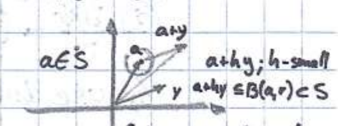
Example: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$ $\quad f$ is countinous in $(0,0)$

If we introduce polar coordinates $x = \rho\cos\theta$; $y = \rho\sin\theta \Rightarrow f(x,y) = \rho\cos\theta\cdot\sin\theta = g(\rho,\theta)$
$|f(x,y) - f(0,0)| = |f(x,y)| = |g(\rho,\theta)| \leq |\rho\cdot\cos\theta\cdot\sin\theta| \leq |\rho| \to 0$ as $(x,y)\to(0,0)$ ✓ok

Example: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad \lim_{(x,y)\to(0,0)} \frac{x^3 - 2xy + y^2}{x^2+y^2} \overset{?}{=}$ the limit does-not exist. To exist it must be the same, from any direction we aproch to $(0,0)$.
Compute the limit along $y = mx$: we get $(-2m+m^2)/(1+m^2) \Rightarrow$ doesen't exist because ↲

Example: $\lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{x} = \left(\lim_{x\to 0} \frac{\sin(2x)}{x} = 2 \right) = \lim_{(x,y)\to(0,2)} \frac{\sin(x\cdot y)}{(x\cdot y)}\cdot y = 2$; $(x\cdot y)\to 0$ as $(x,y)\to(0,2)$

DERIVATIVE OF SCALAR FIELD



Consider S-open; $f: S \subset \mathbb{R}^n \to \mathbb{R}$; S must be open so that you can allways find ball around point a that is cointained in S

S is open $\Rightarrow \exists r$ s.t. $B(a,r)\subset S$, $\|a+hy-a\| < r$

Def: The derivative of f at a with respect to y is $f'(a,y)$, and it is defined as: $\boxed{f'(a,y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h}}$

Example: $f: \mathbb{R}^n \to \mathbb{R}$ - LINEAR $\Rightarrow f(ax_1 + bx_2) = a f(x_1) + b f(x_2) \Rightarrow \boxed{f'(a,y) = \lim_{h\to 0} \frac{f(a+hy) - f(a)}{h} = \frac{h f(y)}{h} = f(y)}$!

Theorem: Define $g(t) = f(a+ty)$; $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ scalar field; $g: A \subseteq \mathbb{R} \to \mathbb{R}$! Now we have $g'(t) = f'(a+ty, y)$ → the derivative at $(a+ty)$ with respect to y!
In parcticular $g'(0) = f'(a,y)$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$; $x \to \|x\|^2$; $\Rightarrow g(t) = f(a+ty) = \|a+ty\|^2 = (a+ty, a+ty) \overset{\text{by the properties of scalar pr.}}{=} (a,a) + t^2(y,y) + 2t(a,y) \Rightarrow$
$\Rightarrow \boxed{g'(t) = dg/dt = 2t(y,y) + 2(a,y) = f'(a,y)}$!

Theorem: MEAN-VALUE THEOREM - in higher dimensions
Assume that $f'(a+ty, y)$ exists for any $t \in [0,1]$, than $\exists\, \theta \in \mathbb{R}$; $0 < \theta < 1$ s.t.
$\boxed{f(a+y) - f(a) = f'(z,y) ;\ z = a + \theta y}$

Pf: $g(t) = f(a+ty) \Rightarrow$ because $f'(..)$ exists also g' exists; $g \in C'((0,1))$. Now we can apply the mean-value theorem (Analiza I) for $g \Rightarrow g(1) - g(0) = g'(\theta)(1-0)$; $\theta \in (0,1)$
$g(1) = f(a+y)$; $g(0) = f(a)$; $g'(\theta) = f'(a+\theta y, y) \Rightarrow f(a+y) - f(a) = f'(a+\theta y, y)$ ✓ok.

Def: When y is a unit vector ($\|y\| = 1$) the $f'(a,y)$ is called DIRECTIONAL DERIVATIVE of f at a!
↦ When $y = e_k = (0, \dots, 0, \overset{k\text{-th position}}{1}, 0 \dots 0)$, than $f'(a, e_k)$ is the PARTIAL DERIVATIVE of f at a with respect to e_k!

Example: $f(x,y) = x^2 + y^2 \sin(xy)$; $\frac{\partial}{\partial x} f(x,y) = 2x + y^2\cos(xy)\cdot y$; $\frac{\partial}{\partial y} f(x,y) = 2y\cdot\sin(xy) + y^2\cos(xy)\cdot x$

$f(x,y) = \frac{x+y}{x-y}$; $\frac{\partial f}{\partial x} = \frac{\cancel{1}(x-y) - (x+y)\cancel{(1)}}{(x-y)^2}$; $\frac{\partial f}{\partial y} = \frac{1\cdot(x-y) - (x+y)\cdot(-1)}{(x-y)^2} = \frac{2x}{(x-y)^2}$

{ In 1-dimension we have $f: S \subseteq \mathbb{R} \to \mathbb{R}$; f differentiable in $a \Rightarrow f$ is countinous at a!
What about in higher dimensions? }

Example: $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & ; x \neq 0 \\ 0 & ; (0,y), y \in \mathbb{R} \end{cases}$ $\quad$ Let $v = (a,b)$; $a \neq 0 \Rightarrow \frac{f(a+hv) - f(a)}{h} = \frac{f(hv)}{h} = \frac{1}{h} \frac{(ha)(hb)^2}{(ha)^2 + (hb)^4} = \frac{ab^2}{a^2 + h^2 b^4} \xrightarrow{h\to 0} \frac{b^2}{a}$

(tujenghe 0 (ničla))

$-\ v = (0,b) \Rightarrow \frac{0-0}{h} = 0 \Rightarrow$ directional derivatives exist for all direction!

But is f countinous? NO
Check along parabola $x = y^2$; $f(x,y) = f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2} \neq 0$

→ We want to EXTEND the concept of differentiation

Def: We say that a scalar field $f: S \subset \mathbb{R}^n \to \mathbb{R}$ is DIFFERENTIABLE at a if exists a LINEAR TRANSFORMATION $T_a: \mathbb{R}^n \to \mathbb{R}$; $v \mapsto T_a(v)$, and a SCALAR FUNCTION $E(a,v): \mathbb{R}^n \to \mathbb{R}$ s.t. $f(a+v) = f(a) + T_a(v) + \|v\| \cdot E(a,v)$ for $\|v\| \leq r$ and $E(a,v) \to 0$ as $\|v\| \to 0$!

$\nabla f(a)$

T_a is called TOTAL DERIVATIVE of f at a! (or FRECHET DERIVATIVE)

Theorem: Assume that $f: S \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a with total derivative T_a, than exists $f'(a,y)$ for $\forall y \in \mathbb{R}^n$ and we have $\boxed{T_a(y) = f'(a,y)}$

$y = (y_1, \dots, y_n)$ and $f'(a,y) = \sum_{k=1}^{n} \frac{\partial f(a)}{\partial x_k} \cdot y_k$

Pf: By hypotesis we know f is differentiable $\Leftrightarrow$ $f(a+v) = f(a) + T_a(v) + \|v\| \cdot E(a,v)$; $v = h \cdot y \Rightarrow$
$f(a+hy) - f(a) = T_a(hy) + |h| \cdot \|v\| \cdot E(a,v) = h \cdot T_a(y) + |h| \cdot \|v\| \cdot E(a,hy) \to$ divide by $h \Rightarrow$
$(1/h) \cdot [f(a+hy) - f(a)] = T_a(y) + \|y\| \cdot E(a,hy) \to f'(a,y)$ as $h \to 0$; $E(a,v) \to 0$ as $h \to 0 \Rightarrow f'(a,y) = T_a(y)$

$$y = \sum_{k=1}^{n} (y_k e_k);\ T_a(y) = T_a\left(\Sigma (y_k e_k)\right) \overset{T_a\text{-LINEAR}}{=} \sum_{k=1}^{n} y_k \cdot \underbrace{T_a(e_k)}_{\frac{\partial}{\partial x_k} f(a)} \overset{\text{STEP BEFORE}}{=} \sum_{k=1}^{n} y_k \cdot \partial_{x_k} f(a) = (y, \nabla f(a))$$

THE GRADIENT FIELD

Def: The GRADIENT FIELD of $f: \mathbb{R}^n \to \mathbb{R}$ at a is a VECTOR in $\mathbb{R}^n$: $\nabla f(a) = (\partial_{x_1} f(a), \dots, \partial_{x_n} f(a))$.
$T_a(v) = (\nabla f(a), v)$ - scalar product.

Theorem: If $f: S \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a than f is continuous at a!

$$\text{Pf}: |f(a+v) - f(a)| = |\nabla f(a) \cdot v + \|v\| \cdot E(a,v)| \overset{\Delta\text{-inequality}}{\leq} |\nabla f(a) \cdot v| + |\|v\| \cdot E(a,v)| \overset{\text{C-S inequality}}{\leq} \|\nabla f(a)\| \cdot \|v\| + \|v\| \cdot E(a,v) \xrightarrow{\|v\| \to 0} 0$$

Theorem: A SUFFICIENT CONDITION FOR DIFFERENTIAL

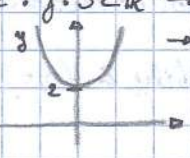
Assume $f: S \subset \mathbb{R}^n \to \mathbb{R}$ and $\partial_{x_i} f(a)$; $i = 1, \dots, n$ exist and is countinuous than f is differentiable!

