18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 1. – Thu, Sept 6, 2007

Handouts: syllabus; PS1; flashcards.

Goal of multivariable calculus: tools to handle problems with several parameters – functions of several variables.

Vectors. A vector (notation: $\vec{A}$) has a direction, and a length ($|\vec{A}|$). It is represented by a directed line segment. In a coordinate system it's expressed by components: in space, $\vec{A} = \langle a_1, a_2, a_3 \rangle = a_1 \hat{\boldsymbol{\imath}} + a_2 \hat{\boldsymbol{\jmath}} + a_3 \hat{\boldsymbol{k}}$. (Recall in space x-axis points to the lower-left, y to the right, z up).

Scalar multiplication

Formula for length? Showed picture of $\langle 3, 2, 1 \rangle$ and used flashcards to ask for its length. Most students got the right answer ($\sqrt{14}$).

You can explain why $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ by reducing to the Pythagorean theorem in the plane (**Draw a picture,** showing $\vec{A}$ and its projection to the xy-plane, then derived $|\vec{A}|$ from length of projection + Pythagorean theorem).

Vector addition: $\vec{A}+\vec{B}$ by head-to-tail addition: **Draw a picture** in a parallelogram (showed how the diagonals are $\vec{A}+\vec{B}$ and $\vec{B}-\vec{A}$); **addition works componentwise, and it is true that** $\vec{A} = 3\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ on the displayed example.

Dot product.

Definition: $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (a scalar, not a vector).

Theorem: geometrically, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$.

Explained the theorem as follows: first, $\vec{A} \cdot \vec{A} = |\vec{A}|^2 \cos 0 = |\vec{A}|^2$ is consistent with the definition. Next, consider a triangle with sides $\vec{A}$, $\vec{B}$, $\vec{C} = \vec{A} - \vec{B}$. Then the law of cosines gives $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$, while we get

$$|\vec{C}|^2 = \vec{C} \cdot \vec{C} = (\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}.$$

Hence the theorem is a vector formulation of the law of cosines.

Applications. 1) computing lengths and angles: $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$.

Example: triangle in space with vertices $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 2)$, find angle at P:

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle -1, 1, 0 \rangle \cdot \langle -1, 0, 2 \rangle}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \qquad \theta \approx 71.5^\circ.$$

Note the sign of dot product: positive if angle less than 90°, negative if angle more than 90°, zero if perpendicular.

2) detecting orthogonality.

Example: what is the set of points where $x + 2y + 3z = 0$? (possible answers: empty set, a point, a line, a plane, a sphere, none of the above, I don't know).

Answer: plane; can see "by hand", but more geometrically use dot product: call $\vec{A} = \langle 1, 2, 3 \rangle$, $P = (x, y, z)$, then $\vec{A} \cdot \overrightarrow{OP} = x + 2y + 3z = 0 \Leftrightarrow |\vec{A}||\overrightarrow{OP}| \cos\theta = 0 \Leftrightarrow \theta = \pi/2 \Leftrightarrow \vec{A} \perp \overrightarrow{OP}$. So we get the plane through O with normal vector $\vec{A}$.

18.02 Lecture 2. – Fri, Sept 7, 2007

We've seen two applications of dot product: finding lengths/angles, and detecting orthogonality. A third one: finding components of a vector. If $\hat{\boldsymbol{u}}$ is a unit vector, $\vec{A}\cdot\hat{\boldsymbol{u}} = |\vec{A}|\cos\theta$ is the component of $\vec{A}$ along the direction of $\hat{\boldsymbol{u}}$. E.g., $\vec{A}\cdot\hat{\boldsymbol{\imath}}$ = component of $\vec{A}$ along x-axis.

Example: pendulum making an angle with vertical, force = weight of pendulum $\vec{F}$ pointing downwards: then the physically important quantities are the components of $\vec{F}$ along tangential direction (causes pendulum's motion), and along normal direction (causes string tension).

Area. E.g. of a polygon in plane: break into triangles. Area of triangle $= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$ ($= 1/2$ area of parallelogram). Could get $\sin\theta$ using dot product to compute $\cos\theta$ and $\sin^2 + \cos^2 = 1$, but it gives an ugly formula. Instead, reduce to complementary angle $\theta' = \pi/2 - \theta$ by considering $\vec{A}' = \vec{A}$ rotated 90° counterclockwise (drew a picture). Then area of parallelogram $= |\vec{A}||\vec{B}|\sin\theta = |\vec{A}'||\vec{B}|\cos\theta' = \vec{A}'\cdot\vec{B}$.

Q: if $\vec{A} = \langle a_1, a_2\rangle$, then what is $\vec{A}'$? (showed picture, used flashcards). Answer: $\vec{A}' = \langle -a_2, a_1\rangle$. (explained on picture). So area of parallelogram is $\langle b_1, b_2\rangle \cdot \langle -a_2, a_1\rangle = a_1b_2 - a_2b_1$.

Determinant. Definition: $\det(\vec{A},\vec{B}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

Geometrically: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \pm$ area of parallelogram.

The sign of 2D determinant has to do with whether $\vec{B}$ is counterclockwise or clockwise from $\vec{A}$, without details.

Determinant in space: $\det(\vec{A},\vec{B},\vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.

Geometrically: $\det(\vec{A},\vec{B},\vec{C}) = \pm$ volume of parallelepiped. Referred to the notes for more about determinants.

Cross-product. (only for 2 vectors in space); gives a vector, not a scalar (unlike dot-product).

Definition: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

(the 3x3 determinant is a *symbolic* notation, the actual formula is the expansion).

Geometrically: $|\vec{A}\times\vec{B}|$ = area of space parallelogram with sides $\vec{A}$, $\vec{B}$; direction = normal to the plane containing $\vec{A}$ and $\vec{B}$.

How to decide between the two perpendicular directions = right-hand rule. 1) extend right hand in direction of $\vec{A}$; 2) curl fingers towards direction of $\vec{B}$; 3) thumb points in same direction as $\vec{A}\times\vec{B}$.

Flashcard Question: $\hat{\boldsymbol{\imath}}\times\hat{\boldsymbol{\jmath}} =$? (answer: $\hat{\boldsymbol{k}}$, checked both by geometric description and by calculation).

Triple product: volume of parallelepiped = area(base) $\cdot$ height $= |\vec{B}\times\vec{C}|\,(\vec{A}\cdot\hat{n})$, where $\hat{n} = \vec{B}\times\vec{C}/|\vec{B}\times\vec{C}|$. So volume $= \vec{A}\cdot(\vec{B}\times\vec{C}) = \det(\vec{A},\vec{B},\vec{C})$. The latter identity can also be checked directly using components.

18.02 Lecture 3. – Tue, Sept 11, 2007

Remark: $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$, $\boldsymbol{A} \times \boldsymbol{A} = 0$.

Application of cross product: equation of plane through P_1, P_2, P_3: $P = (x, y, z)$ is in the plane iff $\det(\overrightarrow{P_1P}, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}) = 0$, or equivalently, $\overrightarrow{P_1P} \cdot \boldsymbol{N} = 0$, where $\boldsymbol{N}$ is the normal vector $\boldsymbol{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$. I explained this geometrically, and showed how we get the same equation both ways.

Matrices. Often quantities are related by linear transformations; e.g. changing coordinate systems, from $P = (x_1, x_2, x_3)$ to something more adapted to the problem, with new coordinates (u_1, u_2, u_3). For example

$$\begin{cases} u_1 = 2x_1 + 3x_2 + 3x_3 \\ u_2 = 2x_1 + 4x_2 + 5x_3 \\ u_3 = x_1 + x_2 + 2x_3 \end{cases}$$

Rewrite using matrix product: $\begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, i.e. $AX = U$.

Entries in the matrix product = dot product between rows of A and columns of X. (here we multiply a 3x3 matrix by a column vector = 3x1 matrix).

More generally, matrix multiplication AB:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \begin{bmatrix} . & 0 \\ . & 3 \\ . & 0 \\ . & 2 \end{bmatrix} = \begin{bmatrix} . & 14 \\ . & . \\ . & . \end{bmatrix}$$

(Also explained one can set up A to the left, B to the top, then each entry of AB = dot product between row to its left and column above it).

Note: for this to make sense, width of A must equal height of B.

What AB means: BX = apply transformation B to vector X, so $(AB)X = A(BX)$ = apply first B then A. (so matrix multiplication is like composing transformations, but from right to left!)

(Remark: matrix product is not commutative, AB is in general not the same as BA – one of the two need not even make sense if sizes not compatible).

Identity matrix: identity transformation $IX = X$. $\quad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ = plane rotation by 90 degrees counterclockwise.

$R\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$, $R\hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}}$, $R^2 = -I$.

Inverse matrix. Inverse of a matrix A (necessarily square) is a matrix $M = A^{-1}$ such that $AM = MA = I_n$.

A^{-1} corresponds to the reciprocal linear relation.

E.g., solution to linear system $AX = U$: can solve for X as function of U by $X = A^{-1}U$.

Cofactor method to find A^{-1} (efficient for small matrices; for large matrices computer software uses other algorithms): $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ ($\mathrm{adj}(A)$ = "adjoint matrix").

Illustration on example: starting from $A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 4 & 5 \\ 1 & 1 & 2 \end{bmatrix}$

1) matrix of minors (= determinants formed by deleting one row and one column from A): $\begin{bmatrix} 3 & -1 & -2 \\ 3 & 1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$ (e.g. top-left is $\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 3$).

2) cofactors = flip signs according to checkerboard diagram $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$: get $\begin{bmatrix} 3 & +1 & -2 \\ -3 & 1 & +1 \\ 3 & -4 & 2 \end{bmatrix}$

3) transpose = exchange rows / columns (read horizontally, write vertically) get the adjoint matrix $M^T = adj(A) = \begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$

4) divide by $\det(A)$ (here = 3): get $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 3 & -3 & 3 \\ 1 & 1 & -4 \\ -2 & 1 & 2 \end{bmatrix}$.

18.02 Lecture 4. – Thu, Sept 13, 2007

Handouts: PS1 solutions; PS2.

Equations of planes. Recall an equation of the form $ax + by + cz = d$ defines a plane.

1) plane through origin with normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: $P = (x, y, z)$ is in the plane $\Leftrightarrow$ $\boldsymbol{N} \cdot \overrightarrow{OP} = 0 \Leftrightarrow \langle 1, 5, 10\rangle \cdot \langle x, y, z\rangle = x + 5y + 10z = 0$. Coefficients of the equation are the components of the normal vector.

2) plane through $P_0 = (2, 1, -1)$ with same normal vector $\boldsymbol{N} = \langle 1, 5, 10\rangle$: parallel to the previous one! P is in the plane $\Leftrightarrow \boldsymbol{N} \cdot \overrightarrow{P_0P} = 0 \Leftrightarrow (x - 2) + 5(y - 1) + 10(z + 1) = 0$, or $x + 5y + 10z = -3$. Again coefficients of equation = components of normal vector.

(Note: the equation multiplied by a constant still defines the same plane).

So, to find the equation of a plane, we really need to look for the normal vector $\boldsymbol{N}$; we can e.g. find it by cross-product of 2 vectors that are in the plane.

Flashcard question: the vector $\boldsymbol{v} = \langle 1, 2, -1\rangle$ and the plane $x + y + 3z = 5$ are 1) parallel, 2) perpendicular, 3) neither?

(A perpendicular vector would be proportional to the coefficients, i.e. to $\langle 1, 1, 3\rangle$; let's test if it's in the plane: equivalent to being $\perp$ $\boldsymbol{N}$. We have $\boldsymbol{v} \cdot \boldsymbol{N} = 1 + 2 - 3 = 0$ so $\boldsymbol{v}$ is parallel to the plane.)

Interpretation of 3x3 systems. A 3x3 system asks for the intersection of 3 planes. Two planes intersect in a line, and usually the third plane intersects it in a single point (picture shown). The unique solution to $AX = B$ is given by $X = A^{-1}B$.

Exception: if the 3rd plane is parallel to the line of intersection of the first two? What can happen? (asked on flashcards for possibilities).

If the line $\mathcal{P}_1 \cap \mathcal{P}_2$ is contained in $\mathcal{P}_3$ there are infinitely many solutions (the line); if it is parallel to $\mathcal{P}_3$ there are no solutions. (could also get a plane of solutions if all three equations are the same)

These special cases correspond to systems with $\det(A) = 0$. Then we can't invert A to solve the system: recall $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$. Theorem: A is invertible $\Leftrightarrow \det A \neq 0$.

Homogeneous systems: $AX = 0$. Then all 3 planes pass through the origin, so there is the obvious ("trivial") solution $X = 0$. If $\det A \neq 0$ then this solution is unique: $X = A^{-1}\,0 = 0$. Otherwise, if $\det A = 0$ there are infinitely many solutions (forming a line or a plane).

Note: $\det A = 0$ means $\det(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3) = 0$, where $\boldsymbol{N}_i$ are the normals to the planes $\mathcal{P}_i$. This means the parallelepiped formed by the $\boldsymbol{N}_i$ has no area, i.e. they are coplanar (showed picture of 3 planes intersecting in a line, and their coplanar normals). The line of solutions is then perpendicular to the plane containing $\boldsymbol{N}_i$. For example we can get a vector along the line of intersection by taking a cross-product $\boldsymbol{N}_1 \times \boldsymbol{N}_2$.

General systems: $AX = B$: compared to $AX = 0$, all the planes are shifted to parallel positions from their initial ones. If $\det A \neq 0$ then unique solution is $X = A^{-1}B$. If $\det A = 0$, either there are infinitely many solutions or there are no solutions.

(We don't have tools to decide whether it's infinitely many or none, although elimination will let us find out).

18.02 Lecture 5. – Fri, Sept 14, 2007

Lines. We've seen a line as intersection of 2 planes. Other representation = parametric equation = as trajectory of a moving point.

E.g. line through $Q_0 = (-1, 2, 2)$, $Q_1 = (1, 3, -1)$: moving point $Q(t)$ starts at Q_0 at $t = 0$, moves at constant speed along line, reaches Q_1 at $t = 1$: its "velocity" is $\vec{v} = \overrightarrow{Q_0Q_1}$; $\overrightarrow{Q_0Q(t)} = t\overrightarrow{Q_0Q_1}$. On example: $\langle x+1, y-2, z-2 \rangle = t\langle 2, 1, -3 \rangle$, i.e.

$$\begin{cases} x(t) = -1 + 2t, \\ y(t) = 2 + t, \\ z(t) = 2 - 3t \end{cases}$$

Lines and planes. Understand where lines and planes intersect.

Flashcard question: relative positions of Q_0, Q_1 with respect to plane $x + 2y + 4z = 7$? (same side, opposite sides, one is in the plane, can't tell).

(A sizeable number of students erroneously answered that one is in the plane.)

Answer: plug coordinates into equation of plane: at Q_0, $x+2y+4z = 11 > 7$; at Q_1, $x+2y+4z = 3 < 7$; so opposite sides.

Intersection of line Q_0Q_1 with plane? When does the moving point $Q(t)$ lie in the plane? Check: at $Q(t)$, $x + 2y + 4z = (-1 + 2t) + 2(2 + t) + 4(2 - 3t) = 11 - 8t$, so condition is $11 - 8t = 7$, or $t = 1/2$. Intersection point: $Q(t = \frac{1}{2}) = (0,\, 5/2,\, 1/2)$.

