Satisifiability and SAT & MaxSAT Solvers

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Topics

1. SAT

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1. SAT
2. Empirical Properties

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3. MaxSAT

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4. The DPLL Algorithm

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6. Conclusion

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   - A Brief History of SAT Solvers
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   - DPLL and Resolution
5. Improvements to DPLL
   - Watched Literals
   - Further Techniques
6. Conclusion

The Setting: the language

- Atoms: $\mathcal{P} = \{p_1, \ldots, p_n\}$
- Literals: p_i and $\neg p_j$
- $\bar{p} = \neg p$, $\overline{\neg p} = p$
- A clause is a set of literals. Ex: $\{p, \bar{q}, r\}$ or $p \vee \neg q \vee r$
- A formula C is a set of clauses

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- Valuation for atoms $v : \mathcal{P} \to \{0, 1\}$

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- A formula C is satisfied ($v(C) = 1$) if all clauses in C are satisfied

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Given a formula C, decide if C is satisfiable.

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Given a formula C, decide if C is satisfiable.

Witnesses: If C is satisfiable, provide a v such that $v(C) = 1$.

- SAT has small witnesses

An NP Algorithm for SAT

NP-SAT(C)

Input: C, a formula in clausal form

Output: v, if $v(C) = 1$; no, otherwise.

```
Guess a v
Show, in polynomial time, that v(C) = 1
return v
if no such v is guessable then
   return no
end if
```

A Naïve SAT Solver

NaiveSAT(C)

Input: C, a formula in clausal form

Output: v, if $v(C) = 1$; no, otherwise.

for every valuation v over $p_1, \dots, p_n$ **do**
    if $v(C) = 1$ **then**
        return v
    end if
end for
return no

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- [Cook 1971] SAT is NP-complete

Incomplete SAT methods

Incomplete methods compute valuation if C is SAT; if C is unSAT, no answer.

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- [Gent & Walsh, 1994] SAT phase transition

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- Applications to planning, microprocessor test and verification, software design and verification, AI search, games, etc.
- Some non-DPLL SAT solvers incorporate all those techniques: [Dixon 2004]

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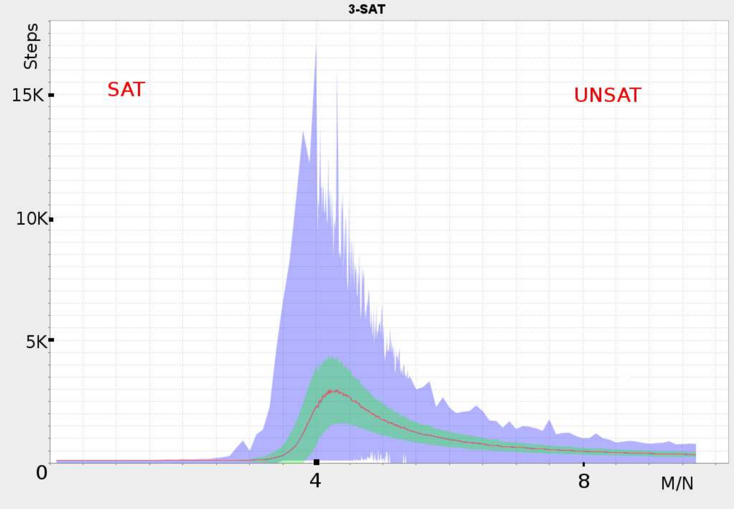
The SAT Phase Transition



The Phase Transition Diagram

- 3-SAT, N is fixed
- Higher N, more abrupt transition
- M/N: low (SAT); high (UNSAT)
- Phase transition point: $M/N \approx 4.3$, 50%SAT [Toby & Walsh 1994]
- Invariant with N
- Invariant with algorithm!

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- Phase transition point: $M/N \approx 4.3$, 50%SAT [Toby & Walsh 1994]
- Invariant with N
- Invariant with algorithm!
- No theoretical explanation
- There is another phase-transition for SAT based on "Impurity" [Lozinskii 2006]

Deolalikar's $P = NP$ Proof Stratgy

- Prove theoretically the existence of phase transition
  - Uses Statistical Phisics
- Model P using Immerman-Vardi LFP-Logic
  - Show that for every p^k, some problem exists in a phase transition above p^k.

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  - Uses Statistical Phisics
  - Critics found a problem here.
- Model P using Immerman-Vardi LFP-Logic
  - Show that for every p^k, some problem exists in a phase transition above p^k.
  - Critics found a problem here.
  - Uses a 2-variable fragment of LFP.

