Trace derivatives $\frac{\partial}{\partial X}\mathrm{Tr}\left[AXB\right] = A^\intercal B^\intercal$
$\frac{\partial}{\partial X}\mathrm{Tr}\left[AX^\intercal B\right] = BA$
$\frac{\partial}{\partial X}\mathrm{Tr}\left[X^\intercal AX\right] = (A + A^\intercal)X$
$\frac{\partial}{\partial X}\mathrm{Tr}\left[AXBX^\intercal C\right] = \frac{\partial}{\partial X}\mathrm{Tr}\left[AXD\right] + \frac{\partial}{\partial X}\mathrm{Tr}\left[EX^\intercal C\right] = A^\intercal D^\intercal + EC$
$D = BX^\intercal C$ and $E = AXB$
$\frac{\partial}{\partial X}\mathrm{Tr}\left[AXBX^\intercal C\right] = A^\intercal C^\intercal XB^\intercal + AXBC$
If C is the identity and A and B are symmetric
$\frac{\partial}{\partial X}\mathrm{Tr}\left[A_{\mathrm{sym}}XB_{\mathrm{sym}}X^\intercal\right] = 2AXB$
Complete data log likelihood
$L_{CD} = -\frac{1}{2}\mathrm{Tr}\left[Q^{-1}\left(M_{(2,T)} + AM_{(1,T-1)}A^\intercal + BU_{(2,T)}B^\intercal - 2AM_\Delta - 2B\tilde{U}_{(2,T)} + 2B\tilde{U}_\Delta A^\intercal\right)\right] - \frac{1}{2}\mathrm{Tr}\left[A\frac{1}{\sigma_A^2}IA^\intercal\right]$
$L_{CD} = -\frac{1}{2}\mathrm{Tr}\left[Q^{-1}M_{(2,T)}\right] - \frac{1}{2}\mathrm{Tr}\left[Q^{-1}AM_{(1,T-1)}A^\intercal\right] - \frac{1}{2}\mathrm{Tr}\left[Q^{-1}BU_{(2,T)}B^\intercal\right] + \mathrm{Tr}\left[Q^{-1}AM_\Delta\right] + \mathrm{Tr}\left[Q^{-1}B\tilde{U}_{(2,T)}\right] - \mathrm{Tr}\left[Q^{-1}B\tilde{U}_\Delta A^\intercal\right] - \frac{1}{2}\frac{1}{\sigma_A^2}\mathrm{Tr}\left[AA^\intercal\right]$
Derivatives
$\frac{\partial L_{CD}}{\partial A} = -Q^{-1}AM_{(1,T-1)} + Q^{-1}M_\Delta^\intercal - Q^{-1}B\tilde{U}_\Delta - \frac{1}{\sigma_A^2}A$
$\frac{\partial L_{CD}}{\partial B} = -Q^{-1}BU_{(2,T)} + Q^{-1}\tilde{U}_{(2,T)}^\intercal - Q^{-1}A\tilde{U}_\Delta^\intercal$
Set to 0
$0 = -Q^{-1}AM_{(1,T-1)} + Q^{-1}M_\Delta^\intercal - Q^{-1}B\tilde{U}_\Delta - \frac{1}{\sigma_A^2}A$
$0 = -Q^{-1}BU_{(2,T)} + Q^{-1}\tilde{U}_{(2,T)}^\intercal - Q^{-1}A\tilde{U}_\Delta^\intercal$
Multiply by Q
$0 = -AM_{(1,T-1)} + M_\Delta^\intercal - B\tilde{U}_\Delta - \frac{1}{\sigma_A^2}QA$
$0 = -BU_{(2,T)} + \tilde{U}_{(2,T)}^\intercal - A\tilde{U}_\Delta^\intercal$
Bring the terms without A and B to the left side and multiply by -1
$M_\Delta^\intercal = AM_{(1,T-1)} + B\tilde{U}_\Delta + \frac{1}{\sigma_A^2}QA$
$\tilde{U}_{(2,T)}^\intercal = +A\tilde{U}_\Delta^\intercal + BU_{(2,T)}$
$$\begin{bmatrix} M_\Delta^\intercal & \tilde{U}_{(2,T)}^\intercal \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} M_{(1,T-1)} & \tilde{U}_\Delta^\intercal \\ \tilde{U}_\Delta & U_{(2,T)} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma_A^2}QA & 0 \end{bmatrix}$$
$$\begin{bmatrix} M_\Delta \\ \tilde{U}_{(2,T)} \end{bmatrix} = \begin{bmatrix} M_{(1,T-1)}^\intercal & \tilde{U}_\Delta \\ \tilde{U}_\Delta^\intercal & U_{(2,T)}^\intercal \end{bmatrix}\begin{bmatrix} A^\intercal \\ B^\intercal \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma_A^2}A^\intercal Q \\ 0 \end{bmatrix}$$
Assume Q is diagonal
$$\begin{bmatrix} M_\Delta \\ \tilde{U}_{(2,T)} \end{bmatrix}_{[:,j]} = \begin{bmatrix} M_{(1,T-1)}^\intercal & \tilde{U}_\Delta \\ \tilde{U}_\Delta^\intercal & U_{(2,T)}^\intercal \end{bmatrix}\begin{bmatrix} A^\intercal \\ B^\intercal \end{bmatrix}_{[:,j]} + \begin{bmatrix} \frac{1}{\sigma_A^2}Q_{jj}A^\intercal \\ 0 \end{bmatrix}_{[:,j]}$$
$$\begin{bmatrix} M_\Delta \\ \tilde{U}_{(2,T)} \end{bmatrix}_{[:,j]} = \begin{bmatrix} M_{(1,T-1)}^\intercal + \frac{1}{\sigma_A^2}Q_{jj}I & \tilde{U}_\Delta \\ \tilde{U}_\Delta^\intercal & U_{(2,T)}^\intercal \end{bmatrix}\begin{bmatrix} A^\intercal \\ B^\intercal \end{bmatrix}_{[:,j]}$$
$$\begin{bmatrix} A^\intercal \\ B^\intercal \end{bmatrix}_{[:,j]} = \begin{bmatrix} M_{(1,T-1)}^\intercal + \frac{1}{\sigma_A^2}Q_{jj}I & \tilde{U}_\Delta \\ \tilde{U}_\Delta^\intercal & U_{(2,T)}^\intercal \end{bmatrix}^{-1}\begin{bmatrix} M_\Delta \\ \tilde{U}_{(2,T)} \end{bmatrix}_{[:,j]}$$
We actually implement the transposed version in the code
$$\begin{bmatrix} A & B \end{bmatrix}_{[:,j]} = \begin{bmatrix} M_\Delta^\intercal & \tilde{U}_{(2,T)}^\intercal \end{bmatrix}_{[:,j]}\begin{bmatrix} M_{(1,T-1)} + \frac{1}{\sigma_A^2}Q_{jj}I & \tilde{U}_\Delta^\intercal \\ \tilde{U}_\Delta & U_{(2,T)} \end{bmatrix}^{-1}$$