(What would happen if the line was parallel to the plane, or inside it. Answer: when plugging the coordinates of $Q(t)$ into the plane equation we'd get a constant, equal to 7 if the line is contained in the plane – so all values of t are solutions – or to something else if the line is parallel to the plane – so there are no solutions.)

General parametric curves.

Example: cycloid: wheel rolling on floor, motion of a point P on the rim. (Drew picture, then showed an applet illustrating the motion and plotting the cycloid).

Position of P? Choice of parameter: e.g., θ, the angle the wheel has turned since initial position. Distance wheel has travelled is equal to arclength on circumference of the circle $= a\theta$.

Setup: x-axis = floor, initial position of P = origin; introduce A = point of contact of wheel on floor, B = center of wheel. Decompose $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP}$.

$\overrightarrow{OA} = \langle a\theta, 0\rangle$; $\overrightarrow{AB} = \langle 0, a\rangle$. Length of $\overrightarrow{BP}$ is a, and direction is θ from the $(-y)$-axis, so $\overrightarrow{BP} = \langle -a\sin\theta, -a\cos\theta\rangle$. Hence the *position vector* is $\overrightarrow{OP} = \langle a\theta - a\sin\theta, a - a\cos\theta\rangle$.

Q: What happens near bottom point? (flashcards: corner point with finite slopes on left and right; looped curve; smooth graph with horizontal tangent; vertical tangent (cusp)).

Answer: use Taylor approximation: for $t \to 0$, $f(t) \approx f(0) + tf'(0) + \frac{1}{2}t^2 f''(0) + \frac{1}{6}t^3 f'''(0) + \dots$. This gives $\sin\theta \approx \theta - \theta^3/6$ and $\cos\theta \approx 1 - \theta^2/2$. So $x(\theta) \simeq \theta^3/6$, $y(\theta) \simeq \theta^2/2$ Hence for $\theta \to 0$, $y/x \simeq (\frac{1}{2}\theta^2)/(\frac{1}{6}\theta^3) = 3/\theta \to \infty$: vertical tangent.

18.02 Lecture 6. – Tue, Sept 18, 2007

Handouts: Practice exams 1A and 1B.

Velocity and acceleration. Last time: position vector $\vec{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}\ [+z(t)\hat{\boldsymbol{k}}]$.

E.g., cycloid: $\vec{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

Velocity vector: $\vec{v}(t) = \dfrac{d\vec{r}}{dt} = \langle \dfrac{dx}{dt}, \dfrac{dy}{dt}\rangle$. E.g., cycloid: $\vec{v}(t) = \langle 1 - \cos t, \sin t\rangle$. (at $t = 0$, $\vec{v} = \vec{0}$: translation and rotation motions cancel out, while at $t = \pi$ they add up and $\vec{v} = \langle 2, 0\rangle$).

Speed (scalar): $|\vec{v}|$. E.g., cycloid: $|\vec{v}| = \sqrt{(1-\cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}$. (smallest at $t = 0, 2\pi, ...$, largest at $t = \pi$).

Acceleration: $\vec{a}(t) = \dfrac{d\vec{v}}{dt}$. E.g., cycloid: $\vec{a}(t) = \langle \sin t, \cos t\rangle$ (at $t = 0$ $\vec{a} = \langle 0, 1\rangle$ is vertical).

Remark: the speed is $|\frac{d\vec{r}}{dt}|$, which is NOT the same as $\frac{d|\vec{r}|}{dt}$!

Arclength, unit tangent vector. s = distance travelled along trajectory. $\dfrac{ds}{dt}$ = speed = $|\vec{v}|$. Can recover length of trajectory by integrating ds/dt, but this is not always easy... e.g. the length of an arch of cycloid is $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$ (can't do).

Unit tangent vector to trajectory: $\hat{T} = \dfrac{\vec{v}}{|\vec{v}|}$. We have: $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt} = \hat{T}\dfrac{ds}{dt} = \hat{T}|\vec{v}|$.

In interval Δt: $\Delta\vec{r} \approx \hat{T}\,\Delta s$, dividing both sides by Δt and taking the limit $\Delta t \to 0$ gives us the above identity.

Kepler's 2nd law. (illustration of efficiency of vector methods) Kepler 1609, laws of planetary motion: the motion of planets is in a plane, and area is swept out by the line from the sun to the planet at a constant rate. Newton (about 70 years later) explained this using laws of gravitational attraction.

Kepler's law in vector form: area swept out in Δt is area $\approx \frac{1}{2}|\vec{r} \times \Delta\vec{r}| \approx \frac{1}{2}|\vec{r} \times \vec{v}|\,\Delta t$

So $\frac{d}{dt}(\text{area}) = \frac{1}{2}|\vec{r} \times \vec{v}|$ is constant.

Also, $\vec{r} \times \vec{v}$ is perpendicular to plane of motion, so $\text{dir}(\vec{r} \times \vec{v})$ = constant. Hence, Kepler's 2nd law says: $\vec{r} \times \vec{v}$ = constant.

The usual product rule can be used to differentiate vector functions: $\frac{d}{dt}(\vec{a} \cdot \vec{b})$, $\frac{d}{dt}(\vec{a} \times \vec{b})$, being careful about non-commutativity of cross-product.

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}.$$

So Kepler's law $\Leftrightarrow \vec{r} \times \vec{v}$ = constant $\Leftrightarrow \vec{r} \times \vec{a} = 0 \Leftrightarrow \vec{a} // \vec{r} \Leftrightarrow$ the force $\vec{F}$ is central.

(so Kepler's law really means the force is directed $//\vec{r}$; it also applies to other central forces – e.g. electric charges.)

18.02 Lecture 7. – Thu, Sept 20, 2007

Handouts: PS2 solutions, PS3.

Review. Material on the test = everything seen in lecture. The exam is similar to the practice exams, or very slightly harder. The main topics are (Problem numbers refer to Practice 1A):

1) vectors, dot product. $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = \sum a_i b_i$. Finding angles. (e.g. Problem 1.)

2) cross-product, area of space triangles $\frac{1}{2}|\boldsymbol{A} \times \boldsymbol{B}|$; equations of planes (coefficients of equation = components of normal vector) (e.g. Problem 5.)

3) matrices, inverse matrix, linear systems (e.g. Problem 3.)

4) finding parametric equations by decomposing position vector as a sum; velocity, acceleration; differentiating vector identities (e.g. Problems 2,4,6).

18.02 Lecture 8. – Tue, Sept 25, 2007

Functions of several variables.

Recall: for a function of 1 variable, we can plot its graph, and the derivative is the slope of the tangent line to the graph.

Plotting graphs of functions of 2 variables: examples $z = -y$, $z = 1 - x^2 - y^2$, using slices by the coordinate planes. (derived carefully).

Contour plot: level curves $f(x, y) = c$. Amounts to slicing the graph by horizontal planes $z = c$.

Showed 2 examples from "real life": a topographical map, and a temperature map, then did the examples $z = -y$ and $z = 1 - x^2 - y^2$. Showed more examples of computer plots ($z = x^2 + y^2$, $z = y^2 - x^2$, and another one).

Contour plot gives some qualitative info about how f varies when we change x, y. (shown an example where increasing x leads f to increase).

Partial derivatives.

$f_x = \dfrac{\partial f}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$; same for f_y.

Geometric interpretation: f_x, f_y are slopes of tangent lines of vertical slices of the graph of f (fixing $y = y_0$; fixing $x = x_0$).

How to compute: treat x as variable, y as constant.

Example: $f(x, y) = x^3 y + y^2$, then $f_x = 3x^2 y$, $f_y = x^3 + 2y$.

18.02 Lecture 9. – Thu, Sept 27, 2007

Handouts: PS3 solutions, PS4.

Linear approximation

Interpretation of f_x, f_y as slopes of *slices* of the graph by planes parallel to xz and yz planes.

Linear approximation formula: $\Delta f \approx f_x \Delta x + f_y \Delta y$.

Justification: f_x and f_y give slopes of two lines tangent to the graph:

$y = y_0,\ z = z_0 + f_x(x_0, y_0)(x - x_0)$ and $x = x_0,\ z = z_0 + f_y(x_0, y_0)(y - y_0)$.

We can use this to get the equation of the tangent plane to the graph:

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Approximation formula = the graph is close to its tangent plane.

Min/max problems.

At a local max or min, $f_x = 0$ and $f_y = 0$ (since (x_0, y_0) is a local max or min of the slice). Because 2 lines determine tangent plane, this is enough to ensure that tangent plane is horizontal (approximation formula: $\Delta f \simeq 0$, or rather, $|\Delta f| \ll |\Delta x|, |\Delta y|$).

Def of critical point: (x_0, y_0) where $f_x = 0$ and $f_y = 0$.

A critical point may be a local min, local max, or saddle.

Example: $f(x, y) = x^2 - 2xy + 3y^2 + 2x - 2y$.

Critical point: $f_x = 2x - 2y + 2 = 0$, $f_y = -2x + 6y - 2 = 0$, gives $(x_0, y_0) = (-1, 0)$ (only one critical point).

Is it a max, min or saddle? (pictures shown of each type). Systematic answer: next lecture.

For today: observe $f = (x-y)^2 + 2y^2 + 2x - 2y = (x-y+1)^2 + 2y^2 - 1 \geq -1$, so minimum.

Least squares.

Set up problem: experimental data (x_i, y_i) $(i = 1, \ldots, n)$, want to find a best-fit line $y = ax + b$ (the unknowns here are a, b, not x, y!)

Deviations: $y_i - (ax_i + b)$; want to minimize the total square deviation $D = \sum_i (y_i - (ax_i + b))^2$.

$\dfrac{\partial D}{\partial a} = 0$ and $\dfrac{\partial D}{\partial b} = 0$ leads to a 2×2 linear system for a and b (done in detail as in Notes LS):

$$\left(\sum x_i^2\right) a + \left(\sum x_i\right) b = \sum x_i y_i$$
$$\left(\sum x_i\right) a + nb = \sum y_i$$

Least-squares setup also works in other cases: e.g. exponential laws $y = ce^{ax}$ (taking logarithms: $\ln y = \ln c + ax$, so setting $b = \ln c$ we reduce to linear case); or quadratic laws $y = ax^2 + bx + c$ (minimizing total square deviation leads to a 3×3 linear system for a, b, c).

Example: Moore's Law (number of transistors on a computer chip increases exponentially with time): showed interpolation line on a log plot.

18.02 Lecture 10. – Fri, Sept 28, 2007

Second derivative test.

Recall critical points can be local min ($w = x^2 + y^2$), local max ($w = -x^2 - y^2$), saddle ($w = y^2 - x^2$); slides shown of each type.

Goal: determine type of a critical point, and find the global min/max.

Note: global min/max may be either at a critical point, or on the boundary of the domain/at infinity.

We start with the case of $w = ax^2 + bxy + cy^2$, at $(0, 0)$.

Example from Tuesday: $w = x^2 - 2xy + 3y^2$: completing the square, $w = (x-y)^2 + 2y^2$, minimum.

If $a \neq 0$, then $w = a(x^2 + \frac{b}{a}xy) + cy^2 = a(x + \frac{b}{2a}y)^2 + (c - \frac{b^2}{4a})y^2 = \frac{1}{4a}(4a^2(x + \frac{b}{2a}y)^2 + (4ac - b^2)y^2)$.

3 cases: if $4ac - b^2 > 0$, same signs, if $a > 0$ then minimum, if $a < 0$ then maximum; if $4ac - b^2 < 0$, opposite signs, saddle; if $4ac - b^2 = 0$, degenerate case.

This is related to the quadratic formula: $w = y^2\big(a(\frac{x}{y})^2 + b(\frac{x}{y}) + c\big)$.

If $b^2 - 4ac < 0$ then no roots, so $at^2 + bt + c$ has a constant sign, and w is either always nonnegative or always nonpositive (min or max). If $b^2 - 4ac > 0$ then $at^2 + bt + c$ crosses zero and changes sign, so w can have both signs, saddle.

General case: second derivative test.

We look at second derivatives: $f_{xx} = \dfrac{\partial^2 f}{\partial x^2}$, f_{xy}, f_{yx}, f_{yy}. Fact: $f_{xy} = f_{yx}$.

Given f and a critical point (x_0, y_0), set $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$, $C = f_{yy}(x_0, y_0)$, then:
– if $AC - B^2 > 0$ then: if $A > 0$ (or C), local min; if $A < 0$, local max.
– if $AC - B^2 < 0$ then saddle.

– if $AC - B^2 = 0$ then can't conclude.

Checked quadratic case ($f_{xx} = 2a = A$, $f_{xy} = b = B$, $f_{yy} = 2c = C$, then $AC - B^2 = 4ac - b^2$).

General justification: quadratic approximation formula (Taylor series at order 2):

$\Delta f \simeq f_x\,(x - x_0) + f_y\,(y - y_0) + \frac{1}{2} f_{xx}\,(x - x_0)^2 + f_{xy}\,(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}\,(y - y_0)^2$.

At a critical point, $\Delta f \simeq \frac{A}{2}(x - x_0)^2 + B(x - x_0)(y - y_0) + \frac{C}{2}(y - y_0)^2$. In degenerate case, would need higher order derivatives to conclude.

NOTE: the global min/max of a function is not necessarily at a critical point! Need to check boundary / infinity.

Example: $f(x, y) = x + y + \frac{1}{xy}$, for $x > 0$, $y > 0$.

$f_x = 1 - \frac{1}{x^2 y} = 0$, $f_y = 1 - \frac{1}{xy^2} = 0$. So $x^2 y = 1$, $xy^2 = 1$, only critical point is $(1, 1)$.

$f_{xx} = 2/x^3 y$, $f_{xy} = 1/x^2 y^2$, $f_{yy} = 2/xy^3$. So $A = 2$, $B = 1$, $C = 2$.

Question: type of critical point? Answer: $AC - B^2 = 2 \cdot 2 - 1 > 0$, $A = 2 > 0$, local min.

What about the maximum? Answer: $f \to \infty$ near boundary ($x \to 0$ or $y \to 0$) and at infinity.

18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 11. – **Tue, Oct 2, 2007**

Differentials.

Recall in single variable calculus: $y = f(x) \Rightarrow dy = f'(x)\,dx$. Example: $y = \sin^{-1}(x) \Rightarrow x = \sin y$, $dx = \cos y\,dy$, so $dy/dx = 1/\cos y = 1/\sqrt{1-x^2}$.

Total differential: $f = f(x,y,z) \Rightarrow df = f_x\,dx + f_y\,dy + f_z\,dz$.

This is a new type of object, with its own rules for manipulating it (df is not the same as Δf! The textbook has it wrong.) It encodes how variations of f are related to variations of x, y, z. We can use it in two ways:

1. as a placeholder for approximation formulas: $\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$.

2. divide by dt to get the **chain rule**: if $x = x(t)$, $y = y(t)$, $z = z(t)$, then f becomes a function of t and $\dfrac{df}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} + f_z \dfrac{dz}{dt}$

Example: $w = x^2 y + z$, $dw = 2xy\,dx + x^2\,dy + dz$. If $x = t$, $y = e^t$, $z = \sin t$ then the chain rule gives $dw/dt = (2te^t)\,1 + (t^2)\,e^t + \cos t$, same as what we obtain by substitution into formula for w and one-variable differentiation.