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The unweighted MaxSAT-Problem

- Given: set of formulas $\Gamma = \{C_i | 1 \leq i \leq k\}$
- $\Gamma' \subseteq \Gamma$, is **maximally satisfiable** if for every Γ'', $\Gamma' \subset \Gamma'' \subset \Gamma$, Γ'' unsatisfiable.

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The (unweigted) MaxSAT Problem

Given $\Gamma = \{C_i | 1 \leq i \leq k\}$

Maximize $|\Gamma'|$

Subject to $\Gamma' \subseteq \Gamma$ is SAT.

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The (unweigted) MaxSAT Problem

Given $\Gamma = \{C_i | 1 \leq i \leq k\}$

Maximize $|\Gamma'|$

Subject to $\Gamma' \subseteq \Gamma$ is SAT.

- Problem: MaxSAT is **not** a decision problem. Class NP only applies do decision problems

NP Version of MaxSAT

NP-MaxSAT(Γ, ℓ)

Given $\Gamma = \{C_i | 1 \leq i \leq k\}$, $\ell \leq k$

Decide if there exists satisfiable $\Gamma' \subseteq \Gamma$ with $|\Gamma'| \geq \ell$.

Witness: valuation that satisfies Γ'

MaxSAT via NP-MaxSAT

Use NP-MaxSAT to solve MaxSAT: binary search on ℓ

MaxSAT via NP-MaxSAT

Given $\Gamma = \{C_i | 1 \leq i \leq k\}$

Fix $M = k, m = 0$. While $m < M$ repeat:

- $\ell = \lceil \frac{M+m}{2} \rceil$
- If NP-MaxSAT(Γ, ℓ), $m = \ell + 1$
- Else $M = \ell - 1$

Return M

Partial Weighted MaxSAT

$\Gamma^w = \{C_i : w_i | 1 \leq i \leq k\}$, w_i: weight of C_i.

$$w(\Gamma^w) = \sum_{C_i : w_i \in \Gamma^w} w_i$$

Partial weighted MaxSAT Problem

Given $\Delta = \{D_i | 1 \leq i \leq d\}$, $\Gamma^w = \{C_i : w_i | 1 \leq i \leq k\}$

Maximize $w(\Gamma')$

Subject to $\Gamma' \subseteq \Gamma^w$ and $\Gamma' \cup \Delta$ is SAT

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DPLL Through Examples

$$
\begin{aligned}
&p \vee q \\
&p \vee \neg q \\
&\neg p \vee t \vee s \\
&\neg p \vee \neg t \vee s \\
&\neg p \vee \neg s \\
&\neg p \vee s \vee \neg a
\end{aligned}
$$

Initial Simplifications

Delete all clauses that contain λ, if $\bar{\lambda}$ does not occur.

$$p \vee q$$
$$p \vee \neg q$$
$$\neg p \vee t \vee s$$
$$\neg p \vee \neg t \vee s$$
$$\neg p \vee \neg s$$
$$\cancel{\neg p \vee s \vee \neg a}$$

Construction of a Partial Valuation

Choose a literal: s. $V = \{\mathbf{s}\}$
Propagate choice: Delete clauses containing s. Delete $\bar{s}$ from other clauses.

$$p \vee q$$
$$p \vee \neg q$$
$$\cancel{\neg p \vee t \vee s}$$
$$\cancel{\neg p \vee \neg t \vee s}$$
$$\neg p \cancel{\vee \neg s}$$

Unit Propagation

Enlarge the partial valuation with unit clauses.
$V = \{\mathbf{s}, \bar{p}\}$
Propagate unit clauses as before.

$$\cancel{p \vee} \; q$$

$$\cancel{p \vee} \; \neg q$$

$$\cancel{\neg p}$$

Another propagation step leads to $V = \{\mathbf{s}, \bar{p}, q, \bar{q}\}$

BACKTRACKING

Unit propagation may lead to contradictory valuation:
$V = \{\mathbf{s}, \bar{p}, q, \bar{q}\}$
Backtrack to the previous choice, and propagate: $V = \{\bar{s}\}$

$$p \vee q$$
$$p \vee \neg q$$
$$\neg p \vee t \cancel{\vee s}$$
$$\neg p \vee \neg t \cancel{\vee s}$$
$$\cancel{\neg p \vee \neg s}$$

New Choice

When propagation finishes, a new choice is made: p.
$V = \{\bar{s}, \mathbf{p}\}$.
This leads to an inconsistent valuation: $V = \{\bar{s}, \mathbf{p}, t, \bar{t}\}$
Backtrack to last choice: $V = \{\bar{s}, \bar{p}\}$

