our problem... RETARDED POTENTIALS

$$\boxed{V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r},t_r)}{R}\,d\tau \; ; \; \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r},t_r)}{R}\,d\tau}$$

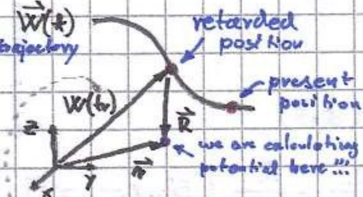
Then in 1966 Japanesse Jefimenko wrote the fields with potentials ...

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \left[\frac{\rho(\vec{r},t_r)}{R^2}\hat{R} + \frac{\dot{\rho}(\vec{r},t_r)}{cR}\hat{R} - \frac{\dot{\vec{J}}(\vec{r},t_r)}{c^2R}\right] d\tau$$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V \left[\frac{\vec{J}(\vec{r},t_r)}{R^2} + \frac{\dot{\vec{J}}(\vec{r},t_r)}{cR}\right] \times \hat{R}\, d\tau$$

Note... static terms decay as $1/R^2$, but time depend. term decay as $1/R$ ... This is because very far away only waves remain (light)

EXAMPLE... charged particle moving and acelerating ...



$$\boxed{t_r = t - R/c}$$

We can writte the potentials for this system ...

$$v(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r},t_r)}{R} d\tau = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\int \rho(\vec{r},t_r)\, d\tau$$

... the R can go out because it is a point particle ... (ne čist point ... ampak dovolj majhen, da lahko damo R ven)

How comes the problem... Giovanni demonstrated that the length of moving object is apperently changed ... $L \to L' = L/(1 - v/c)$

Because of this the volume (that we integrate) will be distorted ...

~ observer $\longrightarrow$ $\Delta x = L' - L \to \frac{L'}{c} = \frac{\Delta x}{v} \Rightarrow \ldots \Rightarrow L' = L/(1 - v/c)$

~ observer 2 $\longrightarrow$ $\frac{L'\cos\theta}{c} = \frac{\Delta x}{v} \Rightarrow L' = L/(1 - \frac{v}{c}\cos\theta)$ $\quad \boxed{L' = \frac{L}{1 - \frac{\vec{v}\cdot\hat{R}}{c}}}$

Only dimensions paralel to speed are modified, dimension paralel to $\vec{v}$ is unchanged!

Using this we evaluate:

$$\boxed{v(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\int_V \rho(\vec{r},t_r)\, d\tau = \frac{1}{4\pi\varepsilon_0}\frac{1}{R}\frac{q}{1 - \frac{\vec{v}\cdot\hat{R}}{c}} = \frac{1}{4\pi\varepsilon_0}\frac{qc}{(Rc - \vec{R}\cdot\vec{v})}}$$

Same for vector pot.:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\overbrace{\rho(\vec{r},t_r)\cdot\vec{v}}^{=\vec{J}(\vec{r},t_r)}\, d\tau}{R} = \frac{\mu_0}{4\pi}\frac{\vec{v}(t_r)}{R}\int \rho(\vec{r},t_r)\, d\tau$$

So we have:

$$\boxed{\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vec{v}}{(Rc - \vec{R}\cdot\vec{v})} = \frac{\vec{v}}{c^2}v(\vec{r},t_r)}$$

LIENARD WIECHERT POTENTIALS

This equations are basis for ciclotron emisions ... if we have many particles ... superposition!

Once we have the expressions for potentials we can calculate the fields ...

$$\vec{E} = -\nabla v - \frac{\partial\vec{A}}{\partial t} = \ldots = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\left[(c^2 - v^2)\vec{u} + \vec{R}\times(\vec{u}\times\vec{a})\right]$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \ldots = \frac{1}{c}\hat{R}\times\vec{E}(\vec{r},t)$$

we used: $\vec{u} = c\hat{R} - \vec{v}$; $\vec{a} = \dot{\vec{v}}$

Note... the 1ST term is decaying as $1/R^2$, while the 2ND term decays as $1/R$!

We can check that if we set $\vec{v} = 0$, $\vec{a} = 0$ we will be left just with equations of electrostatics!

Let's go on with example of single charge ... what is the radiation?

RADIATED POWER

→ total power pasing Σ → $P(r) = \oint_\Sigma \vec{S}\cdot d\vec{a} = \frac{1}{\mu_0}\oint_\Sigma (\vec{E}\times\vec{B})\cdot d\vec{a}$

Than radiated power... $\boxed{P_{rad} = \lim_{r\to\infty} P(r)}$

P.S. From equations for fields (previous page) you can see that we ned acceleration!

