THEORETICAL MECHANICS

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PREFACE

THE study of Mechanics as presented in this volume is founded upon a course in mathematics extending through the Calculus. It is assumed, moreover, that the student has already become familiar with the fundamental ideas of force, energy, and work through such preliminary courses as are included in textbooks on General Physics. In short, this volume presents the subject of Mechanics in that relation to other mathematical subjects which has become established in the curricula of the technical schools of this country. It should be emphasized, however, that the volume includes, for purposes of review, a discussion of the fundamental notions and many simple exercises involving these notions.

Attention may be called to the arrangement in the text. This arrangement is founded upon experience in teaching the subject for many years in the Sheffield Scientific School of Yale University. In 1903 Professor E. R. Hedrick prepared a mimeographed text which followed the conventional arrangement of treating statics first. This text was used for one year. It then developed that an obvious disadvantage existed in not taking up directly upon the conclusion of the study of the Integral Calculus the calculation of the integrals of Mechanics involving centers of gravity and moments of inertia. The point was that this formal integration out of the way, the continuous study of Mechanics proper need not afterwards be interrupted. Acting upon this conviction, the present text was prepared essentially as here published in 1907, and has since that time been used in mimeographed form. The general plan of the arrangement is that a *single* problem may at any one time be under discussion. Thus, when the question of energy of rotation is solved, the appearance of the moment of inertia integral presents no complication. This has been disposed of already. Similarly, the equations of motion presenting themselves as solutions of the force equations have been previously discussed. Another feature is the departure from convention by arranging types of motion under the corresponding fields of force. In this way it is made clear that the emphasis is to be laid upon the force and velocity of projection.

In the case of a book which, like the present volume, has been long in the making, it is difficult to record definite acknowledgments of aid and indebtedness. There are included in the text many problems suggested by past and present members of the mathematical department of the Sheffield Scientific School. Further, the text has been the subject of discussion at frequent departmental conferences, and for all suggestions received on these occasions the authors gratefully here record their thanks. The diagrams were skillfully prepared by Mr. S. J. Berard of the department of mechanical engineering.

New Haven, Connecticut

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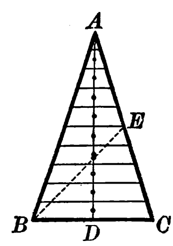
THEORETICAL MECHANICS

CHAPTER I

MOMENTS OF MASS AND INERTIA

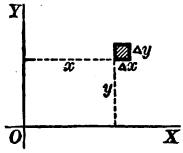
1. Center of gravity. It is shown in a subsequent chapter (Art. 108) that the influence of the weight of a solid in all questions in mechanics is precisely that of a force equal to the weight applied at a point called the center of gravity* of the solid. It is assumed that the student is familiar with simple facts concerning the center of gravity. For example, the center of gravity of a straight line (or thin straight rod) is its middle point. Again, the center of gravity of a triangle is the point of intersection of the medians.



This statement may be proved as follows. Divide the triangle into thin strips by lines parallel to one side. Draw the median AD. The center of gravity of each strip lies on AD. Hence the center of gravity of the triangle lies on AD. Similarly, the center of gravity lies on the median BE. This establishes the statement.

The formulas for the center of gravity introduced in the following sections involve magnitudes called the *moments of area* or *moments of mass*. The student is asked to accept these formulas as *definitions*. Later, in discussing weight the formulas appear as giving the center of gravity.

2. Moment of area. Consider an element of any plane area



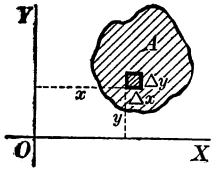
$$\Delta A = \Delta x \Delta y,$$

at the point (x, y). Then the products

$$x\Delta A,\ y\Delta A$$

are called the *moments* of ΔA with respect to the axes OY and OX, respectively.

* Called also center of mass.



This definition is extended to any finite area A in the usual way by summation and taking limits. Hence if M_x and M_y denote the moments of area for the area A with respect to the axes OX and OY, respectively, then

$$\text{(I)} \qquad \begin{aligned} M_x &= \int\int y\,dA = \int\int y\,dx\,dy = \left[\begin{array}{l}\text{limit} \\ \Delta x = 0, \\ \Delta y = 0,\end{array} \sum\sum y\,\Delta x\,\Delta y\right], \\ M_y &= \int\int x\,dA = \int\int x\,dx\,dy = \left[\begin{array}{l}\text{limit} \\ \Delta x = 0, \\ \Delta y = 0,\end{array} \sum\sum x\,\Delta x\,\Delta y\right]. \end{aligned}$$

The *Center of Gravity* of any given area A is the point $(\bar{x}, \bar{y})$ given by the quotients

$$\text{(II)} \qquad \bar{x} = \frac{M_y}{\text{area}} = \frac{\int\int x\,dA}{\text{area}}, \quad \bar{y} = \frac{M_x}{\text{area}} = \frac{\int\int y\,dA}{\text{area}}.$$

In these formulas x and y are the coördinates of any point *within the area.*

The common denominator (the area of the given figure) must be found, if not otherwise known, by integration; that is,

$$\text{Area} = \int\int dx\,dy.$$

In working out examples using (II) calculate the moments $\int\int x\,dA$ and $\int\int y\,dA$, first, and then divide by the area itself.

Dimensions. Whenever it is desirable to express numerically the magnitude of a physical quantity, we do so by choosing a unit of that quantity. It is convenient, when possible, to choose the units of different kinds of quantities so that some of them depend upon others. The units which are chosen arbitrarily are called *fundamental.* The *derived* units are those which are so defined as to depend upon the fundamental units. In mechanics it is customary to choose as fundamental the units of length, mass, and time, and all other units are made to depend upon these. For example, if the unit of length is the foot, the unit of area is defined as the area of a square whose sides are one foot in length. The relation between the derived unit of area and the fundamental unit of length is then expressed by the dimensional equation,

$$\text{Area} = \text{length}^2.$$

Similarly, the dimensional relation between the derived unit of volume and the fundamental unit of length is

$$\text{Volume} = \text{length}^3.$$

The dimensional equation is a concise way of expressing the relation between the units of different quantities, and is not to be interpreted as an ordinary algebraic equation.

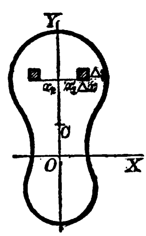
Moment of area has been defined as the product of area by distance, and hence the unit of moment of area is of the third degree in the unit of length.

$$\text{Moment of area} = \text{area} \times \text{length} = \text{length}^3.$$

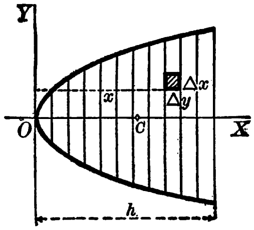
The fact that every term of an equation involving physical quantities must be of the same degree in the fundamental units furnishes a useful check in the problems of mechanics. For example, in (II) $\bar{x}$ is of the first degree in the unit of length, and hence the second member of the first equation must also be of the first degree. This is easily verified, since the dimensional relation gives

$$\frac{M_y}{\text{area}} = \frac{\text{length}^3}{\text{length}^2} = \text{length}.$$

3. Symmetry. The center of gravity will lie upon any axis of symmetry which the figure may possess. For example, if OY is such an axis, we may divide the figure into the equal elements $\Delta x \Delta y$ and sum up, taking two symmetrical pairs at a time. Then the sum of the moments with respect to OY for two such pairs, that is, $x_1 \Delta x \Delta y + x_2 \Delta x \Delta y$, will vanish, since $x_1 = -x_2$. Hence the moment with respect to OY, that is,



$$M_y = \int\int x dA, \text{ also vanishes, and } \bar{x} = 0.$$



Illustrative Example. Find the center of gravity of the area bounded by $y^2 = 2\,px$ and $x = h$.

Solution. Evidently $\bar{y} = 0$.

Calculate the moment of area with respect to OY This is, by (I),

$$M_y = \int\int xdxdy = \int_0^h x \int_{-\sqrt{2px}}^{+\sqrt{2px}} dydx = 2\sqrt{2p}\int_0^h x^{\frac{3}{2}} dx = \tfrac{4}{5}\sqrt{2p}\, h^{\frac{5}{2}}.$$

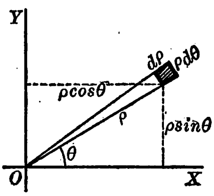
Next find the area. This is

$$\int_0^h \int_{-\sqrt{2px}}^{+\sqrt{2px}} dydx = \tfrac{4}{3}\sqrt{2p}\, h^{\frac{3}{2}}.$$

$$\therefore\ \bar{x} = \tfrac{3}{5} h. \qquad \text{by (II).}$$

PROBLEMS

Note. If the equation of the curve is given in polar coördinates (ρ, θ), place in (I) and (II) $dA = \rho d\rho d\theta$, $x = \rho \cos \theta$, $y = \rho \sin \theta$.



1. Find the center of gravity of the triangle bounded by the lines $y = mx$, $y = 0$, $x = a$. *Ans.* $\bar{x} = \frac{2}{3} a$, $\bar{y} = \frac{ma}{3}$.

2. Find the center of gravity of the triangle bounded by the lines $y = mx$, $y = -mx$, $y = b$. *Ans.* $\bar{x} = 0$, $\bar{y} = \frac{2}{3} b$.

3. Find the center of gravity (1) of a quarter of a circle in the first quadrant; (2) of one sixth of a circle, supposing the x-axis to be an axis of symmetry. *Ans.* (1) $\bar{x} = \bar{y} = \frac{4a}{3\pi}$; (2) $\bar{x} = \frac{2a}{\pi}$, $\bar{y} = 0$.

4. Find the center of gravity of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
Ans. $\bar{x} = \frac{4a}{3\pi}$, $\bar{y} = \frac{4b}{3\pi}$.

5. Find the center of gravity for the area bounded by $y^2 = 4ax$, $y = 0$, $x = b$.
Ans. $\bar{x} = \frac{3}{5} b$, $\bar{y} = \frac{3}{4}\sqrt{ab}$.

6. Find the center of gravity of the area bounded by $y^2 = 4ax$, $x = 0$, $y = b$.
Ans. $\bar{x} = \frac{3b^2}{40a}$, $\bar{y} = \frac{3}{4} b$.

7. Find the center of gravity of the area bounded by the semicubical parabola $ay^2 = x^3$ and $x = a$. *Ans.* $\bar{x} = \frac{5}{7} a$.

8. Find the center of gravity of the area bounded by $y = a \sin \frac{x}{a}$ and the x-axis between $x = 0$ and $x = a\pi$.
Ans. $\bar{x} = \frac{1}{2} a\pi$, $\bar{y} = \frac{1}{8} a\pi$.

9. Find the center of gravity of the area bounded by the hyperbola $xy = c^2$, $x = a$, $x = b$, and $y = 0$. *Ans.* $\bar{x} = \dfrac{b - a}{\log b - \log a}$, $\bar{y} = \dfrac{c^2(b-a)}{2ab(\log b - \log a)}$.

10. Find the center of gravity of the area bounded by the parabola $y^2 = 4ax$ and the straight line $y = mx$. *Ans.* $\bar{x} = \frac{8a}{5m^2}$, $\bar{y} = \frac{2a}{m}$.

11. Find the center of gravity of the area included by the curves $y^2 = ax$ and $x^2 = by$. *Ans.* $\bar{x} = \frac{9}{20} a^{\frac{1}{3}} b^{\frac{2}{3}}$, $\bar{y} = \frac{9}{20} a^{\frac{2}{3}} b^{\frac{1}{3}}$.

12. Find the center of gravity of the area bounded by the cardioid

$$\rho = a(1 + \cos\theta). \qquad \textit{Ans.}\ \bar{x} = \tfrac{5}{6} a.$$

13. Find the center of gravity of the area included by a loop of the curve

$$\rho = a \cos 2\theta. \qquad Ans. \quad \bar{x} = \frac{128\, a\sqrt{2}}{105\,\pi}.$$

14. Find the center of gravity of the area included by a loop of the curve

$$\rho = a \cos 3\theta. \qquad Ans. \quad \bar{x} = \frac{81\, a\sqrt{3}}{80\,\pi}.$$

15. The lengths of the parallel sides of a trapezium are a and b. Show that the center of gravity of the area divides the line joining the middle points of the parallel sides in the ratio $(a + 2b)/(2a + b)$.

16. If the sides of a triangle be 3, 4, and 5 feet, find the distance of the center of gravity from each side. *Ans.* $\frac{4}{3}$, 1, $\frac{4}{5}$ foot.

17. Find the center of gravity of the area bounded by the cissoid

$$y^2(2a - x) = x^3$$

and its asymptote $x = 2a$. *Ans.* $\bar{x} = \frac{5}{3}a$.

18. Find the center of gravity of the area bounded by the witch

$$x^2y = 4a^2(2a - y)$$

and the axis of X. *Ans.* $\bar{y} = \frac{1}{2}a$.

19. Find the center of gravity of the area bounded by the curves $y^2 = ax$ and $y^2 = 2ax - x^2$, which is above the axis of X.

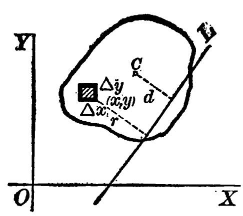
$$Ans. \quad \bar{x} = a\frac{15\pi - 44}{15\pi - 40};\ \bar{y} = \frac{a}{3\pi - 8}.$$

20. Find the distance from the center of the circle to the center of gravity of the area of a circular sector of angle 2θ. *Ans.* $\frac{2}{3}r\frac{\sin\theta}{\theta}$.

21. Find the distance from the center of the circle to the center of gravity of the area of a circular segment, the chord subtending an angle 2θ.

$$Ans. \quad \frac{2}{3}\frac{r\sin^3\theta}{\theta - \sin\theta\cos\theta}.$$

4. Theorem on the center of gravity. The center of gravity of an area is a *fixed point* relative to that area. That is, the position of the center of gravity does not depend upon the axes of coördinates, but upon the area itself only. The proof of this familiar truth is as follows.



Let L be any line, and assume its equation in the normal form (55 (*e*), Chap. XIV)

$$x\cos\omega + y\sin\omega - p = 0.$$

Consider the element of area $\Delta A = \Delta x\Delta y$ at (x, y), and let the distance from L to (x, y) equal r. Then the *product* $r\Delta x\Delta y$ is called the moment of ΔA with respect to the line L. Extending to a finite area as before, the double integral

(1) $$M_L = \int\int r\,dA$$

is called the *moment* * *of area* with respect to L.

This integral may be expressed in terms of the moments M_x and M_y with respect to OX and OY as follows. By Analytic Geometry,† or formula 56, Chapter XIV,

$$r = x\cos\omega + y\sin\omega - p.$$

$$\therefore \int\int r\,dA = \int\int (x\cos\omega + y\sin\omega - p)\,dA$$

$$= \cos\omega\int\int x\,dA + \sin\omega\int\int y\,dA - p\int\int dA$$

$$= \cos\omega\, M_y + \sin\omega M_x - pA.$$

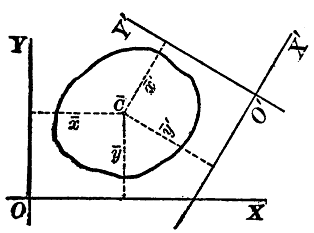
Using formulas (II), putting $A =$ Area of the figure, then

$$M_y = A\bar{x},\quad M_x = A\bar{y},\quad \int\int dA = A. \quad \text{Hence}$$

$$M_L = (\bar{x}\cos\omega + \bar{y}\sin\omega - p)A = \bar{r}A,$$

if $\bar{r} =$ distance from L to the center of gravity $(\bar{x}, \bar{y})$. Hence this

THEOREM. *The moment of area of a plane figure with respect to any line equals the product of the area and the distance from that line to the center of gravity. Hence the moment of area with respect to any line through the center of gravity is zero.*



Now suppose we have worked out the coördinates of the center of gravity $\bar{C}$ for a plane figure with respect to a given set of axes OX and OY. Let $O'X'$, $O'Y'$ be any other set of axes. Let the new coördinates of any point in the area be (x', y'). Also let the new coördinates of $\bar{C}$ be $(\bar{x}', \bar{y}')$. Then, by Art. 2,

* Also called the *first moment*, because of the appearance of the first power of the distance r in the integral.

† Smith and Gale, Elements of Analytic Geometry (Ginn and Company), p. 106. Future references are to this volume.

formulas (II), the coördinates of the center of gravity found by using the new axes are

$$\left[\frac{\int\int x'dA}{\text{area}}, \frac{\int\int y'dA}{\text{area}}\right].$$

By the theorem just given we have, however,

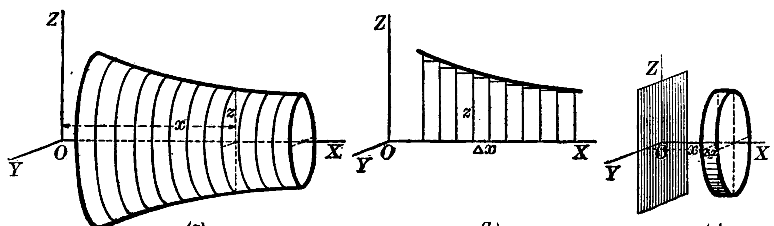
$$\int\int x'dA = \text{moment of area with respect to } OY' = A\bar{x}',$$

$$\int\int y'dA = \text{moment of area with respect to } OX' = A\bar{y}'.$$

Hence $\left[\frac{\int\int x'dA}{A}, \frac{\int\int y'dA}{A}\right] \equiv (\bar{x}', \bar{y}')$; that is, the *same* center of gravity is found by using the new axes. This investigation, therefore, verifies a well-known property of the center of gravity, namely, that it is a fixed point relative to the area.

SOLIDS OF REVOLUTION

5. Moment of mass. The volume of a thin flat plate or lamina equals the product of its surface by the thickness. If of uniform density, its mass is the product of the volume and the density. For the present, the density will be assumed constant and will be denoted by τ. The lamina being thin, its center of gravity is sensibly the same point as the center of gravity of its surface or area. The moment of mass of a lamina with respect to a plane parallel to its surface equals the product of its mass and the distance from the plane to its surface. The plane being parallel to the surface of the lamina, every point of the lamina is at the same



(a) (b) (c)

distance from the plane. Passing now to a homogeneous (of uniform density $= \tau$) solid of revolution, we may slice up such a solid by a series of equidistant parallel planes perpendicular to the axis of revolution (fig. *a*). Assume OX as this axis, and Δx

as the common thickness of the slices. Now consider each slice "trimmed up" into a circular lamina, one face of the slice remaining unchanged, so that the solid is now replaced by a new solid obtained by revolution of the set of rectangles in fig. *b*. The mass Δm of any one of the circular laminæ is (fig. *c*)

$$\Delta m = \tau \Delta v = \tau \cdot \pi y^2 \Delta x,$$

for $y(=z)$ is the radius of the base and Δx the thickness. Since the lamina is parallel to YZ, its moment of mass with respect to YZ is $x\Delta m$ or $\tau \cdot \pi y^2 \Delta x$ times x. The total moment of mass of *all the circular laminæ* may then be represented by $\Sigma x \Delta m$ or also $\Sigma \tau \pi y^2 x \Delta x$. The moment of mass of the solid itself is then defined as the limiting value of this sum when Δx approaches zero. Using for this the symbol M_{yz}, we have

$$\text{(III)} \quad \boldsymbol{M_{yz} = \int x\,dm = \tau\pi \int xy^2 dx \left(= \lim_{\Delta x = 0} \sum \tau\pi x y^2 \Delta x \right)}.$$

The method explained here of slicing the solid of revolution into circular laminæ is very important and should be mastered by the student.

The center of gravity of a solid of revolution whose axis is along OX is defined as the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\text{(IV)} \qquad \bar{x} = \frac{M_{yz}}{\text{mass}} = \frac{\int x\,dm}{\text{mass}} = \frac{\tau\pi \int xy^2 dx}{\text{mass}}, \quad \bar{y} = 0, \ \bar{z} = 0.$$

It is clear that $\bar{y} = \bar{z} = 0$, since the centers of gravity of all the laminæ are on the axis of revolution, and hence the center of gravity of the solid is on the axis of revolution.

In the calculation of $\bar{x}$, we need to find two integrals,

$$M_{yz} = \tau\pi \int xy^2 dx \text{ and Mass} = \int dm = \tau\pi \int y^2 dx,$$

in which *y is to be found in terms of x from the equation of the generating curve.*

Dimensions. The quantity moment of mass has been defined as the product of mass by distance. Hence in terms of the fundamental units of mass and of length the dimensional relation is

$$\text{Moment of mass} = \text{mass} \times \text{length}.$$

PROBLEMS

HOMOGENEOUS SOLIDS OF REVOLUTION

1. Find the center of gravity of the cone formed by revolving the line $hy = ax$ around the x-axis between $x = 0$ and $x = h$. *Ans.* $\bar{x} = \frac{3}{4} h$.

2. Find the center of gravity of a hemisphere.
Ans. Distance from base $= \frac{3}{8}$ radius.

3. Find the center of gravity of the paraboloid of revolution formed by revolving about the x-axis the parabola $y^2 = 4\,ax$ from $x = 0$ to $x = b$.
Ans. $\bar{x} = \frac{2}{3} b$.

4. The area bounded by the lines $y = 0$, $x = a$ and the curve $y^2 = 4\,ax$ is revolved about the y-axis. Find the center of gravity of the solid formed.
Ans. $\bar{y} = \frac{5}{8} a$.

5. The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in the first quadrant, is revolved about the x-axis. Find the center of gravity of the solid formed. *Ans.* $\bar{x} = \frac{3}{8} a$.

6. The area bounded by the lines $y = 0$, $x = 2\,a$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is revolved about the x-axis. Find the center of gravity of the solid formed.

7. The area bounded by the lines $x = 0$, $x = a$, $y = 0$, and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ is revolved about the x-axis. Find the center of gravity of the solid formed.

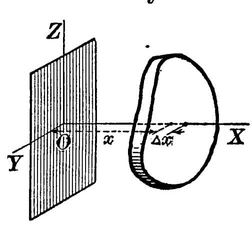
8. The area bounded by the lines $y = 0$, $x = \frac{\pi}{2}$, and the curve $y = \sin x$ is revolved about the x-axis. Find the center of gravity of the solid formed.

9. The area bounded by the lines $x = 0$, $x = a$, $y = 0$, and the curve $y = e^x$ is revolved about the x-axis. Find the center of gravity of the solid formed.

10. Find the center of gravity of the solid generated by a semiparabola bounded by the latus rectum, revolving round the latus rectum.
Ans. Distance from focus $= \frac{5}{32}$ of latus rectum.

PARTICULAR SOLIDS

6. Moment of mass. Certain solids may be divided by a series of parallel planes into laminæ whose surfaces depend in a simple manner only upon their distances from a parallel fixed plane. Taking this plane as YZ and considering a lamina at the distance x, then if A is its surface, by hypothesis, $A = f(x)$,— a known function. Hence $\Delta m = \tau f(x)\,\Delta x$ (if the thickness of the lamina is Δx). The moment of mass of the solid with respect to YZ will then be defined as equal to



$$(1)\quad M_{yz} = \int x\,dm = \tau \int x\, f(x)\,dx \left(= \lim_{\Delta x = 0} \Sigma \tau x f(x)\,\Delta x \right).$$

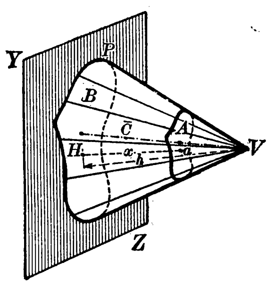
The distance $\bar{x}$ of the center of gravity from the YZ-plane is of course equal to

$$(2)\qquad \bar{x} = \frac{M_{yz}}{\text{Mass}} = \frac{\tau \int x\, f(x)\,dx}{\tau \int f(x)\,dx},$$

since Mass equals $\int dm = \tau \int f(x)\,dx$.

In (2), the uniform density τ cancels out. The function $f(x)$, it is to be remembered, is the area of a cross section parallel to YZ at the distance x.

Illustrative Example. Find the center of gravity of any cone, pyramid, or cylinder of uniform density.



Solution. The definition of a cone or pyramid must be clearly understood. This is the following. Given any plane area B and a point V without it. Draw the line VP through V and any point P on the boundary of the area B. Now let the point P move around the boundary of B, carrying in its motion the line VP. The surface thus generated by the line VP, called a *generator*, and the area B bounds a solid. If B is bounded by straight lines, the solid is a pyramid, otherwise a cone. The area B is called the *base* and V the *vertex*.

The following theorem is now assumed for any cone or pyramid. Take a section A parallel to the base B. *Then the areas of A and B are in the same ratio as the squares of their distances from the vertex V.*

To apply formula (2), let the area B lie in the YZ-plane. Let the section A be at the distance x from the base. Draw the line VH perpendicular to the base B, and let $VH = h =$ *altitude*. Then

distance of the area A from vertex $= h - x$,
distance of the area B from vertex $= h$.

$$\therefore \text{ by the theorem, } \frac{A}{B} = \frac{(h-x)^2}{h^2}, \text{ or } A = \frac{B}{h^2}(h-x)^2.$$

Hence in (2), $f(x) = \dfrac{B}{h^2}(h-x)^2$.

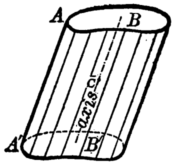
$$\therefore\ M_{yz} = \tau \int_0^h \frac{B}{h^2}\, x\,(h-x)^2 dx = \tfrac{1}{12}\,\tau B h^2,$$

$$M = \tau \int_0^h \frac{B}{h^2}(h-x)^2 dx = \tfrac{1}{3}\,\tau B h.$$

Hence $$\bar{x} = \tfrac{1}{4}\,h.$$

Now it is clear that the centers of gravity of all sections of the cone or pyramid which are parallel to the base B will lie on a line joining V to the center of gravity of the base. This line is called the *axis*. Hence the

THEOREM. *The center of gravity of any homogeneous cone or pyramid is the point on the axis which is one fourth of the distance from the base to the vertex.*



A cylinder is the solid obtained thus. Let a generating line AA' move always parallel to itself, while the point A follows a plane curve inclosing an area B. The solid bounded by this surface, by the area B, and by the section B' parallel to B, is called a cylinder. The line joining the centers of gravity of B and B' is called the *axis*. This line is parallel to the generator AA'. Clearly, the center of gravity of the cylinder is the middle point of the axis.

PROBLEMS

1. Find the center of gravity of a frustum of a pyramid with a square base.

2. Find the center of gravity of an elliptic cone. The equation of an elliptic cone is $\frac{y^2}{a^2}+\frac{z^2}{b^2}=x^2$. Take the plane $x=1$ for the base of the cone.

3. Find the center of gravity of the solid bounded by the elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$, and the plane $z=1$.

4.* Find the center of gravity of a right conoid with circular base, the radius of the base being r and altitude a.

5. A rectangle moves from a fixed point, one side varying as the distance from this point, and the other as the square of this distance. Find the center of gravity of the solid generated while the rectangle moves a distance of 2 feet.

6. On the double ordinates of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, isosceles triangles of vertical angle 90° are described in planes perpendicular to that of the ellipse. Find the center of gravity of the solid generated by supposing such a variable triangle moving from one extremity to the other of the major axis of the ellipse.

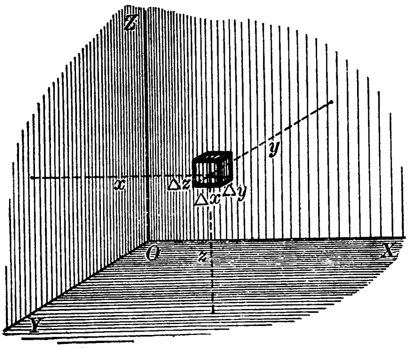
7. Given a right circular cylinder of altitude a and radius of base r. Through a diameter of the upper base pass two planes, which touch the lower base on opposite sides. Find the center of gravity of the solid included between the planes.

8. Two cylinders of equal altitude a have a circle of radius r for their common upper base. Their lower bases are tangent to each other. Find the center of gravity of the solid common to the two cylinders.

9. An anchor ring is cut in two equal parts by a plane through its center, which passes through its axis. Find the center of gravity of one half.

*For the volumes of the solids of examples 4–8, see Granville, Differential and Integral Calculus (Ginn and Company), p. 422. Future references are to this volume.

7. Moment of mass. Any solid. Consider any solid and an interior point (x, y, z). The density of this solid may be variable. In this case, we assume the density τ at any interior point (x, y, z) to be some function of the coördinates, say



(1) density at (x, y, z)
$$= \tau\,(x, y, z).$$

Taking an element of volume
$$\Delta v = \Delta x \, \Delta y \, \Delta z,$$
we have as the element of mass at (x, y, z),
$$\Delta m = \tau\ (x, y, z)\ \Delta v.$$

The moment of mass for this element with respect to the coördinate planes we define thus:

$$\begin{aligned} &\text{with respect to } YZ = x \cdot \Delta m, \\ &\text{“} \quad \text{“} \quad \text{“ } ZX = y \cdot \Delta m, \\ &\text{“} \quad \text{“} \quad \text{“ } XY = z \cdot \Delta m. \end{aligned}$$

The moments of mass of the solid with respect to the coördinate planes are derived from these by summation and passing to the limit as Δx, Δy, and Δz approach zero. That is, we define for any solid,

$$\text{(V)} \qquad M_{zy} = \int\!\!\int\!\!\int x\,dm, \; M_{zx} = \int\!\!\int\!\!\int y\,dm, \; M_{xy} = \int\!\!\int\!\!\int z\,dm,$$

the limits being so chosen that the entire solid is included. Formulas (V) are included in the single formula

$$M = \int\!\!\int\!\!\int r\,dm,$$

where r is the distance from one of the coördinate planes to any interior point of the solid. In these formulas x, y, and z are the coördinates of any point *within the solid.* The center of gravity of the solid is then the point whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are given by

$$\text{(VI)} \qquad \bar{x} = \frac{M_{yz}}{\text{mass}}, \; \bar{y} = \frac{M_{zx}}{\text{mass}}, \; \bar{z} = \frac{M_{xy}}{\text{mass}}.$$

In formulas (V) and (VI) we set

$$dm = \tau(x, y, z)\,dxdydz, \text{ mass} = \int\int\int dm.$$

In determining the center of gravity of a solid, four integrals, namely, the moments with respect to the three coördinate planes and the mass, must be calculated.

Homogeneous solids. In this case the density τ is constant. For such solids, a theorem corresponding to that of Art. 3 holds, namely,

The center of mass of a homogeneous solid lies in any plane of symmetry of the solid. The proof is left to the reader. To derive formula (III) (Art. 5) from (V), proceed thus. We have

$$M_{yz} = \tau\int\int\int xdxdydz = \tau\int\Big[\int\int dydz\Big]_{x=\text{constant}} xdx.$$

But $\Big[\int\int dydz\Big]_{x=\text{constant}}$ = area of cross section in the plane x = constant, and hence equals πy^2 under the conditions of Art. 5.

$$\therefore\ M_{yz} = \tau\int \pi y^2 x dx,$$

which is (III).

THEOREM ON THE CENTER OF MASS. Results analogous to those of Art. 4 are readily derived for solids; namely,

The moment of mass of a solid with respect to any plane equals the product of the mass by the distance from the plane to the center of gravity. The center of gravity is a fixed point relative to the solid. This proof is left to the reader.

PROBLEMS

1. Find the center of gravity of the first octant of the homogeneous ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ *Ans.* $\bar{x}=\frac{3}{8}a,\ \bar{y}=\frac{3}{8}b,\ \bar{z}=\frac{3}{8}c.$

2. Find the center of gravity of the homogeneous solid bounded by the surface $z^2 = xy$, and the planes $x = a$, $y = b$, $z = 0$. *Ans.* $\bar{x}=\frac{3}{5}a,\ \bar{y}=\frac{3}{5}b,\ \bar{z}=\frac{9}{32}\sqrt{ab}.$

3. Find the center of gravity of the paraboloid of revolution formed by revolving about the x-axis the parabola $y^2 = 4ax$ from $x = 0$ to $x = b$, supposing the density to vary as x^2. *Ans.* $\bar{x}=\frac{4}{5}b.$

4. Find the center of gravity of a hemisphere whose density varies as x^2, assuming the base in the YZ-plane and the origin at the center of the base.

Ans. $\bar{x} = \frac{5}{8} a.$

5. Find the center of gravity of the cone formed by revolving the line $hy = ax$ around the x-axis between $x = 0$ and $x = h$, assuming the density varies as x^n.

Ans. $\bar{x} = \frac{n+3}{n+4} h.$

6. Find the center of gravity of the homogeneous solid bounded by the surfaces $x^2 + y^2 = 4z$, $x^2 + y^2 = 3x$ and $z = 0$.

7. The axes of two cylinders each of radius a intersect perpendicularly. Find the center of gravity of the solid included by the two cylinders and a plane through their axes. *Ans.* $\frac{3}{8} a$ from the plane.

8. A thin plate whose density varies as $(h^2 - x^2)^{-\frac{1}{2}}$ is bounded by the lines $y = ax$, $y = 0$, and $x = h$. Find its center of gravity. *Ans.* $\bar{x} = \frac{1}{4} \pi h$; $\bar{y} = \frac{1}{8} \pi a h.$

9. Find the center of gravity of the first quadrant of a circular plate whose density varies as xy. *Ans.* $\bar{x} = \bar{y} = \frac{8}{15} a.$

10. Find the center of gravity of a circular sector (angle $= 2\theta$, radius $= a$) if the density varies as the distance from the center. *Ans.* $\bar{x} = \frac{3a}{4} \cdot \frac{\sin\theta}{\theta}.$

11. Find the center of gravity of a circular sector in which the density varies as the nth power of the distance from the center.

Ans. $\frac{n+2}{n+3} \cdot \frac{ac}{l}$, where a is the radius of the circle, l the length of the arc, and c the length of the chord of the sector.

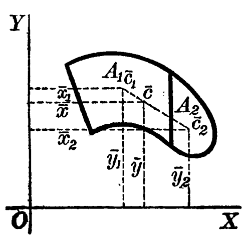
12. Find the center of gravity of a circle in which the density at any point varies as the nth power of the distance from a given point on the circumference.

Ans. It is on the diameter passing through the given point at a distance from this point equal to $\frac{2(n+2)}{n+4} \cdot a$, a being the radius.

13. Find the center of gravity of a quadrant of an ellipse in which the density at any point varies as the distance of the point from the major axis.

Ans. $\bar{x} = \frac{3}{8} a$, $\bar{y} = \frac{3\pi}{16} b.$

8. Principle of combination. Since the moment of area or of mass is a definite integral, if an area or solid is divided into two parts, the moment of the whole equals the sum of the moments of the separate parts. Thus consider the accompanying figure, in which $(\bar{x}_1, \bar{y}_1)$ is the center of gravity of the area A_1, and $(\bar{x}_2, \bar{y}_2)$ the center of gravity of the area A_2. Taking moments with respect to



$$OX: \text{ total moment} = A\bar{y} = A_1 \bar{y}_1 + A_2 \bar{y}_2;$$

$$OY: \text{ total moment} = A\bar{x} = A_1 \bar{x}_1 + A_2 \bar{x}_2.$$

Hence the center of gravity of the *combined* areas is

$$\text{(VII)} \qquad \bar{x} = \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{A_1 + A_2}, \quad \bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}.$$

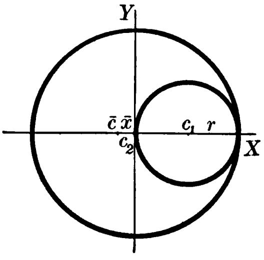
These formulas agree with those for the point of division in formula 50, Chapter XIV, if $\lambda = \frac{A_2}{A_1}$. Hence this

THEOREM. *The center of gravity of a plane figure composed of two parts divides the line joining the centers of gravity of the parts in the inverse ratio of the areas of the parts.*

A similar theorem holds for solids.

The discussion holds for an area (or solid) resulting when a portion of the area (or solid) is removed, if its area or mass be taken *negatively*. The proof, which is left to the reader, comes from (VII) by transposition. In working problems under this head, the line joining the centers of the parts may conveniently be taken for one axis of coördinates.

ILLUSTRATIVE EXAMPLE. To find the center of gravity of the remainder of a circle of radius $2r$ after a circle of radius r has been removed as indicated in the figure.



Solution. Let $\bar{c}$ be the center of gravity sought, and denote the area of the large circle by A_2 and that of the small circle by A_1. Then we have

$$A_1 = -\pi r^2,$$
$$A_2 = 4\pi r^2.$$

Substituting in (VII),

$$\bar{x} = \frac{-\pi r^2 \cdot r + 4\pi r^2 \cdot 0}{3\pi r^2} = -\frac{r}{3}.$$

Evidently $\bar{y}$ is zero by symmetry. Hence the center of gravity $\bar{c}$ lies on the x-axis at a distance of $\frac{1}{3}r$ to the left of the origin. Also $\bar{c}$ divides the line c_1c_2 in the ratio $\lambda = \frac{A_2}{A_1} = -4$.

PROBLEMS

1. A rod of uniform thickness is made up of equal lengths of three substances, the densities of which taken in order are in the proportion of 1, 2, and 3; find the position of the center of mass of the rod.

Ans. At $\frac{7}{18}$ of the whole length from the end of the densest part.

2. If five ninths be cut away from a triangle by a line parallel to the base, show that the center of gravity of the remaining area divides the median in the ratio 4 : 5.

3. One corner of a square plate of side a is cut off by a line joining the middle points of two adjacent sides. Find the center of gravity of the remainder.

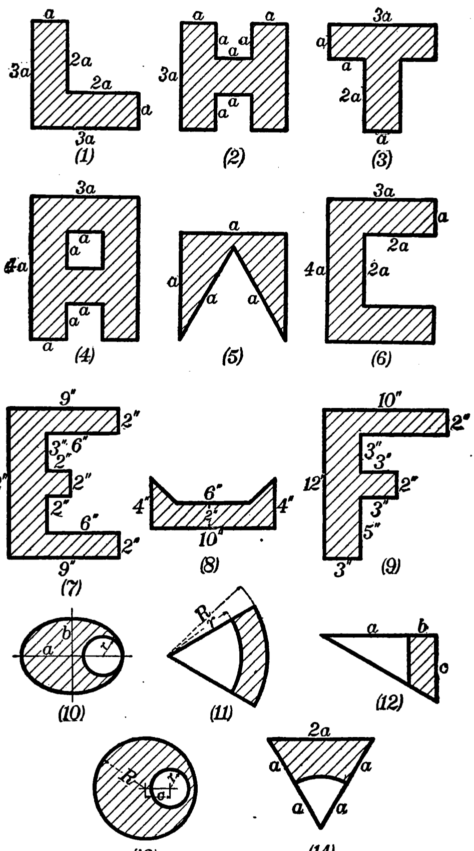
Ans. $\dfrac{\sqrt{2}\,a}{21}$ from the center.

4. An equilateral triangle is formed on one side of a square. Find the center of gravity of the whole area.

Ans. $\dfrac{3\,a}{8+2\sqrt{3}}$ from base of triangle.

5. One corner of a square of side $2\,a$ is cut off by a line drawn from a corner to the middle point of an opposite side. The opposite corner is also cut off by removing a circle of radius ρ having its center at the corner. Find the center of gravity of the remainder.

6. Find the centers of gravity of the shaded portions of the following figures.



7. A cylinder is 12 in. long, and for 8 in. of its length has a diameter of 4 in.; for the remaining 4 in. it has a diameter of 3 in. Find the center of gravity.

Ans. $5\frac{13}{41}$ in. from thick end.

8. A cone having the same base and vertex is cut from the paraboloid of revolution whose generating curve is $y^2 = 4ax$ between $x = 0$ and $x = b$. Find the center of gravity of the remaining solid. *Ans.* $\bar{x} = \frac{b}{2}$.

9. From a sphere of radius R is removed a sphere of radius r, the distance between their centers being c. Find the center of gravity of the remainder.

Ans. It is on the line joining their centers and at a distance $\frac{cr^3}{R^3 - r^3}$ from the center.

10. Find the center of gravity of a cubical box without a lid, the inside edge being 20 in. and the thickness of the wood 1 in.

11. Find the center of gravity of the remainder of an equilateral triangle from which has been cut an isosceles right triangle with hypotenuse coincident with a side of the original triangle.

12. A right circular cone whose base is of radius r is divided into two equal parts by a plane through the axis. Prove that the distance of the center of gravity of either half from the axis is $\frac{r}{\pi}$.

13. Find the center of gravity of half of a regular hexagon.

14. From a hemisphere is cut a cone having the same base and altitude. Find the center of gravity of the remainder. *Ans.* Distance from base $= \frac{1}{2}$ altitude.

15. From a right circular cylinder is cut a cone having the same base and altitude. Find the center of gravity of the remainder.

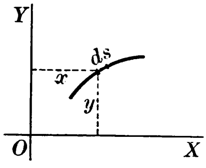
Ans. Distance from base $= \frac{5}{8}$ altitude.

16. From a right circular cone of altitude a is cut a similar cone of altitude b, the bases of the two cones being in the same plane. Find the center of gravity of the remainder. *Ans.* Distance from base $= \frac{1}{4}\frac{a^4 - b^4}{a^3 - b^3}$.

9. Center of gravity of an arc. The center of gravity for any plane curve is given by formulas analogous to (II), Art. 2, obtained by replacing the element of area or mass by the element of arc of the curve, that is, for a plane curve, by 66, Chapter XIV,

$$(1)\quad ds = [(dx)^2 + (dy)^2]^{\frac{1}{2}} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}} dy.$$

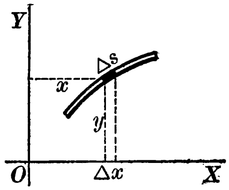
The formulas are



$$\text{(VIII)}\qquad \bar{x} = \frac{\int x\,ds}{\text{arc}},\quad \bar{y} = \frac{\int y\,ds}{\text{arc}},$$

in which ds is found by (1). In these x and y are the coördinates of any point *on the curve*.

Formulas (VIII) are used to find the center of gravity of uniform thin wires. If σ is the area of the cross section, and Δs the length of a piece whose projections on OX and OY are Δx and Δy, respectively, then for the mass of this piece we write



$$\Delta m = \sigma \Delta s \text{ times the density } (= \tau);$$

(1) or $\Delta m = \tau \cdot \sigma \Delta s$.

For the moments of mass with respect to OX and OY of this piece, we have the products

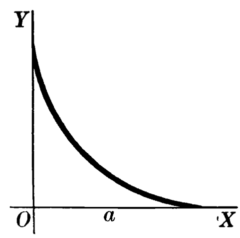
$$y\Delta m \text{ and } x\Delta m,$$

respectively. Thus we obtain for a plane-curve wire as in formulas (II),

$$(2) \quad \begin{cases} \bar{x} = \dfrac{M_y}{\text{mass}} = \dfrac{\int \tau\sigma x ds}{\int \tau\sigma ds} = \dfrac{\int \tau x ds}{\int \tau ds}, \\[2ex] \bar{y} = \dfrac{M_x}{\text{mass}} = \dfrac{\int \tau\sigma y ds}{\int \tau\sigma ds} = \dfrac{\int \tau y ds}{\int \tau ds}, \end{cases}$$

since the constant, σ, divides out. If the wire is uniform, τ is also constant, divides out, and we have (VIII).

ILLUSTRATIVE EXAMPLE. Find the center of gravity of a quadrant of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.



Solution. Consider the part of the curve in the first quadrant.

Then

$$\int xds = \int_0^a x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = a^{\frac{1}{3}}\int_0^a x^{\frac{2}{3}}dx = \frac{3}{5}a^2.$$

$$\int yds = \int_0^a y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \frac{3}{5}a^2.$$

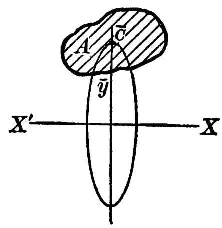
$\displaystyle\int ds = \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{3}{2}a.$ Hence, applying (VIII), $\bar{x} = \bar{y} = \frac{2}{5}a$.

10. Theorems of Pappus. Consider any area in the XY-plane. The distance of the center of gravity from the x-axis is given by the formula (II),

$$(1) \qquad \bar{y} = \frac{\int\int ydydx}{A},$$

where A denotes the area. Let the area be revolved about the x-axis. The volume generated is given by the definite integral

(2) $$V = \pi \int y^2 dx.$$



Consider the numerator in (1). Integrating with respect to y,

(3) $$\int\int y dy dx = \frac{1}{2}\int y^2 dx = \frac{1}{2}\frac{V}{\pi},$$

comparing with (2).

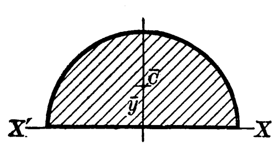
Substituting in the second member of (1), we get

(4) $$\bar{y} = \frac{1}{2}\frac{V}{\pi} \div A, \text{ or } 2\pi\bar{y} = \frac{V}{A}.$$

Now $2\pi\bar{y}$ = length of path described by the center of gravity. Hence the

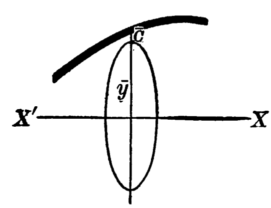
FIRST THEOREM. *If any plane area be revolved about an exterior axis in its plane, the length of the path described by its center of gravity is equal to the volume generated, divided by the area revolved.*

This theorem has two uses: (1) if the area and its center of gravity are known, we may find the volume of the solid of revolution; (2) if the area and volume are known, we may find the center of gravity. For example, to find the distance of the center of gravity of a semicircle from the center, we have



$$2\pi\bar{y} = \frac{\text{volume of sphere}}{\text{semicircle}} = \frac{\frac{4}{3}\pi a^3}{\frac{\pi a^2}{2}}, \text{ whence } \bar{y} = \frac{4}{3}\frac{a}{\pi}.$$

Next consider any curve in the XY-plane. The distance of the center of gravity from the x-axis is given by the formula (VIII),



(5) $$\bar{y} = \frac{\int y ds}{s},$$

where s denotes the length of the curve. Let the curve be revolved about the x-axis. The surface generated is (68, Chap. XIV)

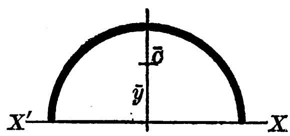
$$S = 2\pi \int y ds.$$

But the length of the path described by the center of arc is (multiplying both members of (5) by 2π)

$$2\pi\bar{y} = \frac{2\pi\int yds}{s} = \frac{S}{s}.$$

Hence the

SECOND THEOREM. *If any plane curve be revolved about an exterior axis in its plane, the length of the path described by its center of gravity is equal to the surface of the solid generated, divided by the length of the arc revolved.*



This theorem has two uses: (1) if the length of the arc and its center are known, we may find the surface of the solid of revolution; (2) if the length of the arc and the surface of the solid are known, we may find the center of gravity of the arc.

For example, to find the distance of the center of gravity of a semicircle from the center, we have

$$2\pi\bar{y} = \frac{\text{surface of sphere}}{\text{semicircumference}} = \frac{4\pi a^2}{\pi a}, \text{ whence } \bar{y} = \frac{2}{\pi}a.$$

PROBLEMS

1. Find the center of gravity of an arc of the circle $\rho = a$ between $-\theta$ and $+\theta$, and from this derive the results for quadrantal and semicircular arcs.

Ans. $\bar{x} = \dfrac{a\sin\theta}{\theta}$. For quadrantal arc $\theta = \dfrac{\pi}{4}$, $\bar{x} = \dfrac{4a}{\pi\sqrt{2}}$.

For semicircular arc $\theta = \dfrac{\pi}{2}$, $\bar{x} = \dfrac{2a}{\pi}$.

2. Find the center of gravity of a thin straight wire of length a whose density varies as the nth power of the distance from one end. *Ans.* $\bar{x} = \dfrac{n+1}{n+2}a$.

3. Find the center of gravity of the perimeter of the cardioid $\rho = a(1+\cos\theta)$. *Ans.* $\bar{x} = \frac{4}{5}a$, $\bar{y} = 0$.

4. Find the center of gravity of the cycloid $x = a \text{ arc vers } \dfrac{y}{a} - (2ay - y^2)^{\frac{1}{2}}$ between two successive cusps. *Hint.* $\dfrac{dx}{dy} = \dfrac{y}{\sqrt{2ay - y^2}}$. *Ans.* $\bar{x} = a\pi$, $\bar{y} = \dfrac{4a}{3}$.

5. Find by the theorem of Pappus the center of gravity of one fourth of a circle in the first quadrant. *Ans.* $\bar{x} = \bar{y} = \dfrac{4a}{3\pi}$.

6. Find by the theorems of Pappus the volume and surface of the torus generated by revolving the circle $(x-b)^2 + y^2 = a^2$ $(b > a)$ about the y-axis.

Ans. $V = 2\pi^2 a^2 b$, $S = 4\pi^2 ab$.

7. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about the line $x = 2a$. Find by the theorem of Pappus the volume generated. *Ans.* $4\pi^2 a^2 b$.

8. An equilateral triangle revolves around its base, whose length is a. Find (1) the area of the surface and (2) the volume of the solid generated.

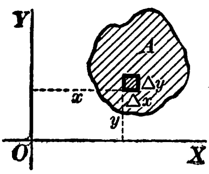
Ans. (1) $\pi a^2 \sqrt{3}$; (2) $\frac{\pi a^3}{4}$.

9. A square of side a is revolved around an axis in its plane, the perpendicular distance of which from the center is c. Find (1) the area of the surface and (2) the volume of the solid generated.

10. A rectangle is revolved around an axis, which lies in its plane and is perpendicular to a diagonal at its extremity. Find the area of the surface and the volume of the solid generated.

11. Moment of inertia. Plane areas. Consider an element of area

$$\Delta A = \Delta x \, \Delta y,$$



at the point (x, y). The products,

$$x^2 \Delta A, \quad y^2 \Delta A,$$

are called the *moments of inertia* or *second moments* of ΔA with respect to the axes OY and OX respectively. The definition is extended to a finite area by summation and passing to the limit. Using I_x, I_y for the moments of inertia with respect to OX and OY, respectively, then

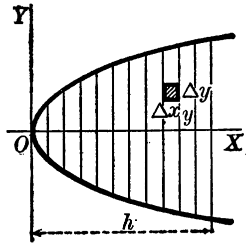
$$\text{(IX)} \quad \begin{cases} I_x = \int\int y^2 dA = \int\int y^2 dx dy = \left[\underset{\substack{\Delta x = 0 \\ \Delta y = 0}}{\text{limit}} \sum y^2 \Delta x \Delta y \right]; \\ I_y = \int\int x^2 dA = \int\int x^2 dx dy = \left[\underset{\substack{\Delta x = 0 \\ \Delta y = 0}}{\text{limit}} \sum x^2 \Delta x \Delta y \right]. \end{cases}$$

In these formulas x and y are the coördinates of any point *within the area.* Formulas (IX) are embraced in the single formula

$$(1) \qquad I = \int\int r^2 dA,$$

where r is the distance from the axis in question to any point within the area. This integral is called also the *second moment* of area, from the *second* power of the distance r.

Since each element $y^2\Delta x\Delta y$ or $x^2\Delta x\Delta y$ is essentially positive, the moment of inertia * is never zero, but a positive number.



Its dimensions are *area times square of a length*, and hence it is of the fourth degree in the fundamental unit of length.

Illustrative Example. Find I_y for the portion of $y^2 = 2\,px$ cut off by $x = h$.

Solution. We have, by (IX),

$$I_y = \int\int x^2 dxdy = \int_0^h \Big[\int_{-\sqrt{2px}}^{\sqrt{2px}} dy\Big] x^2 dx = 2\sqrt{2p}\int_0^h x^{\frac{5}{2}} dx = \tfrac{4}{7}\sqrt{2p}\,h^{\frac{7}{2}}.$$

Since $A = \frac{4}{3}\sqrt{2p}\,h^{\frac{3}{2}}$, we get for I_y the expression

$$I_y = \frac{3A}{7}h^2.$$

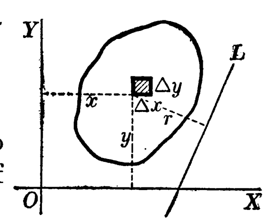
PROBLEMS

Note. If the equation of the curve is given in polar coördinates (ρ, θ), write in (IX)

$$dA = \rho d\rho d\theta, \qquad x = \rho\cos\theta, \qquad y = \rho\sin\theta.$$

1. Find I for a rectangle of sides $2\,a$ and $2\,b$: (1) with respect to an axis through the center of gravity parallel to the side $2\,a$; (2) with respect to the side $2\,a$.
Ans. (1) $\frac{Ab^2}{3}$; (2) $\frac{4}{3}Ab^2$.
2. Find I for a circle with respect to a diameter. *Ans.* $\frac{1}{4}Aa^2$.
3. Find I for an ellipse: (1) with respect to its major axis; (2) with respect to its minor axis. *Ans.* (1) $\frac{1}{4}Ab^2$; (2) $\frac{1}{4}Aa^2$.
4. Find I for a right triangle with respect to one side.
5. Find I for a square with respect to a diagonal. *Ans.* $\frac{1}{12}Aa^2$.
6. Find I for an equilateral triangle with respect to a median.
7. Find I_x for the cardioid $\rho = a(1 + \cos\theta)$.
8. Find I_x and I_y for one loop of the curve $\rho = a\cos 2\,\theta$.
9. Find I_y for the lemniscate $\rho^2 = a^2\cos 2\,\theta$. *Ans.* $I_y = \frac{A}{48}(3\,\pi + 8)\,a^2$.

12. Theorems on moments of inertia. The moment of inertia of the element of area $\Delta A = \Delta x\Delta y$ with respect to any line or axis L equals



$$r^2\Delta A,$$

where r is the distance from the line L to the point (x, y). The moment of inertia of a finite area with respect to L is then

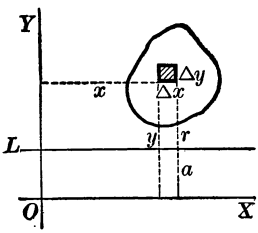
* It appears later that moment of inertia determines the kinetic energy of revolution.

(X) $$I_L = \int\int r^2 dA = \int\int r^2 dx dy,$$

in which r is the perpendicular distance from the line L to any point (x, y) *within the area.*

Let us apply (X) to the case of an axis parallel to OX, whose equation is $y = a$. Then $r = y - a$, and hence

$$I_L = \int\int (y-a)^2 dA = \int\int (y^2 - 2\,ay + a^2) dA$$
$$= \int\int y^2 dA - 2\,a \int\int y dA + a^2 \int\int dA.$$



(1) $\therefore\ I_L = I_x - 2\,aM_x + a^2 A$ (by (IX) Art. 11, and (II) Art. 2).

This formula expresses the moment of inertia I_L in terms of the moment of inertia with respect to *any* parallel axis OX, the moment of area with respect to the latter, and the area itself.

But suppose the center of gravity lies on OX. Then $\bar{y} = 0$ and also $M_x = 0$. Hence

(XI) $$I_L = I_x + a^2 A.$$

An axis passing through the center of gravity is called a *gravity axis.*

This establishes the important

THEOREM. *The moment of inertia of a plane area with respect to any axis equals the moment of inertia with respect to the parallel gravity axis, increased by the product of the area by the square of the distance between the axes.*

This statement shows that the moment of inertia with respect to a gravity axis is *less* than the moment of inertia for any parallel axis.

Radius of gyration. The quotient of the moment of inertia by the area is the square of a length called the radius of gyration. Thus, if $\bar{r}_L$ denote this,

$$\bar{r}_L^2 = \frac{I_L}{A},$$

in which $\bar{r}_L$ is the radius of gyration with respect to the axis L.

PROBLEMS

1. Find the radius of gyration in the problems on page 22.

2. Find I and $\bar{r}$ for a circle with respect to a tangent. *Ans.* $I = \frac{5}{4}Aa^2$, $\bar{r}^2 = \frac{5}{4}a^2$.

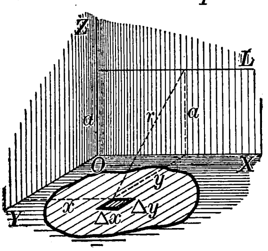
3. Find I and $\bar{r}$ for an ellipse with respect to a tangent (1) at the end of the major axis; (2) at the end of the minor axis. *Ans.* (1) $I = \frac{5}{4}Aa^2$; (2) $I = \frac{5}{4}Ab^2$.

4. Find I for a right triangle with respect to a line through one vertex parallel to the opposite side.

5. Find I for a square with respect to a line through one vertex parallel to the diagonal joining the other two vertices.

6. Find I for an equilateral triangle with respect to a line through one vertex parallel to a median.

13. Further theorems. In the preceding section, the axis L was drawn *in the plane of the given area.* It is necessary, however, to consider moments of inertia with respect to axes *without*, but *parallel* to this plane. Let L be such an axis in the figure. Then if r is the perpendicular distance from the axis L to any point (x, y) within the area, we define in a manner precisely analogous to the foregoing,



$$I_L = \int\int r^2 dA = \int\int r^2 dx dy.$$

Now project the line L upon the plane of the area, and take this projection as the axis OX. Let the distance between L and OX equal a. Then evidently $r^2 = a^2 + y^2$, and hence

$$I_L = \int\int (y^2 + a^2) dA = \int\int y^2 dA + a^2 \int\int dA.$$

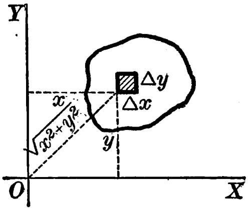
(XII) $$\therefore \boldsymbol{I_L = I_x + a^2 A}.$$

The moment of inertia of an area with respect to an axis parallel to its plane equals the moment of inertia with respect to the projection of the given axis on its plane increased by the product of the area by the square of the distance from the axis to the plane.

14. Polar moment of inertia. The moment of inertia of an area with respect to the origin is defined as equal to

(XIII) $$\boldsymbol{I_0 = \int\int (x^2 + y^2) dA = \int\int x^2 dA + \int\int y^2 dA}.$$

It will be observed that $(x^2 + y^2)\,\Delta A$ is the product of ΔA by the square of the distance from (x, y) to an axis through O, perpendicular to the plane of the area.



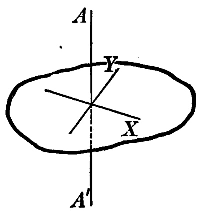
Such an axis is called a *polar axis.*

Comparison with (IX), Art. 11, enables us to write (XIII) in the form

(XIV) $$I_0 = I_x + I_y.$$

Hence the

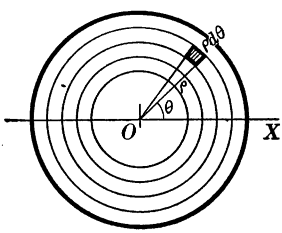
THEOREM. *The moment of inertia of an area with respect to a polar axis (called the polar moment) equals the sum of the moments with respect to two mutually perpendicular axes drawn through its foot.*



If polar coördinates (ρ, θ) are used, the origin O being the pole, I_0, the polar moment of inertia, is given directly by

(XV) $$I_0 = \int\int \rho^2 \cdot \rho d\rho d\theta = \int\int \rho^3 d\rho d\theta.$$

Moments of inertia of a circle. On account of important applications in the next section, the moments of inertia of a circle are now worked out.



Let a = radius. Then, by (XV), the polar moment of inertia with respect to an axis through the center is

(1) $$I_0 = \int_0^a \left[\int_0^{2\pi} d\theta\right] \rho^3 d\rho = \frac{\pi a^4}{2} = \frac{A}{2} a^2,$$

where A = area of the circle.

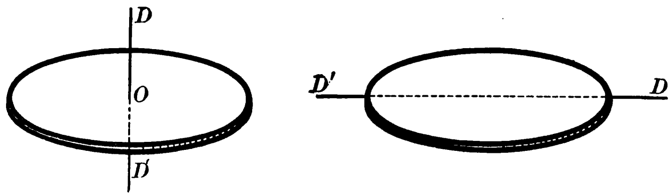
Also since $I_x = I_y$, by symmetry, we have, by (XIV),

(2) $$I_x = \frac{1}{2} I_0 = \frac{A}{4} a^2.$$

In words: *the polar moment of inertia of a circle with respect to its center equals the product of one half the area and the square of the radius; with respect to any diameter — the product of one fourth the area and the square of the radius.*

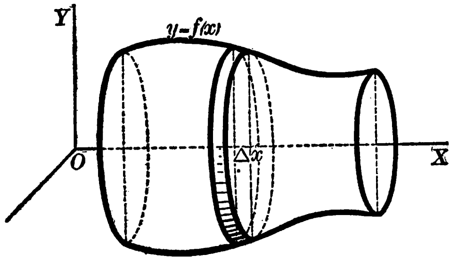
15. Flat thin plates or laminæ. Moments of inertia of laminæ are obtained from the corresponding moments of inertia of their surfaces by replacing the area by the mass of the lamina.

For example, *the polar moment of inertia of a circular lamina with respect to its center equals the product of one half the mass by the square of its radius.*



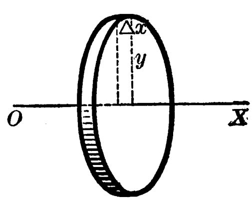
The moment of inertia of a circular lamina with respect to a diameter equals the product of one fourth its mass by the square of the radius.

16. Solids of revolution. Moments of inertia of such solids are obtained by slicing and trimming the solid into circular laminæ by a series of equidistant planes perpendicular to the axis of the surface, and considering the limit of the sum of the moments of inertia of the laminæ. If the axis of revolution be chosen as OX, the common thickness of the laminæ as Δx, and the density as τ, the mass Δm of any lamina is



(1) $$\Delta m = \tau\pi y^2 \Delta x.$$

Moment of inertia of a solid of revolution with respect to the axis of revolution. The moment of inertia of any one lamina with respect to the axis of revolution is the same as the polar moment of a circular lamina with respect to its center. By Art. 15, this moment is equal to



(2) $$\frac{\Delta m}{2} y^2 = \frac{\tau\pi}{2} y^4 \Delta x \qquad \text{by (1).}$$

The moment of inertia of the solid is accordingly

(XVI) $$I_x = \tfrac{1}{2}\int y^2 dm = \int \frac{\tau\pi}{2} y^4 dx,$$

in which y is to be found in terms of x from the equation of the generating curve.

ILLUSTRATIVE EXAMPLE. Find the moment of inertia with respect to the axis of revolution of a cone formed by revolving about the x-axis the line $y = mx$ between $x = 0$ and $x = b$.

Solution. From (XVI),

$$I_x = \frac{\tau\pi}{2}\int_0^b m^4x^4dx = \frac{\tau\pi m^4b^5}{10}.$$

Since the radius of the base $a = mb$ and the volume $= \frac{\pi m^2 b^3}{3}$, we have $I_x = \frac{3}{10} Ma^2$.

PROBLEMS

1. Find I for a rectangle of sides $2a$ and $2b$ with respect to a line perpendicular to the plane and passing through the center. *Ans.* $I = \frac{M}{3}(a^2 + b^2)$.

2. Find I for a right triangle with respect to a line perpendicular to its plane and passing through the vertex of the right angle.
Ans. $\frac{1}{6} Ac^2$, where c is the hypotenuse.

3. Find I for the area of an ellipse with respect to an axis perpendicular to the area and passing through the center. *Ans.* $I = \frac{1}{4} A(a^2 + b^2)$.

4. Find I and $\bar{r}$ for a sphere with respect to a diameter. *Ans.* $I = \frac{2}{5} Ma^2$.

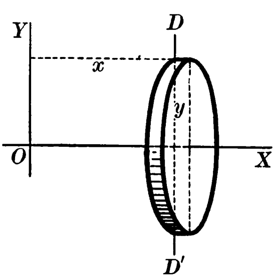
5. Find I_x and $\bar{r}_x$ for an ellipsoid of revolution about the x-axis.
Ans. $I = \frac{2}{5} Mb^2$.

6. Find I and $\bar{r}$ for a right cylinder with respect to its axis.
Ans. $I = \frac{1}{2} Ma^2$.

7. Find I_x for the solids of revolution about the x-axis whose generating curves are

(a) $y^2 = 4ax$ from $x = 0$ to $x = b$;
(b) $y = \sin x$ " $x = 0$ " $x = \pi$;
(c) $y = mx + b$ " $x = 0$ " $x = c$;
(d) $y = e^x$ " $x = 0$ " $x = a$.

Moment of inertia of a solid of revolution with respect to an axis cutting the axis of revolution at right angles. We wish to find the moment of inertia with respect to OY. To do this, we must first find the moment of inertia of one lamina with respect to OY. Now OY is an axis parallel to the surface of the lamina. Let DD' be the projection of OY upon this surface. Then, by (XII), Art. 13, for the lamina we have



(3) $$I_{y\ (\text{for one lamina})} = I_D + x^2\,\Delta m.$$

But I_D is the moment of inertia of the lamina with respect to a diameter. Hence, by Art. 15,

(4) $$I_D = \frac{\Delta m}{4} y^2.$$

Substituting in (3) gives

$$(5)\qquad I_{y\,(\text{for one lamina})} = \frac{\Delta m}{4} y^2 + x^2 \Delta m.$$

Summation and passing to the limit leads to the result

$$(\text{XVII})\qquad I_{y\,(\text{for the solid})} = \int \left(\frac{y^2}{4} + x^2\right) dm = \tau \int \left(\frac{y^2}{4} + x^2\right) \pi y^2 dx,$$

in which y must be expressed in terms of x from the equation of the generating curve.

PROBLEMS

1. Find the moment of inertia of a right cylinder of radius a and altitude h with respect to a diameter of the base. *Ans.* $I = \frac{M}{12}(3a^2 + 4h^2)$.

2. Find the moment of inertia of a right circular cone of altitude h and radius of base a, with respect to an axis through its vertex and perpendicular to its geometrical axis. *Ans.* $I = \frac{3}{20} M (4h^2 + a^2)$.

3. Find the moment of inertia of the cone of problem 2 with respect to a gravity axis perpendicular to its geometrical axis. *Ans.* $I = \frac{3}{80} M (h^2 + 4a^2)$.

4. Find I_y for the solids of problem 7, p. 27.

17. Moments of inertia of solids in general. Consider any solid and an interior point (x, y, z). If the density at this point is $\tau\,(x, y, z)$ (compare Art. 7), the element of mass is

$$(1)\qquad \Delta m = \tau \Delta x \Delta y \Delta z.$$

The moments of inertia of Δm relative to the coördinate planes are defined as

$$(2)\qquad I_{yz} = x^2 \Delta m,\quad I_{zx} = y^2 \Delta m,\quad I_{xy} = z^2 \Delta m.$$

The square of the distance of (x, y, z) from the axis of x being $y^2 + z^2$ (with similar expressions for the other axes), the moments of inertia of Δm with respect to the coördinate axes are

$$(3)\qquad I_x = (y^2 + z^2)\Delta m,\quad I_y = (x^2 + z^2)\Delta m,\quad I_z = (x^2 + y^2)\Delta m.$$

The moments of inertia for the entire solid may now be written down, namely,

$$(\text{XVIII})\quad \left\{ \begin{array}{l} I_{yz} = \displaystyle\iiint x^2 dm = \left[\underset{\substack{\Delta x = 0 \\ \Delta y = 0 \\ \Delta z = 0}}{\text{limit}} \sum x^2 \Delta m \right], \\ I_{zx} = \displaystyle\iiint y^2 dm,\quad I_{xy} = \iiint z^2 dm\,; \end{array} \right.$$

$$(\text{XIX})\quad \left\{ \begin{array}{l} I_x = \displaystyle\iiint (y^2 + z^2) dm,\quad I_y = \iiint (z^2 + x^2) dm, \\ \qquad\qquad\qquad\qquad I_z = \displaystyle\iiint (x^2 + y^2) dm, \end{array} \right.$$

where $dm = \tau(x, y, z)\,dxdydz$, and (x, y, z) is any point within the solid.

Formulas (XVIII) and (XIX) are included in the formula

$$I = \int\int\int r^2 dm,$$

where r is the perpendicular distance from the axis or plane in question to any point within the solid.

Dimensions. The moment of inertia of a solid has been defined as the product of mass by the square of the distance. Hence the derived unit of moment of inertia is expressed in terms of the fundamental units of mass and of distance by the dimensional equation

$$\text{Moment of inertia} = \text{mass} \times \text{length}^2.$$

By the *radius of gyration of a solid* with respect to any axis is understood a length r_l whose *square* is the quotient of the moment of inertia with respect to the axis by the mass. Thus

$$\text{(3a)} \qquad r_x^2 = \frac{I_x}{\text{mass}}, \text{ etc.}$$

The relations

$$\text{(4)} \qquad I_x = I_{zx} + I_{xy}, \quad I_y = I_{yz} + I_{xy}, \quad I_z = I_{yz} + I_{zx},$$

obviously hold. In words,

The moment of inertia of a solid with respect to any axis equals the sum of its moments relative to two mutually perpendicular planes passing through the axis.

Homogeneous solids. For such solids the density τ is everywhere constant. Formulas (XIX) applied in this case to a homogeneous solid of revolution about the x-axis work out as follows:

$$\text{(5)} \; I_x = \tau \int\int\int (y^2 + z^2) dxdydz = \tau \int \left[\int\int (y^2 + z^2) dydz \right]_{x=\text{constant}} dx.$$

But $\int\int (y^2 + z^2) dydz$ calculated for any plane section $x =$ constant is obviously the *polar moment* of a circle with respect to its center. Since the radius of this circle is y, then (Art. 14)

$$\text{(6)} \qquad \int\int (y^2 + z^2)\, dydz = \frac{\pi y^2}{2} \cdot y^2 = \frac{\pi y^4}{2}.$$

(7) $$\therefore I_x = \tau \int \frac{\pi y^4}{2}\, dx,$$

which is (XVI), Art. 16.

Similarly for the same solid,

(8) $$I_y = \tau \iiint (x^2 + z^2)\, dx dy dz$$

$$= \tau \int \Big[\iint dy dz \Big]_{x=\text{constant}} x^2 dx + \tau \int \Big[\iint z^2 dy dz \Big]_{x=\text{constant}} dx.$$

But $\iint dydz$ calculated for any plane section when $x =$ constant, is the area of that section; that is,

(9) $$\Big[\iint dydz \Big]_{x=\text{constant}} = \pi y^2.$$

Again, $\iint z^2 dydz$ for the section $x =$ constant is the moment of inertia of that section with respect to its diameter in the plane XY. Hence (Art. 14),

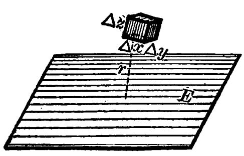
(10) $$\Big[\iint z^2 dydz \Big] = \frac{\pi y^2}{4} \cdot y^2.$$

Substituting in (8) gives

(11) $$I_y = \tau \int \pi x^2 y^2 dx + \tau \int \frac{\pi y^4}{4}\, dx;$$

that is, (XVII).

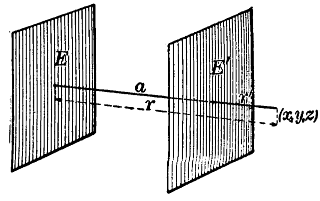
18. Parallel axes. If E is any plane, the moment of inertia of any solid with respect to E is defined as



$$I_E = \iiint r^2 dm,$$

where r is the perpendicular distance from the plane to any point (x, y, z) *within the solid.*

Parallel planes. Let E and E' be two parallel planes, r and r' the distance from them to any interior point (x, y, z) of a solid. Then if a is the common distance apart of E and E', we have



$r = r' + a$, and hence

$$I_E = \int\int\int r^2 dm = \int\int\int (r' + a)^2 dm$$

$$= \int\int\int r'^2 dm + 2a\int\int\int r' dm + a^2\int\int\int dm.$$

But $\int\int\int r'^2 dm = I_{E'}$, $\int\int\int dm = M$,

and $\int\int\int r' dm = M\bar{r}$, where $\bar{r}$ is the distance of the center of gravity of the solid from E'. Hence

(1) $$I_E = I_{E'} + 2aM\bar{r} + a^2M.$$

Suppose E' passes through the center of mass. Then $\bar{r} = 0$, and we have the important result

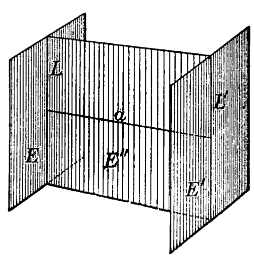
(2) $$I_E = I_{E'} + a^2M.$$

Any plane passing through the center of a mass is called a *gravity plane.*

THEOREM. *The moment of inertia with respect to any plane is equal to the moment of inertia with respect to the parallel gravity plane, increased by the product of the entire mass and the square of the distance between the planes.*

Since in any set of parallel planes one and only one passes through the center of the mass, it follows at once from (2) that of all moments of inertia with respect to parallel planes that with respect to the gravity plane is the least.

Parallel axes. Let L and L' be any two parallel lines. Let E'' be the plane passed through the two lines L and L', and let E and E' be planes through L and L', respectively, perpendicular to E''. Then, from (1),



$$I_E = I_{E'} + 2aM\bar{r} + a^2M.$$

Adding $I_{E''}$ to both members, we have

(3) $$I_E + I_{E''} = I_{E'} + I_{E''} + 2aM\bar{r} + a^2M.$$

But, by (4), Art. 17,

$$I_E + I_{E''} = I_L \text{ and } I_{E'} + I_{E''} = I_{L'}.$$

Also if L' (and consequently E') passes through the center of mass, we have $\bar{r} = 0$, and (3) becomes

(4) $$I_L = I_{L'} + a^2M.$$

Hence the theorem stated above holds when the word "plane" is replaced by "axis."

PROBLEMS

1. Derive formulas for the moments of inertia of plane arcs (or wires).

Ans. $I_x = \int \tau y^2 ds;\ I_y = \int \tau x^2 ds;\ ds = (dx^2 + dy^2)^{\frac{1}{2}}.$

2. Find I for a solid cylinder with respect to an element. *Ans.* $\frac{3}{2} Ma^2$.

3. Find I for a solid sphere with respect to a tangent line. *Ans.* $\frac{7}{5} Ma^2$.

4. Find I for a solid ellipsoid of semiaxes a, b, c with respect to the axis a; with respect to a tangent line at the extremity of the axis b.

Ans. $M\frac{b^2 + c^2}{5};\ M\frac{6\,b^2 + c^2}{5}.$

5. Find I for a uniform wire in the form of an equilateral triangle of side a, (1) with respect to a line perpendicular to the plane of the triangle and equidistant from the vertices; (2) with respect to a line through a vertex perpendicular to the plane. *Ans.* (1) $\frac{Ma^2}{2}$.

6. Find I for a solid cylinder with respect to a line perpendicular to its axis and intersecting it at a distance c from the end, the altitude of the cylinder being h and the radius of the base c. *Ans.* $\frac{1}{4} Mc^2 + \frac{1}{3} M(h^2 - 3\,hc + 3\,c^2)$.

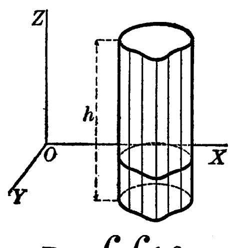
7. Find I for a straight rod of length a with respect to an axis perpendicular to the rod and at a distance d from its middle point. *Ans.* $M\left(\frac{a^2}{12} + d^2\right)$.

8. Find I for an arc of a circle whose radius is a and which subtends an angle $2\,\alpha$ at the center, (1) with respect to an axis through its center perpendicular to its plane; (2) with respect to an axis through its middle point perpendicular to its plane; (3) with respect to the diameter which bisects the arc.