Can justify the chain rule in 2 ways:

1. $dx = x'(t)\,dt$, $dy = y'(t)\,dt$, $dz = z'(t)\,dt$, so substituting we get $dw = f_x dx + f_y dy + f_z dz = f_x\,x'(t)\,dt + f_y\,y'(t)\,dt + f_z\,z'(t)\,dt$, hence dw/dt.

2. (more rigorous): $\Delta w \simeq f_x \Delta x + f_y \Delta y + f_z \Delta z$, divide both sides by Δt and take limit as $\Delta t \to 0$.

Applications of chain rule:

Product and quotient formulas for derivatives: $f = uv$, $u = u(t)$, $v = v(t)$, then $d(uv)/dt = f_u\,u' + f_v\,v' = vu' + uv'$. Similarly with $g = u/v$, $d(u/v)/dt = g_u\,u' + g_v\,v' = (1/v)\,u' + (-u/v^2)\,v' = (u'v - uv')/v^2$.

Chain rule with more variables: for example $w = f(x,y)$, $x = x(u,v)$, $y = y(u,v)$. Then $dw = f_x\,dx + f_y\,dy = f_x(x_u du + x_v dv) + f_y(y_u du + y_v dv) = (f_x x_u + f_y y_u)\,du + (f_x x_v + f_y y_v)\,dv$. Identifying coefficients of du and dv we get $\partial f/\partial u = f_x x_u + f_y y_u$ and similarly for $\partial f/\partial v$. It's not legal to "simplify by ∂x".

Example: polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then $f_r = f_x x_r + f_y y_r = \cos\theta\,f_x + \sin\theta\,f_y$, and similarly f_θ.

18.02 Lecture 12. – **Thu, Oct 4, 2007**

Handouts: PS4 solutions, PS5.

Gradient.

Recall chain rule: $\dfrac{dw}{dt} = w_x \dfrac{dx}{dt} + w_y \dfrac{dy}{dt} + w_z \dfrac{dz}{dt}$. In vector notation: $\dfrac{dw}{dt} = \nabla w \cdot \dfrac{d\vec{r}}{dt}$.

Definition: $\nabla w = \langle w_x, w_y, w_z \rangle$ – GRADIENT VECTOR.

Theorem: ∇w is perpendicular to the level surfaces $w = c$.

Example 1: $w = ax + by + cz$, then $w = d$ is a plane with normal vector $\nabla w = \langle a, b, c \rangle$.

Example 2: $w = x^2 + y^2$, then $w = c$ are circles, $\nabla w = \langle 2x, 2y \rangle$ points radially out so $\perp$ circles.

Example 3: $w = x^2 - y^2$, shown on applet (Lagrange multipliers applet with g disabled).

∇w is a vector whose value depends on the point (x, y) where we evaluate w.

Proof: take a curve $\vec{r} = \vec{r}(t)$ contained inside level surface $w = c$. Then velocity $\vec{v} = d\vec{r}/dt$ is in the tangent plane, and by chain rule, $dw/dt = \nabla w \cdot d\vec{r}/dt = 0$, so $\vec{v} \perp \nabla w$. This is true for every $\vec{v}$ in the tangent plane.

Application: tangent plane to a surface. Example: tangent plane to $x^2 + y^2 - z^2 = 4$ at $(2, 1, 1)$: gradient is $\langle 2x, 2y, -2z\rangle = \langle 4, 2, -2\rangle$; tangent plane is $4x + 2y - 2z = 8$. (Here we could also solve for $z = \sqrt{x^2 + y^2 - 4}$ and use linear approximation formula, but in general we can't.)

(Another way to get the tangent plane: $dw = 2x\,dx + 2y\,dy - 2z\,dz = 4dx + 2dy - 2dz$. So $\Delta w \approx 4\Delta x + 2\Delta y - 2\Delta z$. The level surface is $\Delta w = 0$, its tangent plane approximation is $4\Delta x + 2\Delta y - 2\Delta z = 0$, i.e. $4(x - 2) + 2(y - 1) - 2(z - 1) = 0$, same as above).

Directional derivative. Rate of change of w as we move (x, y) in an arbitrary direction.

Take a unit vector $\hat{u} = \langle a, b\rangle$, and look at straight line trajectory $\vec{r}(s)$ with velocity $\hat{u}$, given by $x(s) = x_0 + as$, $y(s) = y_0 + bs$. (unit speed, so s is arclength!)

Notation: $\dfrac{dw}{ds}_{|\hat{u}}$.

Geometrically: slice of graph by a vertical plane (not parallel to x or y axes anymore). Directional derivative is the slope. Shown on applet (Functions of two variables), with $w = x^2 + y^2 + 1$, and rotating slices through a point of the graph.

Know how to calculate dw/ds by chain rule: $\dfrac{dw}{ds}_{|\hat{u}} = \nabla w \cdot \dfrac{d\vec{r}}{ds} = \nabla w \cdot \hat{u}$.

Geometric interpretation: $dw/ds = \nabla w \cdot \hat{u} = |\nabla w| \cos\theta$. Maximal for $\cos\theta = 1$, when $\hat{u}$ is in direction of ∇w. Hence: direction of ∇w is that of fastest increase of w, and $|\nabla w|$ is the directional derivative in that direction. We have $dw/ds = 0$ when $\hat{u} \perp \nabla w$, i.e. when $\hat{u}$ is tangent to direction of level surface.

18.02 Lecture 13. – Fri, Oct 5, 2007 (estimated – written before lecture)

Practice exams 2A and 2B are on course web page.

Lagrange multipliers.

Problem: min/max when variables are constrained by an equation $g(x, y, z) = c$.

Example: find point of $xy = 3$ closest to origin ? I.e. minimize $\sqrt{x^2 + y^2}$, or better $f(x, y) = x^2 + y^2$, subject to $g(x, y) = xy = 3$. Illustrated using Lagrange multipliers applet.

Observe on picture: at the minimum, the level curves are tangent to each other, so the normal vectors ∇f and ∇g are parallel.

So: there exists λ ("multiplier") such that $\nabla f = \lambda \nabla g$. We replace the constrained min/max problem in 2 variables with equations involving 3 variables x, y, λ:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = c \end{cases} \qquad \text{i.e. here} \qquad \begin{cases} 2x = \lambda y \\ 2y = \lambda x \\ xy = 3. \end{cases}$$

In general solving may be hard and require a computer. Here, linear algebra: $\begin{cases} 2x - \lambda y = 0 \\ -\lambda x + 2y = 0 \end{cases}$ requires either $x = y = 0$ (impossible, since $xy = 3$), or $\det = 4 - \lambda^2 = 0$. So $\lambda = \pm 2$. No solutions for $\lambda = -2$, while $\lambda = 2$ gives $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$. (Checked on applet that $\nabla f = 2\nabla g$ at minimum).

Why the method works: at constrained min/max, moving in any direction along the constraint surface $g = c$ should give $df/ds = 0$. So, for any $\hat{u}$ tangent to $\{g = c\}$, $\frac{df}{ds}_{|\hat{u}} = \nabla f \cdot \hat{u} = 0$, i.e. $\hat{u} \perp \nabla f$. Therefore ∇f is normal to tangent plane to $g = c$, and so is ∇g, hence the gradient vectors are parallel.

Warning: method doesn't say whether we have a min or a max, and second derivative test doesn't apply with constrained variables. Need to answer using geometric argument or by comparing values of f.

Advanced example: surface-minimizing pyramid.

Triangular-based pyramid with given triangle as base and given volume V, using as little surface area as possible.

Note: $V = \frac{1}{3} A_{base}\, h$, so height h is fixed, top vertex moves in a plane $z = h$.

We can set up problem in coordinates: base vertices $P_1 = (x_1, y_1, 0)$, P_2, P_3, and top vertex $P = (x, y, h)$. Then areas of faces $= \frac{1}{2} |\vec{PP_1} \times \vec{PP_2}|$, etc. Calculations to find critical point of function of (x, y) are very hard.

Key idea: use variables adapted to the geometry, instead of (x, y): let $a_1, a_2, a_3 =$ lengths of sides of the base triangle; $u_1, u_2, u_3 =$ distances in the xy-plane from the projection of P to the sides of the base triangle. Then each face is a triangle with base length a_i and height $\sqrt{u_i^2 + h^2}$ (using Pythagorean theorem).

So we must minimize $f(u_1, u_2, u_3) = \frac{1}{2} a_1 \sqrt{u_1^2 + h^2} + \frac{1}{2} a_2 \sqrt{u_2^2 + h^2} + \frac{1}{2} a_3 \sqrt{u_3^2 + h^2}$.

Constraint? (asked using flashcards; this was a bad choice, very few students responded at all.) Decomposing base into 3 smaller triangles with heights u_i, we must have $g(u_1, u_2, u_3) = \frac{1}{2} a_1 u_1 + \frac{1}{2} a_2 u_2 + \frac{1}{2} a_3 u_3 = A_{base}$.

Lagrange multiplier method: $\nabla f = \lambda \nabla g$ gives

$$\frac{a_1}{2} \frac{u_1}{\sqrt{u_1^2 + h^2}} = \lambda \frac{a_1}{2}, \quad \text{similarly for } u_2 \text{ and } u_3.$$

We conclude $\lambda = \dfrac{u_1}{\sqrt{u_1^2 + h^2}} = \dfrac{u_2}{\sqrt{u_2^2 + h^2}} = \dfrac{u_3}{\sqrt{u_3^2 + h^2}}$, hence $u_1 = u_2 = u_3$, so P lies above the incenter.

18.02 Lecture 14. – Thu, Oct 11, 2007

Handouts: PS5 solutions, PS6, practice exams 2A and 2B.

Non-independent variables.

Often we have to deal with non-independent variables, e.g. $f(P, V, T)$ where $PV = nRT$.

Question: if $g(x, y, z) = c$ then can think of $z = z(x, y)$. What are $\partial z/\partial x$, $\partial z/\partial y$?

Example: $x^2 + yz + z^3 = 8$ at $(2, 3, 1)$. Take differential: $2x\,dx + z\,dy + (y + 3z^2)\,dz = 0$, i.e. $4\,dx + dy + 6\,dz = 0$ (constraint $g = c$), or $dz = -\frac{4}{6}\,dx - \frac{1}{6}\,dy$. So $\partial z/\partial x = -4/6 = -2/3$ and $\partial z/\partial y = -1/6$ (taking the coefficients of dx and dy). Or equivalently: if y is held constant then we substitute $dy = 0$ to get $dz = -4/6\,dx$, so $\partial z/\partial x = -4/6 = -2/3$.

In general: $g(x, y, z) = c \Rightarrow g_x\,dx + g_y\,dy + g_z\,dz = 0$. If y held fixed, get $g_x\,dx + g_z\,dz = 0$, i.e. $dz = -g_x/g_z\,dx$, and $\partial z/\partial x = -g_x/g_z$.

Warning: notation can be dangerous! For example:

$f(x, y) = x + y$, $\partial f/\partial x = 1$. Change of variables $x = u$, $y = u + v$ then $f = 2u + v$, $\partial f/\partial u = 2$.

$x = u$ but $\partial f/\partial x \neq \partial f/\partial u$!!

This is because $\partial f/\partial x$ means change x keeping y fixed, while $\partial f/\partial u$ means change u keeping v fixed, i.e. change x keeping $y - x$ fixed.

When there's ambiguity, we must precise what is held fixed: $\left(\frac{\partial f}{\partial x}\right)_y$ = deriv. / x with y held fixed, $\left(\frac{\partial f}{\partial u}\right)_v$ = deriv. / u with v held fixed.

We now have $\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_v \neq \left(\frac{\partial f}{\partial x}\right)_y$.

In above example, we computed $(\partial z/\partial x)_y$. When there is no risk of confusion we keep the old notation, by default $\partial/\partial x$ means we keep y fixed.

Example: area of a triangle with 2 sides a and b making an angle θ is $A = \frac{1}{2}ab\sin\theta$. Suppose it's a right triangle with b the hypothenuse, then constraint $a = b\cos\theta$.

3 ways in which rate of change of A w.r.t. θ makes sense:

1) view $A = A(a, b, \theta)$ independent variables, usual $\frac{\partial A}{\partial \theta} = A_\theta$ (with a and b held fixed). This answers the question: a and b fixed, θ changes, triangle stops being a right triangle, what happens to A?

2) constraint $a = b\cos\theta$, keep a fixed, change θ, while b does what it must to satisfy the constraint: $\left(\frac{\partial A}{\partial \theta}\right)_a$.

3) constraint $a = b\cos\theta$, keep b fixed, change θ, while a does what it must to satisfy the constraint: $\left(\frac{\partial A}{\partial \theta}\right)_b$.

How to compute e.g. $(\partial A/\partial\theta)_a$? [treat A as function of a and θ, while $b = b(a, \theta)$.]

0) Substitution: $a = b\cos\theta$ so $b = a\sec\theta$, $A = \frac{1}{2}ab\sin\theta = \frac{1}{2}a^2\tan\theta$, $(\frac{\partial A}{\partial \theta})_a = \frac{1}{2}a^2\sec^2\theta$. (Easiest here, but it's not always possible to solve for b)

1) Total differentials: $da = 0$ (a fixed), $dA = A_\theta d\theta + A_a da + A_b db = \frac{1}{2}ab\cos\theta\,d\theta + \frac{1}{2}b\sin\theta\,da + \frac{1}{2}a\sin\theta\,db$, and constraint $\Rightarrow da = \cos\theta\,db - b\sin\theta\,d\theta$. Plugging in $da = 0$, we get $db = b\tan\theta\,d\theta$

and then

$$dA = (\frac{1}{2}\, ab\cos\theta + \frac{1}{2} a\sin\theta\, b\tan\theta) d\theta, \quad \left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} ab\cos\theta + \frac{1}{2} a\sin\theta\, b\tan\theta = \frac{1}{2} ab\sec\theta.$$

2) Chain rule: $(\partial A/\partial\theta)_a = A_\theta(\partial\theta/\partial\theta)_a + A_a(\partial a/\partial\theta)_a + A_b(\partial b/\partial\theta)_b = A_\theta + A_b(\partial b/\partial\theta)_a$. We find $(\partial b/\partial\theta)_a$ by using the constraint equation. [Ran out of time here]. Implicit differentiation of constraint $a = b\cos\theta$: we have $0 = (\partial a/\partial\theta)_a = (\partial b/\partial\theta)_a \cos\theta - b\sin\theta$, so $(\partial b/\partial\theta)_a = b\tan\theta$, and hence

$$\left(\frac{\partial A}{\partial \theta}\right)_a = \frac{1}{2} ab\cos\theta + \frac{1}{2} a\sin\theta\, b\tan\theta = \frac{1}{2} ab\sec\theta.$$

The two systematic methods essentially involve calculating the same quantities, even though things are written differently.

18.02 Lecture 15. – Fri, Oct 12, 2007

Review topics.

– Functions of several variables, contour plots.

– Partial derivatives, gradient; approximation formulas, tangent planes, directional derivatives.

Note: *partial differential equations* (= equations involving partial derivatives of an unknown function) are very important in physics. E.g., heat equation: $\partial f/\partial t = k(\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2)$ describes evolution of temperature over time.

– Min/max problems: critical points, 2nd derivative test, checking boundary.
(least squares won't be on the exam)

– Differentials, chain rule, change of variables.