(samo 2^{ND} term nas bo najbož zanimal pri $r\to\infty$...) $\vec{B} = \frac{1}{c}\hat{R}\times\vec{E}$

look previous page

Poynting vector: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}\vec{E}\times(\hat{R}\times\vec{E})$ ... use $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$\vec{S} = \frac{1}{\mu_0 c}\left[E^2\hat{R} - (\hat{R}\cdot\vec{E})\vec{E}\right]$$ ... we put only 2^{ND} term... $\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{R}{(\vec{R}\cdot\vec{u})^3}\left[\vec{R}\times(\vec{u}\times\vec{a})\right]$

So we have... $\boxed{\vec{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\hat{R}}$ ... this term is not contributing, because of scalar product...

Now look at... charge at rest $(\vec{v}=0;\ \vec{a}\neq 0)$

... so we get... $\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 R}\left[\hat{R}\times(\hat{R}\times\vec{a})\right] = \frac{\mu_0 q}{4\pi R}\left[(\hat{R}\cdot\vec{a})\hat{R} - \vec{a}\right]$

$\left[(\hat{R}\cdot\vec{a})\hat{R} - \vec{a}\right]^2 = \dots = (\hat{R}\cdot\vec{a})^2 + a^2 - 2(\hat{R}\cdot\vec{a})\hat{R}\cdot\vec{a}$

Now for $\vec{S}_{rad}$...

$$\vec{S}_{rad} = \frac{1}{\mu_0 c}E_{rad}^2\hat{R} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi R}\right)^2\left[a^2 - (\hat{R}\cdot\vec{a})^2\right]\hat{R}$$

$$\vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c R^2}\underbrace{(1-\cos^2\Theta)}\hat{R}$$ ... Θ - angle between $\hat{R}$ and $\vec{a}$

Finally...

$$\boxed{\vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\Theta}{R^2}\hat{R}}$$

→ accelerated charge at rest $(\vec{a}\neq 0;\ \vec{v}=0)$

→ shaped as "donut" (glej knjigo...)

So total radiated power...

$$P_{rad} = \lim_{R\to\infty} P(R) = \oint \vec{S}_{rad}\cdot d\vec{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\Theta}{R^2} R^2 \sin\Theta\, d\Theta\, d\phi = \underline{\frac{\mu_0 q^2 a^2}{6\pi c}}$$

Generally...
For $\vec{v}\neq 0;\ \vec{a}\neq 0$

$$\boxed{P_{rad} = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)}$$ LARMOR FORMULA ... $\gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$

EXAMPLE 11.3 (in the book)... $\vec{v}$ and $\vec{a}$ are instantaneously collinear

RADIATED POWER PER UNIT SOLID ANGLE → $$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\Theta}{(1-\beta\cos\Theta)^5}}\ ;\ \beta = v/c$$

↳ FOR $\vec{v}\parallel\vec{a}$

→ the "donut" is deformed into "cone" directed forward...

Other example in the book is when $\vec{v}$ and $\vec{a}$ are instantaneusly perpendicular...

$$\boxed{\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\left[(1-\beta\cos\Theta)^2 - (1-\beta^2)\sin^2\Theta\cos^2\phi\right]}{(1-\beta\cos\Theta)^5}}$$ → FOR $\vec{v}\perp\vec{a}$

ELECTRODYNAMICS IN MATTER

can be roughly divided in CONDUCTORS and INSULAT.

INSULATORS (DIELECTRICS)

ko ni E-polja težišči ⊕ in ⊖ naboja sovpadata v atomu

$E_0 = 0$

ELECTRIC DIPOLE

zaradi zunanjega E-polja se težišči ⊕ in ⊖ naboja razmakneta

$$\boxed{\vec{p} = q\vec{d}}$$ ELECTRIC DIPOLE MOMENT

$$\boxed{\vec{p} = \alpha \vec{E}_0}$$ ATOMIC POLARIZIBILITY

Holds if E_0 it's not too strong!

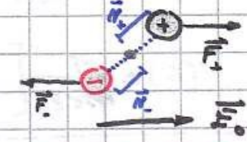
In molecules there can be some dependence on direction ...

$$\left.\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned}\right\} \text{IN MOLECULES}$$

PS. Molecules can also have some intrinsic dipole moment, without and E-field... !

↓

POLARIZED MOLECULES



$\vec{F}_+ = q\vec{E}$

$\vec{F}_- = -q\vec{E}$

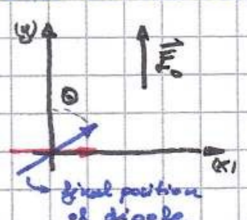
...navor na polarizirano molekulo...