Ans. (1) Ma^2; (2) $2\,M\left(1 - \frac{\sin\alpha}{\alpha}\right)a^2$; (3) $M\left(1 - \frac{\sin 2\,\alpha}{2\,\alpha}\right)\frac{a^2}{2}$.

9. Find I for the arc of the cycloid $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$ with respect to the base. *Ans.* $\frac{32}{15} Ma^2$.

19. Relation between moment of inertia of a beam and polar moment of a right section. Consider any homogeneous straight beam (density $= \tau$) whose elements are parallel to OZ. Then, by (XIX),



$$(1)\quad I_z = \tau \int\int\int (x^2 + y^2)\, dxdydz$$

$$= \tau \int \Big[\int\int (x^2 + y^2)\, dxdy\Big]_{z=\text{constant}} dz.$$

But $\int\int (x^2 + y^2)\, dxdy$, worked out for any section $z =$ constant, is the polar moment of that section for the axis OZ (Art. 14). Hence (1) becomes

(2) $I_{z\,(\text{for the beam})} = I_{0\,(\text{for a right section})} \times$ height of cylinder $(= h) \times \tau$.

Let r_s and r_0 be the radii of gyration of beam and right section, respectively. Then

$$r_s^2 = \frac{I_z}{\text{mass}}, \quad r_0^2 = \frac{I_0}{\text{area}}, \quad \text{or also}$$

$$I_z = Mr_s^2, \quad I_0 = r_0^2 A.$$

Substituting in (2) gives

(3) $$Mr_s^2 = Ar_0^2 h\tau.$$

But $M = Ah\tau$, and hence

(4) $$r_s = r_0.$$

THEOREM. *The radius of gyration of any homogeneous beam with respect to an axis parallel to its elements equals the radius of gyration of a right section with respect to the same axis.*

From (4) we may write

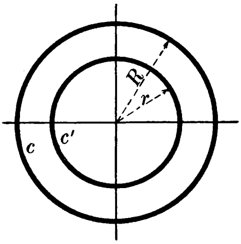
$$I_z = Mr_s^2, \quad I_0 = Ar_s^2,$$

and hence the change from I_0 to I_z is accomplished by replacing the area by the mass of the cylinder. In this form the result is useful and gives this

RULE. To find the moment of inertia of a straight beam or column with respect to an axis parallel to its elements (or edges), work out the corresponding polar moment for any right section and replace in this result the area by the mass of the beam or column.

20. Combined solids and areas. Since the moment of inertia is a definite integral, it follows that if a solid or area is composed of two or more parts, the moment of inertia of the whole with respect to any plane or axis is equal to the sum of the moments of inertia of its parts with respect to that plane or axis. Also, if a portion be removed from a solid or area, the moment of inertia of the remainder equals the moment of inertia of the whole *minus* the moment of inertia of the part removed.

As an example, consider the polar moment of inertia with respect to its center of the circular ring formed by removing from a circle c of radius R, a concentric circle c' of radius r. Denoting the area of c by A, and that of c' by A', the polar moment of inertia of c by I_0, and that of c' by I_0', we have



$$I_0 = \frac{AR^2}{2}, \quad I_0' = \frac{A'r^2}{2}.$$

Hence the polar moment of inertia of the remaining ring is

$$\bar{I} = \tfrac{1}{2}(AR^2 - A'r^2) = \frac{\pi}{2}(R^4 - r^4)$$

$$= \frac{\pi}{2}(R^2 + r^2)(R^2 - r^2).$$

The area of the ring $\bar{A}$ is

$$\bar{A} = \pi R^2 - \pi r^2.$$

Hence

$$\bar{I} = \frac{\bar{A}}{2}(R^2 + r^2).$$

That is, the polar moment of inertia with respect to its center of a circular ring lying between two concentric circles of radii R and r is equal to one half the product of its area by the sum of the squares of the radii.

By the principle of Art. 19, we may at once extend this result to apply to a hollow circular column of outer radius R and inner radius r. Denoting by I the moment of inertia of the column with respect to its axis, we have

$$I = \frac{M}{2}(R^2 + r^2).$$

THEOREM. *The moment of inertia of a homogeneous hollow circular column with respect to its axis is equal to one half the product of its mass by the sum of the squares of the inner and outer radii.*

21. Routh's rules. The following moments of inertia occur frequently and should be committed to memory:

The moment of inertia of

(1) a rectangle whose sides are $2a$ and $2b$ with respect to an axis through its center in its plane perpendicular to the side $2a$ $\Bigg\} = M\dfrac{a^2}{3}$;

with respect to an axis through its center perpendicular to its plane $\Bigg\} = M\dfrac{a^2 + b^2}{3}$;

(2) an ellipse of semiaxes a and b with respect to the major axis (a) $\Big\} = M\dfrac{b^2}{4}$;

with respect to the minor axis (b) $= M\dfrac{a^2}{4}$;

(a circle is an ellipse with semiaxes each equal to a)

(3) an ellipsoid of semiaxes a, b, c, with respect to the axis (a) $\Big\} = M\dfrac{b^2+c^2}{5}$;

(a sphere is an ellipsoid with $a = b = c$)

(4) a parallelopiped whose edges are $2a$, $2b$, $2c$, with respect to an axis through its center perpendicular to the plane containing the sides b and c $\Big\} = M\dfrac{b^2+c^2}{3}$;

(5) a circular cone the radius of whose base is a with respect to its axis $\Big\} = \dfrac{3}{10}Ma^2$.

As an aid to the memory, the first four rules may be combined into one known as Routh's rule:

$$\left.\begin{array}{l}\text{Moment of inertia}\\ \text{with respect to an}\\ \text{axis of symmetry}\end{array}\right\} = \text{Mass} \times \frac{\begin{pmatrix}\text{Sum of squares of per-}\\ \text{pendicular semiaxes}\end{pmatrix}}{3,\ 4,\ \text{or}\ 5}.$$

The denominator is to be 3, 4, or 5, according as the body is rectangular, elliptical, or ellipsoidal.

As an example of the application of Routh's rule, suppose it is required to find the moment of inertia of a circle of radius a with respect to a diameter. We notice that the perpendicular semiaxis in its plane is a and the semiaxis perpendicular to its plane is zero. Hence the moment of inertia is $M\dfrac{a^2}{4}$. Again, let it be required to find the moment of inertia with respect to a line through the center of the circle and perpendicular to its plane. The perpendicular semiaxes are each equal to a and the moment of inertia is

$$M\frac{a^2+a^2}{4} = M\frac{a^2}{2}.$$

PROBLEMS

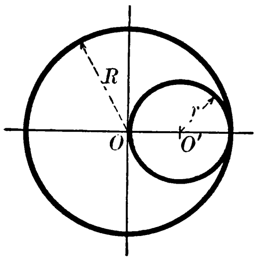
1. Find the moment of inertia of the hollow column of Art. 20 with respect to a line perpendicular to the XY-plane, (1) through the outer circumference; (2) through the inner circumference. *Ans.* (1) $\dfrac{M}{2}(3R^2 + r^2)$; $\dfrac{M}{2}(R^2 + 3r^2)$.

2. Find the moment of inertia of the circular ring, Art. 20, relative to OX.

$$\textit{Ans. } \frac{\bar{A}}{4}(R^2 + r^2).$$

3. Find the moment of inertia of the ring with respect to the tangents to the circles c and c'.

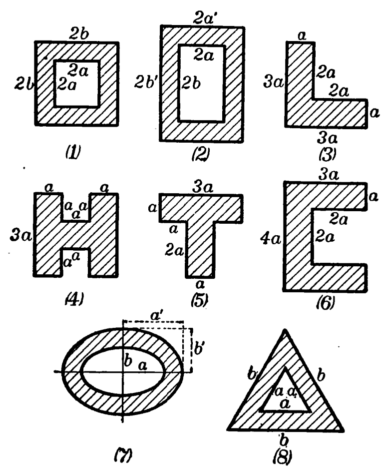
4. Find the moment of inertia of a circular area having a smaller circular area cut from it as in the figure, (1) with respect to a line through O perpendicular to the plane of the circle; (2) with respect to a diameter of the larger circle perpendicular to OO'; (3) with respect to a line through O' perpendicular to the plane of the circle; (4) with respect to the diameter OO'.



Ans. (1) $\frac{13}{24} MR^2$; (2) $\frac{11}{48} MR^2$; (3) $\frac{23}{24} MR^2$; (4) $\frac{15}{48} MR^2$.

5. A square is removed from a circle, the diagonals of the square intersecting at the center of the circle. Find I with respect to (1) an axis passing through the center of the circle perpendicular to its plane; (2) an axis perpendicular to the plane and passing through one corner of the square; (3) a diameter which is also a diagonal of the square.

6. Find the moment of inertia with respect to the gravity axis parallel to an edge of the beams whose cross sections are shown in the following figures.



22. System of material particles. By a material particle, or simply particle, is meant a portion of matter of so small a volume that the volume is regarded as reduced to a point. In other words, it is a weighted point or point mass. The moment of mass of a particle of mass m at the point P with respect to any line or plane equals the product of m by the perpendicular distance to P from the line or plane.

The center of mass of any system of particles of mass m_1 at $P_1(x_1, y_1, z_1)$, m_2 at $P_2(x_2, y_2, z_2)$, etc., is defined by the equations

$$(1)\qquad \begin{cases} \bar{x} = \dfrac{m_1x_1 + m_2x_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{\Sigma mx}{\Sigma m}, \\ \bar{y} = \dfrac{m_1y_1 + m_2y_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{\Sigma my}{\Sigma m}, \\ \bar{z} = \dfrac{m_1z_1 + m_2z_2 + \cdots}{m_1 + m_2 + \cdots} = \dfrac{\Sigma mz}{\Sigma m}. \end{cases}$$

Similarly, the moment of inertia of a particle of mass m at P with respect to any axis equals the product of m by the *square* of the perpendicular distance from P to the axis.

Thus for a system of particles lying in one plane whose masses are m_1 at $P_1(x_1, y_1)$, m_2 at $P_2(x_2, y_2)$, etc., we have

$$I_x = m_1y_1^2 + m_2y_2^2 + \cdots = \Sigma my^2,$$
$$I_y = m_1x_1^2 + m_2x_2^2 + \cdots = \Sigma mx^2,$$
$$I_0 = I_x + I_y = \Sigma m(x^2 + y^2) = \Sigma m\rho^2.$$

PROBLEMS

1. Three edges of a unit cubical frame without weight are taken as the coördinate axes, and particles are placed at all the corners except at the origin. Find I with respect to each face, edge, and vertex of the cube, (1) when the particles are of equal mass; (2) when the masses vary as the squares of their distances from the origin.

2. A straight rod of negligible mass and length a has five particles of equal mass situated on it at equal intervals of $\frac{1}{4}a$. Find I and r_0^2, (1) with respect to one end; (2) with respect to the middle point; (3) find I when the masses increase in arithmetical progression from the end.

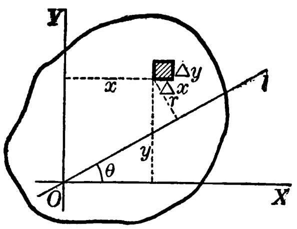
3. Given three particles of equal mass, situated at the vertices of an equilateral triangle. Find (1) I and r_0^2 with respect to one side; (2) with respect to a line parallel to one side passing through the opposite vertex.

4. A regular hexagon has particles at middle points of five of its sides. The masses of the particles taken in order are as 1, 2, 3, 4, 5. Find I and r_0^2 with respect to the unweighted side. *Ans.* $I = 20.25\,a^2$; $r_0^2 = 1.35\,a^2$.

23. Ellipse of inertia. This section is concerned with the solution of the problem,

To determine the moment of inertia of an area with respect to any gravity axis. Let O be the center of mass of a given area; OX,

OY any two mutually perpendicular axes through it, and l any other gravity axis making with OX the angle θ. Then, by Art. 11, (1),



(1) $$I_l = \int\int r^2 dA,$$

where r is the perpendicular distance from l to any interior point (x, y) of the area. The equation of l may be written

(2) $$-x \sin\theta + y \cos\theta = 0,$$

and hence (56, Chapter XIV) we have

(3) $$r = -x \sin\theta + y \cos\theta,$$

when (x, y) is the interior point in question. Substituting in (1),

(4) $$I_l = \int\int (-x \sin\theta + y \cos\theta)^2 dA, \text{ or,}$$

$$I_l = \sin^2\theta \int\int x^2 dA - 2 \sin\theta \cos\theta \int\int xy dA + \cos^2\theta \int\int y^2 dA.$$

The second integral in the right-hand member has not thus far been discussed. If we set this equal to P_{xy}, we may write

(5) $$I_l = I_x \cos^2\theta - 2 P_{xy} \sin\theta \cos\theta + I_y \sin^2\theta,$$

where

(XX) $$P_{xy} = \int\int xy \, dA,$$

and is called the *product of inertia* with respect to the axes OX and OY. It is easy to see that I_l assumes a maximum and a minimum value as the axis l rotates about O. In fact, since

(6) $$\frac{dI_l}{d\theta} = -2 I_x \cos\theta \sin\theta - 2 P_{xy}(\cos^2\theta - \sin^2\theta) + 2 I_y \sin\theta \cos\theta,$$

setting the right-hand member equal to zero gives

(7) $$(I_y - I_x) \sin 2\theta - 2 P_{xy} \cos 2\theta = 0,$$

from which $$\tan 2\theta = \frac{2 P_{xy}}{I_y - I_x}.$$

The values of θ determined by this equation will give axes l_1 and l_2 for which I_l is a maximum and a minimum respectively. Moreover, since these values of θ differ by $\frac{\pi}{2}$, l_1 and l_2 are perpendicular. They are called the *principal axes of inertia.*

Obviously, if $P_{xy} = 0$, the roots of (7) are $\theta = 0$, $\theta = \frac{\pi}{2}$, and hence OX and OY are already the principal axes. Let us now assume this to be the case. Then (5) becomes

$$(8) \qquad I_l = I_x \cos^2 \theta + I_y \sin^2 \theta.$$

Introducing the radii of gyration by setting

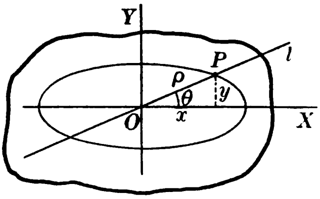
$$I_l = Ar_l^2, \quad I_x = Ar_x^2, \quad I_y = Ar_y^2,$$

then (8) becomes

$$\text{(XXI)} \qquad \boldsymbol{r_l^2 = r_x^2 \cos^2 \theta + r_y^2 \sin^2 \theta}.$$

This equation gives the radius of gyration with respect to any axis in terms of the *principal radii of gyration*, r_x and r_y. For convenience we now write

$$(9) \qquad r_x = \frac{1}{a}, \quad r_y = \frac{1}{b}.$$



Thus (XXI) becomes

$$(10) \qquad r_l^2 = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}.$$

Let us now draw the ellipse,

$$(11) \qquad 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

If (ρ, θ) are the polar coördinates of the point P where the axis l cuts this ellipse, then in (11)

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \text{ and we get}$$

$$(12) \qquad 1 = \frac{\rho^2 \cos^2 \theta}{a^2} + \frac{\rho^2 \sin^2 \theta}{b^2}, \text{ or also}$$

$$\frac{1}{\rho^2} = \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}.$$

Comparison with (10) gives the result

$$(13) \qquad r_l^2 = \frac{1}{\rho^2}.$$

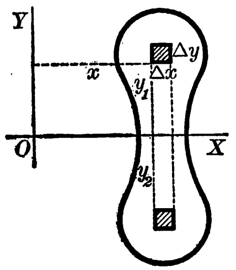
The ellipse (11) is called the *ellipse of inertia*, and the result indicated by (13) may be stated thus:

If the ellipse of inertia is drawn for any plane area, the radius of gyration for any gravity axis equals the reciprocal of the radius vector of the point in which the axis intersects the ellipse.

The principal axes of inertia are those for which the product of inertia is zero; that is,

$$P_{xy} = \int\int xy\,dx\,dy = 0.$$

It is easy to see that $P_{xy} = 0$ if either OX or OY is an axis of symmetry. For example, if OX is such an axis, then in the sum of the products



$$xy\Delta x\Delta y$$

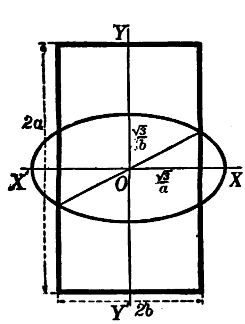
the terms will occur in pairs with the same x and with y's differing only in sign. The terms in each such pair will cancel, and hence the limit of the sum is also zero. This consideration gives the result:

Any axis of symmetry is necessarily a principal axis.

The process, then, of determining the moment of inertia with respect to any gravity axis is the following:

(1) If the figure has no axis of symmetry, choose any pair of rectangular axes, calculate I_x and I_y by (IX), and P_{xy} by (XX). Then use equation (5), or solve equation (7) for θ and determine the principal axes and the principal radii of gyration. Choose these axes for the new axes of coördinates and draw the ellipse of inertia (11). Then apply the theorem just stated to find r_l or use (XXI).

(2) If the figure has an axis of symmetry, choose this for OX or OY, calculate r_x and r_y, and draw the ellipse of inertia (11) or use (XXI).



Illustrative Example. Find the moment of inertia for any gravity axis of a rectangle.

Solution. Taking OX and OY as in the figure, then

$$I_x = \frac{Aa^2}{3}, \quad I_y = \frac{Ab^2}{3},$$

and

$$\therefore r_x^2 = \frac{a^2}{3}, \quad r_y^2 = \frac{b^2}{3},$$

and the equation of the ellipse of inertia is

(1) $$a^2x^2 + b^2y^2 = 3.$$

The radius of gyration for any gravity axis is then the reciprocal of the radius vector of its point of intersection with the ellipse, or also, by (XXI),

$$r_i^2 = \tfrac{1}{3}(a^2 \cos^2 \theta + b^2 \sin^2 \theta).$$

PROBLEMS

1. Show that the ellipse of inertia for any regular polygon is a circle. What is the conclusion regarding the moment of inertia with respect to any gravity axis?

2. Find I for a rectangle whose sides are $2a$ and $2b$ with respect to a diagonal.

Ans. $\dfrac{2}{3} M \dfrac{a^2 b^2}{a^2 + b^2}$.

3. Find I for an isosceles triangle with respect to an axis through its center of area and inclined at an angle α to its axis of symmetry, a being its altitude and $2b$ its base.

Ans. $\frac{1}{6} M(\frac{1}{3} a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)$.

4. Find I for an ellipse with respect to a diameter making an angle α with the major axis.

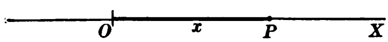
Ans. $I = \dfrac{1}{4} M(b^2 \cos^2 \alpha + a^2 \sin^2 \alpha) = \dfrac{1}{4} M \dfrac{a^2 b^2}{r^2}$, where $r = \dfrac{1}{2}$ diameter.

CHAPTER II

KINEMATICS OF A POINT. RECTILINEAR MOTION

That portion of mechanics which is concerned with the study of motion is called *dynamics*. The subject matter of dynamics is divided into two parts, *kinematics* and *kinetics*. Kinematics treats of pure motion, that is, motion without reference to the mass of the body which is moving or the forces producing the motion. It has to do solely with the relations of time and space. Kinetics treats of motion, including consideration of the mass of the body moved and the forces acting upon it. This chapter treats of the kinematics of a point which moves on a straight line.

24. Motion on a straight line. In order to indicate the position of a point upon a line, we select on that line a fixed point O,



called the origin. The position of any point P with respect to O may then be determined by the length OP and its direction from the origin. For the application of mathematical analysis to the rectilinear motion of a point, it is necessary to regard the path as a directed line;* that is, we must assume an origin, a unit of length, and a direction. If the measure of the length OP be denoted by x, then it is obvious that x is variable if P is a moving point. The motion of P is said to be completely determined when the position of P is known at every instant of time; that is, when the variable x is a function † of the time t, since the position is determined by the value of x. Hence for rectilinear motion we have the relation

$$\text{(I)} \qquad x = \phi(t).$$

This equation is called the *equation of motion*. Its significance is this, that from it we may find the position of the moving point at any instant of time.

* Analytic Geometry, p. 23. † Calculus, p. 12.

In order to indicate instants of time it is necessary to select some fixed instant from which the time may be reckoned, forward and backward.

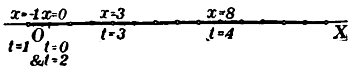
This fixed instant, called the origin of time, is denoted by $t = 0$, and time *before* is indicated by a *minus* sign, time *after* by a *plus* sign.

The position of the moving point when $t = 0$ is called the *initial position*. The corresponding value of x is called the *initial value of* x and is denoted by x_0. From (I) we have,

$$x_0 = \phi(0).$$

For example, if the equation of motion of a moving point is $x = t^2 - 2t$, we find the table of values of t and x as given. We see, therefore, that the point

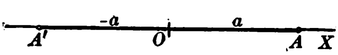
t	x
0	0
1	-1
2	0
3	3
4	8
etc.	etc.



moves from the initial position O to the left and, after reaching the extreme position $x = -1$, thereafter moves continuously to the right.

As a second example, consider the motion defined by the equation $x = a \cos \frac{1}{2}\pi t$. Remembering from trigonometry that the cosine of a variable, increasing angle varies from 1 to -1 inclusive, it is plain that with increasing time, x varies from a to $-a$ inclusive; that is,

t	x
0	a
1	0
2	$-a$
3	0
4	a
etc.	etc.



the moving point P oscillates between the points A and A' of the figure. The initial position is A, since $x_0 = a$, and the point is again at A after the lapse of four seconds.

The vibratory motion just discussed is an example of simple harmonic motion, and will appear frequently in these pages.

25. Velocity. By the velocity at any instant of a point in rectilinear motion is meant the *time-rate of change* of its position at that instant. When the equation of motion is $x = \phi(t)$, the velocity is the rate of change of x with respect to t; that is,* the

* Calculus, p. 148.

derivative of x with respect to t. Hence, denoting the velocity at any instant by v, we have

$$\text{(II)} \qquad v = \frac{dx}{dt} = \phi'(t).$$

The value of the velocity at the origin of time is called the *initial velocity*, and is denoted by v_0. From (II), we have

$$v_0 = \phi'(0).$$

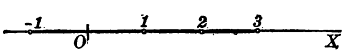
Dimensions. Velocity is defined as the limit of the quotient of distance Δx by time Δt. The derived unit of velocity is therefore expressed in terms of the fundamental units of length and of time by the dimensional equation

$$\text{Velocity} = \frac{\text{length}}{\text{time}}.$$

If $v = \phi'(t)$ is positive for the value $t = t_1$, we know that at the instant $t = t_1$, $x = \phi(t)$ is an increasing function* of t, and the point is moving towards the right along the directed line OX. If v is negative, x is a decreasing function of t, and the point is moving towards the left. If for $t = t_1$, $v_1 = \phi'(t_1) = 0$, the point at the instant $t = t_1$ is at rest. If the velocity is constant, the motion is said to be *uniform*. The numerical value of the velocity is called the *speed*.

For example, to discuss the velocity of a point when its equation of motion is $x = t^2 - 2t$, we find, by differentiation, $v = 2t - 2$. Giving t successive values, we obtain the values in the table. The point 0 is the

t	x	v
0	0	-2
1	-1	0
2	0	2
3	3	4
etc.	etc.	etc.



initial position, and -2 the initial velocity. The point is therefore moving in the negative direction along the line OX with a speed of 2 units of distance per unit of time.† At the instant $t = 1$ the velocity is zero and the point is at rest. For values of t greater than 1 the velocity is positive, and the point moves in the positive direction along OX.

* Calculus, p. 116.

† That is, two feet per second, if the unit of distance is one foot and the unit of time one second.

26. Acceleration. If the velocity of a moving point is variable, the point is said to have acceleration. In mathematical terms, acceleration is the time-rate of change of velocity. That is, acceleration at any instant is the derivative of velocity with respect to the time. Hence, denoting acceleration at any instant by f, we have

$$\text{(III)} \qquad f=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{d^2x}{dt^2};$$

or, acceleration is the second derivative of the distance with respect to the time. When the equation of motion is $x = \phi(t)$, we obtain, by differentiation,

$$f=\frac{d^2x}{dt^2}=\phi''(t).$$

The acceleration may be expressed in another form, which is frequently useful in the solution of problems in mechanics. We have $x = \phi(t)$, and this may be solved for t, giving

$$(1) \qquad t = \psi(x).$$

The velocity is a function of t; namely, $v = \phi'(t)$. When the value of t from (1) is substituted in this expression for the velocity, we have v expressed as a function of x.

$$(2) \qquad v = F(x).$$

This expression determines the velocity when the position is known. We have, from calculus,*

$$\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}.$$

Since $\qquad v=\dfrac{dx}{dt},$ therefore, $f=v\dfrac{dv}{dx}.$

For convenience, the preceding results may be summarized:

I. Equation of motion,† $\qquad x=\phi(t).$

II. Velocity at any instant, $\qquad v=\dfrac{dx}{dt}=\phi'(t).$

III. Acceleration at any instant, $\qquad f=\dfrac{dv}{dt}=\dfrac{d^2x}{dt^2}=\phi''(t)=v\dfrac{dv}{dx}.$

* p. 57.

† Other letters, *e.g.* y, s, will be used also to denote the position of the point P.

The physical meaning of the algebraic sign of the acceleration is made apparent by the following consideration. If the point P moves along OX towards the right, the velocity is positive; if towards the left, the velocity is negative. The acceleration is positive if v increases *algebraically*, and negative if v decreases *algebraically*. Hence, if

P	*moves*	*to*	*the*	*right*	*with*	*increasing*	*speed*, $v > 0, f > 0$;
P	"	"	"	"	"	*decreasing*	" $v > 0, f < 0$;
P	"	"	"	*left*	"	*increasing*	" $v < 0, f < 0$;
P	"	"	"	"	"	*decreasing*	" $v < 0, f > 0$.

If the acceleration is constant, the motion is said to be *uniformly accelerated*. The special case when the acceleration is zero, and hence the velocity constant, has been already referred to in Art. 25 as that of uniform motion.

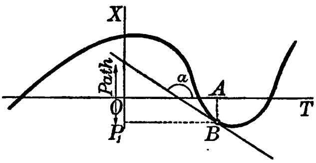
Dimensions. Acceleration is defined as the limit of the quotient of velocity Δv by time Δt. Its dimensions are therefore velocity divided by time, or distance divided by the square of the time. The relation between the derived unit of acceleration and the fundamental units of length and of time is expressed by the dimensional equation

$$\text{Acceleration} = \frac{\text{length}}{\text{time}^2}.$$

Two systems of units are in common use, the English and French. These are given in the table:

Units	*English*	*French*
distance	foot	centimeter
time	second	second
velocity	1 ft. per sec.	1 cm. per sec.
acceleration	1 ft. per sec. in 1 sec.	1 cm. per sec. in 1 sec.

27. Distance-time diagram. Discussion. The preceding discussion has shown that distance, velocity, and acceleration of a moving point are functions of the time. The determination of the variation of these variables with the time constitutes the discussion of the motion. The graph of the equation of motion is very useful in making the discussion. Since x is a function of t, we may plot the curve represented by the equation $x = \phi(t)$, where t is the abscissa and x the ordinate. This curve is called the distance-time diagram. For a given instant of time, t_1, we have a given value of the abscissa; *e.g.* OA in the figure. The corresponding value of x is the ordinate AB and the position* on the path OX is P_1.



Since the velocity is the derivative of x with respect to t, its value is given geometrically by the slope of the tangent at B; that is, by tan α. The numerical value of the acceleration is not given directly by the figure, but its sign is determined by noticing the form of the curve. If the curve is concave upwards, the sign of the acceleration is positive; if concave downwards, the sign is negative.† Maximum and minimum points on the graph of the equation of motion indicate extreme‡ positions of the point moving on the straight line; that is, positions where the velocity is zero. At such a point the velocity changes sign. With reference to the moving point P this means that it ceases to move in one direction and begins to move in the opposite direction. A maximum point corresponds to an extreme position upwards, since the first derivative changes from plus to minus. For a maximum point the second derivative is negative; hence for an extreme upward position the acceleration is negative. Similarly, a minimum point corresponds to an extreme downward position and the acceleration is positive. A point of inflection on the graph of the equation of motion indicates that at the corresponding instant of time the acceleration (which is the second derivative of x with respect to t) is zero. When the characteristics of the motion have been ascertained from this discussion, it will be convenient to take the path along a horizontal line. The properties already known of the motion on the X-axis are readily interpreted on the horizontal path.

* The student must be careful not to confuse the *distance-time* curve with the *path* of the point. The path lies on OX, and the position of the point at any instant, t_1, is found by constructing the point B in the diagram whose abscissa equals t_1, and then projecting this point on to the distance axis, as P_1 in the figure.

† Calculus, Chapter IX.

‡ The word "extreme" here means *relative extreme,* just as in geometry the word "maximum" means *relative maximum.*

ILLUSTRATIVE EXAMPLES

1. Discuss and draw the distance-time diagram for the motion defined by

(1) $$x = t^3 - 3t^2 + 2t.$$

Solution. By differentiation, we obtain

(2) $$v = 3t^2 - 6t + 2,$$

(3) $$f = 6t - 6.$$

The extreme positions of the moving point, and consequently the maximum and minimum points on the graph, are given by the condition,

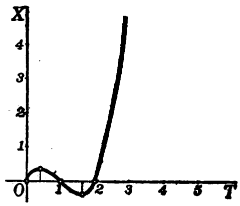
$$v = 3t^2 - 6t + 2 = 0,$$

whence $$t = 1 \pm \frac{\sqrt{3}}{3}, \text{ or approximately}$$

$$t_1 = 0.4, \; t_2 = 1.6.$$

The corresponding values of x are approximately

$$x_1 = 0.38, \; x_2 = -0.38.$$



For $t < 0.4$ the velocity is positive.

For $0.4 < t < 1.6$ the velocity is negative.

For $t > 1.6$ the velocity is positive.

The acceleration is zero, and consequently there is a point of inflection on the graph when $t = 1$. The corresponding value of x is 0. For $t < 1$ the acceleration is negative, and since $f = \frac{dv}{dt}$, the velocity is decreasing (algebraically). For $t > 1$, the acceleration is positive and the velocity is increasing. The distance-time diagram may now be drawn. We may summarize the results obtained in the following table :

t	x	v	f
0.	0.	2.	$-6.$
0.4	0.38	0.	-3.6
1.	0.	$-1.$	0.
1.6	-0.38	0.	$+3.6$
2.	0.	2.	6.
3.	6.	11.	12.
increases	increases	increases	increases

This table and the preceding graph are equivalent. From either we may make the discussion of the motion, and in the solution of problems each should serve as an aid to and a check upon the other. The discussion of the motion on a horizontal line follows. When t is zero the point P is at 0. As t increases from 0 to 0.4, P moves to the right with a velocity which is decreasing numerically. When $t = 0.4$, the velocity is zero and the point P is at an extreme position

$x = 0.38$ to the right (the acceleration is negative). As t increases from 0.4 to 1.6, the point moves to the left. When $t = 1.6$, the velocity is again zero and the point

is at an extreme position $x = -0.38$ to the left (the acceleration is positive). As t increases from the value 1.6, the point moves always to the right with increasing velocity and acceleration.

2. Discuss and draw the distance-time diagram for the motion defined by

(1) $$x = a \cos kt.$$

Solution. Differentiating, we obtain

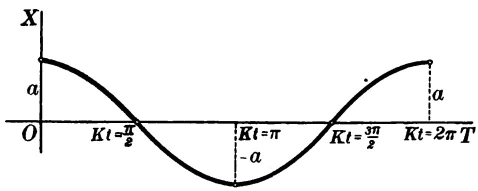
(2) $$v = -ak \sin kt,$$

and for the acceleration, differentiating (2),

(3) $$f = -ak^2 \cos kt = -k^2 x \text{ [from (1)]}.$$

Hence the acceleration and distance are proportional and *differ in sign.* Such a motion is called a *simple harmonic motion.*

The locus of (1) is a cosine curve, the properties of which are well known. The graph of the equation of motion has maxima when $kt = 2n\pi$ (n any integer),

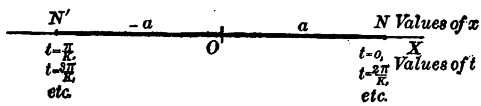


minima when $kt = (2n+1)\pi$, and points of inflection when $kt = \dfrac{(2n+1)}{2}\pi$. At any maximum point the ordinate is equal to a, and at any minimum point it is equal to $-a$. The variation of x, v, and f is exhibited in the table.

Angle kt	t	x	v	f
0	0	a	0	$-ak^2$
$\frac{\pi}{2}$	$\frac{\pi}{2k}$	0	$-ak$	0
π	$\frac{\pi}{k}$	$-a$	0	ak^2
$\frac{3\pi}{2}$	$\frac{3\pi}{2k}$	0	ak	0
2π	$\frac{2\pi}{k}$	a	0	$-ak^2$
etc.	etc.	etc.	etc.	etc.

It is now easy to make the discussion of the motion. When $t = 0$, $x_0 = a$, $v_0 = 0$, f_0 is negative, and the point starts from an extreme position to the right. As t

increases from 0 to $\frac{\pi}{k}$, the figure [see also (2)] shows that the slope of the tangent line (and consequently also the velocity) is negative, and the point moves towards the left. When $t = \frac{\pi}{k}$, $x = -a$, $v = 0$, $f > 0$, and the point is at an extreme position to the left. As t increases from $\frac{\pi}{k}$ to $\frac{2\pi}{k}$, the velocity is positive and the point moves to the right. When $t = \frac{2\pi}{k}$, we have again the initial values of x, v, and f. As t increases from the value $\frac{2\pi}{k}$, the motion just described is repeated again and again. The motion is a *vibration* or *oscillation* between the points N and N' of the figure.



The distance a is called the *amplitude* of the vibration. The time required to move from N to N' and back to N again is $\frac{2\pi}{k}$. This is called the *period* of the vibration. The point midway between N and N' (the point 0 in the figure) is called the *center* of the vibration.

The periodicity of the motion may be best established by reasoning thus. We note first that the series of values of *any* trigonometric function is repeated when the angle has increased 2π radians. Since x, v, and f are in this case dependent in their variation upon sine or cosine, then it is plain that they assume their original values when kt has increased to $kt + 2\pi$. But

$$kt + 2\pi = k\left(t + \frac{2\pi}{k}\right).$$

Hence t has changed to $t + \frac{2\pi}{k}$, and the increment $\frac{2\pi}{k}$ is accordingly the *period.*

3. Discuss the motion defined by

(1) $$x = A\cos(kt + B).$$

Solution. The distance-time diagram is again a cosine curve with A for maximum displacement. The difference from the preceding example consists in this: the initial position on the path is not at an extreme position, but at $x_0 = A\cos B$. The conclusion is, therefore:

The equation (1) *represents a harmonic motion with the period* $\frac{2\pi}{k}$, *and this is true for all values of* A *and* B.

Equation (1) is the general solution of the equation 71, Chapter XIV, which is called the differential equation of harmonic motion. The statement just made explains the designation.

4. Discuss the motion defined by

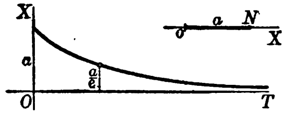
(1) $$x = ae^{-t}.$$

Solution. Differentiating and using (1), we obtain

(2) $$v = -ae^{-t} = -x,$$

(3) $$f = ae^{-t} = x.$$

In this case, therefore, the acceleration and distance are proportional and *agree in sign.* From (1) $x = \frac{a}{e^t}$, and therefore x, which is always positive, decreases numerically as t increases; v is always negative and decreases numerically (that is, the speed decreases).



The graph is now readily drawn and exhibits the motion of a point from N towards O with constantly diminishing speed and acceleration. The motion dies away as O is approached. There is obviously no period, and the motion is called *aperiodic.*

5. Discuss and draw the graph of the equation of motion,

(1) $$x = ae^{-\alpha t}\cos kt,$$

a, α, k being arbitrary, positive constants.

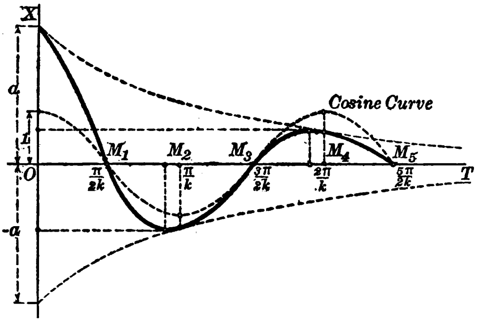
Solution. Differentiating, we obtain for v and f the expressions,

(2) $$v = -ae^{-\alpha t}(\alpha\cos kt + k\sin kt),$$

(3) $$f = ae^{-\alpha t}[2\,\alpha k\sin kt + (\alpha^2 - k^2)\cos kt].$$

The graph of (1) is readily constructed and the characteristics of the motion appear from it. Write (1) in the form of a product,

(4) $x = ae^{-\alpha t}\cdot\cos kt.$



The factor $\cos kt$ varies from -1 to $+1$. Hence the distance x varies from $-ae^{-\alpha t}$ to $+ae^{-\alpha t}$; that is, the graph of (4) is *bounded* by the curves

(5) $x = -ae^{-\alpha t}$, $x = ae^{-\alpha t}$.

These are the dotted lines of the figure.

Again, the product in (4) vanishes only when one of the factors is zero. But $e^{-\alpha t}$ is never zero for finite time. Hence $x = 0$ when and only when $\cos kt = 0$.

Furthermore, the graph touches * the boundary curve when $\cos kt = \pm 1$. We therefore draw also the *auxiliary curve* $x_1 = \cos kt$. We now observe that

(1) *The points of contact with the boundary curves are directly over (or under) the maximum and minimum points on the cosine curve.*

(2) *The required curve crosses the T-axis at the same points as the cosine curve.*

The graph may now be drawn, for we have merely to construct a winding curve from the initial point $t = 0$, $x = a$, which shall cross OT at M_1, M_3, M_5, etc., and *touch* the boundary curves at points corresponding to M_2, M_4, etc.

From this construction it is obvious that maximum and minimum values of X occur *between* each intersection on OX and the succeeding point of contact with

* For when $\cos kt = \pm 1$, then $\sin kt = 0$, and we find from (2) $v = \mp\alpha ae^{-\alpha t}$. This equals $\frac{dx}{dt}$, from (5). Hence the slope of (4) at M_2, M_4, etc., is the slope of the proper boundary curve (5).

the boundary curve; that is, for a value of t between successive odd and even multiples of $\frac{\pi}{2k}$. In fact, from (2), $v = 0$, when

$$(6) \qquad \alpha \cos kt + k \sin kt = 0 \text{ or } \tan kt = -\frac{\alpha}{k}.$$

Now the tangent is negative when the angle is of the second or fourth quadrants. Hence kt must lie between $\frac{\pi}{2}$ and $\frac{2\pi}{2}$, or $\frac{3\pi}{2}$ and $\frac{4\pi}{2}$, etc., or t is between successive odd and even multiples of $\frac{\pi}{2k}$.

The characteristics of the motion are now obvious. It may be described as a vibration with constantly diminishing amplitude. Remembering that the simple harmonic motion represented by the factor $a \cos kt$ of (1) has the constant amplitude a, it is plain that the presence of the second factor $e^{-\alpha t}$ accounts for the diminishing amplitude.

This factor diminishes as t increases, and is called the *damping factor*. The motion is called *damped vibration*.

From (6), it appears that a period of time equal to $\frac{2\pi}{k}$ (from $kt = 2\pi$) must elapse between successive maxima. The motion is accordingly said to have a period equal to $\frac{2\pi}{k}$, the same, namely, as the period of the undamped harmonic vibration (Ex. 2).

The successive amplitudes obey a simple law. For such positions differ by a semi-period, and hence two such values of x may be written in the form

$$x_1 = a'e^{-\alpha t}, \quad x_2 = a'e^{-\alpha\left(t + \frac{\pi}{k}\right)}.$$

Taking natural logarithms and subtracting, we obtain

$$\log x_1 - \log x_2 = \frac{\alpha\pi}{k}.$$

That is, *the logarithms of successive amplitudes form a decreasing arithmetical progression.*

This is otherwise expressed by the statement that the *logarithmic decrement of the amplitude is constant.*

6. Discuss the motion whose equation is

$$(1) \qquad x = Ce^{-\mu t} \cos (lt + \gamma),$$

in which C, μ, l, and γ are arbitrary constants, μ being positive.

Solution. The construction of the graph is precisely as in the previous example; namely, the *boundary* curves are

$$(2) \qquad x = \pm\, Ce^{-\mu t},$$

and the *auxiliary cosine curve* is

$$(3) \qquad x_1 = \cos (lt + \gamma).$$

The difference from the preceding case is in the *initial position*, which is now $x_0 = C \cos \gamma$, an arbitrary point on the path, not necessarily ($\gamma \neq 0$) an *extreme* position.

The result is then this:

The motion defined by (1) *is a damped vibration with the period* $\frac{2\pi}{l}$, *and this is true for all values of* C, γ, l, *and* μ, *provided* $\mu > 0$.

Equation (1) has the form of the general solution of the equation 73, Chapter XIV, which is called the differential equation of damped vibration. The theorem just stated explains this designation.

PROBLEMS

1. Show that each of the following motions is uniform or uniformly accelerated, draw the distance-time diagrams and discuss the motion:

(a) $x = 2 - 4t$;
(b) $y = at + b$;
(c) $s = 6t - 16t^2$;
(d) $y = 10 - t - 3t^2$;
(e) $x = a + bt + ct^2$;
(f) $s = \frac{1}{2}gt^2 + v_0 t$;
(g) $y = v_0 t - \frac{1}{2}gt^2$;
(h) $s = v_0 t + h$;
(i) $s = \frac{1}{2}gt^2 + v_0 t + s_0$;
(j) $y = 50 + 10t - 16t^2$;
(k) $s = \frac{1}{2}g \sin\alpha \cdot t^2$;
(l) $s = v_0 t - \frac{1}{2}g \sin\alpha \cdot t^2$;
(m) $x = 1000t - 16t^2$;
(n) $y = -1000t + 10t^2$.

2. Show that the distance-time diagrams of uniform and of uniformly accelerated motion are respectively a straight line and a parabola.

3. Show that each of the following is a simple harmonic motion.* Draw the distance-time diagrams, discuss the motion, and find the amplitude a and period T in each case.

(a) $x = 5 \sin t$;
(b) $y = 10 \cos t$;
(c) $s = 2 \cos \frac{1}{2}\pi t$;
(d) $x = 5 \sin \frac{3}{2}\pi t$;
(e) $y = a \sin kt$;
(f) $x = 5 \cos (t + \frac{1}{6}\pi)$;
(g) $y = 10 \sin (\frac{1}{2}\pi t - \frac{1}{4}\pi)$;
(h) $y = \sin t + \cos t$;
(i) $s = a \sin (kt + \alpha)$;
(j) $x = b \cos (\mu t - \beta)$;
(k) $x = 2 \sin t + 3 \cos t$.
(l) $x = a \cos kt + b \sin kt$.

Answers denoting *amplitude* by a and *period* by T.

(a) $a = 5,\ T = 2\pi$;
(b) $a = 10,\ T = 2\pi$;
(c) $a = 2,\ T = 4$;
(d) $a = 5,\ T = \frac{4}{3}$;
(e) $a = a,\ T = \dfrac{2\pi}{k}$;
(f) $a = 5,\ T = 2\pi$;
(g) $a = 10,\ T = 4$;
(h) $a = \sqrt{2},\ T = 2\pi$;
(i) $a = a,\ T = \dfrac{2\pi}{k}$;
(j) $a = b,\ T = \dfrac{2\pi}{\mu}$;
(k) $a = \sqrt{13},\ T = 2\pi$;
(l) $a = \sqrt{a^2 + b^2},\ T = \dfrac{2\pi}{k}$.

* Show that the given equation is obtained from $x = A \cos (kt + B)$ by replacing the constants A, k, B by particular values. Thus for (a), $x = 5 \sin t$, we set $A = 5, k = 1, B = -\frac{\pi}{2}$.

For $x = 5 \cos \left(t - \frac{\pi}{2}\right) = 5 \cos \left(\frac{\pi}{2} - t\right) = 5 \sin t$.

4. Show that the acceleration and the distance are proportional and differ in sign for each of the following motions (simple harmonic):

(a) $x = A \sin (kt + \alpha) + B \cos kt$;
(b) $y = A \sin kt + a \cos (kt + \beta)$;
(c) $s = a \sin (\mu t - \alpha) + b \cos (\mu t - \beta)$.

Reduce each to the form $A \cos (kt + B)$.

5. Discuss and draw the distance-time diagram of each of the following motions and show that each is a damped vibration.*

(a) $x = 5 e^{-\frac{1}{8}t} \cos \frac{1}{2} \pi t$;
(b) $y = 2 e^{-\frac{1}{10}t} \sin \frac{1}{2} \pi t$;
(c) $s = 10 e^{-\frac{1}{2}t} \cos t$;
(d) $x = 5 e^{-\frac{1}{2}t} \sin t$;
(e) $x = e^{-t} \cos\left(t + \frac{\pi}{2}\right)$;
(f) $y = 3 e^{-\frac{1}{3}t} \sin \left\{\frac{\pi}{2}(t+1)\right\}$;
(g) $x = ae^{-\beta t} \sin kt$;
(h) $y = 5 e^{-\frac{1}{2}t} \sin (t + \frac{1}{4}\pi)$;
(i) $s = e^{-at}(a \sin kt + b \cos kt)$;
(j) $x = 10 e^{-\frac{1}{\sqrt{3}}t} \cos t$;
(k) $s = e^{-at} \sin (kt + \beta)$;
(l) $y = e^{-at} \cos (kt + \beta)$.

6. Discuss and draw † the distance-time diagrams of the following equations of motion:

(a) $x = \sin t + \cos 2t$;
(b) $x = a \log (1 - t)$;
(c) $y = \frac{1}{2}(e^{t} + e^{-t})$;
(d) $y = e^{-\frac{1}{2}t} \cos t + \sin t$;
(e) $y = \sin \frac{1}{2} t + \sin \frac{1}{3} t$;
(f) $y = e^{-\frac{1}{10}t} \cos t + 10 \sin t$.

7. Show that every solution of $\frac{d^2s}{dt^2} + \mu s = \lambda$, where μ and λ are constants and $\mu > 0$, defines a harmonic motion. Find the period and the center.

Ans. $T = \frac{2\pi}{\sqrt{\mu}}$; $\left(\frac{\lambda}{\mu}, 0\right)$.

8. When will solutions of $\frac{d^2s}{dt^2} + 2\mu\frac{ds}{dt} + \lambda s = 0$ define damped vibrations?

Ans. $\lambda > \mu^2$.

9. Discuss and draw the distance-time diagrams of the following equations of motion:

(a) $x = t \sin t$;
(b) $y = e^{t} \cos t$;
(c) $s = (t + 1) \cos t$;
(d) $x = \frac{\sin t}{t}$;
(e) $y = \frac{\cos t}{t+1}$;
(f) $s = t \sin \left(t + \frac{\pi}{4}\right)$.

10. Discuss and draw the distance-time diagrams of the following equations of motion:

(a) $x = \sin t + 2$;
(b) $y = \cos t - 10$;
(c) $s = e^{-t} \cos t + 1$;
(d) $z = 10e^{-\frac{1}{5}t} \cos t + 5$;
(e) $x = ae^{-at} \cos kt + b$;
(f) $x = a \cos kt + b$;
(g) $y = a \cos kt + b \sin kt + c$;
(h) $s = A \sin (kt + \beta) + b$.

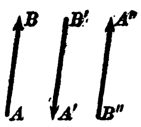
* Show that the given equation is obtained from (1), p. 52, by giving to C, μ, l, and γ particular values.

† When the function of the time is the sum of two simple functions, we may draw the graphs of the latter and *add* the corresponding ordinates. For example, in (a), add the ordinates of $x_1 = \sin t$ and $x_2 = \cos 2t$.

CHAPTER III

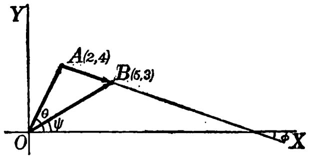
KINEMATICS OF A POINT. CURVILINEAR MOTION

28. Position in a plane or in space. Vectors. In the discussion of the rectilinear motion of a point the quantities involved were time t, position on the straight line x, velocity v, speed s, and acceleration f. Any value of t, x, v, or f is indicated by a single number (positive or zero or negative), and any value of s is indicated by a single number (positive or zero). Quantities which take on values that can be indicated by single numbers are called *scalar* quantities. Such quantities have magnitude ($+$ or $-$) only. A *vector* quantity is one which has *magnitude* and *direction*. For example, (1) the position of a point $P(\rho, \theta)$ in a plane is indicated by its distance from the origin (magnitude) and the angle which OP makes with the initial line; (2) the position of a point $P(\rho, \phi, \theta)$ in space is indicated by its distance from the origin and the direction of the line OP.* Since a scalar quantity has magnitude only, any value which it may take on can be represented graphically by the *length* of a line taken in the proper algebraic sense. To represent a vector quantity graphically the line must have *length* and *direction*. By indicating the direction properly the length may always be taken as positive. Hence we make the definition, *a vector is a straight line having length and direction.* From this definition we conclude that two vectors AB and $A''B''$ are equal if the lines AB and $A''B''$ are parallel, equal in length, and taken in the same sense. If the lines are parallel and equal in length, but taken in the opposite sense, that is, *if the directions differ by* 180°, as AB and $A'B'$, we say $AB = - A'B'$. A vector is zero if, and only if, its length is zero. In solving problems involving vectors we may always replace a vector by an equal vector, which is equivalent to saying that a vector may be moved providing it is kept always parallel to its original position.



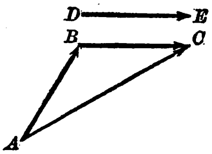
* Analytic Geometry, p. 394.

29. Addition of two vectors. If a point is moved in a plane, the *displacement* is a vector quantity. Suppose a point is moved from the origin to the position $A(2, 4)$. The displacement is represented by a vector whose length $OA = \sqrt{20}$ and whose direction is indicated by the angle θ which the line OA makes with the X-axis. Suppose the point is given a second displacement from A (2, 4) to B (5, 3). This displacement is represented by the vector AB, whose magnitude is $\sqrt{(2-5)^2+(4-3)^2}=\sqrt{10}$, and whose direction is given by the angle ϕ. These two displacements taken in order are evidently equivalent to a single displacement from O to B, which is represented by the vector OB, the magnitude of which is $\sqrt{34}$ and whose direction is given by the angle ψ. Hence we say that the vector OB is the sum of the vectors OA and AB.



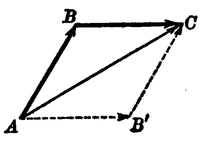
$$OB = OA + AB.$$

If two vectors AB and DE are given, we obtain the sum $AB+DE$ in the following manner. From the point B construct a vector $BC=DE$. The vector AC is now defined as the sum of AB and BC, and, therefore, as the sum of AB and DE.



$$AB + BC = AC.$$
$$\therefore AB + DE = AC.$$

The process of adding two vectors is essentially this. Bring the two vectors into such a position that they form a broken line ABC. Their sum is then equal to the closing line AC. It is readily seen that the order of addition can be changed without altering the sum.



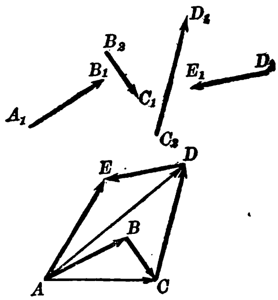
$$AB + BC = BC + AB.$$

The figure is a parallelogram and the proof is obvious.

Addition of any number of vectors. The preceding process is applicable to the addition of any number of vectors. Suppose it is required to find the sum of the vectors A_1B_1, B_2C_1, C_2D_1, and D_2E_1. This is accomplished by repeated application of the process of adding two vectors.

(1) Construct the vectors AB and BC equal respectively to A_1B_1 and B_2C_1. The sum of these two vectors is AC.

$$\therefore A_1B_1 + B_2C_1 = AC.$$



(2) Construct $CD = C_2D_1$. The sum of AC and CD is AD. That is,

$$A_1B_1 + B_2C_1 + C_2D_1 = AD.$$

(3) Construct $DE = D_2E_1$. The sum of AD and DE is AE. Therefore,

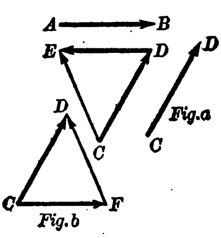
$$A_1B_1 + B_2C_1 + C_2D_1 + D_2E_1 = AE.$$

The process is applicable to any number of vectors and is essentially this. To add any number of vectors, form a broken line having its segments equal, respectively, to the given vectors; the sum is then the closing line.* Since the order of addition of two vectors may be changed without changing the sum, the order of addition of any number of vectors may be changed without changing the sum.

The sum of any number of vectors is called the *resultant* of those vectors.

30. Subtraction of vectors. Any vector AB may be subtracted from the vector CD by adding to CD the negative of AB. In the figure $DE = -AB$ and $CE = CD + DE = CD - AB$.

For practical purposes it is more convenient to obtain the difference of two vectors as follows: To subtract AB from CD, lay off the two vectors from the same origin; that is, construct $CF = AB$. Then



$$CF + FD = AB + FD = CD.$$

Whence, by transposing the term AB,

$$FD = CD - AB.$$

The results of the two methods are equal, as can be shown by comparing the equal triangles, figure *a* and figure *b*.

31. Multiplication of a vector by a scalar. If a vector AB is multiplied by a positive scalar W, the result is a vector $A'B'$

* Analytic Geometry, p. 47.

having the same direction as AB, while the magnitude of $A'B'$ is W times the magnitude of AB. For example, in the figure (a),

$$A'B' = 2\,AB.$$

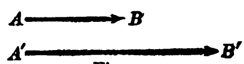


Fig. a

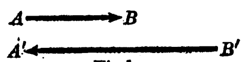


Fig. b

If a vector AB is multiplied by a negative scalar $-W$, the result is a vector $B'A'$ which has a direction opposite to that of AB and a magnitude equal to W times the magnitude of AB. For example, in the figure (b),

$$B'A' = -\,2\,AB.$$

To divide a vector by a scalar W, we multiply the vector by $\frac{1}{W}$. For example, in figure a,

$$AB = \tfrac{1}{2}\,A'B'$$

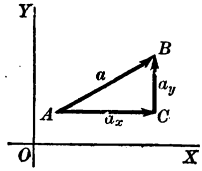
and in figure b,

$$AB = -\,\tfrac{1}{2}\,B'A'.$$

32. Resolution of plane vectors. Suppose a vector AB is given and it is required to find two vectors which are equivalent to AB, that is, whose sum is equal to AB. This may be done in an infinite number of ways. For, suppose C is *any* point, and the lines AC and CB are drawn. Then, by the definition of a vector sum,

$$AC + CB = AB.$$

The point C may be determined so that the vectors AC and CB are parallel to the X- and Y-axes respectively. This is accomplished by drawing through A a line parallel to the X-axis and through B a line parallel to the Y-axis. These two lines intersect in the required point C. For convenience we will denote the vector AB by **a**, the vector AC by* a_x, and CB by a_y. The vector a_x is called the component of **a** in the direction of the X-axis and the vector **a** is said to be *resolved* along the line OX; a_y is the component of **a** in the direction of the Y-axis, and **a** is said to be resolved along the line OY. It is evident that a_x is the projection of **a** on the X-axis and a_y is the projection of **a** on the Y-axis.

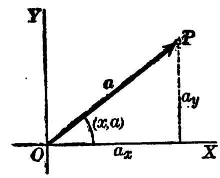


A vector may be resolved along any directed line by projecting the vector on that line; that is, the component of a vector along

* In using the components of a vector **a**, we need to give only the numerical values. The directions are indicated by the subscripts.

any directed line equals its magnitude multiplied by the cosine of the angle its direction makes with the given line.

In solving problems involving vectors it is usually more convenient to deal with the components. If the axial components of a vector **a** are a_x and a_y, it is evident from the figure that the magnitude* of **a** is given by $a = \sqrt{a_x^2 + a_y^2}$ and the direction of **a** is the same as the direction of a line from the origin to the point (a_x, a_y). If we denote the angle which the vector **a** makes with the X-axis by (x, a), we have



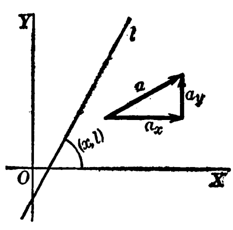
$$\cos(x, a) = \frac{a_x}{a}; \quad \sin(x, a) = \frac{a_y}{a}.$$

Hence we have the formulas:

$$\text{(I)} \qquad \begin{cases} a \ = +\sqrt{a_x^2 + a_y^2}, \\ a_x = a \cos(x, a), \\ a_y = a \sin(x, a) = a \cos(y, a). \end{cases}$$

In particular the components of the vector which represents the position of a point P in a plane are the rectangular coördinates of that point.

When the axial components a_x, a_y of a vector **a** are known, its component in the direction of any line l is readily found. Denote the angle which l makes with the X-axis by (x, l). The projection a_l of **a** upon l is equal to the sum of the projections of a_x and a_y upon l. The projection of a_x upon l is $a_x \cos(x, l)$ and the projection of a_y upon l is $a_y \sin(x, l)$. Therefore the component of **a** in the direction l is



$$\text{(II)} \qquad a_l = a_x \cos(x, l) + a_y \sin(x, l).$$

Components of the resultant of any number of plane vectors. Let **a**, **b**, **c** ⋯ be given vectors with components a_x, a_y; b_x, b_y; c_x, c_y ⋯ respectively. Let **R** (components R_x, R_y) be the resultant of **a**, **b**, **c** ⋯. By the definition of a vector sum, we regard **a**, **b**, **c** ⋯ as the segments of a broken line, while **R** is the

* The letter a represents the magnitude of the vector **a**.

closing line. By the second theorem of projection,* the sum of the projections of **a**, **b**, **c** ⋯ upon any line is equal to the projection of **R** upon that line. Therefore,

$$\text{(III)} \qquad \begin{cases} R_x = a_x + b_x + c_x + \cdots, \\ R_y = a_y + b_y + c_y + \cdots. \end{cases}$$

33. Vectors in space. The results for plane vectors may be extended at once to vectors in space. Any vector in space may be resolved along three mutually perpendicular lines by projecting the vector upon each of the lines.

If the three mutually perpendicular lines are the X-, Y-, and Z-axes, the components of the vector **a** are denoted by a_x, a_y, a_z. The magnitude of **a** is $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$, and its direction is the same as the direction of a line from the origin to the point (a_x, a_y, a_z). The direction cosines of the vector are

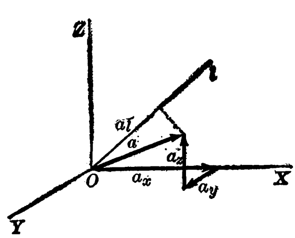
$$\cos(x, a) = \frac{a_x}{a}; \ \cos(y, a) = \frac{a_y}{a}; \ \cos(z, a) = \frac{a_z}{a}.$$

For space we have the formulas:

$$\text{(IV)} \qquad \begin{cases} a \ = +\sqrt{a_x^2 + a_y^2 + a_z^2}, \\ a_x = a \cos(x, a), \\ a_y = a \cos(y, a), \\ a_z = a \cos(z, a). \end{cases}$$

Since the second theorem of projection holds also in space,† the components of the resultant **R** of any number of vectors **a**, **b**, **c** ⋯ are

$$\text{(V)} \qquad \begin{cases} R_x = a_x + b_x + c_x + \cdots, \\ R_y = a_y + b_y + c_y + \cdots, \\ R_z = a_z + b_z + c_z + \cdots. \end{cases}$$



When the axial components a_x, a_y, a_z of a vector **a** in space are known, its component in the direction of any line l may be found. Let the direction angles of l be (x, l), (y, l), (z, l). The projection a_l of **a** upon l is equal to the sum of the projections of a_x, a_y, and a_z upon l.

The projection of a_x on l is $a_x \cos(x, l)$.

* Analytic Geometry, p. 47. † Analytic Geometry, p. 328.

The projection of a_y on l is $a_y \cos(y, l)$.

The projection of a_z on l is $a_z \cos(z, l)$.

Therefore,

$$\text{(VI)} \qquad \boldsymbol{a_l = a_x \cos(x, l) + a_y \cos(y, l) + a_z \cos(z, l).}$$

ILLUSTRATIVE EXAMPLES

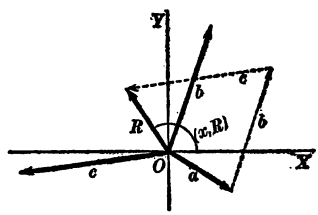
1. Find the resultant of the three plane vectors a, b, c whose components are (3, − 2), (2, 6), (− 7, − 1), respectively.

Solution.

By (III), $\begin{cases} R_x = 3 + 2 - 7 = -2, \\ R_y = -2 + 6 - 1 = 3. \end{cases}$

By (I), $\quad R = \sqrt{13}, \ \cos(x, R) = -\dfrac{2}{\sqrt{13}},$

$$\sin(x, R) = \frac{3}{\sqrt{13}}.$$



In the figure, this result is checked by graphical construction of the resultant.

2. Three vectors a, b, c in space have magnitudes equal to 12, 8, and 6, respectively, and their direction angles are as follows:

$$(a)\ (x, a) = \tfrac{3}{4}\pi,\ (y, a) = \tfrac{1}{2}\pi,\ (z, a) = \tfrac{1}{4}\pi;$$
$$(b)\ (x, b) = \tfrac{5}{6}\pi,\ (y, b) = \tfrac{2}{3}\pi,\ (z, b) = \tfrac{1}{2}\pi;$$
$$(c)\ (x, c) = \tfrac{1}{4}\pi,\ (y, c) = \tfrac{2}{3}\pi,\ (z, c) = \tfrac{2}{3}\pi.$$

Determine the resultant.

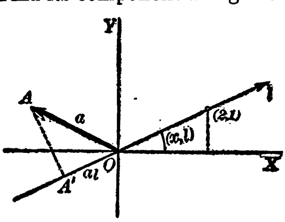
Solution. Finding the axial components by (IV), we have

$$a_x = -6\sqrt{2},\ a_y = 0,\ a_z = 6\sqrt{2},$$
$$b_x = -4\sqrt{3},\ b_y = -4,\ b_z = 0,$$
$$c_x = 3\sqrt{2},\ c_y = -4,\ c_z = -3.$$

Hence, applying (V), the resultant has the components

$$\begin{cases} R_x = -3\sqrt{2} - 4\sqrt{3}, \\ R_y = -8, \\ R_z = 6\sqrt{2} - 3. \end{cases}$$

3. Given in the XY-plane a vector a whose axial components are (− 2, 1). Find its component along the directed line from the origin to the point (2, 1).



Solution. In the figure, l represents the given line, OA the given vector, OA' the projection of OA on l.

By geometry, $\cos(x, l) = \dfrac{2}{\sqrt{5}}$; $\sin(x, l) = \dfrac{1}{\sqrt{5}}$.

By (II),

$$a_l = -2\frac{2}{\sqrt{5}} + 1\frac{1}{\sqrt{5}} = -\frac{3}{\sqrt{5}} = -\frac{3}{5}\sqrt{5}.$$

The negative sign indicates that a_l has the negative direction on l.

PROBLEMS

1. Determine in direction and magnitude the resultant of each of the following groups of vectors in a plane, given by their axial components. Verify the result in each case by a graphical construction.

(*a*) $(2, 0)$, $(-2, -6)$, $(5, 3)$.
(*b*) $(-1, 5)$, $(2, -1)$, $(8, 2)$.
(*c*) $(0, 1)$, $(5, 6)$, $(-2, -8)$, $(-3, -4)$.
(*d*) $(9, 0)$, $(10, 5)$, $(6, 2)$, $(1, 4)$, $(2, 3)$.
(*e*) $(0, -9)$, $(-1, -6)$, $(2, 5)$, $(-1, -8)$.

2. In the following examples the magnitude and angle made with OX of certain vectors are given. Determine the resultant in each case.

(*a*) $5, \frac{1}{3}\pi;\ 8, \frac{5}{6}\pi$. *Ans.* Components are $[(\frac{5}{2} - 4\sqrt{3}), (\frac{5}{2}\sqrt{3} + 4)]$.
(*b*) $2, \frac{1}{2}\pi;\ 9, \frac{7}{6}\pi$. *Ans.* $(-\frac{9}{2}\sqrt{3}, -\frac{5}{2})$.
(*c*) $3, \frac{3}{4}\pi;\ 1, \frac{3}{2}\pi$. *Ans.* $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}} - 1\right)$.
(*d*) $4, \frac{3}{2}\pi;\ 10, \frac{5}{6}\pi$. *Ans.* $(-5\sqrt{3}, 1)$.
(*e*) $4, \frac{7}{4}\pi;\ 9, \frac{11}{6}\pi$. *Ans.* $\left(\frac{4}{\sqrt{2}} + \frac{9}{2}\sqrt{3}, -\frac{4}{\sqrt{2}} - \frac{9}{2}\right)$.

3. In problem 2 find the component of the first vector along the directed line determined by the second.

Ans. (*a*) 0; (*b*) -1; (*c*) $-\frac{3}{\sqrt{2}}$; (*d*) -2; (*e*) $\sqrt{2}(\sqrt{3} + 1)$.

4. Given the axial components of the following vectors in space. Find the resultant of each group:

(*a*) $(1, 1, 5)$, $(2, -1, 6)$.
(*b*) $(1, 0, 8)$, $(-1, -1, 0)$.
(*c*) $(0, 6, 5)$, $(1, 9, -8)$.
(*d*) $(3, -4, 9)$, $(6, 2, 3)$.
(*e*) $(-1, 2, 8)$, $(4, 6, -2)$, $(9, 10, 11)$.

5. The magnitude and direction angles of certain vectors in space are as follows. Determine the resultant in direction and magnitude.

(*a*) $10, \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{5}{6}\pi;\ 5, \frac{1}{3}\pi, \frac{1}{3}\pi, \frac{3}{4}\pi$.

Ans. Components are $\left[\frac{5}{2}, \frac{15}{2}, 5\left(-\frac{1}{\sqrt{2}} - \sqrt{3}\right)\right]$.

(*b*) $6, \frac{1}{4}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi;\ 4, \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{2}{3}\pi$.

Ans. Components are $[(3\sqrt{2} + 2\sqrt{3}), -3, -5]$.

6. Determine the component of each of the pair of vectors in problem 5 (*a*), (*b*), along the other.

7. A point has uniform motion along OX with a velocity of 10 ft. per second. Find the component of the velocity along the directed line from $(0, 0)$ to $(3, 4)$.

Ans. 6 ft. per second.

8. Find the resultant of the following velocities, the capital letters indicating points of the compass as usual, and the numbers the magnitude:

15 N., 20 E., $20\sqrt{2}$ N.W., 35 W. *Ans.* $35\sqrt{2}$ N.W.

9. Find the resultant of:

(a) accelerations 5, 8, 10 parallel to the sides of an equilateral triangle taken in order. *Ans.* (Taking first component along X-axis)$(-4, -\sqrt{3})$.

(b) velocities 2, 5, 6, 3 parallel to the sides of a square taken in order.
Ans. $(-4, 2)$.

10. A point undergoes three displacements of 1, 2, and 3 units, respectively, in directions parallel to the sides of an equilateral triangle taken in order. What is the resulting displacement?
Ans. $\sqrt{3}$ in a direction perpendicular to the second side.

11. A ship is carried by the wind 3 mi. due north, by the current 4 mi. due west, and by her screw 20 mi. southeast. What is her actual displacement?

12. A mail bag is thrown from a train with speed of 20 ft. per second perpendicular to the track. If the speed of the train is 40 mi. per hour, what is the direction and speed of the bag relative to the earth?

13. A particle is kept at rest by forces of 6, 8, 11 units. Find the angle between the forces 6 and 8. *Ans.* 77° 21′ 52″.

14. A boat is carried southwest by the current with a speed of 5 mi. per hour and 30° south of east by the wind at the rate of 12 mi. per hour. What must be the direction and magnitude of the speed due to her screw if she remains at rest? Ans 11.7+ mi/hr N35° 50′ W.

15. Three posts are placed in the ground so as to form an equilateral triangle, and an elastic string is stretched around them, the tension of which is 6 lb. Find the pressure on each post. *Ans.* $6\sqrt{3}$.

16. $ABCD$ is a square, and the middle point of BC is E. Find the resultant of three velocities represented by AB, AE, and AC.

17. The angle between two unknown forces is 62°, and their resultant divides this angle into 40° and 22°. Find the ratio of the component forces.

18. Three forces act at a point and include angles of 90° and 45°. The first two forces are each equal to 2 units and the resultant of them all is $\sqrt{10}$ units. Find the third force. *Ans.* $\sqrt{2}$ units.

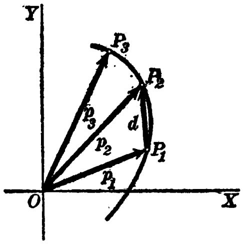
19. If three forces of 99, 100, and 101 units, respectively, act on a point at angles of 120°, find the magnitude of their resultant and its inclination to the second force.
Ans. $\sqrt{3}$, 90°.

20. A weight of 40 lb. is suspended by two strings, inclined to the vertical at angles of 45° and 30°, respectively. Find the tension in each string.
Ans. $20(\sqrt{6}-\sqrt{2})$, $40(\sqrt{3}-1)$.

21. Given the vectors $\mathbf{a}(3, -2)$, $\mathbf{b}(5, 0)$, $\mathbf{c}(-10, 6)$, $\mathbf{d}(7, 7)$. Construct the figures and find the resultants of the following:

(a) $\mathbf{a} + 2\mathbf{b} - 3\mathbf{c}$;
(b) $2\mathbf{a} - \mathbf{b} + \mathbf{c} + 2\mathbf{d}$;
(c) $3\mathbf{a} + 4\mathbf{c} - \mathbf{d}$;
(d) $4\mathbf{b} - 2\mathbf{c} + 5\mathbf{d}$;
(e) $10\mathbf{a} + 5\mathbf{b} + 4\mathbf{c}$;
(f) $2\mathbf{a} + 3\mathbf{b} + \mathbf{c} - \mathbf{d}$;
(g) $2\mathbf{a} - 3\mathbf{b} - 2\mathbf{c} - 2\mathbf{d}$.

34. Displacement in a plane. Path. Suppose a point moves in a plane. Then its position vector changes as the time changes. If the law of the motion is known, the position vector $\mathbf{p}$ is a known function of the time, and its components ($p_x = x$, $p_y = y$) are known functions of the time; that is,



$$\text{(VII)} \qquad x = \phi(t), \; y = \psi(t).$$

Equations (VII) are called the equations of motion. By assuming values t_1, t_2, t_3, etc., for the time we may compute the corresponding position vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, etc. *The locus of the extremities of the position vectors is the path of the moving point P.* Since the components of the position vector are the ordinary rectangular coördinates, the equations (VII) may be regarded as the parametric equations* of the path. The rectangular equation of the path may be obtained by eliminating t from the two equations (VII).

If $\mathbf{p}_1$ is the position vector at the instant t_1, and if $\mathbf{p}_2$ is the position vector at the instant t_2, the total displacement during the interval of time from t_1 to t_2 is represented by the vector $\mathbf{d} = P_1P_2$. This displacement is evidently equal to the difference of the vectors $\mathbf{p}_2$ and $\mathbf{p}_1$ (Art. 30); that is,

$$\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1.$$

The position of the point at the instant $t = 0$ is called the initial position. It is represented by the position vector $\mathbf{p}_0$ whose components are $x_0 = \phi(0)$, $y_0 = \psi(0)$. The length s of the arc described in the interval of time from 0 to t is a function of t. The expression for s is given by (66, Chap. XIV)

$$s = \int_0^t \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}} dt = \int_0^t [\{\phi'(t)\}^2 + \{\psi'(t)\}^2]^{\frac{1}{2}} dt,$$

and $$\therefore \frac{ds}{dt} = \sqrt{\{\phi'(t)\}^2 + \{\psi'(t)\}^2}.$$

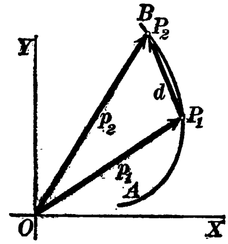
The sign of the radical is always taken as positive. The derivative $\frac{ds}{dt}$ is the time-rate of change of s and is called the *speed*.

* Calculus, p. 93.

35. Velocity in the plane. Velocity curve. Suppose a point P moves along a path AB in the XY-plane and that the equations of its motion are

$$x = \phi(t), \quad y = \psi(t).$$

Suppose that at the instant $t = t_1$ the point is at the position P_1 represented by the position vector $\mathbf{p}_1$, and at the instant $t = t_2$ it is at the position P_2 represented by the position vector $\mathbf{p}_2$. During the interval of time $t_2 - t_1$ the displacement is represented by the vector $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1$.



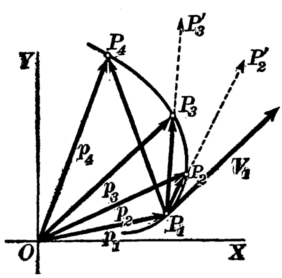
The quotient

$$\frac{\mathbf{p}_2 - \mathbf{p}_1}{t_2 - t_1}$$

is called the *average velocity during the interval of time* $t_2 - t_1$. The average velocity is a vector, since it is the quotient of a vector and a scalar. It has the same direction as the displacement vector $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1$, and its magnitude is equal to the magnitude of $\mathbf{d}$ divided by $t_2 - t_1$.

Let us now consider a fixed instant $t = t_1$, the corresponding position vector being $\mathbf{p}_1$, and denote an interval of time immediately following t_1 by Δt, and the displacement during the interval Δt by $\Delta\mathbf{p}$. The average velocity during the interval of time Δt is therefore $\dfrac{\Delta\mathbf{p}}{\Delta t}$.

To fix the ideas, let us consider some particular values for Δt, the unit of time being 1 second.



(1) Let $\Delta t = 1$; the displacement vector $\Delta\mathbf{p} = P_1P_4$ (see figure), and the average velocity during the interval of one second immediately following the instant $t = t_1$ is $\dfrac{P_1P_4}{1}$. The vector representing the average velocity is therefore equal to the displacement vector, that is, equal to the chord P_1P_4.

(2) Let $\Delta t = \frac{1}{2}$; the displacement vector $\Delta\mathbf{p} = P_1P_3$, and the average velocity during the interval of one half second immediately following the instant $t = t_1$ is

$$\frac{P_1P_3}{\frac{1}{2}} = 2\,P_1P_3 = P_1P_3'.$$

The vector representing the average velocity has the direction of the chord and its magnitude is equal to twice the length of the chord.

(3) Let $\Delta t = \frac{1}{4}$; the displacement vector $\Delta \mathbf{p} = P_1P_2$, and the average velocity during the interval of one fourth of a second immediately following the instant $t = t_1$ is

$$\frac{P_1P_2}{\frac{1}{4}} = 4\, P_1P_2 = P_1P_2'.$$

The vector representing the average velocity has the direction of the chord P_1P_2 and its magnitude is equal to four times the length of the chord.