– Non-independent variables: Lagrange multipliers, and constrained partial derivatives.

Re-explanation of how to compute constrained partials: say $f = f(x,y,z)$ where $g(x,y,z) = c$. To find $(\partial f/\partial z)_y$:

1) using differentials: $df = f_x\,dx + f_y\,dy + f_z\,dz$. We set $dy = 0$ since y held constant, and want to eliminate dx. For this we use the constraint: $dg = g_x\,dx + g_y\,dy + g_z\,dz = 0$, so setting $dy = 0$ we get $dx = -g_z/g_x\,dz$. Plug into df: $df = -f_x g_z/g_x\,dz + g_z\,dz$, so $(\partial f/\partial z)_y = -f_x g_z/g_x + g_z$.

2) using chain rule: $\left(\frac{\partial f}{\partial z}\right)_y = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial z}\right)_y + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y = f_x\left(\frac{\partial x}{\partial z}\right)_y + f_z$, while

$$0 = \left(\frac{\partial g}{\partial z}\right)_y = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial g}{\partial y}\left(\frac{\partial y}{\partial z}\right)_y + \frac{\partial g}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y = g_x\left(\frac{\partial x}{\partial z}\right)_y + g_z$$

which gives $(\partial x/\partial z)_y$ and hence the answer.

18.02 Multivariable Calculus
Fall 2007

18.02 Lecture 16. – **Thu, Oct 18, 2007**

Handouts: PS6 solutions, PS7.

Double integrals.

Recall integral in 1-variable calculus: $\int_a^b f(x)\,dx$ = area below graph $y = f(x)$ over $[a, b]$.

Now: double integral $\iint_R f(x, y)\,dA$ = volume below graph $z = f(x, y)$ over plane region R.

Cut R into small pieces $\Delta A \Rightarrow$ the volume is approximately $\sum f(x_i, y_i)\,\Delta A_i$. Limit as $\Delta A \to 0$ gives $\iint_R f(x, y)\,dA$. (picture shown)

How to compute the integral? By taking slices: $S(x)$ = area of the slice by a plane parallel to yz-plane (picture shown): then

$$\text{volume} = \int_{x_{min}}^{x_{max}} S(x)\,dx, \quad \text{and for given } x,\ S(x) = \int f(x, y)\,dy.$$

In the inner integral, x is a fixed parameter, y is the integration variable. We get an *iterated integral.*

Example 1: $z = 1 - x^2 - y^2$, region $0 \le x \le 1,\ 0 \le y \le 1$ (picture shown):

$$\int_0^1 \int_0^1 (1 - x^2 - y^2)\,dy\,dx.$$

(note: $dA = dy\,dx$, limit of $\Delta A = \Delta y\,\Delta x$ for small rectangles).

How to evaluate:

1) inner integral (x is constant): $\displaystyle\int_0^1 (1 - x^2 - y^2)\,dy = \left[(1 - x^2)y - \frac{1}{3}y^3\right]_0^1 = (1 - x^2) - \frac{1}{3} = \frac{2}{3} - x^2.$

2) outer integral: $\displaystyle\int_0^1 (\frac{2}{3} - x^2)\,dx = \left[\frac{2}{3}x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$

Example 2: same function over the quarter disc $R : x^2 + y^2 < 1$ in the first quadrant.

How to find the bounds of integration? Fix x constant: what is a slice parallel to y-axis? bounds for y = from $y = 0$ to $y = \sqrt{1 - x^2}$ in the inner integral. For the outer integral: first slice is $x = 0$, last slice is $x = 1$. So we get:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy\,dx.$$

(note the inner bounds depend on the outer variable x; the outer bounds are constants!)

Inner: $\displaystyle\left[(1 - x^2)y - y^3/3\right]_0^{\sqrt{1-x^2}} = \frac{2}{3}(1 - x^2)^{3/2}.$

Outer: $\displaystyle\int_0^1 \frac{2}{3}(1 - x^2)^{3/2}\,dx = \cdots = \frac{\pi}{8}.$

($\ldots$ = trig. substitution $x = \sin\theta$, $dx = \cos\theta\,d\theta$, $(1 - x^2)^{3/2} = \cos^3\theta$. Then use double angle formulas... complicated! I carried out part of the calculation to show how it would be done but then stopped before the end to save time; students may be confused about what happened exactly.)

Exchanging order of integration.

$\int_0^1 \int_0^2 dx\,dy = \int_0^2 \int_0^1 dy\,dx$, since region is a rectangle (shown). In general, more complicated!

Example 3: $\int_0^1 \int_x^{\sqrt{x}} \frac{e^y}{y}\, dy\, dx$: inner integral has no formula. To exchange order:

1) draw the region (here: $x < y < \sqrt{x}$ for $0 \le x \le 1$ – picture drawn on blackboard).

2) figure out bounds in other direction: fixing a value of y, what are the bounds for x? here: left border is $x = y^2$, right is $x = y$; first slice is $y = 0$, last slice is $y = 1$, so we get

$$\int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\, dx\, dy = \int_0^1 \frac{e^y}{y}(y - y^2)\, dy = \int_0^1 e^y - ye^y\, dy = [-ye^y + 2e^y]_0^1 = e - 2.$$

(the last integration can be done either by parts, or by starting from the guess $-ye^y$ and adjusting;).

18.02 Lecture 17. – **Fri, Oct 19, 2007**

Integration in polar coordinates. ($x = r\cos\theta$, $y = r\sin\theta$): useful if either integrand or region have a simpler expression in polar coordinates.

Area element: $\Delta A \simeq (r\Delta\theta)\,\Delta r$ (picture drawn of a small element with sides Δr and $r\Delta\theta$). Taking $\Delta\theta, \Delta r \to 0$, we get $dA = r\, dr\, d\theta$.

Example (same as last time): $\iint_{x^2+y^2\le 1,\ x\ge 0,\ y\ge 0} (1 - x^2 - y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^1 (1 - r^2)\, r\, dr\, d\theta.$

Inner: $\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{1}{4}$. Outer: $\int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{2}\frac{1}{4} = \frac{\pi}{8}$.

In general: when setting up $\iint f\, r\, dr\, d\theta$, find bounds as usual: given a fixed θ, find initial and final values of r (sweep region by rays).

Applications.

1) The area of the region R is $\iint_R 1\, dA$. Also, the total mass of a planar object with density $\delta = \lim_{\Delta A = 0} \Delta m / \Delta A$ (mass per unit area, $\delta = \delta(x, y)$ – if uniform material, constant) is given by:

$$M = \iint_R \delta\, dA.$$

2) recall the average value of f over R is $\bar{f} = \frac{1}{Area} \iint_R f\, dA$. The *center of mass*, or *centroid*, of a plate with density δ is given by weighted average

$$\bar{x} = \frac{1}{mass} \iint_R x\, \delta\, dA, \qquad \bar{y} = \frac{1}{mass} \iint_R y\, \delta\, dA$$

3) **moment of inertia:** physical equivalent of mass for rotational motion. (mass = how hard it is to impart translation motion; moment of inertia about some axis = same for rotation motion around that axis)

Idea: kinetic energy for a single mass m at distance r rotating at angular speed $\omega = d\theta/dt$ (so velocity $v = r\omega$) is $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$; $I_0 = mr^2$ is the moment of inertia.

For a solid with density δ, $I_0 = \iint_R r^2 \delta\, dA$ (moment of inertia / origin). (the rotational energy is $\frac{1}{2}I_0\omega^2$).

Moment of inertia about an axis: $I = \iint_R (\text{distance to axis})^2 \delta\, dA$. E.g. about x-axis, distance is $|y|$, so

$$I_x = \iint_R y^2 \delta\, dA.$$

Examples: 1) disk of radius a around its center ($\delta = 1$):

$$I_0 = \int_0^{2\pi} \int_0^a r^2\, r\, dr\, d\theta = 2\pi \left[\frac{r^4}{4}\right]_0^a = \frac{\pi a^4}{2}.$$

2) same disk, about a point on the circumference?

Setup: place origin at point so integrand is easier; diameter along x-axis; then polar equation of circle is $r = 2a\cos\theta$ (explained on a picture). Thus

$$I_0 = \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} r^2\, r\, dr\, d\theta = \ldots = \frac{3}{2}\pi a^4.$$

18.02 Lecture 18. – **Tue, Oct 23, 2007**

Change of variables.

Example 1: area of ellipse with semiaxes a and b: setting $u = x/a$, $v = y/b$,

$$\iint_{(x/a)^2+(y/b)^2<1} dx\,dy = \iint_{u^2+v^2<1} ab\,du\,dv = ab \iint_{u^2+v^2<1} du\,dv = \pi ab.$$

(substitution works here as in 1-variable calculus: $du = \frac{1}{a}\,dx$, $dv = \frac{1}{b}\,dy$, so $du\,dv = \frac{1}{ab} dx\,dy$.

In general, must find out the scale factor (ratio between $du\,dv$ and $dx\,dy$)?

Example 2: say we set $u = 3x-2y$, $v = x+y$ to simplify either integrand or bounds of integration. What is the relation between $dA = dx\,dy$ and $dA' = du\,dv$? (area elements in xy- and uv-planes).

Answer: consider a small rectangle of area $\Delta A = \Delta x \Delta y$, it becomes in uv-coordinates a parallelogram of area $\Delta A'$. Here the answer is independent of which rectangle we take, so we can take e.g. the unit square in xy-coordinates.

In the uv-plane, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so this becomes a parallelogram with sides given by vectors $\langle 3, 1 \rangle$ and $\langle -2, 1 \rangle$ (picture drawn), and area = det = $\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} = 5 \left(= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \right)$.

For any rectangle $\Delta A' = 5\Delta A$, in the limit $dA' = 5dA$, i.e. $du\,dv = 5dx\,dy$. So $\iint \ldots dx\,dy = \iint \ldots \frac{1}{5} du\,dv$.

General case: approximation formula $\Delta u \approx u_x \Delta x + u_y \Delta y$, $\Delta v \approx v_x \Delta x + v_y \Delta y$, i.e.

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

A small xy-rectangle is approx. a parallelogram in uv-coords, but scale factor depends on x and y now. By the same argument as before, the scale factor is the determinant.

Definition: the Jacobian is $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$. Then $du\,dv = |J|\,dx\,dy$.

(absolute value because area is the absolute value of the determinant).

Example 1: polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

So $dx\,dy = r\,dr\,d\theta$, as seen before.

Example 2: compute $\int_0^1 \int_0^1 x^2 y\,dx\,dy$ by changing to $u = x$, $v = xy$ (usually motivation is to simplify either integrand or region; here neither happens, but we just illustrate the general method).

1) Area element: Jacobian is $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ y & x \end{vmatrix} = x$, so $du\,dv = x\,dx\,dy$, i.e. $dx\,dy = \frac{1}{x} du\,dv$.

2) Express integrand in terms of u, v: $x^2 y\,dx\,dy = x^2 y\,\frac{1}{x}\,du\,dv = xy\,du\,dv = v\,du\,dv$.

3) Find bounds (picture drawn): if we integrate $du\,dv$, then first we keep $v = xy$ constant, slice looks like portion of hyperbola (picture shown), parametrized by $u = x$. The bounds are: at the top boundary $y = 1$, so $v/u = 1$, i.e. $u = v$; at the right boundary, $x = 1$, so $u = 1$. So the inner

integral is $\int_v^1$. The first slice is $v = 0$, the last is $v = 1$; so we get

$$\int_0^1 \int_v^1 v \, du \, dv.$$

Besides the picture in xy coordinates (a square sliced by hyperbolas), I also drew a picture in uv coordinates (a triangle), which some students may find is an easier way of getting the bounds for u and v.

18.02 Lecture 19. – **Thu, Oct 25, 2007**

Handouts: PS7 solutions; PS8.

Vector fields.

$\vec{F} = M\hat{\imath} + N\hat{\jmath}$, where $M = M(x,y)$, $N = N(x,y)$: at each point in the plane we have a vector $\vec{F}$ which depends on x, y.

Examples: velocity fields, e.g. wind flow (shown: chart of winds over Pacific ocean); force fields, e.g. gravitational field.

Examples drawn on blackboard: (1) $\vec{F} = 2\hat{\imath} + \hat{\jmath}$ (constant vector field); (2) $\vec{F} = x\hat{\imath}$; (3) $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ (radially outwards); (4) $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ (explained using that $\langle -y, x\rangle$ is $\langle x, y\rangle$ rotated 90° counterclockwise).

Work and line integrals.

$W =$ (force).(distance) $= \vec{F} \cdot \Delta\vec{r}$ for a small motion $\Delta\vec{r}$. Total work is obtained by summing these along a trajectory C: get a "line integral"

$$W = \int_C \vec{F} \cdot d\vec{r} \ \left(= \lim_{\Delta\vec{r}\to 0} \sum_i \vec{F} \cdot \Delta\vec{r}_i \right).$$

To evaluate the line integral, we observe C is parametrized by time, and give meaning to the notation $\int_C \vec{F} \cdot d\vec{r}$ by

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} (\vec{F} \cdot \frac{d\vec{r}}{dt}) \, dt.$$

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$, C is given by $x = t$, $y = t^2$, $0 \le t \le 1$ (portion of parabola $y = x^2$ from (0,0) to (1,1)). Then we substitute expressions in terms of t everywhere:

$$\vec{F} = \langle -y, x\rangle = \langle -t^2, t\rangle, \quad \frac{d\vec{r}}{dt} = \langle \frac{dx}{dt}, \frac{dy}{dt}\rangle = \langle 1, 2t\rangle,$$

so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt = \int_0^1 \langle -t^2, t\rangle \cdot \langle 1, 2t\rangle \, dt = \int_0^1 t^2 \, dt = \frac{1}{3}$. (in the end things always reduce to a one-variable integral.)

In fact, the definition of the line integral does not involve the parametrization: so the result is the same no matter which parametrization we choose. For example we could choose to parametrize the parabola by $x = \sin\theta$, $y = \sin^2\theta$, $0 \le \theta \le \pi/2$. Then we'd get $\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \dots \, d\theta$, which would be equivalent to the previous one under the substitution $t = \sin\theta$ and would again be equal to $\frac{1}{3}$. In practice we always choose the simplest parametrization!

New notation for line integral: $\vec{F} = \langle M, N\rangle$, and $d\vec{r} = \langle dx, dy\rangle$ (this is in fact a differential: if we divide both sides by dt we get the component formula for the velocity $d\vec{r}/dt$). So the line integral becomes

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy.$$

The notation is dangerous: this is not a sum of integrals w.r.t. x and y, but really a line integral along C. To evaluate one must express everything in terms of the chosen parameter.

In the above example, we have $x = t$, $y = t^2$, so $dx = dt$, $dy = 2t\,dt$ by implicit differentiation; then

$$\int_C -y\,dx + x\,dy = \int_0^1 -t^2\,dt + t\,(2t)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$

(same calculation as before, using different notation).

Geometric approach.

Recall velocity is $\dfrac{d\vec{r}}{dt} = \dfrac{ds}{dt}\,\hat{\boldsymbol{T}}$ (where $s =$ arclength, $\hat{\boldsymbol{T}} =$ unit tangent vector to trajectory).

So $d\vec{r} = \hat{\boldsymbol{T}}\,ds$, and $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{\boldsymbol{T}}\,ds$. Sometimes the calculation is easier this way!