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-) =$$

$$= \left(\frac{\vec{d}}{2} \times q\vec{E}\right) + \left(-\frac{\vec{d}}{2} \times (-q\vec{E})\right) = q\,\vec{d} \times \vec{E} = \boxed{\vec{p} \times \vec{E} = \vec{N}}$$ TORQUE ON POLARIZED MOLECULE

Torque will try to align $\vec{p}$ with $\vec{E}_0$! To show this we will look at energy of a system... and see that energy is lowest when $\vec{p}$ in aligned with $\vec{E}_0$.

...what is the energy of dipole in an external electric field?...



dipole at ∞ (moral bi ga narisat na x-osi)

ko pripeljemo dipol do koor. izhodišča ne naredimo nobenega dela... ker sta pot in sila ⊥ !

nato moramo dipol še zavrteti za nek kot... to je neko delo...

$$\text{WORK: } U = \int_{\pi/2}^{\theta} N\, d\theta' \quad \dots \vec{N} = \vec{p} \times \vec{E}_0 = pE_0 \sin\theta\, \hat{z}$$

$$\dots\text{so}\dots\; U = \int_{\pi/2}^{\theta} pE_0 \sin\theta'\, d\theta' = pE_0 \cos\theta' \Big|_{\pi/2}^{\theta} = \dots$$

$$\dots = -pE_0\cos\theta = -\vec{p}\cdot\vec{E}_0$$ → MINIMUM when $\vec{p} \parallel \vec{E}_0$???

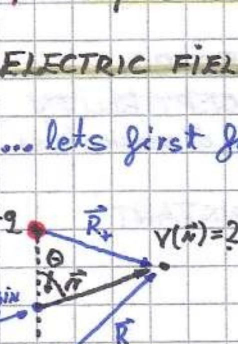
When dipoles orient in direction of $\vec{E}_0$ there is also macro-scopic effect!...

$\vec{P}$ = dipole moment per unit volume

without external $\vec{E}_0$ polarization of molecules is random ⇒ cancels (macro-scopicly)

ELECTRIC FIELD GENERATED BY POLARIZED DIELECTRIC MATERIAL

... lets first find the potential created by dipole moment!...



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{R_+} - \frac{q}{R_-}\right] \quad ; \quad R_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta \; ; \; R_-^2 = r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta$$

$$R_{+-}^2 = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right) \longrightarrow r^2\left(1 \mp \frac{d}{r}\cos\theta\right) \dots \text{FOR } r >> d\,!$$

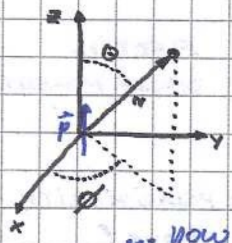
$$\Rightarrow \frac{1}{R_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \approx \frac{1}{r}\left(1 \pm \frac{d}{r}\cos\theta\right)^{1/2} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \frac{1}{R_+} - \frac{1}{R_-} \approx \frac{d}{r^2}\cos\theta$$

...F COSINUS ... $a^2 + b^2 - 2ab\cos\theta$

... or approximetly ...

... in general for $r >> d$ ... $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{R}\cdot\vec{p}}{r^2} \rightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$



$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\ ;\ E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\ ;\ E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi}\cdot\frac{1}{\sin\theta} = 0$$

... now we go back to finding the polarization of ...

$\vec{p} = \vec{P}\,d\tau'$ ... TOTAL POTENTIAL → $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\hat{R}\cdot\vec{P}}{R^2}d\tau'$ P.S. $\vec{P} = \vec{P}(\vec{r}')$

REMEMBER:

$\nabla'\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$ $(\vec{R} = \vec{r} - \vec{r}')$ $\Longrightarrow V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)d\tau'$

..."per partes"... $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \nabla'\cdot\left(\frac{\vec{P}}{R}\right)d\tau' - \int_V \frac{1}{R}\left(\nabla'\cdot\vec{P}\right)d\tau'\right] =$ "use divergence theorem"

$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\oint_S \frac{1}{R}\vec{P}\cdot d\vec{a}' - \int_V \frac{1}{R}\left(\nabla'\cdot\vec{P}\right)d\tau'\right]$ ... we can write the two components...

POTENTIAL

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{\sigma_b}{R}da' + \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_b}{R}d\tau'}\ ;\ \sigma_b = \vec{P}\cdot\hat{u}\ ;\ \rho_b = -\nabla'\cdot\vec{P}$$

vector normal to surface

P.S. index b → "bound" charges

P.S. To je potencial zaradi polarizacije...
zunanje E-polje, ki je povzročilo polarizacijo ni vključeno!