Theorem: CHAIN RULE

Let $f: S \subset \mathbb{R}^n \to \mathbb{R}$ and $r: J \subset \mathbb{R} \to S$! Define $g = f \circ r$ and assume that $f \in C^1$, $r \in C^1 \Rightarrow$
$\Rightarrow g \in C^1$ $\quad g: \mathbb{R} \xrightarrow{r} S \subset \mathbb{R}^n \xrightarrow{f} \mathbb{R}$; $\boxed{g'(t) = \nabla f(a) \cdot r'(t), \text{ where } r(t) = a!}$
$g'(t) = ? \Rightarrow$ $\quad t \longmapsto g(t)$

at point $(1,2)$

Example: $f: S \subset \mathbb{R}^2 \to \mathbb{R}$; $(x,y) \mapsto x^2 - 3xy$ Find the directional derivative of f along $y = x^2 - x + 2$!



→ the parabola can be represented parametically

$r(t) = (t, t^2 - t + 2) \Rightarrow$ the tangent vector $r'(t) = (1, 2t-1) \to$ we have to normalize it

$\Rightarrow$ normalized tangent vec. $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(1, 2t-1)}{\sqrt{1 + (2t-1)^2}} \Rightarrow \|T(t)\| = 1$

We are intrested at point $(1,2) \Rightarrow T(1) = \frac{(1,1)}{\sqrt{2}}$ and $\nabla f(x,y) = (2x - 3y, -3x)$; $\nabla f(1,2) = (-4,-3) \Rightarrow$
$\Rightarrow$ the directional derivative is $\nabla f(1,2) \cdot T(1) = (-4,-3) \cdot (1,1)/\sqrt{2} = \underline{\underline{-7/\sqrt{2}}}$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$; $f \in C^1$ and assume that $\underbrace{f(x,y) = \text{const.}}_{\text{level surface or level set of } f}$ on a curve C! (curve C is also differentiable)
f - temperature

$\Rightarrow$ than ∇f is normal to C.

Pf: We consider the unit tangent vector T to C and let $r(t) = (x(t), y(t))$ - parametisation of C!
Cousider $g(t) = f[r(t)] = f(x(t), y(t)) = \text{const.} \Rightarrow g'(t) = 0$ (ker je f konst. na krivulji C)
$g'(t) \underset{\text{CHAIN RULE}}{=} \nabla f[r(t)] \cdot r'(t) \Rightarrow \nabla f[r(t)] \perp r'(t)$

Posploši

GENERALIZATION TO VECTOR FIELD

Def: We say that $f: S \subset \mathbb{R}^n \to \mathbb{R}^m$; $(x_1 \dots x_n) \mapsto (f_1(x_1 \dots x_n), \dots, f_m(x_1 \dots x_n))$ is DIFFERENTIABLE if exists a linear transformation $T_a: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(a+v) = f(a) + T_a(v) + \|v\| \cdot E(a,v)$, where $E(a,v) \to 0$ as $\|v\| \to 0$ $\quad T_a(y) = \sum_{k=1}^{m} \nabla f_k(a) \cdot y = (\nabla f_1(a) y, \dots \nabla f_m(a, y))$

Theorem: If a vector field is DIFFERENTIABLE $\Rightarrow$ f is COUNTINOUS!

GRADIENT OF VECTOR FIELD

$f: \mathbb{R}^n \to \mathbb{R}^m$ $\quad \boxed{T_a(y) = Df(a) \cdot y}$ - matrix product; $y \in \mathbb{R}^n$

$$Df(a) = \begin{pmatrix} \partial_{x_1} f_1(a), \dots, \partial_{x_n} f_1(a) \\ \partial_{x_1} f_2(a), \dots, \partial_{x_n} f_2(a) \\ \vdots \quad \ddots \quad \vdots \\ \partial_{x_1} f_m(a), \dots, \partial_{x_n} f_m(a) \end{pmatrix}$$

$\to$ The JACOBIAN MATRIX of f at a!

$\to$ $(m \times n)$ - matrix

$$Df(a) = \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_m(a) \end{pmatrix}$$

CHAIN RULE

Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^l \to \mathbb{R}^n$; $f, g \in C^1$! Define $h = f \circ g = f(g(x))$.
If f, g are differentiable at a and $g(a)$ respectively than h is differentiable at a!
$\boxed{Dh(a) = Df(b) \cdot Dg(a) \; ; \; b = g(a)}$ [matrix product]

Example: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad h = f \circ g \quad Dh(a) = Df(g(a)) \cdot Dg(a)$
$g: \mathbb{R}^2 \to \mathbb{R}^2$
$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ $\quad Dg(a) = \begin{pmatrix} \partial_r g_1 & \partial_\theta g_1 \\ \partial_r g_2 & \partial_\theta g_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$; $Df(g(a)) = (\partial_x f(g(a)), \partial_y f(g(a)))$

$$Dh(a) = (\partial_x f(g(a)), \partial_y f(g(a)))_{1\times 2} \cdot \begin{pmatrix} \cos\theta, & -r\sin\theta \\ \sin\theta, & r\cos\theta \end{pmatrix}_{2\times 2} = \begin{pmatrix} \underbrace{\partial_x f(g(a)) \cdot \cos\theta + \partial_y f(g(a)) \cdot \sin\theta}_{= \partial_r h} \\ \underbrace{-\partial_x f(g(a)) \cdot r\sin\theta + \partial_y f(g(a)) \cdot r\cos\theta}_{= \partial_\theta h} \end{pmatrix}_{1\times 2}$$

Remark: $f: S \subset \mathbb{R}^n \to \mathbb{R}$
Mixed partial derivative $\partial_{x_i} \partial_{x_k} f$; $i \neq k$! The order of derivation can be changed if $\partial_{x_i} \partial_{x_k} f$ is COUNTINOUS $(i = 1, \dots, n; k = 1, \dots n)$!

Example: $f(x,y) = xy \cdot \frac{x^2 - y^2}{x^2 + y^2}$; $(x,y) \neq 0$ and $f(0,0) = (0,0)$

$\partial_y \partial_x f(0,0) = -1$ and $\partial_x \partial_y f(0,0) = 1$ $\Rightarrow$ because the derivative is not countinious!

Example: $f(x,y) = \begin{cases} \frac{2x^2 y - xy^2}{x^2 + y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$ $\quad$ Contuinity in $(0,0)$ and differentiabily in $(0,0)$?

$\mapsto$ We have that $\left| \frac{2x^2 y - xy^2}{x^2 + y^2} - 0 \right| = \frac{|xy| \, |2x - y|}{x^2 + y^2} \overset{(x-y)^2 \geq 0 \Rightarrow 2xy \leq x^2 + y^2}{\leq} \frac{1}{2} \frac{(x^2 + y^2)|2x - y|}{x^2 + y^2} = \frac{1}{2} |2x - y| \leq \frac{1}{2}(|2x| - |y|)$

$\Rightarrow f(x,y) \longrightarrow (0,0)$ as $(x,y) \to (0,0)$

$\mapsto (\partial_x f, \partial_y f) = \nabla f$; compute $\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \frac{0 - 0}{x} = 0$
by the same way we can find $\frac{\partial f}{\partial y}(0,0) = 0$
$\Rightarrow \nabla f(0,0) = (0,0)$

$\Rightarrow$ If f is differentiable than T_0 must be identicaly 0! $\quad$ $T_0: \mathbb{R}^n \to \mathbb{R}$ is linear tran!

$f(a+v) = f(0+v) = \overset{0}{f(0)} + \overset{0}{T_0(v)} + \|v\| \cdot E(0,v)$ where $E(0,v) \to 0$ as $\|v\| \to 0$

$\Rightarrow f(0+v) = \|v\| \cdot E(0,v) \Rightarrow$ Let's check if $E(0,v) \to 0$ as $\|v\| \to 0$

$\lim_{\|v\| \to 0} E(0,v) = \lim_{\|v\| \to 0} \frac{f(0+v)}{\|v\|} \to v = (x,y) \Rightarrow \frac{\frac{2x^2 y - xy^2}{x^2 + y^2}}{\sqrt{x^2 + y^2}} = \frac{2x^2 y - xy^2}{(x^2 + y^2)^{3/2}} \overset{?}{=} 0$ as $(x,y) \to (0,0)$ NO!

MAXIMA, MINIMA AND SADLE POINTS

Def: Scalar field $f: S \subset \mathbb{R}^n \to \mathbb{R}$: has an ABSOLUTE MAX. at $a \in S$ if $f(x) \leq f(a); \forall x \in S$
: has an ABSOLUTE MIN. at $a \in S$ if $f(x) \geq f(a); \forall x \in S$

f has a RELATIVE MAX. in $a \in S$ if $\exists B(a,r)$ s.t. $f(x) \leq f(a)$ for $\forall x \in B(a,r)$
f has a RELATIVE MIN. in $a \in S$ if $\exists B(a,r)$ s.t. $f(x) \geq f(a)$ for $\forall x \in B(a,r)$

Scalar field $f: S \subset \mathbb{R}^n \to \mathbb{R}$; $f \in C^1(S)$! Point $a \in S$ is a STACIONARY POINT for f if $\nabla f(a) = 0$ $(=(0,0,\dots 0))$.
A stacionary point is a SADDLE POINT if $\forall B(a,r)$ contains x s.t. $f(x) < f(a)$ and $f(x) > f(a)$

Example: $f(x,y) = 2 - x^2 - y^2$; $f: \mathbb{R}^2 \to \mathbb{R}$; $f \in C^1$. The surface $(x, y, f(x,y)) = (x, y, 2-x^2-y^2)$ is a PARABOLID OF REVOLUTION! $\Rightarrow$ The LEVEL LINE of f; $f(x,y) = C \Rightarrow x^2+y^2 = C \to$ CIRCLE

In $(0,0)$ we have ABS. MAX. $f(x,y) = 2 - x^2 - y^2 \leq 2 = f(0,0)$; $\forall x, y \in \mathbb{R}$
$\partial_x f = -2x \Rightarrow \partial_x f(0,0) = 0$
$\partial_y f = -2y \Rightarrow \partial_y f(0,0) = 0$

Example: $f(x,y) = x \cdot y \to$ HYPERBOLIC PARABELOID $\Rightarrow f(0,0) = 0$ for $\forall r$ we have that $f(x,y) = xy > 0$ in $B(0,r)$ and $f(x,y) = xy < 0$ in $B(0,r)$

$$\left.\begin{array}{l} \partial_x f = y \Rightarrow \partial_x f(0,0) = 0 \\ \partial_y f = x \Rightarrow \partial_y f(0,0) = 0 \end{array}\right\} \Rightarrow (0,0) \text{ is a STACIONARY POINT!}$$

THE 1^{ST} ORDER TAYLOR FORMULA FOR $f: S \subset \mathbb{R}^n \to \mathbb{R}$

$f(a+v) = f(a) + T_a(v) + \|v\| \cdot E(a,v) = f(a) + \overbrace{(\nabla f(a), v)}^{\text{SCALAR PROD.}} + \|v\| \cdot E(a,v)$. If $a \in S$ is STACIONARY POINT for $f \Rightarrow \nabla f(a) = 0! \Rightarrow f(a+v) - f(a) = \|v\| \cdot E(a,v)$!
If we know that $\|v\| \cdot E(a,v) > 0 \Rightarrow f(a+v) - f(a) > 0 \Rightarrow f(a)$ is a MINIMA (and revers for MAXIMA)
To know that we must look at 2nd order Taylor formula!...