Example: $C =$ circle of radius a centered at origin, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, then $\vec{F} \cdot \hat{\boldsymbol{T}} = 0$ (picture drawn), so $\int_C \vec{F} \cdot \hat{\boldsymbol{T}}\,ds = \int 0\,ds = 0$.

Example: same C, $\vec{F} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, then $\vec{F} \cdot \hat{\boldsymbol{T}} = |\vec{F}| = a$, so $\int_C \vec{F} \cdot \hat{\boldsymbol{T}}\,ds = \int a\,ds = a\,(2\pi a) = 2\pi a^2$; checked that we get the same answer if we compute using parametrization $x = a\cos\theta$, $y = a\sin\theta$.

18.02 Lecture 20. – **Fri, Oct 26, 2007**

Line integrals continued.

Recall: line integral of $\vec{F} = M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}$ along a curve C: $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \vec{F} \cdot \hat{\boldsymbol{T}}\,ds.$

Example: $\vec{F} = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$, $\int_C \vec{F} \cdot d\vec{r}$ for $C = C_1 + C_2 + C_3$ enclosing sector of unit disk from 0 to $\pi/4$. (picture shown). Need to compute $\int_{C_i} y\,dx + x\,dy$ for each portion:

1) x-axis: $x = t$, $y = 0$, $dx = dt$, $dy = 0$, $0 \le t \le 1$, so $\int_{C_1} y\,dx + x\,dy = \int_0^1 0\,dt = 0$. Equivalently, geometrically: along x-axis, $y = 0$ so $\vec{F} = x\hat{\boldsymbol{\jmath}}$ while $\hat{\boldsymbol{T}} = \hat{\boldsymbol{\imath}}$ so $\int_{C_1} \vec{F} \cdot \hat{\boldsymbol{T}}\,ds = 0$.

2) C_2: $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, $0 \le \theta \le \frac{\pi}{4}$. So

$$\int_{C_2} y\,dx + x\,dy = \int_0^{\pi/4} \sin\theta(-\sin\theta)d\theta + \cos\theta\cos\theta\,d\theta = \int_0^{\pi/4} \cos(2\theta)d\theta = \left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = \frac{1}{2}.$$

3) C_3: line segment from $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ to $(0,0)$: could take $x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}t$, $y =$ same, $0 \le t \le 1$, ... but easier: C_3 backwards ("$-C_3$") is $y = x = t$, $0 \le t \le \frac{1}{\sqrt{2}}$. Work along $-C_3$ is opposite of work along C_3.

$$\int_{C_3} y\,dx + x\,dy = \int_{1/\sqrt{2}}^0 t\,dt + t\,dt = -\int_0^{1/\sqrt{2}} 2t\,dt = -[t^2]_0^{1/\sqrt{2}} = -\frac{1}{2}.$$

If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f = f_x\hat{\boldsymbol{\imath}} + f_y\hat{\boldsymbol{\jmath}}$ (f is called "potential function"), then we can simplify evaluation of line integrals by using the fundamental theorem of calculus.

Fundamental theorem of calculus for line integrals:

$\displaystyle\int_C \nabla f \cdot d\vec{r} = f(P_1) - f(P_0)$ when C runs from P_0 to P_1.

Equivalently with differentials: $\displaystyle\int_C f_x\,dx + f_y\,dy = \int_C df = f(P_1) - f(P_0)$. Proof:

$$\int_C \nabla f \cdot d\vec{r} = \int_{t_0}^{t_1} (f_x \frac{dx}{dt} + f_y \frac{dy}{dt})\,dt = \int_{t_0}^{t_1} \frac{d}{dt}(f(x(t), y(t))\,dt = [f(x(t), y(t))]_{t_0}^{t_1} = f(P_1) - f(P_0).$$

E.g., in the above example, if we set $f(x, y) = xy$ then $\nabla f = \langle y, x\rangle = \vec{F}$. So $\int_{C_i}$ can be calculated just by evaluating $f = xy$ at end points. Picture shown of C, vector field, and level curves.

Consequences: for a gradient field, we have:

• Path independence: if C_1, C_2 have same endpoints then $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ (both equal to $f(P_1) - f(P_0)$ by the theorem). So the line integral $\int_C \nabla f \cdot d\vec{r}$ depends only on the end points, not on the actual trajectory.

• Conservativeness: if C is a closed loop then $\int_C \nabla f \cdot d\vec{r} = 0$ $(= f(P) - f(P))$.
(e.g. in above example, $\int_C = 0 + \frac{1}{2} - \frac{1}{2} = 0$.)

WARNING: this is only for gradient fields!

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ is not a gradient field: as seen Thursday, along C = circle of radius a counterclockwise ($\vec{F} // \hat{\boldsymbol{T}}$), $\int_C \vec{F} \cdot d\vec{r} = 2\pi a^2$. Hence $\vec{F}$ is not conservative, and not a gradient field.

Physical interpretation.

If the force field $\vec{F}$ is the gradient of a potential f, then work of $\vec{F}$ = change in value of potential.

E.g.: 1) $\vec{F}$ = gravitational field, f = gravitational potential; 2) $\vec{F}$ = electrical field; f = electrical potential (voltage). (Actually physicists use the opposite sign convention, $\vec{F} = -\nabla f$).

Conservativeness means that energy comes from change in potential f, so no energy can be extracted from motion along a closed trajectory (conservativeness = conservation of energy: the change in kinetic energy equals the work of the force equals the change in potential energy).

We have four equivalent properties:

(1) $\vec{F}$ is conservative ($\int_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve C)

(2) $\int F \cdot d\vec{r}$ is path independent (same work if same end points)

(3) $\vec{F}$ is a gradient field: $\vec{F} = \nabla f = f_x\hat{\imath} + f_y\hat{\jmath}$.

(4) $M\,dx + N\,dy$ is an exact differential $(= f_x\,dx + f_y\,dy = df.)$

((1) is equivalent to (2) by considering C_1, C_2 with same endpoints, $C = C_1 - C_2$ is a closed loop. (3) $\Rightarrow$ (2) is the FTC, $\Leftarrow$ will be key to finding potential function: if we have path independence then we can get $f(x, y)$ by computing $\int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$. (3) and (4) are reformulations of the same property).

18.02 Lecture 21. – **Tue, Oct 30, 2007**

Test for gradient fields.

Observe: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field then $N_x = M_y$. Indeed, if $\vec{F} = \nabla f$ then $M = f_x$, $N = f_y$, so $N_x = f_{yx} = f_{xy} = M_y$.

Claim: Conversely, if $\vec{F}$ is defined and differentiable at every point of the plane, and $N_x = M_y$, then $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is a gradient field.

Example: $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$: $N_x = 1$, $M_y = -1$, so $\vec{F}$ is not a gradient field.

Example: for which value(s) of a is $\vec{F} = (4x^2 + axy)\hat{\imath} + (3y^2 + 4x^2)\hat{\jmath}$ a gradient field? Answer: $N_x = 8x$, $M_y = ax$, so $a = 8$.

Finding the potential: if above test says $\vec{F}$ is a gradient field, we have 2 methods to find the potential function f. Illustrated for the above example (taking $a = 8$):

Method 1: using line integrals (FTC backwards):

We know that if C starts at $(0,0)$ and ends at (x_1, y_1) then $f(x_1, y_1) - f(0,0) = \int_C \vec{F} \cdot d\vec{r}$. Here $f(0,0)$ is just an integration constant (if f is a potential then so is $f + c$). Can also choose the simplest curve C from $(0,0)$ to (x_1, y_1).

Simplest choice: take C = portion of x-axis from $(0,0)$ to $(x_1, 0)$, then vertical segment from $(x_1, 0)$ to (x_1, y_1) (picture drawn).

Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_{C_1 + C_2} (4x^2 + 8xy)\,dx + (3y^2 + 4x^2)\,dy$:

Over C_1, $0 \le x \le x_1$, $y = 0$, $dy = 0$: $\displaystyle\int_{C_1} = \int_0^{x_1} (4x^2 + 8x \cdot 0)\,dx = \left[\frac{4}{3}x^3\right]_0^{x_1} = \frac{4}{3}x_1^3.$

Over C_2, $0 \le y \le y_1$, $x = x_1$, $dx = 0$: $\displaystyle\int_{C_2} = \int_0^{y_1} (3y^2 + 4x_1^2)\,dy = \left[y^3 + 4x_1^2 y\right]_0^{y_1} = y_1^3 + 4x_1^2 y_1.$

So $f(x_1, y_1) = \dfrac{4}{3}x_1^3 + y_1^3 + 4x_1^2 y_1$ (+constant).

Method 2: using antiderivatives:

We want $f(x, y)$ such that (1) $f_x = 4x^2 + 8xy$, (2) $f_y = 3y^2 + 4x^2$.

Taking antiderivative of (1) w.r.t. x (treating y as a constant), we get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y +$ integration constant (independent of x). The integration constant still depends on y, call it $g(y)$.

So $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + g(y)$. Take partial w.r.t. y, to get $f_y = 4x^2 + g'(y)$.

Comparing this with (2), we get $g'(y) = 3y^2$, so $g(y) = y^3 + c$.

Plugging into above formula for f, we finally get $f(x, y) = \frac{4}{3}x^3 + 4x^2 y + y^3 + c$.

Curl.

Now we have: $N_x = M_y \Leftrightarrow^* \vec{F}$ is a gradient field $\Leftrightarrow \vec{F}$ is conservative: $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve.

(*): $\Rightarrow$ only holds if $\vec{F}$ is defined everywhere, or in a "simply-connected" region – see next week.

Failure of conservativeness is given by the *curl* of $\vec{F}$:

Definition: $\mathrm{curl}(\vec{F}) = N_x - M_y$.

Interpretation of curl: for a velocity field, curl = (twice) angular velocity of the rotation component of the motion.

(Ex: $\vec{F} = \langle a, b\rangle$ uniform translation, $\vec{F} = \langle x, y\rangle$ expanding motion have curl zero; whereas $\vec{F} = \langle -y, x\rangle$ rotation at unit angular velocity has curl $= 2$).

For a force field, curl $\vec{F} =$ torque exerted on a test mass, measures how $\vec{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

18.02 Lecture 22. – **Thu, Nov 1, 2007**

Handouts: PS8 solutions, PS9, practice exams 3A and 3B.

Green's theorem.

If C is a positively oriented closed curve enclosing a region R, then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\, dA \quad \text{which means} \quad \oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA.$$

Example (reduce a complicated line integral to an easy $\iint$):
Let $C =$ unit circle centered at (2,0), counterclockwise. $R =$ unit disk at $(2, 0)$. Then

$$\oint_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \iint_R N_x - M_y\,dA = \iint_R (x + e^{-x}) - e^{-x}\,dA = \iint_R x\,dA.$$

This is equal to area $\cdot\,\bar{x} = \pi \cdot 2 = 2\pi$ (or by direct computation of the iterated integral). (Note: direct calculation of the line integral would probably involve setting $x = 2 + \cos\theta$, $y = \sin\theta$, but then calculations get really complicated.)

Application: proof of our criterion for gradient fields.

Theorem: if $\vec{F} = M\hat{\imath} + N\hat{\jmath}$ is defined and continuously differentiable in the whole plane, then $N_x = M_y \Rightarrow \vec{F}$ is conservative ($\Leftrightarrow \vec{F}$ is a gradient field).

If $N_x = M_y$ then by Green, $\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R 0\,dA = 0$. So $\vec{F}$ is conservative.

Note: this only works if $\vec{F}$ and its curl are defined everywhere inside R. For the vector field on PS8 Problem 2, we can't do this if the region contains the origin – for example, the line integral along the unit circle is non-zero even though curl$(\vec{F})$ is zero wherever it's defined.

Proof of Green's theorem. 2 preliminary remarks:

1) the theorem splits into two identities, $\oint_C M\,dx = -\iint_R M_y\,dA$ and $\oint_C N\,dy = \iint_R N_x\,dA$.

2) additivity: if theorem is true for R_1 and R_2 then it's true for the union $R = R_1 \cup R_2$ (picture shown): $\oint_C = \oint_{C_1} + \oint_{C_2}$ (the line integrals along inner portions cancel out) and $\iint_R = \iint_{R_1} + \iint_{R_2}$.

Main step in the proof: prove $\oint_C M\,dx = -\iint_R M_y\,dA$ for "vertically simple" regions: $a < x < b$, $f_0(x) < y < f_1(x)$. (picture drawn). This involves calculations similar to PS5 Problem 3.

LHS: break C into four sides (C_1 lower, C_2 right vertical segment, C_3 upper, C_4 left vertical segment); $\int_{C_2} M\,dx = \int_{C_4} M\,dx = 0$ since $x =$ constant on C_2 and C_4. So

$$\oint_C = \int_{C_1} + \int_{C_3} = \int_a^b M(x, f_0(x))\,dx - \int_a^b M(x, f_1(x))\,dx$$

(using along C_1: parameter $a \le x \le b$, $y = f_0(x)$; along C_2, x from b to a, hence $-$ sign; $y = f_1(x)$).

RHS: $-\iint_R M_y\,dA = -\int_a^b \int_{f_0(x)}^{f_1(x)} M_y\,dy\,dx = -\int_a^b (M(x,f_1(x)) - M(x,f_0(x))\,dx$ (= LHS).

Finally observe: any region R can be subdivided into vertically simple pieces (picture shown); for each piece $\oint_{C_i} M\,dx = -\iint_{R_i} M_y\,dA$, so by additivity $\oint_C M\,dx = -\iint_R M_y\,dA$.

Similarly $\oint_C N\,dy = \iint_R N_x\,dA$ by subdividing into horizontally simple pieces. This completes the proof.

Example. The area of a region R can be evaluated using a line integral: for example, $\oint_C x\,dy = \iint_R 1dA = area(R)$.

This idea was used to build mechanical devices that measure area of arbitrary regions on a piece of paper: planimeters (photo of the actual object shown, and principle explained briefly: as one moves its arm along a closed curve, the planimeter calculates the line integral of a suitable vector field by means of an ingenious mechanism; at the end of the motion, one reads the area).

18.02 Lecture 23. – **Fri, Nov 2, 2007**

Flux. The flux of a vector field $\vec{F}$ across a plane curve C is $\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, where $\hat{\boldsymbol{n}}$ = normal vector to C, rotated 90° clockwise from $\hat{\boldsymbol{T}}$.

We now have two types of line integrals: work, $\int \vec{F}\cdot\hat{\boldsymbol{T}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{T}}$ = component of $\vec{F}$ in direction of C, along the curve C. Flux, $\int \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$, sums $\vec{F}\cdot\hat{\boldsymbol{n}}$ = component of $\vec{F}$ perpendicular to C, along the curve.

If we break C into small pieces of length Δs, the flux is $\sum_i (\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta s_i$.

Physical interpretation: if $\vec{F}$ is a velocity field (e.g. flow of a fluid), flux measures how much matter passes through C per unit time.

Look at a small portion of C: locally $\vec{F}$ is constant, what passes through portion of C in unit time is contents of a parallelogram with sides Δs and $\vec{F}$ (picture shown with $\vec{F}$ horizontal, and portion of curve = diagonal line segment). The area of this parallelogram is $\Delta s\cdot$height $= \Delta s\,(\vec{F}\cdot\hat{\boldsymbol{n}})$. (picture shown rotated with portion of C horizontal, at base of parallelogram). Summing these contributions along all of C, we get that $\int(\vec{F}\cdot\hat{\boldsymbol{n}})\,ds$ is the total flow through C per unit time; counting positively what flows towards the right of C, negatively what flows towards the left of C, as seen from the point of view of a point travelling along C.