P.S. To je potencial zunaj in znotraj polariziranega materiala!

How to put this into Maxwell equations? ...

$\varepsilon_0\nabla\cdot\vec{E} = \rho_f + \rho_b = \rho_f - \nabla\cdot\vec{P} \Longrightarrow \nabla\cdot\left[\varepsilon_0\vec{E} + \vec{P}\right] = \rho_f$ FREE CHARGES

define... $\boxed{\vec{D} = \varepsilon_0\vec{E} + \vec{P}}$ ELECTRIC DISPLACEMENT

So we get... $\boxed{\nabla\cdot\vec{D} = \rho_f}$ → $\oint_S \vec{D}\cdot d\vec{a} = (Q_{free})_{int}$ Modified Gauss theorem

To "know" everything we need to know curl of $\vec{D}$ ...

$\nabla\times\vec{D} = \varepsilon_0\underbrace{\nabla\times\vec{E}}_{=0} + \nabla\times\vec{P} \neq 0$ (in general)

linear aproximation (small $\vec{E}$)

For not to strong E-field we can write... $\vec{P} = \varepsilon_0\chi_e\vec{E}$ ELECTRON SUSCEPTIBILITY (it's dimensionless)

From this we get... $\vec{D} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} = \varepsilon_0\underbrace{(1+\chi_e)}_{\varepsilon_r}\vec{E}$

usualy denoted by ε_r → DIELECTRIC CONSTANT

$\vec{D} = \varepsilon_r\varepsilon_0\vec{E}$

For linear matterials we also have... $\nabla\times\vec{D} = 0$ (to sledi iz $\vec{D} = \varepsilon\varepsilon_0\vec{E}$)

example:

ENERGY IN DIELECTRIC MATERIALS

remember... energy stored in capacitor... $W = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{V}$ → in vacuum

Now we put linear dielectric material between the plates of capacitor!

What happens to capacitance C? (or... what happens to potential V?)

vacuum: $\left.\begin{matrix} \vec{\nabla}\cdot\vec{E} = \rho_8/\varepsilon_0 \\ \vec{\nabla}\times\vec{E} = 0 \end{matrix}\right\} \xrightarrow{\text{(in matter)}} \left.\begin{matrix} \vec{\nabla}\cdot\vec{D} = \rho_8 \\ \vec{\nabla}\times\vec{D} = 0 \end{matrix}\right\} \Longrightarrow \vec{D} = \varepsilon_0 \vec{E}_{VAC}$ (in vacuum)

So we get... $\vec{E} = \frac{1}{\varepsilon_r \varepsilon_0} \vec{D} = \frac{1}{\varepsilon_r \varepsilon_0} \varepsilon_0 \vec{E}_{VAC} \Longrightarrow \boxed{\vec{E} = \frac{1}{\varepsilon_r} \vec{E}_{VAC}}$

Same is true for the potential... $\boxed{V = \frac{1}{\varepsilon_r} V_{vac}}$

Capacitance... look at definition... $C = Q/V \Longrightarrow \boxed{C = \varepsilon_r C_{VAC}}$

In vacuum... energy stored in electrostatic system... $W = \frac{1}{2} \varepsilon_0 \int_{\text{all space}} E^2 d\tau$

in linear dielectric material... $\boxed{W = \frac{1}{2} \varepsilon_0 \varepsilon_r \int_{\text{all space}} E^2 d\tau = \frac{1}{2} \int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau}$

This law can be demonstrated in general... not only for capacitor!

Lets demonstrate this in general case... remember... like in vacuum we also now "bring" charges from infinity... (one by one)...

... to increase free charge in system... $\Delta W = \int (\Delta \rho_8) v\, d\tau$

... $\vec{\nabla}\cdot\vec{D} = \rho_8 \longrightarrow \Delta\rho_8 = \vec{\nabla}\cdot(\Delta\vec{D})$ (from Maxwell eq. in matter)

So we have... $\Delta W = \int [\vec{\nabla}\cdot(\Delta\vec{D})] v\, d\tau$ → we do integral "per partes"

"per partes" → $\Delta W = \underbrace{\int \vec{\nabla}\cdot[\Delta\vec{D}\, v]\, d\tau}_{0} + \int (\Delta\vec{D})\cdot\vec{E}\, d\tau$ ... now like allways: Volume → ∞

integral over all space...