$f: S \subset \mathbb{R}^n \to \mathbb{R}$; $f \in C''$; The HESSIAN MATRIX of f at a!

$$Hf(a) = \begin{pmatrix} \partial_{x_1}\partial_{x_1} f & \dots & \partial_{x_1}\partial_{x_n} f \\ \vdots & \ddots & \vdots \\ \partial_{x_n}\partial_{x_1} f & \dots & \partial_{x_n}\partial_{x_n} f \end{pmatrix} \to (n \times n) \text{ matrix}$$

$\Rightarrow Hf(a)$ is a SYMETRIC MATRIX because since $f \in C^2 \Rightarrow$
$\Rightarrow \partial_{x_i}\partial_{x_j} f = \partial_{x_j}\partial_{x_i} f$!

THE 2^{ND} ORDER TAYLOR FORMULA FOR $f: S \subset \mathbb{R}^n \to \mathbb{R}$

$f(a+v) = f(a) + T_a(v) + \frac{1}{2!} \underset{(1\times n)}{v} \cdot \underset{(n\times n)}{H(a+cv)} \cdot \underset{(n\times 1)}{v^T}$; $0 < c < 1$ We can write it in another way $\Rightarrow$

$f(a+v) = f(a) + T_a(v) + \frac{1}{2!} v \cdot H(a) v^T + \|v\|^2 \cdot E_2(a,v)$ where $E_2(a,v) \to 0$ as $\|v\| \to 0$!
If a is a STACIONARY POINT $\Rightarrow \nabla f(a) = 0 \Rightarrow T_a(v) = (\nabla f(a), v) = 0$ (SCALAR PRODUCT)

$f(a+v) = f(a) + \frac{1}{2!} v \cdot H(a) \cdot v^T + \boxed{\|v\|^2 \cdot E_2(a,v)} \to$ If $\|v\|$ is smal $\Rightarrow$ error term $\|v\|^2 \cdot E_2(a,v) \to 0$!

{ If $\|v\|$ is small, than the error term $\|v\|^2 \cdot E_2(a,v)$ tends to zero faster than $\|v\|^2$, hence we can expect that for small v the sign of $f(a+v) - f(a)$ is the same as the one of $v \cdot H(a) \cdot v^T$! }

$\boxed{v \cdot H(a) \cdot v^T = (\underbrace{H(a) \cdot v}_{\substack{(n\times n)\,(n\times 1) \\ (n\times 1)}}, \underset{(n\times 1)}{v}) - \text{scalar product}}$

$|(H(a)v, v)| \overset{\text{C-S INEQ.}}{\leq} \|H(a)v\| \cdot \|v\| \leq C \cdot \|v\| \cdot \|v\| = C \cdot \|v\|^2$; $C \to \max\{|\partial_{x_i}\partial_{x_j} f(a)|\}$

Theorem: Let $A = (a_{ij})$ be a $(n\times n)$ real symetric matrix and let $Q(y) = y \cdot A \cdot y^T =$
$= \sum_{i=1}^n \sum_{j=1}^n a_{ij} y_i y_j = (Ay, y)$ – SCALAR PRODUCT.
Than we have:

a) $Q(y) > 0$; $\forall y \neq 0 \iff$ all the EIGENVALUES of A are positive (A is a positive definit matrix)
b) $Q(y) < 0$; $\forall y \neq 0 \iff$ all the EIGENVALUES of A are negative (A is a negative definite matrix)

Def: If A is a $(n\times n)$ matrix than if $Ax=\lambda x$ for $x\neq 0$: λ – EIGENVALUE and x – EIGENVECTOR associated to the EIGENVALUE λ! P.S.: $\lambda\in\mathbb{R}$; $Ax=\lambda\cdot(Id)\cdot x \Rightarrow (A-\lambda Id)x$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn}\end{pmatrix}\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}=\lambda\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix} \Rightarrow \begin{pmatrix} a_{11}-\lambda & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn}-\lambda\end{pmatrix}\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}=0$$

Theorem: Scalar field $f: S\subset\mathbb{R}^n\to\mathbb{R}$ and $\partial_{x_i}\partial_{x_j} f\in C(B(a,r))$ $i=1,\dots n$; $j=1,\dots n$ and assume that $\nabla f(a)=0$ (a is stacionary point)

a) If all the EIGENVALUES of $H(a)$ are positive $\Rightarrow f(a)$ is a RELATIVE MINIMUM!

b) If all the EIGENVALUES of $H(a)$ are negative $\Rightarrow f(a)$ is a RELATIVE MAXIMUM!

c) If some EIGENVALUES are positive and some negative $\Rightarrow f(a)$ is a SADDLE POINT!

P.S. if some eigenvalues are zero we can not say nothing (test is inconclusive)

Example: EXTREM?
$$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}; & (x,y)\neq(0,0)\\ 3; & (x,y)=(0,0)\end{cases}$$
If $(x,y)\neq(0,0)$ $|f(x,y)|=\left|\frac{xy}{x^2+y^2}\right|\leq\left|\frac{1/2\cdot(x^2+y^2)}{x^2+y^2}\right|=\frac{1}{2}\Rightarrow$
$\Rightarrow f(0,0)=3$ is a MAXIMUM!

For $(x,y)\neq(0,0)$; $\nabla f(x,y)=(\partial_x f,\partial_y f)=\left(\frac{y^3-x^2y}{(x^2+y^2)^2},\frac{x^3-xy^2}{(x^2+y^2)^2}\right)$! We impose $\nabla f(x,y)=0\Rightarrow$

$\Rightarrow \nabla f(x,y)=(0,0)$ along the lines (x,x) and $(x,-x)$; $x\neq 0$! We see $f(x,x)=\frac{1}{2}$ and $f(x,-x)=-\frac{1}{2}$

By step before we saw $|f(x,y)|\leq\frac{1}{2}\Rightarrow \boxed{-\frac{1}{2}\leq f(x,y)\leq\frac{1}{2}}$ $\Rightarrow$ points $(x,-x); x\neq 0$ are MINIMA!

Example: We have a box (without the lid) with rectangular basis! Volume $=32\,000\ cm^3$!
We want to find the dimensions x,y,z of box, which minimize the amount of cartbord emplayed!

$A(x,y,z)=xy+2xz+2yz$ and Volume $=x\cdot y\cdot z=32\,000\ cm^3 \Rightarrow z=32000/xy$

$\Rightarrow A(x,y)=xy+64000\cdot\frac{(x+y)}{xy}=xy+64000\left(\frac{1}{x}+\frac{1}{y}\right)$

$\nabla A(x,y)=(y-64000/x^2, x-64000/y^2)=0\Rightarrow 64000=yx^2=xy^2\Rightarrow \boxed{x=y=\sqrt[3]{64000}=40; z=20}$

P.S.: In 1-dim. we had if $f:[a,b]\to\mathbb{R}$; f-countinous $\Rightarrow f$ has MAX. and MIN. in $[a,b]$

→ A set S in $\mathbb{R}^n$ is **bounded** if $\exists R$ s.t. $S\subset B(0,R)$

→ A set S in $\mathbb{R}^n$ is **closed** if $S^c=\mathbb{R}^n\setminus S$ is an open set.

! [$f: C\subset\mathbb{R}^n\to\mathbb{R}$, ; f-countinous; C is compact in $\mathbb{R}^n$ (= closed and bounded) $\Rightarrow$
$\Rightarrow f$ admits MAXIMUM and MINIMUM in C! (WEIERSTRASS THEOREM)] !

Example: Find extremums of $f(x,y)=x^4+y^4-4xy$! $\to \nabla f(x,y)=(4x^3-4y, 4y^3-4x)=(0,0)\Rightarrow x^3=y; y^3=x$

$\Rightarrow$ stacionary points $=(0,0);(1,1);(-1,-1)$! Since $f\in C^2(\mathbb{R}^2)$ we compote $\to Hf(x,y)=\begin{pmatrix}12x^2 & -4\\ -4 & 12y^2\end{pmatrix}\to$ we put in the stacionary points

$\Rightarrow Hf(0,0)=\begin{pmatrix}0 & -4\\ -4 & 0\end{pmatrix}$ and $Hf(1,1)=Hf(-1,-1)=\begin{pmatrix}12 & -4\\ -4 & 12\end{pmatrix}\to$ we must look for the eigenvalues of this matrixes!