(4) Let Δt approach zero as a limit. The vector which represents the average velocity $\frac{\Delta \mathbf{p}}{\Delta t}$ has the same direction as the chord, and hence when Δt approaches zero its direction approaches the direction of the tangent to the curve. Multiplying $\frac{\Delta p}{\Delta t}$ by $\frac{\Delta s}{\Delta s}$ (Δs represents the increment of arc along the curve in the time Δt), we may write:

$$\text{Magnitude of average velocity} = \frac{\Delta p}{\Delta t} \cdot \frac{\Delta s}{\Delta s} = \frac{\Delta s}{\Delta t} \cdot \frac{\Delta p}{\Delta s}.$$

As Δt approaches zero as a limit, Δs also approaches zero as a limit, and $\frac{\Delta p}{\Delta s} = \frac{\text{chord}}{\text{arc}}$ approaches 1 as a limit, while $\frac{\Delta s}{\Delta t}$ approaches $\frac{ds}{dt}$ as a limit. Therefore the magnitude of the average velocity approaches $\frac{ds}{dt}$ as a limit.

We now make the definition: *The velocity of the moving point at the instant $t = t_1$ is equal to the limit of the average velocity as Δt approaches zero.* The magnitude* of the velocity is therefore

$$v = \frac{ds}{dt} = +\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

and its direction is the direction of the tangent to the path. The cosine and sine of the inclination of the tangent are respectively:†

$$\frac{dx}{ds}, \frac{dy}{ds}.$$

* The magnitude of the velocity is the *speed*. The *velocity* is a vector quantity and possesses magnitude and direction.

† Calculus, p. 142.

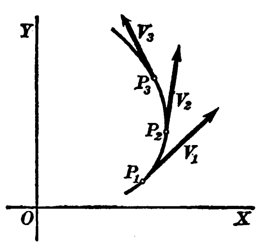
Therefore, applying (I), we have for the axial components of the velocity:

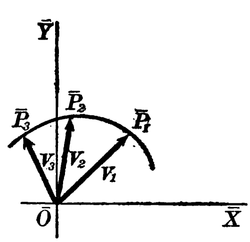
$$\text{(VIII)} \qquad \begin{aligned} v_x &= \frac{ds}{dt} \cdot \frac{dx}{ds} = \frac{dx}{dt} = \phi'(t), \\ v_y &= \frac{ds}{dt} \cdot \frac{dy}{ds} = \frac{dy}{dt} = \psi'(t). \end{aligned}$$

From (VIII) we see that *the component of the velocity of the moving point P in the direction of the X-axis is obtained by differentiating the abscissa of P with respect to the time*, and *the component of the velocity in the direction of the Y-axis is obtained by differentiating the ordinate of P with respect to the time.* In other words, v_x is the velocity of the projection of the moving point on the X-axis, and v_y is the velocity of the projection of the moving point on the Y-axis. Hence the

THEOREM. *The axial components of the velocity in curvilinear motion are equal to the velocities of the axial components of the motion.*

In general the velocity is different at different points of the path. At the point P_1 of the curve the velocity will be represented by a vector $\mathbf{v}_1$; at the point P_2 by a vector $\mathbf{v}_2$, etc. Let a new system of rectangular axes be chosen, $\bar{O}$, $\bar{X}$, $\bar{Y}$, and from the origin $\bar{O}$ lay off the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, etc.





The locus of the extremities of the velocity vectors in the $\overline{XY}$-plane is a curve which is called the *velocity curve* of the motion defined by (VII). The position vector of any point $\bar{P}$ of the velocity curve is equal to the velocity vector of the corresponding point P of the path. The rectangular coördinates of $\bar{P}(\bar{x}, \bar{y})$ are equal respectively to v_x and v_y.

Illustrative Example. Construct the path and the velocity curve for the plane motion defined by the equations

(1) $$x = t, \quad y = \cos 2t.$$

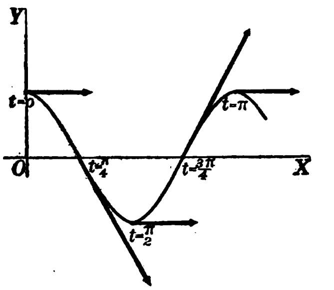
t	x	y	v_x	v_y
0	0	1	1	0
$\frac{\pi}{4}$	$\frac{\pi}{4}$	0	1	-2
$\frac{\pi}{2}$	$\frac{\pi}{2}$	-1	1	0
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	0	1	2
π	π	1	1	0

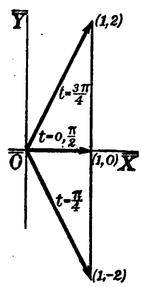
Solution. Eliminating t, the equation of the path is found to be $y = \cos 2x$.

Differentiating (1), the components of velocity are

(2) $$v_x = 1, \quad v_y = -2 \sin 2t.$$

From equations (2), it is seen that the velocity curve consists of a portion of the straight line $\bar{x} = 1$. Since the sine cannot be numerically greater than 1, we have no points on the velocity curve for which $\bar{y}$ is numerically greater than 2.





The coördinates of the moving point P and the components of velocity for certain values of t are shown in the table.

PROBLEMS

1. Construct the path and the velocity curve for the plane motions defined by the following equations:

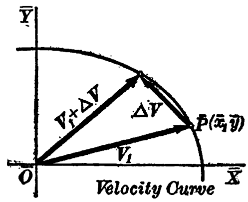
(*a*) $x = \cos t, \ y = \sin t$;
(*b*) $x = a \cos t, \ y = b \sin t$;
(*c*) $x = 2 \sin t, \ y = \cos 2t$;
(*d*) $x = \cos t, \ y = 2 \sin \frac{1}{2} t$;
(*e*) $x = a \cos t, \ y = a \cos 2t$;
(*f*) $x = a \sin 2t, \ y = a \sin t$;
(*g*) $x = a(t - \sin t), y = a(1 - \cos t)$;
(*h*) $x = a(t + \sin t), y = a(1 - \cos t)$;
(*i*) $x = a(t - \sin t), y = b(1 - \cos t)$;
(*j*) $x = a \cos^3 t, \ y = a \sin^3 t$;
(*k*) $x = a \cos^3 t, \ y = b \sin^3 t$;
(*l*) $x = 6t - t^2, \ y = 3t$;
(*m*) $x = at, \ y = bt + ct^2$;
(*n*) $x = t, \ y = 1 + t^3$;
(*o*) $x = 1 - t^2, \ y = t^3$;
(*p*) $x = at^2, \ y = a(1 - t)^2$;
(*q*) $x = at, \ y = b \sin t$;
(*r*) $x = a(1 - \cos t), \ y = a \sin t$;
(*s*) $x = a(1 - \cos t), \ y = b \sin t$.

2. A point describes any curved path with constant speed. What is the form of the velocity curve?

3. Two points are describing free paths in one plane such that each path is the velocity curve of the other. If the moving points be always at corresponding positions, prove that the paths are conic sections.

36. Acceleration in a plane. In plane motion velocity may be defined as the time-rate of change of the position vector, and the acceleration as the time-rate of change of the velocity vector.

Since the velocity is a vector quantity, the acceleration is also a vector quantity. Let $\mathbf{v}_1$ be the velocity vector at the instant $t = t_1$; and $\mathbf{v}_1 + \Delta\mathbf{v}$ the velocity vector at $t = t_1 + \Delta t$. The change in velocity during the interval of time Δt is represented by the vector $\Delta\mathbf{v}$ and the average acceleration during the interval Δt is the quotient $\dfrac{\Delta\mathbf{v}}{\Delta t}$.



The average acceleration is a vector quantity, since it is the quotient of a vector and a scalar. The acceleration *at any instant* $(t = t_1)$ is defined as the *limit of the average acceleration* $\dfrac{\Delta\mathbf{v}}{\Delta t}$ *as* Δt *approaches zero.* This corresponds to the definition of velocity given in Art. 35. Hence the acceleration can be obtained from the velocity curve in the same manner as the velocity is obtained from the path curve. Denoting the acceleration by $\mathbf{f}$, it follows that its direction is the direction of the tangent to the velocity curve at the point corresponding to $t = t_1$. The components of the acceleration in the directions of the coördinate axes are given by formulas similar to (VIII). That is, if $(\bar{x}, \bar{y})$ are the coördinates of the point $\bar{P}$ on the velocity curve, then $f_x = \dfrac{d\bar{x}}{dt}$, $f_y = \dfrac{d\bar{y}}{dt}$, and, since $\bar{x} = v_x$, $\bar{y} = v_y$, we have

$$
\text{(IX)} \qquad \begin{cases} f_x = \dfrac{dv_x}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2} = \phi''(t), \\[2ex] f_y = \dfrac{dv_y}{dt} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d^2y}{dt^2} = \psi''(t). \end{cases}
$$

The axial components of the vector acceleration are therefore obtained from the equations of motion by differentiating twice. Furthermore, a statement similar to the theorem of Art. 35 may be made:

THEOREM. *The axial components of the vector acceleration in curvilinear motion are equal to the accelerations of the axial components of the motion.*

The magnitude of the acceleration is obtained from its components by applying (I),

$$f = +\sqrt{f_x^2 + f_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{\{\phi''(t)\}^2 + \{\psi''(t)\}^2}.$$

37. Motion in space. The discussion of Arts. 34–36 is extended easily to the motion of a point in space. The difference amounts to the consideration of the additional coördinate z. Thus the equations of any motion in space will have the form

$$\text{(X)} \qquad x = \phi(t), \quad y = \psi(t), \quad z = \chi(t),$$

in which the independent variable represents the time. By elimination of t from the two pairs of equations (X), the path will be determined in rectangular coördinates as the intersection of two cylinders.

The velocity is a vector determined as in Art. 35, and if the axial components are v_x, v_y, v_z, then in agreement with (VIII),

$$\text{(XI)} \qquad v = \frac{ds}{dt}; \quad v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}; \quad v_z = \frac{dz}{dt}.$$

Finally, the vector acceleration is defined as in Art. 36, and if f_x, f_y, f_z are its axial components, then

$$\text{(XII)} \qquad f_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}; \quad f_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}; \quad f_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}.$$

The equations of the path being given, the axial components of velocity and acceleration are obtained by differentiation, and from these components **v** and **f** are determined in magnitude and direction by (IV).

38. Discussion of any motion. Given the equations of any motion, the determination of its characteristics involves the following:

1. Notice the nature of the component motions and draw any conclusions as to the general nature of the motion (periodic, etc.).

2. Plot the path either by assuming values of t and computing x, y (and z), or by eliminating t and plotting from the rectangular equation (or equations). Find the initial position.

3. Differentiate and find the axial components of the velocity and acceleration. Determine v and f.

4. Draw the velocity curve and discuss the variation of v with the time.

5. Discuss the variation of f with the time, both in magnitude and direction.

ILLUSTRATIVE EXAMPLES

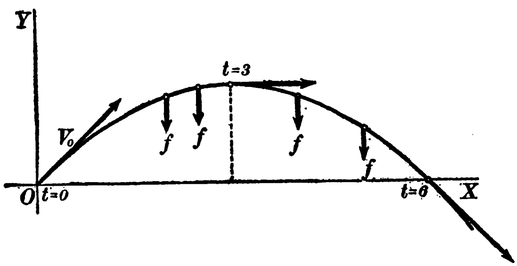
1. Discuss the motion whose equations are

$$x = 2t, \quad y = 2t - \tfrac{1}{3}t^2. \tag{1}$$

Solution. Following out the discussion:

1. The motion is not periodic.

2. The path is a parabola. For, from (1), $t = \frac{1}{2}x$, and $\therefore\ y = x - \frac{1}{12}x^2$, or $x^2 - 12x + 12y = 0$, which is a parabola. The initial position is the origin.



t	x	y
0	0	0
1	2	$1\frac{2}{3}$
2	4	$2\frac{2}{3}$
3	6	3
4	8	$2\frac{2}{3}$
5	10	$1\frac{2}{3}$
6	12	0
etc.	etc.	etc.

3. Differentiating (1), we obtain,

$$v_x = \frac{dx}{dt} = 2, \quad v_y = \frac{dy}{dt} = 2 - \frac{2}{3}t. \tag{2}$$

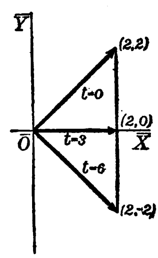
$$\therefore\ v = \sqrt{v_x^2 + v_y^2} = \sqrt{8 - \tfrac{8}{3}t + \tfrac{4}{9}t^2}.$$

$$f_x = \frac{d^2x}{dt^2} = 0, \quad f_y = \frac{d^2y}{dt^2} = -\frac{2}{3}. \tag{3}$$

$$\therefore\ f = \sqrt{f_x^2 + f_y^2} = \tfrac{2}{3}.$$

4. The velocity curve is the straight line $v_x = 2$. The initial velocity has the components (2, 2). Hence at 0, the point is moving in a direction making an angle of 45° with OX. The vertical component diminishes from 2 when $t = 0$, to zero when $t = 3$, and thereafter increases numerically but is negative. Hence the speed diminishes from its initial value $v_0 = 2\sqrt{2}$ to a minimum value 2 when $t = 3$, and thereafter constantly increases. When $t = 3$, the highest point (6, 3) is reached; $v_y = 0$, and hence the tangent to the path is parallel to the X-axis.

t	v_x	v_y	v
0	2	2	$2\sqrt{2}$
1	2	$1\frac{1}{3}$	$\frac{2}{3}\sqrt{13}$
3	2	0	2
6	2	-2	$2\sqrt{2}$
etc.	etc.	etc.	etc.



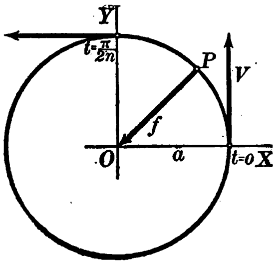
5. From (3) it appears that the acceleration is constant and has a downward direction.

2. Discuss the motion whose equations are

(1) $$x = a \cos nt,\ y = a \sin nt.$$

Solution. 1. Both axial components are periodic with the same period, namely $\frac{2\pi}{n}$. Hence the moving point will return to any position in its path after an interval of time equal to $\frac{2\pi}{n}$ and the motion is periodic.

2. Eliminating t by squaring and adding, the path is found to be the circle



$$x^2 + y^2 = a^2.$$

The initial position is $(a, 0)$.

3. Differentiating (1),

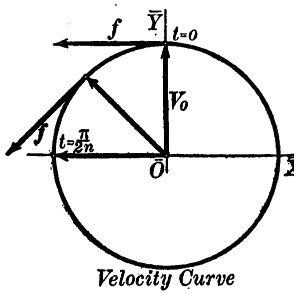
(2) $$v_x = -an \sin nt,\ v_y = an \cos nt.$$

$$\therefore v = \sqrt{v_x^2 + v_y^2} = an.$$

(3) $$f_x = -an^2 \cos nt,\ f_y = -an^2 \sin nt.$$

$$\therefore f = \sqrt{f_x^2 + f_y^2} = an^2 = \frac{v^2}{a}.$$

4. The velocity curve is a circle of radius an. Hence the *speed is constant.* Also when $t = 0$ the components of the velocity are $(0, an)$. Hence the point describes the circle in a counter-clockwise direction.



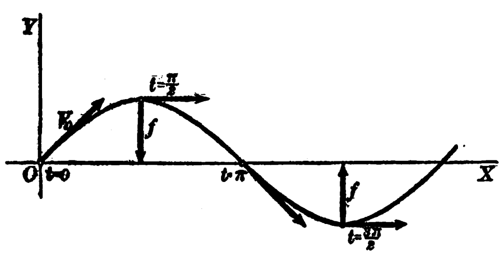
Velocity Curve

5. From (3) the magnitude of the acceleration is constant. To determine its direction, we observe by comparing (1) and (3) that

(4) $$f_x = -n^2x,\ f_y = -n^2y.$$

If in the figure, P is (x, y), then the point $(-n^2x, -n^2y)$, lies on the line OP produced through O. Hence the *vector acceleration at P is directed towards the center.*

The motion just described is called *uniform circular motion.* The axial components (1) are both simple harmonic motions with the same amplitude a and the same period $\frac{2\pi}{n}$. (Compare example 2, p. 49.)



3. Discuss the motion whose equations are

(1) $x = at,\ y = b \sin t.$

Solution. 1. The component of the motion in the direction of the Y-axis is periodic, while the motion in the direction of the X-axis is uniform.

2. Eliminating t, the path is the sine curve

$$y = b \sin \frac{x}{a},$$

whose period is $2\pi a$ and maximum ordinate is b. The initial position is (0, 0).

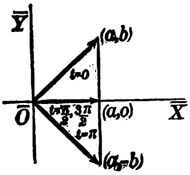
3. Differentiating (1),

(2) $$v_x = a, \quad v_y = b \cos t. \quad \therefore v = \sqrt{a^2 + b^2 \cos^2 t}.$$

(3) $$f_x = 0, \quad f_y = - b \sin t = - y. \quad \therefore f = \sqrt{y^2}.$$

4. The velocity curve is the portion of the straight line $\bar{x} = a$ between the points $\bar{y} = b$ and $\bar{y} = - b$.

From (2) the velocity at 0 has the components (a, b). The speed varies between a $\left(\text{when } t = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc.}\right)$ and $\sqrt{a^2 + b^2}$ (when $t = 0, \pi$, etc.). That is, the speed is least at the highest and lowest points, and greatest at the point of intersection with OX.



5. From (3) the acceleration equals the ordinate numerically but differs in sign. Its direction is parallel to the axis of Y.

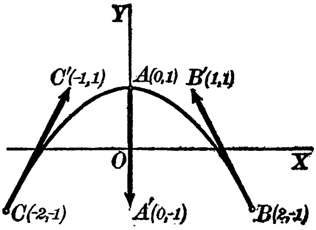
The motion here discussed may be thus described. The point moves with constant speed a parallel to OX and simultaneously executes simple harmonic motion parallel to OY.

4. Discuss the motion represented by

(1) $$x = 2 \sin t, \quad y = \cos 2t.$$

Solution. 1. Both components are periodic, and it is apparent that the moving point will return to any position in its path after an interval of time equal to 2π.

t	x	y
0	0	1
$\frac{1}{2}\pi$	2	-1
π	0	1
$\frac{3}{2}\pi$	-2	-1
2π	0	1
etc.	etc.	etc.



2. Since $\cos 2t = 1 - 2 \sin^2 t$, we find, on eliminating t,

$$y = 1 - 2 \sin^2 t = 1 - \tfrac{1}{2} x^2.$$

That is, the path is a portion of the parabola $x^2 + 2y - 2 = 0$. The initial position is A (0, 1).

3. Differentiating (1),

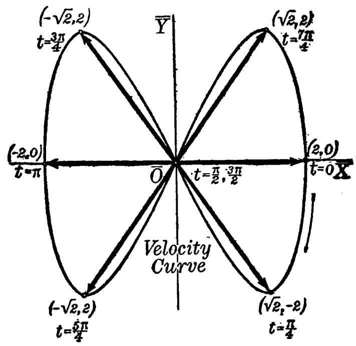
(2) $$v_x = 2 \cos t, \quad v_y = - 2 \sin 2t.$$
$$\therefore v = 2 \sqrt{\cos^2 t + \sin^2 2t}.$$

(3) $$f_x = - 2 \sin t = - x, \quad f_y = - 4 \cos 2t = - 4y.$$
$$\therefore f = \sqrt{x^2 + 16 y^2}.$$

4. The velocity curve, plotted from the parametric equations, has the form of the figure 8.

From (2), when $t=0$, $v_x=2$, $v_y=0$. The point initially at A moves to the right to the extreme position $B(2, -1)$, at which point ($t=\frac{1}{2}\pi$) the velocity is zero. It then returns through A to $C(-2, -1)$, at which point v is again zero ($t=\frac{3}{2}\pi$). The point is again at A when $t=2\pi$, and the vibration is then repeated.

t	v_x	v_y	v
0	2	0	2
$\frac{1}{2}\pi$	0	0	0
π	-2	0	2
$\frac{3}{2}\pi$	0	0	0
2π	2	0	2
etc.	etc.	etc.	etc.

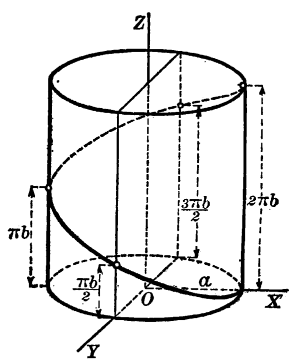


t	f_x	f_y	f
0	0	-4	4
$\frac{1}{2}\pi$	-2	4	$\sqrt{20}$
π	0	-4	4
$\frac{3}{2}\pi$	2	4	$\sqrt{20}$
2π	0	-4	4
etc.	etc.	etc.	etc.

5. From (3), the acceleration has the components $(-x, -4y)$. At A the acceleration is downwards; at B or C it is tangent to the path. For, differentiating the equation of the path in 2, we get $\frac{dy}{dx}=-x$, and hence the slope at B is -2. But the slope of the vector whose components are $(-2, 4)$ is $\frac{4}{-2}=-2$. Therefore the acceleration at B is tangential. Similarly for C. In the figure the vector acceleration is drawn to scale.

AA', BB', and CC' are the vectors representing the acceleration at the points A, B, C, respectively. The unit of length for the acceleration vector is $\frac{1}{2}$ the unit on the X-axis.

The motion just discussed is therefore an *oscillation with parabolic path*, the period being 2π. The components (1) are simple harmonic motions with different periods, namely, 2π and π. Their resultant motion is that of a point executing simultaneously simple harmonic motions parallel to perpendicular axes, the ratio of the periods being 2.



5. Discuss the motion in space defined by

(1) $\quad x=a\cos t,\ y=a\sin t,\ z=bt.$

Solution. 1. The x- and y-components of the motion are harmonic vibrations, and the z-component is uniform motion.

2. The path is a helix on the cylinder $x^2+y^2=a^2$ (Calculus, p. 272). The initial position is $(a, 0, 0)$.

3. From (1), we obtain, by differentiation,

(2) $$v_x = -a \sin t, \quad v_y = a \cos t, \quad v_z = b.$$

$$\therefore v = \sqrt{a^2 + b^2}.$$

(3) $$f_x = -a \cos t = -x, \quad f_y = -a \sin t = -y, \quad f_z = 0.$$

$$\therefore f = a.$$

4. When $t = 0$, $v_x = 0$, $v_y = a$, $v_z = b$.

Hence $(b > 0)$ the point describes the helix with constant speed in the upward direction.

5. The acceleration is constant in magnitude, is parallel to the XY-plane, and is directed towards the Z-axis, since the direction from (0, 0, 0) to $(-x, -y, 0)$, when drawn from (x, y, z), will pass through the axis OZ.

By comparison with example 2, it is seen that (1) may be regarded as the motion of a point having simultaneously uniform circular motion around OZ and constant speed along OZ. Such a motion is obviously that of any point on the periphery of a screw which is forced inward at constant speed. For this reason the motion defined by (1) is called a *screw motion.*

PROBLEMS

1. Discuss each of the following motions:

(*a*) $x = 3t, \ y = 2 - t$;

(*b*) $x = 1 - 3t, \ y = 6 + t$;

(*c*) $x = a + bt, \ y = c + dt$;

(*d*) $x = t^2, \ y = \frac{1}{2}t$;

(*e*) $x = 1 - t, \ y = t^2$;

(*f*) $x = 3t, \ y = 6t - t^2$;

(*g*) $x = at, \ y = bt - \frac{1}{2}gt^2$;

(*h*) $x = at^2 + bt, \ y = ct$;

(*i*) $x = t, \ y = t^3$;

(*j*) $x = t^2, \ y = t^3$;

(*k*) $x = ae^{kt}, \ y = be^{-kt}$;

(*l*) $x = a(t - \sin t), \ y = a(1 - \cos t)$;

(*m*) $x = a\cos^3 t, \ y = a\sin^3 t$;

(*n*) $x = a(t + \sin t), \ y = a(1 - \cos t)$;

(*o*) $x = a(t - \sin t), \ y = b(1 - \cos t)$;

(*p*) $x = a(t + \sin t), \ y = b(1 - \cos t)$;

(*q*) $x = a\sin^3 t, \ y = b\cos^3 t$;

(*r*) $x = at^2, \ y = a(1 - t)^2$;

(*s*) $x = a(1 - \cos t), \ y = a\sin t$;

(*t*) $x = a(1 - \cos t), \ y = b\sin t$;

(*u*) $x = \cos t, \ y = 4\sin\frac{1}{2}t$;

(*v*) $x = a\cos t, \ y = a\cos 2t$;

(*w*) $x = a\sin 2t, \ y = a\sin t$.

2. Discuss each of the following motions, the components in each case being simple harmonic motions:

(*a*) $x = 2\sin t, \ y = 2\cos t$;

(*b*) $x = 2\sin t, \ y = 3\cos t$;

(*c*) $x = \sin t, \ y = \cos 2t$;

(*d*) $x = a\sin kt, \ y = b\cos kt$;

(*e*) $x = a\sin t, \ y = b\sin(t + \beta)$;

(*f*) $x = 2\sin\frac{1}{2}t, \ y = \cos t$;

(*g*) $x = a\cos t, \ y = b\cos 2t$;

(*h*) $x = \sin\frac{1}{2}t, \ y = a\sin t$;

(*i*) $x = a\cos(kt + \beta), \ y = b\sin(kt + \beta)$;

(*j*) $x = a\cos(kt + \beta), \ y = b\cos(kt + \beta)$.

3. Discuss each of the following motions in space:

(*a*) $x = t, \ y = t + 1, \ z = 3 - t$;

(*b*) $x = 1 - 2t, \ y = 2t - 5, \ z = t - 6$;

(*c*) $x = at, \ y = bt, \ z = ct$;

(d) $x = at + a_1,\ y = bt + b_1,\ z = ct + c_1$;

(e) $x = \sin t,\ y = t,\ z = \cos t$;

(f) $x = bt,\ y = a \sin t,\ z = a \cos t$;

(g) $x = a \cos t,\ y = bt,\ z = a \sin t$;

(h) $x = a \cos t,\ y = b \sin t,\ z = A \cos t + B \sin t$;

(i) $x = t,\ y = 1 - t^2,\ z = 3t^2 + 4t$;

(j) $x = t^2 + 8t + 1,\ y = t^2 - 2,\ z = 1 - 8t$;

(k) $x = 2 \cos t,\ y = 3 \cos t,\ z = t$;

(l) $x = \sin t,\ y = \cos 2t,\ z = \sin t$;

(m) $x = \sin t,\ y = \dfrac{1}{\sqrt{2}} \cos t,\ z = \dfrac{1}{\sqrt{2}} \cos t$;

(n) $x = a \cos (kt + \beta),\ y = b \sin (kt + \beta),\ z = t$;

(o) $x = a \cos^3 t,\ y = bt,\ z = a \sin^3 t$;

(p) $x = a(t - \sin t),\ y = t,\ z = a(1 - \cos t)$.

39. Motion in a prescribed path. The question may be raised: What characteristics must any motion on an ellipse possess? Certain points are readily settled. If the path is

$$(1) \qquad b^2x^2 + a^2y^2 = a^2b^2,$$

either axial component of the motion (VII) may be chosen, and the other is then determined. Thus, if we choose the x component as the simple harmonic motion,

$$x = a \cos kt,$$

then, from (1), by substitution,

$$a^2b^2 \cos^2 kt + a^2y^2 = a^2b^2, \text{ or}$$

$$y = b \sin kt.$$

In general, *on a prescribed path one axial component may be chosen arbitrarily, and the other is then found by substitution and solving.* That is, we set $x = \phi(t)$, where $\phi(t)$ is assumed, substitute in the given rectangular equation, and solve for y.

Further useful equations are the following:

From $\qquad v_x = \dfrac{dx}{dt},\ v_y = \dfrac{dy}{dt}$, we obtain

$$(2) \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{v_y}{v_x}.$$

When the path is given, $\frac{dy}{dx}$ is found by differentiation, and (2) gives a relation between the components of the velocity which holds for each point of the path. For example, for the ellipse (1), this relation is

$$-\frac{b^2x}{a^2y}=\frac{v_y}{v_x}.$$

Differentiating (2) with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{v_y}{v_x}\right)\div\frac{dx}{dt}=\frac{v_x\frac{dv_y}{dt}-v_y\frac{dv_x}{dt}}{v_x^{\,3}}.$$

(3) $$\therefore\ \frac{d^2y}{dx^2}=\frac{v_xf_y-v_yf_x}{v_x^{\,3}}\ \text{(using (IX))}.$$

In this equation the value of the second derivative of y with respect to x is found from the equation of the prescribed path.

From (2), it has been seen that one of the axial components of the velocity may be chosen and the other is then determined. Knowing v_x and v_y, we may obtain f_x and f_y by differentiation, and then *check* the results by equation (3).

ILLUSTRATIVE EXAMPLE. If the path is the equilateral hyperbola $xy = c$, and $v_y = k$ (constant), find v_x and f_x.

Solution. From the equation of the path, $\frac{dy}{dx}=-\frac{y}{x}$.

Hence, from (2),

(4) $$v_x=v_y\div\frac{dy}{dx}=-\frac{kx}{y}.$$

From the equation of the path, we find $y=\frac{c}{x}$.

By substituting, (4) becomes

(5) $$v_x=-\frac{k}{c}x^2.$$

Differentiating with respect to t,

(6) $$f_x=-\frac{2k}{c}xv_x=\frac{2k^2}{c^2}x^3.$$

From the equation of the path $y=\frac{c}{x}$, we find $\frac{d^2y}{dx^2}=\frac{2c}{x^3}$, and substituting in (3) from (5) and remembering that $f_y = 0$, the results check.

PROBLEMS

1. In the following problems the path is given. Find (1) f_x if $v_y = \beta$; (2) f_y if $v_x = \alpha$.

(a) $xy = a^2$. *Ans.* $\dfrac{2\beta^2}{a^4}x^3$; $\dfrac{2\alpha^2}{a^4}y^3$.

(b) $y = a^x$. *Ans.* $\dfrac{-\beta^2}{a^{2x}\log a}$; $\alpha^2(\log a)^2 y$.

(c) $y^2 = 4ax$. *Ans.* $\dfrac{\beta^2}{2a}$; $-\dfrac{4a^2\alpha^2}{y^3}$.

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(e) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(f) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$;

(g) $x = a \text{ arc vers}\dfrac{y}{a} - (2ay - y^2)^{\frac{1}{2}}$. $\left(\text{Here } \dfrac{dx}{dy} = \dfrac{y}{(2ay-y^2)^{\frac{1}{2}}}\right)$.

Ans. $\dfrac{\beta^2 ay}{(2ay - y^2)^{\frac{3}{2}}}$; $-\dfrac{\alpha a^2}{y^2}$.

(h) $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. *Ans.* $-\dfrac{\beta^2 ay}{(y^2-a^2)^{\frac{3}{2}}}$; $\dfrac{\alpha^2 y}{a^2}$.

2. In the following problems the path and the component of velocity along one axis are given; to find the component of acceleration along the other axis.

(a) $x + y = 1$; $v_x = \cos kt$. *Ans.* $f_y = k \sin kt$.

(b) $Ax + By + C = 0$; $v_x = t^2 - t$. *Ans.* $f_y = \dfrac{A}{B}(1 - 2t)$.

(c) $y^2 = 4ax$; $v_y = ct$. *Ans.* $f_x = \dfrac{c}{2a}(ct^2 + y)$.

(d) $y^2 = 4ax$; $v_x = \sin t$. *Ans.* $f_y = \dfrac{2a}{y^3}(y^2 \cos t - 4a^2 \sin^2 t)$.

(e) $x^2 + y^2 = a^2$; $v_y = \dfrac{a}{2}\sin 2t$. *Ans.* $f_x = -\dfrac{a}{2}\left\{\dfrac{2y\cos 2t}{x} + \dfrac{a^3 \sin^2 2t}{2x^3}\right\}$.

(f) $x^2 + y^2 = a^2$; $v_x = t^2$. *Ans.* $f_y = -\dfrac{2tx}{y} - \dfrac{a^2t^4}{y^3}$.

(g) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $v_y = nb \cos nt$.

(h) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; $v_y = ct$. *Ans.* $f_x = -\dfrac{ca^2}{b^2}\left(\dfrac{y}{x} + \dfrac{a^2ct^2}{x^3}\right)$.

(i) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$; $v_x = ct^2$.

(j) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + 1 = 0$; $v_x = a \sec^2 t$.

(k) $xy = a^2$; $v_x = a \sec^2 t$.

3. In the following problems the path and one component of the motion are given; to find the components of velocity and acceleration and to discuss the motion.

(*a*) $y^2 = 4\,ax$; $y = ct$.

(*b*) $y^2 = 4\,ax$; $x = a\cos t$.

(*c*) $x^2 + y^2 = a^2$; $x = b\cos nt$; $(b \leq a)$.

(*d*) $x^2 + y^2 = a^2$; $y = ct^2$.

(*e*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $x = ct$.

(*f*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$; $x = a\tan t$.

(*g*) $xy = a^2$; $y = a\tan t$.

4. A point describes the curve given with constant speed; to determine the components of velocity and acceleration.

(*a*) $Ax + By + C = 0$.

(*b*) $x^2 + y^2 = a^2$.

(*c*) $y^2 = 4\,ax$.

(*d*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(*e*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(*f*) $xy = a^2$.

(*g*) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

(*h*) $y = a\log\sin x$.

(*i*) $y = b\log\cos x$.

5. Given $v_x = kx$, $v_y = ky$. Show that the path is a straight line passing through the origin, and find the components of acceleration.

6. Given $v_x = ky$, $v_y = kx$. Find f_x, f_y, and the equation of the path.

7. A wheel rolls on a horizontal plane so that its center has constant speed. Compare the speed at any instant of a point on the circumference with the speed of the center.

8. Find the axial components of the acceleration in problem 7, show that the acceleration at any instant of a point on the circumference is constant in magnitude $\left(= \frac{c^2}{a}\right)$ and is directed towards the center of the wheel.

9. A wheel rolls upon the inside of a second wheel whose diameter is twice its diameter. If the center of the smaller wheel moves with constant speed, show that a point upon its circumference will execute simple harmonic motion.

10. The pin of a crank moves in a groove in a vertical bar whose extremities move in horizontal grooves. If the crank pin rotates with constant speed, show that any point of the vertical bar will execute simple harmonic motion.

11. A point describes a curve with an acceleration parallel to OY. Show that $f = c^2\frac{d^2y}{dx^2}$, where c is the constant speed parallel to OX.

12. A particle describes the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. Show that $\frac{v^2}{a} = \frac{2\,v_x^{\,2}}{y}$. If the acceleration is at right angles to the line joining the cusps, show that it varies inversely as the square of the distance from this line, or also directly as v^4.

13. A point moves on the catenary $y = \frac{1}{2} a (e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

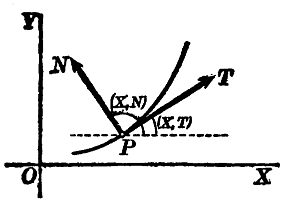
Show that
$$v^2 = \frac{1}{a^2} v_x^2 y^2.$$
If the acceleration is parallel to OY, show that it varies directly as the velocity.

14. What points on the rim of the wheel in problem 7 have the same speed as the center? *Ans.* Points of an arc of 60° described about the lowest point.

40. Tangential and normal accelerations. For plane motion the components of the acceleration vector in the directions of the coördinate axes are given by (IX). The components in the directions of the tangent and normal to the path are obtained by applying (II).

We first adopt a convention as to the *positive* direction along the tangent and normal. The positive direction along the tangent PT shall agree with the direction of the velocity. The positive direction along the normal PN shall agree with the direction obtained by rotating PT counter-clockwise through a right angle. Hence, by the definition,

$$(1) \qquad (x, N) = \frac{\pi}{2} + (x, T).$$



If f_t and f_n are the tangential and normal components of $\mathbf{f}$, from (II),

$$(2) \quad f_t = f_x \cos(x, T) + f_y \sin(x, T).$$

$$(3) \quad f_n = f_x \cos(x, N) + f_y \sin(x, N).$$

The second member of (2) is reduced as follows. Since by assumption $(x, T) = (x, v)$, we have

$$(4) \qquad \begin{cases} \cos(x, T) = \cos(x, v) = \dfrac{v_x}{v}; \\ \sin(x, T) = \sin(x, v) = \dfrac{v_y}{v}. \end{cases}$$

Hence, by substitution in (2), using (IX), we get

$$f_t = \frac{dv_x}{dt} \cdot \frac{v_x}{v} + \frac{dv_y}{dt} \cdot \frac{v_y}{v} = \frac{1}{v}\left(v_x \frac{dv_x}{dt} + v_y \frac{dv_y}{dt}\right) = \frac{dv}{dt}.$$

[Since $v^2 = v_x^2 + v_y^2$, by differentiation,

$$2 v \frac{dv}{dt} = 2 v_x \frac{dv_x}{dt} + 2 v_y \frac{dv_y}{dt}.$$

$$\therefore \frac{dv}{dt} = \frac{1}{v}\left(v_x \frac{dv_x}{dt} + v_y \frac{dv_y}{dt}\right).]$$

Similarly, to transform (3), we have, from (1) and (4),

$$\begin{cases} \cos(x, N) = -\sin(x, T) = -\dfrac{v_y}{v}, \\ \sin(x, N) = \cos(x, T) = \dfrac{v_x}{v}. \end{cases}$$

Substituting in (3), we obtain

$$(5) \qquad f_n = \frac{v_x f_y - v_y f_x}{v}.$$

If R denotes the radius of curvature of the path (formula 64, Chapter XIV), we have, by (VIII) and (IX),

$$R = \frac{v^3}{v_x f_y - v_y f_x}.$$

Hence, we write (5) in the form

$$(6) \qquad f_n = \frac{v^2}{R}.$$

For reference later we give also another form for f_n. From (3), Art. 39, we have

$$\frac{d^2y}{dx^2} = \frac{v_x f_y - v_y f_x}{v_x^{\,3}}.$$

Hence, from (5),

$$(7) \qquad f_n = \frac{v_x^{\,3}}{v}\frac{d^2y}{dx^2}.$$

The results found give the

THEOREM. *If the vector acceleration at any point of the path is resolved along tangent and normal, its components are*

$$(\text{XIII}) \qquad \boldsymbol{f_t = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}}; \qquad \boldsymbol{f_n = \frac{v^2}{R}},$$

where R is the radius of curvature.

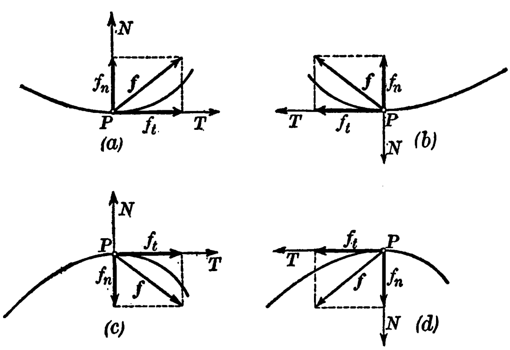
Since f_t and f_n are at right angles, we have obviously

$$f = \sqrt{f_t^2 + f_n^2}.$$

Two important results follow from (XIII). 1. If the path is a straight line, $\dfrac{1}{R} = 0$, and $\therefore f_n = 0$. That is, in rectilinear motion the vector acceleration is directed *along the path.* 2. If the speed is constant, $\dfrac{ds}{dt} = c$, whence $\dfrac{d^2s}{dt^2} = 0$ and $\therefore f_t = 0$. Hence in curvilinear motion with constant speed the acceleration is directed *along the normal.*

Furthermore, in any curvilinear motion ($f_n \neq 0$), the acceleration is directed *towards the concave side of the path.* To show this, four cases must be considered. From formula (7), for f_n it is plain that f_n is positive or negative according as v_x and $\frac{d^2y}{dx^2}$ have like or unlike signs. By Calculus, p. 137, the path is concave upwards or downwards according as $\frac{d^2y}{dx^2}$ is positive or negative. The four cases to be considered are :

1. The path is concave upwards and the point is moving towards the right. Therefore, $\frac{d^2y}{dx^2}$ and v_x are positive ; hence f_n is positive and the resultant of f_t and f_n, that is, f (fig. *a*) is directed towards the concave side of the curve.



2. The path is concave upwards and the point is moving towards the left. Therefore, $\frac{d^2y}{dx^2}$ is positive and v_x is negative ; hence f_n is negative. By definition (1), the normal PN is directed downwards, hence f_n is directed upwards and the resultant f is directed towards the concave side of the curve (fig. *b*).

Similar results follow for the two cases when the curve is concave downwards, as in figures *c* and *d*.

Since the direction of the tangent agrees with the direction of the velocity, we have from the figures the criterion: *The velocity vector is rotating counter-clockwise when f_n is positive, clockwise when f_n is negative.*

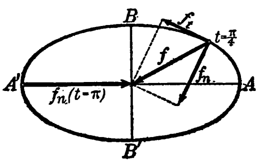
The significance of the algebraic sign of f_t is easily determined. Since $f_t = \frac{d}{dt}\left(\frac{ds}{dt}\right)$, it is seen that when f_t is positive the speed is increasing; when negative, the speed is decreasing.

When the equations of motion are given, we proceed as follows to find f_t and f_n:

1. Differentiate and find v_x, v_y, f_x, f_y.
2. Find v from $v = \sqrt{v_x^2 + v_y^2}$.
3. Differentiate this last result, giving $f_t = \frac{dv}{dt}$.
4. Find f_n by (5), p. 81.

Illustrative Example. Determine the normal and tangential accelerations in the motion defined by

$$x = a\cos t,\quad y = b\sin t.$$



Solution. Eliminating t, the path is the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Following the directions given, we find

$v_x = -a\sin t$, $v_y = b\cos t$, $f_x = -a\cos t$, $f_y = -b\sin t$.

Hence $v = +\sqrt{a^2\sin^2 t + b^2\cos^2 t}$. From these values we obtain

$$f_t = \frac{dv}{dt} = \frac{(a^2 - b^2)\sin t \cdot \cos t}{+\sqrt{a^2\sin^2 t + b^2\cos^2 t}},$$

$$f_n = \frac{v_x f_y - v_y f_x}{v} = \frac{ab}{+\sqrt{a^2\sin^2 t + b^2\cos^2 t}}.$$

t	x	y	f_t	f_n
0	a	0	0	a
$\frac{1}{2}\pi$	0	b	0	b
π	$-a$	0	0	a
$\frac{3}{2}\pi$	0	$-b$	0	b
2π	a	0	0	a

We note the following table of values. The point describes the ellipse counter-clockwise. The normal acceleration is always positive, agreeing with the fact that the velocity vector rotates always counter-clockwise. Since $f_t > 0$ when the point lies in the first and third quadrants, the speed increases from A to B and A' to B'. Similarly, from B to A' and B' to A the speed decreases.

PROBLEMS

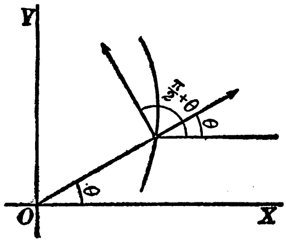
Find f_t and f_n for problems 1 and 2, p. 75, and discuss the results.

41. Equations in polar coördinates. In many cases it is more advantageous to employ polar coördinates in studying motion in a plane. If (ρ, θ) are the polar coördinates of a moving point P, the equations of motion have the form

$$\text{(XIV)}\qquad \rho = \phi(t),\quad \theta = \psi(t),$$

since obviously ρ and θ are now functions of t.

In rectangular coördinates the derivatives of x and y with respect to the time were of fundamental importance. Similarly in using polar coördinates we shall expect their derivatives to appear. The time-rate of change of the radius vector ρ is called the *radial velocity* of the point (ρ, θ). The time-rate of change of the vectorial angle θ is called the *angular velocity* ω of the point (ρ, θ). That is,



$$(1)\quad \begin{cases} \dfrac{d\rho}{dt} = \text{radial velocity}, \\ \dfrac{d\theta}{dt} = \omega = \text{angular velocity}. \end{cases}$$

We desire to obtain the components of the velocity vector and of the acceleration vector when resolved *along and perpendicular to the radius vector.* We first adopt a convention as to the positive directions along these lines.

The positive direction along the radius vector is defined as in Analytic Geometry, p. 149. The positive direction perpendicular to the radius vector is the direction obtained by *increasing* the vectorial angle θ by a right angle.

Denoting the components of the velocity and acceleration vectors along the radius vector by v_ρ and f_ρ and perpendicular to the radius vector by v_θ and f_θ, respectively, we have, (applying (II)),

$$(2)\quad \begin{cases} v_\rho = v_x \cos\theta + v_y \sin\theta, \\ v_\theta = v_x \cos\left(\dfrac{\pi}{2} + \theta\right) + v_y \sin\left(\dfrac{\pi}{2} + \theta\right) = -v_x \sin\theta + v_y \cos\theta, \end{cases}$$

and

$$(3)\quad \begin{cases} f_\rho = f_x \cos\theta + f_y \sin\theta, \\ f_\theta = f_x \cos\left(\dfrac{\pi}{2} + \theta\right) + f_y \sin\left(\dfrac{\pi}{2} + \theta\right) = -f_x \sin\theta + f_y \cos\theta. \end{cases}$$

To transform the derivatives of the rectangular coördinates into the derivatives of the polar coördinates, we have the relation,

$$(4)\quad \begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta. \end{cases}$$

By differentiating (4), we obtain

$$(5)\quad \begin{cases} v_x = \cos\theta \dfrac{d\rho}{dt} - \rho\sin\theta \dfrac{d\theta}{dt}, \\ v_y = \sin\theta \dfrac{d\rho}{dt} + \rho\cos\theta \dfrac{d\theta}{dt}, \end{cases}$$

and

$$(6)\quad \begin{cases} f_x = \cos\theta\dfrac{d^2\rho}{dt^2} - 2\sin\theta\dfrac{d\rho}{dt}\cdot\dfrac{d\theta}{dt} - \rho\cos\theta\left(\dfrac{d\theta}{dt}\right)^2 - \rho\sin\theta\dfrac{d^2\theta}{dt^2}, \\ f_y = \sin\theta\dfrac{d^2\rho}{dt^2} + 2\cos\theta\dfrac{d\rho}{dt}\cdot\dfrac{d\theta}{dt} - \rho\sin\theta\left(\dfrac{d\theta}{dt}\right)^2 + \rho\cos\theta\dfrac{d^2\theta}{dt^2}. \end{cases}$$

Substituting (5) and (6) in (2) and (3), respectively, we obtain, after simplifying,

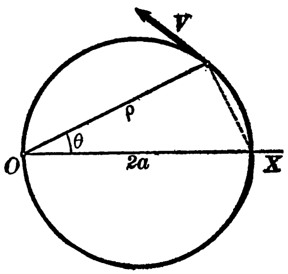
$$(\mathrm{XV})\quad \begin{cases} v_\rho = \dfrac{d\rho}{dt}, \\ v_\theta = \rho\dfrac{d\theta}{dt} = \rho\omega, \\ f_\rho = \dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{dv_\rho}{dt} - \rho\omega^2, \\ f_\theta = \rho\dfrac{d^2\theta}{dt^2} + 2\dfrac{d\rho}{dt}\cdot\dfrac{d\theta}{dt} = \dfrac{1}{\rho}\dfrac{d}{dt}(\rho^2\omega), \end{cases}$$

$$\left[\begin{array}{l} \text{since } \dfrac{1}{\rho}\dfrac{d}{dt}(\rho^2\omega) = \dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\theta}{dt}\right) = \rho\dfrac{d^2\theta}{dt^2} + 2\dfrac{d\rho}{dt}\dfrac{d\theta}{dt}. \\ \text{Of course } v = \sqrt{v_\rho^2 + v_\theta^2},\quad f = \sqrt{f_\rho^2 + f_\theta^2}, \text{ as usual.} \end{array}\right]$$

ILLUSTRATIVE EXAMPLE. A point describes a circle whose equation is given in polar coördinates. Discuss formulas (XV) for this case (compare Art. 38).

Solution. If the origin is on the circumference, the equation is

(1) $\rho = 2a\cos\theta.$



Differentiating with respect to t,

(2) $\dfrac{d\rho}{dt} = -2a\sin\theta\dfrac{d\theta}{dt}$, or $v_\rho = -2a\sin\theta\cdot\omega.$

$$\therefore v^2 = v_\rho^2 + v_\theta^2 = 4a^2\sin^2\theta\cdot\omega^2 + \rho^2\omega^2$$
$$= (4a^2\sin^2\theta + 4a^2\cos^2\theta)\omega^2.$$

Hence

(3) $v^2 = 4a^2\omega^2$, or $\omega = \dfrac{v}{2a}$, and $v_\rho = -v\sin\theta.$

Equations (3) express angular and radial velocity in terms of speed, and are easily found directly from the figure. This verification is left to the student.

To find the component accelerations, differentiate (2) again. This gives

$$\frac{dv_\rho}{dt} = -2a\cos\theta\,\omega^2 - 2a\sin\theta\frac{d\omega}{dt}$$
$$= -\rho\omega^2 - 2a\sin\theta\frac{d\omega}{dt}.$$

(4) $$\therefore f_\rho = -2\,\rho\omega^2 - 2\,a\sin\theta\frac{d\omega}{dt}$$
$$= -2\,\rho\omega^2 + \frac{v_\rho}{\omega}\cdot\frac{d\omega}{dt}$$
$$= -2\,\rho\omega^2 + \frac{v_\rho}{\omega}\alpha \text{ (by (4), Art. 42).}$$

Similarly,

(5) $$f_\theta = \frac{1}{\rho}\frac{d}{dt}(\rho^2\omega) = 2\,\omega v_\rho + \rho\frac{d\omega}{dt}$$
$$= 2\,\omega v_\rho + \rho\alpha.$$

Substituting in $f^2 = f_\rho^2 + f_\theta^2$, we find after reducing,

(6) $$f^2 = v^2\left(4\,\omega^2 + \frac{\alpha^2}{\omega^2}\right) = 4\,a^2(4\,\omega^4 + \alpha^2).$$

This equation expresses the total acceleration in terms of the angular velocity and acceleration.

In particular, assume $f_\theta = 0$, that is, let the acceleration be directed towards the origin. Then, from (5),

(7) $$\frac{d}{dt}(\rho^2\omega) = 0. \quad \therefore \rho^2\omega = c, \text{ and } \omega = \frac{c}{\rho^2}.$$

Also, from (5),

$$2\,\omega v_\rho + \rho\alpha = 0. \quad \therefore \alpha = -\frac{2\,\omega v_\rho}{\rho}.$$

Then (6) becomes

$$f^2 = 4\,a^2\left(4\,\omega^4 + \frac{4\,\omega^2 v_\rho^{\,2}}{\rho^2}\right) = \frac{16\,a^2\omega^2v^2}{\rho^2} = \frac{64\,a^4\omega^4}{\rho^2}.$$

(8) $$\therefore f = -\frac{8\,a^2\omega^2}{\rho} = -\frac{8\,a^2c^2}{\rho^5},$$

the negative sign being used since the acceleration must be directed *towards* 0.

This result is due to Newton, and may be stated as follows: *If a particle describes a circle with an acceleration directed towards a point on the circumference, the acceleration must be inversely proportional to the fifth power of the distance.*

PROBLEMS

1. Plot the path,* find v_ρ, v_θ, f_ρ, f_θ, and discuss the motion defined by the equation:

(*a*) $\rho = 2\,a\sin t^2$, $\theta = t^2$;
(*b*) $\rho = 2\,a\,t$, $\theta = \text{arc}\cos t$ $(0 \leq t \leq 1)$;
(*c*) $\rho = a\cos t$, $\theta = a\sec t$;
(*d*) $\rho = a\sin t$, $\theta = \sin t$;
(*e*) $\rho = \tan t$, $\theta = \cot t$;
(*f*) $\rho = a\tan t$, $\theta = \cot^2 t$;
(*g*) $\rho = e^{at}$, $\theta = t$;
(*h*) $\rho = a\,(1-t)$, $\theta = \text{arc}\cos t$ $(0 \leq t \leq 1)$;
(*i*) $\rho = a\sin t$, $\theta = \frac{1}{2}t$;
(*j*) $\rho = a\cos t$, $\theta = \frac{1}{3}t$.

* The path may be plotted from the parametric form as given, or the ordinary polar equation may be obtained by eliminating t.

2. A point describes the ellipse $\rho = \dfrac{ep}{1 - e\cos\theta}$.

Let θ be given in terms of E by the relations,

$$\sin\theta = \frac{\sqrt{1-e^2}\cdot\sin E}{1+e\cos E},\quad \cos\theta = \frac{\cos E + e}{1 + e\cos E},$$

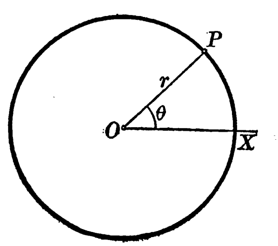
and E be given in terms of t by

$$nt = E + e\sin E.$$

Prove $$f_\rho = -\frac{n^2a^3}{\rho^2},\ f_\theta = 0, \text{ where } a = \frac{ep}{1-e^2}.$$

42. Rotation. When the path is a circle, the motion is called rotation. If the radius is r, the equations of motion are

(1) $\rho = r,\ \theta = \psi(t)$.



The position of the point is completely determined if θ is known. For this reason, the equation $\rho = r$ is unimportant and it is customary to call the second equation,

(2) $\theta = \psi(t)$,

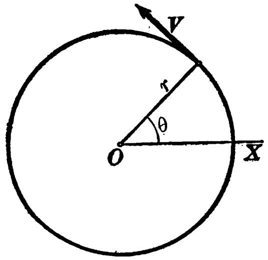
the *equation of the rotation*.

From (2), we obtain by differentiation,

(3) $\dfrac{d\theta}{dt} = \textit{angular velocity} = \omega$;

(4) $\dfrac{d^2\theta}{dt^2} = \dfrac{d\omega}{dt} = \textit{angular acceleration} = \alpha$.

That is, the angular acceleration is the time-rate of change of angular velocity. Angles being measured in radians, angular velocity is measured in *radians per second.* For example, if $\omega = \frac{1}{2}\pi$ and is constant, the radius OP rotates through a right angle in each second. In the same way angular acceleration is measured in radians per second *in each second.* For example, if $\alpha = 1$ and is constant, the radius OP rotates with increasing angular velocity, the gain being one radian per second in each second.



The speed in rotation is readily found. For if $\theta - \theta_0$ is the angle turned through in the time t, and s the length of the corresponding arc, we have

$$s = r(\theta - \theta_0).$$

Hence, by differentiation,

$$(5) \qquad \frac{ds}{dt} = r\frac{d\theta}{dt}, \text{ or } v = r\omega.$$

In rotation the speed equals the angular velocity times the radius.

Next, consider the tangential and normal accelerations. These are also readily expressible in terms of ω and α. For by (XIII) and (5),

$$(6) \qquad \begin{cases} f_t = \dfrac{dv}{dt} = r\dfrac{d\omega}{dt} = r\alpha, \\ f_n = \dfrac{v^2}{R} = \dfrac{r^2\omega^2}{r} = r\omega^2. \end{cases}$$

The same are found from (XV) by noting that in rotation $f_\rho = -f_n, f_\theta = f_t$.

THEOREM. *If ω and α are respectively the angular velocity and angular acceleration in rotation, the speed and acceleration are determined from*

$$(\text{XVI}) \qquad v = r\omega, \; f_t = r\alpha, \; f_n = r\omega^2.$$

ILLUSTRATIVE EXAMPLE. A fly wheel is making 120 revolutions per minute (R.P.M.). If the angular velocity diminishes at a constant rate, find the number of revolutions if the wheel stops in one minute.

Solution. The motion of the wheel is determined by the motion of one of its points. Let ω_0 be the initial angular velocity.

Then, since * 120 R.P.M. $= 2$ R.P.S. $= 4\pi$ radians per second, we have $\omega_0 = 4\pi$.

Since the angular acceleration is constant,

$\alpha = \dfrac{d\omega}{dt} = k$, $\therefore \omega = kt + c$, where c is the constant of integration. But $\omega = \omega_0$ when $t = 0$.

$$(1) \qquad \therefore \omega = kt + \omega_0 \text{ or } \omega = kt + 4\pi.$$

Since the wheel comes to rest in 60 sec., $\omega = 0$ when $t = 60$.

$$\therefore 0 = 60k + 4\pi, \text{ and } k = -\tfrac{1}{15}\pi.$$

$$(2) \qquad \therefore \omega = -\tfrac{1}{15}\pi t + 4\pi.$$

Writing $\omega = \dfrac{d\theta}{dt}$, integrating and assuming $\theta = 0$ if $t = 0$, we obtain from (2)

$$(3) \qquad \theta = -\tfrac{1}{30}\pi t^2 + 4\pi t,$$

which gives the angle turned through in any time. If $t = 60$, $\theta = 120\pi$, and hence the number of revolutions is $\dfrac{\theta}{2\pi} = 60$.

* In general, angular velocity $= \dfrac{2\pi}{60} \cdot$ R.P.M.

PROBLEMS

1. In the following problems the equation of motion of a point describing a circle is given. Discuss the motion.

$$(a)\quad \theta = at + b;$$
$$(b)\quad \theta = at^2 + bt + c;$$
$$(c)\quad \theta = \sin t;$$
$$(d)\quad \theta = t^3 - t;$$
$$(e)\quad \theta = \pi \sin \frac{\pi t}{2}.$$

2. A fly wheel making 360 R.P.M. is subject to a constant retardation of 1 radian per second per second. How many revolutions does it make before stopping? What time is required? *Ans.* $36\,\pi$ revolutions; $12\,\pi$ sec.

3. A fly wheel starting from rest is subject to a constant angular acceleration of $\frac{1}{2}$ radian per second per second for two minutes. Find the angular velocity and the number of revolutions made at the end of the first minute; at the end of the second minute. *Ans.* $\frac{900}{\pi}$ R.P.M., $\frac{450}{\pi}$ rev.; $\frac{1800}{\pi}$ R.P.M., $\frac{1800}{\pi}$ rev.

4. A fly wheel starting from rest and subject to a constant angular acceleration for 3 minutes makes 5000 revolutions. Find the acceleration.

Ans. $\alpha = \frac{50\,\pi}{81}$ rad. per second^2.

5. A fly wheel making 500 R.P.M. and subject to a constant retardation comes to rest after making 2000 revolutions. What time is required? *Ans.* 8 min.

CHAPTER IV

KINETICS OF A MATERIAL PARTICLE

43. Momentum. In the preceding chapters motion of a material particle has been studied without reference to mass or force. The latter are now to be taken into consideration. We begin with the definition:

Momentum or quantity of motion is the product of mass and velocity, or

$$(1) \qquad \text{Momentum at any instant} = m\mathbf{v}.$$

From the definition it is plain that momentum is a vector quantity, being the product of the vector velocity by the positive number m. The direction of the vector momentum is the same as that of $\mathbf{v}$, but its magnitude equals the product of mass and speed.

44. Force. The science of Mechanics is founded upon laws or axioms which sum up the results of experience in the observation of motion. A set of three Laws of Motion was proposed by Sir Isaac Newton (1642–1727), the statement of which is general enough for present purposes. Considering these laws as needed in the development of our subject, we begin with the

First Law of Motion. Every body persists in its state of rest or of uniform motion in a straight line, except in so far as it may be compelled by force to change that state.

Remembering that uniform motion in a straight line means motion with *constant vector velocity*, it is plain that uniform motion means *constant vector momentum*. The First Law is often expressed by saying that the *body has inertia*. A body has no power of itself to change its state of rest or motion, but continues to move with constant momentum when not acted upon by an impressed force. That is, by the First Law we conclude that *no force is acting upon a body if the body is at rest or moving with constant momentum.*

If, however, the momentum is variable, then the existence of forces acting upon the body is inferred. We thus come to the

SECOND LAW OF MOTION. Change in momentum is caused by forces acting upon the body. Force and change in momentum agree in direction, and the magnitude of the force at any instant is proportional to the time-rate of change in momentum.

In this statement of the Second Law is contained the definition of force. For consider the motion of a material particle of mass m. Its momentum at any instant equals $m\mathbf{v}$. Since m is constant, *change* in momentum means change in vector velocity, and the direction of change in velocity we know agrees with the direction of the acceleration. By the Second Law, therefore, force and acceleration agree in direction. Furthermore, the magnitude of the force at any instant is proportional to the time-rate of change in momentum; that is,

$$(2) \qquad \textit{Force at any instant} = k\frac{d}{dt}\left(mv\right) = km\frac{dv}{dt} = kmf,^*$$

where k is a constant factor of proportionality. Hence the Second Law leads to the result:

The force acting at any instant upon a material particle has the direction of the vector acceleration and in magnitude is proportional to the product of the mass and acceleration. Force is therefore the *cause of acceleration.*

The value of the factor k in (2) depends upon the units assumed. Evidently for analytical purposes it is convenient to assume $k = 1$. This is shown below to be equivalent to assuming that force is measured in so-called *scientific units.* For theoretical purposes, therefore, we may assume as the magnitude of force,

$$(\text{I}) \qquad \textbf{Force} = \boldsymbol{m\frac{dv}{dt} = mf.}$$

In Applied Mechanics, however, it is found more convenient to select k not equal to unity. (See Art. 45.)

Observation of falling bodies makes familiar the phenomenon of changing momentum. The force in question is then called the *weight* of the body, or also, the force of gravity. That is, weight is the force of attraction exerted by the earth upon other bodies. The acceleration caused by weight is nearly constant in

* In equation (2) the differentiation is made on the assumption that the mass is constant. If the mass is variable, a special investigation is required. See Routh, Dynamics of a Particle, p. 80.

a small region near the earth's surface and is denoted by g. This acceleration is also called the *intensity of gravity*. The numerical value of g varies from place to place and also depends upon the units of length and time adopted. In the English and French systems, respectively, as an average value,

$$g = 32.2 \text{ ft. per sec. in 1 sec. (English)},$$

$$g = 983 \text{ cm. per sec. in 1 sec. (French)}.$$

Dimensions. From the definition of force it follows that its dimensions are mass times acceleration. The derived unit of force is therefore expressed in terms of the fundamental units of mass, distance, and time by the dimensional equation

$$\text{Force} = \frac{\text{mass} \times \text{length}}{\text{time}^2}.$$

45. Units of force. Scientific units. For theoretical purposes it is convenient to define unit force as that force which will produce unit acceleration in unit mass. With this definition it is apparent that in equation (2), Art. 44, the factor of proportionality, k, is unity. Hence, in *scientific units*,

(1) *Force* = *mass* times *acceleration.*

In the English system, the unit of mass is the pound and the scientific unit of force is the poundal. Hence, *one poundal is that force which will give to a mass of one pound an acceleration of one foot per second in one second.* In the French system, the unit of mass is the gram and the unit of force is the dyne. Hence, *one dyne is that force which will give to a mass of one gram an acceleration of one centimeter per second in one second.*

Technical units. In engineering practice the English unit of force is equal to the weight of unit mass and is called the *pound.* Referring to (2), Art. 44, since the force in question is weight, we must replace f by g, and thus obtain

$$F = kmg.$$

By hypothesis, when m is unity, so also is F,

$$\therefore\ 1 = kg \text{ and } \therefore\ k = 1 \div g.$$

Substituting in (2), Art. 44, gives as the magnitude of force in *technical units,*

(2) *Force* = *mass* times *acceleration* divided by g.

Comparison of the two systems of units. Mass, time, and length are measured by the same units in both systems. As just explained, however, force is measured by different units. To find the relation between the latter, we may apply (1) to the case of weight, whence, in *scientific* English *units*,

$$\text{(3)} \qquad \text{Weight} = mg \text{ (poundals).}$$

Since, by definition of the technical unit, the weight of a 1-lb. mass equals 1 lb. of force, hence the equivalence,

$$\text{(4)} \qquad \textit{One pound of force} = g \textit{ poundals.}$$

The student will observe that in *technical units* weight and mass are *numerically equal.* The difference is one of dimensions only.

The following Table of Equivalents, together with equation (4), will be found useful :

English	French
1 foot	= 30.48 centimeters;
1 pound (mass)	= 453.6 grams;
1 poundal	= 13,825 dynes;
1 pound (force)	= 4.45 $(10)^5$ dynes.

46. Rectilinear motion. If the path of a material particle is a straight line, the expressions for the acceleration are given by (III), Chapter II. Hence, applying (I), and denoting the force* by F, we have

$$F = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = mv\frac{dv}{dx}.$$

Dividing by m, we have the *force equation* or the *differential equation of motion* in a straight line.

$$\text{(II)} \qquad (a)\ \frac{F}{m} = \frac{d^2x}{dt^2}, \text{ or } (b)\ \frac{F}{m} = \frac{dv}{dt}, \text{ or } (c)\ \frac{F}{m} = v\frac{dv}{dx}.$$

Suppose the mass m is given and the force is known. It is required to discuss the motion. For this purpose we must determine x from equations (II) by integration. If F is a function of the time only, (a) should be used; if F is a function of the

* The discussion of the text assumes scientific units in all cases.

velocity only, (b) should be used; and if F is a function of the distance only, it is usually more convenient to use (c). However, in case of a linear function of v and x, that is, if $F = Av + Bx + C$, where A and B are constants and C is constant or involves t, use (a) (see equations 71, 72, 73, 74, Chapter XIV). If F is a constant, either form may be used.

The force alone is not sufficient to determine the motion completely. For example, let us consider the case of a particle projected vertically in a vacuum. Obviously the motion will depend upon the position (on the vertical line OX) from which the particle is started, and upon the velocity with which it is projected. The initial position x_0 and the initial velocity v_0 are called the *initial conditions*, and it will be shown that when known they determine the motion completely. The only force acting is the weight, whose magnitude is mg. The direction of the force is downward, and if we choose the positive direction along OX downward, we have, from (II), (b),

$$\frac{F}{m} = \frac{mg}{m} = \frac{dv}{dt},$$

or

$$\frac{dv}{dt} = g.$$

Multiplying by dt and integrating,

$$(1) \qquad v = gt + c_1,$$

where c_1 is a constant of integration; and since $v = \frac{dx}{dt}$, we may multiply by dt and integrate again, obtaining

$$(2) \qquad x = \tfrac{1}{2} gt^2 + c_1 t + c_2,$$

where c_2 is a second constant of integration.

To determine the constants of integration, we make use of the initial conditions. Suppose the particle is started at the point x_0 with the velocity of projection v_0. Then when $t = 0$, $x = x_0$, and $v = v_0$. Hence, substituting in (1) and (2), we have

$$(1') \qquad v_0 = c_1,$$

$$(2') \qquad x_0 = c_2.$$

Hence the equation of motion is

$$x = \tfrac{1}{2} gt^2 + v_0 t + x_0.$$

The discussion may be made according to the directions given in Chapter II.

47. Resultant force in rectilinear motion. If a particle moving in a straight line be acted upon by two or more forces directed along the line of motion, the resultant acceleration is the algebraic sum of the accelerations due to the separate forces. Suppose the particle is acted upon by n forces, $F_1, F_2, \cdots, F_n$. The acceleration due to F_1 is $f_1 = \frac{F_1}{m}$, to F_2 is $f_2 = \frac{F_2}{m}$, $\cdots$, to F_n is $f_n = \frac{F_n}{m}$, and the resultant acceleration, f, is given by

$$(1) \qquad f = f_1 + f_2 + \cdots f_n = \frac{F_1 + F_2 + \cdots F_n}{m}.$$

Hence, if F denotes the algebraic sum or resultant of the collinear forces $(F = F_1 + F_2 + \cdots F_n)$, we have, from (1),

$$(2) \qquad F = mf.$$

That is, *if a particle moving in a straight line be acted upon by any number of forces directed along the line of motion, the product of the mass and the acceleration is equal to the resultant force.*

ILLUSTRATIVE EXAMPLES

1. A heavy body is projected in a vertical direction. Determine the equation of motion if the resistance of the air is proportional to the speed.

Solution. We take the X-axis vertical with positive direction downwards. There are two cases: (a) when the body is falling; (b) when it is rising.

(a) The weight, acting downwards, is positive and equal to mg. The resistance of the air always opposes the motion, and hence, when the body is falling, this force is negative. Since the velocity is positive, we have

$$\text{Resistance} = -\mu m v,$$

where μ is a factor of proportionality.

The resultant force is $F = mg - \mu m v$.

(b) When the body is rising, the resistance of the air acts in the same direction as the weight, and is, therefore, positive. Since the velocity is negative, we have

$$\text{Resistance} = -\mu m v,$$

and the resultant force has the same form as in case (a).

Hence, in this problem, the force equation is the same when the body is falling as when it is rising.

Since the force is a linear function of v, we use (II) (a),

$$\frac{d^2x}{dt^2} = \frac{F}{m} = g - \mu v,$$

or

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} = g. \tag{1}$$

The solution of the homogeneous equation (see Calculus, p. 440),

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} = 0,$$

is

$$x = c_1 + c_2e^{-\mu t}.$$

We see by inspection that a particular solution of (1) is $x = \frac{g}{\mu}t$, and hence the general solution is

$$x = \frac{g}{\mu}t + c_1 + c_2e^{-\mu t}. \tag{2}$$

The constants of integration are determined if the initial position, x_0, and the velocity of projection, v_0, are known. Differentiating (2), we find the velocity,

$$v = \frac{g}{\mu} - \mu c_2 e^{-\mu t}. \tag{3}$$

If $x = x_0$, $v = v_0$, when $t = 0$, we find, from (2) and (3), $c_2 = \frac{g}{\mu^2} - \frac{v_0}{\mu}$, $c_1 = x_0 + \frac{v_0}{\mu} - \frac{g}{\mu^2}$, and hence the equation of motion is

$$x = \frac{g}{\mu}t + x_0 + \frac{v_0}{\mu} - \frac{g}{\mu^2} + \left(\frac{g}{\mu^2} - \frac{v_0}{\mu}\right)e^{-\mu t}.$$

2. A box of mass 100 lb. is placed on an elevator which ascends with an acceleration of 10 ft. per second per second. What pressure does the elevator exert upon the box?

Solution. Taking the positive direction upwards, and denoting the pressure of the elevator on the box by P, we have for the resultant force,

$$F = P - mg = mf.$$

Substituting the values of m and f, we find

$$P = 100(10 + 32) = 4200 \text{ poundals}.$$

PROBLEMS

1. Find the equation of each of the following rectilinear motions under the given conditions:

(a) $F_x = mt$; $x = 1$, $v = 0$, when $t = 1$.

Ans. $x = \frac{1}{6}t^3 - \frac{1}{2}t + \frac{4}{3}$.

(b) $F_y = m(t - 1)$; $y = 0$, $v = 1$, when $t = 0$.

Ans. $y = \frac{1}{6}t^3 - \frac{1}{2}t^2 + t$.

(c) $F_x = \frac{m}{t-1}$; $x = 0$, $v = 0$, when $t = 2$.

Ans. $x = (t - 1)[\log(t - 1) - 1] + 1$

(*d*) $F_y = m \cos t$; $y = 0$, $v = 0$, when $t = 0$.
Ans. $y = 1 - \cos t$.

(*e*) $F_x = -mx$; $x = a$, $v = 0$, when $t = \frac{\pi}{2}$.
Ans. $x = a \sin t$.

(*f*) $F_y = -my$; $y = 0$, $v = 1$, when $t = \pi$.
Ans. $y = -\sin t$.

(*g*) $F_x = -mx$; $x = a\cos\beta$, $v = -a\sin\beta$, when $t = 0$.
Ans. $x = a\cos(t + \beta)$.

(*h*) $F_y = -mk^2y$; $y = a\cos\beta$, $v = -ak\sin\beta$, when $t = 0$.
Ans. $y = a\cos(kt + \beta)$.

(*i*) $F_x = -mn^2x$; $x = a\sin\nu$, $v = an\cos\nu$, when $t = 0$.

(*j*) $F_y = my$; $y = 0$, $v = 1$, when $t = 0$.
Ans. $y = \frac{1}{2}(e^t - e^{-t})$.

(*k*) $F_y = -2my - 2mv$; $y = 0$, $v = 10$, when $t = 0$.
Ans. $y = 10\,e^{-t}\sin t$.

(*l*) $F_x = -25mx - 6mv$; $x = a$, $v = 0$, when $t = 0$.
Ans. $x = \frac{a}{4}e^{-3t}(4\cos 4t + 3\sin 4t)$.

(*m*) $F_x = -2\mu mv - k^2mx$; $x = 0$, $v = b$, when $t = 0$, $(k > \mu)$.
Ans. $x = \frac{b}{\sqrt{k^2 - \mu^2}}e^{-\mu t}\sin\sqrt{k^2 - \mu^2}\,t$.

(*n*) $F_x = -4mx + 2m\cos t$; $x = 0$, $v = 0$, when $t = 0$.
Ans. $x = \frac{2}{3}(\cos t - \cos 2t)$.

(*o*) $F_y = -my - m\sin t$; $y = 0$, $v = 0$, when $t = 0$.
Ans. $y = \frac{1}{2}t\cos t - \frac{1}{2}\sin t$.

(*p*) $F_x = -k^2mx + m\sin nt$; $x = 0$, $v = 0$, when $t = 0$.
Ans. $x = -\frac{n}{k(k^2 - n^2)}\cos kt + \frac{1}{k^2 - n^2}\sin nt$.

(*q*) $F_y = -k^2my + m\cos nt$; $y = 0$, $v = 0$, when $t = 0$.

(*r*) $F_x = -mx + m\sin t + 3m\cos 2t$; $x = 0$, $v = 0$, when $t = 0$.

2. Discuss the following rectilinear motions, taking into account the initial conditions.

(*a*)	$f = a^2 + x$,	given $v = c$, $x = 0$, when $t = 0$.
(*b*)	$f = x^{-3}$;	given $v = v_0$, $x = x_0$, when $t = t_0$.
(*c*)	$f = v^2$;	given $v = \frac{1}{2}$, $x = 0$, when $t = 0$.
(*d*)	$f = av$;	given $v = b$, $x = \frac{b}{a}$, when $t = 0$.
(*e*)	$f = \frac{g^2 - k^2v^2}{g}$;	given $v = 0$, $x = 0$, when $t = 0$.

(*f*) $F_x = m\left(at^2 - \frac{1}{ct}\right)$; given $v = \frac{a}{c^3}$, $x = \frac{a}{c^4}$, when $t = \frac{1}{c}$.

Answers:

(*c*) $x = -\log(1 - \frac{1}{2}t)$.

(*e*) $t = \frac{1}{2k}\log\frac{g+kv}{g-kv}$; $v = \frac{g}{k}\cdot\frac{e^{kt} - e^{-kt}}{e^{kt} + e^{-kt}}$;

(*f*) $v = \frac{a}{3}\left(t^3 + \frac{2}{c^3}\right) - \frac{1}{c}\log ct$; $x = \frac{1}{12}at^4 + \frac{2a}{3c^3}t - \frac{1}{c}(t\log ct - t) + \frac{a - 4c^2}{4c^4}$.

3. Show that a particle projected with a velocity v_0 and acted upon by a constant force mk will acquire a velocity equal to $\sqrt{2kx + v_0^2}$ in moving the distance x.

4. A body is projected vertically upwards with a velocity V. Prove the formulas $v = V - gt$, $h = Vt - \frac{1}{2}gt^2$, where h is the height at any instant. What is the greatest height?

Ans. $h = \frac{V^2}{2g}$.

5. A body of 25 lb. mass is acted upon by a constant force which in 10 sec. gives it a velocity of 75 ft. per second. What is the magnitude of the force in poundals?

6. A heavy body is projected in a vertical direction. Write the force equation and find the equation of motion if the resistance of the air is proportional to the square of the speed.

Ans. When the body is rising, $F = mg + \mu mv^2$; $x = \frac{1}{\mu}\log\sec(\sqrt{\mu g}\,t + c_1) + c_2$.

When the body is falling, $F = mg - \mu mv^2$; $x = \sqrt{\frac{g}{\mu}}\,t + \frac{1}{\mu}\log(1 - e^{-2\sqrt{\mu g}t + c_1}) + c_2$.