Now... $\Delta\vec{D}\cdot\vec{E} = \varepsilon_r (\Delta\vec{E})\cdot\vec{E} = \frac{1}{2}\Delta(\varepsilon_r E^2) = \frac{1}{2}\Delta(\vec{D}\cdot\vec{E})$

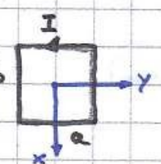
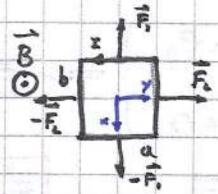
We have found...

$$\Delta W = \Delta \int_{\text{all space}} \frac{1}{2}\vec{D}\cdot\vec{E} \longrightarrow \boxed{W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau}$$

In material we have currents ... electrons moving around nucleus are current $\Rightarrow$ produces $\vec{B}$-field ... also electron spin produces $\vec{B}$-field ...

Usualy this $\vec{B}$-fields cancel each other because of randomness ... EXCEPT IN FERROMAGNETS

But if we apply external $\vec{B}$-field there will be alignment of ...

consider:  we switch $\vec{B}$-field on $\rightarrow$  $F_1 = IBa$

$F_2 = IBb$

... now suppose the circuit is tilted ... torque on tilted circuit

 $\rightarrow$ torque $\vec{N} = aF_2 \sin\theta\, \hat{x} = IabB\sin\theta\, \hat{x} = \vec{m}\times\vec{B}$

... we introduce : $\vec{m} = Iab\,\hat{n}$ MAGNETIC MOMENT

Now ... any general circuit can be sub-divided into small rectangles ...

... this means that this result is general ... $\vec{m} = I\underset{\text{AREA}}{S}\hat{n}$! $\boxed{\vec{N} = \vec{m}\times\vec{B}}$

In presence of external $\vec{B}$-field the contribution of spin is usualy easyly canged compared to contribution because of "orbiting" the e^- around the nucleus (but remember $\rightarrow$ Pauli exclusion principle ... sometimes spin of electron can not be changed) $\Rightarrow$ we have to make difference betwen atoms with odd number of electrons ... and even # of e^- ...

Atoms with odd number of e^- $\rightarrow$ spin contribution to $\vec{B}$-moment dominates !

Atoms with even number of e^- $\rightarrow$ orbital contribution to $\vec{B}$-moment dominates !

Because of torge ... SPIN DOMINATION $\rightarrow$ $\vec{m}$ paralel to $\vec{B}$ (PARAMEGNITISM)

ORBITAL DOMINATION $\rightarrow$ $\vec{m}$ anti-paralel to $\vec{B}$ (DIAMAGNETISM)

Lets try to demonstrate that fot atoms with even number of e^- the magnetic moment (orbital) is anti-paralel to external $\vec{B}$-field !

... atom with 1 e^- ... current $I = \frac{\text{charge}}{\text{period}} = \frac{-e}{T} = \frac{-ev}{2\pi R}$ $\quad T = \frac{2\pi R}{v}$

... so the magnetic moment ... $\vec{m} = \pi R^2 I\,\hat{z} \Rightarrow \vec{m} = -\frac{1}{2}evR\,\hat{z}$

Now ... in absence of $\vec{B}$-field ... $\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$

... and in presence of $\vec{B}$-field ... $\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$ $\quad$ $\bar{v}$ – e^- velocity in presence of $\vec{B}$-field

$\longrightarrow m_e \frac{v^2}{R} + e\bar{v}B = m_e \frac{\bar{v}^2}{R} \Rightarrow e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$

... we assume that effect of $\vec{B}$-field is small... its positive! ⇒ speed of e^- increases

$\Delta v = \bar{v} - v$ SMALL ; $\bar{v} + v \sim 2v \Rightarrow \Delta v = \frac{eRB}{2m_e} > 0$

A change in orbital speed means a change in the magnetic moment...

$$\boxed{\Delta\vec{m} = -\frac{1}{2} e \Delta v R \hat{z} = -\frac{e^2 R}{4 m_e} \vec{B}}$$ ⟶ change of magnetic moment is alligned with $\vec{B}$-field, but in opposite direction! (possible in the oral exam → what if we change $\vec{B} \to -\vec{B}$?)

PS. Magnetizaid material → $\vec{M}$ = magnetic dipole moment per unit volume

Now we will look an effect of magnetized materal...

... vector potential generated by $\vec{m}$: $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{R}}{R^2}$ Note: $\vec{m} = \vec{M}\, d\tau'$

so we get : $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}') \times \hat{R}}{R^2} d\tau'$ → vec. potential generated by magnetized material

... we will use known fact that : $\nabla'\left(\frac{1}{r}\right) = \hat{r}/r^2$ ... put into equation for $\vec{A}$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \times \left(\nabla' \frac{1}{R}\right) d\tau' = \frac{\mu_0}{4\pi}\left[\int \frac{1}{R}\left[\nabla' \times \vec{M}(\vec{r}')\right] d\tau' - \int \nabla' \times \left[\frac{\vec{M}(\vec{r}')}{R}\right] d\tau'\right]$$

It can be shown that : $\int_V (\nabla \times \vec{v})\, d\tau = -\oint_S \vec{v} \times d\vec{a}$ (mathematical excercise!)