$\cancel{p \vee}\ q$

$\cancel{p \vee}\ \neg q$

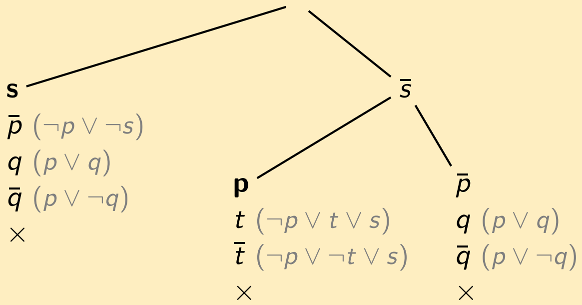
$\cancel{\neg p \vee t}$

$\cancel{\neg p \vee \neg t}$

Propagation leads to another contradiction: $V = \{\bar{s}, \bar{p}, q, \bar{q}\}$

The Formula is UnSAT

There is nowhere to backtrack to now!
The formula is unsatisfiable, with a proof sketched below.



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The Resolution Inference For Clauses

Usual Resolution

$$\frac{C \vee \lambda \quad \bar{\lambda} \vee D}{C \vee D}$$

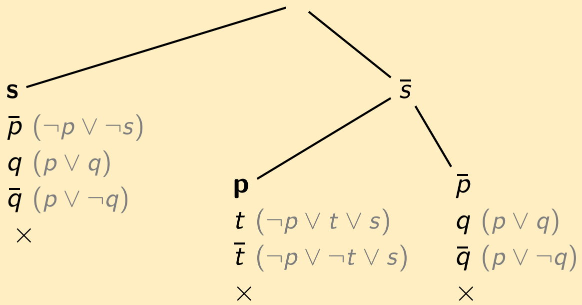
Clauses as Sets

$$\frac{\Gamma \cup \{\lambda\} \quad \{\bar{\lambda}\} \cup \Delta}{\Gamma \cup \Delta}$$

Note that, as clauses are sets

$$\frac{\Gamma \cup \{\mu, \lambda\} \quad \{\bar{\lambda}, \mu\} \cup \Delta}{\Gamma \cup \Delta \cup \{\mu\}}$$

DPLL Proofs and Resolution



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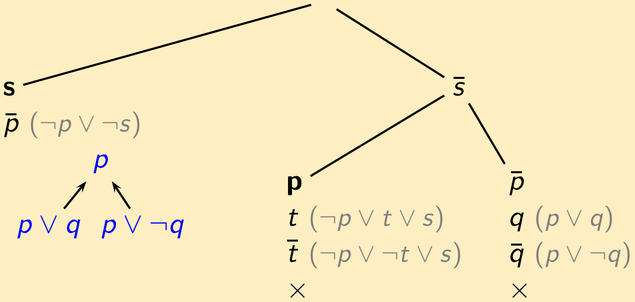

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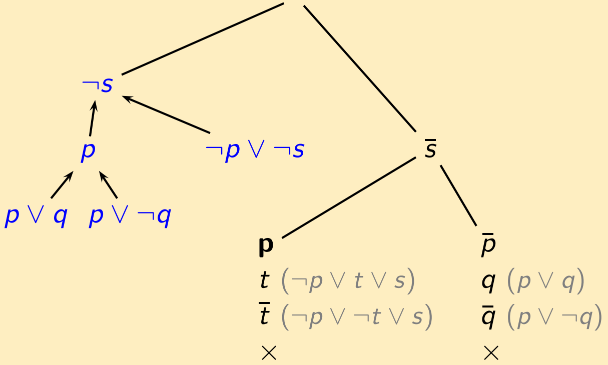

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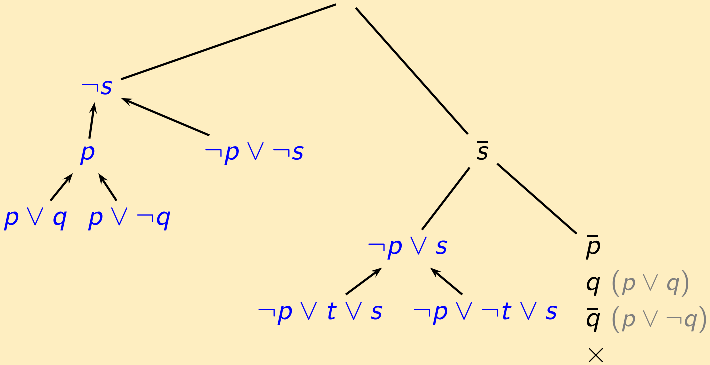