7. An elevator, starting from rest, has a downward acceleration of 16 ft. per second per second for 1 sec., then moves uniformly for 2 sec., then has an upward acceleration of $10\frac{2}{3}$ ft. per second per second until it comes to rest. (*a*) How far does it descend? (*b*) A person whose weight is 150 lb. experiences what pressure from the elevator during each of the three periods of its motion?

Ans. (*a*) 52 ft. (*b*) 75 lb.; 150 lb.; 200 lb.

8. Equal masses of m lb. each rest upon two platforms, one of which has at a certain instant a velocity of a ft. per second upwards and the other a velocity of b ft. per second downwards. Both platforms have an upward acceleration f. Compare the pressures of the platforms on the bodies.

9. A bucket containing 112 lb. of coal is drawn up the shaft of a coal pit and the pressure of the coal on the bottom of the bucket is equal to the weight of 126 lb. Find the acceleration of the bucket.

Ans. $\frac{g}{8}$.

10. While ascending vertically in a balloon with a velocity v, a man drops a stone when h ft. above the ground. Find the time required for the stone to fall to the ground.

Ans. $\frac{v + \sqrt{v^2 + 2gh}}{g}$.

11. A string which can just sustain a mass of 10 lb. against gravity is attached to a mass of 2 lb. which rests upon a horizontal table. Supposing that friction is $\frac{1}{10}$ the weight of the body, find the greatest acceleration that can be given to the body by means of the string.

12. A particle moves in a straight line under a force directed towards the origin and varying inversely as the third power of the distance. Prove $v^2 = \frac{k^2}{x^2} + v_0^2$, if k^2 is the absolute intensity. If the initial distance and velocity are respectively b and $\frac{k}{b}$, show that the equation of motion is $x^2 = b^2 - 2\,kt$. Discuss the motion.

13. A particle is projected with a velocity v in a medium offering a resistance proportional to the square of the velocity. Show that the equation of motion may be written $s = \frac{1}{\mu} \log(\mu vt + 1)$. Discuss the motion.

14. Find the equation of motion if the force is a periodic function of the time. *Hint.* Assume* $F_x = ma \cos kt$. Then F_x varies from ma to $-ma$ with the period $\frac{2\pi}{k}$.

Ans. $x = -\frac{a}{k^2} \cos kt + v_0 t + x_0$.

15. When is the motion in problem 14 periodic and what is the period?

Ans. $v_0 = 0$, period $= \frac{2\pi}{k}$.

16. A particle describes a straight line under the action of two forces, one constant and the other an attractive central force proportional to the distance. Show that the force equation may be written $\frac{d^2y}{dt^2} = -\mu^2 y + f$, where μ and f are constants. Find the equation of motion and discuss it.

Ans. $y = c \cos(\mu t + \nu) + \frac{f}{\mu^2}$, where c and ν are constants of integration.

17. Show that the motion in problem 16 is central motion, the center being at $y = +\frac{f}{\mu^2}$ and attracting directly as the distance. (*a*) What is the period of the motion? (*b*) If $y = a$, $v = 0$, when $t = 0$, find the amplitude.

Ans. (*a*) $\frac{2\pi}{\mu}$.

18. A spring balance is extended $\frac{1}{4}$ in. by a mass of 1 lb. and the force of the spring is proportional to the extension. The spring is then pulled downward and released. Show that the force equation has the same form as in problem 16, namely $\frac{d^2y}{dt^2} = g(1 - 48\,y)$. What is the period of the vibration?

Ans. $\frac{2\pi}{\sqrt{48\,g}} = \frac{1}{11}$ sec. nearly.

* A finite periodic function of the time must have the form $A \sin(bt + \nu)$ or $A \cos(bt + \nu)$, where A, b, and ν are constants.

19. A particle is acted upon by a center of force which attracts directly as the distance and moves in a medium resisting directly as the velocity. Show that the force equation may be written $\frac{d^2x}{dt^2} + 2\mu\frac{dx}{dt} + k^2x = 0$. Find the equation of motion if $\mu < k$.

Ans. $x = Ae^{-\mu t}\cos(\sqrt{k^2 - \mu^2}\, t + \nu)$, where A and ν are constants of integration.

20. Write the force equation for a particle which is acted upon by an attractive center of force proportional to the cube of the distance if the particle moves in a medium offering a resistance proportional to the square of the speed.

21. A central force is attractive and varies as the nth power of the distance. If the particle starts from rest at the distance a from the center, find the time of arriving at the center when (1) $n = 1$, (2) $n = -3$.

Ans. (1) $\frac{\pi}{2\sqrt{\mu}}$, (2) $\frac{a^2}{\sqrt{\mu}}$, where μ is the absolute intensity.

22. In example 1, p. 95, show that the velocity approaches $\frac{g}{\mu}$ as t increases indefinitely. Show also that when the particle is projected downwards with this limiting velocity, the velocity remains constant, and the motion is uniform.

48. Curvilinear motion. Axioms on force action. Concurrent forces. Three things must be known of a force in order to completely determine it, namely, its magnitude, its direction, and its point of application. Forces are therefore not vector quantities in the sense in which a vector was defined in Chapter III, because the line of action of a force cannot be moved without changing the effect of the force. We are, however, familiar from experience with certain properties of force action which at least suggest vector properties. In fact, it is evident that if we confine ourselves to forces *acting simultaneously upon a material particle*, since at any instant such forces have the same point of application, magnitude and direction are now alone significant. Such forces are said to be *concurrent*. For these forces vector resolution and composition have meanings with which we are familiar. These results of experience we state in the form of axioms.

AXIOM 1. The acceleration produced by the simultaneous action of any number of concurrent forces is equal to the acceleration which would be produced by their vector resultant.

In other words, any number of concurrent forces may be replaced by a single force equal to their vector sum.

AXIOM 2. If a force is resolved along any direction, the acceleration due to this component may be found by resolving the original acceleration along that direction.

For example, given a force F which causes at a given instant the vector acceleration f in the motion of a material particle of mass m. Then, by (I),

$$F = mf. \tag{1}$$

If now we resolve F and f along any directed line l, the corresponding components being F_l and f_l, respectively, then f_l is the acceleration caused by F_l, and by (I) and Axiom 2, we shall have

$$F_l = mf_l. \tag{2}$$

That is, *the component of a force along any direction equals the mass times the acceleration along that direction.*

49. Curvilinear motion. Suppose a particle moves in a plane and is acted upon by n forces, $F_1, F_2, \cdots, F_n$. By the first axiom on force action the n forces may be replaced by a single resultant force F obtained by the vector addition of the individual forces $F_1, F_2, \cdots, F_n$. By the second axiom on force action the component of the resultant force F in the direction of the X-axis is equal to the mass times the acceleration in the direction of the X-axis. Similarly, the component of F in the direction of the Y-axis is equal to the mass times the acceleration in the direction of the Y-axis. Hence we have the *rectangular force equations* for plane motion:

$$\text{(III)} \qquad \begin{cases} F_x = m\dfrac{d^2x}{dt^2}, & F_y = m\dfrac{d^2y}{dt^2}, \qquad \text{or} \\ F_x = m\dfrac{dv_x}{dt}, & F_y = m\dfrac{dv_y}{dt}, \qquad \text{or} \\ F_x = mv_x\dfrac{dv_x}{dx}, & F_y = mv_y\dfrac{dv_y}{dy}, \end{cases}$$

where

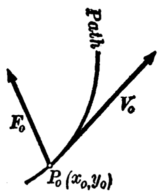
F_x = sum of x-components of all forces acting,

F_y = sum of y-components of all forces acting.

The equations of motion are obtained by integrating the force equations.

Equations (III) are the force equations for motion in the XY-plane. For motion in space of three dimensions the force equations have the same form, the only difference being that the additional coördinate z is introduced. See (XII), Art. 37.

The integration of equations (III) introduces four arbitrary constants. Hence to determine the motion completely we must have four conditions. These conditions may be the two coördinates of the initial position (x_0, y_0) and the two components of the initial velocity (v_{x0}, v_{y0}).

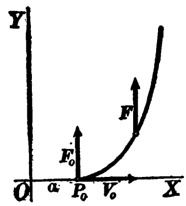


From the discussion of Art. 40, we know that the *resultant acceleration* in any curvilinear motion is directed towards the concave side of the path (in special cases along the tangent). This fact enables us to construct in almost all cases the beginning of the path. For we may plot the initial position (x_0, y_0) and draw the initial velocity (since v_{x0} and v_{y0} are given). Further we may calculate F_x and F_y for the initial conditions and construct the initial force F_0. *The path then starts in the direction of* v_0 *and is concave in the direction of* F_0.

ILLUSTRATIVE EXAMPLES

1. Find the equations of motion and the path if $F_x = 0$, $F_y = mg$; when $t = 0$, $x = a$, $y = 0$, $v_x = b$, $v_y = 0$.

Solution. The force equations are



$$(1) \qquad m\frac{d^2x}{dt^2} = 0, \quad m\frac{d^2y}{dt^2} = mg.$$

Each equation may be integrated separately:

$$(2) \qquad \frac{dx}{dt} = c_1, \quad \frac{dy}{dt} = gt + c_2.$$

A second integration gives

$$(3) \qquad x = c_1t + c_3, \quad y = \tfrac{1}{2}gt^2 + c_2t + c_4.$$

Substituting the initial conditions in (2) and (3), we find

$$(2') \qquad b = c_1,\ 0 = c_2.$$

$$(3') \qquad a = c_3,\ 0 = c_4.$$

The equations of motion are

$$x = bt + a, \quad y = \tfrac{1}{2}gt^2.$$

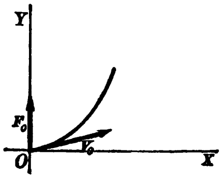
Eliminating t, the equation of the path is found to be

$$y = \tfrac{1}{2}g\left(\frac{x-a}{b}\right)^2.$$

The path is a parabola with its axis parallel to the Y-axis.

2. A particle of mass m is acted upon by two forces: (1) one in the direction of the Y-axis and equal to mk; (2) one in a direction making a constant angle α with the X-axis and equal to mt^2. When $t = 0$, $x = 0$, $y = 0$, $v_x = a$, $v_y = b$. Find the equations of motion.

Solution. The x- and y-components of the first force are zero and mk, respectively. The x- and y-components of the second force are $mt^2 \cos \alpha$ and $mt^2 \sin \alpha$, respectively. Hence the force equations are



(1) $$m\frac{d^2x}{dt^2} = mt^2 \cos \alpha, \quad m\frac{d^2y}{dt^2} = mt^2 \sin \alpha + mk.$$

Integrating once,

(2) $$\frac{dx}{dt} = \tfrac{1}{3} t^3 \cos \alpha + c_1, \quad \frac{dy}{dt} = \tfrac{1}{3} t^2 \sin \alpha + kt + c_2.$$

Integrating a second time,

(3) $$\begin{cases} x = \tfrac{1}{12} t^4 \cos \alpha + c_1 t + c_3, \\ y = \tfrac{1}{12} t^4 \sin \alpha + \tfrac{1}{2} kt^2 + c_2 t + c_4. \end{cases}$$

Substituting the initial conditions in (2) and (3), we find,

(2′) $$a = c_1, \quad b = c_2,$$

(3′) $$0 = c_3, \quad 0 = c_4.$$

Hence the equations of motion are

$$x = \tfrac{1}{12} t^4 \cos \alpha + at, \quad y = \tfrac{1}{12} t^4 \sin \alpha + \tfrac{1}{2} kt^2 + bt.$$

PROBLEMS

1. Find the equations of motion and the path in each of the following:

(*a*) $F_x = mk$, $F_y = 0$; when $t = 0$, $y = 0$, $x = 0$, $v_x = a$, $v_y = b$.
Ans. $x = \frac{1}{2} kt^2 + at$, $y = bt$. *Parabola.*

(*b*) $F_x = mx$, $F_y = 0$; when $t = 0$, $x = 0$, $y = 0$, $v_x = a$, $v_y = b$.
Ans. $x = \frac{1}{2} a(e^t - e^{-t})$, $y = bt$.

(*c*) $F_x = am$, $F_y = bm$; when $t = 0$, $x = 0$, $y = 0$, $v_x = 0$, $v_y = c$.
Ans. $x = \frac{1}{2} at^2$, $y = \frac{1}{2} bt^2 + ct$. *Parabola.*

(*d*) $F_x = mx$, $F_y = my$; when $t = 0$, $x = 0$, $y = 0$, $v_x = a$, $v_y = b$.
Ans. $x = \frac{1}{2} a(e^t - e^{-t})$, $y = \frac{1}{2} b(e^t - e^{-t})$.

(*e*) $F_x = 0$, $F_y = my$; when $t = 0$, $x = 0$, $y = a$, $v_x = 1$, $v_y = 0$.
Ans. $x = t$, $y = \frac{1}{2} a(e^t + e^{-t})$. *Catenary.*

(*f*) $F_x = mx$, $F_y = my$; when $t = 0$, $x = a$, $y = 0$, $v_x = 0$, $v_y = b$.
Ans. $x = \frac{1}{2} a(e^t + e^{-t})$, $y = \frac{1}{2} b(e^t - e^{-t})$. *Hyperbola.*

(*g*) $F_x = 0$, $F_y = mv_y$; when $t = 0$, $x = 0$, $y = 1$, $v_x = 1$, $v_y = 1$.
Ans. $x = t$, $y = e^t$. Curve $y = e^x$.

(*h*) $F_x = -mv_x^2$, $F_y = 0$; when $t = 0$, $x = 0$, $y = 0$, $v_x = 1$, $v_y = 1$.
Ans. $x = \log(t + 1)$, $y = t$. Curve $x = \log(1 + y)$.

(*i*) $F_x = -m \sin t$, $F_y = mv_y^2$; when $t = 0$, $x = 0$, $y = 0$, $v_x = 1$, $v_y = 1$.
Ans. $x = \sin t$, $y = \log\left(\frac{1}{1 - t}\right)$.

(*j*) $F_x = \frac{m}{v_x}$, $F_y = 0$; when $t = 0$, $x = 9$, $y = 9$, $v_x = 3$, $v_y = 2$.
Ans. $x = \frac{(2t + 9)^{\frac{3}{2}}}{3}$, $y = 2t + 9$. Curve $9x^2 = y^3$.

(*k*) $F_x = 0$, $F_y = mv$; when $t = 0$, $x = 0$, $y = 1$, $v_x = 1$, $v_y = 0$.

Ans. $x = t$, $y = \frac{e^t + e^{-t}}{2}$. *Catenary.*

(*l*) $F_x = ma$, $F_y = mb$, $F_z = mc$; when $t = 0$, $x = 0$, $y = 1$, $z = 2$, $v_x = 0$, $v_y = 0$, $v_z = 1$.

Ans. $x = \frac{1}{2}at^2$, $y = \frac{1}{2}bt^2 + 1$, $z = \frac{1}{2}ct^2 + t + 2$.

Parabola in plane $ay = bx + a$.

(*m*) $F_x = 0$, $F_y = 0$, $F_z = 0$. *Ans. Straight line.*

(*n*) $F_x = -mx$, $F_y = -my$, $F_z = 0$.

Ans. Helix if initial position is $(a, 0, 0)$ and velocity $(0, b, c)$.

2. Show that the path is necessarily a straight line or a parabola if the force is constant.

3. A particle is acted upon by two forces: (1) one parallel to the X-axis and equal to $m(t - 1)$; (2) one in a direction making an angle of 30° with the X-axis and equal to $m \sin t$. When $t = 0$, $x = 0$, $y = a$, $v_x = b$, $v_y = 0$. Find the equations of motion.

4. A particle is acted upon by two forces: (1) one parallel to the X-axis and equal to $-mx$; (2) one in a direction making an angle of 135° with the X-axis and equal to mk. When $t = 0$, $x = a$, $y = 0$, $v_x = 0$, $v_y = 0$. Find the equations of motion.

5. A particle is acted upon by three forces: (1) one parallel to the Y-axis and equal to $-mk^2y$; (2) one parallel to the X-axis and equal to $m(t^2 - t)$; (3) one in a direction making an angle of 210° with the X-axis and equal to mk^2. When $t = 0$, $x = 0$, $y = 0$, $v_x = 0$, $v_y = 0$. Find the equations of motion.

6. A particle of mass 12 lb. is moving in a northeast direction with a speed of 6 ft. per second. It is acted upon by two forces, one of 48 poundals towards the north, the other of 72 poundals towards the east. Find its position after the lapse of one second.

7. A particle of mass 10 lb. is moving towards the north with a speed of 20 ft. per second. It is acted upon by three constant forces: (1) 10 poundals towards the northeast, (2) 20 poundals towards the east, (3) 15 poundals towards the south. Find its position and the components of its velocity after the lapse of 3 sec.

8. A particle of mass m free to move in the XY-plane is subject to a force whose axial components are $F_x = -16\,mx$, $F_y = -4\,my$. The initial conditions are $x = 1$, $v_x = 0$, $y = 0$, $v_y = 2$, when $t = 0$. Find the equations of motion and the equation of the path. Discuss the motion.

9. The axial components of the force causing a plane curvilinear motion are $F_x = -mx$, $F_y = -4\,my$. The initial conditions are $x = 0$, $y = 1$, $v_x = 1$, $v_y = 0$, when $t = 0$. Derive the equations of motion, discuss them, and draw the path.

10. A particle of mass m moves in the XY-plane under the action of a force whose axial components are $F_x = -mx$, $F_y = -my$. The initial conditions are $x = 2a$, $y = 0$, $v_x = 0$, $v_y = a$, when $t = 0$. Derive the equations of motion. Discuss the motion.

11. Discuss the motion of the particle in problem 10 if the initial conditions are $x = 0$, $v_x = a$, $y = a$, $v_y = 0$, when $t = 0$.

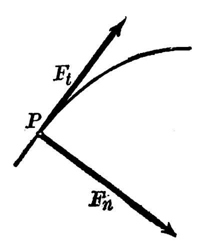
12. A particle of unit mass moves in the XY-plane under the action of a force which is directed always towards the origin, and its magnitude is proportional to the distance of the particle from the origin. (*a*) Denoting the magnitude of the force when the particle is at unit distance by k^2, find and discuss the equations of motion if the initial conditions are $x = a$, $y = 0$, $v_x = 0$, $v_y = kb$, when $t = 0$. (*b*) Prove that for any initial conditions the path is an ellipse with center at the origin. (*c*) Under what initial conditions can the ellipse degenerate into a straight line?

13. A particle of unit mass moves in the XY-plane under the action of a repulsive force from the origin. The magnitude of the force is proportional to the distance from the origin. (*a*) Find and discuss the equations of motion if the initial conditions are $x = a$, $y = 0$, $v_x = 0$, $v_y = b$, when $t = 0$. (*b*) Prove that, for any initial conditions, the path is an hyperbola with center at the origin. (*c*) Under what initial conditions can the hyperbola degenerate into a straight line?

50. Intrinsic force equations. In Art. 40, the tangential and normal components of the acceleration were found. If, then, the resultant force F, producing a motion in the plane, is resolved at any instant along tangent and normal, the corresponding components being F_t and F_n, we shall have

$$\text{(IV)} \qquad F_t = mf_t = m\frac{dv}{dt} = mv\frac{dv}{ds}, \quad F_n = mf_n = \frac{mv^2}{R}.$$

These equations, being entirely independent of coördinates, are called the *intrinsic force equations.*



The component F_n is directed always towards the center of curvature. Motion along any plane curve may therefore be said to be caused by the simultaneous action of a tangential and a normal force. The change of motion involved, that is the change in the vector velocity, may be explained thus. The resultant tangential force F_t changes only the speed, that is the *magnitude* of the vector velocity. The resultant normal force causes change in *direction* of the velocity. If the speed is constant, the resultant tangential force is zero. If the path is rectilinear, the resultant normal force is zero.

51. Polar equations. In many problems the use of polar coordinates is advantageous. If all forces causing a motion are resolved along and perpendicular to the radius vector drawn to any point of the path, we shall have, by Art. 41,

$$\text{(1)}\qquad \begin{cases} F_\rho = mf_\rho = m\left(\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2\right), \\ F_\theta \ = mf_\theta = \dfrac{m}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\theta}{dt}\right). \end{cases}$$

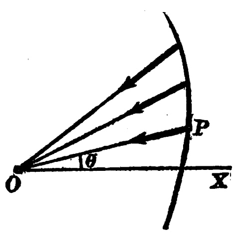
When F_ρ and F_θ are given, the equations (1) become simultaneous differential equations of the second order, from the integration of which ρ and θ are to be determined as functions of t.

By introducing into (1) the radial velocity $\left(v_\rho = \dfrac{d\rho}{dt}\right)$ and the angular velocity $\left(\omega = \dfrac{d\theta}{dt}\right)$ about the origin, we obtain

$$\text{(V)}\qquad \begin{cases} F_\rho = m\left(\dfrac{dv_\rho}{dt} - \rho\omega^2\right); \\ F_\theta = \dfrac{m}{\rho}\dfrac{d}{dt}(\rho^2\omega). \end{cases}$$

ILLUSTRATIVE EXAMPLE. A particle moves under the action of a force *directed always along the radius vector*, the magnitude of the force being inversely proportional to the cube of the distance. Discuss the motion.

Solution. Since the force is radial, we have



$$\text{(1)}\qquad F_\rho = \frac{m\mu}{\rho^3},^* \quad F_\theta = 0.$$

Hence from the force equations (V),

$$\text{(2)}\qquad \frac{dv_\rho}{dt} - \rho\omega^2 = \frac{\mu}{\rho^3}, \quad \frac{d}{dt}(\rho^2\omega) = 0.$$

The second of these gives, by integration,

$$\text{(3)}\qquad \rho^2\omega = \text{constant, or } \omega = \frac{c}{\rho^2}.$$

Substituting in the first, we get after reducing, since $v_\rho = \dfrac{d\rho}{dt}$,

$$\text{(4)}\qquad \frac{d^2\rho}{dt^2} = -\frac{k}{\rho^3}, \text{ where } k = -\mu - c^2.$$

Multiplying both members by $2\dfrac{d\rho}{dt}dt$, and integrating,

$$\text{(5)}\qquad \left(\frac{d\rho}{dt}\right)^2 = c_1 + \frac{k}{\rho^2}.$$

To obtain a simple solution,† assume as one initial condition, when $\rho = a$, $v_\rho = \dfrac{\sqrt{k}}{a}$. Then $c_1 = 0$, and (5) becomes

* When $\rho = 1$ and $m = 1$, the force $= \mu$. Hence μ is the magnitude of the force upon unit mass at unit distance. It is convenient to call μ the *absolute intensity* of the force.

† For the general case see problem 5 below.

$$(6) \qquad \frac{d\rho}{dt} = \frac{\sqrt{k}}{\rho}, \text{ or } dt = \frac{\rho d\rho}{\sqrt{k}}.$$

Evidently k must be positive, that is, since $k = -\mu - c^2$, μ must be negative, and from (2), the force is now an *attraction.*

The equation of the path is found directly from (3) and (6). For,

$$(7) \qquad \omega = \frac{d\theta}{dt} = \frac{c}{\rho^2}, \quad \therefore d\theta = \frac{c}{\rho^2}dt = \frac{c}{\sqrt{k}}\frac{d\rho}{\rho}.$$

Integrating, we have,

$$\theta = \frac{c}{\sqrt{k}}\log\rho + c'_2,$$

or also

$$\frac{\sqrt{k}}{c}\theta = \log c_2\rho, \text{ or } \rho = be^{\frac{\sqrt{k}}{c}\theta},$$

by changing the form of the constant of integration. The path is therefore a logarithmic spiral.

The assumed initial radial velocity $\left(v_\rho = \frac{\sqrt{k}}{a}\right)$ may be interpreted thus. Integrating (4) with the assumption that $v_\rho = 0$ when $\rho = \infty$, we find $c_1 = 0$, and $\therefore v_\rho{}^2 = \frac{k}{\rho^2}$. If $\rho = a$, this gives $v_\rho = \frac{\sqrt{k}}{a}$, the value assumed above. That is, the initial radial velocity is assumed to be that acquired in moving in from infinity. To find the resultant initial velocity, we have from (7) for initial angular velocity, $\omega = \frac{c}{a^2}$. $\therefore v_\theta = \rho\omega = a\omega = \frac{c}{a}$, and $v_0{}^2 = v_\rho{}^2 + v_\theta{}^2 = \frac{k}{a^2} + \frac{c^2}{a^2} = -\frac{\mu}{a^2}$, since $k = -\mu - c^2$. Hence $v_0 = \frac{\sqrt{-\mu}}{a}$, the initial velocity.

The discussion leads to the

THEOREM. *Given an attractive central force varying inversely as the cube of the distance. Under such a force a particle will describe a logarithmic spiral if projected in any direction with a velocity equal to* $\frac{\sqrt{\mu}}{a}$, *where a is the distance from the center and μ the absolute intensity of the force.*

PROBLEMS

1. Show that a particle can move freely in a conic section with focus at the origin when acted upon by an attractive center of force varying as the inverse square of the distance.

2. Show that a particle can move freely in a conic section with center at the origin when acted upon by a center of force varying directly as the distance.

3. Show that a particle can move freely in a circle when acted upon by a constant attractive central force.

4. Show that a particle can move freely in a circle under a center of force directed towards a point on the circumference. Show that the law of the force is the inverse fifth power of the distance.

5. Find the general path under a central force varying inversely as the cube of the distance.

Hint. Integrate equation (5) of the Illustrative Example, p. 106, without specializing c_1.

Ans. $k < 0,\ \dfrac{1}{\rho} = a\cos\left(\dfrac{\sqrt{k}}{c}\theta + \beta\right);\ k = 0,\ \rho\theta = a;$

$$k > 0,\ \frac{1}{\rho} = c_1 e^{\frac{\sqrt{k}}{c}\theta} + c_2 e^{-\frac{\sqrt{k}}{c}\theta}.$$

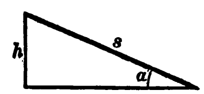
6. A particle is acted upon by two centers of force, one attractive and the other repulsive. Each force varies directly as the distance, and the two have the same absolute intensity. Show that the path is a parabola.

7. A particle is acted upon by a central force proportional to the distance and a constant force. Show that the path is a conic section.

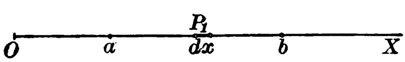
CHAPTER V

WORK, ENERGY, IMPULSE

52. Work. A force is said to do work when its point of application undergoes a displacement. The amount of work done by a constant force is equal to the product of the force and the displacement *in the direction of the force.* For example, consider the force of gravity. If a particle of mass m drops vertically through a distance of h units, the work done by the force of gravity is mgh units. If the particle rises vertically a distance of h units, the work done by gravity is $-mgh$ units.* If the particle slides a distance of s units down a smooth plane whose inclination is α, the work done by the force of gravity is $mgs \sin \alpha$ units, or mgh units, where $h = s \sin \alpha$ is the vertical distance moved through, that is, h is the distance in the direction of the force. If $\alpha = 0$, that is, if the particle slides along a horizontal plane, the work done by the force of gravity is zero.



Suppose the particle moves along the straight line OX under the action of a force directed always along the line, but whose magnitude is variable. The law of the variation of the force being known, its value is known for any position of the particle on the line. The small amount of work dW,† done by the force F, while the particle moves a small distance dx from the point P_1, is approximately equal to $F_1 dx$, where F_1 is the value of the force at the point $P_1(x = x_1)$. Applying the principles of the Calculus, Chapter XXX, it is evident that the total work done by the force F while the particle moves from the point $x = a$ to the point $x = b$, is the definite integral of F with respect to x from a to b; that is,

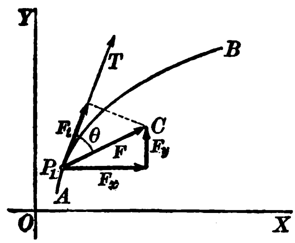


$$(1) \qquad W = \int_a^b F dx.$$

* When the work done by a force is negative, it is sometimes said that the particle does work against the force.

† dW is called the element of work; dx, the element of distance.

Suppose the particle moves in a plane under the action of a resultant force which is variable in magnitude and direction. Let P_1 represent any position of the particle on the path AB. Let P_1C represent the resultant force in magnitude and direction. The element of work dW corresponding to an element of arc ds is equal approximately to the value of the force F at P_1 multiplied by $ds \cos \theta$, where θ is the angle between the direction of the force and the tangent to the curve; that is,



$$(2) \qquad dW = F \cos \theta ds.$$

But $F \cos \theta = F_t$ where F_t is the tangential component of the force F.

The total work done by the force F while the particle moves from an initial position s_0 on the path to any other position s is obtained by integration.

$$(3) \qquad W = \int_0^s F_t \, ds.$$

But by (II), Art. 32, and (4), Art. 40, we have

$$F_t = F_x \frac{dx}{ds} + F_y \frac{dy}{ds}.$$

By substitution (3) becomes the *work integral:*

$$(\mathrm{I}) \qquad W = \int_{x_0, y_0}^{x, y} (F_x dx + F_y dy).$$

An important consequence of formula (I) is the application to plane motion under a constant force. Let the direction of the Y-axis agree with the direction of the force. Then $F_x = 0$, $F_y = F$, and formula (I) becomes

$$W = \int_{y_0}^{y} F dy = F(y - y_0).$$

Hence in any plane motion the work done by a constant force is equal to the product of the force and the distance moved through in the direction of the force. *The work is independent of the path.* For example, if a particle moves on any plane curve the work done by the force of gravity is equal to mgh where h represents the vertical distance moved through.

Dimensions. Since work has been defined as the product of force by distance, the relation between the derived unit of work and the fundamental units of mass, distance, and time is given by the dimensional equation:

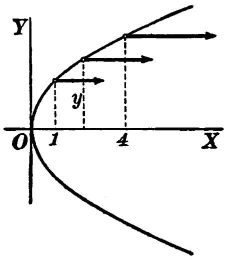
$$\text{Work} = \frac{\text{mass} \times \text{length}^2}{\text{time}^2}.$$

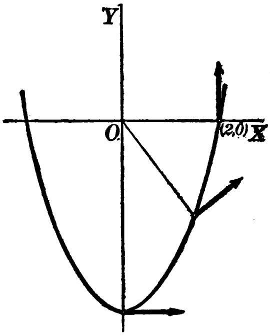
ILLUSTRATIVE EXAMPLES

1. A unit particle describes the parabola $y^2 = 4x$ from $x = 1$ to $x = 4$. Find the work done by the force whose axial components are $F_x = 2my$, $F_y = 0$.

Solution. Since $m = 1$, (I) becomes

$$W = \int_{x=1}^{x=4} 2y\,dx = 4\int_1^4 \sqrt{x}\,dx = \tfrac{8}{3}\left[x^{\frac{3}{2}}\right]_1^4 = \tfrac{56}{3} \text{ units.}$$





2. Find the work done by a force whose axial components are

(1) $$F_x = -mky,\ F_y = mkx,$$

upon a particle moving along the parabola

(2) $$x^2 = 4 + y,$$

from $x = 0$ to $x = 2$.

Solution. The work integral (I) may be obtained as a function of x in the following manner. From (2), $y = x^2 - 4$, $\therefore\ dy = 2x\,dx$, and (I) becomes

$$W = \int_0^2 (-mk(x^2-4)\,dx + mkx \cdot 2x\,dx) = mk\int_0^2 (x^2+4)\,dx = 10\tfrac{2}{3}\,mk.$$

PROBLEMS

1. Show that in polar coördinates work is given by the formula

$$W = \int_{\rho_0,\,\theta_0}^{\rho,\,\theta} (F_\rho d\rho + F_\theta \rho\, d\theta).$$

2. A body whose mass is m falls vertically to the earth's surface from a height equal to the radius R. Compute the work done by the earth's attraction during the fall. *Ans.* $\frac{1}{2}\,mgR$ units.

3. A body is suspended by an elastic string whose unstretched length is 4 ft. Under a pull * of 100 poundals the string stretches to a length of 5 ft. Required the work done on the body by the tension of the string while its length changes from 6 ft. to 4 ft. *Ans.* 200 units.

4. A unit particle describes the circle $x^2 + y^2 = a^2$, from $(0, a)$ to $(a, 0)$. Find the work done by the force whose axial components are $F_x = ky^2$, $F_y = kxy$.

Ans. $\frac{k}{3} a^3$ units.

5. A particle describes the circle $x^2 + y^2 = a^2$ from the point $(0, a)$ to the point $(a, 0)$. Prove that the work done by the force whose axial components are $F_x = mxy$, $F_y = my^2$ is zero.

6. Calculate the work in problem 5 if the particle moves from the point $(0, -a)$ to the point $(a, 0)$.

7. A particle describes the circle $x^2 + y^2 = a^2$ from $(0, a)$ to $(a, 0)$. Find the work done by the force whose axial components are $F_x = my$, $F_y = my^2$.

Ans. $ma^2\left(\frac{\pi}{4} - \frac{a}{3}\right)$.

8. A unit particle describes the curve $x = e^y - e^{-y}$ from $x = 0$ to $x = 2$. Find the work done by the force whose axial components are $F_x = mx$, $F_y = 0$.

Ans. 2 units.

9. A particle of mass m describes the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the point $(a, 0)$ to the point (x_0, y_0). Find the work done by the force whose axial components are $F_x = mx$, $F_y = my$. *Ans.* $\frac{m}{2}(x_0^2 + y_0^2 - a^2)$ units.

10. The equations of motion of a unit particle are $x = t$, $y = e^t$. Compute the work done by the resultant force during the time from $t = 0$ to $t = 3$.

Ans. $\frac{1}{2}(e^6 - 1)$ units.

11. The equations of motion of a particle of mass m are $x = \log(t + 1)$, $y = t$. Compute the work done by the resultant force during the time from $t = 0$ to $t = 3$.

Ans. $\frac{15}{32} m$ units.

12. A particle describes the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ from $(0, a)$ to $(a, 0)$. Find the work done by the force whose axial components are $F_x = kmy$, $F_y = kmxy$.

Ans. $\frac{kma^2}{30}(5 - a)$ units.

13. A unit particle describes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ under the action of a force whose axial components are $F_x = -x$, $F_y = -y$. Compute the work done while the particle moves from $(0, b)$ to $(a, 0)$.

14. Compute the work done when a particle describes one half of the ellipse $\rho = \frac{ep}{1 - e\cos\theta}$, under the action of a force directed always towards the origin and varying inversely as the square of the distance.

* The law of force for an elastic string is HOOKE'S LAW. *The tension of an elastic string is proportional to the extension.*

53. Kinetic energy. A particle is said to possess *energy* when its condition is such that it can do work against a force which may be applied to it. If a particle, acted upon by no forces, has a velocity v, it will continue to move uniformly in a straight line. Suppose a force, acting upon the particle, brings it to rest after moving through a certain distance. The force does work upon the particle, and the particle is said to do work *against the force*. A particle in motion, therefore, possesses energy called *kinetic energy*. The kinetic energy of a particle of mass m, moving with velocity v, is defined as *one half the product of the mass by the square of the speed*. That is,

$$\text{Kinetic energy} = \tfrac{1}{2}mv^2.$$

Dimensions. From the preceding definition it follows that the derived unit of kinetic energy is expressed in terms of the fundamental units of mass, length, and time by the dimensional equation

$$\text{Kinetic energy} = \frac{\text{mass} \times \text{length}^2}{\text{time}^2}.$$

Comparison with Art. 52 shows that the dimensions of kinetic energy are the same as the dimensions of work.

If a particle under the action of a resultant force F moves along the X-axis from the initial position x_0 to any other position x, the work done is given by

$$(1) \qquad W = \int_{x_0}^{x} F dx.$$

Now $F = mv\dfrac{dv}{dx}$, and by substitution, (1) becomes

$$(2) \qquad W = \int_{x_0}^{x} mvdv.$$

If v_0 is the velocity at the point x_0 and v the velocity at the point x, we obtain by integration of (2),

$$(3) \qquad W = \frac{m}{2}(v^2 - v_0^2) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2.$$

The work is therefore expressed in terms of the kinetic energy at the final and initial positions of the particle. If the initial velocity is zero, (3) becomes

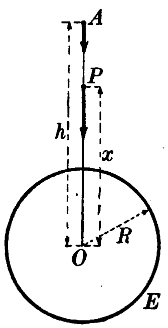
$$W = \tfrac{1}{2}mv^2,$$

and we have proved for rectilinear motion the

THEOREM. *The work done by the resultant force while a particle starting from rest acquires the velocity v is equal to the kinetic energy of the particle.*

If the velocity of a particle changes from the initial value v_0 to the final value v, equation (3) gives for rectilinear motion the

THEOREM. *In the displacement of a particle the work done by the resultant force is equal to the difference of the final kinetic energy and the initial kinetic energy.*



Comparison of equations (1) and (3) gives the *energy equation*

$$\text{(II)} \qquad \frac{m}{2}(v^2 - v_0^2) = \int_{x_0}^{x} F dx,$$

for rectilinear motion. When the force F is a given function of the distance, integration of (II) leads to an expression for the speed as a function of the distance.

ILLUSTRATIVE EXAMPLE. Find the velocity of a body falling from a distance h from the center of the earth.

Solution. The earth's attraction varies inversely as the square of the distance from the center. Hence at P where $OP = x$, we have $F_x = -\frac{mk^2}{x^2}$, where k^2 is the absolute intensity. Using the energy equation, the result is

$$\tfrac{1}{2} mv^2 = \int_h^x F_x dx = \int_h^x -\frac{mk^2}{x^2} dx = mk^2\left(\frac{1}{x} - \frac{1}{h}\right),$$

since $x = OA = h,\ v_0 = 0,$ when $t = 0.$

$$\therefore\ v^2 = 2\,k^2\left(\frac{1}{x} - \frac{1}{h}\right).$$

But $F_x = -mg$ when $x =$ radius of earth $= R.$

$$\therefore\ -\frac{mk^2}{R^2} = -mg, \text{ and } k^2 = R^2 g.$$

Hence

$$(1) \qquad v^2 = 2\,gR^2\left(\frac{1}{x} - \frac{1}{h}\right).$$

When the body reaches the surface, $x = R$, and (1) becomes

$$v^2 = 2\,gR - \frac{2\,gR^2}{h}.$$

If the particle falls from an infinite distance ($h = \infty$), the velocity upon reaching the surface of the earth is $\sqrt{2\,gR}$. Expressing R and g in miles, we have $R = 4000$, $g = \frac{32}{5280}$, and the velocity from infinity is approximately 7 miles per second.

If the particle moves along a plane curve from the initial position s_0 to any other position s, where s denotes the length of arc measured from a fixed point on the curve, the work done upon it by a force F is given by (3), Art. 52:

$$W = \int_{s_0}^{s} F_t \, ds. \tag{4}$$

Now, by (IV), Art. 50, $F_t = mv\frac{dv}{ds}$, and by substitution, (4) becomes

$$W = \int_{s_0}^{s} mvdv. \tag{5}$$

If v_0 is the velocity at the point s_0, and v the velocity at the point s, we obtain by integration of (5)

$$W = \frac{m}{2}(v^2 - v_0^2) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2. \tag{6}$$

Equation (6) shows that the preceding theorems, p. 114, hold also for plane curvilinear motion.

Comparison of (6) and (I) gives the *energy equation* for curvilinear motion:

$$\frac{m}{2}\left(v^2 - v_0^2\right) = \int_{x_0, y_0}^{x, y}(F_x dx + F_y dy). \tag{III}$$

When the components F_x and F_y are given functions of the position (x, y), integration of (III) leads to an expression for the speed as a function of the position.

Illustrative Example. A bullet with an initial velocity of 1500 ft. per second, strikes a target at 1200 yd. distance with a velocity of 900 ft. per second. Supposing the range of the bullet is horizontal, compare the mean resistance of the air with the weight of the bullet.

Solution. Denoting the constant resistance of the air by $-F$, we find that the work done by this force is $-3600\,F$. Hence the work equation gives

$$-3600\,F = \frac{m}{2}\left(v^2 - v_0^2\right) = \frac{m}{2}\left(900^2 - 1500^2\right),$$

whence

$$F = 200\,m.$$

Since

$$\text{the weight} = 32\,m,$$

we have

$$\frac{\text{resistance}}{\text{weight}} = \frac{25}{4}.$$

PROBLEMS

1. A particle is projected in any direction with the velocity v_0 and then falls freely under the action of gravity. Find the energy equation.

Ans. $v^2 - v_0^2 = 2\,gh$, where h = vertical distance.

2. A particle moves in a circle of radius a under a constant tangential force mf. Find the energy equation.

Ans. $a\theta f = \dfrac{v^2 - v_0^2}{2}$ (θ = angle moved through).

3. Find the energy equation for a force parallel to one axis and proportional to the distance from the other. $v^2 - v_0^2 = k(x^2 - x_0^2)$

4. With what velocity must a particle be projected from the surface of the earth in order that it may never return, no force except the earth's attraction being supposed to act?

5. A bullet moving with the speed of 1000 ft. per second has its speed reduced by 100 ft. per second in passing through a plank. How many such planks would the bullet penetrate?

Ans. $5\frac{5}{19}$.

6. A bullet fired with a velocity of 1000 ft. per second penetrates a block of wood to a depth of 12 in. Assuming the resistance of the wood to be constant, prove that if fired through a board 2 in. thick, its velocity would be reduced by about 87 ft. per second.

7. A laborer has to send bricks to a bricklayer at a height of 10 ft. He throws them up so that they reach the bricklayer with a velocity of 10 ft. per second. What proportion of his work could he save if he threw them so as only just to reach the bricklayer?

8. A particle moves under a central attraction proportional to the distance. Find the energy equation.

Ans. $k(\rho^2 - \rho_0^2) = v^2 - v_0^2$.

9. A particle moves under a central attraction inversely proportional to the square of the distance. Find the energy equation.

Ans. $\dfrac{2\,k}{\rho} - \dfrac{2\,k}{\rho_0} = v^2 - v_0^2$.

54. Constrained motion. Dynamic pressure. The motion of a particle is said to be constrained when it is confined to a certain curve or surface; for example, a bead sliding on a wire, or swinging on a string, or moving on an inclined plane. In constrained motion, the forces acting upon a particle may be divided into two classes:* (1) the impressed forces; and (2) the pressure of the constraint.

Two cases are to be distinguished. (1) On a *smooth* curve the tangential component of the force of constraint is zero, that

* The difference between the impressed forces and the force of constraint is that the former are given directly, while the latter is not given directly, but its effect upon the motion is specified.

is, there is no friction. (2) On a *rough* curve the tangential component of the force of constraint is called the sliding friction. We suppose, for the present, that the curve is smooth. The case of a rough inclined plane is treated later, Art. 67.

Making use of the intrinsic force equations, Art. 50, we have

$$\text{(IV)} \qquad \frac{mv^2}{R} = F_n = \left\{\begin{matrix}\textbf{normal}\\ \textbf{impressed}\\ \textbf{force}\end{matrix}\right\} + \left\{\begin{matrix}\textbf{normal}\\ \textbf{pressure}\end{matrix}\right\}.$$

The impressed forces being known, their normal components are known ; R, the radius of curvature, may be calculated from the equation of the path, and v^2 may be found from the energy equation. Hence formula (IV) determines the normal pressure.

The normal pressure is exerted by the path upon the particle. In many practical problems it is important to know the normal pressure exerted by the particle upon the path. This is given by

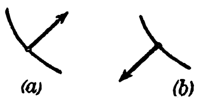
NEWTON'S THIRD LAW OF MOTION. To every action there is a corresponding reaction, equal in magnitude but opposite in direction.

The curve exerts a normal pressure on the particle. Hence the particle exerts a pressure equal in magnitude but opposite in direction on the path. In the case of a bead sliding on a surface, this is called the *dynamic pressure.* In the case of a particle swinging on a string, it is the tension in the string. From (IV) we obtain

$$\text{(V)} \qquad \left\{\begin{matrix}\textbf{dynamic}\\ \textbf{pressure}\\ \textbf{on path}\end{matrix}\right\} = \left\{\begin{matrix}\textbf{normal}\\ \textbf{impressed}\\ \textbf{force}\end{matrix}\right\} - \frac{mv^2}{R}.$$

The motion is said to be free when the dynamic pressure is zero.

It must be remembered that normal forces are resolved along the *directed normal.* The resultant force, $\frac{mv^2}{R}$, acts always towards the center of curvature or inwards. If the normal impressed force acts inwards also, as in (a), then the dynamic pressure equals numerically the *difference* of the other two forces. On the other hand, if the normal impressed force acts outwards, as in (b), then the dynamic pressure equals numerically the sum of the other forces.


(a) (b)

When $v = 0$, that is, when the particle is at rest, the pressure on the path equals the normal impressed force. For this reason, in this case, the latter is also called the *static pressure*. The term $-\frac{mv^2}{R}$, which gives the *change* in pressure due to the motion, is commonly called *centrifugal force*. From (IV) this force is equal and opposite to the resultant normal force and acts always *outwards*.*

In terms of static pressure and centrifugal force, equation (V) may be written

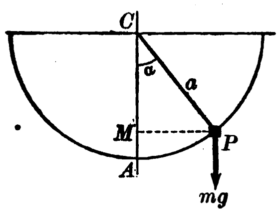
$$\left\{\begin{matrix}\textbf{dynamic}\\ \textbf{pressure}\end{matrix}\right\} = \left\{\begin{matrix}\textbf{static}\\ \textbf{pressure}\end{matrix}\right\} + \left\{\begin{matrix}\textbf{centrifugal}\\ \textbf{force}\end{matrix}\right\}.$$

Since the centrifugal force acts always outwards, it follows that the dynamic pressure is numerically equal to the sum or difference of the centrifugal force and static pressure according as the latter is outwards or inwards.

ILLUSTRATIVE EXAMPLES

1. A heavy particle is constrained to move in a smooth fixed semicircle whose plane is vertical. Find the pressure at the lowest point.

Solution. The impressed force is weight. If the particle falls from P to A, the work done is $mg \cdot AM = mg\,(AC - MC) = mga\,(1 - \cos\alpha)$, if a is the radius. Hence, using the energy equation, and assuming the particle to start from rest at P, we have



(1) $\frac{1}{2}mv^2 = mga(1-\cos\alpha)$, or $v^2 = 2\,ag(1-\cos\alpha)$.

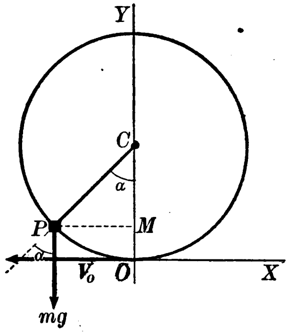
The normal impressed force acting outwards, we obtain for the pressure at the lowest point,

$$mg + \frac{mv^2}{a} = mg + 2\,mg(1-\cos\alpha) = mg(3 - 2\cos\alpha).$$

If the particle starts at the highest point of the semicircle, $\alpha = \frac{\pi}{2}$, then the pressure equals $3\,mg$. That is, *the pressure at the lowest point is trebled by the motion.* This increase of the static pressure by motion is a matter of importance. For example, in a scenic railway the structure must withstand not only the weight of a car and its occupants, but also the added pressure due to motion. This added pressure equals the centrifugal force.

* It must be clearly understood that the centrifugal force is not an actual force acting on the particle. It is the reaction of the particle against the normal component of the resultant force. By the first law of motion the particle tends to move in a straight line. If it moves in a curved path, centrifugal force is a convenient term to designate the *magnitude* of the normal force which must act on a particle of mass m and velocity v, in order to produce the curvature $\frac{1}{R}$ in the path.

2. A heavy particle is suspended from a fixed point by a string of length a. A horizontal velocity v_0 is suddenly imparted to the particle so that it begins to describe a vertical circle. Determine whether the particle will oscillate or the string become slack.



Solution. The work done by the impressed force weight when the particle moves from O to P is negative and equal to $-mgy$, if $OM = y$. Here the energy equation gives

$$(2) \qquad \begin{cases} \frac{1}{2} mv^2 - \frac{1}{2} mv_0^2 = -mgy, \text{ or} \\ v^2 = v_0^2 - 2gy. \end{cases}$$

The normal impressed force is the component of the weight along the radius and equals

$$mg \cos \alpha = mg \frac{MC}{CP} = \frac{a-y}{a} mg.$$

Hence from (V), since the dynamic pressure is in this case the tension, we get

$$\text{Tension} = mg\frac{a-y}{a} + \frac{mv^2}{a} = mg\left(1 - \frac{y}{a}\right) + m\left(\frac{v_0^2 - 2gy}{a}\right)$$

$$(3) \qquad = mg + \frac{mv_0^2}{a} - \frac{3mgy}{a}.$$

If the particle oscillates, the velocity at the highest point must vanish. From (2), if $v_0^2 - 2gy = 0$, we obtain $y = \frac{v_0^2}{2g}$ as the corresponding height. Since this height must be less than a radius, a *necessary condition* for oscillation is $v_0^2 < 2ag$.

If the string becomes slack, the tension must vanish. From (3), if

$$mg + \frac{mv_0^2}{a} - \frac{3mgy}{a} = 0,$$

we have $y = \frac{v_0^2 + ag}{3g}$ as the corresponding height. This must be less than $2a$, and from $\frac{v_0^2 + ag}{3g} < 2a$, we get $v_0^2 < 5ag$ as a *necessary condition* for the tension vanishing.

Subtracting the two heights found, we obtain

$$(4) \qquad \frac{v_0^2 + ag}{3g} - \frac{v_0^2}{2g} = \frac{2ag - v_0^2}{6g}.$$

Comparison of the inequalities and interpretation of (4) gives the criterion: If $v_0^2 > 5ag$, the particle describes the whole circle. If $v_0^2 < 5ag$ and $> 2ag$, the tension vanishes, the string becomes slack and the particle will leave the circle and fall freely. If $v_0^2 < 2ag$, the equation (4) shows that the velocity vanishes *before* the tension, hence the particle will oscillate.

3. A particle is attached to the end of a fine thread which just winds around the circumference of a circle of radius a at whose center there is a repulsive force varying as the distance and of absolute intensity μ. Find the time of unwinding and the tension at any time.

Solution. The path is an involute of the given circle. Since the impressed force acts along the radius vector OP, we have, using polar coördinates,

$$F_\rho = m\mu\rho, \quad F_\theta = 0.$$

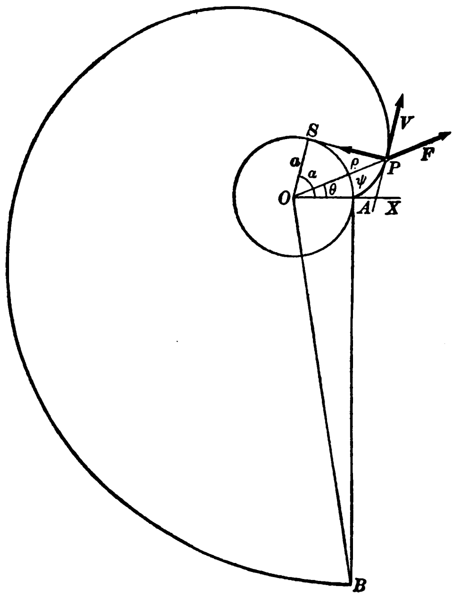
Hence, by the energy equation, we have

$$\tfrac{1}{2} mv^2 = \int_a^\rho m\mu\rho d\rho = \tfrac{1}{2} m\mu (\rho^2 - a^2),$$

since the particle is at rest at A where $\rho = a$.

(1) $$\therefore\ v^2 = \mu (\rho^2 - a^2).$$

Now introduce properties of the involute. PS is the normal at P, $\therefore\ \psi =$ angle



SOP. Also $PS =$ arc SA. Hence $\cos\psi = \cos SOP = \dfrac{OS}{OP} = \dfrac{a}{\rho}$. $PS = a\alpha$. Since also $\overline{OP}^2 = \overline{OS}^2 + \overline{SP}^2$, we have

$$\rho^2 = a^2 + a^2\alpha^2, \text{ or } \rho^2 - a^2 = a^2\alpha^2.$$

Hence (1) becomes

(2) $$v^2 = \mu (\rho^2 - a^2) = \mu a^2\alpha^2.$$

Now $v_\rho = v \cos\psi = v\dfrac{a}{\rho}$. $\therefore$ using (2),

(3) $$v_\rho = \frac{d\rho}{dt} = v\frac{a}{\rho} = \sqrt{\mu}\frac{a\sqrt{\rho^2 - a^2}}{\rho}, \text{ or } \frac{\rho d\rho}{\sqrt{\rho^2 - a^2}} = \sqrt{\mu}\, a dt.$$

Integrating, remembering that $\rho = a$ when $t = 0$, we obtain

(4) $$\sqrt{\rho^2 - a^2} = \sqrt{\mu}\, at, \text{ or also, } \alpha = \sqrt{\mu}\, t, \text{ [by (2)]}.$$

When the string is unwound, $\alpha = 2\pi$, and hence $t = \dfrac{2\pi}{\sqrt{\mu}}$.

The pressure on the path is the tension. In the figure,

$$F_n = F \sin \psi = F \sin POS = F \cdot \frac{PS}{OP} = m\mu\rho \frac{a\alpha}{\rho} = m\mu a\alpha.$$

Also $\dfrac{mv^2}{R} = \dfrac{m\mu a^2\alpha^2}{SP} = m\mu a\alpha$. Since F_n acts outwards, the tension is the sum of F_n and the centrifugal force; that is, tension $= 2\, m\mu a\alpha = 2\, m\mu^{\frac{3}{2}} at$. Hence the tension constantly increases and reaches the maximum value $4\, m\mu a\pi$.

PROBLEMS

1. A heavy particle slides from rest down a smooth curve. If h is the vertical height fallen through, prove $v^2 = 2\, gh$.

2. A heavy particle falls down an inclined plane whose inclination to the horizon is α. Show that the dynamic pressure is constant and equal to $mg \cos \alpha$.

3. A particle slides down a smooth plane whose inclination to the horizon is 30°. What is the velocity when it has traversed a distance of 20 ft.?

Ans. 25.3 ft. per second.

4. A heavy particle slides on the exterior of a vertical circle. If the particle is *just started* at the highest point, show that it will leave the circle and fall freely after sliding through a vertical height equal to one third of the radius.

5. A heavy particle is constrained to move in a circle under a repulsive center of force lying on the circumference and varying as the distance. The particle just starts from rest at the center of force. Find the pressure.

Ans. $\dfrac{3\, m\, \mu\rho^2}{2\, a}$, where ρ = distance from center of force and a = radius.

6. A heavy bead is constrained to slide on a smooth wire of the shape given by one of the following equations (assuming the y-axis vertically upwards). It starts at the point indicated. Find the pressure at the end point given.

(*a*) $y^2 = 4\, x$,	start	$(4, 4)$,	end	$(0, 0)$.
(*b*) $4\, x^2 + y^2 = 16$,	"	$(0, 4)$,	"	$(2, 0)$.
(*c*) $y = \cos x$,	"	$(0, 1)$,	"	$\left(\frac{\pi}{2}, 0\right)$.
(*d*) $x = a \text{ arc vers } \frac{y}{a} - \sqrt{2\, ay - y^2}$,	"	$(\pi, 2\, a)$,	"	$(2\, \pi, 0)$.
(*e*) $y = x^3$,	"	$(2, 8)$,	"	$(0, 0)$.
(*f*) $y = -x^2$,	"	$(0, 0)$,	"	$(1, -1)$.

7. If, in problem 6, the bead slides on the exterior of the given curve, find where it will leave the curve.

8. A heavy bead is constrained to move on a smooth vertical circular wire. Show that the bead will describe the whole circle if projected from the lowest point with a velocity greater than $2\sqrt{ag}$, where a is the radius of the wire.

9. A particle is attached by a string of length l to a point in the same horizontal plane. What is the least upward velocity with which it must be projected so that it shall describe a circle? *Ans.* $\sqrt{3\, gl}$.

10. A particle hangs freely from a string of length l; it is projected horizontally with a velocity $\sqrt{4\,lg}$. Find how high it rises before the string becomes slack.

Ans. $\frac{5}{3}\,l$.

11. A weight of 10 lb. is fastened by a string which passes through a hole in a smooth horizontal table to a weight of 1 lb. which hangs vertically. The first weight is revolving on a table about the hole as center. How many revolutions are there per minute if the horizontal portion of the string is 15 in. long?

12. A ball is hung by a string in a railway carriage which is rounding a curve of 1000 ft. radius with a velocity of 30 mi. an hour. Find the inclination of the string to the vertical. *Ans.* arc tan $\frac{121}{2000}$.

13. A heavy particle is suspended by a string from a fixed point and rotates in a vertical circle. Show that the sum of the tensions of the string when the particle is at opposite ends of a diameter is the same for all diameters.

14. A particle falls down a vertical circle, starting from rest at the highest point. If, when at any point, its velocity be resolved into two components, — one passing through the center, the other through the lowest point of the circle, — prove that the latter is of constant magnitude.

15. A bead is constrained to move on a circular wire and is acted upon by a central force tending to a point on the circumference and varying inversely as the fifth power of the distance. Show that the pressure is constant.

16. A body describes a parabola under the action of two equal forces, one tending to the focus and varying inversely as the distance, the other parallel to the axis. Show that the speed is constant.

17. A particle is constrained to move on a logarithmic spiral $\rho = ae^{m\theta}$ in a central field for which the force varies inversely as the cube of the distance. Show that the pressure varies inversely as the distance. When is the motion free?

18. A particle describes a parabola freely under the action of two forces, one a repulsion from the focus varying as the distance, the other parallel to the axis and equal in magnitude to three times the former. Show that the initial velocity is $2\,\rho_0\sqrt{\mu}$, where ρ_0 is the distance from the focus and μ is the absolute intensity of the repulsive force.

55. Units of work and energy. Power. Work is the product of force and displacement. Hence unit work is done when unit force causes unit displacement. By the energy equation (III), Art. 53, we infer that unit change in kinetic energy arises when unit work is done. The unit of energy is accordingly the same as the unit of work. If scientific units are employed, the unit of distance is the foot, the unit of force is the poundal, and the unit of work and energy is called the *foot-poundal.* If technical units* are employed, the unit of force is the pound and the unit of work and energy is called the *foot-pound*. Since one pound of force $= g$ poundals, we have

$$1 \text{ foot-pound} = g \text{ foot-poundals.}$$

In the French system the unit of distance is the centimeter, the unit of force is the dyne, and the unit of work is called the *erg*.

Power. The question of time does not enter in calculating the amount of work done. Power is defined as the rate of doing work. The unit in the English system is the *horse power*, which is the equivalent of 550 foot-pounds per second, and in the French system is the *force de cheval*, which is the equivalent of 75 meter-kilograms per second.

The relation between the units in the English and French systems is exhibited in the following table of equivalents:

$$1 \text{ foot-poundal} = 4.214(10)^5 \text{ ergs.}$$
$$1 \text{ foot-pound} = 1.356(10)^7 \text{ ergs.}$$
$$1 \text{ horse power} = 1.014 \text{ force de cheval.}$$

PROBLEMS

1. Compute the energy of a body weighing 300 lb. and moving at the rate of 16 ft. per second. *Ans.* 1200 foot-pounds.

2. A body weighing 10 lb. is thrown upward against gravity. Compute the work done upon it by its weight (*a*) while it rises 10 ft., (*b*) while it falls 10 ft. *Ans.* (*a*) -100 foot-pounds; (*b*) $+100$ foot-pounds.

If the resistance of the air amounts to a constant force of 2 lb., compute the work done by it in both cases. *Ans.* -20 foot-pounds in each case.

3. If a body of 10 lb. mass is projected horizontally on a rough plane with a velocity of 50 ft. per second, how far will it move before its velocity is reduced to 20 ft. per second, the retarding force due to friction being constantly 5 lb.? *Ans.* 65.2 ft.

* The energy equation (III), Art. 53, was derived under the assumption that scientific units are employed, that is, force is equal to the product of mass by acceleration. If technical units are used, we have the relation

$$\text{Force} = \frac{\text{mass} \times \text{acceleration}}{g},$$

and the energy equation becomes

$$\text{Work done} = \text{change in kinetic energy} = \frac{m}{2g}(v^2 - v^2_0).$$

4. A body weighing 10 lb. falls vertically under gravity against a constant force of 1 lb. due to the resistance of the air. How far must it fall in order that its velocity may change (*a*) from zero to 20 ft. per second, (*b*) from 10 ft. per second to 20 ft. per second? *Ans.* (*a*) 6.9 ft.; (*b*) 5.2 ft.

5. A mass of 1000 lb. is moving with a velocity of 2 ft. per second. (*a*) What force will stop it in 0.1 second? (*b*) What work is done by the force in stopping it? *Ans.* (*a*) 20,000 poundals; (*b*) 2000 foot-poundals.

6. Water is to be lifted 150 ft. at the rate of 5 cu. ft. per second. What horse power is required? *Ans.* 85.2 HP.
(1 cu. ft. of water weighs 62.5 lb.).

7. Compare the power of two men one of whom can do 4000 foot-pounds of work per minute and the other $(10)^7$ ergs per second.

8. A steam crane lifts 26,280 lb. 150 ft. in 8 min. What is the horse power? *Ans.* 18.45 HP.

9. (1) How long will it take a 3 HP engine to raise 12 T. 42 ft.? (2) From what depth will a 22 HP engine raise 13 T. in one hour?

10. The monkey of a pile driver weighing 1000 lb. is raised 20 ft. and allowed to fall on the head of a pile which is driven into the ground 1 in. by the blow. Find the average force exerted on the head of the pile. *Ans.* 120 tons.

11. A train of 60 T. runs a mile on a level track at constant speed. If the resisting forces are equivalent to the weight of 8 lb. per ton, find the work done by the engine. What must be the minimum HP of the engine to attain a speed of 20 mi. per hour? *Ans.* 1267 foot-tons; 25.6 HP.

12. Suppose the train of the preceding example has steam cut off and brakes applied when running 15 mi. per hour. If it stops $\frac{1}{4}$ mi. from where the brakes were first applied, find (1) the mean resistance; (2) the time taken to stop the train; (3) the work done by the resisting forces.
Ans. (1) 687.5 lb.; (2) 2 min.; (3) -508.2 foot-tons.

13. A train runs (under the action of gravity) from rest for 1 mi. down a plane whose descent is one foot vertically for each 100 ft. of its length; if the resistances be equal to 8 lb. per ton, how far will the train be carried along the horizontal level at the foot of the incline? *Ans.* 1 mi. 1408 yd.

14. In how many hours would an engine of 18 HP empty a vertical shaft full of water, if the diameter of the shaft be 9 ft., the depth 420 ft., and the mass of a cubic foot of water 62.5 lb.? *Ans.* 9.8 hr.

15. The average flow over Niagara Falls is 270,000 cu. ft. per second. The height of the fall is 161 ft. What horse power could be developed from the falls if all the energy were utilized? *Ans.* 4,940,000 nearly.

16. A particle has been falling for 40 sec. Find (*a*) the resultant force which will stop it in 10 sec.; (*b*) in 10 ft. *Ans.* (*a*) 4 m lb.; (*b*) 2560 m lb.

17. A particle whose mass is 8 lb., tied to one end of a fine thread, 6 ft. long, swings in the arc of a semicircle. Find its kinetic energy and velocity as it passes through the lowest point. *Ans.* 48 foot-pounds; $8\sqrt{6}$ ft. per second.

18. The piston of a steam engine is 15 in. in diameter, its stroke is $2\frac{1}{2}$ ft. long, and it makes 40 strokes per minute. If the mean pressure of the steam is 15 lb. per square inch, what work is done by the steam per minute and what is the horse power of the engine? *Ans.* 265,072.5 foot-pounds; 8.03 HP.

19. What must be the length of the stroke of the piston of an engine, the surface of which is 1500 sq. in., which makes 20 strokes per minute, so that with a mean pressure of 12 lb. on each square inch of the piston, the engine may be of 80 horse power? *Ans.* $7\frac{1}{3}$ ft.

20. A hammer weighing a lb., and moving with a velocity v, strikes a nail. How far will the nail be driven if it offers a resistance r? *Ans.* $\dfrac{6\,av^2}{rg}$ in.

21. The diameter of a piston head is 1 ft., the steam pressure 20 lb. per square inch, and the length of the stroke 3 ft. How many strokes per minute must the engine make to raise 2 cu. ft. of water per second from a depth of 400 ft., assuming that 0.02 of the energy is lost by friction? (1 cu. ft. of water weighs 62.5 lb.)

22. A man who weighs 140 lb. walks up a mountain path at a slope of 30° to the horizon at a rate of 1 mi. per hour. Find his rate of working in raising his own weight in horse power.

23. An automobile, weighing 1 T., can run up a hill of 1 in 60 at 8 mi. an hour. Taking the resistance due to friction as $\frac{1}{50}$ of the weight of the car, find at what rate it could run down the same hill, assuming the horse power developed by the engine to remain the same.

24. Assuming that a man in walking raises his center of gravity through a vertical height of 1 in. at every step, find at what horse power a man is working in walking at 4 mi. an hour, his stride being 33 in., and his weight 168 lb.

56. Impulse. A second fundamental equation for rectilinear motion is obtained by integration of the force equation,

$$F = m\frac{dv}{dt}.$$

Multiplying both sides by dt, interchanging members, and integrating between the limits t_0 and t' for the time, we obtain

$$\int_{t_0}^{t'} m\,dv = \int_{t_0}^{t'} F\,dt.$$

The second member is called the *time-integral* of the force F. If v_0 and v' are the values of the velocity for $t = t_0$ and $t = t'$, respectively, then we may write

$$(1)\qquad mv' - mv_0 = \int_{t_0}^{t'} F\,dt,$$

or, introducing momentum,

(VI) **Change in momentum = time-integral of force.**

For a reason now to be explained, the equation (1) is called the *impulse equation.* Changing the limits in (1) from $t=0$ to $t=t$, it becomes

$$(2) \qquad mv - mv_0 = \int_0^t F dt.$$

In this equation the force F is assumed continuous. As a consequence velocity will necessarily change continuously with the time, and therefore a force cannot cause a *sudden* change of momentum. The phenomenon of sudden changes in velocity, such as are produced by blows, is, however, frequently observed. Such changes are said to be caused by *impulses* or impulsive forces. That is, impulses cause changes of velocity in a time too short to be measurable. In this phenomenon the change of momentum, $mV - mv$, where v and V are respectively the initial and final velocities, may be observed. For this reason an impulse is said to be measured by the change of momentum it causes, and in scientific units is set equal to this. That is, using R for impulse,

$$(3) \qquad R = mV - mv.$$

This equation may be regarded as a limiting case of the impulse equation, and the latter designation is derived from this fact. For if $\bar{F}$ is the mean value of the force F in the time t, we have $\int_0^t F dt = \bar{F}t$ (Calculus, p. 358). The impulse equation (2) may now be written $$mv - mv_0 = \bar{F}t.$$

To apply this to a sudden change in velocity, we may let t diminish and $\bar{F}$ increase in such a way that the product $\bar{F}t$ remains finite and approaches a limiting value, namely,

$$\underset{t=0}{\text{limit}}\, \bar{F}t = mV - mv,$$

V and v being as before the final and initial velocities. Comparing with the definition (3), we have

$$R = \underset{t=0}{\text{limit}}\, \bar{F}t,$$

that is, from the present point of view, a sudden change in momentum may be roughly regarded as caused by a very great force acting for a very small time, and the corresponding impulse is measured by their product.

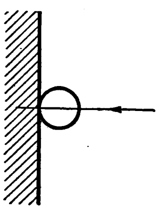
Dimensions of impulse and momentum. Momentum has been defined (Art. 43) as the product of mass by velocity. Its dimensions in terms of the fundamental units of mass, length, and time are expressed by the equation

$$\text{Momentum} = \frac{\text{mass} \times \text{length}}{\text{time}}.$$

From the preceding definition of impulse it is clear that its dimensions are the same as those of momentum.

57. Impact. Problems in impact or collision of solids furnish examples of impulses.

Consider, for example, the impact of a solid elastic sphere upon an elastic plane surface, the direction of motion being along the normal to the plane. The phenomena during contact may be described as follows :



1. The sphere is compressed until its velocity is zero.

2. The sphere then assumes its original shape and a certain final velocity.

Obviously, the change in momentum in each of the two stages described may be regarded as caused by an impulse, and we shall have by definition, if m is the mass of the sphere,

$$R = \textit{impulse of compression} = mv,$$

if v is the original velocity, and

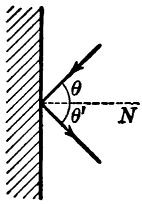
$$R' = \textit{impulse of restitution} = mv',$$

if v' is the final velocity.

Now it is a fact observed by Newton, that the final and initial velocities are in a ratio which depends upon the substances in contact, and not upon the velocity of impact. This constant ratio, called the *coefficient of elasticity*, we denote by e, and hence $v' = -ev$, the negative sign indicating reversal in direction. Consequently, the impulses satisfy the relation

$$R' = -eR. \tag{4}$$

The coefficient e is less than unity, if the solid and plane surfaces are not *perfectly elastic*. There is accordingly a *loss* of energy by the impact. No exception, however, is here afforded to the doctrine of the conservation of energy. The accompanying phenomena of heat, light, etc., indicate a transformation of energy corresponding to the lost energy of motion.



Oblique impact. If the direction of approach be inclined at an angle θ to the normal, and if the sphere and plane are *smooth*, then no change of motion occurs along the surface of the plane, and the preceding discussion applies without change to the *normal* components of the velocities. That is,

$$(5) \qquad v_n' = -ev_n, \quad R' = mv_n', \quad R = mv_n, \quad R' = -eR.$$

Since the tangential components (along the plane surface) are equal, that is, $v_t' = v_t$, we shall have

$$(6) \qquad \tan\theta' = \frac{v_t'}{v_n'} = \frac{v_t}{ev_n} = \frac{1}{e}\tan\theta.$$

If the solids are imperfectly elastic, $\tan\theta' > \tan\theta$, hence $\theta' > \theta$, and the path is bent away from the normal.

ILLUSTRATIVE EXAMPLE. A bullet weighing half an ounce is fired with a speed of 2000 ft. per second from a rifle weighing 10 lb. If the rifle kicks back through 3 in., find the average pressure applied by the shoulder in bringing it to rest.

Solution. Since the impulse acting upon the gun is equal and opposite to the impulse acting upon the bullet, we have the relation

$$mv = -m'v',$$

where m, v and m', v' denote the masses and velocities of the gun and bullet, respectively. Substituting the values of m, $m,'$ v', we find

$$v = \tfrac{25}{4} \text{ ft. per second.}$$

If F denotes the average pressure exerted by the shoulder, we have from the work equation,

$$F \cdot \tfrac{1}{4} = \tfrac{1}{2} \cdot 10 \cdot (\tfrac{25}{4})^2,$$

whence

$$F = \tfrac{3125}{4} \text{ poundals} = 24.42 \text{ lb.}$$

PROBLEMS

1. An arrow shot from a bow starts off with a velocity of 120 ft. per second. With what velocity will an arrow twice as heavy leave the bow if sent off with three times the force? *Ans.* 180 ft. per second.

2. A ball falls from rest at a height of 20 ft. above a fixed horizontal plane. Find the height to which it will rebound, e being $\frac{3}{4}$. *Ans.* $11\frac{1}{4}$ ft.

3. A ball falls from a height h on to a horizontal plane, and then rebounds. Find the height to which it rises in its ascent. *Ans.* e^2h.

4. A ball is projected from the middle point of one side of a billiard table, so as to strike first an adjacent side, and then the middle point of the side opposite to that from which it started. Find where the ball must hit the adjacent side, its length being b. *Ans.* At the distance $\frac{b}{1+e}$ from the end nearest the opposite side.

5. A rifle weighing 3 lb. is discharged while lying on a smooth horizontal table. The weight of the bullet is 2 oz., and it leaves the gun with a velocity of 1400 ft. per second. What is the impulse of the kick?

6. A man weighing 180 lb. jumps from a boat weighing 110 lb. into a boat weighing 160 lb. If the boats are initially at rest compare their velocities after the jump.

7. A ball whose mass is $5\frac{1}{4}$ oz. is moving at the rate of 100 ft. per second when it is struck by a bat in such a way that immediately after the blow it has a velocity of 150 ft. per second in a direction making an angle of 30° upward from the horizontal. Assuming the horizontal velocity to be reversed by the blow, find the value and direction of the impulse.

8. A shot of 700 lb. is fired with a velocity of 1600 ft. per second from a 35-T. gun. Find the velocity with which the gun recoils, neglecting the weight of the powder. If the recoil is resisted by a steady pressure equal to the weight of 10 T., through what space will the gun move? *Ans.* $14\frac{2}{7}$ ft. per second; $11\frac{9}{56}$ ft.

9. A particle falls from a height h upon a fixed horizontal plane. If e be the coefficient of restitution, show that the whole distance described by the particle before it has finished rebounding is $h\frac{1+e^2}{1-e^2}$, and that the time that elapses is

$$\sqrt{\frac{2h}{g}}\frac{1+e}{1-e}.$$

10. A smooth elastic ball is projected horizontally from the top of a tower 100 ft. high with a velocity of 100 ft. per second, and after one rebound describes a horizontal range of 40 ft. Find the coefficient of elasticity. *Ans.* $\frac{2}{25}$.

11. Two equal scale pans, each of mass M, are connected by a string which passes over a smooth peg, and are at rest. A particle of mass m is dropped on one of them from a height $\frac{u^2}{2g}$, the coefficient of elasticity being e. Find the velocity of the scale pan after the first impact. *Ans.* $\frac{mu}{2M+m}(1+e)$.

12. Show that a billiard ball of any elasticity, struck from any point on the table, and returning to the same point after impinging against the four sides in order, describes a parallelogram, with sides parallel to the diagonals of the table.

13. A heavy elastic ball drops from the ceiling of a room and after twice rebounding from the floor reaches a height equal to one half that of the room. Show that its coefficient of restitution is $\sqrt[4]{\frac{1}{2}}$.

14. A heavy elastic ball falls from a height of n ft. and meets a plane inclined at an angle of 60° to the horizon. Find the distance between the first two points at which it strikes the plane. *Ans.* $2\sqrt{3}\,ne(1+e)$.

15. The sides of a rectangular billiard table are of lengths a and b. If a ball of elasticity e be projected from a point in one of the sides, of length b, to strike all four sides in succession and continually retrace its path, show that the angle of projection θ with the side is given by $ae\cot\theta = c + ec'$, where c and c' are the parts into which the side is divided at the point of projection.

16. A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball of elasticity e projected along the table from a point in the rim in a direction making an angle $\arctan\sqrt{\dfrac{e^3}{1+e+e^2}}$ with the radius through the point will return to the point of projection after two impacts on the rim. Prove also that when the ball returns to the point of projection its velocity is to the original velocity as $e^{\frac{3}{2}}:1$.

58. Force-moments in a plane. The discussion of this article will enable us to work out a second* integration of the force equations

$$(1)\qquad m\frac{dv_x}{dt} = F_x,\ m\frac{dv_y}{dt} = F_y.$$

Multiply the first of these equations by y, the second by x, and subtracting, we get

$$(2)\qquad m\left(x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right) = xF_y - yF_x.$$

The second member, $xF_y - yF_x$, has a simple geometrical significance. If the directed line PQ represents the force whose axial components are (F_x, F_y) and whose point of application is $P(x, y)$, then the coördinates of the point Q are at once seen from figure a to be $(x+F_x, y+F_y)$, for

$$ON = OM + MN = x + F_x,\ NQ = MP + SQ = y + F_y.$$

* The energy equation, Art. 53, is to be regarded as a first integration of the force equations. For writing these in the form

$$mv_x\frac{dv_x}{dx} = F_x,\ mv_y\frac{dv_y}{dy} = F_y,$$

multiplying the first by dx, the second by dy, and adding, gives

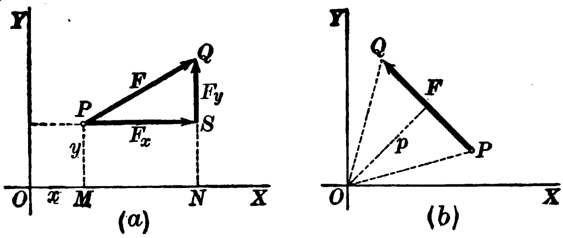
$$m(v_x dv_x + v_y dv_y) = F_x dx + F_y dy.$$

Integrating, we obtain

$$\tfrac{1}{2}m(v_x^2 + v_y^2) + C = \int(F_x dx + F_y dy).$$

But $v^2 = v_x^2 + v_y^2$, and hence the result is a form of the equation in question.