So we get : $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{1}{R}\left[\nabla' \times \vec{M}(\vec{r}')\right] d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{1}{R}\left[\vec{M}(\vec{r}') \times d\vec{a}'\right]$

J_b ~ bound current

Now, similar as for E-field, we define : $\boxed{\vec{J}_b = \vec{\nabla} \times \vec{M} \;;\; \vec{K}_b = \vec{M} \times \hat{u}}$

So we can write : $\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r}')}{R} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r}')}{R} da'}$

Now we can take look at Maxwell equations using this results!

current density can be written as : $\vec{J} = \vec{J}_f + \vec{J}_b$ → "free" current + "bound" current

Static Ampere law: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla} \times \vec{M}$ ... we used def: $\vec{J}_b = \vec{\nabla} \times \vec{M}$

so we get... $\vec{\nabla} \times \left[\frac{1}{\mu_0}\vec{B} - \vec{M}\right] = \vec{J}_f$ ... define new quantaty : $\boxed{\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}}$

So we get modified equation : $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}_f}$ "MODIFIED AMPERE LAW"

This can be written in integral form as : $\boxed{\oint \vec{H} \cdot d\vec{l} = (I_f)_{enc}}$

PS. $\vec{\nabla} \cdot \vec{B} = 0$ but $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \neq 0$ (in general)

Now we look at **LINEAR MATERIALS** for which : $\boxed{\vec{M} = \underbrace{\chi_m}_{\text{magnetic susceptibility (small} \sim 10^{-5})} \vec{H}}$ magnetization due to external field!

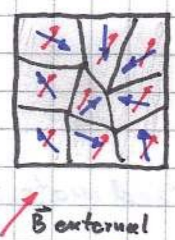
We can write for linear material : $\boxed{\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(1+\chi_m)\vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H}}$

where : $\boxed{\mu = \mu_0(1+\chi_m)}$ PERMEABILITY OF MATERIAL

For materials discused until now the magnetization is very small!

This is not true for **FEROMAGNETICS**... they remain magnetic even when external magnetic field is removed. It is due to the spin of the e^-!

All feromagnetic materials have odd number of e^- on outer shell!



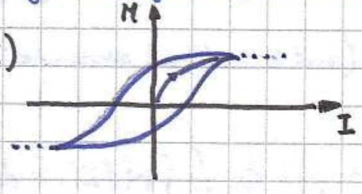
... initialy dipole moments of different regions are random => cancel each other

Now we switch external $\vec{B}$-field... bounderies of regions are more affected by external $\vec{B}$-field ... effect is after some time also in the centre ... when we remove external $\vec{B}$-field the magnetization remains!

Imagine we put feromagnetic material into coil and swich the current on!

we draw $M=M(I)$

HYSTERESIS LOOP



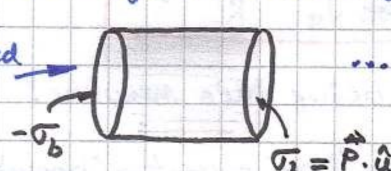
note: even when we have 0 current magnetization remains (if previosly magnetized)

Now we go back to try to find general Maxwell equations!

remember: $\rho_b = -\vec{\nabla}\cdot\vec{P}$; $\vec{J}_b = \vec{\nabla}\times\vec{M}$

PS. Any polarization change involve a flow (current) of bound charges!

piece of polarized material



... suppose $\vec{P}$ increases by small amount!

$$dI = \frac{\partial}{\partial t}\, dq = \frac{\partial}{\partial t}[\sigma_b\, da] = \frac{\partial}{\partial t}[\vec{P}\cdot d\vec{a}]$$

$$dI = \frac{\partial \vec{P}}{\partial t}\cdot d\vec{a} \equiv \vec{J}_p \cdot d\vec{a} \qquad d\vec{a} = \hat{u}\, da$$

We defined new current (due to change of polarization) : $\boxed{\vec{J}_p = \frac{\partial \vec{P}}{\partial t}}$

Now current that can flow in material has 3 contributions... free, bound, polarization

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t} \quad \text{and} \quad \rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}\cdot\vec{P}$$

Gauss law: $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \longrightarrow \vec{\nabla}\cdot\vec{D} = \rho_f$ ... we introduced $\vec{D}$ to write with free charges!