$\Rightarrow Hf(1,1)-\lambda Id=0\Rightarrow\begin{pmatrix}12 & -4\\ -4 & 12\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}=\begin{pmatrix}12-\lambda, & -4\\ -4, & 12-\lambda\end{pmatrix}\to$ to find the eigenvalues we compute the determinant and impose $=0$!

$\Rightarrow \det\begin{pmatrix}12-\lambda, & -4\\ -4, & 12-\lambda\end{pmatrix}=(12-\lambda)^2-16=0\Rightarrow(12-\lambda)^2=16\Rightarrow \boxed{\lambda_1=8}$ and $\boxed{\lambda_2=16}\to$ both $\oplus\Rightarrow(1,1)$ is MINIMA!

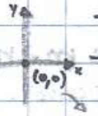
Because $Hf(1,1)=Hf(-1,-1)$, we get the same eigenvalues $\Rightarrow(-1,-1)$ is also MINIMA!

$Hf(0,0)-\lambda Id=\begin{pmatrix}-\lambda, & -4\\ -4, & -\lambda\end{pmatrix}=B$; $\det B=\lambda^2-16=0\Rightarrow\boxed{\lambda_1=4,\lambda_2=-4}\to$ one eigenvalue is $\oplus$ and other $\ominus\Rightarrow$
$\Rightarrow(0,0)$ is A SADDLE!

$f(x,y)=x^4+y^4-4xy\geq x^4+y^4-2x^2-2y^2\dots\to+\infty$ as $|x|,|y|\to+\infty$
$(x,y)\in\mathbb{R}^2$ ↳ for $|x|>4$ and $|y|>4$ $\Rightarrow f$ is not bounded in $\mathbb{R}^2\Rightarrow$ it does not admit maxima!

If we study the same function $f(x,y)=x^4+y^4-4xy$ on bounded and closed interval instead on $\mathbb{R}^2$ (for example $B(0,4)$)!
Because the interval is bounded and closed and f is countinous everywhere we know by WEIERSTRASS THEOREM it admits maxima and minima!
But the maxima can be (and is) on the boundry of $B(0,4)$, so there $\nabla f(x,y)\neq 0$! $\Rightarrow$ maxima are not stacionary points!

Example: To find maxima and minima of $f(x,y)=x^2y-x^4-y^3$; $f:\mathbb{R}^2\to\mathbb{R}$



- $\nabla f(x,y)=(2xy-4x^3,\ x^2-3y^2)=(0,0) \Rightarrow$ we get 3 stacionary points $P_1(0,0)$, $P_2(\sqrt{3}/6, 1/6)$, $P_3(-\sqrt{3}/6, 1/6)$
- since $f\in C^2(\mathbb{R}^2) \Rightarrow Hf(x,y)=\begin{pmatrix} 2y-12x^2, & 2x \\ 2x, & -6y\end{pmatrix}$; for P_1: $(Hf(0,0)-\lambda Id)$ we get $\lambda=0 \Rightarrow$ this method INCONCLUSIVE

$\Rightarrow$ Study $f(x,y)$ along line $x=0$: $f(0,0)=0$; $f(0,y)=-y^3 \Rightarrow$ $P_1(0,0)$ is a SADDLE POINT!

$Hf(P_2)=\begin{pmatrix} -2/3, & \sqrt{3}/3 \\ \sqrt{3}/3, & -1\end{pmatrix} \Rightarrow \det(Hf(P_2)-\lambda Id)=0 \Rightarrow 3\lambda^2+5\lambda+1=0 \Rightarrow$ we get both λ negative $\Rightarrow$ $P_2(\sqrt{3}/6, 1/6)$ is RELATIVE MAXIMA!

→ By the same method we also get that $P_3(-\sqrt{3}/6, 1/6)$ is a relative maxima!

→ This function does not admit minima on $\mathbb{R}^2$! = $f(x,y)$ is not bounded from below on $\mathbb{R}^2$!

Def: THE TANGENT PLANE

$P(x_0, y_0, z_0)$

The TANGENT PLANE to the SURFACE $z=f(x,y)$ at the point $P(x_0,y_0)$ is:

$\boxed{z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)}$; where $\boxed{z_0=f(x_0,y_0)}$!

Example: Determine the tangent plane of $f(x,y)=x^3-y^3$ at $P(0,1,-1)$! Check $-1=f(0,1)$ ✓ $\Rightarrow P$ is on the surface!

$\nabla f(x,y)=(3x^2,-3y^2) \to \nabla f(0,1)=(0,-3) \Rightarrow \underline{z}=-3(y-1)+1=\underline{2-3y}$

Example: Tangent plane for $f(x,y)=x^y+y^2$ at $(1,1,2)$? H.W. dobiš $z=x+2y-1$

Example: Consider the function $f(x,y)=(x^2+y^2)\cdot\sin(1/\sqrt{x^2+y^2})$ for $(x,y)\neq(0,0)$ and $f(0,0)=0$! H.W.
Prove that f is differentiable in $(0,0)$, despite the fact that the partial derivative are not continuous there!

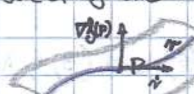
EXTREMA WITH COINSTRAINTS - method of Lagrange multipliers

Assume that g is a differentiable function of x,y,z! We consider the surface M defined as $M=\{(x,y,z)\in\mathbb{R}^3;\ g(x,y,z)=0\}$!

We want to find points P on surface M s.t. $f(P)$ is a maxima or minima on M!

$\Rightarrow$ we want $P\in M$, $g(P)=0$ and $f(P)\geq f(x)$; $\forall x\in M$ (CONSTRAINED MAXIMA) or
$P\in M$, $g(P)=0$ and $f(P)\leq f(x)$; $\forall x\in M$ (CONSTRAINED MINIMA)

Idea of the method of Lagrange multipliers:



$M=\{(x,y,z);\ g(x,y,z)=0$
$P\in M$

Supose that $f:S\subset\mathbb{R}^3\to\mathbb{R}$ and $S\supset M$ has a local maximum at P on M!

Let $r(t)=(x(t),y(t),z(t))$ be an arbitraly parametization of a curve which lies on the surface M and assume that $P=(x(0),y(0),z(0))$!

We define $h(t)=f(r(t))=f(x(t),y(t),z(t))$! $h(t)$ has a local maximum at $t=0 \Rightarrow h'(0)=0$!

By chainrule: $h'(t)=\nabla f(x,y,z)\cdot(\dot{x}(t),\dot{y}(t),\dot{z}(t))$!

$0=h'(0)=\nabla f(x(0),y(0),z(0))\cdot(\dot{x}(0),\dot{y}(0),\dot{z}(0))=\nabla f(P)\cdot\dot{r}(0)$! $\Rightarrow \nabla f(P)$ is ORTHOGONAL to any curve tangent to M at P! $\Rightarrow \nabla f(P)\perp$ (surface M)!

⊕ we know from previos results that $\nabla g(P)$ is ORTHOGONAL to the level surface M $(=g(x,y,z)=0)$

$\Rightarrow \nabla f(P)$ and $\nabla g(P)$ must be PARALLEL! $\Rightarrow \exists\lambda\in\mathbb{R}$ s.t. $\nabla f(P)+\lambda\nabla g(P)=0$ (LINEARLY DEPENDENT)

λ - LAGRANGE MULTIPLIER and $\mathcal{L}(P)=\ f(P)+\lambda\ g(P)=\mathcal{L}(P,\lambda)$ - LAGRANGE FUNCTION $f(P)-\lambda g(P)$

So we have $\partial_x\mathcal{L}(P,\lambda)=0$; $\partial_y\mathcal{L}(P,\lambda)=\lambda$; $\partial_z\mathcal{L}(P,\lambda)=0$ and $\partial_\lambda\mathcal{L}(P,\lambda)=0$!

$\nabla\mathcal{L}(P,\lambda)=\vec{0}$

Example: $f(x,y)=x+y$; $M=\{(x,y)\in\mathbb{R}^2;\ x^2+y^2=1\} \Rightarrow g(x,y)=x^2+y^2-1$

$\mathcal{L}(x,y,\lambda)=f(x,y)-\lambda\, g(x,y)$

$\mathcal{L}(x,y,\lambda)=x+y-\lambda(x^2+y^2-1)$

METHOD OF LAGRANGE MULTIPLIERS

If a scalar field $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ has a relative extremum which is subject to m-constrains, say: $g_1(x_1, \dots, x_n) = 0 \dots g_m(x_1, \dots x_n) = 0$, where $m < n$, then $\exists$ scalar $\lambda_1, \dots, \lambda_m$ s.t. $\nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_m \nabla g_m$ at each extr. point.

Example: Find the absolute max. and min. of $f(x,y) = x+y$ over $M = \{(x,y) \in \mathbb{R}^2; x^2+y^2 = 1\}$
→ since M is CLOSED and BOUNDED in $\mathbb{R}^2 \Rightarrow$ by WEIERSTRASS THEOREM we have existance of minimum and maximum of $f(x,y)$ over M! (P.S. in $\mathbb{R}^n$: closed + bounded set = COMPACT)
→ define $g_1(x,y) = x^2+y^2-1$
→ Lagrangian multiplier: $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y)$! Then compute $\nabla \mathcal{L} = 0$
$\mathcal{L}(x,y,\lambda) = x+y+\lambda x^2 + \lambda y^2 - \lambda$; $\mathcal{L}_x = 1+2x\lambda$; $\mathcal{L}_y = 1+2y\lambda$; $\mathcal{L}_\lambda = x^2+y^2-1 \Rightarrow$
$\Rightarrow \nabla \mathcal{L} = 0 \Rightarrow 2x\lambda = -1$; $2y\lambda = -1$ and $x^2+y^2 = 1 \Rightarrow$
$\Rightarrow x = -1/2\lambda$; $y = -1/2\lambda \Rightarrow x^2+y^2 = 1 \Rightarrow \lambda = \pm\sqrt{2}/2$ → we have the following points
$\Rightarrow$ stacionary points: $(\sqrt{2}/2, \sqrt{2}/2, -\sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2, \sqrt{2}/2)$ for $\mathcal{L}(x,y,\lambda) \Rightarrow$
$\Rightarrow$ calculate $f(\sqrt{2}/2, \sqrt{2}/2) = \sqrt{2}$ and $f(-\sqrt{2}/2, -\sqrt{2}/2) = -\sqrt{2}$ → by WEIERSTRASS THEOREM we know that this are absolute max. and min. over M!