In figure *b* the area of the triangle OPQ is (Analytic Geometry, p. 42)



(3) $$\tfrac{1}{2}[x(y+F_y)-y(x+F_x)]=\tfrac{1}{2}(xF_y-yF_x).$$

But the area of the triangle also equals $\frac{1}{2}pF$, if p is the altitude drawn from the origin upon PQ.

(4) $$\therefore\ xF_y-yF_x=pF.$$

The product pF is called the *moment* of the force F with respect to the origin. The point O is called the *center of moments.* The perpendicular distance p is called the *lever arm.* Hence

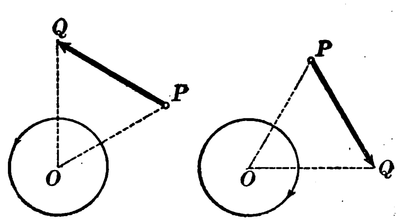
Force $\times$ *lever arm* $=$ *force-moment.*

Equation (4) gives the expression for force-moment in analytic form. That is,

(VII) **Force-moment with respect to the origin** $= xF_y - yF_x$,

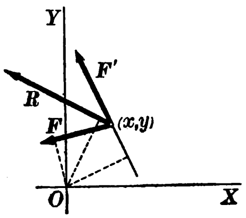
where the axial components of the force are (F_x, F_y) and the point of application is (x, y).

Sign of force-moment. The area of OPQ is positive when the order O, P, Q of the vertices on the perimeter is counter-clockwise; negative, if clockwise. Hence force-moment is *positive* if the force acts to cause *positive* rotation (counter-clockwise) about the point; otherwise, *negative.*



Positive *Negative*

Evidently, if $p=0$, the force-moment vanishes, that is, if the line of action of the force passes through the center of moments the force-moment vanishes. *Force-moment is unchanged if the point of application is displaced along* *the line of action*, the magnitude and direction of the force remaining the same. For F and p in (4) both remain constant if the point of application is so displaced.



Consider next two concurrent forces $F(F_x, F_y)$ and $F'(F_x', F_y')$, the common point of application being (x, y). Then, by (VII),

$$\text{Moment of } F = xF_y - yF_x,$$

$$\text{Moment of } F' = xF_y' - yF_x'.$$

Adding,

$$(5)\quad \text{Moment of } F + \text{Moment of } F' = x(F_y + F_y') - y(F_x + F_x').$$

Let R be the vector resultant of F and F'. Then the axial components of R are $(F_x + F_x', F_y + F_y')$. The second member of (5) is therefore the *moment* of R. Hence

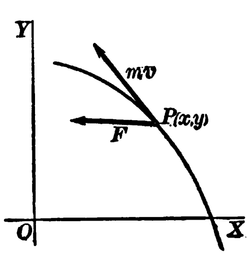
$$(6)\qquad \text{Moment of } F + \text{moment of } F' = \text{moment of } R.$$

That is, the sum of the force-moments of two concurrent forces is equal to the moment of their resultant. This principle is general; it can be extended to any number of concurrent forces and leads to the important

THEOREM OF MOMENTS. *The algebraic sum of the force-moments of any number of concurrent forces with respect to any center equals the force-moment of their resultant.*

59. Moment of momentum. Consider the first member of (2), Art. 58. This expression is a *time-rate*. For it is easily seen that

$$(1)\quad m\left(x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right) = \frac{d}{dt}(x\cdot mv_y - y\cdot mv_x).$$



Hence the equation (2), Art. 58, takes the form

$$(2)\quad \frac{d}{dt}(x\cdot mv_y - y\cdot mv_x) = xF_y - yF_x.$$

This result is called the *moment equation* for a reason now to be explained.

The expressions mv_x, mv_y in the first member of (2) are the axial components of the vector momentum $m\mathbf{v}$. The expression

$$(3)\qquad x\cdot mv_y - y\cdot mv_x,$$

being entirely analogous to

$$x\cdot F_y - y\cdot F_x,$$

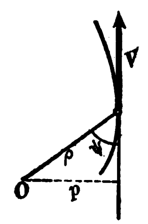
is called the *moment of momentum* with respect to the origin. The moment of momentum is of course variable in any general motion, and is a function of the time. Equation (2) therefore gives the relation,

(VIII)

Time-rate of change of moment of momentum = force-moment.

In the equations of motion, F_x and F_y are the axial components of the complete resultant of all forces acting. Hence in (VIII) the force-moment is the *resultant* force-moment (Theorem of Moments).

60. Angular momentum. The expression for moment of momentum is simple if polar coördinates are used. In the figure, let p be the perpendicular distance from the center of moments O to the tangent to the path. Then p is the lever arm of the vector momentum. Hence, by Art. 59,



(1) Moment of momentum $= mv \cdot p.$

But in the figure, $p = \rho \sin \psi = \rho \cdot \rho \dfrac{d\theta}{ds}.$

Also, since $v = \dfrac{ds}{dt}$, (1) becomes

(2) $$\textit{Moment of momentum} = m\frac{ds}{dt} \cdot \rho^2 \frac{d\theta}{ds} = m\rho^2 \frac{ds}{dt} \cdot \frac{d\theta}{ds} = m\rho^2 \frac{d\theta}{dt}.$$

Now (see Art. 22), $m\rho^2 =$ moment of inertia of the particle with respect to $O(= I_0)$, and since $\dfrac{d\theta}{dt} =$ angular velocity $= \omega$. we obtain the result,

(3) $$\textit{Moment of momentum} = I_0 \omega.$$

If this result is compared with the definition of momentum (mv), it is seen that moment of inertia corresponds to mass, and angular velocity to linear velocity. From the introduction of angular velocity, moment of momentum is often called *angular momentum*, and momentum proper, *linear momentum.* We thus have the definitions

Linear momentum = mass × linear velocity,

Angular momentum = moment of inertia × angular velocity.

Moreover, the results of Arts. 44 and 59 give the equations,

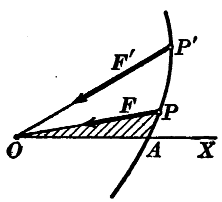
$$\text{(4)} \qquad \text{Force} = \frac{d}{dt}(\text{linear momentum});$$

$$\text{Force-moment} = \frac{d}{dt}(\text{angular momentum}).$$

Integration of the last equation will give the result:

Change in angular momentum = time-integral of force-moment.

This result is to be regarded as a second integral of the force equation, the other being the energy equation already found. Comparison should be made with (VI) for rectilinear motion. In the latter, momentum has been replaced by angular momentum and force by force-moment.



ILLUSTRATIVE EXAMPLE. As an application, consider **central motion**, that is, motion caused by a force constantly directed towards a fixed center O.

Solution. The force-moment with respect to O is zero. Hence, choosing O for origin, (4) gives

$$m\rho^2\frac{d\theta}{dt} = \text{constant, or } \rho^2\frac{d\theta}{dt} = \text{constant.}$$

If u is the area of any sector AOP, then

$$u = \tfrac{1}{2}\int \rho^2 d\theta, \text{ and } \frac{du}{dt} = \tfrac{1}{2}\rho^2\frac{d\theta}{dt} = \text{constant.}$$

The derivative $\frac{du}{dt}$, that is, the time-rate of change of the sectorial area AOP, is called the *areal velocity*. *Hence in any central motion the areal velocity is constant;* or also *the radius vector sweeps over sectors of equal area in equal times.*

61. Fundamental equations. The preceding sections have led to three types of equations, namely:

Rectilinear Motion.	*Curvilinear Motion.*
Force Equation,	Force Equations,
Energy Equation,	Energy Equation,
Impulse Equation.	Moment Equation.

Problems in motion depend for their solution largely upon these equations, and their application has been seen to be of fundamental importance. The moment equation will be used in a later chapter.

62. Formulas in dynamics of a particle. For convenience of reference the chief results in connection with the kinetics of a material particle are collected here. In every case F denotes the resultant of *all forces* acting on the particle.

F_x = sum of x-components of all forces.
F_y = sum of y-components of all forces.
F_z = sum of z-components of all forces.
F_t = sum of tangential components.
F_n = sum of normal components.
F_ρ = sum of components along radius vector.
F_θ = sum of components perpendicular to radius vector.

Force Equations:

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= m\frac{dv_x}{dt} = mv_x\frac{dv_x}{dx} = F_x, \\ m\frac{d^2y}{dt^2} &= m\frac{dv_y}{dt} = mv_y\frac{dv_y}{dy} = F_y, \\ m\frac{d^2z}{dt^2} &= m\frac{dv_z}{dt} = mv_z\frac{dv_z}{dz} = F_z, \end{aligned}\right\}\text{Rectangular Coördinates.}$$

$$m\frac{dv}{dt} = mv\frac{dv}{ds} = F_t, \quad \frac{mv^2}{R} = F_n, \quad \text{Intrinsic Equations.}$$

$$\left.\begin{aligned} m\left\{\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right\} &= F_\rho, \\ \frac{m}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right) &= F_\theta, \end{aligned}\right\}\text{Polar Coördinates.}$$

Work Integrals:

$$\text{Work} = \int (F_x dx + F_y dy), \quad \text{Rectangular Coördinates.}$$

$$\text{Work} = \int F_t ds, \quad \text{Intrinsic Equations.}$$

$$\text{Work} = \int (F_\rho d\rho + F_\theta \rho d\theta), \quad \text{Polar Coördinates.}$$

Energy Equation:

Work done by all forces acting $= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$,
initial velocity $= v_0$, final velocity $= v$.

Impulse Equation:

Impulse $= mV - mv$,

initial velocity $= v$, final velocity $= V$.

Moments:

Force-moment $= xF_y - yF_x$,

point of application $= (x, y)$, center of moments $= (0, 0)$.

Moment Equation:

Force-moment $= \dfrac{d}{dt}$ (angular momentum) $= \dfrac{d}{dt}(I_0\omega)$,

$I_0 =$ moment of inertia and $\omega =$ angular velocity with respect to $(0, 0)$.

CHAPTER VI

MOTION OF A PARTICLE IN A CONSTANT FIELD

63. Field of force. A region in which force is known to exist is called a *field of force.* A body free to move when placed in such a field will in general not remain at rest. The acceleration which will be imparted to a material particle at any point in a field of force must be regarded as characteristic for the field at that point. Upon particles of different masses m_1, m_2, etc., at the *same* point the field will exert different forces. These forces will have the same direction, but different magnitudes, namely, m_1f, m_2f, etc., if f is the acceleration at the point. Consequently, if we know the acceleration due to a field at each of its points, we know the force the field will exert upon any material particle. For this reason *a field of force is said to be determined when the vector acceleration at every one of its points is known.*

In general the acceleration due to a field of force is different at different points of the field. When the acceleration is the same at every point, the field is said to be *constant* or *uniform.* A familiar and important example of a constant field is afforded by considering the earth's attraction, that is, the force of gravity, in any small region. All bodies are attracted towards the center of the earth. Particles falling freely from rest will describe rectilinear paths which may be regarded as parallel if the region under consideration is small, and experiment shows that the acceleration at every point of such a region is constant.

64. Rectilinear motion under a constant force. This type of motion is very important. We consider the motion of a material

particle along a straight line, the force causing the motion being directed along that line.

Assume a positive direction and an origin of distance on the path, and let s denote the distance from the origin to any position

P. The driving force F is then mf or $-mf$, according as its direction is positive or negative. From the force equation we have

$$m\frac{d^2s}{dt^2} = \pm mf, \text{ or } \frac{d^2s}{dt^2} = \pm f.$$

Integrating, using s_0 and v_0 to determine the initial position and velocity, we derive the result:

The velocity and position of a material particle moving along a straight line under a constant force are given by

$$\text{(I)} \qquad \boldsymbol{v = v_0 \pm ft, \;\; s = s_0 + v_0t \pm \tfrac{1}{2}ft^2,}$$

in which f denotes the constant acceleration, and v_0 and s_0 determine the initial velocity and position, respectively. The positive or negative sign holds according as the distance and force have the same or opposite directions.

The energy equation is

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{s_0}^{s} Fds = F(s-s_0) = \pm mf(s-s_0).$$

Hence if $d = s - s_0 =$ distance moved, this equation gives, by solving,

$$\text{(II)} \qquad \boldsymbol{v^2 = v_0^2 \pm 2fd,}$$

expressing the final velocity in terms of the initial velocity and distance moved.

When the initial velocity is opposed to the force, from (II), $v^2 = v_0^2 - 2fd$, and (I), $v = v_0 - ft$. Hence $v = 0$ when $t = \frac{v_0}{f}$, and then $d = \frac{v_0^2}{2f}$, the velocity constantly diminishing until this position is reached, and thereafter constantly increasing.

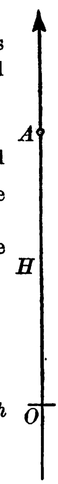
Analysis of (I) is important. If $v_0 = s_0 = 0$, then

$$v = \pm ft, \quad s = \pm\tfrac{1}{2}ft^2, \quad v^2 = \pm 2fs,$$

formulas giving the velocity and distance *due to the force only.* To obtain (I) and (II), we add on the initial velocity to get the final velocity, and the distance ($= v_0t$) due to it and the initial distance to get the total distance.

ILLUSTRATIVE EXAMPLES

1. *Rectilinear motion under gravitation.* In this case $f = g$. We distinguish two problems.



Particle projected vertically upwards. Taking the upward direction as positive, and the origin as the point of projection, we have in (I), $s_0 = 0$, and obtain

(1) $$v = v_0 - gt, \; s = v_0 t - \tfrac{1}{2} gt^2,$$

in which v_0 is the velocity of projection. The highest point A is reached when $v = 0$. Then $t = \frac{v_0}{g}$, and $s = OA = \frac{v_0^2}{2g}$. This is therefore the *greatest height.*

Particle falling freely. Choosing the downward direction as positive, we obtain from (I),

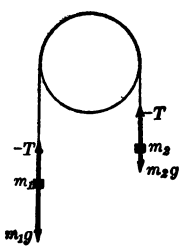
(2) $$v = v_0 + gt, \;\; s = s_0 + v_0 t + \tfrac{1}{2} gt^2.$$

If the particle falls from rest at the origin, then $v_0 = 0$, $s_0 = 0$, and we get

(3) $$v = gt, \;\; s = \tfrac{1}{2} gt^2, \;\; v^2 = 2\,gs.$$

Hence the velocity acquired in falling freely from rest a distance h equals $\sqrt{2\,gh}$.

2. *Atwood's machine.* Let the figure represent two masses m_1 and m_2 suspended by an inextensible thread passing over a smooth pulley. The motion of the system is known if the motion of one of the particles m_2 is known, that is, if m_2 descends with an acceleration f. In other words, if the acceleration of m_2 is f, the acceleration of m_1 is $-f$. Denoting by T the tension in the thread, that is, the pull of the particle on the thread, the resultant force acting on m_2 is $m_2 g - T$, and on m_1 is $m_1 g - T$.



Hence the force equation gives

$$\begin{cases} m_2 g - T = m_2 f, \\ m_1 g - T = - m_1 f. \end{cases}$$

Eliminating T and solving for f, we find

$$f = \frac{m_2 - m_1}{m_2 + m_1} g.$$

The acceleration is constant and the equation of motion is found by substitution in (I).

Eliminating f from the force equations, we obtain the tension in the thread, namely,

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g.$$

PROBLEMS

1. How long will it take a body to fall from rest through 625 ft.? Find the velocity acquired. How far does it fall in the fourth second? ($g = 32$.)

Ans. $\frac{25}{4}$ sec.; 200 ft. per second; 112 ft.

2. How long will it take a body to acquire a velocity of 260 ft. per second falling from rest? *Ans.* $\frac{65}{8}$ sec.

3. (1) How high will a stone rise which is projected upwards with a velocity of 256 ft. per second? (2) What is its position, direction of motion, and velocity at the end of the tenth second? $(g = 32.)$

Ans. (1) 1024 ft. (2) 960 ft. high; $v = 64$ ft. per second downwards.

4. Compare the momentum of a 3-lb. weight after falling 30 ft. with that of a half-ounce bullet having a velocity of 2000 ft. per second. *Ans.* $24\sqrt{30} : \frac{125}{2}$.

5. With what velocity must a body be projected downwards that it may overtake in 10 sec. another which has already fallen through 100 ft.?

Ans. 90 ft. per second.

6. A body is projected upwards with a velocity of 80 ft. per second. How long will it be in returning to the starting place? With what velocity will it return? $(g = 32.)$ *Ans.* 5 sec.; 80 ft. per second.

7. A particle has an initial velocity of 125 cm. per second, and an acceleration (1) of 10 cm. per second each second; (2) of -10 cm. per second each second. How long will it take in each case to move over 420 cm.? Explain the results.

Ans. (1) 3 sec. or -28 sec.; (2) 4 sec. or 21 sec.

8. The velocity of a particle moving in a constant field is a cm. per second at the end of c seconds, and b cm. per second at the end of $(c + 1)$ sec. What was the initial velocity and acceleration? *Ans.* $v_0 = a + (a - b)c, f = b - a$.

9. The sum of the two weights of an Atwood's machine is 12 lb. The heavier weight descends through 128.8 ft. in 8 sec. What are the values of each weight? $(g = 32.2.)$ *Ans.* 6.75 lb., 5.25 lb.

10. A 2-lb. weight carried on a spring balance in a balloon has an apparent weight of 2.4 lb. when the balloon is ascending. What is the acceleration of the balloon? What should the body weigh if the balloon is descending with an acceleration of 10 ft. per second? *Ans.* 6.4 ft. per sec.^{2}; $\frac{11}{8}$ lb.

11. A mass of 12 lb. rests on a smooth horizontal table. A second mass of 1.5 lb. is attached to the first by means of a cord passing over the edge of the table.

Find the following:

(1) The acceleration of the system.	*Ans.* $\frac{32}{9}$ ft. per second.
(2) The space described in 3 sec.	16 ft.
(3) The velocity attained at the end of 5 sec.	$\frac{160}{9}$ ft. per second.
(4) The force on the string.	$\frac{128}{3}$ poundals.
(5) The time required for the system to move 120 ft.	$3\sqrt{\frac{15}{2}}$ sec.

12. Two unequal masses are connected by a weightless inextensible string passing over a smooth peg. What must be the ratio of the masses that the system may move through 24 ft. in 3 sec. from rest? *Ans.* 5 : 7.

13. A train passes another on a parallel track, the former having a velocity of 45 mi. an hour and an acceleration of 1 ft. per second per second, the latter a velocity of 30 mi. an hour and an acceleration of 2 ft. per second per second. How soon will the second be abreast of the first again, and how far will the trains have moved in the meantime?

14. A body is dropped from a balloon at a height of 70 ft. from the ground. Find its velocity on reaching the ground, if the balloon is (*a*) rising, (*b*) falling with a velocity of 30 ft. per second.

15. A stone is dropped into a well, and the sound of the splash reaches the top after 9 sec. Find the depth of the well, the velocity of sound being 1100 ft. per second.

16. A body whose mass is 5 lb., moving with a speed of 160 ft. per second, suddenly encounters a constant resistance equal to the weight of $\frac{1}{4}$ lb., which lasts until the speed is reduced to 96 ft. per second. For what time and through what distance has the resistance acted?

17. A body falls freely from the top of a tower and during the last second of its flight falls $\frac{16}{25}$ of the whole distance. Find the height of the tower.

Ans. 100 ft.

18. Two scale pans of mass 3 lb. each are connected by a string passing over a smooth pulley. Show how to divide a mass of 12 lb. between the two scale pans so that the heavier may descend through a distance of 50 ft. in the first 5 sec.

Ans. In the ratio 19 : 13.

19. A string hung over a pulley has at one end a mass of 10 lb. and at the other end masses of 8 and 4 lb., respectively. After being in motion for 5 sec. the 4-lb. mass is taken off. Find how much farther the masses go before they come to rest.

Ans. 29 ft. 9 in. nearly.

20. If the string in an Atwood's machine can bear a strain of only $\frac{1}{4}$ the sum of the two weights, show that the least possible acceleration is $\frac{g}{\sqrt{2}}$. Find the least ratio of the larger to the smaller weight.

21. A mass m pulls a mass m' up an inclined plane, inclination α, by means of a string passing over a pulley at the top of the plane. Show that the acceleration is $\frac{m - m' \sin \alpha}{m + m'} g$.

22. A mass of 6 oz. slides down a smooth inclined plane, whose height is half its length, and draws another mass from rest over a distance of 3 ft. in 5 sec. along a horizontal table which is level with the top of the plane over which the string passes. Find the mass on the table.

Ans. 24 lb. 10 oz.

23. A weight P is drawn up a smooth plane inclined at an angle of 30° to the horizon, by means of a weight Q which descends vertically, the weights being connected by a string passing over a small pulley at the top of the plane. Find the ratio of Q to P if the acceleration is $\frac{g}{4}$.

Ans. $Q = P$.

24. A juggler keeps three balls going with one hand, so that at any instant two are in the air and one in his hand. Find the time during which a ball stays in his hand if each ball rises to a height of a ft.

25. A stone dropped into a well is heard to strike the bottom in t sec. Find the depth of the well, the velocity of sound being a ft. per second.

Ans. $\left[\sqrt{at + \frac{a^2}{2g}} - \frac{a}{\sqrt{2g}}\right]^2$.

26. The two masses of an Atwood's machine are 8 and 10 lb., respectively, and the string is clamped so that no motion can take place. If the string is suddenly unclamped, find the change in pressure exerted on the pulley.

27. A mass of 10 lb. resting on a smooth inclined plane, inclination 30°, is connected by a string passing over a pulley at the top of the plane to a mass of 10 lb. hanging vertically. Find the tension in the thread (1) when the weight on the plane is held fixed, (2) when the hanging weight rests on a support, (3) when both weights are free to move.

65. Curvilinear free motion. We first prove the

THEOREM. *If a free material particle is projected into a constant field in a direction oblique to the direction of the force of the field, the path will be a parabola.*

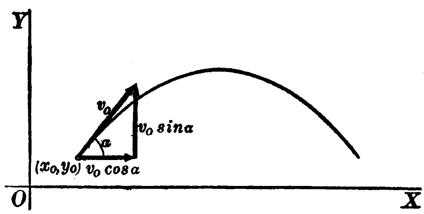
Let the acceleration due to the field be f and its direction opposite to the direction of the Y-axis. The axial components of the force at any point are

$$F_x = 0, \quad F_y = -mf.$$

Hence the rectangular force equations are

$$(1) \qquad m\frac{d^2x}{dt^2} = 0, \quad m\frac{d^2y}{dt^2} = -mf.$$

Let the initial position be (x_0, y_0), the initial velocity v_0, and the angle which v_0 makes with the X-axis α. Then the components of the initial velocity are $v_0 \cos\alpha$ and $v_0 \sin\alpha$, respectively. Integrating equations (1) and determining the constants by the given initial conditions, we obtain



$$(2) \qquad x = x_0 + v_0 \cos\alpha \cdot t, \quad y = y_0 + v_0 \sin\alpha \cdot t - \tfrac{1}{2}ft^2.$$

Eliminating t, the equation of the path is

$$(3) \qquad y - y_0 = \tan\alpha\,(x - x_0) - \frac{f}{2\,v_0^2 \cos^2\alpha}(x - x_0)^2,$$

which is the equation of a parabola with its axis parallel to the Y-axis. Q. E. D.

The distance moved through in the direction of the force is $-(y - y_0)$. Hence the energy equation gives

$$(4) \qquad v^2 = v_0^2 - 2f(y - y_0).$$

The applications of the preceding theorem are mainly to problems of the motion of projectiles near the surface of the earth. Neglecting the resistance of the atmosphere and the variation of the force of gravity, the circumstances of the motion are given by (2) where $f = g$. If the origin of coördinates is the initial position, the X-axis horizontal, and the positive direction of the Y-axis upwards, the equations of motion are

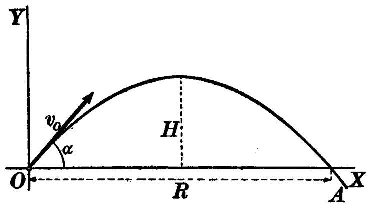
$$\text{(III)} \qquad x = v_0 \cos \alpha \cdot t, \qquad y = v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2.$$

and the equation of the path and the energy equation become, respectively,

$$y = \tan \alpha \cdot x - \frac{g}{2 v_0^2 \cos^2 \alpha} x^2,$$

$$\text{(IV)} \qquad v^2 = v_0^2 - 2 g y.$$

The projectile reaches its greatest height H when the velocity in the direction of the Y-axis is zero. Then $v_y = 0$, and $v = v_x = v_0 \cos \alpha$. Making these substitutions in (IV), and setting $y = H$, we obtain



$$2 g H = v_0^2 - v_0^2 \cos^2 \alpha = v_0^2 \sin^2 \alpha.$$

$$(5) \quad \therefore \quad H = \frac{v_0^2 \sin^2 \alpha}{2 g}.$$

The time of flight T is the time elapsed when the projectile again reaches the X-axis. Setting $y = 0$ in (III), we obtain

$$0 = v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2, \text{ whence } t = 0 \text{ (at } O\text{)},$$

$$\text{or } t = \frac{2 v_0 \sin \alpha}{g}.$$

$$(6) \qquad \therefore \quad T = \frac{2 v_0 \sin \alpha}{g}.$$

The horizontal range R is the intercept OA on the X-axis, the value of x when $t = T$. Setting $t = T$ in (III), we obtain

$$x = v_0 \cos \alpha \cdot T = \frac{2 v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2 \sin 2 \alpha}{g}.$$

$$(7) \qquad \therefore \quad R = \frac{v_0^2 \sin 2 \alpha}{g}.$$

From (7) it is obvious that the *maximum* range for a given velocity of projection results when $\sin 2 \alpha = 1$ or $\alpha = 45°$.

ILLUSTRATIVE EXAMPLES

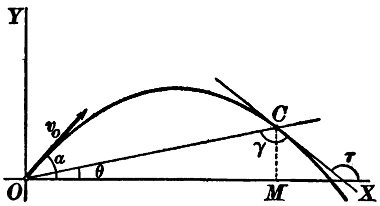
1. Find the range on an inclined plane through the origin with an inclination θ.

Solution. The equation of OC is $y = \tan\theta \cdot x$. Substituting from (III), we have

$$v_0 \sin\alpha \cdot t - \frac{1}{2}gt^2 = \tan\theta\, v_0 \cos\alpha \cdot t,$$

or solving,
$$t = \frac{2\,v_0(\sin\alpha - \tan\theta\cos\alpha)}{g} = \frac{2\,v_0 \sin(\alpha-\theta)}{g\cos\theta},$$

which gives the time when the projectile will strike the plane at C.



Since $OC = OM \sec\theta$, and OM is the value of $x (= v_0 \cos\alpha \cdot t)$ when t has the value just found, we readily deduce the result,

$$OC = \frac{2\,v_0^2 \cos\alpha \cdot \sin(\alpha-\theta)}{g\cos^2\theta}.$$

The velocity at C is (by IV)

$$v^2 = v_0^2 - 2\,g \cdot MC = v_0^2 - 2\,g \cdot OC \sin\theta.$$

The angle of impact with the plane, namely, $\gamma = \tau - \theta$, is readily found. For $\tan\tau$ is the slope of the parabola. Substituting the value of $OM (= x)$ already found, in this, and reducing, gives

$$\tan\gamma = \frac{\sin(\theta-\alpha)}{\cos(\theta+\alpha) + 2\sin\theta\tan\theta\cos\alpha}.$$

2. Required the elevation in order that the projectile may pass through a given point, the velocity of projection being a given constant.

Solution. Let the given point be $Q(x_1, y_1)$. Since this point lies on the parabola (IV), we have

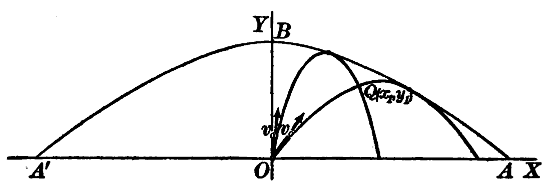
$$y_1 = \tan\alpha\, x_1 - \frac{g}{2\,v_0^2\cos^2\alpha}x_1^2,$$

from which α must be determined. Since $\cos^2\alpha = \frac{1}{\tan^2\alpha + 1}$, substituting and collecting gives the equation

$$(1) \qquad gx_1^2\tan^2\alpha - 2\,v_0^2\,x_1\tan\alpha + (2\,v_0^2 y_1 + gx_1^2) = 0,$$

in which $\tan\alpha$ is the unknown.

Since the equation (1) has in general two roots, the point Q may be reached in



two ways. To cover all cases, find the discriminant of (1). Since $A = gx_1^2$, $B = -2v_0^2x_1$, $C = 2\,v_0^2y_1 + gx_1^2$, we have for the discriminant

$$\Delta = B^2 - 4\,AC = 4\,v_0^4x_1^2 - 8\,v_0^2gx_1^2y_1 - 4\,g^2x_1^4 = 4\,x_1^2(v_0^4 - 2\,v_0^2gy_1 - g^2x_1^2).$$

Hence $\Delta = 0$ if
$$v_0^4 - 2\,v_0^2gy_1 - g^2x_1^2 = 0,$$

or, omitting the subscripts,

$$(2) \qquad x^2 = \frac{v_0^4}{g^2} - \frac{2\,v_0^2}{g}y.$$

The locus of this equation is the parabola $A'BA$, where $OA = 2 \cdot OB = \frac{v_0^2}{g}$. This parabola is called the *bounding* parabola. The final statement obviously is as follows:

If Q is *within* the bounding parabola, two parabolic paths pass through it ($\Delta > 0$ and (1) has real and distinct roots).

If Q is *on* the bounding parabola, one parabolic path passes through it, and this will touch the bounding parabola at Q ($\Delta = 0$ and the roots of (1) are real and equal).

If Q is *without* the bounding parabola, no trajectory passes through it.

Applied to gunnery, the interpretation of the results is as follows: The region covered by a projectile with a given muzzle velocity is the interior of a paraboloid of revolution whose axis is vertical (obtained by revolution of $A'BA$ around OY). Any point within the paraboloid may be hit in two ways.

In terms of the greatest vertical height h, which can be attained with the given initial velocity v_0, the equation of the bounding parabola takes a simple form. The height h is attained when the particle is projected vertically upwards, and is given by

$$v_0^2 = 2\,gh.$$

Substituting this value of v_0^2 in (2), the equation of the bounding parabola becomes

$$(3) \qquad 4\,hy + x^2 = 4\,h^2.$$

Another convenient form of this equation is obtained by introducing the greatest horizontal range r. This is found from (7), Art. 65, by putting $\sin 2\,\alpha = 1$, whence

$$(4) \qquad r = \frac{v_0^2}{g} = 2\,h,$$

and the equation of the bounding parabola may be written

$$(5) \qquad y = \frac{h}{r^2}(r^2 - x^2).$$

3. A man can throw a ball 100 yd. on a horizontal plane. (*a*) Find the highest point that he can hit on a vertical wall 35 yd. away. (*b*) If he stands on a cliff 150 feet high, how far from the base can he throw?

Solution. Evidently in either case the greatest distance will be the point where the bounding parabola cuts the given plane. We have $r = 100$ (yd.), and hence the equation of the bounding parabola is

$$(1) \qquad 200\,y = 10{,}000 - x^2.$$

(*a*) This parabola will cut the vertical line $x = 35$ at the height

$$y = \frac{10000 - 1225}{200} = 43.88 \text{ yd.},$$

which is the highest point on the wall that he can hit.

(*b*) Since the horizontal plane is 50 yd. below the top of the cliff, we substitute in (1) $y = -50$, and find

$$x = 141.4 \text{ yd.},$$

which is the greatest distance from the base that he can throw.

PROBLEMS

1. A gun is fired at an elevation of 30°. If the muzzle velocity is 1000 ft. per second, determine the following: (*a*) equation of path; (*b*) range; (*c*) time of flight; (*d*) position of the projectile after 2 sec.; (*e*) highest point reached.

$$\textit{Ans.}\quad (a)\ y = \frac{x}{\sqrt{3}} - \frac{64}{3000000}x^2;\ (b)\ \frac{1000000\sqrt{3}}{64}\text{ ft.};$$

$$(c)\ \frac{1000}{32}\text{ sec.};\ (d)\ x = 1000\sqrt{3},\ y = 936;\ (e)\ \frac{(10)^6}{256}\text{ ft.}$$

2. In problem 1 find the magnitude and direction of the velocity and its axial components after 20 seconds.

Ans. $v_x = 500\sqrt{3}$ ft. per second; $v_y = -140$ ft. per second.

3. Discuss the circumstances of the motion in problem 1: (*a*) after the projectile has passed over a horizontal distance $x = 5000$ ft., (*b*) when at an elevation of 1000 ft.

4. A projectile moves subject to the equations $x = at$, $y = bt - \frac{1}{2}gt^2$. Discuss its motion fully.

5. Show that a given gun will shoot three times as high when elevated at an angle of 60° as when fired at an angle of 30°, but will carry the same distance on a horizontal plane.

6. The range is 300 ft. and the time of flight 5 sec. Find the initial velocity and the angle of elevation. What is the effect on the range of doubling the initial velocity?

Ans. $v_0 = 100$ ft. per second; $\sin\alpha = \frac{4}{5}$.

7. (*a*) A boy can throw a stone 75 yd. on a level. How far from the base can he throw standing at the top of a cliff 150 ft. high? (*b*) If the stone is thrown horizontally and strikes 450 ft. from the base, what is its initial velocity?

Ans. (*a*) $25\sqrt{21}$ yd.; (*b*) $60\sqrt{6}$ ft. per second.

8. The wheels of an engine running at the rate of 40 mi. per hour encounter a drop of one quarter inch at the rail joint. How far from the joint will the wheels strike the lower rail?

Ans. $\frac{11}{9}\sqrt{3}$ ft.

9. Show that to strike an object at a distance x on the horizontal plane through the starting point, the elevation must be α or $90° - \alpha$ where $\alpha = \frac{1}{2}\sin^{-1}\frac{gx}{v_0^2}$. How do the striking velocities compare in the two cases?

10. A bicyclist riding a wheel 28 in. in diameter notes that a piece of mud flying off the top of his wheel has a range of 12 ft. Find the angular velocity of the wheel and the cyclist's speed per hour.

Ans. $\omega = \frac{144}{7}\sqrt{\frac{3}{7}}$ rad. per second; $\frac{180}{11}\sqrt{\frac{3}{7}}$ mi. per hour.

11. A fountain sends out water horizontally in all directions from a central point a ft. high, with a velocity of c ft. per second. What is the shape of the water surface and the equation of a section made by the XY-plane?

Ans. $x^2 + y^2 = \dfrac{2ac^2}{g}$.

12. A drop of water flies off a grindstone just at the top. The radius of the stone is 2 ft., and it makes 1.5 revolutions per second. Find the velocity of the drop and the point at which it strikes the floor 7 ft. below the axis of revolution. What is the range if the drop flies off at a point such that the angle of elevation is 60°?

Ans. $v = 6\sqrt{16 + \pi^2}$ ft. per sec.; $x = \frac{8}{3}\pi$ ft.; range $= \dfrac{9\sqrt{3}\,\pi^2 + 3\,\pi\sqrt{27\,\pi^2 + 512}}{32}$ ft.

13. Where must the drop of water leave the grindstone in problem 12 in order to fall squarely on top of it? In order to fall tangent to the opposite side?

Ans. $\cos\alpha = \dfrac{8}{9\,\pi^2 - 8}$, $\cos\alpha = \dfrac{16}{9\,\pi^2}$.

14. An emery wheel, 1 ft. in diameter, bursts into small particles when revolving 100 times per second. Which particle will fly the farthest, and what is its initial velocity and range? 45° [illegible]

15. Show that the area of a level plane swept by a gun at a height h above the plane increases proportionally with h, being equal to $A + 2\,h\sqrt{\pi A}$ where A is the area commanded when the gun is at the level of the plane.

16. What must be the elevation α to strike an object 100 ft. above the horizontal plane and 5000 ft. distant, the initial velocity being 1200 ft. per second?

Ans. α is given by the equation $9\cos^2\alpha - 450\sin\alpha\cos\alpha + 25 = 0$.

17. An engine can send a stream of water vertically 125 ft. How much of a vertical wall, distant 200 ft., can the engine wet? *Ans.* 45 ft.

18. Show that the area commanded by a gun on a hillside is an ellipse, with one focus at the gun. Find the area commanded by a gun which has a muzzle velocity of $\frac{1}{2}$ mi. per second, the slope of the hill being 10°.

19. Determine the angle of projection so that the area included between the path and the horizontal plane is a maximum. Find the area.

Ans. $\alpha = 60°$; area $= \dfrac{v^4}{8\,g^2}\sqrt{3}$.

20. Determine the elevation if the range on a given inclined plane is a maximum.

Ans. Direction of projection must bisect the angle between the vertical and the inclined plane.

21. Show that the range R of a projectile fired from a height h above a horizontal plane with velocity v_0 at an angle α is given by

$$2\,v_0^2\,(h + R\tan\alpha) = gR^2\sec^2\alpha.$$

22. A heavy particle descends the outside of a circular arc whose plane is vertical. Prove that when it leaves the circle at some point Q to describe a parabola the circle is the circle of curvature of the parabola at Q.

23. From a train moving at 60 mi. per hour a stone is dropped. The stone starts at a height of 8 ft. above the ground. Through what horizontal distance does the stone pass before it strikes the ground? *Ans.* $44\sqrt{2}$ ft.

24. If the greatest range down an inclined plane be three times the greatest range up, show that the plane is inclined at 30° to the horizon.

25. Two balls are projected from the top of a tower, each with a velocity of 50 ft. per second, the first at an elevation of 30°, and the second at an elevation of 45°. They strike the ground at the same point. Find the height of the tower.

Ans. $(9 - 5\sqrt{3})\dfrac{2500}{g} = 26.5$ ft.

26. The back lines of a tennis court are 78 ft. apart, and the service lines 42 ft. The net is 3 ft. 3 in. high. Find the horizontal velocity of the ball (a) when it is returned from near the ground at one back line so as to graze the net and just strike the other back line; (b) when it is served from a height of 8 feet, grazes the net, and strikes the service line.

Ans. (a) 86.4 ft. per second; (b) 170.64 ft. per second.

27. The Norwegian ski jumping contests in February, 1904, took place on a snow slope at Holmenkollen 186 yards long. The competitors slid down $\frac{2}{3}$ of the slope (which was in this part inclined 15° to the horizon) to a ledge, from which they took off for the jump. Below the ledge the steepness of the slope increased to 24°. Supposing that the lip of the ledge was so curved as to give the jumper an elevation of 6° above the horizon at the take-off, find the speed at the ledge and the length of the leap.

28. A particle is projected with velocity $2\sqrt{ag}$ so that it just clears two walls of equal height a, which are at a distance of $2a$ from each other. Find the time of passing between the walls.

Ans. $2\sqrt{\dfrac{a}{g}}$.

29. A gun is aimed directly at a target suspended to a balloon. Show that the bullet will strike the target if the latter is dropped at the instant the gun is fired.

30. Show that the greatest range on an inclined plane through the point of projection is equal to the distance through which a particle could fall freely during its time of flight.

31. Three bodies are projected simultaneously from the same point in the same horizontal plane, one vertically, another at an elevation of 30°, and the third horizontally. If their velocities be in the ratio $1:1:\sqrt{3}$, show that they are always in a straight line.

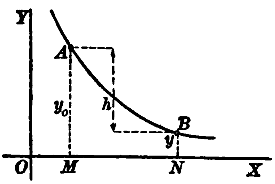
32. A heavy particle is placed very near the vertex of a smooth cycloid having its axis vertical and vertex upwards. Find where the particle runs off the curve and prove that it falls on the base of the cycloid at the distance $\left(\dfrac{\pi}{2} + \sqrt{3}\right)a$ from the center of the base, a being the radius of the generating circle.

66. Constrained motion. On account of its fundamental practical importance, we shall assume the constant field to be the gravitational field. If a particle falls along any smooth path under the action of gravity, we find the acquired velocity as usual by using the energy equation.

If v_0 and v are the velocities at A and B, respectively, the change in kinetic energy is

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2.$$

By Art. 52 the work done equals the force times the total displacement in the direction of the force, that is, equals $mg(y_0 - y)$, since the acting force is weight. Hence we have



$$mg(y_0 - y) = \tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2,$$

from which we find

$$(1) \qquad v^2 = 2g(y_0 - y) + v_0^2.$$

If we set $y_0 - y$ = height fallen = h, then (1) becomes

$$\text{(V)} \qquad \boldsymbol{v^2 = 2\,gh + v_0^2.}$$

The final velocity, therefore, depends upon the initial velocity and the height fallen, and is *independent of the path*. If $v_0 = 0$, that is, if the body falls from rest, then

$$(2) \qquad v^2 = 2\,gh \text{ or } v = \sqrt{2\,gh}.$$

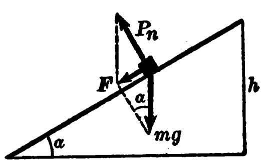
This expression, $\sqrt{2\,gh}$, is called *the speed due to a fall through the height h.*

The time of falling down any smooth curve is, however, not the same for all curves. Examples appear below.

The intrinsic force equations are useful, and are readily written down, since the particle is acted upon by two forces only, weight and the normal pressure P_n of the curve. Hence

$$(3) \quad \begin{cases} m\dfrac{d^2s}{dt^2} = mv\dfrac{dv}{ds} = \text{tangential component of weight,} \\ m\dfrac{v^2}{R} = P_n + \text{normal component of weight.} \end{cases}$$

67. Inclined plane. A particle constrained to move along a straight line oblique to the vertical is said to move along an *inclined plane*. The angle between the inclined plane and a horizontal plane is called the *inclination*.



Smooth plane. The forces acting on the particle are weight and the pressure of the plane. Their resultant F must act along the plane. Taking the positive direction upwards along the inclined plane, we have from the figure

$$(1) \qquad F = -\,mg \sin \alpha,\ P_n = mg \cos \alpha.$$

The particle therefore has an acceleration down the plane equal to $g \sin \alpha$. Consequently formulas (I) and (II) apply by setting

$$(2) \qquad f = -g \sin \alpha.$$

Rough plane. When a particle slides along a rough curve, the tangential component P_t of the pressure of the path is called *sliding friction.* The following laws characterize this force:

1. The direction of friction is opposite to the direction of motion.

2. The magnitude of friction in any problem is directly proportional to the magnitude of the normal pressure.

From the second law we have

$$\text{Friction} = P_t = \mu P_n,$$

where μ is a constant called the *coefficient of friction.*

Hence in the case of an inclined plane,

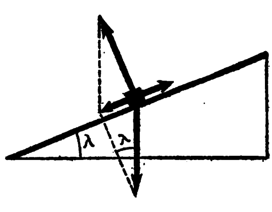
$$\text{Friction} = \mu P_n = \mu \cdot mg \cos \alpha.$$

Since friction is always opposed to the motion, for the resultant acceleration the formulas obviously are

$$(3) \quad \begin{cases} f = -g(\sin \alpha + \mu \cos \alpha), \text{ when the particle is moving } \textit{up} \text{ the plane.} \\ f = -g(\sin \alpha - \mu \cos \alpha), \text{ when the particle is moving } \textit{down} \text{ the plane.} \end{cases}$$

In either case the acceleration is constant and (I) and (II) apply.

The expressions (3) are made more compact by introducing the *angle of friction.* This is defined as an angle λ whose tangent is the coefficient of friction; that is,



$$(4) \qquad \mu = \tan \lambda.$$

To see the significance of λ, consider a particle at rest upon a rough plane. If now the plane be tipped so that the inclination increases, the particle will eventually move. The inclination when the particle is *on the point of moving* equals the angle of friction. For at this instant the resistance of the friction equals the component of weight down the plane; that is,

$$(5) \quad \begin{cases} mg \sin \alpha = \mu \, mg \cos \alpha; \text{ or} \\ \tan \alpha = \mu. \quad \therefore \alpha = \lambda. \end{cases}$$

Substituting $\mu = \tan \lambda$ in (3) gives

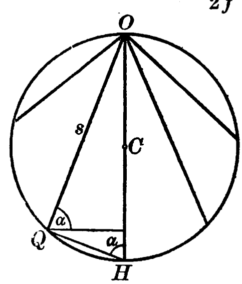
$$(6) \quad \begin{cases} f = -g \sec \lambda \sin (\alpha + \lambda), \text{ when the particle is moving } up \text{ the plane.} \\ f = -g \sec \lambda \sin (\alpha - \lambda), \text{ when the particle is moving } down \text{ the plane.} \end{cases}$$

The numerical value of the coefficient of friction depends upon the character of the substances in contact, and is determined by experiment. The value of μ is slightly less when the particle is in motion than when it is at rest but on the point of moving.

ILLUSTRATIVE EXAMPLES

1. A particle is projected up an inclined plane with the velocity v_0. How far will it ascend?

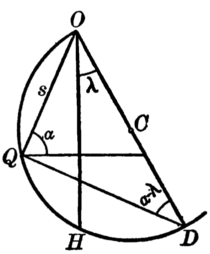
Solution. Since the motion is resisted by a constant force, we have, by the energy equation, $\frac{1}{2} mv_0^2 = mfs$, where s is the distance required. $\therefore s = \frac{v_0^2}{2f}$, where $f = g \sin \alpha$ or $g (\sin \alpha + \mu \cos \alpha)$ according as the plane is smooth or rough.



2. A heavy particle starts from rest at the top of an inclined plane. Required the locus of the foot of the plane if the time of descent is constant and independent of the inclination.

Solution. Smooth Plane. By (I),

$$s = \tfrac{1}{2} g \sin \alpha \, t^2 = \tfrac{1}{2} gt^2 \sin \alpha.$$



In this equation s and α are variable. If O is the starting point, it is easily seen that the required locus is a circle having O as the highest point and a diameter $OH = \frac{1}{2} gt^2$. For $s = OQ = OH \sin OHQ = OH \sin \alpha = \frac{1}{2} gt^2 \sin \alpha$. The result may also be stated thus: *The time of descent along all smooth chords of a vertical circle drawn from its highest point is the same.*

Rough Plane. In this case from (I) and (6) Art. 67,

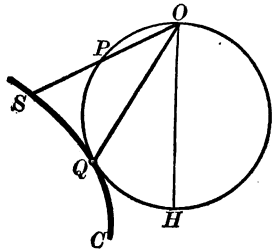
$$s = \tfrac{1}{2} g \sec \lambda \sin (\alpha - \lambda) t^2 = \tfrac{1}{2} gt^2 \sec \lambda \sin (\alpha - \lambda).$$

As before, let the vertical line $OH = \frac{1}{2} gt^2$. Lay off the angle $HOD = \lambda$, and construct the arc of a circle whose center lies on OD passing through O and H.

This arc is the required locus when the particle descends to the left of OH. For $OD = OH \sec\lambda = \frac{1}{2}gt^2 \sec\lambda$. Also

$$s = OQ = OD \cdot \sin ODQ = OD \sin(\alpha - \lambda) = \tfrac{1}{2}gt^2 \sec\lambda \sin(\alpha - \lambda).$$

If the particle moves to the right, the locus is a corresponding equal arc.



Line of quickest descent from a point to a given curve. Given a curve C and a point O. If a vertical circle tangent to C and having O for the upper extremity of a vertical diameter is drawn, then OQ is a line of quickest descent along all smooth straight lines from O to C. For along OS the time is greater than down OQ, since the time along OP equals that along OQ.

PROBLEMS

1. A smooth plane has an inclination of 30°. With what velocity must a particle be projected up the plane, the length of which is 48.3 ft., that it may just reach the top? What must be the initial velocity to reach the top in 1 sec.? ($g = 32.2$.)
Ans. $v_0 = 16.1\sqrt{6}$ ft. per second; $v_0 = 56.35$ ft. per second.

2. A particle falls from rest down a given inclined plane. Compare the times of descending the first and second halves. *Ans.* $1 : \sqrt{2} - 1$.

3. Along what chord of a circle must a particle fall in order to gain half the velocity which it acquires in falling through the vertical diameter?
Ans. Chord inclined 60° to vertical.

4. A particle is projected up a smooth plane which has an inclination of 3 in 5 with a velocity of 40 ft. per second. In what time will it come to rest and how far up the plane will it go? *Ans.* $\frac{25}{12}$ sec.; $\frac{125}{3}$ ft.

5. A weight of 10 lb. falls vertically and draws a 15-lb. weight up a smooth plane having an inclination of 30°. What is the acceleration, pull on the string, and space fallen through in 10 sec.? *Ans.* $f = \frac{1}{10}g$: $T = 9$ lb.; space $= 5g$.

6. A railway train is running at the rate of 30 mi. per hour up a grade of 1 in 50. The coupling breaks, cutting loose part of the train. How long will the detached part continue up the grade, friction being neglected? What is its position with respect to the point where the break occurred and what is the direction and velocity of its motion after 2 minutes? *Ans.* $\frac{275}{4}$ sec.; $\frac{164}{5}$ ft. per second downhill.

7. With what velocity must a particle be projected down an inclined plane of length l so that the time of descent shall be the same as that for a free fall through the height of the plane? *Ans.* $v_0 = g\dfrac{l - h\sin\alpha}{\sqrt{2gh}}$.

8. What is the value of g if a given mass descending vertically draws an equal mass up an incline of 30° a distance of 25 ft. in 2.5 sec.? *Ans.* $g = 32$.

9. Find the position of a point on the circumference of a circle such that the time of descent from it to the center shall be the same as the time of descent from it to the lowest point of the circle.

10. A particle slides down a smooth plane inclined at an angle of 45° and then drops into free space. (1) If the particle has a velocity of 20 ft. per second when it leaves the plane, find the equation of its path. (2) Where will it strike a horizontal plane 100 ft. below? (3) What are the axial components of the velocity when it strikes this plane? (4) At what point will it cut a vertical line 45 ft. distant from the plane?

Ans. (3) $v_x = 10\sqrt{2}$, $v_y = -10\sqrt{66}$; (4) 207 ft. below.

11. A train of 100 T. starts on an up grade of 1 in 50 with a speed of 20 mi. per hour. It is stopped by gravity and the resistance of the brakes in 4 seconds. (1) What is the coefficient μ of the resisting forces? (2) What is the velocity at the end of one second?

Ans. (1) $\mu = .21$.

12. Show that the times of descent down all radii of curvature of the cycloid are equal.

Ans. $T = \sqrt{\frac{8a}{g}}$.

13. A heavy particle starts from rest at the top of an inclined plane. Required the locus of the foot of the plane if the speed at the foot is constant and independent of the inclination.

Ans. A straight line.

14. Give a construction for finding the line of quickest descent from a fixed point to a circle in the same vertical plane.

15. A body begins to slide down an inclined plane from the top, and at the same instant another body is projected upwards from the foot of the plane with such a velocity that the bodies meet in the middle of the plane. Find the velocity of projection and determine the velocity of each body when they meet.

Ans. $\sqrt{gh}$; 0, and $\sqrt{gh}$, where h = vertical height of the plane.

16. A parabola is placed with its axis vertical and vertex upwards. Find the chord of quickest descent from the focus to the curve.

Ans. The chord makes an angle of 60° with the vertical.

17. Through what chord of a vertical circle drawn from the bottom of the vertical diameter must a body descend so as to acquire a velocity equal to $\frac{1}{n}$th part of the velocity acquired in falling down the vertical diameter?

Ans. If θ denote the angle between the required chord and the vertical diameter, $\cos\theta = \frac{1}{n}$.

18. A heavy particle is projected up a smooth inclined plane with a velocity of 36 ft. per second. The inclination of the plane is 30° and its vertical height is 20 ft. It projects into space at the top of the plane. Determine (a) the time in ascending the plane, (b) the velocity at the top, (c) the equations of the free path.

19. A body is projected up a rough inclined plane with the velocity which would be acquired in falling freely through 12 ft., and just reaches the top of the plane. If the inclination of the plane is 60° and the angle of friction is 30°, find the height of the plane.

Ans. 9 ft.

20. A body is projected up a rough inclined plane with the velocity $2g$. If the inclination of the plane is 30° and the angle of friction is 15°, find the distance along the plane which the body will move.

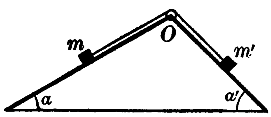
Ans. $g(\sqrt{3} + 1)$ ft.

21. A body is projected up a rough inclined plane, inclination α, and the angle of friction is λ. If m be the time of ascending and n the time of descending, show that

$$\left(\frac{m}{n}\right)^2 = \frac{\sin(\alpha - \lambda)}{\sin(\alpha + \lambda)}.$$

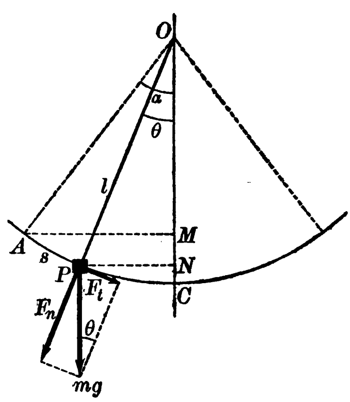
22. A particle descends an inclined plane. If the upper portion be smooth and the lower rough, coefficient of friction being μ, and if the smooth length be to the rough length as $p : q$, show that the particle will just come to rest at the foot of the plane if $\mu = \frac{p+q}{q}\tan\alpha$, where α is the inclination of the plane.

23. Two rough planes, coefficient of friction $= \mu$, inclined respectively at angles α and α' to the horizon, are placed back to back as shown in the figure. Two masses, m and m', are placed upon them, being connected by a string passing over a pulley at O. (a) If $m = m'$, find the limit of the difference $\alpha - \alpha'$, if the acceleration is zero. (b) If $\alpha = \alpha'$, find the limit of the difference $m - m'$ if the acceleration is zero.



24. Particles are sliding down a number of wires which meet in a point, all having started from rest simultaneously at this point. Prove that at any instant their velocities are in the same ratio as the distances they have traversed.

68. Motion on a smooth circle. Simple pendulum. A heavy particle is suspended from a fixed point O by an inextensible thread, and swings under the action of its weight in a vertical circle. Discuss the motion, neglecting the weight of the thread and the influence of the atmosphere.



Let l = length of thread,

A be the initial position ($t = 0$) *at rest*,

P be the position after time t,

s = arc AP.

Then, from the figure,

$$(1) \qquad s = l(\alpha - \theta).$$

Resolving the impressed force mg into tangential and normal components, we have the intrinsic force equations:

$$(2) \qquad m\frac{d^2s}{dt^2} = F_t = mg\sin\theta,$$

$$(3) \qquad \frac{mv^2}{l} = \begin{Bmatrix}\text{normal}\\ \text{impressed}\\ \text{force}\end{Bmatrix} + \begin{Bmatrix}\text{normal}\\ \text{pressure}\end{Bmatrix} = -mg\cos\theta - T,$$

where $T =$ *tension in the thread*, that is, the pull of the particle on the thread.

The work done by weight when the particle descends from A to P is

$$mg \cdot MN = mgl(\cos\theta - \cos\alpha).$$

Hence, the energy equation gives

$$\tfrac{1}{2}v^2 = gl(\cos\theta - \cos\alpha). \tag{4}$$

The tension in the thread at any instant is given by eliminating v^2 from (3) and (4), that is,

$$T = mg(2\cos\alpha - 3\cos\theta). \tag{5}$$

The circumstances of the motion are known if θ is known as a function of the time.

From (2)
$$\frac{d^2s}{dt^2} = g\sin\theta,$$

and from (1)
$$\frac{d^2s}{dt^2} = -l\frac{d^2\theta}{dt^2}.$$

Hence, the differential equation for the determination of θ is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{6}$$

Small amplitudes. The angle α, which is the maximum value of θ, is called the amplitude of the motion. A simple solution of (6) results if α and consequently θ are small. A close approximation is then found by assuming $\sin\theta = \theta$. Equation (6) then becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{7}$$

The general solution of (7) is (see 71, Chap. XIV)

$$\theta = c_1\cos\sqrt{\frac{g}{l}}\,t + c_2\sin\sqrt{\frac{g}{l}}\,t. \tag{8}$$

The constants of integration c_1 and c_2 are to be determined by the initial conditions.

When $t = 0$, $\theta = \alpha$, $\dfrac{d\theta}{dt} = 0$.

Hence, $c_1 = \alpha$, and $c_2 = 0$, and (8) becomes

$$\theta = \alpha\cos\sqrt{\frac{g}{l}}\,t. \tag{9}$$

From (9), the motion is periodic, the time for a complete oscillation being

(VI) $$P = 2\pi\sqrt{\frac{l}{g}}.$$

For any given position on the earth's surface g is a constant. Hence (VI) shows that the period depends only upon the *length* of the pendulum. It must be remembered that this formula holds only when the amplitude is so small that the substitution of the angle for its sine is within the limit of error.

Any amplitude. To obtain an expression for the period which is valid for any value of the amplitude, we proceed as follows. Multiplying equation (6) by $\frac{d\theta}{dt}$, we may integrate each term, obtaining

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{g}{l}\cos\theta = c.$$

Since $\frac{d\theta}{dt} = 0$ when $\theta = \alpha$, $c = -\frac{g}{l}\cos\alpha$, and we may write the equation in the form

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cos\theta - \cos\alpha).$$

It is convenient to set in this equation $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$, $\cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha$. Extracting the square root of both sides, and taking the negative sign with the radical, since θ decreases as t increases, we obtain

$$\frac{d\theta}{dt} = -2\sqrt{\frac{g}{l}}\sqrt{\sin^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\theta}.$$

The time required for the particle to move from the highest point ($\theta = \alpha$) to the lowest ($\theta = 0$), which is one fourth the period, is obtained by integration, namely,

$$\frac{1}{4}P = -\frac{1}{2}\sqrt{\frac{l}{g}}\int_\alpha^0 \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta}},$$

or

(10) $$P = 2\sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta}}.$$

The integral in (10) is transformed into a known form by setting

$$\sin \tfrac{1}{2}\alpha = k, \quad \sin \tfrac{1}{2}\theta = k \sin \phi,$$

where ϕ is the new variable. Differentiating, we get

$$\tfrac{1}{2} \cos \tfrac{1}{2} \theta d\theta = k \cos \phi d\phi.$$

$$\therefore\ d\theta = \frac{2k \cos \phi d\phi}{\cos \frac{1}{2}\theta} = \frac{2k \cos \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Since if $\theta = 0$, $\phi = 0$; $\theta = \alpha$, $\phi = \frac{\pi}{2}$, we obtain for (10) the form

$$P = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \qquad (k = \sin \tfrac{1}{2}\alpha) \tag{11}$$

The integral involved here is known as the *complete elliptic integral of the first species* and is denoted by K. The integral evidently is not independent of α. Indicating this dependence by a subscript, then

$$P_\alpha = 4K\sqrt{\frac{l}{g}}. \tag{12}$$

Values of K may be found tabulated (p. 117) in Peirce's A Short Table of Integrals (Ginn and Company).

A few values of K are set down here in order to make clear the dependence of the exact period P_α upon the amplitudes. Comparing (12) and (VI), we have

$$\frac{P}{P_\alpha} = \frac{\frac{1}{2}\pi}{K}, \quad \text{or,} \quad \frac{P_\alpha - P}{P_\alpha} = \frac{K - \frac{1}{2}\pi}{K}. \tag{13}$$

α	K
0°	$\frac{1}{2}\pi$
2°	1.5709
4°	1.5713
6°	1.5719
8°	1.5727
10°	1.5738

Remembering that P is an approximate value of the period for small amplitudes, (13) gives the percentage of error. For example, when $\alpha = 10°$ this is

$$\frac{1.5738 - 1.5708}{1.5738} = \frac{30}{15738} = \text{about } \tfrac{1}{5} \text{ of } 1\ \%.$$

An approximate form of P_α closer than the value of P is found by writing (11) in the form

$$P_\alpha = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi. \tag{14}$$

Using the Binomial Theorem (1, Chap. XIV) then

$$(1-k^2\sin^2\phi)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}k^2\sin^2\phi - \text{etc.}$$

Substituting and integrating gives

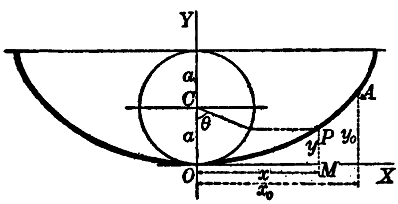
$$P_a = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\pi}{8}k^2 + \text{terms in } k^4, \text{ etc.}\right).$$

Since $k = \sin\frac{1}{2}\alpha = \frac{1}{2}\alpha - \frac{\alpha^3}{48} + \cdots$, if we neglect all powers of α higher than the second, we obtain

$$(15) \qquad P_a = 2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\alpha^2}{16}\right) = P\left(1+\frac{\alpha^2}{16}\right).$$

This formula is useful in practice. P_a may be observed and the value of P then determined.

69. Motion on a smooth cycloid. The discussion of Art. 68 demonstrates that the period of oscillation of a heavy particle on a vertical circle depends upon the amplitude. The question arises: *Does any curve exist upon which the period of the oscillations is the same for all amplitudes?* The cycloid possesses this property, as will now be shown.



The equation of the cycloid of the figure is (Calculus, p. 281),

$$(1) \qquad x = a \text{ arc vers } \frac{y}{a} + \sqrt{2ay - y^2}.$$

We shall show that the time of descent to the lowest point O from rest at A is the same for all positions of A.

The velocity at P is found by the energy equation to be

$$(2) \qquad v^2 = 2g(y_0 - y) \text{ or } \frac{ds}{dt} = -\sqrt{2g}\sqrt{y_0 - y},$$

if s is measured from the lowest point. Hence the time from A to O is given by

$$(3) \qquad t = -\frac{1}{\sqrt{2g}}\int_{y_0}^{0}\frac{ds}{\sqrt{y_0 - y}}.$$

We must now express s in terms of y. To do this, differentiate (1), which gives

$$\frac{dx}{dy}=\frac{2a-y}{\sqrt{2ay-y^2}}=\sqrt{\frac{2a-y}{y}}.$$

(4) $$\therefore\ ds=\left(1+\left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}}dy=\frac{\sqrt{2a}}{\sqrt{y}}\,dy.$$

Equation (3) now becomes, by substitution,

(5) $$t=-\sqrt{\frac{a}{g}}\int_{y_0}^{0}\frac{dy}{\sqrt{y_0y-y^2}}=\pi\sqrt{\frac{a}{g}}.$$

This result depends only upon the height of the cycloid and is therefore the same for all positions of A.

A second demonstration is important. Differentiating (2) with respect to s, we obtain

(6) $$2v\frac{dv}{ds}=-2g\frac{dy}{ds}.$$

Hence by the intrinsic force equations,

$$f_t=-g\frac{dy}{ds}.$$

Let s be measured from the position of equilibrium O. Then we have, from (4),

(7) $$s=OP=\int_0^y\frac{\sqrt{2a}}{\sqrt{y}}\,dy=2\sqrt{2ay}.\quad\therefore\ s^2=8ay.$$

Differentiating this, we obtain $\dfrac{dy}{ds}=\dfrac{s}{4a}$.

Hence from (6), we have,

(8) $$\frac{d^2s}{dt^2}+\frac{g}{4a}s=0.$$

The general solution is (see 71, Chap. XIV)

$$s=c_1\cos\frac{1}{2}\sqrt{\frac{g}{a}}\,t+c_2\sin\frac{1}{2}\sqrt{\frac{g}{a}}\,t.$$

The motion is therefore a *harmonic curvilinear oscillation* about O with the period $4\pi\sqrt{\dfrac{a}{g}}$.

Conversely, it may be shown that the cycloid is the path of a heavy particle which performs oscillations of equal periods on a smooth curve. The oscillations are said to be *tautochronous* (equal times) and the cycloid is the only tautochronous curve when the impressed force is gravity.

70. Seconds pendulum. If the period of a simple pendulum for small amplitudes at a given locality on the earth's surface is two seconds, the pendulum is called a seconds pendulum for that place. Since g varies along the earth's surface, the length of such a pendulum is necessarily variable, but is about 39.11 inches.

For points without the earth, gravity varies inversely as the square of the distance from the earth's center. Hence if g' is the intensity of gravity at a height x above the earth's surface, we shall have

$$(1) \qquad \frac{g'}{g} = \frac{R^2}{(x+R)^2}, \text{ or } g' = \frac{gR^2}{(R+x)^2},$$

where R = radius of the earth. Hence if P_x denotes the period at the height x, then

$$(2) \qquad P_x = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g}}\left(\frac{R+x}{R}\right) = P\left(1+\frac{x}{R}\right).$$

This formula gives the relation between the periods of the same pendulum at the earth's surface and at any height x.

For points in the interior of the earth, gravity varies directly as the distance from the earth's center. Hence, if g' is the intensity of gravity at the distance y below the surface of the earth, we shall have

$$(3) \qquad g' = \frac{R-y}{R}g.$$

PROBLEMS

1. If 39.11 in. be taken as the length of a seconds pendulum, that is, a pendulum which makes one full swing in one second, what is the length of the pendulum which vibrates 25 times per minute? whose period is $\frac{3}{4}\pi$?

2. In what time will a pendulum vibrate whose length is double that of a seconds pendulum?

3. A pendulum which beats seconds in London requires to be shortened by one thousandth of its length if it is to keep time in New York. Compare the values of gravity at London and New York.

4. What is the length of the seconds pendulum where $g = 980$ cm. per second per second?

5. Prove that a seconds pendulum if brought to the height x mi. will lose about $22x$ sec. per day, the radius of the earth being taken as 4000 mi.

6. Find the height at which a pendulum 60 cm. long will beat seconds, taking the radius of the earth as 4000 mi.

7. The mass of a pendulum bob is 100 gm., and the string is 1 m. long. What is the kinetic energy when the string makes an angle of 30° with the vertical if the bob is dropped from a horizontal position? *Ans.* $8.5(10)^6$ ergs.

8. A body whose mass is 1 lb. is suspended from a fixed point by a string 12 ft. long. The string is swung to a position 60° from the vertical and the body released. Determine the velocity when the body is in its lowest position; also when 2 ft. above its lowest position.

9. A clock gains 3 min. per day. How much should the bob be screwed up or down? *Ans.* Down by $\frac{1}{240}$ of its length.

10. Find the time, to four decimal places, of a half vibration of a pendulum 1 m. long at a place where $g = 980.8$ cm. per second per second.

11. The seconds pendulum loses 12 sec. per day when carried to a mountain top. How high is the mountain? *Ans.* About 2900 ft.

12. Find the time of vibration of a seconds pendulum placed in a mine 1.5 mi. deep.

13. Compare g at two places where the rates of the same pendulum differ by 5 vibrations per hour.

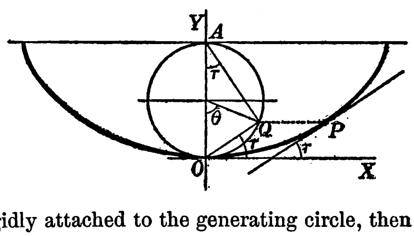
14. A string r ft. long has a mass m attached to the lower end and acts as a simple pendulum. Find the point in the arc where the pull on the string is the same as where the pendulum is at rest.

Ans. $y = \frac{2}{3}h$, where h is the height from which the pendulum has fallen.

15. A heavy particle oscillates in a complete cycloid from cusp to cusp. Prove the following properties:

(1) The velocity at any point P equals the velocity at the lowest point resolved along the tangent at P.

(2) The time of description of any arc OP is proportional to the angle $OAQ = \tau$. In fact $\tau = \sqrt{\frac{g}{4a}} \cdot t$.



(3) If the particle is regarded as rigidly attached to the generating circle, then the center of the latter moves with constant speed.

(4) The pressure on the curve equals twice the normal component of weight.

(5) The acceleration of the particle is equal to g and is directed towards the center of the generating circle.

Hint. Use the equations $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, the properties indicated in the figure, and the relation $R = 2\,AQ$ (R = radius of curvature).

16. In the motion of a particle down a cycloid, prove that the vertical velocity is greatest when it has completed half its vertical descent.

17. When a particle falls from the highest to the lowest point of a cycloid, show that when it has described half of the path, $\frac{2}{3}$ of the time has elapsed and it has passed through $\frac{3}{4}$ of the vertical distance.

18. The bob of a pendulum which is hung close to the face of a vertical cliff is attracted by the cliff with a force which would produce an acceleration f in the bob. Show that the time of a complete oscillation is $2\pi\sqrt[4]{\frac{l^2}{g^2+f^2}}$, where l is the length of the pendulum, and find the center of the arc described by the bob.

19. A railway train is moving uniformly along a curve at the rate of 60 mi. per hour, and in one of the carriages a pendulum, which would ordinarily beat seconds, is observed to oscillate 121 times in 2 min. Show that the radius of the curve is very nearly a quarter of a mile.

CHAPTER VII

CENTRAL FORCES

71. Central field of force. A field of force is called a central field if the direction of the acceleration at every point of the field passes through a fixed point called the center of force. The acceleration may be directed towards the center of force or from it; that is, the force may be attractive or repulsive. The magnitude of the acceleration may vary according to any given law. In the general case it may depend upon the direction and the distance from the center and also upon the time. In many practical problems, however, the magnitude of the acceleration depends only upon the distance from the center, and we shall confine our attention to this case. The term *central force*, therefore, as used here applies to central fields in which the magnitude of the force depends only upon the distance from the center of force. Let the origin of coördinates O be taken at the center of force and P be the position of a material particle subject to the force of the field. Denote the distance OP, which is called the radius vector, by ρ. Then the magnitude of the acceleration exerted upon the particle at P is a function of ρ,

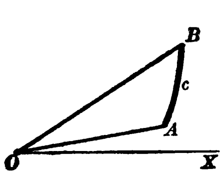
$$f = \phi(\rho),$$

and its direction is along the line OP, in the positive sense if the force is repulsive, and in the negative sense if the force is attractive.

The path of the particle is called the *orbit*. If the initial velocity is along the line OP, the orbit is a straight line, since the force has no component tending to draw the particle out of the line. If the direction of the initial velocity is oblique to the line OP, the orbit is a plane curve, since the force has no component tending to draw the particle out of the plane determined by the direction of the initial velocity and the line OP. The orbit is, then, necessarily a plane curve.

Since the acceleration is directed towards the concave side of the path we may draw the conclusions: (1) if the orbit is concave towards the center of force, then the force is attractive; (2) if the force is attractive, the orbit is concave towards the center of force.

72. Areal velocity. When a point moves along a curve, its radius vector is said to generate area. Thus, if the moving point describes the curve c in the figure from the point A to the point B, its radius vector generates the area AOB. The time-rate at which the radius vector generates area, that is, the derivative of the area with respect to the time, is called the areal velocity of the moving point. By Calculus, p. 377, the differential of area in polar coördinates is



$$dA = \tfrac{1}{2}\rho^2 d\theta.$$

Hence the areal velocity is given in terms of the angular velocity by the relation

$$\frac{dA}{dt} = \frac{1}{2}\rho^2\frac{d\theta}{dt} = \frac{1}{2}\rho^2\omega. \tag{1}$$

To derive an expression for the areal velocity in terms of the rectangular components of velocity, we proceed as follows. The rectangular and polar coördinates of a point are connected by the relations

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta. \end{cases} \tag{2}$$

Differentiating with respect to t,

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{d\rho}{dt}\cos\theta - \rho\sin\theta\dfrac{d\theta}{dt}, \\[2ex] \dfrac{dy}{dt} = \dfrac{d\rho}{dt}\sin\theta + \rho\cos\theta\dfrac{d\theta}{dt}. \end{cases} \tag{3}$$

Multiplying the first of equations (3) by $\sin\theta$, the second by $\cos\theta$, and subtracting, we get

$$\cos\theta\frac{dy}{dt} - \sin\theta\frac{dx}{dt} = \rho\frac{d\theta}{dt}.$$

Multiplying by $\frac{1}{2}\rho$ and taking account of (2), we have finally

$$(4) \qquad \frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \frac{1}{2}(xv_y - yv_x).$$

This equation expresses the areal velocity in terms of the rectangular coördinates and their derivatives with respect to the time.

73. Law of areas for central forces. Since a central force acts in the direction of the radius vector, its component perpendicular to the radius vector is zero. Hence, using polar coördinates, the differential equations of motion, Art. 51, of a particle of mass m are

$$(1) \qquad \begin{cases} m\left[\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2\right] = F_\rho = mf, \\ m\dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2\dfrac{d\theta}{dt}\right) = F_\theta = 0, \end{cases}$$

where f denotes the acceleration. The second equation gives by integration

$$(2) \qquad \rho^2\frac{d\theta}{dt} = h,$$

where h is a constant.

Comparing (2) with (1), Art. 72, we have

$$(3) \qquad 2\frac{dA}{dt} = h.$$

Hence the

THEOREM.* *In the motion of a particle subject to any central force the areal velocity is constant.*

The constant h, which is twice the areal velocity, is called the *constant of areas.*

Integrating equation (3) between the limits $t = t_1$ and $t = t_2$, we have

$$A = \frac{h}{2}(t_2 - t_1);$$

that is, the area generated in any interval of time $t_2 - t_1$ is proportional to the length of the interval. In other words, the radius vector sweeps over equal areas in equal intervals of time.

* Since the law of variation of the force, that is, the function F, does not enter in the derivation of (3), it is evident that this theorem and the two following theorems in this article hold also for the general central field of force, that is, when F may depend upon ρ, θ, and t.

From (2),

(4) $$\frac{d\theta}{dt} = \frac{h}{\rho^2}.$$

Hence the

THEOREM. *In the motion of a particle subject to any central force, the angular velocity is inversely proportional to the square of the distance from the center of force.*

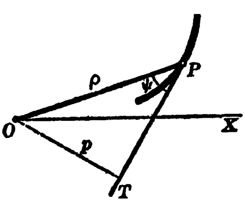
The speed of the particle is

$$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt},$$

or, substituting the value of $\frac{d\theta}{dt}$ from (4),

(5) $$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{h}{\rho^2}.$$

Let p denote the perpendicular distance from the origin to the tangent to the orbit. Then from the figure,



$$\frac{p}{\rho} = \sin\psi = \frac{\rho}{\frac{ds}{d\theta}} \quad \text{(Calculus, p. 98).}$$

Hence $\frac{ds}{d\theta} = \frac{\rho^2}{p}$.