This remains the same: $\vec{\nabla}\cdot\vec{B} = 0$

What abaut this: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left[\vec{J}_f + \vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}\right] + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

Becomes this: $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ ... $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ stays the same!

$$\boxed{\begin{array}{lll} \vec{\nabla}\cdot\vec{D} = \rho_f & \vec{\nabla}\cdot\vec{B} = 0 & \vec{D} = \varepsilon \vec{E} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} & \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} & \vec{H} = \frac{1}{\mu}\vec{B} \\ \vec{P} = \varepsilon_0 \chi_e \vec{E} & \varepsilon = \varepsilon_0 (1+\chi_e) & \\ \vec{M} = \chi_m \vec{H} & \mu = \mu_0 (1+\chi_m) & \end{array}}$$

MODIFIED MAXWELL and CONSTITUTIVE RELATIONS

From now on we will look at some aplications of all this!

We already looked at light propagation in vacuum ... now we look at propagation of waves in matter!

Linear dielectric medium → no free charges, no free currents

So we have: $\vec{\nabla}\cdot\vec{E} = 0$ $\quad \vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\quad \vec{\nabla}\times\vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$

only difference to vacuum is: $\mu_0 \to \mu$; $\varepsilon_0 \to \varepsilon$

Speed of light ... vacuum $c_0 = 1/\sqrt{\mu_0 \varepsilon_0}$ ...matter $c = 1/\sqrt{\mu\varepsilon}$

in matter can be written ... $\boxed{c = c_0/n}$ where $\boxed{n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}}}$ INDEX OF REFRACTION

Now we will take a closer look at index of refraction n!

Remember: DISPERSION  ⇒ wave packet will be distorted!

Look at electron inside material: $\vec{F}_{bound} = -kx$ (bounding force)

$\vec{F}_{damping} = -m\gamma \frac{dx}{dt}$ (damping ... e^- radiates fotons!)

monochromatic wave → $\vec{F}_{driving} = q\vec{E}_0 \cos(\omega t)$ (electric field force)

DRIVING FREQUENCY

Newton law:
$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = qE_0 \cos(\omega t)$$

we have written $\vec{F}_{boun} = -kx = -m\omega_0^2 x$ ... ω_0 - NATURAL FREQUENCY of e^-

we define: $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$ ... insert into equation this ansatz ...

and we get: $\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \cdot E_0$ The "real" e^- displacement is real part of $\tilde{x}(t)$!

oscilation amplitude

Similar like at polarization the dipole moment we can introduce here will be time dependent! → $\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$

To get macroscopic view we switch from dipole moment to polarization!

In multitude of e^- ... ω remains same but ω_0 and γ are e^- dependent!

f_j - fraction of e^- with natural frequency ω_{0j} and damping constant γ_j inside given molecule (or atom) of the material

N - number of molecules (atoms) per unit volume

$\tilde{P}$ - dipole moment per unit moment = POLARIZATION

$$\tilde{P}(t) = \frac{N q^2}{m}\left(\sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\gamma_j\omega}\right) E_0 e^{-i\omega t}$$

We know from before that for linear material: $\tilde{P} = \varepsilon_0 \tilde{\chi}_e \tilde{E}$

We also introduced (here in complex notation): $\tilde{\varepsilon} = \varepsilon_0 (1 + \tilde{\chi}_e) = \varepsilon_0 \tilde{\varepsilon}_r$

Putting all together we can find expression for ε_r!

P.S. Usualy $\mu_r \approx 1 \Rightarrow n \approx \sqrt{\varepsilon_r}$ ... this is why we are intrested in ε_r!

We get... $\boxed{\tilde{\varepsilon}_r = 1 + \frac{N q^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\gamma_j\omega}}$

Wave equation in DISPERSIVE MEDIUM ... $\nabla^2 \tilde{E} = \tilde{\varepsilon}\mu_0 \frac{\partial^2 \tilde{E}}{\partial t^2} \rightarrow \tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z - \omega t)}$

where ... $\tilde{k} = \sqrt{\tilde{\varepsilon}\mu_0}\,\omega = \frac{\omega}{c_0}\sqrt{\tilde{\varepsilon}_r} = Re\{\tilde{k}\} + i\, Im\{\tilde{k}\} \rightarrow k_R + i k_I$

Inserting this to solution of wave equation ...

$\tilde{E}(z,t) = \tilde{E}_0 e^{-k_I z} e^{i(k_R z - \omega t)}$ ... now if we look at intensity ...