Example: Find min. and max. of $f(x,y) = \sqrt{x^2+y^2} + y^2 - 1$ on $M = \{(x,y) \in \mathbb{R}^2; x^2+y^2 = 9\}$
→ M is COMPACT, then by Weierstrass theorem $f(x,y)$ admits max. and min. on M!
→ we can define $\varphi: [-3,3] \to \mathbb{R}$; $y \mapsto f(x(y), y) = y^2+2$ ker $\sqrt{x^2+y^2} + y^2 - 1 = 3 + y^2 - 1 = y^2+2$
→ we know $2 \le \varphi(y) \le 11$ ker $\varphi(y) = y^2+2$ in $y \in [-3,3]$ dobiš $2 \le \varphi \le 11$!
→ we have $\varphi(3) = \varphi(-3) = 11$ (max. for φ) and $\varphi(0) = 2$ (min. for φ)
→ we have $y = 0 \Rightarrow x(y) = \sqrt{9-0} = \pm 3$; $y = 3 \Rightarrow x = 0$ and $y = -3 \Rightarrow x = 0$
$\Rightarrow (\pm 3, 0)$ are points of absolute maximum for $f(x,y)$ on M!
$\Rightarrow (0, \pm 3)$ are points of absolute minimum for $f(x,y)$ on M!

Example: Find max. and min. of $f(x,y) = x^2+y^2$ on $M = \{(x,y) \in \mathbb{R}^2; (x-1)^2 + (y-2)^2 - 20 = 0\}$
→ M is COMPACT ⇒ by Weierstrass theorem $f(x,y)$ admits extremum (absolute) on M!
→ $g(x,y) = (x-1)^2 + (y-2)^2 - 20$
→ $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2+y^2 - \lambda[(x-1)^2 + (y-2)^2 - 20]$; put $\nabla \mathcal{L} = 0$
→ $\mathcal{L}_x = 2x - 2\lambda(x-1)$; $\mathcal{L}_y = 2y - 2\lambda(y-2)$; $\mathcal{L}_\lambda = (x-1)^2 + (y-2)^2 - 20 \Rightarrow$
$\Rightarrow$ we get $x = \lambda/(\lambda-1)$; $y = 2\lambda/(\lambda-1)$ and $\lambda_1 = 1/2$, $\lambda_2 = 3/2 \Rightarrow (-1, -2, 1/2)$ and $(3, 6, 3/2)$ are stacionary points for $\mathcal{L}(x,y,\lambda)$ on M! ⇒
$\Rightarrow f(-1,-2) = 5$ is minimum and $f(3,6) = 45$ is maximum for $f(x,y)$ on M!

Example: Look for max. and min. of $f(x,y) = x^4+y^4 - 8(x^2+y^2)$ on $M = \{(x,y) \in \mathbb{R}^2; x^2+y^2 \le 9\}$
→ By Weierstrass: f is continuous and M is compact ⇒ f admits abs. ex. on M!
We can see $M = \{(x,y) \in \mathbb{R}^2; x^2+y^2 < 9\} \cup \{(x,y) \in \mathbb{R}^2; x^2+y^2 = 9\} = \text{int}M \cup \partial M$
use (Free) min. and max. method ; use the Lagrangian multiplier ; (open set) ; (closed set)

① for (int$M$)
$\nabla f(x,y) = 0 \Rightarrow \nabla f = (4x^3 - 16x, 4y^3 - 16y) = (0,0) \Rightarrow$ stacionary points for $f(x,y)$ in (int$M$) are $(0,0)$; $(0,\pm 2)$; $(\pm 2, 0)$; $(\pm 2, \pm 2)$!
To establish if they are max., min. or saddle points we compute the HESSIAN MATRIX!
we compute: $f_{xx} = 12x^2 - 16$; $f_{yy} = 12y^2 - 16$; $f_{xy} = f_{yx} = 0$!
we get: $H_{f(x,y)} = \begin{pmatrix} 12x^2-16, & 0 \\ 0, & 12y^2-16 \end{pmatrix}$ ... by computing and comparing eigenvalues for all stacion. points we get $(0,0)$ is max.; $(0,\pm 2)$ and $(\pm 2, 0)$ is saddle; $(\pm 2, \pm 2)$ min. (RELATIVE!)

② for (∂M)
$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = x^4+y^4 - 8(x^2+y^2) - \lambda(x^2+y^2-9)$. We look for $\nabla \mathcal{L} = 0$!
$\mathcal{L}_x = 4x^3 - 16x - 2\lambda x$; $\mathcal{L}_y = 4y^3 - 16y - 2\lambda y$; $\mathcal{L}_\lambda = -(x^2+y^2-9)$; $\mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0 \Rightarrow$ we get stacionary points for $\mathcal{L}(x,y,\lambda)$: $(0, \pm 3, 10)$; $(\pm 3, 0, 10)$; $(\pm 3\sqrt{2}/2, \pm 3\sqrt{2}/2, 1) \Rightarrow$
$\Rightarrow$ stacionary points for $f(x,y)$ on (∂M) are $(0,\pm 3)$; $(\pm 3, 0)$ and $(\pm 3\sqrt{2}/2, \pm 3\sqrt{2}/2)$

Now we compare $f(x,y)$ for all stacionary points for (int$M$) and (∂M)!
→ In the end we get $(0,\pm 3)$ and $(\pm 3, 0)$ are points of absolute max. on M!
and $(\pm 2, \pm 2)$ are points of absolute minimum of $f(x,y)$ on M!

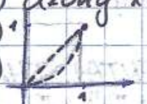
LINE INTEGRALS

Def.: A function $\gamma:[a,b]\to\mathbb{R}^n$ countinous is called a COUNTINUOUS PATH. If $\exists\,\gamma' \in C'$ the path is SMOOTH. If $[a,b]$ can be partitioned into a finite number of subintervals on each of which the path is smooth, the path is called PIECWISE SMOOTH.

DEFINITION OF LINE INTEGRALS (of 2^{ND} KIND)

Def.: Let γ be a PIECEWISE SMOOTH path defined on $[a,b]$ and let f be a vector field defined and bounded on the graph of γ!
The LINE INTEGRAL of f ALONG γ is denoted with: $\boxed{\int_\gamma f\,d\gamma}$
and it is defined: $\boxed{\int_\gamma f\,d\gamma=\int_a^b f[\gamma(t)]\cdot\gamma'(t)\,dt}$
whenever the right integral exists whether as proper or improper integral!

Example: Consider the 2D vector field $f(x,y)=(\sqrt{y},\,x^3+y)$; $y\geq 0$!
Compute the line integrals of f from $(0,0)$ to $(1,1)$ along the following path:
a) $\gamma_1: t\mapsto(t,t)$ $\quad 0\leq t\leq 1 \Rightarrow \gamma_1(0)=(0,0)$ and $\gamma_1(1)=(1,1)$
b) $\gamma_2: t\mapsto(t^2,t^3)$ $\quad 0\leq t\leq 1 \Rightarrow \gamma_2(0)=(0,0)$ and $\gamma_2(1)=(1,1)$



a) $\Rightarrow \int_{(0,0)}^{(1,1)} f\,d\gamma=\int_0^1 f[x(t),y(t)]\cdot\gamma_1'(t)\,dt=\int_0^1(\sqrt{t},t^3+t)\cdot(1,1)\,dt=\int_0^1(\sqrt{t}+t^3+t)\,dt=17/2$

b) $\Rightarrow =\int_0^1(\sqrt{t^3},t^6+t^3)\cdot(2t,3t^2)\,dt=\int_0^1[t^3\cdot 2t+(t^6+t^3)\cdot(3t^2)]\,dt=\ldots=\overset{42}{50}/24$

THE LINE INTEGRAL MAY DEPEND ON THE PATH!

The integral does NOT depend on the parametric represetation: $\gamma(t)=(t^2,t^3)$ and $\beta(t)=(t,t^{3/2})$; $0\leq t\leq 1$ are the two different parametizations for the same curve!

Def.: Let $\alpha:[a,b]\to\mathbb{R}^n$ be a parametization of a curve and let u be a real valued function; $u\in C'$ s.t. u' is never zero. Then the function $\beta(t)=\alpha[u(t)]$ is a parametization of the same curve and α and β are EQUIVALENT!
If: $u'>0 \Rightarrow u$ is ORIENTING PRESERVING
$\quad u'<0\Rightarrow u$ is ORIENTING REVERSING!

Theorem: Let α and β be equivalent piecewise smooth paths (two parametizations of the same curve C). Then we have: $\boxed{\int_C f\cdot d\alpha=\int_C f\cdot d\beta}$ if α and β trace out C in the same direction ($u'>0$)
and: $\boxed{\int_C f\cdot d\alpha=-\int_C f\cdot d\beta}$ if α and β trace out C in the opposite direction ($u'<0$)

Pf: $\alpha:[a,b]\to\mathbb{R}^n \quad u:[c,d]\to[a,b]$
$\beta(t)=\alpha[u(t)]\Rightarrow\beta'(t)=\alpha'[u(t)]\cdot u'(t)$
$\int_C f\cdot d\beta=\int_c^d f[\beta(t)]\cdot\beta'(t)\,dt=\int_c^d f[\alpha(u(t))]\cdot\alpha'[u(t)]\cdot u'(t)\,dt \to$ introduce new variable $v=u(t)$; $dv=u'(t)\,dt$
$u(c)=a$ and $u(d)=b$ if u is perserving...
$\Rightarrow =\int_{u(c)}^{u(d)} f[\alpha(v)]\cdot\alpha'(v)\cdot dv=\int_C f\cdot d\alpha$ ✓ if $u'<0$ is anologus!