Example: C = circle of radius a counterclockwise, $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ (picture shown): along C, $\vec{F}//\hat{\boldsymbol{n}}$, and $|\vec{F}| = a$, so $\vec{F}\cdot\hat{\boldsymbol{n}} = a$. So

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C a\,ds = a\,\text{length}(C) = 2\pi a^2.$$

Meanwhile, the flux of $-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ across C is zero (field tangent to C).

That was a geometric argument. What about the general situation when calculation of the line integral is required?

Observe: $d\vec{r} = \hat{\boldsymbol{T}}\,ds = \langle dx, dy\rangle$, and $\hat{\boldsymbol{n}}$ is $\hat{\boldsymbol{T}}$ rotated 90° clockwise; so $\hat{\boldsymbol{n}}\,ds = \langle dy, -dx\rangle$.

So, if $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ (using new letters to make things look different; of course we could call the components M and N), then

$$\int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \int_C \langle P, Q\rangle\cdot\langle dy, -dx\rangle = \int_C -Q\,dx + P\,dy.$$

(or if $\vec{F} = \langle M, N\rangle$, $\int_C -N\,dx + M\,dy$).

So we can compute flux using the usual method, by expressing x, y, dx, dy in terms of a parameter variable and substituting (no example given).

Green's theorem for flux. If C encloses R counterclockwise, and $\vec{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$, then

$$\oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \iint_R \operatorname{div}(\vec{F})\,dA, \quad \text{where} \quad \operatorname{div}(\vec{F}) = P_x + Q_y \quad \text{is the divergence of } \vec{F}.$$

Note: the counterclockwise orientation of C means that we count flux of $\vec{F}$ *out* of R through C.

Proof: $\displaystyle\oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \oint_C -Q\,dx + P\,dy$. Call $M = -Q$ and $N = P$, then apply usual Green's theorem $\displaystyle\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$ to get

$$\oint_C -Q\,dx + P\,dy = \iint_R (P_x - (-Q_y))\,dA = \iint_R \operatorname{div}(\vec{F})\,dA.$$

This proof by "renaming" the components is why we called the components P, Q instead of M, N. If we call $\vec{F} = \langle M, N\rangle$ the statement becomes $\displaystyle\oint_C -N\,dx + M\,dy = \iint_R (M_x + N_y)\,dA$.

Example: in the above example ($x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ across circle), $\operatorname{div}\vec{F} = 2$, so flux $= \iint_R 2\,dA = 2\,\text{area}(R) = 2\pi a^2$. If we translate C to a different position (not centered at origin) (picture shown) then direct calculation of flux is harder, but total flux is still $2\pi a^2$.

Physical interpretation: in an incompressible fluid flow, divergence measures source/sink density/rate, i.e. how much fluid is being added to the system per unit area and per unit time.

18.02 Lecture 24. – **Tue, Nov 6, 2007**

Simply connected regions. [slightly different from the actual notations used]

Recall Green's theorem: if C is a closed curve around R counterclockwise then line integrals can be expressed as double integrals:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\,dA, \qquad \oint_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds = \iint_R \operatorname{div}(\vec{F})\,dA,$$

where $\operatorname{curl}(M\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}}) = N_x - M_y$, $\operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}) = P_x + Q_y$.

For Green's theorem to hold, $\vec{F}$ must be defined on the *entire* region R enclosed by C.

Example: (same as in pset): $\vec{F} = \dfrac{-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}}{x^2+y^2}$, C = unit circle counterclockwise, then $\operatorname{curl}(\vec{F}) = \dfrac{\partial}{\partial x}(\dfrac{x}{x^2+y^2}) - \dfrac{\partial}{\partial y}(\dfrac{-y}{x^2+y^2}) = \cdots = 0$. So, if we look at both sides of Green's theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = 2\pi \quad \text{(from pset)}, \qquad \iint_R \operatorname{curl}\vec{F}\,dA = \iint_R 0\,dA = 0\,?$$

The problem is that R includes 0, where $\vec{F}$ is not defined.

Definition: a region R in the plane is simply connected if, given any closed curve in R, its interior region is entirely contained in R.

Examples shown.

So: Green's theorem applies safely when the domain in which $\vec{F}$ is defined and differentiable is simply connected: then we automatically know that, if $\vec{F}$ is defined on C, then it's also defined in the region bounded by C.

In the above example, can't apply Green to the unit circle, because the domain of definition of $\vec{F}$ is not simply connected. Still, we can apply Green's theorem to an annulus (picture shown of a curve C' = unit circle counterclockwise + segment along x-axis + small circle around origin clockwise + back to the unit circle allong the x-axis, enclosing an annulus R'). Then Green applies and says $\oint_{C'} \vec{F}\cdot d\vec{r} = \iint_{R'} 0\,dA = 0$; but line integral simplifies to $\oint_{C'} = \int_C - \int_{C_2}$, where C = unit circle, C_2 = small circle / origin; so line integral is actually the same on C and C_2 (or any other curve encircling the origin).

Review for Exam 3.

2 main objects: double integrals and line integrals. Must know how to set up and evaluate.

Double integrals: drawing picture of region, taking slices to set up the iterated integral.

Also in polar coordinates, with $dA = r\,dr\,d\theta$ (see e.g. Problem 2; not done)

Remember: mass, centroid, moment of inertia.

For evaluation, need to know: usual basic integrals (e.g. $\int \frac{dx}{x}$); integration by substitution (e.g. $\displaystyle\int_0^1 \frac{t\,dt}{\sqrt{1+t^2}} = \int_1^2 \frac{du}{2\sqrt{u}}$, setting $u = 1+t^2$). Don't need to know: complicated trigonometric integrals (e.g. $\int cos^4\theta\,d\theta$), integration by parts.

Change of variables: recall method:

1) Jacobian: $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ c_x & v_y \end{vmatrix}$. Its absolute value gives ratio between $du\,dv$ and $dx\,dy$.

2) express integrand in terms of u, v.

3) set up bounds in uv-coordinates by drawing picture. The actual example on the test will be reasonably simple (constant bounds, or circle in uv-coords).

Line integrals: $\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot \hat{\boldsymbol{T}}\, ds = \int_C M\,dx + N\,dy$. To evaluate, express both x, y in terms of a single parameter and substitute.

Special case: gradient fields. Recall: $\vec{F}$ is conservative $\Leftrightarrow \int \vec{F}\cdot d\vec{r}$ is path independent $\Leftrightarrow \vec{F}$ is the gradient of some potential $f \Leftrightarrow \operatorname{curl}\vec{F} = 0$ (i.e. $N_x = M_y$).

If this is the case, then we can look for a potential using one of the two methods (antiderivatives, or line integral); and we can then use the FTC to avoid calculating the line integral. (cf. Problem 3).

Flux: $\int_C \vec{F}\cdot \hat{\boldsymbol{n}}\, ds$ ($= \int_C -Q\,dx + P\,dy$). Geometric interpretation.

Green's theorem (in both forms) (already written at beginning of lecture).

18.02 Lecture 25. – **Fri, Nov 9, 2007**

Handouts: Exam 3 solutions.

Triple integrals: $\displaystyle\iiint_R f\,dV$ (dV = volume element).

Example 1: region between paraboloids $z = x^2 + y^2$ and $z = 4 - x^2 - y^2$ (picture drawn), e.g. volume of this region: $\displaystyle\iiint_R 1\,dV = \int_?^? \int_?^? \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$

To set up bounds, (1) for fixed (x, y) find bounds for z: here lower limit is $z = x^2+y^2$, upper limit is $z = 4-x^2-y^2$; (2) find the shadow of R onto the xy-plane, i.e. set of values of (x, y) above which region lies. Here: R is widest at intersection of paraboloids, which is in plane $z = 2$; general method: for which (x, y) is z on top surface $>$ z on bottom surface? Answer: when $4 - x^2 - y^2 > x^2 - y^2$, i.e. $x^2 + y^2 < 2$. So we integrate over a disk of radius $\sqrt{2}$ in the xy-plane. By usual method to set up double integrals, we finally get:

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} dz\,dy\,dx.$$

Evaluation would be easier if we used polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$: then

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{4-r^2} dz\,r\,dr\,d\theta.$$

(evaluation easy, not done).

Cylindrical coordinates. (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$. r measures distance from z-axis, θ measures angle from xz-plane (picture shown).

Cylinder of radius a centered on z-axis is $r = a$ (drawn); $\theta = 0$ is a vertical half-plane (not drawn).

Volume element: in rect. coords., $dV = dx\,dy\,dz$; in cylindrical coords., $dV = r\,dr\,d\theta\,dz$. In both cases this is justified by considering a small box with height Δz and base area ΔA, then volume is $\Delta V = \Delta A\,\Delta z$.

Applications: Mass: $M = \iiint_R \delta\,dV$.

Average value of f over R: $\bar{f} = \frac{1}{Vol} \iiint_R f\, dV$; weighted average: $\bar{f} = \frac{1}{Mass} \iiint_R f\, \delta\, dV$.

In particular, center of mass: $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x} = \frac{1}{Mass} \iiint_R x\, \delta\, dV$.

(Note: can sometimes avoid calculation using symmetry, e.g. in above example $\bar{x} = \bar{y} = 0$).

Moment of inertia around an axis: $I = \iiint_R (\text{distance from axis})^2\, \delta\, dV$.

About z-axis: $I_z = \iiint_R r^2\, \delta\, dV = \iiint_R (x^2 + y^2)\, \delta\, dV$. (consistent with I_0 in 2D case)

Similarly, about x and y axes: $I_x = \iiint_R (y^2 + z^2)\, \delta\, dV$, $I_y = \iiint_R (x^2 + z^2)\, \delta\, dV$
(setting $z = 0$, this is consistent with previous definitions of I_x and I_y for plane regions).

Example 2: moment of inertia I_z of solid cone between $z = ar$ and $z = b$ ($\delta = 1$) (picture drawn):

$$I_z = \iiint_R r^2\, dV = \int_0^b \int_0^{2\pi} \int_0^{z/a} r^2\, r\, dr\, d\theta\, dz \quad \left(= \frac{\pi b^5}{10 a^4} \right).$$

(I explained how to find bounds in order $dr\, d\theta\, dz$: first we fix z, then slice for given z is the disk bounded by $r = z/a$; the first slice is $z = 0$, the last one is $z = b$).

Example 3: volume of region where $z > 1 - y$ and $x^2 + y^2 + z^2 < 1$? Pictures drawn: in space, slice by yz-plane, and projection to xy-plane.

The bottom surface is the plane $z = 1 - y$, the upper one is the sphere $z = \sqrt{1 - x^2 - y^2}$. So inner is $\int_{1-y}^{\sqrt{1-x^2-y^2}} dz$. The shadow on the xy-plane = points where $1 - y < \sqrt{1 - x^2 - y^2}$, i.e. squaring both sides, $(1 - y)^2 < 1 - x^2 - y^2$ i.e. $x^2 < 2y - 2y^2$, i.e. $-\sqrt{2y - 2y^2} < x < \sqrt{2y - 2y^2}$. So we get:

$$\int_0^1 \int_{-\sqrt{2y-2y^2}}^{\sqrt{2y-2y^2}} \int_{1-y}^{\sqrt{1-x^2-y^2}} dz\, dx\, dy.$$

Bounds for y: either by observing that $x^2 < 2y - y^2$ has solutions iff $2y - y^2 > 0$, i.e. $0 < y < 1$, or by looking at picture where clearly leftmost point is on z-axis ($y = 0$) and rightmost point is at $y = 1$.

18.02 Lecture 26. – **Tue, Nov 13, 2007**

Spherical coordinates (ρ, ϕ, θ).

ρ = rho = distance to origin. $\phi = \varphi$ = phi = angle down from z-axis. θ = same as in cylindrical coordinates. Diagram drawn in space, and picture of 2D slice by vertical plane with z, r coordinates.

Formulas to remember: $z = \rho\cos\phi$, $r = \rho\sin\phi$ (so $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$).

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}$. The equation $\rho = a$ defines the sphere of radius a centered at 0.

On the surface of the sphere, ϕ is similar to *latitude*, except it's 0 at the north pole, $\pi/2$ on the equator, π at the south pole. θ is similar to *longitude*.

$\phi = \pi/4$ is a cone (asked using flash cards) ($z = r = \sqrt{x^2+y^2}$). $\phi = \pi/2$ is the xy-plane.

Volume element: $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

To understand this formula, first study surface area on sphere of radius a: picture shown of a "rectangle" corresponding to $\Delta\phi$, $\Delta\theta$, with sides = portion of circle of radius a, of length $a\Delta\phi$, and portion of circle of radius $r = a\sin\phi$, of length $r\Delta\theta = a\sin\phi\Delta\theta$. So $\Delta S \approx a^2 \sin\phi \, \Delta\phi\Delta\theta$, which gives the surface element $dS = a^2 \sin\phi \, d\phi d\theta$.

The volume element follows: for a small "box", $\Delta V = \Delta S \, \Delta\rho$, so $dV = d\rho \, dS = \rho^2 \sin\phi \, d\rho d\phi d\theta$.

Example: recall the complicated example at end of Friday's lecture (region sliced by a plane inside unit sphere). After rotating coordinate system, the question becomes: volume of the portion of unit sphere above the plane $z = 1/\sqrt{2}$? (picture drawn). This can be set up in cylindrical (left as exercise) or spherical coordinates.

For fixed ϕ, θ we are slicing our region by rays straight out of the origin; ρ ranges from its value on the plane $z = 1/\sqrt{2}$ to its value on the sphere $\rho = 1$. Spherical coordinate equation of the plane: $z = \rho\cos\phi = 1/\sqrt{2}$, so $\rho = \sec\phi/\sqrt{2}$. The volume is:

$$\int_0^{2\pi} \int_0^{\pi/4} \int_{\frac{1}{\sqrt{2}}\sec\phi}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

(Bound for ϕ explained by looking at a slice by vertical plane $\theta =$constant: the edge of the region is at $z = r = \frac{1}{\sqrt{2}}$).

Evaluation: not done. Final answer: $\dfrac{2\pi}{3} - \dfrac{5\pi}{6\sqrt{2}}$.

Application to gravitation.

Gravitational force exerted on mass m at origin by a mass ΔM at (x, y, z) (picture shown) is given by $|\vec{F}| = \dfrac{G\,\Delta M\, m}{\rho^2}$, $dir(\vec{F}) = \dfrac{\langle x, y, z\rangle}{\rho}$, i.e. $\vec{F} = \dfrac{G\,\Delta M\, m}{\rho^3}\langle x, y, z\rangle$. ($G$ = gravitational constant).

If instead of a point mass we have a solid with density δ, then we must integrate contributions to gravitational attraction from small pieces $\Delta M = \delta\,\Delta V$. So

$$\vec{F} = \iiint_R \frac{Gm\,\langle x, y, z\rangle}{\rho^3}\,\delta\, dV, \quad \text{i.e. } z\text{-component is} \quad F_z = Gm \iiint_R \frac{z}{\rho^3}\delta\, dV, \ \ldots$$

If we can set up to use symmetry, then F_z can be computed nicely using spherical coordinates.