Substituting this value in (5), we obtain

$$\frac{ds}{dt} = \frac{h}{p}.$$

Hence the

THEOREM. *In the motion of a particle subject to any central force the speed is inversely proportional to the perpendicular distance from the center of force to the tangent to the orbit.*

74. Converse of the theorem of areas. Suppose a particle moves in a plane in such a manner that its radius vector generates equal areas in equal intervals of time, that is, its areal velocity is constant. Then

$$\frac{dA}{dt} = h,$$

and

$$\rho^2\frac{d\theta}{dt} = 2\,h.$$

Differentiating with respect to t,

(1) $$\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right) = 0.$$

But the first member of (1) is ρ times the component of acceleration perpendicular to the radius vector ((1), Art. 73). Therefore the total acceleration is in the direction of the radius vector, that is, the direction of the acceleration passes always through the fixed point which is the origin of coördinates. Hence the

THEOREM. *If a particle moves in a plane in such a manner that its areal velocity with respect to a fixed point O in the plane is constant, then the particle is subject to the action of a central field of force with the point O as center.*

75. The energy equation. The energy equation in polar coördinates is, Art. 62,

$$W = \frac{m}{2}(v^2 - v_0^2) = \int_{\rho_0, \theta_0}^{\rho, \theta} F_\rho d\rho + F_\theta \rho d\theta.$$

For central forces $F_\theta = 0$, and, under the assumption that the magnitude of the force depends only on the distance, we may write,

$$F_\rho = F(\rho),$$

and the energy equation becomes

(1) $$\frac{m}{2}(v^2 - v_0^2) = \int_{\rho_0}^{\rho} F(\rho)d\rho.$$

Let the function $U(\rho)$ be defined by the equation,

$$-\frac{dU}{d\rho} = +F,$$

Then $$-U = \int_{\rho_0}^{\rho} Fd\rho.$$

Equation (1) may now be written

(2) $$\frac{m}{2}v^2 + U(\rho) = \frac{m}{2}v_0^2 + U(\rho_0).$$

The function U is called the potential function * and the value of U for any given value of ρ is called the potential energy of the

* The subject of the potential function and potential energy is treated in Chapter X.

moving particle. The kinetic energy is $\frac{1}{2} mv^2$, and since the second member of equation (2) is constant, we have the

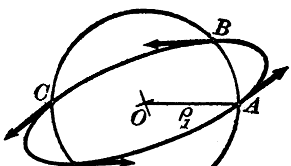
THEOREM. *The sum of the kinetic energy and potential energy of a particle free to move in a central field of force is constant.*

Equation (2) is called the *vis viva* equation. The constant value of the second member depends upon the problem, that is, upon the initial conditions. When this has been determined, equation (2) defines v^2 (the square of the speed) in terms of U which depends upon ρ alone. In physical problems U is a single-valued function of ρ, and from (2) the speed is uniquely determined if the distance from the origin is known. Hence the

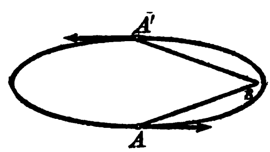
THEOREM. *In any given problem of the motion of a particle in a central field of force, the speed depends only upon the distance from the center of force.*

As examples of the application of the preceding theorem consider the following:

(1) One end of an elastic string is fixed at the point O. To the other end is attached a particle which moves under the action of the elasticity of the string (neglecting the friction of the air). Motion is begun by projecting the particle from a given point (ρ_0, θ_0) with a given speed v_0. These facts determine the constant value c of the second member of equation (2), namely $c = \frac{mv_0^2}{2} + U(\rho_0)$. Then if a particle passes through a point A in *any* direction we can determine its speed v_1 if we know the distance $OA = \rho_1$. Furthermore, if the particle crosses the circle about O with radius OA at *any* point as B, C, or D in *any* direction, its speed is the same as the speed at A, namely v_1.



(2) If a small planet or comet revolves in an ellipse about the sun under the sun's attraction, which is inversely proportional to the square of the distance from its center, the speed of the planet at the distance SA when approaching the sun is the same as the speed at the distance $SA' = SA$ when receding from the sun.



76. Circular orbits. The problem of determining the motion of a particle in a central field of force demands the solution of the differential equations of motion;

$$
(1)\qquad \begin{cases} m\left[\dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2\right] = F(\rho) = mf(\rho), \\ m\,\dfrac{1}{\rho}\left[\dfrac{d}{dt}\left(\rho^2\dfrac{d\theta}{dt}\right)\right] = 0. \end{cases}
$$

Integrating the second equation we have,

$$\rho^2 \frac{d\theta}{dt} = h, \tag{2}$$

where h is a constant of integration. Let us impose the condition that the particle shall move around the center of force in a circle of radius a. Then $\rho = a$ and from (2), we get for the angular velocity,

$$\frac{d\theta}{dt} = \frac{h}{a^2}. \tag{3}$$

Substituting this value in the first of the equations (1) gives, since $\frac{d^2\rho}{dt^2} = 0$,

$$-\frac{h^2}{a^3} = f(a),$$

whence,

$$h = \sqrt{-a^3 f(a)}. \tag{4}$$

The value of h thus determined is real if $f(a)$ is negative, that is, if the force is attractive.

By integration of (3) we obtain

$$\theta = \frac{h}{a^2} t + c = \sqrt{\frac{-f(a)}{a}}\, t + c. \tag{5}$$

Hence a particular solution of the differential equations (1) is

$$\begin{cases} \rho = a, \\ \theta = \sqrt{\dfrac{-f(a)}{a}}\, t + c. \end{cases} \tag{6}$$

The constant c is determined if we know the position of the particle at any given time. For example, if $\theta = \theta_0$ when $t = 0$, we find $c = \theta_0$.

THEOREM. *In an attractive central field of force a particle may move around the center of force in a circle of given radius a. The angular velocity ω is constant and equal to $\sqrt{\frac{-f(a)}{a}}$. The speed is equal to $a\omega$.*

ILLUSTRATIVE EXAMPLE. If the acceleration in a central field is towards the center of force and proportional to the distance, the time of describing a circular orbit is independent of the radius.

Solution. The acceleration is given by

$$f = -k^2 r,$$

where k is a constant.

Hence the angular velocity in a circular orbit of radius a is

$$\omega = \sqrt{\frac{k^2 a}{a}} = k.$$

The constant angular velocity $\omega = k$ is the angle (measured in radians) turned through by the radius vector in one unit of time. Hence the time required to describe the complete circle is $\frac{2\pi}{k}$ units. The time required for the particle to move completely around its orbit is called the period. The period is constant and equal to $\frac{2\pi}{k}$.

77. Differential equation of the orbit. To find the equation of the orbit of a particle in a central field of force we may integrate the differential equations of motion and then eliminate t. Another method which leads to important results is to first eliminate t, obtaining a differential equation involving ρ and θ (or x and y), the integration of which furnishes the equation of the orbit. In this process it is convenient to use, instead of the radius vector ρ, its reciprocal $u = \frac{1}{\rho}$. Then

$$(1) \qquad \frac{d\rho}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}.$$

But from (2), Art. 73,

$$(2) \qquad \frac{1}{u^2}\frac{d\theta}{dt} = h.$$

Hence
$$\frac{d\rho}{dt} = -h\frac{du}{d\theta}.$$

Differentiating with respect to t and taking account of (2),

$$(3) \qquad \frac{d^2\rho}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting the value of $\frac{d^2\rho}{dt^2}$ from (3) and the value of $\frac{d\theta}{dt}$ from (2) in the first of the differential equations of motion [(1), Art. 76], we have

$$m\left[-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4\right] = F\left(\frac{1}{u}\right).$$

Or

$$(4) \qquad h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{F}{m}.$$

Equation (4) is important for the solution of two problems:

(1) Given the orbit, to determine the law of central force.

(2) Given the law of central force, to determine the orbit.

The solution of the first problem is usually quite simple. From the equation of the orbit we know u in terms of θ, $u = \phi(\theta)$. Hence we may compute the first member of (4),

$$(5) \qquad h^2\phi^2(\theta)[\phi''(\theta) + \phi(\theta)] = -\frac{F}{m}.$$

Under the assumption that the law of force shall depend only on the distance, we may find θ from the equation of the orbit and substitute its value in (5), obtaining F in terms of ρ and the constant of areas h. If h is known, the force is uniquely determined.

One exceptional case must be mentioned. The process fails if the orbit is a circle about the center of force. If $u = \frac{1}{a}$ and h is given, equation (4) furnishes a value for the intensity of the force at the distance a from the center, but does not prescribe a law governing the intensity of the force at any other distance. It was shown in the preceding article that a circular orbit about the center is possible for any attractive central force.

ILLUSTRATIVE EXAMPLE. Determine the law of central force if the orbit is the circle $\rho = 2a\cos\theta$.

Solution. $u = \dfrac{1}{\rho} = \dfrac{\sec\theta}{2a}$,

$$\frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a}.$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a} = \frac{\sec\theta\,(2\sec^2\theta - 1)}{2a}.$$

Hence $\qquad \dfrac{d^2u}{d\theta^2} = u(8a^2u^2 - 1).$

Applying (4), we have

$$-\frac{F}{m} = h^2u^2[u(8a^2u^2 - 1) + u] = 8a^2h^2u^5.$$

Since $u = \dfrac{1}{\rho}$, the final expression for the force is

$$F = -\frac{8a^2h^2m}{\rho^5}.$$

The force is attractive and proportional to the fifth power of the distance. Hence the

Theorem. *If a particle describes a circle under the action of a center of force on the circumference, the force is attractive and varies inversely as the fifth power of the distance.*

PROBLEMS

1. Assuming that the planets move around the sun in circles, prove Kepler's Harmonic Law, which states that the squares of the periods are proportional to the cubes of the distances.

2. A particle describes a circular orbit with angular velocity ω about a center of force which is inversely proportional to the distance $\left(F = -\frac{k^2 m}{\rho}\right)$. Determine the radius of the circle.

Ans. $a = \frac{k}{\omega}$.

3. A particle describes an ellipse with the center of force at one focus. Show that the force is inversely proportional to the square of the distance.

4. A particle describes an ellipse with the center of force at its center. Show that the force is proportional to the distance.

Suggestion. The equation of a conic section with center at the origin is $\rho^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}$. Then $F = -\frac{mh^2(1-e^2)}{b^4}\rho$.

For an ellipse $1 - e^2 > 0$ and the force is attractive.

For an hyperbola $1 - e^2 < 0$ and the force is repulsive.

5. The orbit is an hyperbola with the center of force at the right-hand focus. Show that if the particle moves (*a*) on the right-hand branch of the curve, the force is attractive and inversely proportional to the square of the distance; (*b*) on the left-hand branch, the force is repulsive and inversely proportional to the square of the distance.

6. Find the central force under which a particle may describe the orbit given.

(*a*) the reciprocal spiral $\rho\theta = a$. *Ans.* $F = \frac{-mh^2}{\rho^3}$.

(*b*) the logarithmic spiral $\rho = e^{a\theta}$. *Ans.* $F = \frac{-mh^2(a^2+1)}{\rho^3}$.

(*c*) the lituus $\rho^2\theta = a^2$. *Ans.* $F = -mh^2\left(\frac{1}{\rho^3} - \frac{\rho}{4a^4}\right)$.

(*d*) the lemniscate $\rho^2 = a^2 \cos 2\theta$. *Ans.* $F = -\frac{3h^2a^4m}{\rho^7}$.

(*e*) the cardioid $\rho = a(1 + \cos\theta)$. *Ans.* $F = -\frac{3ah^2m}{\rho^4}$.

(*f*) the limaçon $\rho = b - a\cos\theta$. *Ans.* $F = -mh^2\left(\frac{2(a^2-b^2)}{\rho^5} + \frac{3b}{\rho^4}\right)$.

(*g*) the four-leaved rose $\rho = a\cos 2\theta$. *Ans.* $F = -mh^2\left(\frac{8a^2}{\rho^5} - \frac{3}{\rho^3}\right)$.

(*h*) the three-leaved rose $\rho = a\cos 3\theta$. *Ans.* $F = -mh^2\left(\frac{18a^2}{\rho^5} - \frac{8}{\rho^3}\right)$.

(*i*) the rose $\rho = a\cos n\theta$. *Ans.* $F = -mh^2\left(\frac{2n^2a^2}{\rho^5} - \frac{(n^2-1)}{\rho^3}\right)$.

7. Find the law of central force under which a particle may describe the curve whose equation is

$$\rho^k = a \cos k\theta + b,$$

where a, b, and k are constants.

$$\textit{Ans.}\quad F = -mh^2\left(\frac{(k+1)(a^2-b^2)}{\rho^{2k+3}} + \frac{b(k+2)}{\rho^{k+3}}\right).$$

The curve in problem 7 includes many of the common curves as special cases. For example,

when $k = -1$, a conic with origin at the focus;
when $k = -2$, a conic with origin at the center;
when $k = 1$, $b \neq 0$, the limaçon;
when $k = 1$, $b = 0$, a circle;
when $k = 2$, $b = 0$, the lemniscate.

78. Determination of the orbit when the law of force is known. When the law of the force is known as a function of the distance ρ, we may determine the orbit by the integration of equation (4), Art. 77. The differential equation is linear and of the second order. The general solution will contain two arbitrary constants. In general the form of the orbit depends upon the constants of integration, which depend upon the initial conditions of the motion. The method of integration of the differential equation of the orbit depends upon the form of the function F, that is, upon the law of force. We shall consider in detail the case of an *attractive force inversely proportional to the square of the distance.*

In this case $F = -\dfrac{k^2m}{\rho^2} = -k^2mu^2$ and the differential equation of the orbit becomes

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -\frac{F}{m} = k^2u^2,$$

whence

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2}.$$

This is a well-known differential equation of which the solution (74, Chap. XIV) is

$$u = -c_1 \cos(\theta + c_2) + \frac{k^2}{h^2}.$$

Hence

$$(1)\qquad \rho = \frac{1}{\dfrac{k^2}{h^2} - c_1\cos(\theta + c_2)} = \frac{\dfrac{h^2}{k^2}}{1 - c_1\dfrac{h^2}{k^2}\cos(\theta + c_2)}.$$

Equation (1) is the equation of a conic section with focus at the origin.* The principal axis of the conic makes an angle $-c_2$ with the axis of coördinates. The eccentricity is $e = c_1 \frac{h^2}{k^2}$. The distance p from the focus to the directrix is given by the relation $ep = \frac{h^2}{k^2}$, whence $p = \frac{1}{c_1}$.

THEOREM. *The orbit of a particle subject to an attractive central force varying inversely as the square of the distance is a conic section with focus at the center of force.*

The special case of the circular orbit is obtained when we select $c_1 = 0$. The radius of the circle is $\frac{h^2}{k^2}$.

To determine the type of the orbit, we must find e in terms of the initial distance and the initial speed. Since $F = -\frac{k^2 m}{\rho^2}$, the energy equation (Art. 75) gives

$$\frac{m}{2}(v^2 - v_0{}^2) = \int_{\rho_0}^{\rho} -\frac{k^2 m}{\rho^2}\,d\rho.$$

Hence

$$v^2 - \frac{2k^2}{\rho} = v_0{}^2 - \frac{2k^2}{\rho_0}, \tag{2}$$

from which the result is derived that $v^2 - \frac{2k^2}{\rho}$ has a *constant value for any particular orbit.*

Taking for the equation of the orbit

$$\rho = \frac{ep}{1 - e\cos\theta},$$

we find, by differentiation,

$$\frac{d\rho}{dt} = -\frac{e^2 p \sin\theta}{(1 - e\cos\theta)^2}\frac{d\theta}{dt} = -\frac{\rho^2 \sin\theta}{p}\frac{d\theta}{dt} = -\frac{h}{p}\sin\theta,$$

$$\left(\text{since } \rho^2 \frac{d\theta}{dt} = h\right).$$

* The standard form of the equation of a conic section with focus at the origin is (Analytic Geometry, p. 173)

$$\rho = \frac{ep}{1 - e\cos\theta}.$$

This equation takes the form (1) if the polar axis is rotated through an angle c_2.

Also $$v^2 = \left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\theta}{dt}\right)^2$$
$$= \frac{h^2}{p^2}\sin^2\theta + \frac{h^2}{\rho^2}$$
$$= \frac{h^2}{e^2p^2}\left(e^2 - 1 + \frac{2\,ep}{\rho}\right).$$

Hence $$v^2 - \frac{2\,k^2}{\rho} = \frac{h^2}{e^2p^2}(e^2-1) + \frac{2}{\rho}\left(\frac{h^2}{ep} - k^2\right).$$

Since the first member of this equation is constant and independent of ρ, we must determine h so that $\frac{h^2}{ep} - k^2 = 0$, that is, $h^2 = epk^2$. The above equation now becomes

$$v^2 - \frac{2\,k^2}{\rho} = \frac{k^2}{ep}(e^2 - 1). \tag{3}$$

For the three types of conics, (3) gives:

parabola $e = 1$, $\therefore\ v^2 = \frac{2\,k^2}{\rho}$.

ellipse $e < 1$, $a =$ semi-major axis $= \frac{ep}{1-e^2}$,

$$\therefore\ v^2 = k^2\left(\frac{2}{\rho} - \frac{1}{a}\right).$$

hyperbola $e > 1$, $a =$ semi-transverse axis, $= \frac{ep}{e^2-1}$,

$$\therefore\ v^2 = k^2\left(\frac{2}{\rho} + \frac{1}{a}\right).$$

From these results we see that at a given distance ρ from the center of force the speed in an elliptic orbit is less than the speed in a parabolic orbit. Also in an hyperbolic orbit the speed is greater than in a parabolic orbit. When $e = 1$, then, by (2) and (3), if $v_0 = 0$, then $\rho_0 = \infty$ and $v^2 = \frac{2\,k^2}{\rho}$, that is, the speed in a parabolic orbit is the speed which would be acquired by a particle starting from rest at an infinite distance, or, briefly, the speed from infinity. Hence the

THEOREM. *The path of a free particle in an attractive central field of force varying inversely as the square of the distance is an ellipse, parabola, or hyperbola according as the initial speed is less than, equal to, or greater than the speed from infinity.*

79. Position in the orbit. If the orbit of a particle is given or has been determined, it remains to determine the position of the particle in the orbit at any instant. Let the equation of the orbit be $\rho = f(\theta)$. From the law of areas,

$$\rho^2 \frac{d\theta}{dt} = h, \tag{1}$$

whence

$$\int f^2(\theta) d\theta = F(\theta) = ht + c_3. \tag{2}$$

This equation determines the vectorial angle in terms of the time. Solving for θ,

$$\theta = \phi(t).$$

If the position at any instant is known, the constant of integration may be determined and the position at any other instant may be found.

Since θ is known as a function of t, this value may be substituted in the equation of the orbit and ρ will be expressed in terms of t: $\rho = \psi(t)$. We may now find the speed at any instant from the energy equation, or from the third theorem of Art. 73.

The time T required to describe any given arc from $\theta = \theta_1$ to $\theta = \theta_2$ may be found from (1) by integration. This gives

$$hT = h(t_2 - t_1) = \int_{\theta_1}^{\theta_2} \rho^2 d\theta,$$

$$\text{or } T = \frac{1}{h} \int_{\theta_1}^{\theta_2} \rho^2 d\theta. \tag{3}$$

This result might have been anticipated from the law of areas. The integral in (3) is twice the area bounded by the curve and the radii vectores $\theta = \theta_1$ and $\theta = \theta_2$. The constant h is twice the areal velocity. Hence (3) may be written

$$T = \frac{\text{area}}{\text{areal velocity}}.$$

For example, if the orbit is an ellipse, the period, that is the time to describe the complete curve, is

$$T = \frac{2\pi ab}{h}.$$

Illustrative Example. A particle describes a logarithmic spiral under a center of force at the pole. Find the time of describing any arc. Determine the coördinates and speed in terms of the time.

Solution. The equation of the curve is $\rho = e^{a\theta}$. From (3),

$$T = \frac{1}{h}\int_{\theta_1}^{\theta_2} e^{2a\theta}d\theta = \frac{1}{2\,ah}(e^{2a\theta_2} - e^{2a\theta_1}) = \frac{1}{2\,ah}(\rho_2^2 - \rho_1^2).$$

From (2),
$$\int e^{2a\theta}d\theta = ht + c_3,$$

whence
$$\frac{1}{2\,a}e^{2a\theta} = \frac{1}{2\,a}\rho^2 = ht + c_3,$$

and
$$2\,a\theta = \log 2\,a(ht + c_3).$$

We may find the speed directly from the relation

$$v^2 = \left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{d\rho}{d\theta}\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\theta}{dt}\right)^2 = \left[\frac{1}{\rho^4}\left(\frac{d\rho}{d\theta}\right)^2 + \frac{1}{\rho^2}\right]\rho^4\left(\frac{d\theta}{dt}\right)^2.$$

Now,
$$\frac{d\rho}{d\theta} = ae^{a\theta} = a\rho,$$

and
$$\rho^4\left(\frac{d\theta}{dt}\right)^2 = h^2.$$

Hence
$$v^2 = h^2\left(\frac{a^2+1}{\rho^2}\right) = h^2\left[\frac{a^2+1}{2\,a(t+c_3)}\right].$$

80. Complete solution of a problem in central motion. We have seen (Art. 76) that the problem of determining the motion of a particle in a central field of force demands the solution of a system of two simultaneous differential equations each of the second order. The complete solution must contain four constants of integration. For the determination of the constants we must have four initial conditions, for example, the two coördinates of position ρ_0, θ_0 and the two components of velocity $\left(\frac{d\rho}{dt}\right)_0$, $\left(\frac{d\theta}{dt}\right)_0$ at the instant $t = t_0$. We must be able to express the constants of integration in terms of the initial conditions.

Given the law of force, and the initial conditions $\rho = \alpha$, $\theta = \beta$, $\frac{d\rho}{dt} = \gamma$, $\frac{d\theta}{dt} = \delta$, *when* $t = 0$, *to determine the motion completely.* The solution of the differential equations of motion (1), Art. 73, is accomplished by three steps.

I. Integrating the second equation, we have

$$\rho^2\frac{d\theta}{dt} = h. \tag{1}$$

II. Integrating the differential equation of the orbit,

$$(2) \qquad h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=-\frac{F}{m},$$

we obtain

$$(3) \qquad \rho=f(\theta;\ c_1,\ c_2),$$

which is the polar equation of the orbit and involves two constants of integration.

III. Substituting in (1) the value of ρ from (3), we have, by integration,

$$(4) \qquad \int\rho^2 d\theta=F(\theta)=ht+c_3.$$

To determine the four constants of integration (h, c_1, c_2, c_3), we impose the initial conditions.

I. From (1),

$$(5) \qquad h=\alpha^2\delta.$$

II. To determine c_1 and c_2, we differentiate (3) with respect to t,

$$(3') \qquad \frac{d\rho}{dt}=\frac{df(\theta)}{d\theta}\frac{d\theta}{dt}=f'(\theta;\ c_1,\ c_2)\frac{d\theta}{dt}.$$

From (3) and (3′),

$$(6) \qquad \begin{cases}\alpha=f(\beta;\ c_1,\ c_2),\\ \gamma=f'(\beta;\ c_1,\ c_2)\,\delta.\end{cases}$$

We find c_1 and c_2 by solving the simultaneous equations (6).

III. From (4),

$$(7) \qquad c_3=F(\beta).$$

When the values of the constants of integration given by (5), (6), and (7) are substituted in (3) and (4), we have the finite equations of motion, by which ρ and θ are expressed in terms of t and the given constants.

Illustrative Example. A particle is subject to an attractive central force inversely proportional to the square of the distance $\left(F=-\frac{m}{\rho^2}\right)$. Determine the motion completely if $\rho=a$, $\theta=0$, $\frac{d\rho}{dt}=0$, $\frac{d\theta}{dt}=b$, when $t=0$. Discuss the form of the orbit for various values of b.

Solution. By (5) the constant of areas is $h = a^2b$.
Hence the differential equation of the orbit is

$$a^4b^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^2.$$

The polar equation of the orbit is (see (1), Art. 78)

$$(8) \qquad \rho = \frac{a^4b^2}{1 - c_1a^4b^2\cos(\theta + c_2)}.$$

Differentiating (8) with respect to t, we have after simplifying,

$$(9) \qquad \frac{d\rho}{dt} = -c_1\rho^2\sin(\theta + c_2)\frac{d\theta}{dt}.$$

Substituting the given initial values in (8) and (9), the equations for the determination of c_1 and c_2 become

$$(10) \qquad \begin{cases} a = \dfrac{a^4b^2}{1 - c_1a^4b^2\cos c_2}, \\ 0 = -c_1a^2b\sin c_2. \end{cases}$$

The solution of (10) gives

$$c_1 = \frac{1 - a^3b^2}{a^4b^2}, \quad c_2 = 0.$$

Substituting these values in (8), the equation of the orbit becomes

$$(11) \qquad \rho = \frac{a^4b^2}{1 - (1 - a^3b^2)\cos\theta}.$$

To express θ in terms of t, we have, from (4),

$$(12) \qquad \int\rho^2 d\theta = \int\frac{a^8b^4 d\theta}{\{1 - e\cos\theta\}^2} = a^2bt + c_3,$$

where $e = 1 - a^3b^2$.

Integration of (12) gives

$$\frac{a^8b^4}{1 - e^2}\left[\frac{e\sin\theta}{1 - e\cos\theta} + \frac{2}{\sqrt{1 - e^2}}\arctan\left\{\sqrt{\frac{1 + e}{1 - e}}\tan\frac{\theta}{2}\right\}\right] = a^2bt + c_3.$$

Substituting the initial values of θ and t, we find

$$c_3 = 0.$$

Hence θ is expressed in terms of t by the relation*

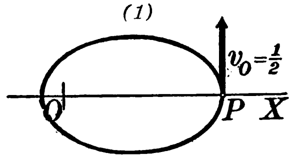
$$\frac{a^6b^3}{1-e^2}\left[\frac{e\sin\theta}{1 - e\cos\theta} + \frac{2}{\sqrt{1 - e^2}}\arctan\left\{\sqrt{\frac{1 + e}{1 - e}}\tan\frac{\theta}{2}\right\}\right] = t.$$

Equation (11) shows that the orbit is a conic section with focus at the origin. For various values of the initial angular velocity b, the following cases occur.

* The solution of this equation for θ is not simple. For practical purposes it is customary to employ infinite series. The student is referred to Moulton's Celestial Mechanics, Chapter V.

(1) If $b^2 < \frac{1}{a^3}$, the orbit is an ellipse ($e = 1 - a^3b^2$) with the left-hand focus at the origin.

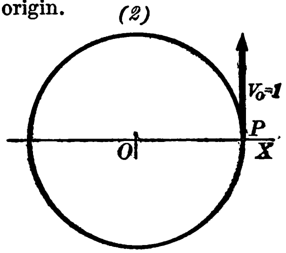
(1)



$OP = a = 1$, $b^2 = \frac{1}{4}$, $V_0 = ab = \frac{1}{2}$.

(2) If $b^2 = \frac{1}{a^3}$, the orbit is a circle with center at the origin.

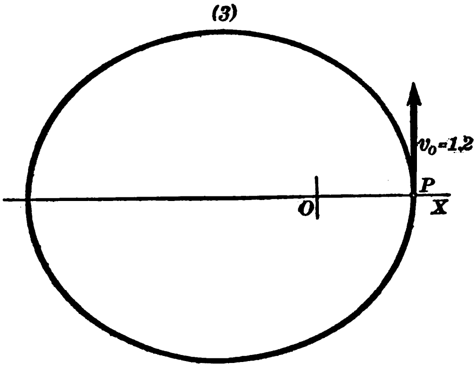
(2)



$OP = a = 1$, $b^2 = 1$, $V_0 = ab = 1$.

(3) If $\frac{1}{a^3} < b^2 < \frac{2}{a^3}$, the orbit is an ellipse ($e = a^3b^2 - 1$) with the right-hand focus at the origin.

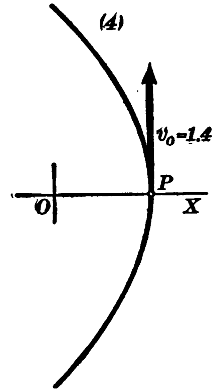
(3)



$OP = a = 1$, $b^2 = \frac{3}{2}$, $V_0 = ab = 1.2$.

(4) If $b^2 = \frac{2}{a^3}$, the orbit is a parabola.

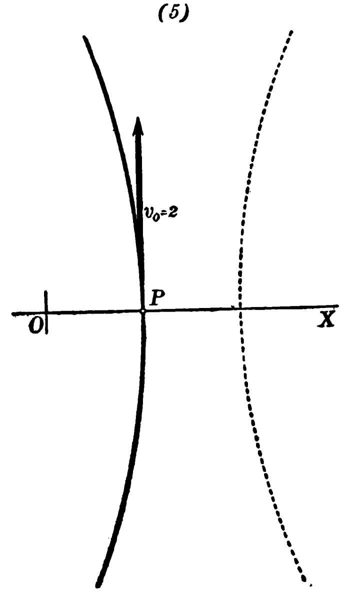
(4)



$OP = a = 1$, $b^2 = 2$, $V_0 = ab = 1.4$.

(5) If $b^2 > \frac{2}{a^3}$, the orbit is an hyperbola ($e = a^3b^2 - 1$), with the left-hand focus at the origin.

(5)



$OP = a = 1$, $b^2 = 4$, $V_0 = ab = 2$.

PROBLEMS

1. Determine the various orbits for the law of inverse cube of the distance, $F = -\frac{mk^2}{\rho^3}$.

Ans. When $k^2 < h^2$, $\frac{1}{\rho} = a \cos (c\theta + \beta)$, where $c^2 = 1 - \frac{k^2}{h^2}$.

When $k^2 = h^2$, $\frac{1}{\rho} = a\theta + \beta$.

When $k^2 > h^2$, $\frac{1}{\rho} = ae^{c\theta} + be^{-c\theta}$, where $c^2 = \frac{k^2}{h^2} - 1$.

2. Determine the coördinates and the speed in terms of the time, and the time of describing any arc when the orbit is the curve given:

(*a*) the reciprocal spiral $\rho\theta = a$;
(*b*) the curve $\rho = ae^{-a\theta} + be^{a\theta}$;
(*c*) the lituus $\rho^2\theta = a^2$;
(*d*) the lemniscate $\rho^2 = a^2 \cos 2\theta$;
(*e*) the cardioid $\rho = a(1 + \cos \theta)$;
(*f*) the limaçon $\rho = b - a \cos \theta$;
(*g*) the four-leaved rose $\rho = a \cos 2\theta$;
(*h*) the three-leaved rose $\rho = a \cos 3\theta$;
(*i*) the rose $\rho = a \cos n\theta$.

3. When the force is $F = m\left(\frac{\mu}{\rho^2} + \frac{\nu}{\rho^3}\right)$ show that, if $\nu < h^2$, the general equation of the orbit described has the form

$$\rho = \frac{a}{1 - e \cos (k\theta)},$$

where a, e, and k are constants.

4. Suppose the law of force is

$$F = m\left(\frac{\mu + \nu \cos 2\theta}{\rho^2}\right).$$

Find the equation of the orbit and show, by transforming it to rectangular coördinates, that the orbit will be an algebraic curve of the fourth degree for all initial conditions.

5. Show that in the case of a central force the motion along the radius vector is defined by the equation

$$\frac{d^2\rho}{dt^2} = f - \frac{h^2}{\rho^3}.$$

6. A particle is subject to an attractive central force proportional to the distance $(F = -m\rho)$. Determine the motion completely if $\rho = a$, $\theta = 0$, $\frac{d\rho}{dt} = 0$, $\frac{d\theta}{dt} = b$, when $t = 0$.

Ans. $\begin{cases} \tan \theta = b \tan t, \\ \rho^2 = \dfrac{a^2b^2}{b^2 \cos^2 \theta + \sin^2 \theta}. \end{cases}$

7. A particle is subject to an attractive force inversely proportional to the fifth power of the distance. Determine the motion completely in the two cases:

(a) $\rho = a,\ \theta = 0,\ \frac{d\rho}{dt} = 0,\ \frac{d\theta}{dt} = \frac{1}{a^3}$, when $t = 0$.

(b) $\rho = a,\ \theta = 0,\ \frac{d\rho}{dt} = 0,\ \frac{d\theta}{dt} = \frac{1}{\sqrt{2}\,a^3}$, when $t = 0$.

Ans. (a) $\theta = t/a^3,\ \rho = a.$
(b) $2\theta + \sin 2\theta = \frac{2\sqrt{2}\,t}{a^3},\quad \rho = a\cos\theta.$

81. Planetary motion. The law of gravitation. The astronomer Kepler (1571–1630) was led to formulate the following empirical laws of planetary motion, his conclusions resulting from the study of a great number of observations made by his predecessors and himself.

I. *The radius vector of each planet with respect to the sun as origin sweeps over equal areas in equal times.*

II. *The orbit of each planet is an ellipse with the sun at a focus.*

III. *The square of the period of revolution is proportional to the cube of the major semiaxis.*

Upon the basis of Kepler's laws Newton proved that the planets move under the action of a force directed towards the sun, and varying inversely as the square of the distance, thus. By the first law the theorem of areas holds, and we conclude, by Art. 74, that the planets are subject to a central field of force with center at the sun. There is no evidence that the intensity of the force of the field is different for different directions, and we assume that the law of force depends only upon the distance. From the second law the equation of the orbit is (Analytic Geometry, p. 173)

$$\rho = \frac{ep}{1 - e\cos\theta}.$$

Therefore, since
$$u = \frac{1}{\rho},$$

$$u + \frac{d^2u}{d\theta^2} = \frac{1}{ep},$$

and equation (4), Art. 77, for the determination of the law of force, gives

$$F = -\frac{h^2 m}{ep\rho^2}. \tag{1}$$

We therefore see that, assuming the force depends only on the distance, the first two laws of Kepler lead to the conclusion that any one planet is attracted by the sun with a force inversely proportional to the square of the distance.

By means of the third law we show that the factor $\frac{h^2}{ep}$ is the same for all the planets. By Art. 79, the period $T = \frac{2\pi ab}{h}$,

$$\text{or } h = \frac{2\pi ab}{T},$$

$$\text{and } \frac{h^2}{ep} = \frac{4\pi^2 a^2 b^2}{ep T^2}. \tag{2}$$

But $p = \frac{b^2}{ae}$ (Analytic Geometry, p. 185), and (2) becomes

$$\frac{h^2}{ep} = \frac{4\pi^2 a^3}{T^2}.$$

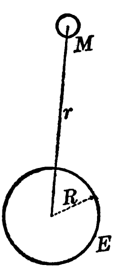
Since by the third law $\frac{a^3}{T^2}$ is constant for all the planets, we obtain for the law of force,

$$F = -c\frac{m}{\rho^2}, \tag{3}$$

where c has the same value for all the planets. From (3) we may conclude, as did Newton, that the sun exerts upon a planet a force of attraction which is directly proportional to the mass of the planet, and inversely proportional to the square of its distance from the sun.

Law of universal gravitation. It is shown by observations that laws corresponding to those of Kepler hold for the motion of the moon around the earth, and also for the motion of every family of satellites in the solar system. It follows, therefore, that each satellite is subject to a central force directed towards the primary, and varying inversely as the square of the distance. It has been shown also in every case in which the motion of a comet has been observed that the path is a conic section with the sun at a focus, and that the law of areas holds. These bodies, therefore, move under the same law of force as the planets. The laws of Kepler and the preceding statements concerning satellites and comets, although of immense importance, are only approximately true. The errors are comparatively small, but easily perceptible by observation, and readily explained theoretically upon the basis of *Newton's law of universal gravitation.* This is: *every particle of matter in the universe attracts every other particle with a force which acts in a line joining them, and whose intensity is directly proportional to the product of their masses, and inversely proportional to the squares of the distances apart.*

Observations show that the orbit of the moon about the earth is not an exact ellipse. This is due to the fact that its motion is influenced by the attractions of the sun and every other member of the solar system. The orbit is approximately an ellipse because the moon is very near the earth, and the central force directed towards the earth is much greater than all the other forces acting. The proper computation shows that Newton's law accounts for the motions of all the planets and satellites in the solar system, and not a single fact is known to dispute its truth. By means of it and the appropriate mathematical processes, we are able to predict the positions of the planets and satellites many years in advance. We therefore consider its truth to be established as far as the solar system is concerned.



Newton's verification that the force which holds the moon in its orbit is the same as that which makes an apple fall to the ground is historically important. To work this out, we need the theorem that the attraction of the earth upon an exterior object is the same as if its mass were concentrated at its center. Next we assume that the attraction of the earth for relatively small masses is the same as if the latter were material particles. Consider, therefore, the attraction of the earth (*a*) upon the moon; (*b*) upon a material particle upon its own surface. By (3) these are

$$(4) \qquad F_1 = c\frac{Mm_1}{r^2} = \frac{4\,\pi^2 a^3 m_1}{T^2 r^2}; \quad F_2 = c\frac{Mm}{R^2} = mg,$$

where $M =$ mass of earth, $m_1 =$ mass of moon, $m =$ mass of particle, $r =$ mean distance of moon, $R =$ radius of earth, $a =$ major semiaxis of moon's orbit, $T =$ moon's period, $g =$ acceleration due to gravity at earth's surface.

Comparing the values of cM in F_1 and F_2, we get from (4)

$$cM = \frac{4\,\pi^2 a^3}{T^2} = R^2 g.$$

Or,

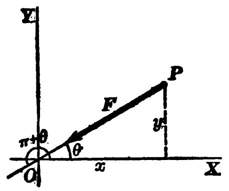
$$g = \frac{4\,\pi^2 a^2}{R^2 T^2}. \tag{5}$$

This value of g is accordingly the condition that the law of universal gravitation shall hold for the influence of the earth upon the moon and falling bodies at the earth's surface. The value of g computed from (5) is, in the C. G. S. system, 975, which compares with the observed value of 981 as closely as is to be expected under the approximate conditions assumed.

CHAPTER VIII

HARMONIC FIELD

82. Harmonic central field. We begin with the study of free motion in a central field due to a center of force attracting directly as the distance. That is, if O is the center of force, then at any point P,



$$(1) \qquad \text{Force} = mk^2 \cdot OP,$$

where k^2 is the absolute intensity of the field, that is, the force on unit mass at unit distance.

The axial components of F are

$$F_x = F\cos(x, F) = -F\cos\theta, \quad F_y = F\sin(x, F) = -F\sin\theta,$$

since $(x, F) = \pi + \theta$. But $\cos\theta = \dfrac{x}{OP}$, $\sin\theta = \dfrac{y}{OP}\cdot$ Hence we have

$$(2) \qquad F_x = -mk^2x, \quad F_y = -mk^2y.$$

Consider now the question of the path of a free particle projected with any velocity into such a field. It is obvious that the path is a straight line if the initial velocity is *along* a line of force. In the general case, however, the path is curvilinear, and we can at least foresee that it must be *everywhere concave towards the center of force*, since the force causing the motion has that direction. The general statement is contained in the

THEOREM. *The path of a free particle in a harmonic central field is elliptic if it is projected with a velocity oblique to the lines of force.*

Proof. The force equations in rectangular coördinates reduce by (2) to

$$(3) \qquad \frac{d^2x}{dt^2} + k^2x = 0, \quad \frac{d^2y}{dt^2} + k^2y = 0.$$

Each equation is harmonic (71, Chap. XIV), and the solutions may be written

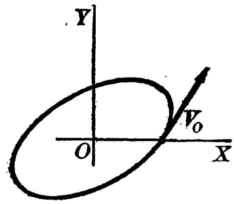
(4) $$x = c_1 \sin kt + c_2 \cos kt, \quad y = c_3 \sin kt + c_4 \cos kt,$$

in which c_1, c_2, c_3, c_4 are constants of integration. The rectangular equation of the path is obtained from (4) by eliminating t. We may, however, simplify the problem if we draw the axis of x through the initial position. Then if $t = 0$, we have $y = 0$, and hence $c_4 = 0$. We must now eliminate t from

(5) $$x = c_1 \sin kt + c_2 \cos kt, \quad y = c_3 \sin kt.$$

This is readily done by solving the second equation for $\sin kt$ and substituting in the first. The result is, after reduction, found to be

(6) $$c_3^2x^2 - 2\, c_1c_3xy + (c_1^2 + c_2^2)y^2 = c_2^2c_3^2,$$

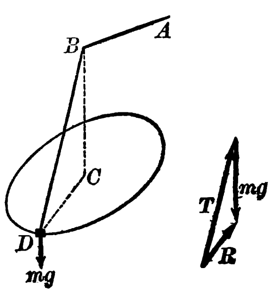


which is the equation of a central conic with center at the origin. But since this conic must be everywhere concave towards the center O, the locus must be an *ellipse*. Q. E. D.

Since the rectangular component motions (4) are both periodic with the same period $2\pi \div k$, the particle completely describes the ellipse in the time $2\pi \div k$, and we have the important

THEOREM. *A free particle projected in a central harmonic field in any direction will describe a periodic orbit whose period depends only upon the absolute intensity of the field.*

ILLUSTRATIVE EXAMPLE. An elastic string AB is fastened at A and the other extremity is pulled through a ring at B and attached to a heavy particle. The latter is at rest in the position C. If the particle is now displaced obliquely a small distance and then projected, determine the motion.



Solution. Let T represent the pull of the string when the particle is at D. Then, by Hooke's Law (footnote, p. 112),

(1) $$T : mg :: BD : BC,$$

since at C the pull and weight are equal. The resultant force R is the vector sum of T and mg. The vector triangle is, however, similar to BDC by virtue of the proportion (1) and the equality of the angle at B to that between T and mg. Hence R acts towards C and is proportional to DC. That is, the particle moves as if attracted towards C with a force proportional to the distance. The particle therefore describes a vertical ellipse about C as a center when projected from D in the plane BCD in any direction oblique to CD. The period of the motion is $2\pi\sqrt{\frac{d}{g}}$, if $BC = d$. For, since $R : mg :: DC : BC$, we have $R = \frac{mg}{BC} \cdot CD$. Hence, if m and CD are unity, the absolute intensity of R is $\frac{g}{BC}$ or $\frac{g}{d}$, and this is k^2.

83. Energy equation. The energy equation in rectangular coördinates gives, using (2), Art. 82,

$$\tfrac{1}{2} mv^2 - \tfrac{1}{2} mv_0^2 = \int_{x_0, y_0}^{x, y} (-mk^2 x dx - mk^2 y dy)$$

$$= -\frac{mk^2}{2}\Big[x^2 + y^2\Big]_{x_0, y_0}^{x, y} = -\frac{mk^2}{2}(\rho^2 - \rho_0^2),$$

if ρ = distance from center of force.

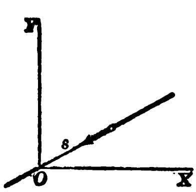
(I) $$\therefore v^2 - v_0^2 = -k^2(\rho^2 - \rho_0^2).$$

The path (6), Art. 82, is circular, when and only when $c_1 = 0$, $c_3 = c_2$. The equations of motion now are, if $c_2 = c_3 = a$,

(1) $$x = a \cos kt, \quad y = a \sin kt.$$

Since ρ is now constant and equal to ρ_0, (I) gives $v = v_0$, that is, the motion is *uniform circular motion.*

84. Simple harmonic motion. The path is clearly rectilinear when the direction of the velocity of projection is along a line of force. Let s be the distance from the origin at any instant. Then $F = -mk^2 s$, and the force equation is



(1) $$\frac{d^2 s}{dt^2} + k^2 s = 0,$$

that is, the harmonic equation (71, Chap. XIV).

The solution, or equation of motion, may be written

(2) $$s = a \cos (kt + \beta),$$

in which a and β are arbitrary constants. The characteristics of the motion have been discussed in example 3, p. 50.

The following terminology is in common use for simple harmonic motion.

The attraction F is called the *force of restitution*, the distance s the *displacement*, and the constant β, upon which the initial position ($s_0 = a \cos \beta$) depends, is named the *epoch*. As already pointed out, the maximum displacement ($= a$) is the amplitude, and the period of the motion is

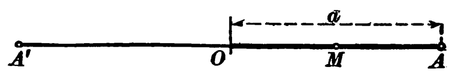
$$(3) \qquad T = \frac{2\pi}{k}.$$

Since in any given harmonic field the period T is constant, we may write in (2), $k = \frac{2\pi}{T}$, and thus obtain the equation of motion in the *standard form*

$$\text{(II)} \qquad s = a \cos\left(\frac{2\pi t}{T} + \beta\right).$$

Frequency. The reciprocal of the period is named the frequency of the vibration. Obviously, the frequency gives the number of total vibrations (integral and fractional) performed in unit time.

Phase. The extreme position A is reached when $\frac{2\pi t}{T} + \beta = 0$, or $t = -\frac{\beta}{2\pi}T$. Let M be a subsequent position in the path at the time t. Then the *elapsed* time from A to M is $t + \frac{\beta}{2\pi}T$. The



ratio of this interval to a complete period is called the *phase;* that is,

$$(4) \qquad \textit{Phase} \text{ at the time } t = \frac{t}{T} + \frac{\beta}{2\pi}.$$

For example, if the phase $= \frac{1}{4}$, the particle is at O since a quarter period has elapsed from A; if the phase $= \frac{1}{2}$, the position is A', etc.

Difference in phase. Given the simple harmonic motions with the same period,

$$(5) \qquad x = a \cos\left(\frac{2\pi t}{T} + \beta\right),\ x' = a' \sin\left(\frac{2\pi t}{T} + \beta'\right),$$

to determine the difference in phase. We must first write the second equation in the standard form, thus:

$$x' = a' \sin\left(\frac{2\pi t}{T} + \beta'\right) = a' \cos\left(\frac{2\pi t}{T} + \beta' + \frac{\pi}{2}\right).$$

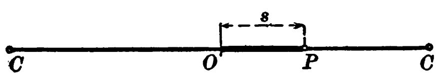
The respective phases are now (by (4))

$$\frac{t}{T} + \frac{\beta}{2\pi} \quad \text{and} \quad \frac{t}{T} + \frac{\beta' + \frac{\pi}{2}}{2\pi}.$$

Their difference is accordingly equal to $\dfrac{\beta - \beta' - \frac{\pi}{2}}{2\pi}$;

that is, a constant independent of the time. This gives the result: *If two simple harmonic motions have the same period, their difference in phase is constant.*

Illustrative Example. A heavy particle is at rest at O on a rough horizontal plane midway between two points C and C'. The extremities of an elastic string



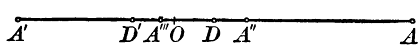
of length less than OC are attached to C and to the particle, and a like elastic string is attached to C' and to the particle. The particle is now displaced a small distance from the position O in the direction CC' and then released; determine the motion.

Solution. Let d = elongation of each string when the particle is at O. Then at P, if $s = OP$, the elongations are respectively $d + s$, $d - s$. Since the pull of each string is proportional to the elongation, the *force of restitution*, towards O, is numerically $m\lambda(d + s) - m\lambda(d - s) = 2\,m\lambda s$, where λ is a constant factor of proportionality. This is resisted, however, by the friction μmg, where μ is the coefficient of friction. The resultant force is the difference. The force of restitution must be written $-2\,m\lambda s$, since its direction is opposite to s. The friction, however, must remain indeterminate in sign, $\pm\,\mu mg$, since its direction reverses with the change in direction of the motion. The force equation therefore gives, after division by m,

$$(1) \qquad \frac{d^2s}{dt^2} = -2\,\lambda s \pm \mu g,$$

the plus or minus sign being used according as the direction of motion is negative or positive.

Let the particle be displaced from O to A. Then, for motion to begin, the force of restitution $2 \cdot \lambda m \cdot OA$ must exceed the friction. That is, $2\,\lambda \cdot OA > \mu g$, or



$OA > \frac{1}{2}\,\mu g \div \lambda$. If then the points D and D' be marked such that $OD = OD' = \dfrac{\mu g}{2\lambda}$, the initial position must be beyond D or D'.

Separating the two cases in (1), we have for motion in the *negative* direction,

$$\frac{d^2s}{dt^2} + 2\,\lambda s = \mu g, \text{ or } \frac{d^2s}{dt^2} + 2\,\lambda\left(s - \frac{\mu g}{2\,\lambda}\right) = 0.$$

If in this equation we write $s - \frac{\mu g}{2\lambda} = s'$, we obtain the harmonic equation $\frac{d^2s'}{dt^2} + 2\lambda s' = 0$. We therefore have a harmonic oscillation whose center is $s = \frac{\mu g}{2\lambda}$ or D; that is, on the positive side.

The solution of the problem is now clear. The effect of friction is to pull the center of force in its own direction from O to D or D'. Thus, if motion begins at A, the particle moves to A', where $DA' = DA$. The particle next moves to A'' such that $D'A' = D'A''$, etc. We note that friction reduces the amplitude each time by DD' or $\mu g \div \lambda$. Motion ceases when an extreme position falls within DD', in the figure at A'''.

PROBLEMS

1. Integrate the following harmonic equations, under the given conditions:

(a) $\frac{d^2x}{dt^2} + x = 0;\ x_0 = 2,\ v_0 = 0.$ *Ans.* $x = 2\cos t.$

(b) $\frac{d^2y}{dt^2} + 4y = 0;\ a = 2,\ \beta = \frac{\pi}{6}.$ *Ans.* $y = 2\cos(2t + \frac{1}{6}\pi).$

(c) $\frac{d^2x}{dt^2} + 3x = 0;\ a = 1,\ \beta = \frac{\pi}{2}.$ *Ans.* $x = \sin\sqrt{3}\,t.$

(d) $\frac{d^2y}{dt^2} + n^2y = 0;\ y_0 = c,\ v_0 = nc.$

(e) $\frac{d^2\theta}{dt^2} + \frac{g}{e}\theta = 0;\ \theta_0 = \alpha,\ \omega_0 = 0.$

(In this problem, $\omega = \frac{d\theta}{dt}$ = angular velocity.) *Ans.* $\theta = \alpha\cos\sqrt{\frac{g}{e}}\,t.$

(f) $\frac{d^2\theta}{dt^2} + \frac{g}{e}\theta = 0;\ \theta_0 = 0,\ \omega_0 = k.$ *Ans.* $\theta = k\sqrt{\frac{e}{g}}\sin\sqrt{\frac{g}{e}}\,t.$

(g) $\frac{d^2z}{dt^2} + bz = 0;\ a = a,\ \beta = \frac{3}{2}\pi.$

(h) $\frac{d^2y}{dt^2} + y = 0;\ a = 1,\ v_0 = \frac{1}{2}.$

2. Show that each of the following defines a simple harmonic motion, by writing each in the standard form (II). Find the amplitude, epoch, and period.

(a) $x = \sin t - 2\cos t.$ *Ans.* $a = \sqrt{5},\ \sin\beta = -\frac{1}{\sqrt{5}},\ \cos\beta = -\frac{2}{\sqrt{5}},\ T = 2\pi.$

(b) $x = -\cos 2t + 3\sin 2t.$ *Ans.* $a = \sqrt{10},\ \sin\beta = -\frac{3}{\sqrt{10}},\ \cos\beta = -\frac{1}{\sqrt{10}}.$

(c) $s = 2\sin\left(t - \frac{\pi}{6}\right).$ *Ans.* $s = 2\cos(t - \frac{1}{3}\pi).$

(d) $s = \cos\pi t - 4\sin\pi t.$ *Ans.* $a = \sqrt{17},\ \sin\beta = \frac{4}{\sqrt{17}},\ \cos\beta\frac{1}{\sqrt{17}}.$

(e) $z = 5\cos\left(\pi t - \frac{3\pi}{2}\right).$

(f) $x = \cos\left(\pi t + \frac{\pi}{6}\right) - \sin\left(\pi t - \frac{\pi}{6}\right).$

(g) $x = a_1 \cos(kt + \beta_1) + a_2 \cos(kt + \beta_2)$.

(h) $y = b_1 \sin kt + b_2 \sin(kt - \pi)$.

(i) $y = \sin t - \cos t + \sin\left(t + \frac{\pi}{6}\right)$.

(j) $\theta = \alpha_1 \cos(nt - \beta_1) + \alpha_2 \cos(nt + \beta_2)$.

(k) $x = a_1 \cos(kt + \beta_1) + a_2 \cos(kt + \beta_2) + a_3 \cos(kt + \beta_3)$.

(l) $x = -\cos\frac{\pi}{2}t - 3\sin\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.

(m) $x = 2\sin\frac{1}{2}\pi t - \cos\left(\frac{1}{2}\pi t - \frac{3}{4}\pi\right)$.

(n) $x = \sin\frac{2}{3}\pi t - 5\cos\left(\frac{2}{3}\pi t - \frac{\pi}{2}\right)$.

Note the characteristic thing: The functions are sines or cosines and when occurring together the coefficient of t is the same.

3. Show that

$$x = a_1 \cos(kt + \beta_1) + a_2 \cos(kt + \beta_2) + \cdots + a_n \cos(kt + \beta_n)$$

defines a harmonic motion. What is the period?

4. Draw the distance-time diagram for each solution of problem 1, and discuss the figure.

5. Construct the positions of a particle having simple harmonic motion, if the amplitude is 2, and phase equals $\frac{1}{3}$, $\frac{1}{2}$, 2, 1, $1\frac{1}{2}$, $1\frac{1}{4}$, $2\frac{3}{4}$, 5, $\frac{1}{8}$, $1\frac{1}{8}$.

6. An elastic string supporting a heavy particle hangs in equilibrium, the elongation due to the weight of the particle being equal to d. The particle is now depressed below the position of equilibrium through a distance c greater than d, and then released. Find the height to which the particle will rise after the string ceases to be taut. *Ans.* $\frac{c^2 - d^2}{2d}$.

7. If the heavy particle in problem 6 be acted upon by an impulse sufficient to project it downward from the position of equilibrium with the velocity v, find the maximum extension of the string. *Ans.* $d + \sqrt{\frac{d}{g}}\, v$.

8. Find the velocity of the impulse in problem 7 if the string just resumes its original length when the particle rises. *Ans.* $\sqrt{dg}$.

9. What is the nature of a field of force due to two centers of equal absolute intensity and each attracting directly as the distance?

Ans. A harmonic central field of double intensity whose center is the middle point of those given.

10. Two material particles act as centers of force attracting as the distance, the absolute intensity of each equaling the mass of the particle. Determine the nature of the field.

Ans. A similar field due to the entire mass placed at the center of mass.

11. Generalize problem 10.

12. Find the path under a *repulsive* center of force, the law of direct distance still holding as in Art. 82. *Ans. Hyperbola.*

13. Two heavy particles of masses m and m', respectively, hang at rest, being attached to the lower extremity of an elastic thread, whose upper end is fixed. Supposing the second particle drops off, determine the subsequent motion of the other.

Ans. If the separate particles cause elongations d and d', respectively, and if l = length of string, the equation of motion is $x = l + d + d' \cos\sqrt{\frac{g}{d}}\,t$, where x = distance of the particle from the fixed point of suspension.

85. Composition of simple harmonic motions in a given field. Many problems in mechanics depend for their solution upon the following simple principle.

Consider two simple harmonic motions occurring simultaneously on XX' in the given field. Their equations may be written

$$(1) \qquad x_1 = a_1 \cos(kt + \beta_1), \quad x_2 = a_2 \cos(kt + \beta_2).$$

By composition (or addition) we derive from these the motion whose equation is

$$(2) \qquad x = x_1 + x_2 = a_1 \cos(kt + \beta_1) + a_2 \cos(kt + \beta_2).$$

But this motion is also simple harmonic *with the same center and period* as the components (1). For the equations (1) are solutions of the harmonic equation,

$$(3) \qquad \frac{d^2x}{dt^2} + k^2 x = 0,$$

and their sum is also a solution.

Hence the equation (2) *must* be in the form

$$(4) \qquad x = a \cos(kt + \beta).$$

In order to find a and β, expand (2) and (4) and compare the coefficients of $\cos kt$ and $\sin kt$. We obtain

$$(5) \quad a \cos\beta = a_1 \cos\beta_1 + a_2 \cos\beta_2, \quad a \sin\beta = a_1 \sin\beta_1 + a_2 \sin\beta_2.$$

Squaring and adding these gives

$$(6) \qquad a^2 = a_1^2 + a_2^2 + 2\, a_1 a_2 \cos(\beta_1 - \beta_2).$$

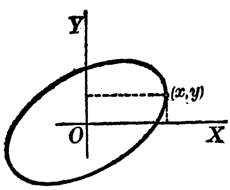
This is the amplitude of the resultant motion. Knowing a, then $\sin\beta$ and $\cos\beta$ are given by (5) and hence the epoch can be found.

The preceding discussion may be generalized and gives the result:

The resultant of any number of simple harmonic motions on the same line in a given field is also a simple harmonic motion in the same field.

Consider next the composition of simple harmonic motions in a given field along lines mutually perpendicular. Let these equations be

(7) $$x = a \cos(kt + \beta_1), \quad y = b \cos(kt + \beta_2).$$



The resultant motion is that of the point (x, y), and, as already proved, the motion is in general elliptic. The path may, however, be rectilinear, namely, if the difference in phase is $\frac{1}{2}$ or any multiple thereof; that is, if

(8) $$\frac{\beta_1 - \beta_2}{2\pi} = \frac{n}{2}, \text{ and hence } \beta_1 = \pi n + \beta_2,$$

we get in (7), by substitution,

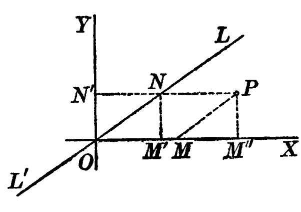
(9) $$x = a \cos(kt + \beta_2 + n\pi) = \pm a \cos(kt + \beta_2) = \pm \frac{ay}{b},$$

and the path is accordingly a straight line.

This result is important and gives the

THEOREM. *Two simultaneous simple harmonic motions along perpendicular lines in the same field compound into a simple harmonic motion when the difference in phase is a multiple of one half. Conversely, any simple harmonic motion in a given plane field may be resolved in two simple harmonic motions along perpendicular lines, one of which may be chosen arbitrarily.*

Consider finally the composition of simultaneous simple harmonic motions along oblique lines LL' and XX' in a given field. Let P be any position resulting from composition of the motions of M along OX and N along OL. By the theorem, the motion of N along LL' may be resolved into simultaneous simple harmonic motions of M' along OX and N' along OY. But since in the figure $OM'' = OM' + OM$, the simultaneous simple harmonic motions of M' and M compound into a simple harmonic motion of M''.



Hence the resultant of the motions of M along OX and N along OL is now compounded of simple harmonic motions along the perpendicular lines OX and OY, and is therefore an elliptic harmonic motion in general.

The conclusion of the whole matter may now be stated in the

THEOREM. *The resultant of any number of simple harmonic motions having the same center and period is either an elliptic harmonic motion or a simple harmonic motion.*

This theorem finds numerous applications in Physics in connection with elastic media and the theory of wave motion. By Hooke's Law, any particle of such a medium, when the stress causing a displacement is removed, performs small oscillations under the action of a force of restitution proportional to the displacement from a normal position, and therefore executes simple harmonic motion. The composition of such motions is accordingly of importance in studying the effect of simultaneous disturbances in such media.

PROBLEMS

1. Find the equation of the resultant of the following simultaneous motions. In all cases determine the resultant amplitude, epoch, and the difference of phase.

(a) $x_1 = \sin t$; $x_2 = \cos t$. *Ans.* $x = \sqrt{2}\cos(t - \frac{1}{4}\pi)$.

(b) $x_1 = 2\cos t$; $x_2 = \sin\left(t + \frac{\pi}{2}\right)$. *Ans.* $x = 3\cos t$.

(c) $x_1 = 2\cos\frac{1}{2}\pi t$; $x_2 = \sin\left(\frac{1}{2}\pi t + \frac{\pi}{6}\right)$. *Ans.* Difference of phase $= \frac{1}{6}$.

(d) $x_1 = -\cos\pi t$; $x_2 = 2\sin(\pi t + \frac{1}{4}\pi)$. *Ans.* Difference of phase $= \frac{1}{8}$.

(e) $y_1 = \sin\frac{1}{2}t$; $y_2 = 2\cos\frac{1}{2}t$. *Ans.* Difference of phase $= \frac{1}{4}$.

(f) $y_1 = -\cos(\frac{3}{2}\pi t + \frac{1}{4}\pi)$; $y_2 = 2\sin(\frac{3}{2}\pi t + \frac{3}{4}\pi)$.
Ans. Difference of phase $= 0$.

(g) $x_1 = a\cos kt$; $x_2 = a\cos\left(kt + \frac{2\pi}{3}\right)$. *Ans.* $x = a\cos\left(kt + \frac{\pi}{3}\right)$.

2. Find the equation of motion of the resultant of

$$x_1 = a\cos kt,\ \ x_2 = a\cos(kt + \tfrac{2}{3}\pi),\ \ x_3 = a\cos(kt + \tfrac{4}{3}\pi).\quad \textit{Ans. } x = 0.$$

3. Find the amplitude and epoch of the motion whose equations are

$$x = a\cos(kt + \beta),\ \ y = b\cos(kt + \beta).$$

Ans. Amplitude $= \sqrt{a^2 + b^2}$, epoch $= \beta$.

4. What is the theorem concerning difference of phase when the resultant of two simple harmonic motions along perpendicular lines in the same field is a uniform circular motion?

Ans. Difference of phase must be an odd multiple of one fourth.

5. Discuss the motion defined by each of the following. Find the equation of the path in each case, and plot the locus.

(a) $x = \cos t$; $y = 2\sin(t + \frac{1}{4}\pi)$. *Ans.* $4x^2 - 2\sqrt{2}\,xy + y^2 = 2$.

(b) $x = 2\sin t$; $y = 3\cos t$. *Ans.* $9x^2 + 4y^2 = 36$.

(c) $x = \sin(\frac{1}{2}\pi t + \frac{1}{4}\pi)$; $y = -\cos(\frac{1}{2}\pi t)$. *Ans.* $x^2 + \sqrt{2}xy + y^2 = \frac{1}{2}$.

(d) $x = \cos kt$; $y = \cos(kt + \frac{2}{3}\pi)$. *Ans.* $x^2 + xy + y^2 = \frac{3}{4}$.

6. A heavy particle is suspended from a fixed point by a fine elastic thread and is hanging at rest. Motion is set up by an impulse imparting a velocity v in a vertical plane through the thread but inclined at an angle α to the latter. Determine the motion.

Ans. Simple harmonic, amplitude $= v \div k$, where k has same value as before (Art. 82).

7. If the particle in problem 6 is not originally at rest but performing vertical vibrations, determine the motion when the same impulse acts upon it when in any position. Work out the equations of motion by composition.

8. Solve problem 3 by rotating the axes through the angle $\theta = \tan^{-1}\frac{b}{a}$, and show that the new equations of motion are

$$x' = x\cos\theta - y\sin\theta = \sqrt{a^2 + b^2}\cos(kt + \beta),\quad y' = x\sin\theta + y\cos\theta = 0.$$

9. A particle is projected from the point (3, 4) with a velocity v_0 of 30 ft. per second in the direction given by $(v_0, x) = \frac{1}{2}\pi + \sin^{-1}\frac{4}{5}$. The force acting is an attraction from the origin varying as the distance and in magnitude equaling 1 lb. per unit mass at the distance of 2 ft. Discuss the motion.

Ans. Path is $9\frac{1}{9}x^2 - 23\frac{5}{6}xy + 16\frac{1}{16}y^2 = 625$.

10. Discuss the resultant of two simple harmonic motions on the same line in a given field if the difference of phase is $\frac{1}{4}$; $\frac{1}{3}$; $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{4}$.

11. If the fly wheel of an engine revolves with constant speed, show that the motion of the piston is more nearly simple harmonic the greater the length of the connecting rod.

86. Composition with different periods. Forced vibrations. Consider again the motion of a heavy particle performing small vertical oscillations by virtue of being suspended by an elastic thread. Let the particle be acted upon by a *periodic* vertical force, that is, a force whose magnitude and direction vary periodically. Such a force is given by

$$(1) \qquad F = F_0 \cos \lambda t = mf_0 \cos \lambda t.$$

Its maximum value is F_0 and its period is $2\pi \div \lambda$.

It is clear that the original simple vibration will be altered. In particular, if the force varies in such a manner that its direction always coincides with the direction of motion, the amplitude will increase with each oscillation. The periods of the harmonic motion and the force must in this case be equal. On the contrary, if the periods are *nearly* equal, the direction of the force will eventually oppose the motion and reduce the amplitude. Thus a great oscillation will be at first produced, then reduced, and subsequently renewed, etc.

Mathematical verification of these observed facts is readily made. The differential equation of motion is

$$(2) \qquad \frac{d^2s}{dt^2} + k^2s = f_0 \cos \lambda t,$$

since the forces are F and the attraction of the field $(= -mk^2s)$.

CASE 1. *Periods unequal* $(\lambda \neq k)$. A particular integral of (2) is without difficulty seen to be $\frac{f_0}{k^2 - \lambda^2} \cos \lambda t$. Hence, the equation of motion is (see 75 (a), Chap. XIV)

$$(3) \qquad s = a \cos(kt + \beta) + \frac{f_0}{k^2 - \lambda^2} \cos \lambda t.$$

This equation is obviously obtained by *composition of simple harmonic motions of different periods*, namely,

$$(4) \qquad s_1 = a \cos(kt + \beta), \quad s_2 = \frac{f_0}{k^2 - \lambda^2} \cos \lambda t.$$

The first of these is the undisturbed harmonic motion. The second has the same period as the disturbing force $(2\pi \div \lambda)$, and the amplitude is

$$(5) \qquad b = \frac{f_0}{k^2 - \lambda^2}.$$

The resultant motion (3) is, of course, an oscillation, but not harmonic. The case is important when k and λ are *nearly equal*, that is, when the disturbing force F has a period differing slightly from the period of the field. The amplitude b is now very large and the oscillations of the motion (3) consequently become very great. This result may be formulated:

If a vibrating body is acted upon by a periodic force of frequency nearly equal to that of the undisturbed vibrations, the forced oscillations will be of great amplitude.

We have here an illustration of the principle of resonance.

The conclusion may now be drawn that a small force with the proper period may produce remarkable effects, and an explanation is arrived at of the danger to bridges from the steady marching of troops, the heavy rolling of ships caused by waves of proper period, etc. The phenomenon of "beats" in acoustics rests upon this principle also.

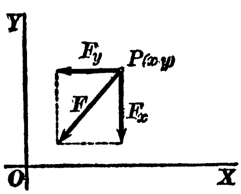
CASE 2. *Periods equal* ($\lambda = k$). A particular solution of (2) is now found to be $\frac{f_0}{2k} t \sin kt$, and the general solution may therefore be written in the form (see 75 (*b*), Chap. XIV)

$$(6) \qquad s = a \cos(kt + \beta) + \frac{f_0}{2k} t \sin kt.$$

The presence of t in the second term destroys the harmonic character of the component. It is plain, however, that this term determines the numerical magnitude of s when t is large, and consequently it is seen that the amplitude of this vibration becomes and remains very great.

87. General harmonic field. If the rectangular components of a plane field are

$$(1) \qquad F_x = -k^2 mx, \quad F_y = -l^2 my,$$



the field is a *general harmonic* field.

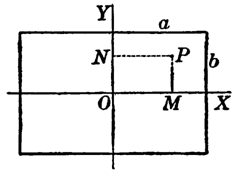
The force equations for a free particle are in this case

$$(2) \qquad \frac{d^2x}{dt^2} = -k^2 x, \quad \frac{d^2y}{dt^2} = -l^2 y.$$

Each of these equations is harmonic, and therefore *the equations of motion of a free particle in a general harmonic field* are

$$(3) \qquad x = a \cos(kt + \beta), \quad y = b \cos(lt + \gamma).$$

The path evidently lies within and touches the sides of a rectangle whose sides are $2a$ and $2b$. Furthermore, the path will be a closed curve when k and l are commensurable. For if



$$\frac{k}{l} = \frac{m}{n} \text{ (} m \text{ and } n \text{ integers),}$$

then, since $k = 2\pi \div T_1$, $l = 2\pi \div T_2$, where T_1 and T_2 are the periods of the component motions (3), we have

$$(4) \qquad mT_1 = nT_2.$$

Hence when a period of time equal to $T = mT_1 = nT_2$ has elapsed, x and y have their original values, and the particle has returned to its original position.

The curves defined by (3) are known in Physics as Lissajous' Curves, from the name of the scientist who first studied them. They may be defined as the path of a free point whose motion is compounded of simple harmonic motions along perpendicular lines.*

PROBLEMS

1. Obtain the equations of motion of a free particle in the harmonic fields for which $k=1$, $l=\sqrt{2}$; $k=\sqrt{2}$, $l=1$; $k=\sqrt{3}$, $l=1$; $k=1$, $l=2$; $k=2$, $l=2$.

2. Determine the path of a free particle in a general harmonic field under the following conditions:

$$(a)\quad k=1,\ l=2,\ a=2,\ b=1,\ \beta=\gamma=0.$$

$$(b)\quad k=\tfrac{1}{2},\ l=1,\ a=1,\ b=1,\ \beta=\frac{\pi}{2},\ \gamma=0.$$

$$(c)\quad k=2,\ l=1,\ a=b,\ \beta=\gamma=\frac{\pi}{2}.$$

* See General Physics, Hastings and Beach (Ginn and Company), p. 529.

CHAPTER IX

MOTION IN A RESISTING MEDIUM

88. Law of resistance. In the preceding sections the characteristics of motion in various fields have been determined without reference to any resistance to the motion which might be offered by the medium in which motion takes place. The law of resistance must necessarily be established by experiment. For air, a study of this law under given conditions of temperature, pressure, etc., has been made by numerous investigators. The following table will exhibit results found for rotating projectiles of the standard Krupp form, the assumption having been made by the experimenter in each case that the resistance varies as some *power* of the speed. The first line gives the speed in meters per second, the second line the resistance, a, b, c, d, e, f, g being constants of proportionality.

Speed	50		240		295		375		419		550		800		1000
Resistance		av^2		bv^3		cv^5		dv^3		ev^2		$fv^{1.7}$		$gv^{1.55}$	

The arrangement of the table is intended to indicate that the law of resistance holds for all speeds in the interval under which is written the expression for the law. Thus for speeds between 50 and 240 m. per second, the resistance varies as the square of the speed, etc. It will be observed that the resistance involves a complicated exponent for very large speeds. In any given case the law necessarily depends upon the shape of the moving body, the medium (water, air, etc.), and the changing physical conditions of the latter (temperature, pressure, etc.). We consider in this chapter the effect upon the motion of a resistance assumed to follow a given law.

89. Constant field. Resistance proportional to the square of the velocity. We may begin by studying the effect of the presence of a resisting medium upon motion in a constant field, and, as an example, consider the case of a falling body, assuming the resistance to be proportional to v^2. The equation of motion is therefore, if distance is measured downwards,

$$(1) \qquad \frac{dv}{dt} = g - \mu v^2,$$

where μ is a positive factor of proportionality, called the *coefficient of resistance.* Evidently μ equals the resistance offered to unit mass when moving with unit speed.

Integrating (1), we have

$$\frac{1}{2\sqrt{g}} \log \frac{\sqrt{g} + \sqrt{\mu} v}{\sqrt{g} - \sqrt{\mu} v} = \sqrt{\mu} t + C.$$

For initial conditions, assume $s = 0$, $v = 0$ when $t = 0$. Hence $C = 0$, and solving for v, we obtain, after simple transformations,

$$(2) \qquad v = \sqrt{\frac{g}{\mu}} \frac{e^{\sqrt{\mu g} t} - e^{-\sqrt{\mu g} t}}{e^{\sqrt{\mu g} t} + e^{-\sqrt{\mu g} t}}.$$

Writing $v = \frac{ds}{dt}$ and integrating, we obtain

$$(3) \qquad s = \frac{1}{\mu} \log\left(\frac{e^{\sqrt{\mu g} t} + e^{-\sqrt{\mu g} t}}{2}\right).$$

This is therefore the desired equation of motion. The formula is applicable to motion in any constant field under the given conditions, if g is replaced by the acceleration of that field. It should be observed in equation (1) that the initial acceleration under the given conditions ($v = 0$ when $t = 0$) equals g. The acceleration then diminishes and will be zero if $v = \sqrt{\frac{g}{\mu}}$. Examination of (2), however, shows that the speed approaches this value as t increases indefinitely. For this reason this value is called the *limiting speed.* In words: *the speed increases constantly and approaches the limiting value* $\sqrt{\frac{g}{\mu}}$.

If the resistance offered by water to the motion of a ship is proportional to the square of the velocity, then (3) may be applied by replacing g by the acceleration due to the propelling force of the engines. In the same example, if the engines be stopped when the velocity is v_0, the equation of the further motion of the ship is

$$(4) \qquad \frac{dv}{dt} = - \mu v^2,$$

since the resistance is the only force acting. Integrating with the conditions $s = 0$, $v = v_0$ when $t = 0$, we obtain

$$(5) \qquad v = \frac{v_0}{\mu v_0 t + 1}, \quad s = \frac{1}{\mu} \log (\mu v_0 t + 1).$$

The equations (5) may be applied to the problem in question for small values of t, this limitation being necessary on account of complications (drifting, etc.) which must soon arise.

90. Damped harmonic motion. Resistance varying as velocity. The motion of a material particle in a central harmonic field has already been discussed. We now investigate the effect of the presence of a resisting medium in such a field. For small speeds the resistance may be assumed proportional to the first power of the speed. Further, since the resistance and the velocity have opposite directions, we may set

$$\text{Resistance} = -2\,\mu m v,$$

where μ is a positive constant, called the *damping factor.* The force due to the field being equal to $-mk^2s$, the resultant force F acting upon the particle is

$$F = -2\,\mu m v - mk^2 s,$$

and the force equation $\left(\frac{F}{m} = \frac{d^2s}{dt^2}\right)$ consequently may be written in the form

$$(1) \qquad \frac{d^2s}{dt^2} + 2\,\mu \frac{ds}{dt} + k^2 s = 0.$$

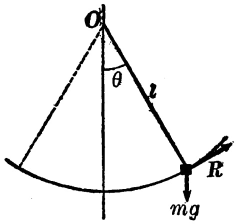
Two important cases present themselves for discussion.

(*a*) *Damping factor small,* $\mu < k$. Equation (1) is now the equation of damped vibration (73, Chap. XIV), the equation of motion being

$$s = Ae^{-\mu t} \cos (\sqrt{k^2 - \mu^2}\, t + \beta),$$

in which A and β are arbitrary constants. The characteristics of this motion have been discussed at length in example 5, p. 51. The motion is therefore a damped vibration with the period $\frac{2\,\pi}{\sqrt{k^2 - \mu^2}}$. Obviously the effect of the damping factor is to *increase* the period, since the latter for the undamped vibration equals $\frac{2\,\pi}{k}$.

A close approximation to damped vibration is afforded by the simple pendulum for small vibrations. To obtain the desired result we may use the moment equation (Art. 62). Taking for center of moments the center of suspension, the total force-moment is



$$Rl - mgl \sin \theta,$$

where

$$R = -2\,m\mu v = -2\,m\mu l\frac{d\theta}{dt}.$$

The angular momentum $= I_0\omega = ml^2\dfrac{d\theta}{dt}$. Hence the moment equation gives

$$-2\,m\mu l^2\frac{d\theta}{dt} - mgl \sin \theta = \frac{d}{dt}\left(ml^2\frac{d\theta}{dt}\right) = ml^2\frac{d^2\theta}{dt^2};$$

that is,

$$\frac{d^2\theta}{dt^2} + 2\,\mu\frac{d\theta}{dt} + \frac{g}{l}\sin \theta = 0.$$

For small amplitudes, $\sin \theta = \theta$, approximately, and this equation now agrees with (1). The discussion confirms the observed facts of the *constancy* in the period of a simple pendulum and the *decrease* in the amplitude.

(*b*) *Damping factor large*, $\mu > k$. The solution of (1) is now (Calculus, p. 437)

(2) $$s = Ae^{r_1t} + Be^{r_2t},$$

in which r_1 and r_2 are the roots of the characteristic equation

$$r^2 + 2\,\mu r + k^2 = 0.$$

Let us discuss (2) for the initial conditions $s = a$, $v = 0$ when $t = 0$. These conditions are nearly met if a large vertical damping vane be affixed to a magnetic needle and if the latter is then slightly turned from its position of equilibrium and released.