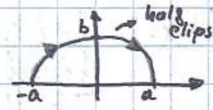
LENGTH OF A CURVE

Def.: Let $\gamma:[a,b]\to\mathbb{R}^n$; γ countinous and piecewise smooth: $\boxed{l(\gamma)=\int_a^b\|\gamma'(t)\|\,dt}$
where $\gamma'(t)=(\gamma_1'(t),\ldots,\gamma_n'(t))$ and $\|\gamma'(t)\|=\sqrt{\gamma_1'(t)^2+\ldots+\gamma_n'^2(t)}$!

Example: Compute the length of $\gamma:[0,1]\to\mathbb{R}^3$, where $\gamma(t)=(t-1,1-t^2,2+\frac{2}{3}t^3)$; $t\in[0,1]$
Compare the length of such a curve with the one of the segment of end points $A=\gamma(0)$ and $B=\gamma(1)$!

$$\Rightarrow \ell(\gamma)=\int_0^1\|\gamma'(t)\|dt=\int_0^1\sqrt{1^2+(-2t)^2+(2t^2)^2}\,dt=\int_0^1\sqrt{4t^4+4t^2+1}\,dt=\int_0^1\sqrt{(2t^2+1)^2}\,dt=$$
$$=\int_0^1|2t^2+1|\,dt=\frac{2}{3}t^3+t\Big]_0^1=\boxed{5/3}$$ Now: $\gamma(0)=(-1,1,2)=A$; $\gamma(1)=(0,0,8/3)=B$

$\Rightarrow$ Segment $\overline{AB}$ has length $\sqrt{22}/3\approx 1'56$!



Example: Vector field $F(x,y)=(y^2,x^2)$ and $\gamma(t)=(-a\cos t, b\sin t)$; $0\le t\le\pi$

$\Rightarrow \gamma'(t)=(a\sin t, b\cos t)$

$\Rightarrow F[\gamma(t)]\cdot\gamma'(t)=F[-a\cos t, b\sin t]\cdot(a\sin t, b\cos t)=(b^2\sin^2 t, a^2\cos^2 t)\cdot(a\sin t, b\cos t)=$
$=ab^2\sin^3 t+a^2b\cos^3 t$

$\Rightarrow \int_\gamma F\,d\gamma=\int_0^\pi ab(b\sin^3 t+a\cos^3 t)\,dt=\dots$

we can compute $\int_0^\pi\cos^3 t\,dt=\int_0^{\pi/2}\cos^3 t\,dt+\int_{\pi/2}^{\pi}\cos^3 dt$; new variable $\tau=\pi-t\Rightarrow d\tau=-dt\Rightarrow$
$\Rightarrow\int_0^{\pi/2}\cos^3 t\,dt-\int_{\pi/2}^{0}\cos^3(\pi-\tau)\,d\tau=\int_0^{\pi/2}\cos^3 t\,dt-\int_0^{\pi/2}\cos^3\tau\,d\tau=0$

and for sinus we get... finish

FUNDAMENTAL THEOREM OF LINE INTEGRALS

$\gamma:[t_0,t_1]\to\mathbb{R}^n$

Theorem: If $F=\nabla f$ ($\exists f:\mathbb{R}^n\to\mathbb{R}$ s.t. $F_1=\partial_{x_1}f\dots F_n=\partial_{x_n}f$) and γ is curve with end points $P_0(x_0,y_0)$ and $P_1(x_1,y_1)$ then: $\boxed{\int_\gamma F\,d\gamma=f(x,y)\Big|_{P_0}^{P_1}=f(x_1,y_1)-f(x_0,y_0)}$!

For gradient field the line integral is independent of the path!

Pf: $\int_\gamma F\,d\gamma=\int_{t_0}^{t_1}F[\gamma(t)]\gamma'(t)\,dt$, now we use $F=\nabla f\Rightarrow=\int_{t_0}^{t_1}\nabla f[\gamma(t)]\cdot\gamma'(t)\,dt$

$\Rightarrow$ we can put $\psi(t)=f(\gamma(t))=(f\circ\gamma)(t)\Rightarrow$ by chain rule $=\int_{t_0}^{t_1}\frac{d}{dt}[f\circ\gamma(t)]\,dt\Rightarrow$

$\Rightarrow$ by fundamental theorem of calculus $\Rightarrow=(f\circ\gamma)(1)-(f\circ\gamma)(0)=f(P_1)-f(P_0)$ ✓

P.S.: LINE INTEGRAL OF 2^{ND} KIND: $f:\mathbb{R}^n\to\mathbb{R}^n$: $\int_c f\,d\gamma=\int_a^b f[\gamma(t)]\gamma'(t)\,dt$

Def.: LINE INTEGRAL OF 1^{ST} KIND: $\boxed{f:\mathbb{R}^n\to\mathbb{R} : \int_c f\cdot d\gamma=\int_a^b f[\gamma(t)]\cdot\|\gamma'(t)\|\,dt}$!

Example: Compute the line integral of 1^{st} kind $\int_\gamma x\,d\gamma$; $f(x,y)=x$ where $\gamma(t)=(t,t^2)$, $t\in[0,a]$, $a>0$

$\Rightarrow f(\gamma(t))=t$; $\|\gamma'(t)\|=\sqrt{1+4t^2}$

$\Rightarrow\int_\gamma x\,d\gamma=\int_0^a t\cdot\sqrt{1+4t^2}\,dt=\left[\frac{1}{12}(1+4t^2)^{3/2}\right]_0^a=\frac{1}{12}\left[(1+4a^2)^{3/2}-1\right]$

Example: Compute the integrals of the 1^{st} kind! $f(x,y)=x/(1+y^2)$; $\gamma(t)=(\cos t,\sin t)$; $t\in[0,\pi/2]$

$\Rightarrow\gamma'(t)=(-\sin t,\cos t)$; $\|\gamma'(t)\|^2=\sin^2 t+\cos^2 t=1$

$\Rightarrow\int_\gamma f\,d\gamma=\int_\gamma\frac{x}{1+y^2}\cdot 1\,dt=\int_0^{\pi/2}\frac{\cos t}{1+\sin^2 t}\,dt=\int_0^1\frac{dz}{1+z^2}=\arctan\Big|_0^1=\boxed{\frac{\pi}{4}}$

MULTIPLE INTEGRALS

Example: $\Omega = \{(x,y)\in\mathbb{R}^2;\ 0<y<\sqrt{2}/2;\ y<x<\sqrt{1-y^2}\}$

$\int_\Omega (x+y)\,dx\,dy = ?$

$$= \int_0^{\sqrt2/2}\left[\int_y^{\sqrt{1-y^2}}(x+y)\,dx\right]dy = \int_0^{\sqrt2/2}\left[\frac12 x^2 + xy\right]_y^{\sqrt{1-y^2}}dy = \int_0^{\sqrt2/2}\left[\frac12(1-y^2)+y\sqrt{1-y^2}-\frac32 y^2\right]dy =$$

$$= \left[\frac12 y - \frac23 y^3 - \frac13(1-y^2)^{3/2}\right]_0^{\sqrt2/2} = \frac13 !$$

Example: $\Omega=\{(x,y)=\mathbb{R}^2;\ 0\le x\le 1;\ 1\le y\le 2\}$ Compute $\int_\Omega (x^2+y^2)\,dx\,dy$!

$$\Rightarrow \int_0^1\left[\int_1^2 (x^2+y^2)\,dy\right]dx = \int_0^1\left[x^2y+\frac{y^3}{3}\right]_1^2 dx = \int_0^1 \left(x^2+\frac73\right)dx = \frac13x^3+\frac73 x\Big|_0^1 = 8/3 !$$

Example: $\Omega=\{(x,y)\in\mathbb{R}^2;\ 0<x<1;\ x^2<y<\sqrt x\}$

Compute $\int_\Omega xy\,dx\,dy = \int_0^1\left[\int_{x^2}^{\sqrt x} xy\,dy\right]dx = \int_0^1 x\left[\int_{x^2}^{\sqrt x} y\,dy\right]dx = \int_0^1 \frac12 x\left[x-x^4\right]dx = \ldots = \frac1{12}$

FUBINI THEOREM

Theorem: If f is a countinous function on a rectangle $Q=[a,b]\times[c,d]$, then f is INTEGRABLE over Q!
Moreover, the value of the integral can be obtained as an iterated integration:

$$\boxed{\iint_Q f = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx}$$

Contraexample of Fubini theorem:

Let $f: R=[0,2]\times[0,1]\to\mathbb{R}$; $f(x,y) = \frac{xy(x^2-y^2)}{(x^2+y^2)^3}$ if $(x,y)\neq(0,0)$ and $=0$ if in $(0,0)$!