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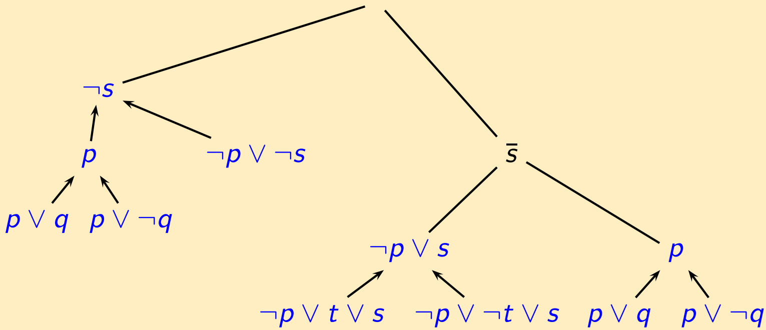

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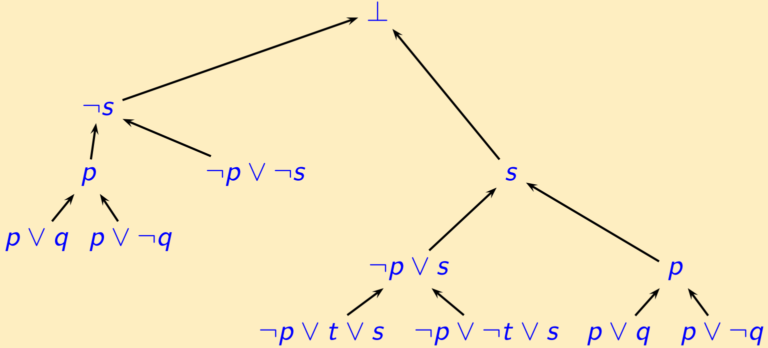

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- DPLL inherits all properties of this (restricted form of resolution
- In particular, DPLL inherits the exponential lower bounds

Enhancing DPLL

For the reasons discussed, DPLL needs to be improved to achieve better efficiency. Several techniques have been applied:

- Learning
- Unlearning
- Backjumping
- Watched literals
- Heuristics for choosing literals

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Watched Literals

The Cost of Unit Propagation

- Empirical measures show that **80% of time** DPLL is doing Unit Propagation
- Propagation is the main target for optimization
- Chaff introduced the technique of **Watched Literals**
  - Unit Propagation speed up
  - No need to delete literals or clauses
  - No need to watch all literals in a clause
  - Constant time backtracking (very fast)

DPLL and 3-valued Logic

- DPLL underlying logic is 3-valued
- Given a partial valuation

$$V = \{\lambda_1, \ldots, \lambda_k\}$$

- Let λ be any literal.

$$V(\lambda) = \begin{cases} 1(\text{true}) & \text{if } \lambda \in V \\ 0(\text{false}) & \text{if } \bar{\lambda} \in V \\ *(\text{undefined}) & \text{otherwise} \end{cases}$$

The Watched Literal Data Structure

- Every clause c has two selected literals: $\lambda_{c1}, \lambda_{c2}$
- For each c, $\lambda_{c1}, \lambda_{c2}$ are dynamically chosen and varies with time
- $\lambda_{c1}, \lambda_{c2}$ are **properly watched** under partial valuation V if:
  - they are both undefined; or
  - at least one of them is true

Dynamics of Watched Literals

- Initially, $V = \emptyset$
- A pair of watched literals is chosen for each clause. It is proper.
- Literal choice and unit propagation expand V
- One or both watched literals may be falsified
- If $\lambda_{c1}, \lambda_{c2}$ become improper then
  - The falsified watched literal is changed
- if no proper pair of watched literals can be found, two things may occur to alter V
  - Unit propagation (V is expanded)
  - Backtracking (V is reduced)

Example

clause	λ_{c1}	λ_{c2}
$p \vee q \vee r$	$p = *$	$q = *$
$p \vee \neg q \vee s$	$p = *$	$\bar{q} = *$
$p \vee r \vee \neg s$	$p = *$	$r = *$

Initially $V = \emptyset$
A pair of literals was elected for each clause
All are undefined, all pairs are proper