General setup: place the mass m at the origin (so integrand is as above), and place the solid so that the z-axis is an axis of symmetry. Then $\vec{F} = \langle 0, 0, F_z\rangle$ by symmetry, and we have only one component to compute. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \iiint_R \frac{\rho\cos\phi}{\rho^3}\,\delta\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = Gm \iiint_R \delta\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta.$$

Example: Newton's theorem: the gravitational attraction of a spherical planet with uniform density δ is the same as that of the equivalent point mass at its center.

[[Setup: the sphere has radius a and is centered on the positive z-axis, tangent to xy-plane at the origin; the test mass is m at the origin. Then

$$F_z = Gm \iiint_R \frac{z}{\rho^3}\,\delta\,dV = Gm \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\phi} \delta\,\cos\phi\,\sin\phi\,d\rho\,d\phi\,d\theta = \cdots = \frac{4}{3}Gm\delta\,\pi a = \frac{GMm}{a^2}$$

where M = mass of the planet $= \frac{4}{3}\pi a^3\delta$. (The bounds for ρ and ϕ need to be explained carefully, by drawing a diagram of a vertical slice with z and r coordinate axes, and the inscribed right triangle with vertices the two poles of the sphere + a point on its surface, the hypothenuse is the diameter $2a$ and we get $\rho = 2a\cos\phi$ for the spherical coordinate equation of the sphere).]]

18.02 Lecture 27. – **Thu, Nov 15, 2007**

Handouts: PS10 solutions, PS11

Vector fields in space.

At every point in space, $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, where P, Q, R are functions of x, y, z.

Examples: force fields (gravitational force $\vec{F} = -c\langle x,y,z\rangle/\rho^3$; electric field $\mathbf{E}$, magnetic field $\mathbf{B}$); velocity fields (fluid flow, $\boldsymbol{v} = \boldsymbol{v}(x,y,z)$); gradient fields (e.g. temperature and pressure gradients).

Flux.

Recall: in 2D, flux of a vector field $\vec{F}$ across a curve $C = \int_C \vec{F}\cdot\hat{\boldsymbol{n}}\,ds$.

In 3D, flux of a vector field is a *double* integral: flux through a *surface*, not a curve!

$\vec{F}$ vector field, S surface, $\hat{\boldsymbol{n}}$ unit normal vector: Flux $= \iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$.

Notation: $d\vec{S} = \hat{\boldsymbol{n}}\,dS$. (We'll see that $d\vec{S}$ is often easier to compute than $\hat{\boldsymbol{n}}$ and dS).

Remark: there are 2 choices for $\hat{\boldsymbol{n}}$ (choose which way is counted positively!)

Geometric interpretation of flux:

As in 2D, if $\vec{F}$ = velocity of a fluid flow, then flux = flow per unit time across S.

Cut S into small pieces, then over each small piece: what passes through ΔS in unit time is the contents of a parallelepiped with base ΔS and third side given by $\vec{F}$.

Volume of box = base × height $= (\vec{F}\cdot\hat{\boldsymbol{n}})\,\Delta S$.

• Examples:

1) $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ through sphere of radius a centered at 0.

$\hat{\boldsymbol{n}} = \frac{1}{a}\langle x,y,z\rangle$ (other choice: $-\frac{1}{a}\langle x,y,z\rangle$; traditionally choose $\hat{\boldsymbol{n}}$ pointing out).

$\vec{F}\cdot\hat{\boldsymbol{n}} = \langle x,y,z\rangle\cdot\hat{\boldsymbol{n}} = \frac{1}{a}(x^2+y^2+z^2) = a$, so $\iint_S \vec{F}\cdot\hat{\boldsymbol{n}}dS = \iint_S a\,dS = a\,(4\pi a^2)$.

2) Same sphere, $\vec{H} = z\hat{\boldsymbol{k}}$: $\vec{H}\cdot\hat{\boldsymbol{n}} = \frac{z^2}{a}$.

$$\iint_S \vec{H}\cdot d\vec{S} = \iint_S \frac{z^2}{a}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{a^2\cos^2\phi}{a}\,a^2\sin\phi\,d\phi d\theta = 2\pi a^3 \int_0^{\pi}\cos^2\phi\sin\phi\,d\phi = \frac{4}{3}\pi a^3.$$

Setup. Sometimes we have an easy geometric argument, but in general we must compute the surface integral. The setup requires the use of two parameters to describe the surface, and $\vec{F}\cdot\hat{\boldsymbol{n}}\,dS$ must be expressed in terms of them. How to do this depends on the type of surface. For now, formulas to remember:

0) plane $z = a$ parallel to xy-plane: $\hat{\boldsymbol{n}} = \pm\hat{\boldsymbol{k}}$, $dS = dx\,dy$. (similarly for planes // xz or yz-plane).

1) sphere of radius a centered at origin: use ϕ, θ (substitute $\rho = a$ for evaluation); $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, z\rangle$, $dS = a^2\sin\phi\,d\phi\,d\theta$.

2) cylinder of radius a centered on z-axis: use z, θ (substitute $r = a$ for evaluation): $\hat{\boldsymbol{n}}$ is radially out in horizontal directions away from z-axis, i.e. $\hat{\boldsymbol{n}} = \frac{1}{a}\langle x, y, 0\rangle$; and $dS = a\,dz\,d\theta$ (explained by drawing a picture of a "rectangular" piece of cylinder, $\Delta S = (\Delta z)\,(a\Delta\theta)$).

3) graph $z = f(x, y)$: use x, y (substitute $z = f(x, y)$). We'll see on Friday that $\hat{\boldsymbol{n}}$ and dS separately are complicated, but $\hat{\boldsymbol{n}}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy$.

18.02 Lecture 28. – Fri, Nov 16, 2007

Last time, we defined the flux of $\vec{F}$ through surface S as $\iint \vec{F}\cdot\hat{\boldsymbol{n}}\,dS$, and saw how to set up in various cases. Continue with more:

Flux through a graph. If S is the graph of some function $z = f(x, y)$ over a region R of xy-plane: use x and y as variables. Contribution of a small piece of S to flux integral?

Consider portion of S lying above a small rectangle $\Delta x\,\Delta y$ in xy-plane. In linear approximation it is a parallelogram. (picture shown)

The vertices are $(x, y, f(x, y))$; $(x + \Delta x, y, f(x + \Delta x, y))$; $(x, y + \Delta y, f(x, y + \Delta y))$; etc. Linear approximation: $f(x + \Delta x, y) \simeq f(x, y) + \Delta x\, f_x(x, y)$, and $f(x, y + \Delta y) \simeq f(x, y) + \Delta y\, f_y(x, y)$.

So the sides of the parallelogram are $\langle \Delta x, 0, \Delta x\, f_x\rangle$ and $\langle 0, \Delta y, \Delta y\, f_y\rangle$, and

$$\Delta\vec{S} = (\Delta x\,\langle 1, 0, f_x\rangle) \times (\Delta y\,\langle 0, 1, f_y\rangle) = \Delta x\Delta y \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1\rangle\Delta x\Delta y.$$

So $d\vec{S} = \pm\langle -f_x, -f_y, 1\rangle dx\,dy$.

(From this we can get $\hat{\boldsymbol{n}} = \mathrm{dir}(d\vec{S}) = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}$ and $dS = |d\vec{S}| = \sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$. The conversion factor $\sqrt{\cdots}$ between dS and dA relates area on S to area of projection in xy-plane.)

• Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ through S = portion of paraboloid $z = x^2 + y^2$ above unit disk, oriented with normal pointing up (and into the paraboloid): geometrically flux should be > 0 (asked using flashcards). We have $\hat{\boldsymbol{n}}\,dS = \langle -2x, -2y, 1\rangle\,dx\,dy$, and

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S z\,dx\,dy = \iint_S (x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^2\,r\,dr\,d\theta = \pi/2.$$

Parametric surfaces. If we can describe S by parametric equations $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ (i.e. $\vec{r} = \vec{r}(u, v)$), then we can set up flux integrals using variables u, v. To find $d\vec{S}$, consider a small portion of surface corresponding to changes Δu and Δv in parameters, it's a parallelogram with sides $\vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx (\partial\vec{r}/\partial u)\,\Delta u$ and $(\partial\vec{r}/\partial v)\,\Delta v$, so

$$\Delta\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\Delta u\right)\times\left(\frac{\partial\vec{r}}{\partial v}\Delta v\right), \qquad d\vec{S} = \pm\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)du\,dv.$$

(This generalizes all formulas previously seen; but won't be needed on exam).

Implicit surfaces: If we have an implicitly defined surface $g(x,y,z)=0$, then we have a (non-unit) normal vector $\mathbf{N} = \nabla g$. (similarly for a slanted plane, from equation $ax+by+cz=d$ we get $\mathbf{N} = \langle a,b,c\rangle$).

Unit normal $\hat{\boldsymbol{n}} = \pm\mathbf{N}/|\mathbf{N}|$; surface element $\Delta S = ?$ Look at projection to xy-plane: $\Delta A = \Delta S\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}}/|\mathbf{N}|)\,\Delta S$ (where α = angle between slanted surface element and horizontal: projection shrinks one direction by factor $\cos\alpha = (\mathbf{N}\cdot\hat{\boldsymbol{k}})/|\mathbf{N}|$, preserves the other).

Hence $dS = \dfrac{|\mathbf{N}|}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dA$, and $\hat{\boldsymbol{n}}\,dS = \dfrac{|\mathbf{N}|\hat{\boldsymbol{n}}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy = \pm\dfrac{\mathbf{N}}{\mathbf{N}\cdot\hat{\boldsymbol{k}}}\,dx\,dy.$

(In fact the first formula should be $dS = \dfrac{|\mathbf{N}|}{|\mathbf{N}\cdot\hat{\boldsymbol{k}}|}\,dA$, I forgot the absolute value).

Note: if S is vertical then the denominator is zero, can't project to xy-plane any more (but one could project e.g. to the xz-plane).

Example: if S is a graph, $g(x,y,z) = z - f(x,y) = 0$, then $\mathbf{N} = \langle g_x, g_y, g_z\rangle = \langle -f_x, -f_y, 1\rangle$, $\mathbf{N}\cdot\hat{\boldsymbol{k}} = 1$, so we recover the formula $d\vec{S} = \langle -f_x, -f_y, 1\rangle dx\,dy$ seen before.

Divergence theorem. ("Gauss-Green theorem") – 3D analogue of Green theorem for flux.

If S is a closed surface bounding a region D, with normal pointing outwards, and $\vec{F}$ vector field defined and differentiable over all of D, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\,dV, \quad \text{where} \quad \operatorname{div}(P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}} + R\hat{\boldsymbol{k}}) = P_x + Q_y + R_z.$$

Example: flux of $\vec{F} = z\hat{\boldsymbol{k}}$ out of sphere of radius a (seen Thursday): $\operatorname{div}\vec{F} = 0+0+1 = 1$, so $\iint_S \vec{F}\cdot d\vec{S} = 3\,\mathrm{vol}(D) = 4\pi a^3/3$.

Physical interpretation (mentioned very quickly and verbally only): $\operatorname{div}\vec{F}$ = source rate = flux generated per unit volume. So the divergence theorem says: the flux outwards through S (net amount leaving D per unit time) is equal to the total amount of sources in D.

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Recall statement of divergence theorem: $\iint_S \mathbf{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\mathbf{F}\,dV$.

Del operator. $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$ (symbolic notation!)

$\nabla f = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z\rangle$ = gradient.

$\nabla\cdot\mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z$ = divergence.

Physical interpretation. $\operatorname{div}\mathbf{F}$ = source rate = flux generated per unit volume. Imagine an incompressible fluid flow (i.e. a given mass occupies a fixed volume) with velocity $\mathbf{F}$, then $\iiint_D \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ = flux through S is the net amount leaving D per unit time = total amount of sources (minus sinks) in D.

Proof of divergence theorem. To show $\iint_S \langle P,Q,R\rangle\cdot d\vec{S} = \iiint_D (P_x+Q_y+R_z)\,dV$, we can separate into sum over components and just show $\iint_S R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iiint_D R_z\,dV$ & same for P and Q.

If the region D is vertically simple, i.e. top and bottom surfaces are graphs, $z_1(x,y)\le z\le z_2(x,y)$, with (x,y) in some region U of xy-plane: r.h.s. is

$$\iiint_D R_z\,dV = \iint_U \Big(\int_{z_1(x,y)}^{z_2(x,y)} R_z\,dz\Big)\,dx\,dy = \iint_U (R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\,dx\,dy.$$

Flux through top: $d\vec{S} = \langle -\partial z_2/\partial x, -\partial z_2/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{top}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint R(x,y,z_2(x,y))\,dx\,dy$.

Bottom: $d\vec{S} = -\langle -\partial z_1/\partial x, -\partial z_1/\partial y, 1\rangle dx\,dy$, so $\iint_{\text{bottom}} R\hat{\boldsymbol{k}}\cdot d\vec{S} = \iint -R(x,y,z_1(x,y))\,dx\,dy$.

Sides: sides are vertical, $\hat{\mathbf{n}}$ is horizontal, $\mathbf{F}$ is vertical so flux = 0.

Given any region D, decompose it into vertically simple pieces (illustrated for a donut). Then $\iiint_D$ = sum of pieces (clear), and $\iint_S$ = sum of pieces since the internal boundaries cancel each other.

Diffusion equation: governs motion of smoke in (immobile) air (dye in solution, ...)

The equation is: $\dfrac{\partial u}{\partial t} = k\nabla^2 u = k\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right)$,

where $u(x,y,z,t)$ = concentration of smoke; we'll also introduce $\mathbf{F}$ = flow of the smoke. It's also the heat equation (u = temperature).

Equation uses two inputs:

1) Physics: $\mathbf{F} = -k\nabla u$ (flow goes from highest to lowest concentration, faster if concentration changes more abruptly).

2) Flux and quantity of smoke are related: if D bounded by closed surface S, then $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\frac{d}{dt}\iiint_D u\,dV$. (flux out of D = - variation of total amount of smoke inside D)

By differentiation under integral sign, the r.h.s. is $-\iiint_D \frac{\partial}{\partial t}u\,dV$ (This can be explained in terms of integral as a sum of $u(x_i,y_i,z_i,t)\Delta V_i$ and derivative of sum is sum of derivatives) and by divergence theorem the l.h.s. is $\iiint_D \operatorname{div}\mathbf{F}\,dV$. Dividing by volume of D, we get

$$-\frac{1}{vol(D)}\iiint_D \frac{\partial u}{\partial t}\,dV = \frac{1}{vol(D)}\iiint_D \operatorname{div}\mathbf{F}\,dV.$$

Same average values over any region; taking limit as D shrinks to a point, get $\partial u/\partial t = -\operatorname{div}\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.

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Handouts: practice exams 4A and 4B.

Clarification from end of last lecture: we derived the diffusion equation from 2 inputs. $u =$ concentration, $\mathbf{F} =$ flow, satisfy:

1) from physics: $\mathbf{F} = -k\nabla u$,

2) from divergence theorem: $\partial u/\partial t = -\operatorname{div}\mathbf{F}$.

Combining, we get the diffusion equation: $\partial u/\partial t = -\operatorname{div}\mathbf{F} = +k\operatorname{div}(\nabla u) = k\nabla^2 u$.

Line integrals in space.