We can see that f is not countinous in $(0,0)$, because $f(t,2t)=\frac{-6}{125t}\to+\infty$ as $t\to0$

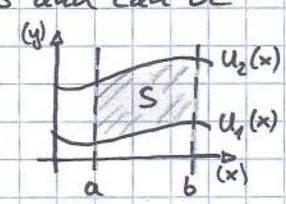
1) If we integrate first in the y-direction:
$\Rightarrow$ for any x: $A(x)=\int_0^1 f(x,y)\,dy$ assume $x\neq0$ and put $u=x^2+y^2$; $du=2y\,dy$
$x^2-y^2 = x^2-(u-x^2) = 2x^2-u \Rightarrow A(x)=\int_{x^2}^{x^2+1}\frac{x(2x^2-u)}{2u^3}\,du = \frac{x}{2(x^2+1)^2}$ $\forall x$; If $x=0$, $A(0)=0$!
Now $\int_0^2 A(x)\,dx = \int_0^2\frac{x}{2(x^2+1)^2}\,dx = \frac15$

2) If we integrate first in x-direction: $B(y)=\int_0^2 f(x,y)\,dx = \ldots$
we put $u=x^2+y^2$; $du=2x\,dx$; $y\neq0$
$\Rightarrow \quad = \left[\frac{-y}{2u}+\frac{y^3}{2u^2}\right]_{u=y^2}^{u=4+y^2} = \frac{-2y}{(4+y^2)^2}$
Now $\int_0^1 B(y)\,dy = \int_0^1 \frac{-2y}{(4+y^2)^2}\,dy = -\frac1{20}$ — NOT THE SAME AS BEFORE in (1)!

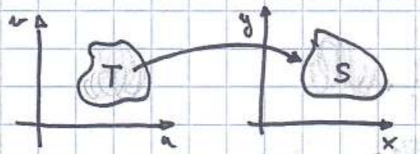
Theorem: Let f be DEFINED and BOUNDED on S and such that it is COUNTINOUS in the interior of S. Then the double integral $\iint_S f$ EXISTS and can be evaluated as:

$$\iint_S f(x,y)\,dx\,dy = \int_a^b\left[\int_{u_1(x)}^{u_2(x)} f(x,y)\,dy\right]dx$$

where $S=\{(x,y)\in\mathbb{R}^2;\ a<x<b;\ u_1(x)\le y\le u_2(x)\}$!



CHANGE OF VARIABLES in DOUBLE INTEGRAL



$$\begin{cases} x = X(u,v) \\ y = Y(u,v) \end{cases} \qquad \begin{cases} u = U(x,y) \\ v = V(x,y) \end{cases}$$

$$\boxed{\iint_S f(x,y)\,dx\,dy = \iint_T f(X(u,v), Y(u,v)) \cdot |J(u,v)|\,du\,dv}$$

where $J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u}, & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v}, & \frac{\partial Y}{\partial v} \end{vmatrix}$ DETERMINANT OF THE JACOBIAN MATRIX

CHANGE OF VARIABLES IN AN N-DIM. INTEGRAL

$$\begin{cases} x_1 = X_1(u_1, \dots u_n) \\ \vdots \\ x_n = X_n(u_1, \dots u_n) \end{cases} \qquad \begin{aligned} x &= (x_1, \dots x_n) \\ u &= (u_1, \dots u_n) \\ X &= (X_1, \dots X_n) \end{aligned}$$

$$X : T \to S$$

$$(u_1 \dots u_n) \mapsto (X_1(u_1, \dots u_n), \dots, X_n(u_1, \dots, u_n))$$

$$\boxed{\int_S f(x)\,dx = \int_T f[X(u)] \cdot |\det DX(u)|\,du \qquad DX(u) = \begin{pmatrix} \frac{\partial X_1}{\partial u_1} & \cdots & \frac{\partial X_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial u_1} & \cdots & \frac{\partial X_n}{\partial u_n} \end{pmatrix}}$$

Example: $\int_\Omega \frac{xy}{x^2+y^2}\,dx\,dy = ?$ $\quad \Omega = \{(x,y) \in \mathbb{R}^2;\ 1 < x^2+y^2 < 4;\ x>0;\ y>0\}$

Most natural is to change into polar coordinates: $\begin{cases} x = \rho \cdot \cos\theta \\ y = \rho \cdot \sin\theta \end{cases}$ $\quad \rho > 0;\ 0 \le \theta < 2\pi$

Now $\int_\Omega \frac{xy}{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{\rho\cos\theta\,\rho\sin\theta}{\rho^2(\cos^2\theta + \sin^2\theta)} |\det J|\,d\rho\,d\theta$

$\Rightarrow \begin{cases} x = X(\rho,\theta) = \rho\cos\theta \\ y = Y(\rho,\theta) = \rho\cdot\sin\theta \end{cases}$ Now $J(\rho,\theta) = \begin{pmatrix} \frac{\partial X}{\partial \rho} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial \rho} & \frac{\partial Y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta, & -\rho\sin\theta \\ \sin\theta, & \rho\cos\theta \end{pmatrix}$

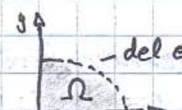
ni pomembno če smo zamenjali vrstice in stolpce, ker det. je enaka.

$\Rightarrow \det|J(\rho,\theta)| = \rho\cos^2\theta + \rho\sin^2\theta = \rho$

$\Rightarrow$ Integral $= \int_0^{\pi/2} \int_1^2 \rho\cos\theta\sin\theta\,d\rho\,d\theta = \dots = \boxed{\frac{3}{4}}$!

Example:

$\int_\Omega xy\,dx\,dy;\quad \Omega = \{(x,y) \in \mathbb{R}^2;\ x^2+2y^2 < 1;\ x>0;\ y>0\}$

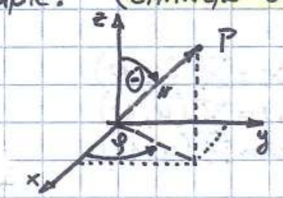


- del elipse $\frac{x^2}{1} + \frac{y^2}{(1/2)} = 1$ $\quad \left(\frac{y}{1/\sqrt{2}}\right)^2$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\frac{1}{\sqrt{2}}\sin\theta \end{cases} \Rightarrow \det|J(\rho,\theta)| = \frac{1}{\sqrt{2}}\rho$

$$\Rightarrow \int_\Omega xy\,dx\,dy = \frac{1}{2}\int_0^1 \int_0^{\pi/2} \rho^3 \cos\theta\sin\theta\,d\theta\,d\rho = \boxed{\frac{1}{16}}$$

Example: (CHANGE OF VARIABLE IN 3D)



r - radial distance of point P from $(0,0)$
θ - polar angle
φ - azimuth angle of the orthogonal projection on the (x,y)-plane

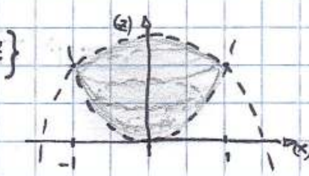
$$\begin{cases} x = r\sin\theta\cos\varphi & r \geq 0 \\ y = r\sin\theta\sin\varphi & 0 \leq \varphi < 2\pi \\ z = r\cos\theta & 0 \leq \theta < \pi \end{cases} \quad X: (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$$

$$\boxed{|\det JX| = r^2 \cdot \sin\theta}$$

Compute volume $Vol(B_R(0))$

$$Vol(B_R(0)) = \int 1\,dx\,dy\,dz = \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta \, dr\, d\varphi\, d\theta = \ldots = \boxed{\frac{4}{3}\pi R^3}$$

Example: $\int_\Omega (x^2+y^2+z^2-1)\,dx\,dy\,dz \quad ; \Omega = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2+z^2 < 2, x^2+y^2 < z\}$



Consider CYLINDRICAL COORDINATES

$$\phi \begin{cases} x = \rho\cos\theta & \rho > 0 \\ y = \rho\sin\theta & 0 \leq \theta < 2\pi \\ z = z \end{cases} \quad \text{Now } \boxed{|\det J\phi(\rho,\theta,z)| = \rho}$$

$$\Omega = \phi(\Omega') \; ; \Omega' = \{(\rho,\theta,z) \in \mathbb{R}^3 ; 0 < \rho < 1 ; 0 \leq \theta < 2\pi , \rho^2 < z < \sqrt{2-\rho^2}\}$$

$$\int_\Omega (x^2+y^2+z^2-1)\,dx\,dy\,dz = \int_{\Omega'} (\rho^2+z^2-1)\rho \, d\rho\, d\theta\, dz = (*) \quad D = \{(\rho,\theta) \in \mathbb{R}^2 ; 0<\rho<1 ; 0 \leq \theta < 2\pi\}$$

$$(*) = \int_D \left[\int_{\rho^2}^{\sqrt{2-\rho^2}} (\rho^2+z^2-1)\rho\, dz \right] d\rho\, d\theta = \ldots = \pi\left(\frac{4}{15}\sqrt{2} - \frac{17}{60}\right)$$

GREEN'S THEOREM

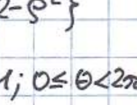
Def: JORDAN CURVE - is a CLOSED curve NOT SELF-INTERSECTING!

Let P, Q two C^1 SCALAR FIELD on an open set S in the plane. Let C be a PIECEWISE SMOTH JORDAN curve and R denotes the union of C and it's interior. Assume $R \subset S$, then we have:

$$\boxed{\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy}$$

the line integral are taken around C in the counter clock-wise direction!