Differentiating (2), we obtain

(3) $$v = r_1Ae^{r_1t} + r_2Be^{r_2t}.$$

We have now for $t = 0$,

$$a = A + B, \quad 0 = r_1A + r_2B,$$

and hence

$$A = \frac{ar_2}{r_2 - r_1}, \quad B = \frac{ar_1}{r_1 - r_2},$$

and (2) and (3) become

$$(4)\quad s=\frac{a}{r_1-r_2}(r_1e^{r_2t}-r_2e^{r_1t}),\quad v=\frac{ar_1r_2}{r_1-r_2}(e^{r_2t}-e^{r_1t}).$$

From these equations the following characteristics of the motion are obvious — results agreeing with experience. The distance * diminishes and approaches zero as a limit. The speed increases to a maximum and then diminishes to zero.

PROBLEMS

1. A particle is projected with velocity v_0 into a medium offering a resistance proportional to the velocity $(=kv)$. Show that the particle would come to rest after describing the finite space $\frac{v_0}{k}$ in an infinite time.

2. If the resistance of a medium is kv^2, show that a particle projected with a velocity v_0 would describe an infinite space in an infinite time before coming to rest.

3. If the resistance of the medium per unit mass is kv^2, and a particle slides under the action of gravity on a smooth straight wire inclined at an angle α to the horizontal, prove that the space s described in time t from rest is given by

$$ks=bt-\log\left(\frac{e^{2bt}+1}{2}\right),$$

where $b^2=kg\sin\alpha$.

4. A heavy particle is projected upwards with a velocity V in a medium resisting as the nth power of the velocity. Prove that the whole space (up and down) described when the velocity downwards is V is equal to LT where L is the limiting velocity and T is the time in which the particle falling from rest in the medium will acquire a velocity $\frac{V^2}{L}$.

* Since $r_1(=-\mu+\sqrt{\mu^2-k^2})$ and $r_2(=-\mu-\sqrt{\mu^2-k^2})$ are both negative, e^{r_1t} and e^{r_2t} both approach zero as t increases indefinitely.

CHAPTER X

POTENTIAL AND POTENTIAL ENERGY

91. A constant, harmonic, or general central field has the property: There exists a function U of the coördinates x and y such that the rectangular components of the force of the field are the negative partial derivatives of this function. That is,

$$\frac{\partial U}{\partial x} = -F_x, \quad \frac{\partial U}{\partial y} = -F_y. \tag{1}$$

The function U is called the *potential* of the field. Obviously the potential is given as the integral

$$U = -\int (F_x dx + F_y dy). \tag{2}$$

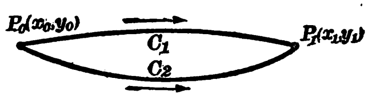
Constant field. Then F_x and F_y are constants, say $F_x = A$, $F_y = B$; hence $U = -(Ax + By)$, and conversely, equations (1) hold.

Harmonic field. Here $F_x = -mk^2x$, $F_y = -mk^2y$; hence $U = -\frac{1}{2}mk^2(x^2 + y^2)$, and conversely, equations (1) hold.

Let it be understood, then, that the potential of a field (*if the field have a potential*) is a function of the coördinates satisfying equations (1).

92. Conservative field. For a field in which there exists a potential the following theorem is characteristic:

The work done by the force of the field upon a material particle moving from one position to another is the same for all paths between those positions.



Proof. If a particle moves from $P_0(x_0, y_0)$ to $P_1(x_1, y_1)$ along a path C_1 in the field whose rectangular components are F_x and F_y, the work done is (Art. 62)

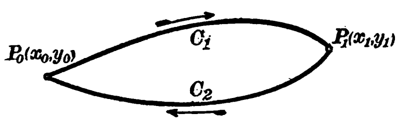
$$\int_{x_0, y_0}^{x_1, y_1} (F_x dx + F_y dy),$$

the integral being worked out for the given path. If, however, the field has a potential, that is, if equation (2) of the preceding section holds, then

$$(1) \qquad \int_{x_0, y_0}^{x_1, y_1} (F_x dx + F_y dy) = -(U_1 - U_0),$$

U_1 and U_0 being the potential at $P_1(x_1, y_1)$ and $P_0(x_0, y_0)$, respectively. But this equation shows that the work done equals the *difference* of the potential at P_1 and at P_0 taken negatively. Hence the work done along any other path C_2 is the same, and the theorem is proved.

Again, let the particle describe any *closed* path from $P_0(x_0, y_0)$. *The total work done is now zero.* For take a second point $P_1(x_1, y_1)$ on the path and denote the path from P_0 to P_1 by C_1, and from P_1 to P_0 by C_2. Then by the result just found we have



$$\text{Work done along } C_1 = -(U_1 - U_0);$$
$$\text{Work done along } C_2 = -(U_0 - U_1).$$

Adding gives the work done along the closed path as zero. The designation *conservative* is applied to a field possessing a potential, and also to the force of such a field. The discussion may be summarized thus:

The work done by a conservative force along any path equals the negative difference of the potential at its extremities.

93. Potential energy. Conservation of energy. Comparison of the energy equation (Art. 62) with the theorem just stated gives the result:

When a material particle describes any path in a conservative field, the change in kinetic energy equals the change in potential taken negatively.

In the form of an equation, this statement reads

$$(1) \qquad \tfrac{1}{2} m v_1^2 - \tfrac{1}{2} m v_0^2 = -(U_1 - U_0),$$

if v_1 and v_0 are the speeds at (x_1, y_1) and (x_0, y_0), respectively.

By transposing in (1), remarking that $\frac{1}{2} m v_0^2$ is a constant, and dropping the subscript 1, we obtain

$$(2) \qquad \tfrac{1}{2} m v^2 + (U - U_0) = \text{constant}.$$

Now $\frac{1}{2}mv^2$ gives the kinetic energy of the particle at any instant. Hence, since each term in this equation must be of the dimensions of energy, we may give to $(U - U_0)$ the name of *potential energy*, that is, we define

(3) $$\text{Potential Energy} = U - U_0;$$

or in words: *The potential energy at any point in a conservative field equals the change in the value of the potential from an arbitrarily chosen point of reference.*

The essential difference between kinetic and potential energy is this: Kinetic energy is due to *motion* — depends upon mass and velocity. Potential energy is due to *relative position* — depends upon the position relative to an assumed point of reference.

In (2), writing

$$E_k = \tfrac{1}{2}mv^2, \quad E_p = U - U_0,$$

we obtain the *equation of energy* for a conservative field:

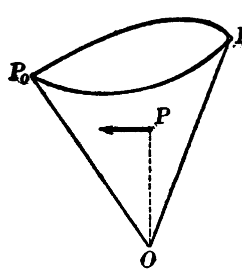
(I) $$\boldsymbol{E_k + E_p = \textbf{constant.}}$$

Equation (I) illustrates the PRINCIPLE OF THE CONSERVATION OF ENERGY for a material particle in a conservative field, namely, this:

If a material particle describes any path in a conservative field, the sum of the kinetic and potential energy remains constant.

A simple illustration of a *non-conservative* field is afforded by the following example. Using polar coördinates, let

$$F_\rho = 0, \quad F_\theta = \tfrac{1}{2}m\rho.$$



The work integral is now (Art. 62)

$$\text{Work} = \int_c \tfrac{1}{2}m\rho^2 d\theta = \frac{m}{2}\int_c \rho^2 d\theta;$$

that is, the work done now equals the mass times the area swept over by the radius vector of the curve in moving from OP_0 to OP_1, a number obviously dependent on the path c between the points. The field in question is a simple one, the force at any point P being perpendicular to the radius vector OP, and proportional to it.

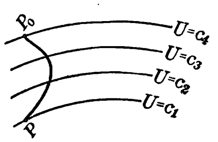
94. Equipotential lines and lines of force. If $U(x, y)$ is the potential function for a certain field, then at every point of the locus of

(1) $$U(x, y) = \text{constant}$$

the potential is the same.

By assigning to the constant in (1) different values, we derive a series of curves called equipotential lines, such that the potential is the same at all points on one of these curves.

For example, in the constant field for which $U = Ax + By$, the equipotential lines consist of the system of parallel lines $Ax + By =$ constant.



If the equipotential lines are drawn in any field, the work done along any path joining two points P_0 and P equals the difference of the potential of the equipotential lines through P and P_0, or equals $(c_1 - c_4)$ in the figure. The slope of the equipotential line (1) at any point (x, y) is (Calculus, p. 202),

$$(2) \qquad \frac{dy}{dx} = -\frac{\dfrac{\partial U}{\partial x}}{\dfrac{\partial U}{\partial y}} = -\frac{F_x}{F_y} \text{ (by (1), Art. 91).}$$

Let us now find the direction of the force F of the field at the point $P(x, y)$. Since the axial components of F are F_x and F_y, then the

$$(2') \qquad \text{slope of the force } F = \frac{F_y}{F_x}.$$

Comparing with (2), it is clear (Analytic Geometry, p. 36) that the direction of F is *perpendicular* to the equipotential line through the point of application.

The system of curves drawn in a field of force such that the direction of the curve through any point is the same as the direction of the force of the field at that point are called *lines of force*. Clearly, the differential equation of these lines is

$$(2'') \qquad \frac{dy}{dx} = \frac{F_y}{F_x}, \text{ or } F_x dy - F_y dx = 0.$$

If this equation can be integrated, the lines of force may be constructed. Or, if the equipotential lines have been drawn, we may construct the lines of force by drawing the orthogonal trajectories. For, as is clear from the above discussion, we have the

THEOREM. *Equipotential lines and lines of force intersect everywhere at right angles.*

The coördinates x and y of any point P on a curve are functions of the arc $s(=P_0P)$ measured from an assumed initial point P_0. Hence, in a conservative field, the potential U along a curve may be considered a function of the length of arc. Then, since now U along the path P_0P is a function of s, we have (Calculus, p. 199)

$$(3) \qquad \frac{dU}{ds}=\frac{\partial U}{\partial x}\frac{dx}{ds}+\frac{\partial U}{\partial y}\frac{dy}{ds}.$$

This becomes, by substitution from (1), Art. 91,

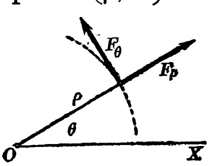
$$(4) \qquad \frac{dU}{ds}=-\left(F_x\frac{dx}{ds}+F_y\frac{dy}{ds}\right).$$

Remembering that $\frac{dx}{ds}$ and $\frac{dy}{ds}$ are the direction cosines of the tangent to the curve C at P, we see (Art. 40) that the second member of (4) gives the tangential component along C of the force due to the field taken negatively. That is, from (4),

$$(5) \qquad \frac{dU}{ds}=-F_t.$$

The derivative of the potential with respect to the arc of a curve equals the tangential component along that curve of the force due to the field, with sign changed.

For example, the components of the force of the field parallel and perpendicular to the radius vector of the point (ρ, θ) are found thus:



Taking the path along the radius vector ($\theta =$ constant), we have

$$ds=d\rho, \therefore \frac{dU}{d\rho}=-F_\rho.$$

Taking the path as the circle $\rho =$ constant, then $ds=\rho d\theta$, and hence

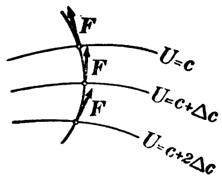
$$\frac{1}{\rho}\frac{dU}{d\theta}=-F_\theta.$$

In particular, if we consider the variation of the potential along a *line of force*, then, in (5), F_t is the force of the field, or F itself.

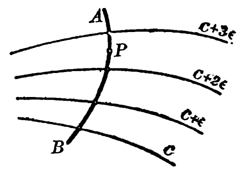
$$(6) \qquad \therefore \textit{along a line of force, } \frac{dU}{ds}=-F.$$

In this equation we may assume the direction of increasing arc to agree with the direction of the line of force. Then, by (6),

$\dfrac{dU}{ds}$ is always negative (or zero) and hence U is a decreasing function.* That is, the potential along a line of force diminishes in the direction of the force. Otherwise expressed thus: *The force at any point in a conservative field is directed towards the region of lower potential.* If, therefore, the equipotential lines are drawn in a field, the direction of the force of the field at each point is uniquely determined.



Finally, the equipotential lines (1), when drawn for equal increments of the potential, for example, c, $c \pm \epsilon$, $c \pm 2\,\epsilon$, $c \pm 3\,\epsilon$, etc., will, if the increment ϵ is small enough, indicate by their degree of proximity the relative magnitude of the force of the field. For, by the theorem of mean value,† we have, using (6),



$$(7) \qquad \Delta U = -\,(F)_{at\,P} \cdot \Delta s,$$

where F is the force of the field at some point P between successive equipotential lines, and Δs is the distance apart of these lines measured along the line of force AB. Since $\Delta U = \epsilon$, (7) becomes, by solving,

$$(F)_{at\,P} = -\frac{\epsilon}{\Delta s}.$$

Hence the force at P is inversely proportional to the normal distance ($=\Delta s$) between consecutive equipotential lines.

Our results are summarized as follows:

THEOREM. *When the equipotential lines are drawn in a conservative field for equal small increments of the potential, the force of the field at any point is inversely proportional to the normal distance between consecutive lines.*

That is, the force is greatest where the equipotential lines are most dense, etc. In a field for which the force is everywhere of

* The function of s increases or decreases with s according as its derivative with respect to s is positive or negative (Calculus, p. 116).

† This theorem may be stated: Given a function $U(s)$, for which $\dfrac{dU}{ds} = -F$, then $U(s) - U(s_0) = -(F(s'))(s - s_0)$, where s' is a value of s between s_0 and s, and $F(s')$ is the value of F when $s = s'$.

the same magnitude, the equipotential lines drawn for equal increments are equidistant.

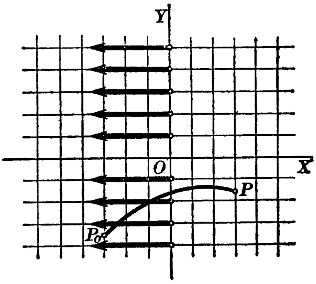
We now illustrate the preceding by examples.

ILLUSTRATIVE EXAMPLES

1. Discuss the conservative field for which the potential is $U = a^2x$.

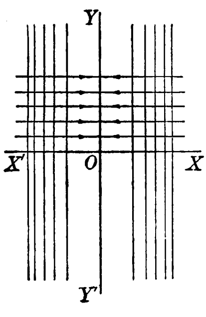
The equipotential lines are $a^2x = c$, that is, lines parallel to YY'. Since

$$F_x = -\frac{\partial U}{\partial x} = -a^2, \quad F_y = -\frac{\partial U}{\partial y} = 0,$$



the lines of force are parallel to XX' and are directed to the left. Setting $c = 0, \pm\epsilon, \pm 2\epsilon$, etc., we obtain the equidistant equipotential lines of the figure, $x = 0$, $x = \pm\frac{\epsilon}{a^2}$, $x = \pm\frac{2\epsilon}{a^2}$, etc. The work done by the field along any path from P_0 to P equals $-\frac{3\epsilon}{a^2} - \frac{3\epsilon}{a^2} = -\frac{6\epsilon}{a^2}$, a negative number, since the motion is against the field.

2. Discuss the conservative field for which the potential is $U = \frac{1}{2}k^2x^2$.

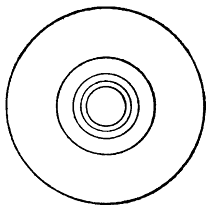


The equipotential lines are $\frac{1}{2}k^2x^2 = c$, lines parallel to YY'. We find

$$F_x = -\frac{\partial U}{\partial x} = -k^2x, \quad F_y = 0,$$

so that the force at any point is directed towards YY', and is proportional to the distance of that point from YY' Setting $c = 0, \pm\epsilon, \pm 2\epsilon, \pm 3\epsilon$, etc., we obtain the equipotential lines of the figure. It is observed that these lines are closer together as we recede from YY', an indication of increasing force. The potential is a minimum when $x = 0$, and the necessary condition for this, namely, $\frac{dU}{dx} = 0$ when $x = 0$, is seen to be satisfied.

3. Discuss the conservative field for which the potential is $U = \frac{k^2}{\rho}$.



The equipotential lines are the concentric circles $\frac{k^2}{\rho} = c$, or $\rho = k^2 \div c$. We find $F_\rho = -\frac{\partial U}{\partial \rho} = -\frac{k^2}{\rho^2}$, $F_\theta = -\frac{1}{\rho}\frac{\partial U}{\partial \theta} = 0$. In the figure the equipotential lines are drawn for $c = k^2$, $k^2 \pm \epsilon$, $k^2 \pm 2\epsilon$, etc. The decreasing distance between the circles as the center is approached indicates increasing force.

4. Discuss the Principle of the Conservation of Energy for the motion of a projectile when the angle of elevation is $\frac{1}{2}\pi$.

Choose the initial position O as the point of reference for determining potential energy. Then in the equation of energy (I) we have, since at O

$$E_p = 0, \quad E_k = \tfrac{1}{2} mv_0^2, \text{ the result}$$

$$(1) \qquad E_k + E_p = \tfrac{1}{2} mv_0^2.$$

The lines of force being directed downwards, E_p increases during ascent, and diminishes during descent. Hence during ascent E_k must constantly decrease; that is, the velocity must decrease, and become zero, namely, at the highest point. The maximum value of E_p is therefore $\frac{1}{2} mv_0^2$. In descent, since E_p constantly diminishes, E_k increases without limit, that is, the speed increases.

Analytically, we have for the field

$$F_x = 0, \quad F_y = - mg.$$

$$\therefore \; E_p = U - U_0 = - \int_0^y F_y dy = mgy.$$

Hence the equation of energy is

$$\tfrac{1}{2} mv^2 + mgy = \tfrac{1}{2} mv_0^2.$$

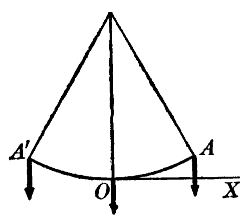
If h is the greatest height, then

$$mgh = \tfrac{1}{2} mv_0^2, \quad \text{or } h = v_0^2 \div 2\,g.$$

5. Discuss the equation of energy for a simple pendulum.

Taking the lowest point O for reference, then in (I), if v_0 is the speed at O, we have for the equation of energy

$$(1) \qquad E_k + E_p = \tfrac{1}{2} mv_0^2.$$

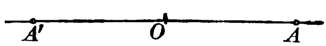


Since the lines of force are directed downwards, E_p increases in ascent and diminishes during descent. Hence, as in the preceding example, E_k must diminish when the particle moves from O towards A; that is, the speed must decrease to zero at A. At this extreme position, E_p is a maximum. In descent from A, E_p diminishes towards zero; E_k increases, reaching the maximum value ($= \frac{1}{2} mv_0^2$) again at O. This cycle is repeated from O to A' and A' to O. The motion is therefore a ceaseless vibration from A to A'.

Analytically, as before, $E_p = mgy$, and (1) is $\frac{1}{2} mv^2 + mgy = \frac{1}{2} mv_0^2$. The greatest vertical height is therefore $v_0^2 \div 2\,g$.

6. Discuss the equation of energy for simple harmonic motion.

Assuming the center O as point of reference and v_0 as velocity at O, the equation of energy is



$$(1) \qquad E_k + E_p = \tfrac{1}{2} mv_0^2.$$

From example 2, we know that E_p increases when the motion is away from the center, and diminishes for motion towards the center. Hence E_k must constantly diminish and become zero when the particle moves away from the center, and must subsequently increase until the center is again reached. We see, therefore, as in example 5, that the motion must be a ceaseless vibration.

Analytically, since $F_x = - mk^2 x$, $F_y = 0$, we have

$$E_p = - \int_0^x F_x dx = \tfrac{1}{2} mk^2 x^2.$$

Hence (1) is $\frac{1}{2} mv^2 + \frac{1}{2} mk^2 x^2 = \frac{1}{2} mv_0^2$, and for the amplitude we find

$$x = \pm \, v_0 \div k.$$

PROBLEM

1. Discuss the conservative fields for which the potential function U is that given:

(a) by; (b) $ax + by$; (c) ax^2; (d) by^2; (e) $ax^2 + by^2$; (f) cxy; (g) $ax^2 + by^2 + 2\,dx + 2\,ey$.

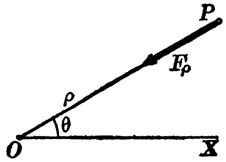
95. Non-conservative forces. Friction. The work done by the force in a conservative field changes sign when the direction of the motion is reversed. It is therefore obvious that friction is not a conservative force. For the direction of the force of friction is reversed when the direction of motion changes, and consequently the work done by a frictional resistance does not change sign. For example, if the resistance offered by the air is considered in the motion of a particle projected vertically upwards, the frictional resistance is opposed to the motion in all positions, and consequently the work done is negative in both ascent and descent. In the illustrative example 4 of Art. 94, therefore, we see that *in nature* the velocity of the projectile will be lessened, a conclusion agreeing with the observed fact that the velocity is *less* when the projectile returns to the initial position. Again, in the discussion of the simple pendulum (example 5, Art. 94), if the frictional resistances present in nature are considered, the motion will not be perpetual. Friction will act to diminish the velocity.

In general, non-conservative forces are said to be dissipative, since by their action kinetic energy is lessened without an equivalent increase of potential energy. In nature, non-conservative forces are always present, and accordingly the principle of the conservation of energy does not apply in the form enunciated in Art. 93. To cover the actual facts, thermal and chemical energy must be considered, matters with which we are not concerned in this volume.

96. Newtonian potential. According to Newton's Law of Universal Gravitation (Art. 81), two particles attract each other with a force varying directly as the mass of each and inversely as the square of their distance apart. The force is therefore, with proper notation and units,

$$(1) \qquad F = -\frac{mm'}{\rho^2}.$$

Consider, now, the plane field of force due to the attraction of a particle of mass m situated at the origin. The force upon *unit* mass at any point is, therefore,



(2) $$F_\rho = -\frac{m}{\rho^2}, \quad F_\theta = 0.$$

The potential function is, accordingly,

(3) $$U = C - \int F_\rho d\rho = C - \frac{m}{\rho}.$$

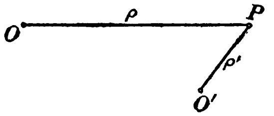
If, for $\rho = \infty$, we assume $U = 0$, then $C = 0$, and (3) becomes

(4) $$U = -\frac{m}{\rho}.$$

The *newtonian potential* at P is defined as equal to $m \div \rho$, or, denoting this by N,

(5) $$N = -U = \frac{m}{\rho}.$$

From the result of Art. 93, we may make the definition: *The newtonian potential at any point equals the work done in moving up to that point from infinite distance.*



To study the field due to two attracting centers of masses m and m', we merely have to observe that the newtonian potential at any point P must equal the sum of the potentials due to the separate masses. This appears at once from the above definition. Hence,

(6) $$N = \frac{m}{\rho} + \frac{m'}{\rho'}.$$

Similarly for any number of centers. The newtonian potential due to the attraction of a continuous solid is readily defined, for if the solid is divided into elements of mass, and a point chosen in each element, then, proceeding as usual, we define the potential N at P as

(7) $$N = \int \frac{dm}{\rho},$$

in which ρ denotes the distance from P to any point of the solid.

Illustrative Example. Find the potential due to an attracting thin homogeneous spherical shell of density τ. Find the force of the field at any point.

Solution. Take the center of the sphere as origin and draw OX through the point P. Let $OP = c$. We may consider the shell divided into strips by planes passed through it perpendicularly to OX. The mass dm of one of these strips* is $2\pi a dx$ times τ. The distance ρ from the strip to P is $\sqrt{y^2 + (c - x)^2}$. Hence

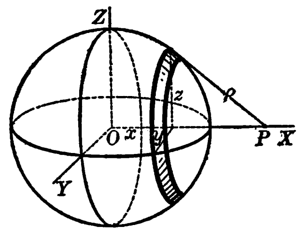
$$N = \int_{-a}^{a} \frac{2\pi a \cdot \tau dx}{\sqrt{y^2 + (c - x)^2}} = 2\pi a\tau \int_{-a}^{a} \frac{dx}{\sqrt{a^2 + c^2 - 2cx}}.$$

The radical must be taken with the positive sign, and this makes it necessary to distinguish two cases.

(I) $c > a$, *exterior point.* Then

$$N = -\frac{2\pi a\tau}{c}[c - a - (c + a)] = \frac{4\pi a^2\tau}{c} = \frac{\text{mass of shell}}{c}.$$

Hence the newtonian potential for a spherical shell at an exterior point is the same as if its mass were concentrated at its center.



(II) $c < a$, *interior point.* Now

$$N = -\frac{2\pi a\tau}{c}[a - c - (a + c)] = 4\pi a\tau,$$

that is, is the same at *every* interior point. *Hence the force of the field is zero within the shell.*

The result for an exterior point is extended at once to a solid sphere by conceiving it to be made up of concentric shells. Hence the

THEOREM. *The newtonian potential for a solid sphere at an external point is the same as if its mass were concentrated at its center.*

Consider now an interior point of a solid sphere, at the distance of x from its center. We conceive this sphere as consisting of (a) a spherical shell of interior radius x and exterior radius equal to that of the solid sphere; (b) a solid sphere of radius x.

Take these up in order.

(a) Since the potential is everywhere constant within the shell, its attraction is zero. Consequently, the attraction exerted is that due to the solid sphere of radius x.

(b) The mass of this sphere is $\frac{4}{3}\pi\tau x^3$, and hence the attraction is by the theorem $-\frac{4}{3}\pi\tau x^3 \div x^2$, or equal to $-\frac{4}{3}\pi\tau x$. That is, the attraction exerted by a solid sphere upon a point within it is proportional to the distance of that point from the center of the sphere.

Finally, we readily see that the mutual force of attraction of two solid spheres is the same as if their masses were concentrated at their respective centers.

PROBLEMS

1. Find the newtonian potential due to a line distribution of matter (thin uniform rod) at a point on the line produced.

Ans. $N = \tau \log\left(1 + \frac{l}{d}\right)$, where l = length of line, d = distance of point from nearest end.

* The surface of a zone equals the product of its altitude by the circumference of a great circle.

2. Find the force of attraction of the rod in the first example upon unit mass at the given point.

$$Ans.\quad F=\frac{dN}{dx}=-\frac{\tau}{d(l+d)}.$$

3. Find the newtonian potential due to a homogenous circular disk at a point on the line through the center of the disk and perpendicular to its plane.

Ans. $N=\frac{2m}{a^2}(\sqrt{a^2+x^2}-x)$, where $m=$ mass of disk, $x=$ distance of point from disk.

4. Find the force of attraction exerted by the disk in problem 3 upon unit mass at the given point.

$$Ans.\quad F=\frac{2m}{a}\left(\frac{x}{\sqrt{a^2+x^2}}-1\right).$$

5. Find the newtonian potential due to the attraction exerted by a homogeneous right cylinder or cone at a point upon the axis.

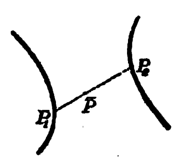
6. Find the newtonian potential due to a homogeneous square plate at a point on a side of the square produced.

CHAPTER XI

SYSTEM OF MATERIAL PARTICLES

97. System in a plane. The preceding chapters have been devoted to the study of motion of a single particle. We shall now study the simultaneous motion of two or more particles, and in this manner prepare ourselves for the study of motion of a solid.

Consider a system of two particles moving in a plane. Let m_1 and m_2 be their masses, and the positions at any instant $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, respectively. Then if (F_{1x}, F_{1y}) are the axial components of the *complete resultant* of the forces acting upon the particle at P_1, and (F_{2x}, F_{2y}) the same at P_2, the force equations (Art. 62) are, for the first particle,



$$(1) \qquad m_1 \frac{d^2x_1}{dt^2} = F_{1x}, \quad m_1 \frac{d^2y_1}{dt^2} = F_{1y},$$

and for the second particle,

$$(2) \qquad m_2 \frac{d^2x_2}{dt^2} = F_{2x}, \quad m_2 \frac{d^2y_2}{dt^2} = F_{2y}.$$

Motion of the center of gravity. Adding the first equations in (1) and (2), we obtain

$$(3) \qquad m_1 \frac{d^2x_1}{dt^2} + m_2 \frac{d^2x_2}{dt^2} = F_{1x} + F_{2x};$$

also from the second equations we get

$$(4) \qquad m_1 \frac{d^2y_1}{dt^2} + m_2 \frac{d^2y_2}{dt^2} = F_{1y} + F_{2y}.$$

If $\bar{P}(\bar{x}, \bar{y})$ is the center of gravity of the given particles, then (Art. 22)

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \quad \bar{y} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2},$$

or also

$$(5) \quad m_1x_1 + m_2x_2 = (m_1 + m_2)\bar{x}, \quad m_1y_1 + m_2y_2 = (m_1 + m_2)\bar{y}.$$

Differentiating (5) twice with respect to t, and substituting in the first members of (3) and (4), we obtain

$$(6)\quad (m_1+m_2)\frac{d^2\bar{x}}{dt^2}=F_{1x}+F_{2x},\quad (m_1+m_2)\frac{d^2\bar{y}}{dt^2}=F_{1y}+F_{2y}.$$

The second member of the first of equations (6) is the sum of the X-components of the resultant forces acting on the first and second particles which comprise the system. It is therefore the sum of the X-components of all forces acting upon the system, and will be denoted by F_x. Similarly the second member of the second of equations (6) is the sum of the Y-components of all forces acting upon the system, and will be denoted by F_y. Denoting the mass of the system, which is the sum of the masses of the individual particles, by M, equations (6) may be written

$$(7)\qquad M\frac{d^2\bar{x}}{dt^2}=F_x,\quad M\frac{d^2\bar{y}}{dt^2}=F_y.$$

These are the fundamental equations of motion of the center of gravity of the system. Their discussion shows that this point moves *as if* forces equal and parallel to the given forces were acting at that point upon a mass equal to the combined mass.

That is, *the center of gravity moves as if the total mass of the system were concentrated at that point and acted upon by forces equal and parallel to the given forces.*

98. System in space. Let the masses of n particles moving in space of three dimensions be denoted by $m_1, m_2, \cdots m_n$. Let the positions at any instant be given by $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $\cdots P_n(x_n, y_n, z_n)$, and the axial components of the *complete resultant* of the forces acting upon the individual particles by (F_{1x}, F_{1y}, F_{1z}), (F_{2x}, F_{2y}, F_{2z}), $\cdots (F_{nx}, F_{ny}, F_{nz})$. The force equations for the separate particles are

$$(1)\qquad \begin{cases} m_1\dfrac{d^2x_1}{dt^2}=F_{1x}, & m_1\dfrac{d^2y_1}{dt^2}=F_{1y}, \quad m_1\dfrac{d^2z_1}{dt^2}=F_{1z}, \\ m_2\dfrac{d^2x_2}{dt^2}=F_{2x}, & m_2\dfrac{d^2y_2}{dt^2}=F_{2y}, \quad m_2\dfrac{d^2z_2}{dt^2}=F_{2z}, \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ m_n\dfrac{d^2x_n}{dt^2}=F_{nx}, & m_n\dfrac{d^2y_n}{dt^2}=F_{ny}, \quad m_n\dfrac{d^2z_n}{dt^2}=F_{nz}. \end{cases}$$

Motion of the center of gravity. Adding the first column of equations (1), we get

$$(2) \quad m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + \cdots m_n\frac{d^2x_n}{dt^2} = F_{1x} + F_{2x} + \cdots F_{nx}.$$

If $\overline{P}(\bar{x}, \bar{y}, \bar{z})$ is the center of gravity of the system, then

$$(3) \qquad \bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots m_nx_n}{m_1 + m_2 + \cdots m_n}.$$

If $M (= m_1 + m_2 + \cdots m_n)$ is the total mass of the system, and $F_x(= F_{1x} + F_{2x} + \cdots F_{nx})$ is the sum of the X-components of all forces acting on the system, it is evident that equation (2) may be written

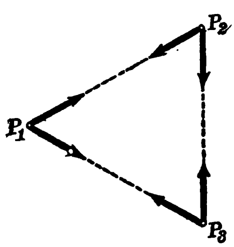
$$M\frac{d^2\bar{x}}{dt^2} = F_x.$$

Similar equations in y and z respectively follow from the second and third columns of (1). Hence the fundamental equations for the motion of the center of gravity of the system are

$$(4) \qquad M\frac{d^2\bar{x}}{dt^2} = F_x, \quad M\frac{d^2\bar{y}}{dt^2} = F_y, \quad M\frac{d^2\bar{z}}{dt^2} = F_z.$$

Equations (4) show that the statement of the preceding article concerning two particles moving in a plane holds also for n particles moving in space. Hence the

THEOREM. *The center of gravity of a system of material particles moves as if the total mass of the system were concentrated at that point, and acted upon by forces equal and parallel to the given forces.*



In particular, if the vector sum of all forces acting is zero, then

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0,$$

and the center of gravity moves with uniform rectilinear motion. For example, in the case of three particles moving under the action of their mutual gravitational attraction, the forces are in pairs equal in magnitude but opposite in direction, and their vector sum is zero. Hence the

THEOREM. *The center of gravity of a system of particles moving under the action of their mutual gravitational attraction will describe a straight line with constant speed.*

This result is called the *principle of the conservation of the motion of the center of gravity.*

99. Moment equation for a system of particles. Let the first of equations (1), Art. 97, be multiplied by y_1 and the second by x_1. Then, by subtraction,

$$(1) \qquad m_1\left(x_1\frac{d^2y_1}{dt^2} - y_1\frac{d^2x_1}{dt^2}\right) = x_1F_{1y} - y_1F_{1x}.$$

It may be easily verified (see Art. 59) that

$$x_1\frac{d^2y_1}{dt^2} - y_1\frac{d^2x_1}{dt^2} = \frac{d}{dt}\left(x_1\frac{dy_1}{dt} - y_1\frac{dx_1}{dt}\right).$$

Hence equation (1) may be written (see Art. 59)

$$(2) \qquad \frac{d}{dt}\left(x_1m_1\frac{dy_1}{dt} - y_1m_1\frac{dx_1}{dt}\right) = x_1F_{1y} - y_1F_{1x}.$$

Now the second member is the moment (Art. 58) of F_1 with respect to the origin, and the quantity in parenthesis is the moment of momentum (Art. 59) of m_1 with respect to the origin. Denoting the latter quantity by H_1, equation (2) becomes

$$\frac{dH_1}{dt} = \text{moment of } F_1.$$

Also from equations (2), Art. 97, similarly,

$$\frac{dH_2}{dt} = \text{moment of } F_2.$$

If $H = H_1 + H_2$ denotes the *total moment of momentum* of the system, and if the *total force-moment* of the system is defined as the sum of the moments of the resultant forces acting on each particle, we have, by addition,

$$\frac{dH}{dt} = \text{total force-moment.}$$

This result is clearly a generalization of (VIII), Art. 59.*

* This result, proved for two particles in the XY-plane, may be extended by a similar process to any number of particles in space of three dimensions.

In particular, if the forces acting are as before mutual gravitational attractions only, the forces and consequently their moments cancel in pairs, and

$$\frac{dH}{dt} = 0, \text{ or}$$

$$H = \text{constant.}$$

THEOREM. PRINCIPLE OF THE CONSERVATION OF MOMENT OF MOMENTUM. *The total moment of momentum in any system of particles moving under the action of their mutual gravitation is invariable.*

100. Work and energy of the system. The total work done upon a system of material particles is obtained by summing up the work done upon each individual particle by the resultant of the forces acting upon it. The kinetic energy of the system at any instant is the sum of the kinetic energy of the individual particles. Referring to the two particles of Art. 97, we have for any displacement,

Work done by $F_1 = \frac{1}{2}(m_1v_1^2 - m_1v_1'^2) =$ change in K.E. of m_1.

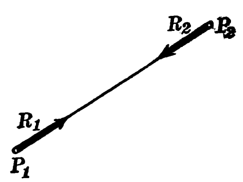
Work done by $F_2 = \frac{1}{2}(m_2v_2^2 - m_2v_2'^2) =$ change in K.E. of m_2.

Hence, by addition,

(1) **Total work done** $= \frac{1}{2}[m_1v_1^2 + m_2v_2^2 - (m_1v_1'^2 + m_2v_2'^2)]$
$=$ **change in K. E. of system.**

As in the preceding articles the result expressed by equation (1) is general.

101. Rigid system of particles. An especially important example of motion of a system of particles arises when the system is *rigid;* that is, when the mutual distance of each pair of points is invariable. The importance of this case is due primarily to its application in the case of a rigid solid body, which is then regarded as a continuous rigid system.



The rigidity is to be regarded as maintained by a constraint which exerts upon any pair of particles P_1 and P_2 reactions equal in magnitude but opposite in direction. These reactions are unknown, but cancel out in the equations of motion, as shall now appear.

The forces acting in the motion of any rigid system may be classified as

(1) reactions due to the rigidity;
(2) impressed forces.

Let (R_{1x}, R_{1y}) be the axial components of the reaction due to rigidity at P_1, and (R_{2x}, R_{2y}) be the components of this force at P_2. Then

$$R_{1x} + R_{2x} = 0; \quad R_{1y} + R_{2y} = 0. \tag{1}$$

Let the sum of the axial components of the impressed forces at P_1 be (F_{1x}, F_{1y}) and at P_2 be (F_{2x}, F_{2y}). Then the equations of motion are, for the first particle,

$$m_1\frac{d^2x_1}{dt^2} = F_{1x} + R_{1x}, \quad m_1\frac{d^2y_1}{dt^2} = F_{1y} + R_{1y}, \tag{2}$$

and for the second particle,

$$m_2\frac{d^2x_2}{dt^2} = F_{2x} + R_{2x}, \quad m_2\frac{d^2y_2}{dt^2} = F_{2y} + R_{2y}. \tag{3}$$

Then the method of Art. 97 gives at once the

THEOREM. *The center of gravity of any rigid system moves as if the mass of the entire system were concentrated at that point and acted upon by forces equal and parallel to all impressed forces.*

In analytic form this theorem is for space

$$\boldsymbol{M}\frac{d^2\bar{x}}{dt^2} = \boldsymbol{F}_x, \quad \boldsymbol{M}\frac{d^2\bar{y}}{dt^2} = \boldsymbol{F}_y, \quad \boldsymbol{M}\frac{d^2\bar{z}}{dt^2} = \boldsymbol{F}_z, \tag{I}$$

where M is the total mass of the system, F is the resultant of the impressed forces, acting at the center of gravity $(\bar{x}, \bar{y}, \bar{z})$.

For a rigid system of particles in the XY-plane we may obtain by the process of Art. 99 the

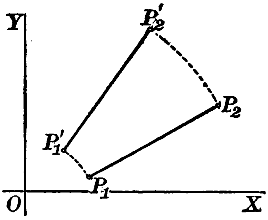
THEOREM. *The time-rate of change of the total moment of momentum of any rigid system equals the total force-moment of the impressed forces.*

In analytic form, this is

$$\frac{d}{dt}\sum_{i=1}^{n} m_i\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right) = \sum_{i=1}^{n}\left(x_iF_{iy} - y_iF_{ix}\right). \tag{II}$$

In applying this theorem, any point whatever may be chosen as the center of moments.

Consider next the question of work and energy, supposing the two particles of Art. 97 are rigidly connected. Let the equations of motion of P_1 and P_2 be, respectively,



$$(4) \qquad \begin{aligned} x_1 &= \phi_1(t), & x_2 &= \phi_2(t), \\ y_1 &= \psi_1(t), & y_2 &= \psi_2(t). \end{aligned}$$

At the instant $t = 0$ let the position of the system be P_1P_2 in the figure. At any other instant, t, let the position be $P_1'P_2'$. The total work done by the reactions due to rigidity is then given by the sum of the definite integrals

$$\int_0^t (R_{1x}dx_1 + R_{1y}dy_1) + \int_0^t (R_{2x}dx_2 + R_{2y}dy_2).$$

By virtue of equations (4) the integrands are functions of t alone, and since from (1), $R_{1x} = -R_{2x}$, $R_{1y} = -R_{2y}$, we may write the expression for the work in the form

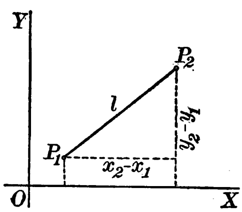
$$(5) \qquad \int_0^t [R_{1x}(dx_1 - dx_2) + R_{1y}(dy_1 - dy_2)].$$

Now

$$(6) \qquad (x_1 - x_2)^2 + (y_1 - y_2)^2 = l^2,$$

where l is *constant*. Differentiating (6), we obtain

$$(7) \qquad (x_1 - x_2)(dx_1 - dx_2) + (y_1 - y_2)(dy_1 - dy_2) = 0.$$



Now $\quad R_{1x} = R_1 \cos(x, l) = R_1 \dfrac{x_2 - x_1}{l},$

$$R_{1y} = R_1 \sin(x, l) = R_1 \frac{y_2 - y_1}{l}.$$

$$\therefore x_2 - x_1 = \frac{l}{R_1} R_{1x}, \quad y_2 - y_1 = \frac{l}{R_1} R_{1y}.$$

Hence (7) becomes after dividing out $\dfrac{l}{R_1}$,

$$R_{1x}(dx_1 - dx_2) + R_{1y}(dy_1 - dy_2) = 0.$$

Hence the definite integral in (5) is zero, and the reactions R_1 and R_2 *do no work*. We see therefore that in a rigid system the work done is contributed by the impressed forces only. This result can be extended at once to a system of n particles, and we have

(III)

$$\textbf{Work done by impressed forces} = \textbf{change in kinetic energy}$$

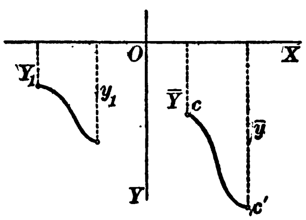
$$= \sum_{i=1}^{n} (m_i v_i^2 - m_i v_{i0}^2).$$

Work done in a constant field upon a rigid system. Let the constant force be weight, and draw the axis of Y vertically downwards. Then for a single particle the work done is

$$m_1 g\ (y_1 - Y_1),$$

if y_1 and Y_1 are the final and initial ordinates. Hence the total work is

$$(8) \qquad \sum_{i=1}^{n} m_i g(y_i - Y_i) = g\left[\sum m_i y_i - \sum m_i Y_i\right].$$



If $\bar{Y}$ and $\bar{y}$ are the initial and final ordinates of the center of gravity, then

$$\bar{Y} = \frac{\sum m_i Y_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

Hence (8) becomes, if $\sum m_i = M$,

$$(IV) \qquad \textbf{Total work done by gravity} = Mg(\bar{y} - \bar{Y}).$$

This gives the important

THEOREM. *If a rigid system is in motion under the action of weight only, the total work done equals the total weight of the system multiplied by the vertical displacement of the center of gravity.*

PROBLEMS

1. Two particles of masses 50 lb. and 40 lb. are acted upon at a certain instant by parallel forces of 75 poundals and 60 poundals, respectively, whose lines of action are 4 ft. apart and perpendicular to the line joining the particles. Determine (a) the position of the center of gravity and (b) its acceleration at the instant named.

Ans. (b) 1.5 ft. per second per second, if the forces have the same direction.

2. If the two particles of problem 1 attract each other with forces of 40 poundals, the remaining data being as before, compute (a) the acceleration of each particle and (b) the acceleration of the center of gravity.

Ans. (b) 1.5 ft. per second per second.

3. A particle of mass m slides down a smooth inclined plane of angle α, the plane itself (mass M) being free to slide on a smooth table. Find the acceleration of the particle and of the plane.

4. A system consists of two particles, of which one (m_1) moves always on the X-axis with an acceleration $-k^2x$, and the other (m_2) along the Y-axis with an acceleration $-k^2y$. Discuss the motion of the center of gravity.

5. To the system of problem 4 is added a third particle (m_3) which moves along a line whose inclination to the X-axis is 45° with a constant acceleration α. Discuss the motion of the center of gravity.

6. Three particles in the XY-plane are acted upon by forces as follows:

m_1 by a force equal to kt whose inclination to X-axis is 45°,
m_2 by a force equal to kt whose inclination to X-axis is 135°,
m_3 by a force equal to $-\sqrt{2}\,kt$ in the direction of Y-axis.

Show that the center of gravity moves uniformly in a straight line.

CHAPTER XII

DYNAMICS OF A RIGID BODY

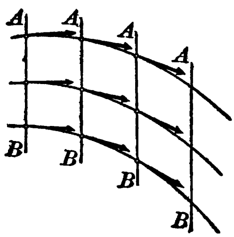
KINEMATICS

102. Rigid body. A rigid body is defined mathematically as a continuous system of material particles whose mutual distances remain unchanged. The motion of a rigid body is known if the motion of each point of the body is known. More explicitly, we say the motion of a rigid body is completely determined if we know:

(1) the position of the body at any instant;
(2) the velocity of each point at any instant;
(3) the acceleration of each point at any instant.

The position, velocity, and acceleration of each point are known, if the position, velocity, and acceleration * of three of the points not on the same straight line are known. Hence in the general case, the discussion of the motion of a rigid body may be reduced to the discussion of the motion of a system of three particles, forming an invariable triangle. For practical purposes we confine our attention to the simple types of motion treated below.

103. Translation. The motion of a body is a translation if every line in it remains parallel to its original position. Such a motion is observed in the driving rod of a locomotive or in the motion of a book sliding upon a table so that one edge of the book remains parallel to one edge of the table. At any instant *every point of the body has the same velocity* both in direction and magnitude. The motion is completely determined if the motion of a single point is known, *e.g. the motion of the center of gravity.*



* Since the mutual distances of these points are invariable, these quantities are not independent.

In *uniform translation* the velocity is constant and the path of any point is a straight line.

In *uniformly accelerated translation* the acceleration is constant and the path of any point is rectilinear.

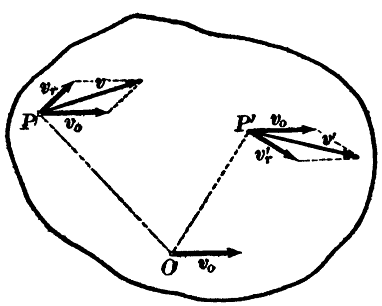
104. Rotation. The motion is a rotation if the body turns around a fixed axis, its points describing circles which lie in planes perpendicular to the axis and have their centers on the axis. An example of rotation is furnished by a fly wheel. At any instant, *every point of the body has the same angular velocity about the axis.* The motion is completely determined by the motion of a single point.

In *uniform rotation* the angular velocity is always constant.

In *uniformly accelerated rotation* the angular acceleration is always constant.

105. Uniplanar motion. In this type of motion the body moves so that all its points move parallel to a fixed plane. Examples are furnished by a rolling cylinder and the connecting rod of an engine.

The velocity of each point is parallel to this fixed plane, which is called the *directing plane.* Each line in the body perpendicular to the directing plane moves parallel to itself, and *at any instant* every point of this line has the same velocity. Consequently, we need to study the motion only of all points in the body lying in a plane parallel to the directing plane. Let the plane of the paper be such a plane. Let O and P be any two points.



Let $\mathbf{v}_0$ = velocity of O; $\mathbf{v}$ = velocity of P.

Since the distance OP is invariable, the motion of P *relative to* O must be a rotation about O. If $\mathbf{v}_r$ is the velocity of P relative to O, then $\mathbf{v}_r$ must be perpendicular to OP. The actual velocity ($\mathbf{v}$) of P is then compounded of the velocity ($\mathbf{v}_0$) of O and the velocity ($\mathbf{v}_r$) of P relative to O. That is in the sense of vector addition $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_r$.

The velocity of any other point P' in the section may be treated in the same way. The velocity ($\mathbf{v}'_r$) of P' relative to O must be perpendicular to OP' and, since the body is rigid, we have the proportion $v_r : OP :: v'_r : OP'$. The actual velocity ($\mathbf{v}'$) of P' is given by $\mathbf{v}' = \mathbf{v}'_r + \mathbf{v}_0$. Hence the motion of the body is at any instant compounded of:

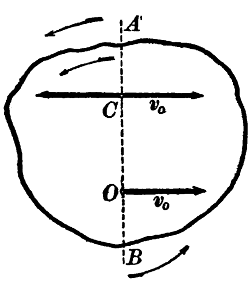
(*a*) a rotation about a temporary axis chosen arbitrarily perpendicular to the directing plane; and

(*b*) a translation parallel to the directing plane with a velocity equal to that of any point on the temporary axis.

In the notation used,

$$(1)\qquad \begin{cases} \text{Velocity of translation} = \mathbf{v}_0\,; \\ \text{Angular velocity of rotation } \omega = \dfrac{v_r}{OP}. \end{cases}$$

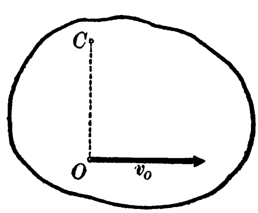
THEOREM. INSTANTANEOUS AXIS. *There is at each instant an axis perpendicular to the directing plane which is at rest.*



Proof. Draw AB in the plane of the section at right angles to $\mathbf{v}_0$. Assume the direction of rotation of AB about O as in the figure. Lay off on AB

$$(2)\qquad OC = \frac{v_0}{\omega},$$

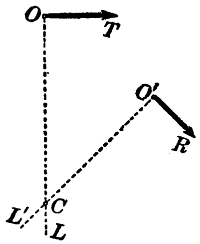
where v_0 = speed of O, ω = angular velocity about O. Then the velocity of rotation of C about $O = -\mathbf{v}_0$. Since the actual velocity of C is the resultant of $\mathbf{v}_0$ and $-\mathbf{v}_0$, it is zero. Q.E.D.



The point C is the *instantaneous center* of the section. The locus of the instantaneous centers is the instantaneous axis. Since the velocity of O (any point) is $v_0 = \omega \cdot OC$ by (2), the *angular velocity* of the body about the instantaneous axis is ω also, that is:

THEOREM. *In the resolution of a uniplanar motion into a translation and a rotation, the angular velocity about the axis is the same for all axes.*

To construct the instantaneous center. Given the actual velocities of two points O and O' in a section parallel to the directing plane. Let OT and $O'R$ be the vectors representing the velocities. From the preceding proof of the existence of an instantaneous center C, it is seen that C must lie on a line perpendicular to OT, and also on a line perpendicular to $O'R$. Draw OL and $O'L'$ perpendicular to OT and OR, respectively. Then the intersection of OL and $O'L'$ is the instantaneous center. To construct the instantaneous center in any uniplanar motion it is necessary to know only the *directions* of the motion of two of the points.

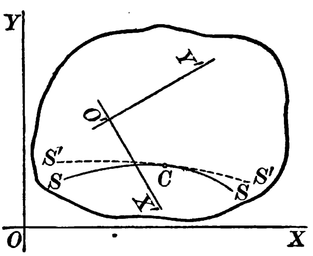


Q.E.F.

In the preceding discussion we have proved the

THEOREM. *Uniplanar motion may at any instant be regarded as a rotation about the instantaneous axis.*

106. Centrodes. The instantaneous center moves both relative to the body and in space. The locus of its various positions relative to the body is called the *body centrode* SS. The locus of its various positions in space is called the *space centrode* $S'S'$. The body centrode is *fixed relative to the body*. The space centrode is *fixed in space*. These two loci are tangent at any instant. The motion of the body may be arrived at by rolling the body centrode upon the space centrode.

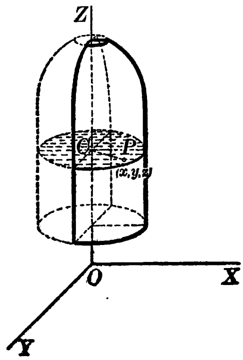


107. Screw motion. In this type the motion is compounded of a translation and rotation. The body rotates about an axis in space and at the same time undergoes a translation along the axis. The motion of the points in the axis of rotation is clearly a translation. The path of any other point is a curve traced on a cylinder about the axis of rotation. This curve is a helix if the angular velocity bears a constant ratio to the velocity of translation. The position of the body at any instant is given by the position of one of its points (provided this does not lie on the axis of rotation).

ILLUSTRATIVE EXAMPLES

1. The motion of a projectile is compounded of a uniform translation along its axis and a uniform rotation around its axis. Find the equations of motion of any point.

Solution. Let the Z-axis be the line of motion of the axis of the projectile. Consider the motion of the point $P(x, y, z)$. A plane section through P perpendicular to the Z-axis has for its instantaneous center the point $C(0, 0, z)$. The motion of C is uniform translation. Hence,



$$z = a + bt.$$

The motion of P relative to C is uniform rotation. Hence, if d denote the distance CP and if the initial position of P is in the XZ-plane, we have

$$\begin{cases} x = d\cos kt, \\ y = d\sin kt. \end{cases}$$

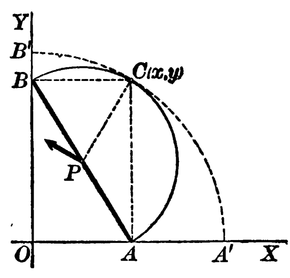
Hence the equations of motion of the point P are

$$\begin{cases} x = d\cos kt, \\ y = d\sin kt, \\ z = a + bt. \end{cases}$$

The path of the point is a helix (Calculus, p. 272).

2. A line AB moves with its extremities on two perpendicular lines. Find the centrodes and the direction of motion of any point at any instant.

Solution. Let the point A move along the X-axis and the point B along the Y-axis. The instantaneous center corresponding to any position of the line AB is found by erecting a perpendicular to the X-axis at A and a perpendicular to the Y-axis at B. These lines intersect in the point C, which is the instantaneous center. The body centrode is the locus of the point C *relative to* AB. Since C is the vertex of a right triangle constructed on AB as a hypotenuse, the body centrode is a semicircle (ACB) with AB as diameter. The space centrode is the locus of C in space, that is, relative to the XY-plane. Denoting the coördinates of C by (x, y) and the length of AB by l, we have, for any position of C,



$$x^2 + y^2 = l^2.$$

Hence the space centrode is a circle ($A'CB'$), with center at the intersection of the two fixed lines and radius equal to the length of AB. It is readily seen that the motion of AB under the conditions stated in the problem may be accomplished by rolling the circle ACB, of which AB is the diameter, on the inside of the circle $A'CB'$, of which AB is the length of the radius.

At any instant the motion of the line is a rotation about C. Hence the direction of motion of any point P is perpendicular to the line PC.

PROBLEMS

1. The center of a fly wheel moves in a straight line (in the plane of the wheel) with constant velocity b, while the wheel turns with constant angular velocity ω. Find the equations of motion of a point on the circumference.

2. In the preceding problem suppose the center moves with constant acceleration f, and the wheel turns with constant angular acceleration α. Find the equations of motion of a point on the circumference.

3. A circular disk rolls on the X-axis. If the center moves with a constant acceleration f, find the equations of motion of a point on the circumference.

4. A circular disk A rolls on the exterior of a second circular disk B. Determine the centrodes.

5. A pole slides through a fixed ring while one end moves along a horizontal line a feet below the ring. Determine the centrodes.

6. A chord of a circle moves around the circumference. What are the centrodes?

7. Construct the centrodes for the connecting rod of an engine.

8. A point P of a plane figure moves with constant speed along a straight line while the figure rotates with constant angular velocity. Show that the body and space centrodes are respectively a circle whose center is P and a line parallel to the path of P.

9. A coin of radius a rolls down a plane. What is the locus at any instant of all points having the same speed as the center.

10. A plane figure moves in its own plane so that a point P moves on a curve C with constant speed, the figure meanwhile rotating with constant angular velocity. Show that the motion may be obtained by rolling a circle with P as a center upon a *parallel curve* of C.

11. Construct the centrodes for problems 8 and 10; (a) when the acceleration of P along its path is constant and the angular velocity is constant: (b) when the acceleration of P is constant and the angular acceleration is constant.

KINETICS

108. Force equations. Work and energy. In Art. 101 certain theorems on the motion of a rigid system of material particles were proved when the number of particles is finite. These theorems can be extended to cover the motion of a rigid body, which has been defined as a continuous rigid system of material particles. In the case of a finite number of particles the theorems were proved (see, for example, Art. 97) by forming the sum of a finite number of expressions. In the case of an infinite number of particles forming a continuous system, the limit of the sum is considered. In other words, the ordinary finite sum is replaced by the definite integral. This process, which is carried out in detail in the following article on kinetic energy, shows that the theorems of Art. 101 are applicable to the motion of a rigid body. For the motion of the center of gravity we have the

THEOREM. *When a rigid body is subjected to the action of any forces, its center of gravity moves as if the entire mass of the body were concentrated at the center of gravity and the given forces applied there parallel to their former directions.*

For example, when no forces are acting, the center of gravity has uniform motion in a straight line. When the forces acting are all equal and parallel, the center of mass has uniformly accelerated rectilinear motion, or else describes a parabola. The center of mass of a projectile describes a parabolic orbit.

From the theorem stated above we may write the force equations for the motion of the center of mass. For plane motion of the center of gravity these are:

$$\text{(I)} \qquad M\frac{d^2x}{dt^2} = F_x, \quad M\frac{d^2y}{dt^2} = F_y,$$

where M is the total mass of the body, (x, y) the coördinates of the center of mass, and F is the resultant of all applied forces.

From Art. 101 we have for a rigid body the *energy equation*

$$\text{(II)} \qquad \left.\begin{matrix}\textbf{Work done on a rigid body}\\ \textbf{by all impressed forces}\end{matrix}\right\} = \textbf{Change in kinetic energy.}$$

In particular

$$\text{Work done by weight} = Mgh,$$

where M is the mass of the body and h is the vertical distance described by the center of gravity.

109. Kinetic energy. The kinetic energy of a system of material particles was defined as the sum of the kinetic energy of each particle,

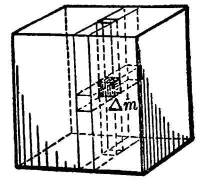
$$\text{K.E.} = \sum_{i=1}^{n} \tfrac{1}{2} m_i v_i^2.$$

We now apply this definition to the continuous system of particles forming a rigid body. Dividing the whole mass in any way into n small elements $\Delta_1 m, \Delta_2 m, \cdots \Delta_i m, \cdots \Delta_n m$, we have as an approximate value of the kinetic energy of the small element $\Delta_i m$ the expression

$$\tfrac{1}{2} \Delta_i m v_i^2,$$

where v_i is the speed of some point P_i within the element. As an approximate value of the kinetic energy of the whole mass we have

$$A_n = \sum_{i=1}^{n} \tfrac{1}{2}\,\Delta_i m v_i^2.$$



Let the number of elements into which the whole mass is divided be increased indefinitely in such a way that $\Delta_i m$ (for every i) approaches zero as a limit. Then the definition of the kinetic energy of the rigid body is

$$\text{(III)}\qquad \mathbf{K.E.} = \lim_{n=\infty} A_n = \lim_{n=\infty} \sum_{i=1}^{n} \frac{1}{2}\Delta_i m v_i^2 = \frac{1}{2}\int v^2 dm,$$

where the definite integral is understood to extend over the entire mass.

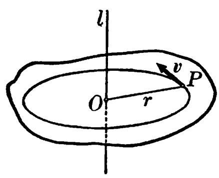
We shall consider the calculation of the kinetic energy in the four types of motion treated in Arts. 103–107.

(i) *Translation.* For every point of the body the velocity is the same at any instant. Hence in (III) v is a constant, and

$$(1)\qquad \text{K.E.} = \frac{1}{2}\int v^2 dm = \frac{v^2}{2}\int dm = \frac{1}{2}Mv^2,$$

where M is the total mass of the body.

(ii) *Rotation.* For every point of the body the angular velocity ω about the axis of rotation is the same at any instant. Consider an element of mass dm at P, moving with velocity v in the circle whose center is O. Then if $\omega =$ angular velocity about the axis l,



$$v = r\omega.$$

Hence

$$(2)\qquad \text{K.E.} = \tfrac{1}{2}\int v^2 dm = \tfrac{1}{2}\,\omega^2\int r^2 dm = \tfrac{1}{2}I_l\omega^2,$$

where $I_l =$ *moment of inertia with respect to* l.

(iii) *Uniplanar motion.* Referring to the figure on page 227, in the triangle at P, $v^2 = v_0^2 + v_r^2 + 2\,v_0 v_r \sin\theta$, if θ is the angle between OP and $\mathbf{v}_0$. Let $OP = r$. Then $v_r = r\omega$, and by (III)

$$(3)\quad \text{K.E.} = \tfrac{1}{2}\int v^2 dm = \tfrac{1}{2}\,Mv_0^2 + \tfrac{1}{2}\,I_0\omega^2 + v_0\omega\int r\sin\theta\, dm.$$

If the temporary axis is the instantaneous axis, then $v_0 = 0$ and

$$\text{(4)} \qquad \text{K.E.} = \tfrac{1}{2} I_c\omega^2,$$

where I_c is the moment of inertia about the instantaneous axis.

An important formula for the kinetic energy is obtained if the temporary axis passes through the center of gravity of the body.

The integral in (3) is now zero. To see this it is only necessary to observe that the integral is the moment of mass of the solid (Art. 7) with respect to the plane drawn through O, the vector v_0, and the perpendicular at O to the directing plane.

Then (3) becomes

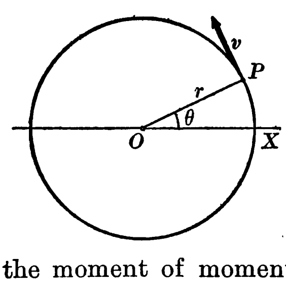
$$\text{(5)} \qquad \text{K.E.} = \tfrac{1}{2} Mv_g^2 + \tfrac{1}{2} I_g\omega^2.$$

This formula exhibits the kinetic energy as made up of the energy of translation of the entire body with a velocity equal to that of the center of gravity and the energy of rotation about an axis through the center of gravity.

(iv) *Screw motion.* Since screw motion is compounded of a translation along an axis in space and a rotation about this axis, the total kinetic energy is the sum of the energy of translation and the energy of rotation. If v denotes the velocity of a point in the axis, the kinetic energy of translation is $\frac{1}{2} Mv^2$. If I is the moment of inertia with respect to the axis along which the body is moving, and ω is the angular velocity about this axis, the kinetic energy of rotation is $\frac{1}{2} I\omega^2$. Hence the total kinetic energy of a body executing screw motion is given by

$$\text{K.E.} = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2.$$

110. Moment equation in rotation. It was shown in Art. 99 for any system of material particles that the time derivative of the moment of momentum with respect to a point is equal to the total force-moment with respect to that point. This result may be stated in a new form when the system of particles is rigid and rotates about an axis. Consider first a single particle in the XY-plane moving in a circle of radius r about the origin. Then the moment of momentum, Art. 59, with respect to the origin is



mrv where v is the linear velocity, or $mr^2\omega$ where ω is the angular velocity. Since m and r are constants, the moment equation of the particle is

$$(1) \qquad mr^2\frac{d\omega}{dt} = mr^2\frac{d^2\theta}{dt^2} = l,$$

where l is the moment with respect to O of the resultant of all forces acting upon P. Now mr^2 is the moment of inertia, i, of m with respect to O. Hence (1) may be written

$$(2) \qquad i\frac{d^2\theta}{dt^2} = l.$$

Equation (2) holds for each particle of a rigid system. Suppose, for simplicity, the system consists of two particles, P_1 and P_2, rigidly connected, and rotating about the origin. Since the system is rigid, the angular acceleration $\frac{d^2\theta}{dt^2}$ is the same for each particle. Hence we have

$$i_1\frac{d^2\theta}{dt^2} = l_1, \quad i_2\frac{d^2\theta}{dt^2} = l_2,$$

and by addition,

$$(3) \qquad (i_1 + i_2)\frac{d^2\theta}{dt^2} = l_1 + l_2.$$

Since the moments of the internal forces cancel in pairs, the second member of equation (3) contains only the sum of the moments of the impressed forces, which is denoted by L. Further $i_1+i_2=I$ is the moment of inertia of the system. Hence we have the *moment equation in rotation*,

$$\text{(IV)} \qquad \boldsymbol{I\frac{d^2\theta}{dt^2} = L}.$$

By the process of integration employed in deriving (III), it is readily shown that (IV) holds when the system of particles forms a rigid body. Hence the

THEOREM. *The product of the angular acceleration and the moment of inertia of a rigid body with respect to an axis about which it is rotating is equal to the sum of the moments of the impressed forces with respect to that axis.*

111. Comparison of formulas in translation and rotation.

Uniform Translation.

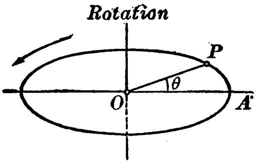
$$v = v_0,$$
$$s = s_0 + v_0 t.$$

Uniformly Accelerated.

$$v = v_0 + ft,$$
$$s = s_0 + v_0 t + \tfrac{1}{2} ft^2.$$
$$f = \text{acceleration.}$$
$$\text{K.E.} = \tfrac{1}{2} Mv^2.$$

Force Equation

$$m\frac{d^2 s}{dt^2} = F.$$



Uniform Rotation.

$$\omega = \omega_0,$$
$$\theta = \theta_0 + \omega_0 t.$$

Uniformly Accelerated.

$$\omega = \omega_0 + \alpha t,$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2} \alpha t^2.$$
$$\alpha = \text{angular acceleration.}$$
$$\text{K.E.} = \tfrac{1}{2} I_l \omega^2.$$

Moment Equation

$$I\frac{d^2 \theta}{dt^2} = L.$$

Hence, to change formulas in

Translation	to	*Rotation*
Replace linear velocity	by	angular velocity.
Replace linear acceleration	by	angular acceleration.
Replace mass	by	moment of inertia.
Replace distance	by	angle.
Replace force	by	moment of force.

112. Fundamental equations for uniplanar motion. It has been shown (Art. 105) that uniplanar motion may be regarded as compounded of a translation with the velocity of an arbitrarily chosen point O, and a rotation about an axis through O perpendicular to the directing plane. Simple formulas result when the point O is chosen at the center of gravity of the solid. The motion of translation of the solid is then determined by the force equations (I). Furthermore, it may readily be shown that the moment equation (IV) applies to uniplanar motion for a gravity axis perpendicular to the directing plane. Hence the following fundamental equations for uniplanar motion.

$$
\text{(V)}\quad\left\{\begin{array}{ll}
\boldsymbol{M}\dfrac{d^2x}{dt^2} = \boldsymbol{F_x}, \quad \boldsymbol{M}\dfrac{d^2y}{dt^2} = \boldsymbol{F_y}. & \textbf{(Force Equations)}\\
\boldsymbol{I_g}\dfrac{d^2\theta}{dt^2} = \boldsymbol{L}. & \textbf{(Moment Equation)}\\
\textbf{Work done by all impressed forces} & \\
\qquad = \textbf{change in kinetic energy.} & \\
 & \textbf{(Energy Equation)}\\
\textbf{K.E.} = \tfrac{1}{2}\boldsymbol{M}\boldsymbol{v_g}^2 + \tfrac{1}{2}\boldsymbol{I_g}\boldsymbol{\omega}^2. &
\end{array}\right.
$$

In equations (V) the point (x, y) is the center of gravity of the body, v_g is the velocity of the center of gravity, I_g is the moment of inertia with respect to the gravity axis perpendicular to the directing plane, L is the resultant moment of all the impressed forces with respect to the gravity axis perpendicular to the directing plane, and I is the moment of inertia with respect to the same axis.

ILLUSTRATIVE EXAMPLES

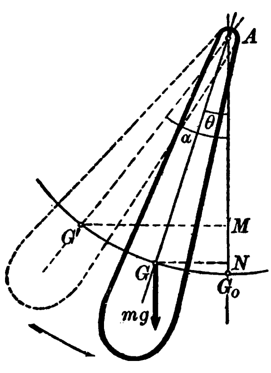
1. *Compound Pendulum.* A heavy body is suspended on a horizontal axis and swings under the action of weight. Determine the motion.

Solution. Let G' be the extreme position of the center of gravity, G_0 be the lowest position of the center of gravity, and G be any position of the center of gravity. Consider the motion from G' to G.

From (2), Art. 109,

$$K.E. = \tfrac{1}{2} I_A\omega^2,$$

where I_A is moment of inertia about the axis of suspension, and ω is the angular velocity when the center of mass is at G.



The work done by weight when the center of mass falls from G' to G is

$$\text{Work done} = mg \cdot \overline{MN}.$$

But $\quad \overline{MN} = AN - AM = AG \cos\theta - AG' \cos\alpha.$

Let $\quad AG = AG' = d,$

$$\therefore \text{Work} = mg \cdot d(\cos\theta - \cos\alpha).$$

By the energy equation,

$$mg \cdot d(\cos\theta - \cos\alpha) = \tfrac{1}{2} I_A\omega^2.$$

Hence $$\omega^2 = \frac{2\,mg \cdot d}{I_A}(\cos\theta - \cos\alpha).$$

Differentiating with respect to t, since $\omega = \dfrac{d\theta}{dt}$,

$$2\,\omega\frac{d^2\theta}{dt^2} = \frac{2\,mg \cdot d}{I_A}\left(-\sin\theta\frac{d\theta}{dt}\right).$$

After division by $\omega = \dfrac{d\theta}{dt}$, this equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{mgd}{I_A}\sin\theta = 0.$$

This agrees with (6), Art. 68, if

$$l = \frac{I_A}{md}.$$

Hence the

THEOREM. *A compound pendulum moves precisely like a simple pendulum whose length is given by the formula* $l = \dfrac{I_A}{md}$.

The corresponding simple pendulum is called the *equivalent simple pendulum.*

2. A homogeneous circular cylinder of mass M and radius r, rotating about its axis a times per second, falls from rest through a vertical distance of h feet under the action of its weight. Compute the total kinetic energy.

Solution. The cylinder is executing uniplanar motion and hence the kinetic energy is given by (V). The moment of inertia of the cylinder with respect to its axis is $I_g = \frac{1}{2} Mr^2$. The angular distance moved in one second is $2\pi a$ radians. Hence $\omega = 2a\pi$. The kinetic energy of translation is found at once from the energy equation to be

$$\tfrac{1}{2} Mv^2 = Mgh.$$

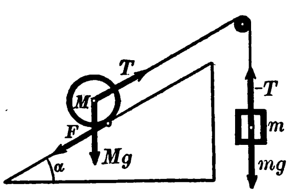
Applying (V),

$$\text{K.E.} = \tfrac{1}{2}\cdot\tfrac{1}{2} Mr^2 (2a\pi)^2 + Mgh.$$

$$\therefore \text{K.E.} = M(r^2a^2\pi^2 + gh).$$

In many problems involving a system of two or more connected bodies acted upon by given external forces the motion may be discussed by the previous methods if we take into account the reactions due to the connections. From the fundamental equations thus obtained for each body the reactions or internal forces may be eliminated and the motion determined. The process is illustrated in the following example.

3. A cord passes over a smooth peg as shown in the figure. To one end of the cord is attached a mass m, which falls vertically, and the other end is fastened to the axle of a solid disk of mass M and radius R which rolls on a plane inclined at an angle α to the horizon. Discuss the motion.



Solution. The motion of the system is known if the acceleration of m is known. If T denotes the pull of m on the string, then the resultant force acting on m is $mg - T$. The forces acting on the disk are (1) the pull of the string T, (2) the weight Mg, (3) the friction F at the point of contact with the plane, (4) the normal pressure N (not indicated in the figure) of the plane on the disk. The pressure N acts normal to the plane at its point of contact with the disk. We suppose that the friction is sufficient to prevent slipping. Then its point of application is the instantaneous center. Since at any instant this point is at rest, the work done by the friction is zero. The work done by the normal pressure N is also zero.

Supposing that m starts from rest, moves through a distance s, and acquires the velocity v, the energy equation gives

$$(1) \qquad \tfrac{1}{2}\, mv^2 = (mg - T)s.$$

Applying the energy equation to M, we get (since v is the velocity of the center of gravity of M)

$$(2) \qquad \tfrac{1}{2}\, Mv^2 + \tfrac{1}{2}\, I_g\omega^2 = (-\, Mg \sin\alpha + T)s.$$

Now $$I_g = \tfrac{1}{2}\, MR^2, \text{ and } \omega = \frac{v}{R}.$$

Hence $$I_g\omega^2 = \tfrac{1}{2}\, Mv^2.$$

By substitution of this value (2) becomes

$$(3) \qquad \tfrac{3}{4}\, Mv^2 = (-\, Mg \sin\alpha + T)s.$$

Adding (1) and (3) to eliminate the tension T, we have

$$(4) \qquad (\tfrac{1}{2}\, m + \tfrac{3}{4}\, M)v^2 = (m - M \sin\alpha)gs.$$

In deriving equation (4) we added (in the second member) the work done by all the forces acting on m and the work done by all the forces acting on M. The work done by T and $-T$ cancels; that is, for the system under consideration the total work done by the internal forces is zero. Hence for this system we have the

THEOREM. *The change in the total kinetic energy of the system is equal to the work done by the external or impressed forces.*

Differentiating (4) with respect to t,

$$(5) \qquad (m + \tfrac{3}{2}\, M)v\frac{dv}{dt} = (m - M \sin\alpha)g\frac{ds}{dt}.$$

Since $v = \dfrac{ds}{dt}$, and $\dfrac{dv}{dt} = f$, equation (5) gives the acceleration of m, namely,

$$(6) \qquad f = \frac{(m - M \sin\alpha)g}{m + \frac{3}{2}\, M}.$$

To find the tension T we differentiate (1) with respect to t, cancel $v = \dfrac{ds}{dt}$, and substitute the value of $f = \dfrac{dv}{dt}$ from (6). The result is

$$T = mg - mf = mg\left(1 - \frac{m - M \sin\alpha}{m + \frac{3}{2}\, M}\right).$$

The magnitude of the frictional force which is necessary to prevent slipping may be found from the moment equation. Taking moments with respect to the axis of the cylinder, the moments of the forces Mg, N, and T are zero, and the moment equation becomes

$$(7) \qquad I\frac{d^2\theta}{dt^2} = -\, F \cdot R.$$

Since $$\theta = \frac{s}{R}, \text{ then } \frac{d^2\theta}{dt^2} = \frac{1}{R}\frac{d^2s}{dt^2} = \frac{1}{R}f.$$

Also $$I = \tfrac{1}{2}\, MR^2.$$

Substituting in (7), we obtain

$$F = -\,\tfrac{1}{2}\, Mf = -\,\frac{(m - M \sin\alpha)Mg}{2\,m + 3\,M}.$$

PROBLEMS

1. A uniform cylindrical shot weighing 200 lb. is fired from a rifled gun with a velocity of 1000 ft. per second. Find the total kinetic energy at the muzzle if it rotates 25 times per second, the diameter of the shot being 6 in. What must be the average pressure during the discharge if the length of the gun is 7 ft.?

Ans. K.E. $= [\frac{25}{2}(25\,\pi)^2 + 10^8]$ foot-poundals.
Pressure $= \frac{1}{7}$ (K.E.) poundals.

2. A uniform circular disk rotates about an axis through its center perpendicular to its plane. The disk weighs 16 T. and its radius is 3 ft. (*a*) What is the kinetic energy when it is revolving at the rate of 200 revolutions per minute? (*b*) What constant tangential force must be applied to a crank 18 in. long to give the disk this speed from rest in 1 min.? (*c*) If the disk lifts a weight of 2 T. through 10 ft., what part of its K.E. is lost?

Ans. (*a*) $\dfrac{1600\,\pi^2}{g}$ foot-tons, (*c*) 20 foot-tons.

3. A circular disk of mass m_1 is suspended by a horizontal axis passing through its center. A flexible thread is wound around its exterior and carries a mass m_2 attached to its free extremity. Show that the angular acceleration is $\dfrac{2\,m_2 g}{r(2\,m_2 + m_1)}$, where $r =$ radius of disk. What distance will m_2 fall from rest in t seconds?

Ans. $h = \dfrac{m_2 g t^2}{m_1 + 2\,m_2}$.