Force field $\mathbf{F} = \langle P, Q, R\rangle$, curve C in space, $d\vec{r} = \langle dx, dy, dz\rangle$

$\Rightarrow$ Work $= \int_C \mathbf{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$.

Example: $\mathbf{F} = \langle yz, xz, xy\rangle$. C: $x = t^3$, $y = t^2$, $z = t$. $0 \le t \le 1$. Then $dx = 3t^2dt$, $dy = 2tdt$, $dz = dt$ and substitute:

$$\int_C \mathbf{F}\cdot d\vec{r} = \int_C yzdx + xzdy + xydz = \int_0^1 6t^5 dt = 1$$

(In general, express (x, y, z) in terms of a *single* parameter: 1 degree of freedom)

Same $\mathbf{F}$, curve $C' =$ segments from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$. In the xy-plane, $z = 0 \implies \mathbf{F} = xy\hat{\boldsymbol{k}}$, so $\mathbf{F}\cdot d\vec{r} = 0$, no work. For the last segment, $x = y = 1$, $dx = dy = 0$, so $\mathbf{F} = \langle z, z, 1\rangle$ and $d\vec{r} = \langle 0, 0, dz\rangle$. We get $\int_0^1 1\,dz = 1$.

Both give the same answer because $\mathbf{F}$ is conservative, in fact $\mathbf{F} = \nabla(xyz)$.

Recall the fundamental theorem of calculus for line integrals:

$$\int_{P_0}^{P_1} \nabla f\cdot d\vec{r} = f(P_1) - f(P_0).$$

Gradient fields.

$\mathbf{F} = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$?

Then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$, so $P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$.

Theorem: $\mathbf{F}$ is a gradient field if and only if these equalities hold (assuming defined in whole space or simply connected region)

Example: for which a, b is $axy\hat{\boldsymbol{\imath}} + (x^2 + z^3)\hat{\boldsymbol{\jmath}} + (byz^2 - 4z^3)\hat{\boldsymbol{k}}$ a gradient field?

$P_y = ax = 2x = Q_x$ so $a = 2$; $P_z = 0 = 0 = R_x$; $Q_z = 3z^2 = bz^2 = R_y$ so $b = 3$.

Systematic method to find a potential: (carried out on above example)

$f_x = 2xy$, $f_y = x^2 + z^3$, $f_z = 3yz^2 - 4z^3$:

$f_x = 2xy \Rightarrow f(x, y, z) = x^2y + g(y, z)$.

$f_y = x^2 + g_y = x^2 + z^3 \Rightarrow g_y = z^3 \Rightarrow g(y, z) = yz^3 + h(z)$, and $f = x^2y + yz^3 + h(z)$.

$f_z = 3yz^2 + h'(z) = 3yz^2 - 4z^3 \Rightarrow h'(z) = -4z^3 \Rightarrow h(z) = -z^4 + c$, and $f = x^2y + yz^3 - z^4 + c$.

Other method: $f(x_1, y_1, z_1) = f(0,0,0) + \int_{P_0}^{P_1} \mathbf{F}\cdot d\vec{r}$ (use a curve that gives an easy computation, e.g. 3 segments parallel to axes).

Curl: encodes by how much $\mathbf{F}$ fails to be conservative.

$\operatorname{curl}\langle P, Q, R\rangle = (R_y - Q_z)\hat{\boldsymbol{\imath}} + (P_z - R_x)\hat{\boldsymbol{\jmath}} + (Q_x - P_y)\hat{\boldsymbol{k}}$.

How to remember the formula? Use the del operator $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$.

Recall from last week that $\nabla \cdot \mathbf{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle P, Q, R\rangle = P_x + Q_y + R_z = \operatorname{div} \mathbf{F}$.

Now we have: $\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \operatorname{curl} \mathbf{F}$.

Interpretation of curl for velocity fields: curl measures *angular velocity.*

Example: rotation around z-axis at constant angular velocity ω (trajectories are circles centered on z-axis): $\boldsymbol{v} = \langle -\omega y, \omega x, 0\rangle$.

Then $\nabla \times \boldsymbol{v} = ... = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + (\omega + \omega)\hat{\boldsymbol{k}} = 2\omega\hat{\boldsymbol{k}}$. So length of curl = twice angular velocity, and direction = axis of rotation.

A general motion can be complicated, but decomposes into various effects.

• curl measures the *rotation* component of a complex motion.

18.02 Lecture 31. – **Thu, Nov 29, 2007**

Handouts: PS11 solutions, PS12.

Stokes' theorem is the 3D analogue of Green's theorem for work (in the same sense as the divergence theorem is the 3D analogue of Green for flux).

Recall $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$.

Stokes' theorem: if C is a *closed curve*, and S *any* surface bounded by C, then

$$\oint_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Orientation: compatibility of an orientation of C with an orientation of S (changing orientation changes sign on both sides of Stokes).

Rule: if I walk along C in positive direction, with S to my left, then $\hat{\mathbf{n}}$ is pointing up. (Various examples shown.)

Another formulation (right-hand rule): if thumb points along C (1-D object), index finger towards S (2-D object), then middle finger points along $\hat{\mathbf{n}}$ (3-D object).

More examples shown.

Example: Stokes vs. Green. If S is a portion of xy-plane bounded by a curve C counterclockwise, then $\oint_C \mathbf{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$, by Green this is equal to $\iint_S (Q_x - P_y)\,dx\,dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\boldsymbol{k}}\,dx\,dy = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,dS$, so Green and Stokes say the same thing in this example.

Remark. In Stokes' theorem we are free to choose any surface S bounded by C! (e.g. if C = circle, S could be a disk, a hemisphere, a cone, ...)

"Proof" of Stokes.

1) if C and S are in the xy-plane then the statement follows from Green.

2) if C and S are in an arbitrary plane: this also reduces to Green in the given plane. Green/Stokes works in any plane because of *geometric invariance* of work, curl and flux under rotations of space. They can be defined in purely geometric terms so as not to depend on the coordinate system (x, y, z); equivalently, we can choose coordinates (u, v, w) adapted to the given plane, and work

with those coordinates, the expressions of work, curl, flux will be the familiar ones replacing x, y, z with u, v, w.

3) in general, we can decompose S into small pieces, each piece is nearly flat (slanted plane); on each piece we have approximately work = flux by Green's theorem. When adding pieces, the line integrals over the inner boundaries cancel each other and we get the line integral over C; the flux integrals add up to flux through S.

Example: verify Stokes for $\mathbf{F} = z\hat{\imath} + x\hat{\jmath} + y\hat{\boldsymbol{k}}$, C = unit circle in xy-plane (counterclockwise), S = piece of paraboloid $z = 1 - x^2 - y^2$.

Direct calculation: $x = \cos\theta$, $y = \sin\theta$, $z = 0$, so $\oint_C \mathbf{F}\cdot d\vec{r} = \int_C z\,dx + x\,dy + y\,dz = \oint_C x\,dy = \int_0^{2\pi} \cos^2\theta\,d\theta = \pi$.

By Stokes: $\operatorname{curl}\mathbf{F} = \langle 1,1,1\rangle$, and $\hat{\mathbf{n}}\,dS = \langle -f_x, -f_y, 1\rangle dx\,dy = \langle 2x, 2y, 1\rangle\,dx\,dy$.

$$\iint_S (\nabla\times\mathbf{F})\cdot d\vec{S} = \iint \langle (2x+2y+1)\,dx\,dy = \iint 1\,dx\,dy = \text{area(disk)} = \pi.$$

($\iint x\,dx\,dy = 0$ by symmetry and similarly for y).

18.02 Lecture 32. – **Fri, Nov 30, 2007**

Stokes and path independence.

Definition: a region is simply connected if every closed loop C inside it bounds some surface S inside it.

Example: the complement of the z-axis is not simply connected (shown by considering a loop encircling the z-axis); the complement of the origin is simply connected.

Topology: uses these considerations to classify for example surfaces in space: e.g., the mathematical proof that a sphere and a torus are "different" surfaces is that the sphere is simply connected, the torus isn't (in fact it has two "independent" loops that don't bound).

Recall: if $\mathbf{F}$ is a gradient field then $\operatorname{curl}(\mathbf{F}) = 0$.

Conversely, Theorem: if $\nabla\times\mathbf{F} = 0$ in a *simply connected* region then $\mathbf{F}$ is conservative (so $\int \mathbf{F}\cdot d\vec{r}$ is path-independent and we can find a potential).

Proof: Assume R simply connected, $\nabla\times\mathbf{F} = 0$, and consider two curves C_1 and C_2 with same end points. Then $C = C_1 - C_2$ is a closed curve so bounds some S; $\int_{C_1}\mathbf{F}\cdot d\vec{r} - \int_{C_2}\mathbf{F}\cdot d\vec{r} = \oint_C \mathbf{F}\cdot d\vec{r} = \iint_S \nabla\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 0$.

Orientability. We can apply Stokes' theorem to any surface S bounded by C... provided that it has a well-defined normal vector! Counterexample shown: the Möbius strip. It's a one-sided surface, so we can't compute flux through it (no possible consistent choice of orientation of $\hat{\mathbf{n}}$). Instead, if we want to apply Stokes to the boundary curve C, we must find a two-sided surface with boundary C. (pictures shown).

Stokes and surface independence.

In Stokes we can choose any S bounded by C: so if a same C bounds two surfaces S_1, S_2, then $\oint_C \mathbf{F}\cdot d\vec{r} = \iint_{S_1} \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_2} \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$? Can we prove directly that the two flux integrals are equal?

Answer: change orientation of S_2, then $S = S_1 - S_2$ is a closed surface with $\hat{\mathbf{n}}$ outwards; so we can apply the divergence theorem: $\iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_D \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV$. But $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$, always. (Checked by calculating in terms of components of $\mathbf{F}$; also, symbolically: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$, much like $u \cdot (u \times v) = 0$ for genuine vectors).

Review for Exam 4.

We've seen three types of integrals, with different ways of evaluating:

1) $\iiint f\, dV$ in rect., cyl., spherical coordinates (I re-explained the general setup and the formulas for dV); applications: center of mass, moment of inertia, gravitational attraction.

2) surface integrals, flux. Setting up $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, by knowing formulas for $\hat{\mathbf{n}}\, dS$.

We have seen: planes parallel to coordinate planes (e.g. yz-plane: $\hat{\mathbf{n}} = \pm\hat{\boldsymbol{\imath}}$, $dS = dy\, dz$); spheres and cylinders ($\hat{\mathbf{n}}$ = straight out/in from center or axis; $dS = a\, dz\, d\theta$ for cylinders, $a^2 \sin\phi\, d\phi d\theta$ for spheres); if we can express $z = f(x, y)$, $\hat{\mathbf{n}}\, dS = \pm\langle -f_x, -f_y, 1\rangle dx\, dy$ (recall $\langle \dots \rangle$ is not $\hat{\mathbf{n}}$ and $dx\, dy$ is not dS); if S has a given normal vector $\vec{N}$ (e.g. if S is given by $g(x, y, z) = 0$), $\hat{\mathbf{n}}\, dS = \pm\vec{N}/(\vec{N} \cdot \hat{\boldsymbol{k}})\, dx\, dy$.

3) line integrals $\int_C \mathbf{F} \cdot d\vec{r}$ ($=\int_C P\, dx + Q\, dy + R\, dz$), evaluate by parameterizing C and expressing in terms of a single variable.

While these various types of integrals are completely different in terms of interpretation and method of evaluation, we've seen some theorems that establish bridges between them:

a) ($\iint$ vs $\iiint$) divergence theorem: S closed surface, $\hat{\mathbf{n}}$ outwards, then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_D (\operatorname{div} \mathbf{F})\, dV$.

b) ($\int$ vs $\iint$) Stokes' theorem: C closed curve bounding S compatibly oriented, then $\int_C \mathbf{F} \cdot d\vec{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$.

Both sides of these theorems are integrals of the types discussed above, and are evaluated by the usual methods! (even if the integrand happens to be a div or a curl).

In fact, another conceptually similar bridge exists between no integral at all and line integral: the fundamental theorem of calculus, $f(P_1) - f(P_0) = \int_C \nabla f \cdot d\vec{r}$.

One more topic: given $\mathbf{F}$ with $\operatorname{curl} \mathbf{F} = 0$, finding a potential function.

18.02 Lecture 33. – Thu, Dec 6, 2007

Handouts: PS12 solutions; exam 4 solutions; review sheet and practice final.

Applications of div and curl to physics.

Recall: curl of velocity field = 2· angular velocity vector (of the rotation component of motion).

E.g., for uniform rotation about z-axis, $\boldsymbol{v} = \omega(-y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}})$, and $\nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{k}}$.

Curl singles out the rotation component of motion (while div singles out the stretching component).

Interpretation of curl for force fields.

If we have a solid in a force field (or rather an acceleration field!) $\mathbf{F}$ such that the force exerted on Δm at (x, y, z) is $\mathbf{F}(x, y, z)\,\Delta m$: recall the *torque* of the force about the origin is defined as $\tau = \vec{r} \times \mathbf{F}$ (for the entire solid, take $\iiint \ldots\, \delta\, dV$), and measures how $\mathbf{F}$ imparts rotation motion.

For translation motion: $\dfrac{\text{Force}}{\text{Mass}} = \text{acceleration} = \dfrac{d}{dt}(\text{velocity})$.

For rotation effects: $\dfrac{\text{Torque}}{\text{Moment of inertia}} = \text{angular acceleration} = \dfrac{d}{dt}(\text{angular velocity})$.

Hence: $\text{curl}(\dfrac{\text{Force}}{\text{Mass}}) = 2\,\dfrac{\text{Torque}}{\text{Moment of inertia}}$.

Consequence: if $\mathbf{F}$ derives from a potential, then $\nabla\times\mathbf{F} = \nabla\times(\nabla f) = 0$, so $\mathbf{F}$ does not induce any rotation motion. E.g., gravitational attraction by itself does not affect Earth's rotation. (not strictly true: actually Earth is deformable; similarly, friction and tidal effects due to Earth's gravitational attraction explain why the Moon's rotation and revolution around Earth are synchronous).

Div and curl of electrical field. – part of Maxwell's equations for electromagnetic fields.

1) Gauss-Coulomb law: $\nabla \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$ (ρ = charge density, ϵ_0 = physical constant).

By divergence theorem, can reformulate as: $\displaystyle\iint_S \vec{E} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \vec{E}\, dV = \frac{Q}{\epsilon_0}$, where Q = total charge inside the closed surface S.

This formula tells how charges influence the electric field; e.g., it governs the relation between voltage between the two plates of a capacitor and its electric charge.

2) Faraday's law: $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ($\vec{B}$= magnetic field).

So in presence of a varying magnetic field, $\vec{E}$ is no longer conservative: if we have a closed loop of wire, we get a non-zero voltage ("induction" effect). By Stokes, $\displaystyle\oint_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot \hat{\mathbf{n}}\, dS$.

This principle is used e.g. in transformers in power adapters: AC runs through a wire looped around a cylinder, which creates an alternating magnetic field; the flux of this magnetic field through another output wire loop creates an output voltage between its ends.

There are two more Maxwell equations, governing div and curl of $\vec{B}$: $\nabla \cdot \vec{B} = 0$, and $\nabla \times \vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\dfrac{\partial \vec{E}}{\partial t}$ (where $\vec{J}$ = current density).