Example: compute the line integral $\int_\gamma y^2 dx + x dy$; γ parametization of a circle



Now $R = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 \leq 1\}$ and $P(x,y) = y^2$; $Q(x,y) = x$

$\int_\gamma y^2 dx + x dy \xrightarrow{\text{(Green's th.)}} \iint_R (1-2y)\,dx\,dy$! now we use polar $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$

so we have $\rightarrow \int_0^1 \int_0^{2\pi} (1 - 2\rho\sin\theta) \cdot \underbrace{\rho}_{\det|J|!} \, d\rho\, d\theta = (\ldots) = \boxed{\pi}$

for practice compute the line integral ... $\gamma \rightarrow x(t) = \cos t$, $y(t) = \sin(t) \rightarrow \ldots$

SURFACE INTEGRAL

A SURFACE CAN BE REPRESENTED in a parametric form: $\begin{cases} x = X(u,v) \\ y = Y(u,v) \\ z = Z(u,v) \end{cases}$

We can associate THE VECTOR EQUATION OF THE SURFACE:

$$\vec{r}(u,v) = X(u,v)\vec{i} + Y(u,v)\vec{j} + Z(u,v)\vec{k}$$

Example: the parametization of the sphere of radius $1 \Rightarrow$ $x = \sin v \cos u$ $\quad 0 \le u < 2\pi$; $y = \sin v \sin u$ $\quad 0 \le v < \pi$; $z = \cos v$

$S = \{(x,y,z) \in R^3;\ x^2+y^2+z^2=1\}$;

$T = \{(u,v) \in R^2;\ 0 \le u < 2\pi;\ 0 \le v < \pi\}$ $\qquad S = \vec{r}(T)$

FUNDAMENTAL VECTOR PRODUCT

$$S = \vec{r}(T)\ ;\ T = \{(u,v)\} \Rightarrow \frac{\partial \vec{r}}{\partial u} = \frac{\partial X}{\partial u}\vec{i} + \frac{\partial Y}{\partial u}\vec{j} + \frac{\partial Z}{\partial u}\vec{k}$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial X}{\partial v}\vec{i} + \frac{\partial Y}{\partial v}\vec{j} + \frac{\partial Z}{\partial v}\vec{k}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \end{vmatrix} \cdot \vec{i} + \begin{vmatrix} \frac{\partial Z}{\partial u} & \frac{\partial X}{\partial u} \\ \frac{\partial Z}{\partial v} & \frac{\partial X}{\partial v} \end{vmatrix} \cdot \vec{j} + \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix} \vec{k} =$$

$$= \frac{\partial(Y,Z)}{\partial(u,v)}\vec{i} + \frac{\partial(Z,X)}{\partial(u,v)}\vec{j} + \frac{\partial(X,Y)}{\partial(u,v)}\vec{k}$$

DEFINITION OF THE SURFACE INTEGRAL

Let $S = \vec{r}(T)$, be a parametric surface described by $\vec{r}$ over T in the u-v-plane and let f be a scalar field. Then the SURFACE INTEGRALS of f OVER S: $\iint_{\vec{r}(T)} f\, dS = \iint_T f[\vec{r}(u,v)] \cdot \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\, dv$

STOKE'S THEOREM

Assume that S is a smooth parametric surface say $S = \vec{r}(T)$, where T is a region in the u-v-plane bounded by a Jordan curve Γ. Let C denote the image of Γ under $\vec{r}$ and let P, Q, R be C^1 scaler fields on S. Then we have:

$$(*)\quad \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \int_C P dx + Q dy + R dz$$

where: $\iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz = \iint_T \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \frac{\partial(x,y)}{\partial(u,v)}$ and analogus for others...

The L.H.S. of $(*)$ is denoted by: $\iint_S \overbrace{\nabla \times F}^{\text{CURL of } F} dS = \oint_C F\ ;\ F = (P,Q,R)$

Properties: The curl $(= \nabla \times F)$ of a vector field is a SELENOIDAL FIELD $\iint_S \nabla \times F = 0$ when S is a closed surface!

Examples: given
$$F(x,y,z) = \overbrace{xy}^{P}\cdot\vec{i} + \overbrace{x^2}^{Q}\vec{j} + \overbrace{yz}^{R}\vec{k}$$ Compute the flux $\nabla\times F$ though the surface $S=\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2+z^2=1;\ z\geq 0\}$! $\to$ (upper half-sphere)

By stoke's th. we know: $\iint_S \nabla\times F = \int_C F$ ... we parametize C. $\begin{cases} x(t)=\cos t \\ y(t)=\sin t \\ z(t)=0\end{cases}$

$$\int_C F = \int_C xy\,dx + x^2dy + y\underbrace{z}_{0}dz = \int_0^{2\pi}\overbrace{(\cos t)}^{x}\cdot\overbrace{(\sin t)}^{y}\cdot\overbrace{(-\sin t)\,dt}^{dx} + $$
$$+ \int_0^{2\pi}\cos^2 t\cdot\cos t\,dt = (\ldots) = \boxed{0}$$

Now to check this ... $D=\{(x,y,z);\ x^2+y^2\leq 1;\ z=0\}$ – disk; $S\cup D$ – is a closed surface!

$$\iint_{S\cup D}\nabla\times F = 0 \Rightarrow \iint_S \nabla\times F = -\iint_D \nabla\times F$$

First we need to find $\nabla\times F = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) = (\ldots) = (z,0,x)$

$$\iint_D \nabla\times F = \iint_{D\cap z=0} x\,dx\,dy = \int_0^1\int_0^{2\pi}\rho^2\cos t\,dt\,d\rho = \int_0^1\rho^2[-\sin t]_0^{2\pi}\,d\rho = \ldots = \boxed{0}$$

ORIENTABLE SURFACE – glej str. 455 Apostol II

THE DIVERGENCE THEOREM

Let V be a solid in 3D, bounded by an orientable surface (closed) and let $\vec{n}$ be the unit outer normal to S. If F is a C^1 vector field on V, we have:
$$\iiint_V \operatorname{div}(F)\,dx\,dy\,dz = \iint_S \vec{F}\cdot\vec{n}\,dS$$

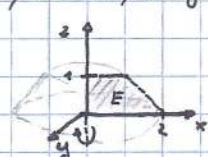
where $F=(F_1,F_2,F_3) \Rightarrow \operatorname{div}(F) = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$

$$F=\nabla f = (\partial_x f, \partial_y f, \partial_z f) \Rightarrow \operatorname{div}(F) = \operatorname{div}(\nabla f) = \Delta f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$$

$$\nabla f\cdot n = \frac{\partial f}{\partial n}\ ;\ \iint_V \Delta f = \iint_S \frac{\partial f}{\partial n}$$

Example: Verify the divergence theorem for $F(x,y,z)=(x,y,z)$ on $V=\{(x,y,z)\in\mathbb{R}^3;\ 0\leq z\leq 1;\ \sqrt{x^2+y^2}\leq 2-z\}$

V is obtained by the revolution of angle 2π around z-axis of an surface $E=\{(x,z)\in\mathbb{R}^2;\ 0\leq z\leq 1;\ 0\leq x\leq 2-z\}$



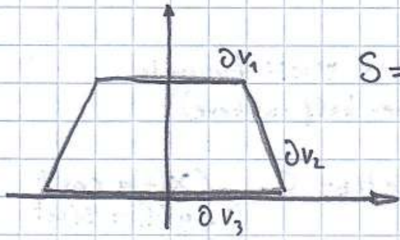
$\iiint_V \operatorname{div}(F) = ?$ Perform change of var.: $\begin{array}{ll} x=(2-z)\cos\theta & \\ y=(2-z)\sin(\theta) & 0\leq\theta<2\pi \\ z=z & 0\leq z\leq 1\end{array}$

We use also:

PAPPUS – GULDINO THEOREM

Let V be a solid of revolution, obtained by the revolution of a domain E around the z-axis of an angle $\alpha\in(0,2\pi]$!
Then we have for the volume of V: $\quad Vol(V) = \alpha\cdot\iint_E x\,dx\,dz$

$\to$ ... glej knjigo



$S = \partial V_1 \cup \partial V_2 \cup \partial V_3$ Normal vector for ∂V_1 is $n_1 = (0,0,1)$ and normal vector for ∂V_3 is $n_3 = (0,0,-1)$!

The hard part is to find normal vector for ∂V_2!

We need to parametrivize ∂V_2 ... $\varphi(z,\theta) \to \begin{cases} x = (2-z)\cos\theta \\ y = (2-z)\sin\theta \\ z = z \end{cases}$

Now:

$$\frac{\partial \varphi(z,\theta)}{\partial(z,\theta)} = \begin{pmatrix} \partial x/\partial z & \partial x/\partial\theta \\ \partial y/\partial z & \partial y/\partial\theta \\ \partial z/\partial z & \partial z/\partial\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta & -(2-z)\sin\theta \\ -\sin\theta & (2-z)\cos\theta \\ 1 & 0 \end{pmatrix}$$

$$\frac{\partial(y,z)}{\partial(z,\theta)} = \begin{vmatrix} -\sin\theta & (2-z)\cos\theta \\ 1 & 0 \end{vmatrix} = (2-z)\cos\theta$$

– iz matrike zbrišeš elemente, ki nimajo x-a!

$$\frac{\partial(z,x)}{\partial(z,\theta)} = \begin{vmatrix} \partial x/\partial z & \partial x/\partial\theta \\ \partial z/\partial z & \partial z/\partial\theta \end{vmatrix} = (z-2)\sin\theta$$

$$\frac{\partial(x,y)}{\partial(z,\theta)} = \begin{vmatrix} \partial x/\partial z & \partial x/\partial\theta \\ \partial y/\partial z & \partial y/\partial\theta \end{vmatrix} = (z-2)$$

With this we can calculate $n_2 = \left(\frac{\partial(y,z)}{\partial(z,\theta)}, \frac{\partial(z,x)}{\partial(z,\theta)}, \frac{\partial(x,y)}{\partial(z,\theta)}\right) =$

$$\left((z-2)\cos\theta, (z-2)\sin\theta, (z-2)\right)$$

$$\iint_{\partial V = S} F\cdot n = \underbrace{\int_{\partial V_1} F\cdot n}_{0} + \int_{\partial V_2} F\cdot n + \underbrace{\int_{\partial V_3} F\cdot n}_{0} \quad \dots \text{ker } F(x,y,z) = (x,y,0) \text{ in } n_{1,3} = (0,0,\pm 1)$$

We are left with $\int_{\partial V_2} F\cdot n \; \dots = \iint_{[0,1]\times(0,2\pi]} \left[(2-z)^2\cos^2\theta + (2-z)^2\sin^2\theta\right] d\theta\, dz =$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 (2-z)^2 dz = \boxed{\underline{\underline{\frac{14}{3}\pi}}}$$