$\bar{p}$ is chosen

$V = \{\bar{\mathbf{p}}\}$
All watched literals become $(0, *)$, improper
New literals are chosen to be watched

clause	λ_{c1}	λ_{c2}
$p \vee q \vee r$	$r = *$	$q = *$
$p \vee \neg q \vee s$	$s = *$	$\bar{q} = *$
$p \vee \neg r \vee \neg s$	$\bar{s} = *$	$r = *$

$\bar{r}$ IS CHOSEN

$V = \{\bar{\mathbf{p}}, \bar{\mathbf{r}}\}$
WL in clauses 1,3 become improper
No other *- or 1-literal to be chosen
Unit propagation: $q, \bar{s}$ become true

clause	λ_{c1}	λ_{c2}
$p \lor q \lor r$	$r = 0$	$q = \cancel{*}\ 1$
$p \lor \neg q \lor s$	$s = *$	$\bar{q} = *$
$p \lor \neg r \lor \neg s$	$\bar{s} = \cancel{*}\ 1$	$r = 0$

Unit propagation leads to backtracking

$V = \{\bar{\mathbf{p}}, \bar{\mathbf{r}}, q, \bar{s}\}$
WL in clause 2 becomes improper
No other *- or 1-literal to be chosen
No unit propagation is possible: clause 2 is false

clause	λ_{c1}	λ_{c2}
$p \vee q \vee r$	$r = 0$	$q = 1$
$p \vee \neg q \vee s$	$s = 0$	$\bar{q} = 0$
$p \vee r \vee \neg s$	$\bar{s} = 1$	$r = 0$

Fast Backtracking

V is contracted to last choice point

$V = \cancel{\{\bar{\mathbf{p}}, \bar{\mathbf{r}}, q, \bar{s}\}}\ \{\bar{\mathbf{p}}, r\}$

clause	λ_{c1}	λ_{c2}
$p \vee q \vee r$	$r = 1$	$q = *$
$p \vee \neg q \vee s$	$s = *$	$\bar{q} = *$
$p \vee r \vee \neg s$	$\bar{s} = *$	$r = 1$

Only affected WLs had to be recomputed

No need to reestablish previous context from a stack of contexts

Very quick backtracking

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Improvements Techniques from Chaff

- Learning new clauses
- VSDIS Heuristics
- Random restarts
- Backjumping

Learning

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Sages learn from their bad choices

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  - We learn that a choice of literals lead to contradiction

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- We can added learned information to the problem

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- ... or *learn* from our mistakes:
  - We learn that a choice of literals lead to contradiction
  - We learn that the clauses involved have enough information to avoid the mistake
- We can added learned information to the problem
- **Learning means adding new clauses**, without adding new variables

Learning 1: Bad Choices

Suppose we had choices

$$V = \{\bar{\mathbf{p}}, \bar{\mathbf{r}}\}$$

which after propagation led to

$$V = \{\bar{\mathbf{p}}, \bar{\mathbf{r}}, q, \bar{s}, s\}$$

That is, in that context we learned that we cannot have both $\bar{\mathbf{p}}$ and $\bar{\mathbf{r}}$.

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$$V = \{\bar{\mathbf{p}}, \bar{\mathbf{r}}, q, \bar{s}, s\}$$

That is, in that context we learned that we cannot have both $\bar{\mathbf{p}}$ and $\bar{\mathbf{r}}$.
Add the new clause:

$$p \vee r$$

Learning 1: Properties

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- Theorem: proofs with DPLL + Learning2 can polynomially simulate full resolution proofs

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  - **SATO Heuristics** is a variation of MOM. Let $f(p)$ be 1 plus the number of clauses of smallest size which contain p. Choose p that maximizes $f(p) * f(\neg p)$. Choose p if $f(p) > f(\neg p)$; $\neg p$ otherwise.

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- Low overheads: counters updated only during learning

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- This problem is avoided if learned formulas are kept

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- Empirically checked: this brings efficiency gains

Backjumping

Backjumping via Examples

- Suppose we have the partial valuation
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- Backjumping always brings efficiency gains

Next Issue

1. SAT
   - A Brief History of SAT Solvers
2. Empirical Properties
3. MaxSAT
4. The DPLL Algorithm
   - DPLL and Resolution
5. Improvements to DPLL
   - Watched Literals
   - Further Techniques
6. Conclusion

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- SAT can also be enhanced: **SAT Modulo Theories**

SAT is not a Panacea

- Reduction to SAT may be ok for some NPc problems, but ...
  - ... some NPc problems with **no known** polynomial SAT-reduction.

    E.g. Answer Set Programming
  - ... some NPc problems with **no efficient** polynomial SAT-reduction.

    E.g. Probabilistic SAT