$I \sim |E|^2 \rightarrow \boxed{\alpha = 2k_I}$ ABSORPTION COEFFICIENT

$\boxed{n = \frac{c}{\omega} k_R}$ REFRACTION INDEX

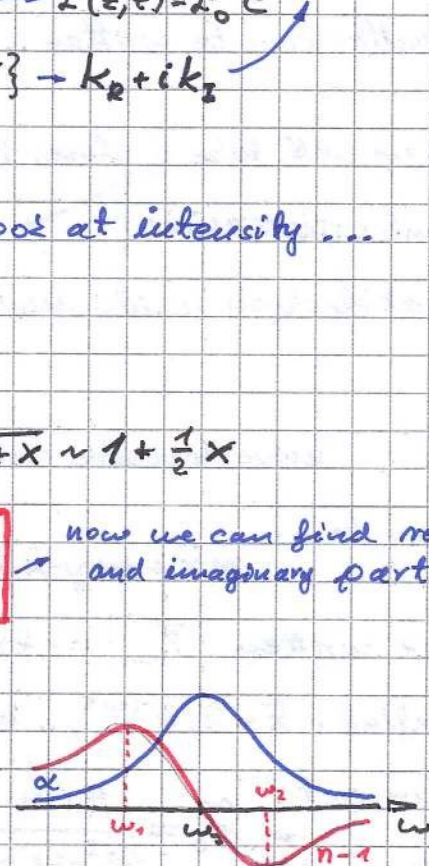
$\tilde{k}$ depends on $\sqrt{\varepsilon_r}$ ~ for gas we aproximate $\sqrt{1+x} \sim 1 + \frac{1}{2}x$

$$\tilde{k} = \frac{\omega}{c}\sqrt{\varepsilon_r} \approx \frac{\omega}{c}\left[1 + \frac{N q^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 - i\gamma_j\omega}\right]$$

now we can find real and imaginary part!

$$n = \frac{c}{\omega} k_R \approx 1 + \frac{N q^2}{2m\varepsilon_0}\sum_j \frac{f_j(\omega_{0j}^2 - \omega^2)}{(\omega_{0j}^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

$$\alpha = 2k_I \approx \frac{N q^2 \omega^2}{m\varepsilon_0 c_0}\sum_j \frac{f_j \gamma_j}{(\omega_{0j}^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$



$[\omega_1, \omega_2]$ - ANOMALIS DISPERSION

same region as peak of α

Now we go back to wave packet speed! ...

wave packet → $\tilde{E}(z,t) = \int_{\Delta\omega} a(\omega)\, e^{i(kz-\omega t)}\, d\omega = \tilde{A}(z,t)\, e^{i(k_0 z-\omega_0 t)}$

where → $\tilde{A}(z,t) = \int_{\Delta\omega} a(\omega)\, e^{i(\omega-\omega_0)\left[\frac{dk}{d\omega} z - t\right]}\, d\omega$; $k-k_0 \approx \frac{dk}{d\omega}(\omega-\omega_0)$

Phase velocity → $\boxed{v_\varphi = \omega_0/k_0}$ (c_0/n) Group velocity → $\boxed{v_g = \frac{d\omega_0}{dk} = \frac{d}{dk}\left(k_0 \frac{c}{n}\right)}$

PS. Phase velocity can be greater than c_0, but group velocity is smaller than c_0!

In non-dispersive media the two velocities are the same!

What about a wave propagating in a conductor?

EM-wave in CONDUCTOR $\rho_f, \vec{J}_f$

Ohm law → $\vec{J}_f = \sigma \vec{E}$ Maxwell → $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon}\rho_f$ $\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ $\vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\frac{\partial}{\partial t}\vec{E}$

Continuity equation → $\vec{\nabla}\cdot\vec{J}_f = -\frac{\partial \rho_f}{\partial t}$

↳ $\frac{\partial \rho_f}{\partial t} = -\sigma\,\vec{\nabla}\cdot\vec{E} = -\frac{\sigma}{\varepsilon}\rho_f \rightarrow \rho_f = \rho_f(0)\, e^{-(\frac{\sigma}{\varepsilon})t}$ Fast damping!

We wait for damping to do it's effect and we get simplyfied set of equations: $\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t}$; $\nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t}$

Solutions → $\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0\, e^{i(\tilde{k}z-\omega t)}$; $\tilde{\vec{B}} = \tilde{\vec{B}}_0\, e^{i(\tilde{k}z-\omega t)}$

where ... $\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \rightarrow k_R + i k_I$

skin depth → $\boxed{d = 1/k_I}$ SKIN EFFECT

Wave can not go deper than some SKIN DEPTH!