4. A solid cylinder rolls down an inclined plane whose inclination is α. Show that the linear acceleration of the center is constant and equal to $\frac{2}{3}\,g \sin\alpha$.

5. Compare the time of descent of the rolling cylinder in problem 4 with that of a body which slides without friction.

Ans. $\sqrt{3} : \sqrt{2}$.

6. Two equal particles revolve in a horizontal plane around a vertical axis at distances a and b. At what distance from the axis must both particles be placed together in order that the K.E. may remain unchanged?

Ans. $r^2 = \frac{1}{2}(a^2 + b^2)$.

7. A solid fly wheel weighs W lb. and makes N revolutions per minute. Its radius is r ft. and that of the axle c in. If the frictional retarding force on the axle is F units per pound, find the number of revolutions before stopping.

Ans. $\dfrac{\pi r^2 N^2}{600\,Fcg}$.

8. A sphere weighing 100 lb. rotates about a horizontal diameter, making 80 revolutions per minute. Find the K.E.

Ans. $\dfrac{40\,\pi^2 r^2}{9}$ foot-pounds.

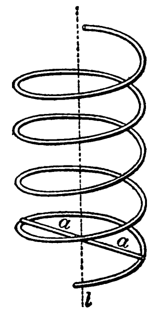
9. A solid sphere rolls down an inclined plane. Show that the acceleration of the center is constant and equal to $\frac{5}{7}\,g \sin\alpha$.

10. Compare the times of descent of the cylinder (problem 4) and sphere (problem 9).

Ans. $\sqrt{15} : \sqrt{14}$.

11. A uniform rod of length $2a$ turns about a screw as in the figure. How high will it rise if an angular velocity ω about the axis l is imparted to it?

Ans. $h = \dfrac{a^2\omega^2}{6g}$.



12. Show that the acceleration of the center of a hoop rolling down a hill is $\frac{1}{2} g \sin \alpha$.

13. Show that the acceleration of the center of a hollow sphere rolling down a hill is $\frac{3}{5} g \sin \alpha$.

14. Compare the times of descent of a hollow and a solid sphere rolling down an inclined plane. *Ans.* $5 : \sqrt{21}$.

15. Two bicyclists, riding exactly similar machines, coast down a hill, starting with equal velocities at the top. Neglecting the forces of friction and the resistance of the air, show that the heavier rider will reach the bottom first.

16. A straight piece of uniform wire is stood vertically on end and allowed to fall over. With what velocity does its extremity strike the ground?

Ans. $\sqrt{3lg}$, $l =$ length of wire.

17. A train of T tons descends an incline of s ft. in length having a total fall of h ft. What will be the velocity at the bottom, friction being p lb. per ton?

Ans. $v^2 = 2gh - \dfrac{1}{1000} gps$.

18. A uniform cylindrical rod 6 ft. long, radius 2 in., and density 5, is suspended so that it swings freely in a vertical plane about one end. It is dropped from a position making an angle of 30° with the vertical. Find the K.E. and angular velocity as it passes through the vertical position.

Ans. K.E. $= \dfrac{5\pi}{2}\left(1 - \dfrac{\sqrt{3}}{2}\right)$ foot-pounds, $\omega^2 = \dfrac{g}{2}\left(1 - \dfrac{\sqrt{3}}{2}\right)$.

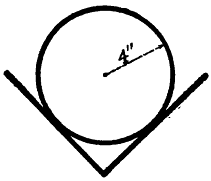
19. A homogeneous cylinder of mass M and radius a can turn around its axis, which is horizontal. A fine thread supporting a mass m is wound around it. Find the angular velocity of the cylinder when m has descended a distance h.

Ans. $\omega^2 = \dfrac{4mgh}{a^2(M + 2m)}$.

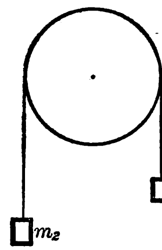
20. Show that a cylinder of altitude a and radius b rotating about its axis has enough energy to raise a weight equal to its own through a vertical distance $\dfrac{b^2\omega^2}{4g}$.

21. A sphere whose radius is a ft. rolls without sliding down an incline of 30°. If the mass of the sphere is m, find its velocity after rolling a distance s. Find the time required to roll this distance. If the plane is 200 ft. long, what is the velocity of the center of the sphere at the bottom of the incline? How far up an incline of 45° would it run? *Ans.* $v^2 = \dfrac{5}{7} gs$; $t^2 = \dfrac{28}{5g} s$; $v^2 = \dfrac{1000g}{7}$; $\dfrac{200}{\sqrt{2}}$ ft.

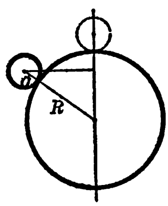
22. A bowling alley has a return trough 60 ft. long with a slope of 1 ft. in 15 ft. (*a*) Discuss the motion of a ball whose radius is 4 in. and mass 8 lb. when allowed to roll, without slipping, from rest at the higher end. (*b*) Suppose the trough is triangular and the angle 90°. The ball then rolls on two points on its sides. Discuss the motion.



Ans. (*a*) $f = \frac{g}{21}$, (*b*) $f = \frac{g}{27}$.



23. Two masses m_1 and m_2 ($m_1 > m_2$) hang over a pulley by means of a flexible, inelastic thread whose mass can be disregarded. Discuss the motion, (*a*) leaving out the mass m_3 of the pulley, (*b*) taking account of the pulley's mass.



24. A marble of radius a starting practically from rest at the upper end of the vertical diameter, rolls off a sphere of radius R. At what point will it cease to touch the sphere?

Ans. After the center of mass has descended a vertical distance $= \frac{7}{17}(R + a)$.

25. A uniform rod whose length is $2b$ oscillates in a vertical plane about a horizontal axis distant a from its center of mass. Find the length of the equivalent simple pendulum. Find also the center of oscillation when the rod is suspended from one end.

Ans. $l = a + \frac{b^2}{3a}$; $OC = \frac{4}{3}b$.

26. A pendulum formed of a right circular cylinder of radius r and length h oscillates about a diameter of one of its bases as a fixed horizontal axis. Find the period.

Ans. $T = 2\pi\sqrt{\frac{3r^2 + 4h^2}{6gh}}$.

27. Suppose the cylinder in problem 26 falls from the vertical position above the point of support. What is its angular velocity when it has turned through an angle of $\theta = \frac{2}{3}\pi$? What should be its angular velocity at the lowest point in order that it may just rise to its original position?

Ans. $\omega^2 = \frac{18\,hg}{3r^2 + 4h^2}$; $\omega^2 = \frac{24\,gh}{3r^2 + 4h^2}$.

28. A circular lamina of radius r oscillates (*a*) about a tangent lying in its plane, (*b*) about a line through the circumference perpendicular to its plane. Find in each case the length of the equivalent simple pendulum.

Ans. (*a*) $l = \frac{5}{4}r$; (*b*) $l = \frac{3}{2}r$.

29. A cube whose edge is a swings as a pendulum about a horizontal edge. Find the length of the equivalent simple pendulum and the period.

Ans. $l = \frac{2}{3}a\sqrt{2}$.

30. A circular arc oscillates about an axis through its middle point perpendicular to its plane. Show that the equivalent simple pendulum is independent of the length of the arc and equal to twice the radius.

31. A given sphere has a radius of 6 in. If the axis of revolution is a horizontal tangent, find the moment of inertia and show that the length of the equivalent simple pendulum is 8.4 in.

32. A right circular cone of height h and radius r oscillates about a horizontal axis perpendicular to its own axis at the vertex. Show that the length of the equivalent simple pendulum is $\frac{4h^2 + r^2}{5h}$.

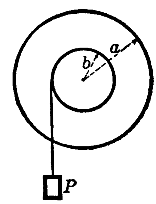
33. A square whose side is a oscillates about a horizontal axis perpendicular to its plane. How far from the center of the square is this axis when the period has a minimum value?

Ans. $\frac{a}{\sqrt{6}}$.

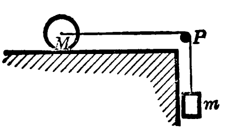
34. Find the axis about which an elliptic lamina must oscillate that the time of an oscillation may be a minimum.

Ans. The axis must be parallel to the major axis and bisect the semiminor axis.

35. A heavy disk weighing W lb. is set in motion by a weight P as indicated in the figure. The radius of the disk is a and that of the axle is b. The mass of the axle may be disregarded in comparison with that of the disk. (a) What is the angular velocity of the disk when P has descended h ft.? What time is required?



✓ **36.** A cord passes over a smooth peg as shown in the figure. To one end of the cord is attached a mass m which falls vertically, and the other end is fastened to the axle of a solid cylinder of mass M and radius R which rolls on a horizontal plane. Find the acceleration of m and the frictional force.

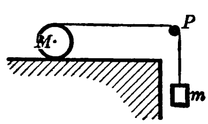


Ans. $f = \frac{mg}{m + \frac{3}{2}M}$.

37. In problem 36 suppose M is a solid sphere.

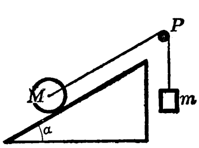
38. In problem 36 suppose M is a hollow sphere.

39. A cord passes over a smooth peg as shown in the figure. To one end of the cord is attached a mass m which falls vertically and the other end is wrapped around a solid cylinder which rolls on a horizontal plane. Find the acceleration of m.



Ans. $f = \frac{mg}{m + \frac{3}{8}M}$.

40. In problem 39 suppose M is a hollow cylinder of negligible thickness.

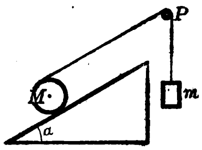


41. Suppose the cylinder of problem 36 rolls on a plane whose inclination is α. Find the acceleration of m and the frictional force.

Ans. $\frac{(m - M\sin\alpha)g}{m + \frac{3}{2}M}$.

42. In problem 41 suppose M is a solid sphere.

43. In problem 41 suppose M is a hollow sphere.



44. Suppose the cylinder of problem 39 rolls on a plane whose inclination is α. Find the acceleration of m.

Ans. $\dfrac{(2m - M\sin\alpha)g}{2m + \frac{3}{4}M}$.

45. In problem 44 suppose M is a hollow cylinder of negligible thickness.

46. In problem 41 suppose m and M given. Determine the inclination α so that the acceleration of m is zero.

Ans. $\sin\alpha = \dfrac{m}{M}$.

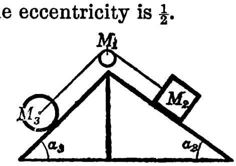
47. Two masses m and m' suspended from a wheel and axle do not balance. The radius of the wheel is a, and that of the axle is b. Show that the acceleration of m is $\dfrac{(ma - m'b)ag}{ma^2 + m'b^2 + I}$, where I is the moment of inertia of the machine about its axis.

48. The handle of a wheel and axle is let go just as a bucket full of water weighing 60 lb. reaches the top of a well 18 ft. deep, and the bucket gets to the bottom again in 6 sec. If the axle is 6 in. in diameter, find the moment of inertia of the wheel and axle. *Ans.* 116.25.

49. A prism whose cross section is a square, each side being a, and whose height is h, oscillates about one of its upper edges. Find the length of the equivalent simple pendulum. *Ans.* $\frac{2}{3}\sqrt{a^2 + h^2}$.

50. A uniform cylinder has coiled around its central section a light, perfectly flexible string. One end of the string is attached to a fixed point, and the cylinder is allowed to fall. Show that it will fall with acceleration $\frac{2}{3}g$.

51. An elliptic lamina swings about a horizontal axis which passes through one focus, is perpendicular to the major axis, and lies in the plane of the ellipse. The other focus is the center of oscillation. Prove that the eccentricity is $\frac{1}{2}$.



52. Two planes are placed back to back, as shown in the figure. The body M_2 slides down the smooth plane of inclination α_2, and by means of a cord passing over a pulley of mass M_1 draws the cylinder of mass M_3 up the plane of inclination α_3. Supposing the cylinder rolls without slipping, determine the acceleration.

Ans. $\dfrac{2g(M_2\sin\alpha_2 - M_3\sin\alpha_3)}{M_1 + 2M_2 + 3M_3}$.

CHAPTER XIII

EQUILIBRIUM OF COPLANAR FORCES

113. Equilibrium of forces. If a system of forces acting upon a body produces no change of motion, the forces are said to be in equilibrium; and if the body is initially at rest, it will remain at rest under the action of a system of forces in equilibrium. The part of mechanics which deals with systems of forces in equilibrium is called *statics*. Assuming that the body is initially at rest, the problem of statics is the determination of the conditions upon the forces acting in order that the body shall remain at rest.

114. Analytic conditions for equilibrium of coplanar forces. Consider a uniplanar motion with the following characteristics: (*a*) the center of gravity moves in a straight line with constant speed; (*b*) the angular velocity remains constant. Referring to the fundamental equations ((V), Chapter XII),

$$(1) \qquad M\frac{d^2x}{dt^2} = F_x, \quad M\frac{d^2y}{dt^2} = F_y, \quad I_g\frac{d^2\theta}{dt^2} = L,$$

then, by the hypothesis (*a*), $\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = 0$, and, by (*b*), $\frac{d^2\theta}{dt^2} = 0$. Hence for the motion described

$$(2) \qquad F_x = 0, \; F_y = 0, \; L = 0,$$

that is, *the sum of the axial components of all forces acting is zero*, and *the resultant moment with respect to the center of gravity of all forces acting also vanishes.*

Under condition (*a*) it can easily be shown that the moment equation holds for an origin chosen arbitrarily, and, consequently, that *the resultant moment with respect to any origin of all forces acting vanishes.*

Next assume a system of coplanar forces (that is, whose lines of action lie in a plane) such that equations (2) are satisfied.

Suppose these forces act upon a rigid body whose previous motion ($t = 0$) possessed the characteristics (a) and (b), assuming that the directing plane is parallel to the plane of the forces. Then, since by (1) we have to integrate

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = 0,$$

it is clear that the center of gravity will continue to move uniformly in a straight line, and further that the angular velocity will be unchanged. *Hence the motion is entirely unchanged.* It is also clear that such a system of forces will not disturb the body if it is initially at rest.

Hence the

THEOREM. *A system of coplanar forces is in equilibrium if and only if* (1) *the sum of the X-components of all the forces is zero,* (2) *the sum of the Y-components of all the forces is zero,* (3) *the sum of the moments with respect to the origin of all the forces is zero.*

If the forces be denoted by $F_1, F_2, \cdots F_n$, the angles which their lines of action make with the X-axis by $\alpha_1, \alpha_2, \cdots \alpha_n$, and the lever arms with respect to the origin by $d_1, d_2, \cdots d_n$, the conditions for equilibrium may be written

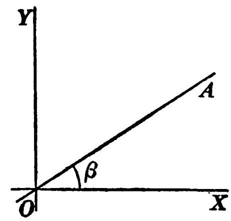
$$(3) \qquad \begin{cases} F_x = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \cdots F_n \cos \alpha_n = 0, \\ F_y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + \cdots F_n \sin \alpha_n = 0, \\ L \;= F_1 d_1 + F_2 d_2 + \cdots F_n d_n = 0. \end{cases}$$

The first two equations are the conditions that the acceleration of the center of gravity shall be zero, and the third is the condition that the angular acceleration shall be zero.

Since any point in the plane may be chosen for the origin and since any line through the origin may be taken for the X-axis, we see that if a system of forces is in equilibrium, (a) the sum of the components in any direction is zero, (b) the sum of the moments with respect to any point is zero. And conversely, if (a) and (b) are satisfied, the system is in equilibrium.

From (3) we may deduce other forms of the conditions of equilibrium which are convenient in applications.

I. *A system of forces is in equilibrium if the sum of the components along any two intersecting (not coincident) straight lines is zero, and the sum of the moments with respect to one origin is zero.*



Proof. The second part of I is the condition $L = 0$. To prove the first part, let the two lines be OX and OA, and denote the angle XOA by β. The sum of the components along OX is

$$(4) \qquad F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + \cdots F_n \cos \alpha_n = F_x = 0.$$

The sum of the components along OA is

$$(5) \quad F_1 \cos (\alpha_1 - \beta) + F_2 \cos (\alpha_2 - \beta) + \cdots F_n \cos (\alpha_n - \beta) = 0.$$

Equation (5) may be written in the form

$$F_1 \cos \alpha_1 \cos \beta + F_1 \sin \alpha_1 \sin \beta + F_2 \cos \alpha_2 \cos \beta + F_2 \sin \alpha_2 \sin \beta \\ + \cdots F_n \cos \alpha_n \cos \beta + F_n \sin \alpha_n \sin \beta = 0.$$

Since $F_x = \Sigma F_i \cos \alpha_i$, $F_y = \Sigma F_i \sin \alpha_i$, this equation becomes

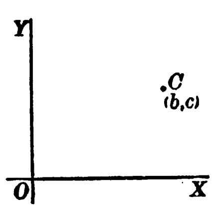
$$F_x \cos \beta + F_y \sin \beta = 0.$$

Since by (4) $F_x = 0$ and $\beta \neq 0$ by hypothesis, this equation gives

$$F_y = 0.$$

Hence (4) and (5) are equivalent to the first two of equations (3). Q.E.D.

II. *A system of forces is in equilibrium if the sum of the moments is zero for each of two origins, O and C, and the sum of the components is zero in any direction not perpendicular to OC.*



Proof. Take the point O for origin of coördinates, and the X-axis parallel to the direction of resolution.

Let the point of application of F_1 be (x_1, y_1). Then (Art. 62),

$$(6) \qquad \text{moment of } F_1 \text{ with respect to } O = L_1 = \\ x_1 F_1 \sin a_1 - y_1 F_1 \cos a_1,$$

and similarly,

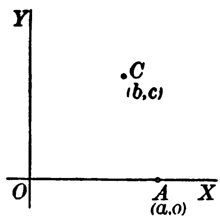
(7) $$\text{moment of } F_1 \text{ with respect to } C(b, c) = (x_1 - b)F_1 \sin a_1 - (y_1 - c)F_1 \cos a_1 = L_1 - bF_1 \sin a_1 + cF_1 \cos a_1.$$

Hence by summing up, if L = sum of moments with respect to O, we shall have, using (3),

(8) Sum of moments with respect to $C = L - bF_y + cF_x$.

This vanishes by hypothesis. Also $L = O$, $F_x = 0$ by hypothesis. Hence $F_y = 0$ and (3) hold. Q.E.D.

III. *A system of forces is in equilibrium if the sum of the moments is zero for each of three origins not on the same straight line.*



Let the three centers of moments be $O(0, 0)$, $A(a, 0)$, and $C(b, c)$. Then if the moments of the force F_1 are, respectively, L_1, L_1', L_1'', we shall have

$$L_1 = x_1F_1 \sin a_1 - y_1F_1 \cos a_1,$$

$$L_1' = (x_1 - a)F_1 \sin a_1 - y_1F_1 \cos a_1 = L_1 - aF_1 \sin a_1,$$

$$L_1'' = (x_1 - b)F_1 \sin a_1 - (y_1 - c)F_1 \cos a_1 = L_1 - bF_1 \sin a_1 + cF_1 \cos a_1.$$

Summing up for all the forces, and denoting these sums by L, L', L'', then, using (3),

$$L' = L - aF_y, \qquad L'' = L - bF_y + cF_x.$$

The hypothesis $L = L' = L'' = 0$ leads to the condition $F_x = 0$, $F_y = 0$. Q.E.D.

In any problem of statics either I, II, or III may be used. The choice depends upon the convenience for the particular problem.

A special condition which is important in the applications may be derived when the number of forces is three. Let the forces be denoted by F_1, F_2, and F_3. There are two cases to be considered.

(1) Suppose the lines of action of F_1 and F_2 intersect in the point O. Taking moments with respect to O, we have

$$\text{moment of } F_1 = \text{moment of } F_2 = 0,$$

and, for equilibrium,

$$\text{moment of } F_1 + \text{moment of } F_2 + \text{moment of } F_3 = 0.$$

Hence moment of $F_3 = 0$, which means that the line of action of F_3 must pass through O. The first two conditions of equilibrium (3) simply assert that the *vector sum* of the three forces is zero.

(2) Suppose the lines of action of F_1 and F_2 are parallel. Let the axis OX be parallel to the lines of action of F_1 and F_2. Taking components in the direction of the Y-axis, we have

$$F_{1y} = F_{2y} = 0,$$

and, for equilibrium,

$$F_{1y} + F_{2y} + F_{3y} = 0.$$

Hence $F_{3y} = 0$, which means that F_3 is parallel to the X-axis. Hence the

THEOREM. *If three coplanar forces are in equilibrium, their lines of action are concurrent or parallel and their vector sum is zero.*

115. General method of solving problems in equilibrium. The general problem of equilibrium of a system of forces is the following: given a body or system of bodies acted upon by a system of forces of which some are known, to determine the unknown forces so that the system is in equilibrium. For the solution we have the three conditions of equilibrium as expressed by I, II, or III and such geometric conditions as may be implied in the statement of the problem. Most problems of statics can be solved in different ways and the best method is to be found only by experience. The general method of procedure is indicated in the following steps:

(1) Draw a figure showing the body acted upon and represent by vectors all the forces acting.

(2) Enumerate all the forces acting, specifying the magnitude and direction of each so far as known.

(3) Write the three conditions of equilibrium, using I, II, or III to make the equations as simple as possible.

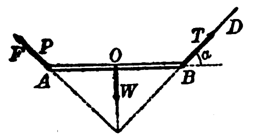
(4) If the equations of equilibrium are sufficient to determine the unknown quantities, solve them.

(5) If not, write as many equations as possible from the geometric conditions.

(6) If the problem is determinate, the number of static and geometric equations is sufficient to determine the unknown quantities by algebraic solution.

ILLUSTRATIVE EXAMPLES

1. A heavy uniform rod, AB, is fastened at A with a smooth hinge and is supported in a horizontal position by a string attached at B and making an angle α with the horizon. Determine the tension in the string and the magnitude and direction of the force exerted by the hinge.



Solution. Following the steps indicated above we (1) draw the figure.

(2) The forces acting on the rod are three : (i) the known weight W acting downwards at O, the middle point of AB, (ii) the tension T of unknown magnitude, acting at B in a direction indicated by the angle α, (iii) the force P of the hinge at A, unknown in magnitude and direction.

(3) Since the number of forces is three, we may conclude from the theorem of Art. 114 that DB and FA intersect in a point E vertically under O. This determines the direction of the force P, since the angle FAO is $\pi - \alpha$.

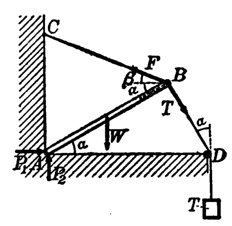
Resolving the forces in directions parallel and perpendicular to OB, we have

$$(1) \qquad \begin{aligned} T\cos\alpha - P\cos\alpha &= 0, \\ T\sin\alpha + P\sin\alpha - W &= 0. \end{aligned}$$

The moment equation has been used in applying the theorem of Art. 114.

(4) The solution of equations (1) gives $T = P = \dfrac{W}{2}\operatorname{cosec}\alpha$.

2. A uniform rod AB rests with the end A against the corner of a smooth* horizontal floor and a smooth vertical wall. At the end B two strings are attached of which one is fastened to a point C in the wall. The other passes over a smooth peg at D in the floor, making ABD a right angle, and supports a weight T. The weight of the rod is W pounds and the tension in BC is F pounds. Determine the weight T and the pressures at A.



Solution. (1) In the figure the middle point of AB is the center of gravity of the rod, α is the inclination of AB to the horizon, and β is the inclination of BC to the horizon.

(2) The forces acting on the rod are five in number: (i) the weight W acting downwards at the middle point of AB, (ii) the known tension F in BC, (iii) the tension T in BD, unknown in magnitude, (iv) the pressure P_1 of the vertical wall at A, unknown in magnitude, (v) the pressure P_2 of the floor at A, unknown in magnitude.

(3) For the equations of equilibrium we shall use II. Resolving in a horizontal direction,

$$(1) \qquad -F\cos\beta + P_1 + T\sin\alpha = 0.$$

Taking moments about A and denoting the length of the rod by l,

$$-\tfrac{1}{2}Wl\cos\alpha - Tl + Fl\sin(\alpha+\beta) = 0,$$

* A surface is defined as *smooth* if it can exert pressure only in the direction of the normal to the surface.

or,

(2) $$-\tfrac{1}{2}W\cos\alpha - T + F\sin(\alpha+\beta) = 0.$$

Taking moments about B,

$$\tfrac{1}{2}Wl\cos\alpha - P_2 l\cos\alpha + P_1 l\sin\alpha = 0,$$

or,

(3) $$\tfrac{1}{2}W\cos\alpha - P_2\cos\alpha + P_1\sin\alpha = 0.$$

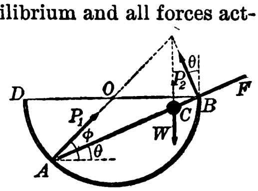
(4) The three equations of equilibrium are sufficient to determine the three unknown quantities P_1, P_2, and T. From (1), (2), and (3) we find

$$T = F\sin(\alpha+\beta) - \tfrac{1}{2}W\cos\alpha.$$

$$P_1 = [F\cos(\alpha+\beta) + \tfrac{1}{2}W\sin\alpha]\cos\alpha.$$

$$P_2 = F\cos(\alpha+\beta)\sin\alpha + \tfrac{1}{2}W(1+\sin^2\alpha).$$

3. A light rigid rod rests partly within and partly without a hemispherical smooth bowl, which is fixed in space. A weight W is clamped on to the rod at a point C within the bowl. Determine the position of equilibrium and all forces acting on the rod.



Solution. (1) The figure represents a section cut from the bowl by the vertical plane determined by the center O and the rod AF. The position of equilibrium is known if the inclination θ of AB to the horizontal is known.

(2) The forces acting on the rod are three in number: (i) the weight W acting downwards at C, (ii) the unknown pressure P_1 of the surface of the bowl at A, acting in the direction of the normal, (iii) the unknown pressure P_2 of the edge of the bowl at B, acting in a direction perpendicular to AF.

(3) For the equations of equilibrium we shall use I. Resolving in a horizontal direction,

(1) $$-P_2\sin\theta + P_1\cos\phi = 0.$$

Resolving in a vertical direction,

(2) $$P_2\cos\theta + P_1\sin\phi - W = 0.$$

Taking moments about A, denoting the known distance AC by l and the unknown distance AB by x,

(3) $$-Wl\cos\theta + P_2 x = 0.$$

(4) The unknown quantities in (1), (2), and (3) are P_1, P_2, θ, ϕ, and x. Hence we require some geometric conditions.

(5) Since the curve DAB is a semicircle, the normal at A passes through O and it follows that $\phi = 2\theta$. Also, since AB is a chord of the circle,

$$AB = x = 2r\cos\theta,$$

where r denotes the radius.

(6) Substituting the values of ϕ and x in (1), (2), and (3), respectively, we obtain

(4) $$P_2\sin\theta - P_1\cos 2\theta = 0,$$

(5) $$P_2\cos\theta + P_1\sin 2\theta = W,$$

(6) $$2rP_2 - Wl = 0.$$

Solving (4), (5), and (6), we find

$$\cos\theta = \frac{l \pm \sqrt{32\,r^2 + l^2}}{8\,r},$$

$$P_1 = W \tan\theta, \quad P_2 = \frac{Wl}{2\,r}.$$

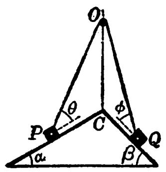
It may be remarked that the angle of equilibrium does not depend on the magnitude of the weight attached to the rod.

PROBLEMS

1. A rod AB is hinged at A and supported in a horizontal position by a string BC making an angle of 45° with the rod. A weight of 10 lb. is suspended from B and the weight of the rod may be neglected. Find the tension in the string and the force at the hinge. *Ans.* $10\sqrt{2}$, 10 lb.

2. A wheel capable of turning freely about a horizontal axis has a weight of 2 lb. fixed to the end of a spoke which makes an angle of 60° with the horizon. What weight must be attached to the end of a horizontal spoke to prevent motion taking place?

3. Two weights, P and Q, rest on a smooth double inclined plane as shown in the figure, and are attached to the extremities of a string which passes over a smooth peg O at a point vertically over the intersection of the planes, the peg and the weights being in a vertical plane. Find the position of equilibrium.



Ans. The position of equilibrium is given by the equations

$$P\,\frac{\sin\alpha}{\cos\theta} = Q\,\frac{\sin\beta}{\cos\phi},$$

$$\frac{\cos\alpha}{\sin\theta} + \frac{\cos\beta}{\sin\phi} = \frac{l}{h},$$

where l is the length of the string and $h = CO$.

4. A bar of mass 15 lb., whose center of gravity is at its middle point, rests with its ends upon two smooth planes inclined to the horizon at angles of 36° and 45° respectively. Determine the inclination of the bar to the horizon when in equilibrium, and the pressures exerted upon it by the supporting planes.

Ans. 10° 39′, 8.93 lb., 10.74 lb.

5. A uniform rod 15 in. long and weighing 12 lb. has a weight of 10 lb. suspended from one end. At what point must the rod be supported that it may just balance? *Ans.* $4\frac{1}{11}$ in. from the weight.

6. Prove that three forces acting at the middle points of the sides of a triangle perpendicularly inwards, and proportional to the lengths of the sides, are in equilibrium.

7. Extend the theorem of problem 6 to a plane polygon of any number of sides.

8. $ABCD$ is a plane quadrilateral, P and Q are the middle points of the opposite sides AB and CD, and O is the middle point of PQ. Prove that the four forces represented by OA, OB, OC, OD, respectively, are in equilibrium.

9. A uniform beam of weight W and length 3 ft. rests in equilibrium with its upper end A against a smooth vertical wall, while its lower end B is supported by a string, 5 ft. long, whose other end is attached to a point C in the wall. Find AC and the tension in the string.

Ans. $AC = \frac{4}{\sqrt{3}}$ ft., $T = \frac{5\sqrt{3}}{8} W$.

10. $ABCD$ is a plane quadrilateral. Forces act along the sides AB, BC, CD, DA, measured by α, β, γ, δ times those sides respectively. Show that if these forces are in equilibrium, then $\alpha\gamma = \beta\delta$.

11. A bar AB, whose center of gravity is at its middle point and whose mass is 12 lb., is supported in a horizontal position by strings attached to the ends, and sustains loads of 16 lb. and 20 lb. at A and B respectively. If the string at A is inclined 45° to the horizon, what is the inclination of the string B? Find the tensions in the strings.

Ans. 49° 47′, 31.12 lb., 34.05 lb.

12. Three smooth pegs A, B, C stuck in a wall are the vertices of an equilateral triangle, A being the highest and the side BC horizontal. A light string passes once around the pegs and its ends are fastened to a weight W which hangs in equilibrium below BC. Find the pressure on each peg.

13. A weightless string is suspended from two fixed points and at given points on the string equal weights are attached. Prove that the tangents of the inclinations to the horizon of different portions of the string form an arithmetic progression.

14. A uniform yardstick weighing 10 oz. is supported in a horizontal position by the thumb at one end and the forefinger at a point 3 in. from the end. What is the pressure on the thumb and on the finger?

Ans. 50 oz., 60 oz.

15. A beam AB weighing $\frac{1}{2}$ T. per running foot and 18 ft. long is loaded with 4 T. at A and 5 T. at B. It is supported at points 4 ft. from A and 6 ft. from B. Find the supporting forces P and Q.

Ans. $P = 5\frac{5}{8}$ T., $Q = 12\frac{3}{8}$ T.

16. A uniform rod of weight 50 lb. and length 18 ft. is carried on the shoulders of two men who walk at distances of 2 and 3 ft., respectively, from the two ends. A weight of 50 lb. is suspended from the middle point of the rod. Find the total weight carried by each man.

17. A uniform plank 20 ft. long, weighing 42 lb., is placed over a rail, and two boys weighing 75 and 99 lb., respectively, stand each at a distance of 1 ft. from each end. Find the position of the plank for equilibrium.

Ans. 1 ft. from the middle point.

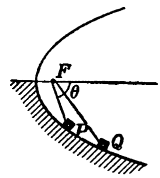
18. Two equal weights, P, Q, are connected by a string which passes over two smooth pegs A, B, situated in a horizontal line, and supports a weight W which hangs from a smooth ring through which the string passes. Find the position of equilibrium.

Ans. The depth of the ring below the line AB is $\frac{W}{2\sqrt{4P^2 - W^2}} \cdot AB$.

19. A light rod rests wholly inside a smooth hemispherical bowl whose radius is r, and a weight W is clamped on to the rod at a point whose distances from the ends are a and b. Show that, if θ be the inclination of the rod to the horizon in the position of equilibrium, then

$$\sin\theta = \frac{a - b}{2\sqrt{r^2 - ab}}.$$

20. Two weights, P and Q, rest on the concave side of a parabola whose axis is horizontal, as shown in the figure, and are connected by a light string, of length l, which passes over a smooth peg at the focus F. Find the position of equilibrium.



Ans. If θ is the angle which FP makes with the axis, and $4\,m$ is the latus rectum of the parabola, then

$$\cot\frac{\theta}{2} = \frac{P\sqrt{l-2\,m}}{\sqrt{m(P^2+Q^2)}}.$$

21. In problem 20 show that the depths of the weights below the axis are proportional to their masses.

22. A particle is placed on the convex side of a smooth ellipse, and is acted upon by two forces F and F', towards the foci, and a force F'', towards the center. Find the position of equilibrium.

Ans. $r = \dfrac{b}{\sqrt{1-n^2}}$, where r is the distance of the particle from the center of the ellipse, b is the semiminor axis, and $n = \dfrac{F-F'}{F''}$.

116. Friction. A *smooth* surface is defined as one which can exert upon a body in contact with it only a pressure in a direction normal to the surface. Such surfaces do not exist in nature. Suppose a heavy box is at rest upon a horizontal table. If the table were smooth, the box could be moved by any horizontal pull, the acceleration being, by Newton's Second Law of Motion, directly proportional to the force and inversely proportional to the mass of the box. Experiment shows, however, that this is not true. If the horizontal pull is slight, no motion ensues, and consequently the forces acting on the box are in equilibrium. The forces acting are three in number: (i) the weight acting vertically downwards, (ii) the horizontal pull H, and (iii) the pressure of the table. From the principles of equilibrium it follows that the pressure of the table must be made up of two components, of which one, numerically equal to the weight, acts vertically upwards, while the other is numerically equal to H but opposite in direction. A *rough* surface can exert upon bodies in contact with it a pressure made up of (1) a component normal to the surface called the *normal pressure*, and (2) a component tangent to the surface called the *friction*. All physical surfaces are more or less rough.

Our knowledge of frictional forces is obtained by experiment and is expressed in the following

LAWS OF FRICTION. 1. *If the body is in equilibrium, the friction is equal and opposite to the tangential component of the applied forces.* In the preceding example there is no friction if there is no horizontal pull.

2. *No more than a certain amount of friction can be called into play.* The value of the friction when sliding is just about to take place is called the limiting friction.

3. *The magnitude of the limiting friction bears a constant ratio to the normal pressure.* This constant ratio, μ, is called the *coefficient of friction*, and its value depends upon the nature of the surfaces in contact.

4. *The coefficient of friction is independent of the area of contact of the two bodies if the touching surfaces are uniform in character.*

The *angle of friction*, λ, is defined by (Art. 67)

$$\tan \lambda = \mu.$$

The coefficient of friction for various substances has been determined by experiment and some of the results are given in the following table of values for μ:

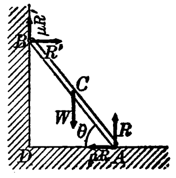
Wood on wood, dry	0.25 to 0.5
Wood on wood, soaped	0.2
Metals on oak, dry	0.5 to 0.6
Metals on oak, wet	0.24 to 0.26
Metals on oak, soaped	0.2
Leather on oak, wet or dry	0.27 to 0.35
Metals on metals, dry	0.15 to 0.2
Metals on metals, wet	0.3
Smooth surfaces, occasionally lubricated	0.07 to 0.08
Smooth surfaces, thoroughly lubricated	0.03 to 0.036

The values of μ given above are the coefficients of static friction as defined in 3. The coefficient of dynamic friction (see Art. 67) is slightly less in numerical value than the coefficient of static friction.

ILLUSTRATIVE EXAMPLE. A uniform rod rests with one end on a rough horizontal floor and the other against a rough vertical wall. Supposing the coefficient of friction to be the same at both ends, determine the least inclination it can make with the horizon.

Solution. (1) In the figure, AB is the rod which is just about to slide down.

(2) The forces acting on the rod are five in number: (i) the known weight W acting vertically downwards at C, (ii) the normal pressure R of the floor, unknown in magnitude, acting vertically upwards at A, (iii) the friction at A, of magnitude μR and acting in the direction AD, (iv) the normal pressure R' of the wall at B, unknown in magnitude, (v) the friction $\mu R'$ at B acting vertically upwards.



(3) For the conditions of equilibrium we shall use I, Art. 114. Resolving in a horizontal direction,

$$(1) \qquad R' = \mu R.$$

Resolving in a vertical direction,

$$(2) \qquad R + \mu R' = W.$$

Taking moments about A, and denoting the length of the rod by l,

$$\tfrac{1}{2} Wl \cos\theta - \mu R' l \cos\theta - R' l \sin\theta = 0,$$

or,

$$(3) \qquad \tfrac{1}{2} W \cos\theta = R'(\mu\cos\theta + \sin\theta).$$

(4) From equations (1), (2), and (3) we eliminate R and R', and solve for θ. The result is

$$\tan\theta = \frac{1-\mu^2}{2\mu}.$$

PROBLEMS

1. A body of mass W pounds is at rest upon a plane making an angle θ with the horizon. A cord attached to this body runs parallel to the plane, passes over a smooth pulley, and sustains a weight of P pounds. Determine the magnitude and direction of the friction, the normal pressure, and the total pressure exerted by the plane upon the body.

2. In problem 1 let $W = 50$ lb., $P = 40$ lb., $\theta = 32°$, and suppose the body is just about to slide up the plane. Determine the coefficient of friction.

Ans. $\mu = 0.318$.

3. The roughness of a plane of inclination α is such that a body of mass W can rest on it. Find the least force required to draw the body up the plane.

Ans. $W \sin 2\alpha$, inclined at an angle α to the plane.

4. A uniform beam rests with one end on a rough horizontal plane and the other against a rough vertical wall, and, when inclined to the horizon at an angle of 30°, is on the point of slipping down. Supposing that the surfaces are equally rough, determine the coefficient of friction.

Ans. $\mu = \dfrac{1}{\sqrt{3}}$.

5. A body of 30 lb. mass, resting on a plane inclined 45° to the horizon, is pulled horizontally by a force P. If the coefficient of friction is 0.2, between what limits may the value of P vary and still permit the body to remain at rest?

Ans. 20 and 45 lb.

6. On a rough plane of inclination θ the greatest value of the force acting along the plane and producing equilibrium is double the least. What is the coefficient of friction?

Ans. $\mu = \tfrac{1}{3}\tan\theta$.

7. If the angle of friction is 30°, what is the least force which will sustain a weight of 100 lb. on a plane whose inclination is 60° ? *Ans.* 50 lb.

8. A ladder inclined at an angle of 60° to the horizon rests with one end on a rough pavement and the other against a smooth vertical wall. The ladder begins to slide down when a weight is put at its middle point. Show that the coefficient of friction is $\frac{\sqrt{3}}{6}$.

9. A uniform ladder weighing 100 lb. and 50 ft. long rests against a rough vertical wall and a rough horizontal plane, making an angle of 45° with each. If the coefficient of friction at each end is $\frac{2}{3}$, how far up the ladder can a man weighing 200 lb. ascend before the ladder begins to slip ? *Ans.* 47 ft.

10. A heavy body is placed on a rough plane whose inclination to the horizon is arc sin $\frac{3}{5}$, and is connected by a string passing over a smooth pulley with a body of equal weight which hangs freely. Supposing that motion is on the point of ensuing up the plane, find the inclination of the string to the plane, the coefficient of friction being 0.5. *Ans.* $\theta = 2 \text{ arc tan } \frac{1}{2}$.

11. Two weights rest on a rough inclined plane and are connected by a string which passes over a smooth peg in the plane. If the angle of inclination α is greater than the angle of friction ϵ, show that the least ratio of the less to the greater is $\sin(\alpha - \epsilon)/\sin(\alpha + \epsilon)$.

12. Two equal weights are attached to a string laid over the top of two inclined planes, having the same altitude, and placed back to back, the angles of inclination of the planes being 30° and 60° respectively. Show that the weights will be on the point of moving if the coefficient of friction between each plane and weight be $\frac{1}{2+\sqrt{3}}$.

13. A body is supported on a rough inclined plane by a force acting along it. If the least magnitude of the force, when the plane is inclined at an angle α to the horizon, be equal to the greatest magnitude when the plane is inclined at an angle β, show that the angle of friction is $\frac{1}{2}(\alpha - \beta)$.

14. A cubical block rests on a rough plank with its edges parallel to the edges of the plank. If, as the plank is gradually raised, the block turns over on it before slipping, what is the least value of the coefficient of friction ?

15. It is observed that a body whose weight is known to be W can be just sustained on a rough inclined plane by a horizontal force P, and that it can also be just sustained on the same plane by a force Q up the plane. Express the angle of friction in terms of these known forces. *Ans.* $\lambda = \text{arc cos } \frac{PW}{Q\sqrt{P^2 + W^2}}$.

16. It is observed that a force Q_1 acting up a rough inclined plane will just sustain on it a body of weight W, and that a force Q_2 acting up the plane will just drag the same body up. Find the angle of friction.

Ans. $\lambda = \text{arc sin } \frac{Q_2 - Q_1}{2\sqrt{W^2 - Q_1 Q_2}}$.

17. A heavy uniform rod rests with its extremities on the interior of a rough vertical circle. Find the limiting position of equilibrium.

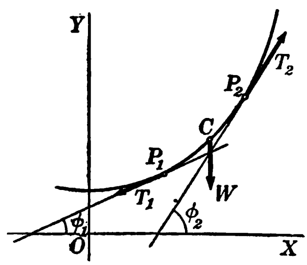
Ans. If 2α is the angle subtended at the center by the rod, and λ the angle of friction, the limiting inclination of the rod to the horizon is given by the equation

$$\tan\theta = \frac{\sin 2\lambda}{\cos 2\lambda + \cos 2\alpha}.$$

18. An insect tries to crawl up the inside of a hemispherical bowl of radius a. How high can it get if the coefficient of friction between its feet and the bowl is $\frac{1}{3}$?

19. Two equal rings of weight W are movable along a curtain pole, the coefficient of friction being μ. The rings are connected by a loose string of length l, which supports by means of a smooth ring a weight W_1. How far apart must the rings be so that they will not come together?

117. Equilibrium of flexible cords. It is assumed that the cords discussed in this article are inextensible and perfectly flexible. The cross section is supposed to be small so that we may consider the *curve* formed by the cord. For a perfectly flexible cord in equilibrium it is evident that the resultant force at any point must act in the direction of the tangent to the curve formed by the cord. We wish to investigate the form of the curve assumed by a cord which is fastened at both ends and which sustains a weight distributed according to a given law. Since the cord is in equilibrium it is evident that, if any segment be replaced by a rigid wire of the same shape and bearing the same load, the system would still be in equilibrium. In order to determine the form of the curve we may consider any segment and treat it as a rigid body.



Let the plane of the cord be the XY-plane with the Y-axis directed vertically upwards, and let ω be a function (of the coördinates or length of arc) representing the distribution of weight along the cord. Consider any segment P_1P_2. This segment is in equilibrium under the action of three forces: (i) the tension T_1, directed along the tangent to the curve at P_1; (ii) the tension T_2, directed along the tangent at P_2; (iii) the weight W acting vertically downwards at C, the center of gravity of the load of the segment. The weight W along the segment P_1P_2 is the difference of values of the function ω at P_2 and P_1, that is $W = \omega_2 - \omega_1$.

Let ϕ_1 and ϕ_2 denote the inclinations of the tangents to the curve at P_1 and P_2 respectively. Resolving in a horizontal direction,

$$(1) \qquad T_1 \cos \phi_1 = T_2 \cos \phi_2.$$

Resolving in a vertical direction,

$$(2) \qquad T_2 \sin \phi_2 = T_1 \sin \phi_1 + W.$$

Since P_1 and P_2 are any points on the curve, equation (1) shows that the horizontal component of the tension is the same at every point of the curve, and this is evidently equal to the tension at the lowest point. Denoting the constant horizontal component of the tension by H, we have, from (1),

$$T_1 = \frac{H}{\cos \phi_1}, \quad T_2 = \frac{H}{\cos \phi_2}.$$

With these values of T_1 and T_2 we may write equation (2) in the form

$$(3) \qquad \tan \phi_2 - \tan \phi_1 = \frac{W}{H} = \frac{\omega_2 - \omega_1}{H}.$$

The function ω is supposed to be known for every point of the curve. If H and the slope ϕ_1 at some one point P_1 has been determined, equation (3) may be used to determine the slope ϕ_2 at any second point P_2.

In order to determine the shape of the curve we must find the differential equation which characterizes it. Let s denote the length of arc measured from some fixed point on the curve, s_1 and s_2 being the distances to P_1 and P_2 respectively. Dividing both members of equation (3) by $s_2 - s_1$, we have

$$(4) \qquad \frac{\tan \phi_2 - \tan \phi_1}{s_2 - s_1} = \frac{1}{H} \frac{\omega_2 - \omega_1}{s_2 - s_1}.$$

Now let P_2 approach P_1 along the curve. Then $s_2 - s_1$ approaches zero as a limit, the first member of equation (4) approaches $\frac{d(\tan \phi)}{ds}$, and the second member approaches $\frac{1}{H}\frac{d\omega}{ds}$. Hence the differential equation of the curve is

$$(5) \qquad \frac{d(\tan \phi)}{ds} = \frac{1}{H}\frac{d\omega}{ds}.$$

When ω is given, the ordinary equation of the curve is found by integrating equation (5) and determining the constants by means of the initial conditions. In the following articles we consider the two cases which are most important in applications.

118. The common catenary. The curve assumed by a heavy cord or by a cord carrying a weight distributed uniformly along the cord, is called a *catenary*. If w denotes the weight supported by unit length of the cord, then

$$\omega = ws,$$

and the differential equation of the curve [(5), Art. 117] becomes

$$\frac{d(\tan\phi)}{ds} = \frac{w}{H}. \tag{1}$$

Since $\tan\phi$ is the slope of the curve we have (Calculus, p. 86)

$$\tan\phi = \frac{dy}{dx}.$$

Now,

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = \frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}. \quad \text{(Calculus, p. 142)}$$

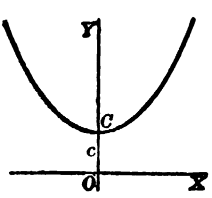
Writing $\frac{dy}{dx} = p$, and $\frac{w}{H} = \frac{1}{c}$, equation (1) becomes

$$\frac{dp}{\sqrt{1+p^2}} = \frac{dx}{c}. \tag{2}$$

Integrating,

$$\log(p + \sqrt{1+p^2}) = \frac{x}{c} + c_1. \tag{3}$$

To determine the constant of integration c_1, we select the axes so that the Y-axis passes through C, the lowest point of the catenary. The distance of the origin below C will be determined later. Since the tangent to the curve at C is horizontal, $\frac{dy}{dx} = p = 0$, when $x = 0$. Hence $c_1 = 0$, and equation (3) may be written in the form



$$p + \sqrt{1+p^2} = e^{\frac{x}{c}}.$$

Solving for p,

$$p = \frac{dy}{dx} = \tfrac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

Integrating,

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + c_2.$$

We now determine the distance OC so that the constant of integration c_2 is zero. If, when $x = 0$, $y = OC = c$, then $c_2 = 0$. Hence the

THEOREM. *The equation of the catenary is*

$$(4) \qquad y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right),$$

where c, the intercept on the Y-axis, is the ratio of the horizontal tension to the weight per unit length.

119. Load distributed uniformly along the horizontal. This is the case of the cables supporting a suspension bridge if the weight of the cables is neglected in comparison with that of the bridge. When a cord supports a load distributed uniformly along the horizontal, the weight supported by any segment is proportional to the length of the projection of the segment on a horizontal line. If w' denotes the weight per horizontal unit, then

$$\omega = w'x,$$

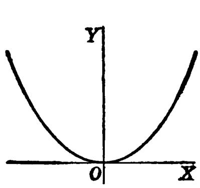
and the differential equation of the curve [(5), Art. 117] becomes

$$(1) \qquad \frac{d(\tan\phi)}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = \frac{w'}{H}\frac{dx}{ds}.$$

Setting $\frac{w'}{H} = c'$, equation (1) takes the form

$$(2) \qquad \frac{d^2y}{dx^2} = c'.$$

In order to determine the constants of integration, we choose the origin at the lowest point of the curve. The initial conditions are $y = 0$, $\frac{dy}{dx} = 0$, when $x = 0$. Integrating (2) and imposing these conditions, we find for the equation of the curve



$$(3) \qquad y = \frac{c'}{2}x^2.$$

This equation represents a parabola with its axis vertical and latus rectum $= \frac{1}{4}c'$. Hence the

THEOREM. *The curve assumed by a cord carrying a weight distributed uniformly among the horizontal is a parabola with its axis vertical and latus rectum equal to $\frac{1}{4}c'$, where c' is the ratio of the weight per unit horizontal distance to the horizontal tension.*

PROBLEMS

1. If s denotes the arc of the catenary measured from the lowest point, and ϕ is the inclination of the tangent to the horizon, prove the following relations:

$$(a)\quad s = c \tan \phi,$$
$$(b)\quad y = c \sec \phi,$$
$$(c)\quad y^2 - s^2 = c^2.$$

2. A cord hanging in the form of a catenary [(4), Art. 118] sustains a load of 50 lb. per foot, and the tension at the lowest point is 1000 lb. The points of suspension are in the same horizontal line 100 ft. apart. Find (a) the coördinates of the points of suspension, (b) the length of the cord, (c) the direction of the cord at the points of suspension. *Ans.* (a) $(\pm\ 50,\ 122.6)$, (b) 241.8, $\phi = 80°\ 37'$.

3. A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation k above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $\frac{8k^2}{3l}$.

4. A chain 110 ft. long is suspended from two points in the same horizontal plane, 108 ft. apart. Show that the tension at the lowest point is nearly 1.477 times the weight of the chain.

5. A heavy chain hangs over two smooth fixed pegs. The two ends of the chain are free and the central portion hangs in a catenary [(4), Art. 118]. Show that the free ends are on the X-axis.

6. A heavy uniform chain is suspended from two fixed points A and B in the same horizontal line, and the tangent at A makes an angle of 45° with the horizon. Prove that the depth of the lowest point of the chain below AB is to the length of the chain as $\frac{\sqrt{2}-1}{2}$.

7. If α and β are the angles which a uniform heavy string of length l makes with the vertical at the points of support show that the height of one point above the other is

$$\frac{l \cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}.$$

120. Stability. Suppose a heavy bead is constrained to slide on a wire in the form of a vertical circle. If the bead is at rest at the highest point A or at the lowest point B of the circle, the forces acting upon it are in equilibrium. If the bead is given a small displacement from the highest point A, the forces are no longer in equilibrium and the bead will move away from A. The position A is said to be a position of *unstable* equilibrium. If the bead is given a small displacement from the lowest point B, the forces are no longer in equilibrium, but the bead will return to its original position, and, if the wire is smooth, will perform small oscillations about B. The position B is said to be a position of *stable* equilibrium.

Suppose the bead is constrained to slide on a horizontal wire. At any point on the wire the forces acting on the bead are in equilibrium. If the bead is given a small displacement from a position P, it will remain in the new position. The position P is said to be a position of *neutral* equilibrium.

To derive the analytic conditions for stability we make use of the potential function (Chapter X). Suppose a particle is constrained to move without friction along a given path of any shape in a plane conservative field of force. Assume that the potential U is a known function of the coördinates x and y. In addition to the force of the field the particle is acted upon by a force of constraint (the pressure of the path) which, at any point of the path, is equal in magnitude but opposite in direction to the normal component of the force of the field. The resultant force acting on the particle is therefore the tangential component of the force of the field, and this is (Art. 94)

$$F_t = \frac{dU}{ds} = -\left(F_x\frac{dx}{ds} + F_y\frac{dy}{ds}\right).$$

The necessary and sufficient condition that any position $A\ (x_1, y_1)$ on the path shall be a position of equilibrium, is that

$$F_t(x_1, y_1) = \frac{dU}{ds} = 0.$$

But this is the condition (Calculus, p. 118) that the function U shall be a maximum or minimum.* Hence the

THEOREM. *For a position of equilibrium of a particle in a conservative field of force, the potential energy is either a maximum or a minimum.*

* It may happen that the graph of the function $U(s)$ has a point of inflection for the value $s = s_1$, corresponding to the point $A\ (x_1, y_1)$ of the path of the particle. For example, suppose a heavy bead slides on a smooth wire in a vertical plane, and that the point A is a point of inflection where the tangent is horizontal. The point A is then a position of equilibrium for the bead, but the potential function U is neither a maximum nor a minimum. This special case is excluded in the statement of the theorems which follow.

The point A is a position of stable equilibrium if, when given a small displacement from A in either direction, the particle tends to return to A. From Art. 94 the force at any point in a conservative field is directed towards the region of lower potential. Hence, if the particle returns to A, we may conclude that the value of the potential at A is smaller than at neighboring points of the path. In other words, at a position of stable equilibrium the potential function is a minimum. Similarly, at a position of unstable equilibrium the potential function is a maximum.

THEOREM. *For a position of stable (unstable) equilibrium of a particle in a conservative field of force, the potential energy is a minimum (maximum).*

Since maximum and minimum values of a continuous function of one variable occur alternately, we have the

THEOREM. *Along any given path in a conservative field of force, positions of stable and unstable equilibrium occur alternately.*

In solving problems to find the positions of equilibrium we may either (1) express the potential in terms of s, the length of arc along the curve, and use the condition $\frac{dU}{ds} = 0$; or (2) we may find the components of force and use the condition $F_x \frac{dx}{ds} + F_y \frac{dy}{ds} = 0$ where y is expressed in terms of x by the equation of the given curve; or (3) we may choose the direction along the curve so that s is an increasing function of x (or y), and examine the conditions under which U, as a function of x (or y), is a maximum or minimum.

ILLUSTRATIVE EXAMPLES

1. A bead of mass m is constrained to move on a smooth curve $y = f(x)$ in a field of force of which the potential function is $U = -\frac{1}{2} m\omega^2 x^2 + mgy$. Find the positions of equilibrium.*

* This problem in the plane is equivalent to the following:

A heavy bead slides on a wire in the form of the curve $y = f(x)$, the Y-axis being directed vertically upwards. The plane of the wire rotates about the Y-axis with constant angular velocity ω. Determine the position of equilibrium of the bead.

Suppose the bead is in a position of equilibrium $A(x, y)$. It then revolves around the Y-axis in a circle of radius x with angular velocity ω. The bead exerts a horizontal pressure on the wire equal to $\frac{mv^2}{x} = m\omega^2 x$ (Art. 54). For motion of the bead along the wire the resultant of the forces acting is equivalent to that of the field specified above.

Solution. The components of the force acting at any point of the field are given by (Art. 91)

$$(1)\qquad \begin{cases} F_x = -\dfrac{\partial U}{\partial x} = m\omega^2 x, \\ F_y = -\dfrac{\partial U}{\partial x} = -mg. \end{cases}$$

For a position of equilibrium we have

$$F_x \frac{dx}{ds} + F_y \frac{dy}{ds} = 0,$$

or,

$$F_x + F_y \frac{dy}{dx} = 0,$$

where $\frac{dy}{dx}$ is found from the equation of the curve.

Substituting the values from (1), the positions of equilibrium are found by solving the equations

$$(2)\qquad \begin{cases} \omega^2 x = g\dfrac{dy}{dx}, \\ y = f(x). \end{cases}$$

2. Suppose the curve of example 1 is the straight line $y = x \tan\alpha$. **Find the position of equilibrium and determine whether it is stable or unstable.**

Solution. Substituting in equations (2) we have

$$\omega^2 x = g \tan\alpha,$$
$$y = x \tan\alpha.$$

This set of equations has one solution, namely,

$$x = \frac{g}{\omega^2}\tan\alpha, \quad y = \frac{g}{\omega^2}\tan^2\alpha.$$

To determine whether the equilibrium is stable or unstable, we express the potential in terms of the length of arc measured from the origin. Since the curve is a straight line, we have

$$x = s\cos\alpha, \quad y = s\sin\alpha.$$

Hence

$$U = -\tfrac{1}{2} m\omega^2 s^2 \cos^2\alpha + mgs\sin\alpha.$$

Differentiating,

$$\frac{dU}{ds} = -m\omega^2 s\cos^2\alpha + mg\sin\alpha,$$
$$\frac{d^2U}{ds^2} = -m\omega^2\cos^2\alpha.$$

Since the second derivative of U is negative, the function U is (Calculus, p. 124) a maximum, and the position is one of unstable equilibrium.

PROBLEMS

1. Suppose the curve of illustrative example 1 is the parabola $x^2 = 2py$. Show that there is no position of neutral equilibrium unless $g = p\omega^2$. If this condition is satisfied, then every point on the curve is a position of equilibrium.

2. If the curve is the circle $x^2 + y^2 = a^2$, find the position of equilibrium.

Ans. $y = -\frac{g}{\omega^2}$.

3. Suppose the curve is the cubical parabola $3y = x^3$. Find the position of equilibrium and prove that it is stable. *Ans.* $x = \frac{\omega^2}{g}$, $y = \frac{\omega^6}{3g^3}$.

4. Suppose the curve is the semicubical parabola $9y^2 = 4x^3$. Find the position of equilibrium and prove that it is unstable. *Ans.* $x = \frac{g^2}{\omega^4}$, $y = \frac{2}{3}\frac{g^3}{\omega^6}$.

5. A heavy bead slides on a smooth wire of any shape in a vertical plane. Discuss the positions of equilibrium.

6. A unit particle is constrained to move along the curve $ny = x^n$, in the field of force of which the potential function is $U = -\omega^2 x^2 + 2gy$. Show that the position of equilibrium is unstable, neutral, or stable according as n is less than, equal to, or greater than 2.

7. A unit particle is constrained to move along the circle $x^2 + y^2 = a^2$, in the field of force of which the potential function is $U = Ax^2 + By^2$. Discuss the positions of equilibrium.

8. A unit particle is constrained to move along the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, in the field of force of which the potential function is $U = x^2 + y^2$. Discuss the positions of equilibrium.

9. In the preceding problem suppose the potential of the field is $U = \frac{1}{\sqrt{x^2 + y^2}}$.

CHAPTER XIV

COLLECTION OF FORMULAS

For the convenience of the student we give the following list of elementary formulas from Algebra, Geometry, Trigonometry, Analytic Geometry, and Calculus.

FORMULAS FROM ALGEBRA

1. Binomial Theorem (n being a positive integer):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-r+2)}{\underline{|r-1}}a^{n-r+1}b^{r-1} + \cdots.$$

Also written:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots$$
$$+ \binom{n}{r-1}a^{n-r+1}b^{r-1} + \cdots.$$

2. $n! = \underline{|n} = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n.$

3. In the quadratic equation $ax^2 + bx + c = 0$,
when $b^2 - 4ac > 0$, the roots are real and unequal;
when $b^2 - 4ac = 0$, the roots are real and equal;
when $b^2 - 4ac < 0$, the roots are imaginary.

4. When a quadratic equation is reduced to the form $x^2 + px = q$, p = sum of roots with sign changed,
and q = product of roots with sign changed.

5. In an arithmetic series,

$$l = a + (n-1)d; \quad s = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n-1)d].$$

6. In a geometric series,

$$l = ar^{n-1}; \quad s = \frac{rl - a}{r - 1} = \frac{a(r^n - 1)}{r - 1}.$$

7. $\log ab = \log a + \log b.$ 9. $\log a^n = n \log a.$

8. $\log \frac{a}{b} = \log a - \log b.$ 10. $\log \sqrt[n]{a} = \frac{1}{n} \log a.$

11. $\log 1 = 0.$ 12. $\log_a a = 1.$ 13. $\log \frac{1}{a} = -\log a.$

FORMULAS FROM GEOMETRY

14. Circumference of circle $= 2\pi r$.*

15. Area of circle $= \pi r^2$. 16. Volume of prism $= Ba$.

17. Volume of pyramid $= \frac{1}{3} Ba$.

18. Volume of right circular cylinder $= \pi r^2 a$.

19. Lateral surface of right circular cylinder $= 2\pi ra$.

20. Total surface of right circular cylinder $= 2\pi r(r + a)$.

21. Volume of right circular cone $= \frac{1}{3} \pi r^2 a$.

22. Lateral surface of right circular cone $= \pi rs$.

23. Total surface of right circular cone $= \pi r(r + s)$.

24. Volume of sphere $= \frac{4}{3} \pi r^3$. 25. Surface of sphere $= 4\pi r^2$.

FORMULAS FROM TRIGONOMETRY

26. $\sin x = \frac{1}{\csc x}$; $\cos x = \frac{1}{\sec x}$; $\tan x = \frac{1}{\cot x}$.

27. $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$.

28. $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $1 + \cot^2 x = \csc^2 x$.

29. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$; 30. $\sin(\pi - x) = \sin x$;

$\cos x = \sin\left(\frac{\pi}{2} - x\right)$; $\cos(\pi - x) = -\cos x$;

$\tan x = \cot\left(\frac{\pi}{2} - x\right)$. $\tan(\pi - x) = -\tan x$.

31. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

32. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

33. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

* In formulas 14–25 r denotes radius, a altitude, B area of base, and s slant height.

34. $\cos(x-y)=\cos x\cos y+\sin x\sin y.$

35. $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}.$

36. $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}.$

37. $\sin 2x=2\sin x\cos x;$ $\cos 2x=\cos^2 x-\sin^2 x;$ $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}.$

38. $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2};$ $\cos x=\cos^2\dfrac{x}{2}-\sin^2\dfrac{x}{2};$ $\tan x=\dfrac{2\tan\dfrac{x}{2}}{1-\tan^2\dfrac{x}{2}}.$

39. $\cos^2 x=\frac{1}{2}+\frac{1}{2}\cos 2x;$ $\sin^2 x=\frac{1}{2}-\frac{1}{2}\cos 2x.$

40. $1+\cos x=2\cos^2\dfrac{x}{2};$ $1-\cos x=2\sin^2\dfrac{x}{2}.$

41. $\sin\dfrac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{2}};$ $\cos\dfrac{x}{2}=\pm\sqrt{\dfrac{1+\cos x}{2}};$

$\tan\dfrac{x}{2}=\pm\sqrt{\dfrac{1-\cos x}{1+\cos x}}.$

42. $\sin x+\sin y=2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y).$

43. $\sin x-\sin y=2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y).$

44. $\cos x+\cos y=2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y).$

45. $\cos x-\cos y=-2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y).$

46. $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C};$ Law of Sines.

47. $a^2=b^2+c^2-2bc\cos A;$ Law of Cosines.

FORMULAS FROM ANALYTIC GEOMETRY

48. $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2};$ distance between points (x_1, y_1) and (x_2, y_2).

49. $d=\dfrac{Ax_1+By_1+C}{\pm\sqrt{A^2+B^2}};$ distance from line $Ax+By+C=0$ to (x_1, y_1).

50. $x=\dfrac{x_1+\lambda x_2}{1+\lambda},\ y=\dfrac{y_1+\lambda y_2}{1+\lambda};$ (x, y) is the point dividing the line P_1P_2 in the ratio λ.

51. $x = x_0 + x'$, $y = y_0 + y'$; transforming to new origin (x_0, y_0).

52. $x = x' \cos\theta - y' \sin\theta$, $y = x' \sin\theta + y' \cos\theta$; transforming to new axes making the angle θ with old.

53. $x = \rho\cos\theta$, $y = \rho\sin\theta$; transforming from rectangular to polar coördinates.

54. $\rho = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$; transforming from polar to rectangular coördinates.

55. Different forms of equation of a straight line:

(*a*) $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$, two-point form;

(*b*) $\frac{x}{a} + \frac{y}{b} = 1$, intercept form;

(*c*) $y - y_1 = m(x - x_1)$, slope-point form;

(*d*) $y = mx + b$, slope-intercept form;

(*e*) $x\cos\alpha + y\sin\alpha = p$, normal form;

(*f*) $Ax + By + C = 0$, general form.

56. Distance from the line $x\cos\alpha + y\sin\alpha - p = 0$ to the point $(x_1, y_1) = x_1\cos\alpha + y_1\sin\alpha - p$.

57. $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$; angle between two lines whose slopes are m_1 and m_2.

$$m_1 = m_2 \text{ when lines are parallel,}$$

and $$m_1 = -\frac{1}{m_2} \text{ when lines are perpendicular.}$$

58. $(x - \alpha)^2 + (y - \beta)^2 = r^2$; equation of circle with center (α, β) and radius r.

59. $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$; distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2).

60. $d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}$; distance from plane $Ax + By + Cz + D = 0$ to point (x_1, y_1, z_1).

61. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$; α, β, γ being the direction angles of a line in space.

62. $x\cos\alpha + y\cos\beta + z\cos\gamma$; projection of the line joining $(0, 0, 0)$ and (x, y, z) upon a line whose direction angles are α, β, γ.

63. $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$; equation of sphere with center (α, β, γ) and radius r.

FORMULAS FROM CALCULUS

64. Radius of curvature.

(*a*) Rectangular coördinates.

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

(*b*) Polar coördinates.

$$R = \frac{\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}.$$

(*c*) Parametric form.

$$R = \frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}.$$

65. Plane area.

(*a*) Rectangular coördinates.

$$A = \int y\,dx = \int\int dy\,dx.$$

(*b*) Polar coördinates.

$$A = \tfrac{1}{2}\int \rho^2 d\theta = \int\int \rho\,d\rho\,d\theta.$$

66. Length of arc.

(*a*) Rectangular coördinates.

$$s = \int [dx^2 + dy^2]^{\frac{1}{2}} = \int\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx = \int\left[\left(\frac{dx}{dy}\right)^2+1\right]^{\frac{1}{2}} dy.$$

(*b*) Polar coördinates.

$$s = \int\left[\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta = \int\left[\rho^2\left(\frac{d\theta}{d\rho}\right)^2+1\right]^{\frac{1}{2}} d\rho.$$

67. Volume of solid of revolution about the X-axis.

$$V = \pi \int y^2 dx.$$

68. Area of surface of revolution about the X-axis.

$$A = 2\pi \int y ds = 2\pi \int y \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} dx = 2\pi \int y \left[\left(\frac{dx}{dy} \right)^2 + 1 \right]^{\frac{1}{2}} dy.$$

69. Area of any surface, $z = f(x, y)$.

$$A = \int \int \left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right]^{\frac{1}{2}} dy dx.$$

70. Volume of any solid.

$$V = \int \int \int dz dy dx.$$

DIFFERENTIAL EQUATIONS

71. The differential equation of HARMONIC MOTION.

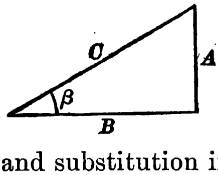
$$\frac{d^2x}{dt^2} + k^2 x = 0.$$

The general solution may be written in the following forms:

(a) $$x = c_1 e^{kt\sqrt{-1}} + c_2 e^{-kt\sqrt{-1}},$$

(b) $$x = A \cos kt + B \sin kt.$$

We give to (b) another form, thus :
Draw a right triangle with sides A and B. Since A and B are arbitrary constants, this right triangle is arbitrary, and hence also the hypotenuse C and the angle β. Now,



$$A = C \sin \beta,\ B = C \cos \beta,$$

and substitution in (b) gives

$$x = C(\sin \beta \cos kt + \cos \beta \sin kt), \text{ or,}$$

(c) $$x = C \sin (kt + \beta).$$

If in (c) we write for β, $\beta' + \frac{\pi}{2}$, we obtain

$$x = C \sin \left(kt + \beta' + \frac{\pi}{2} \right), \text{ or}$$

(d) $$x = C \cos (kt + \beta').$$

In these formulas c_1, c_2, A, B, C, β, β' denote arbitrary constants.

72. $$\frac{d^2x}{dt^2} - k^2x = 0.$$

The general solution is

$$x = c_1e^{kt} + c_2e^{-kt}.$$

73. The differential equation of DAMPED VIBRATION.

$$\frac{d^2x}{dt^2} + 2\mu\frac{dx}{dt} + k^2x = 0,\ \mu<k.$$

The general solution is

$$x = e^{-\mu t}(A\cos\sqrt{k^2-\mu^2}\,t + B\sin\sqrt{k^2-\mu^2}\,t),\ \text{or}$$

$$x = Ce^{-\mu t}\cos(\sqrt{k^2-\mu^2}\,t+\beta).$$

74. The differential equation of harmonic motion with a constant disturbing force.

$$\frac{d^2x}{dt^2} + k^2x = c.$$

The general solution is

$$x = A\cos kt + B\sin kt + \frac{c}{k^2},\ \text{or}$$

$$x = C\sin(kt+\beta) + \frac{c}{k^2}.$$

75. The differential equation of FORCED VIBRATION.

(*a*) $$\frac{d^2x}{dt^2} + k^2x = L\cos nt + M\sin nt,\ \text{where } n\neq k.$$

The general solution is

$$x = A\cos kt + B\sin kt + \frac{L}{k^2-n^2}\cos nt + \frac{M}{k^2-n^2}\sin nt,$$

where A and B are arbitrary constants.

(*b*) $$\frac{d^2x}{dt^2} + k^2x = L\cos kt + M\sin kt.$$

The general solution is

$$x = A\cos kt + B\sin kt + \frac{L}{2k}t\sin kt - \frac{M}{2k}t\cos kt.$$

FORMULAS FOR DIFFERENTIATION

In these formulas, u, v, and w denote *variable* quantities which are functions of x.

I $$\frac{dc}{dx}=0.$$

II $$\frac{dx}{dx}=1.$$

III $$\frac{d}{dx}(u+v-w)=\frac{du}{dx}+\frac{dv}{dx}-\frac{dw}{dx}.$$

IV $$\frac{d}{dx}(cv)=c\frac{dv}{dx}.$$

V $$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}.$$

VI $$\frac{d}{dx}(v_1v_2\cdots v_n)=(v_2v_3\cdots v_n)\frac{dv_1}{dx}+(v_1v_3\cdots v_n)\frac{dv_2}{dx}+\cdots+(v_1v_2\cdots v_{n-1})\frac{dv_n}{dx}.$$

VII $$\frac{d}{dx}(v^n)=nv^{n-1}\frac{dv}{dx}.$$

VII *a* $$\frac{d}{dx}(x^n)=nx^{n-1}.$$

VIII $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}.$$

VIII *a* $$\frac{d}{dx}\left(\frac{u}{c}\right)=\frac{\dfrac{du}{dx}}{c}.$$

VIII *b* $$\frac{d}{dx}\left(\frac{c}{v}\right)=-\frac{c\dfrac{dv}{dx}}{v^2}.$$

IX $$\frac{d}{dx}(\log_a v)=\log_a e\cdot\frac{\dfrac{dv}{dx}}{v}.$$

IX *a* $$\frac{d}{dx}(\log v)=\frac{\dfrac{dv}{dx}}{v}.$$

X $$\frac{d}{dx}(a^v)=a^v\log a\frac{dv}{dx}.$$

X *a* $$\frac{d}{dx}(e^v)=e^v\frac{dv}{dx}.$$

XI $$\frac{d}{dx}(u^v)=vu^{v-1}\frac{du}{dx}+\log u\cdot u^v\frac{dv}{dx}.$$

XII $$\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}.$$

XIII $$\frac{d}{dx}(\cos v) = -\sin v \frac{dv}{dx}.$$

XIV $$\frac{d}{dx}(\tan v) = \sec^2 v \frac{dv}{dx}.$$

XV $$\frac{d}{dx}(\cot v) = -\csc^2 v \frac{dv}{dx}.$$

XVI $$\frac{d}{dx}(\sec v) = \sec v \tan v \frac{dv}{dx}.$$

XVII $$\frac{d}{dx}(\csc v) = -\csc v \cot v \frac{dv}{dx}.$$

XVIII $$\frac{d}{dx}(\text{vers } v) = \sin v \frac{dv}{dx}.$$

XIX $$\frac{d}{dx}(\text{arc } \sin v) = \frac{\frac{dv}{dx}}{\sqrt{1-v^2}}.$$

XX $$\frac{d}{dx}(\text{arc } \cos v) = -\frac{\frac{dv}{dx}}{\sqrt{1-v^2}}.$$

XXI $$\frac{d}{dx}(\text{arc } \tan v) = \frac{\frac{dv}{dx}}{1+v^2}.$$

XXII $$\frac{d}{dx}(\text{arc } \cot v) = -\frac{\frac{dv}{dx}}{1+v^2}.$$

XXIII $$\frac{d}{dx}(\text{arc } \sec v) = \frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}.$$

XXIV $$\frac{d}{dx}(\text{arc } \csc v) = -\frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}.$$

XXV $$\frac{d}{dx}(\text{arc vers } v) = \frac{\frac{dv}{dx}}{\sqrt{2v-v^2}}.$$

XXVI $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx},$$ y being a function of v.

XXVII $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$ y being a function of x.

INTEGRALS FOR REFERENCE

SOME ELEMENTARY FORMS

1. $\int (du \pm dv \pm dw \pm \cdots) = \int du \pm \int dv \pm \int dw \pm \cdots.$

2. $\int a\,dv = a\int dv.$

3. $\int df(x) = \int f'(x)dx = f(x) + C.$

4. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C,\ n \neq 1.$

5. $\int \dfrac{dx}{x} = \log x + C.$

Forms containing Integral Powers of $a + bx$

6. $\int \dfrac{dx}{a+bx} = \dfrac{1}{b}\log(a+bx) + C.$

7. $\int (a+bx)^n dx = \dfrac{(a+bx)^{n+1}}{b(n+1)} + C,\ n \neq 1.$

8. $\int F(x,\ a+bx)dx.$ Try one of the substitutions, $z = a + bx$, $xz = a + bx$.

9. $\int \dfrac{x\,dx}{a+bx} = \dfrac{1}{b^2}[a + bx - a\log(a+bx)] + C.$

10. $\int \dfrac{x^2dx}{a+bx} = \dfrac{1}{b^3}[\tfrac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)] + C.$

11. $\int \dfrac{dx}{x(a+bx)} = -\dfrac{1}{a}\log\dfrac{a+bx}{x} + C.$

12. $\int \dfrac{dx}{x^2(a+bx)} = -\dfrac{1}{ax} + \dfrac{b}{a^2}\log\dfrac{a+bx}{x} + C.$

13. $\int \dfrac{x\,dx}{(a+bx)^2} = \dfrac{1}{b^2}\left[\log(a+bx) + \dfrac{a}{a+bx}\right] + C.$

14. $\int \dfrac{x^2dx}{(a+bx)^2} = \dfrac{1}{b^3}\left[a + bx - 2a\log(a+bx) - \dfrac{a^2}{a+bx}\right] + C.$

15. $\int \dfrac{dx}{x(a+bx)^2} = \dfrac{1}{a(a+bx)} - \dfrac{1}{a^2}\log\dfrac{a+bx}{x} + C.$

16. $\int \dfrac{x\,dx}{(a+bx)^3} = \dfrac{1}{b^2}\left[-\dfrac{1}{a+bx} + \dfrac{a}{2(a+bx)^2}\right] + C.$

Forms containing $a^2 + x^2$, $a^2 - x^2$, $a + bx^n$, $a + bx^2$

17. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C$; $\int \dfrac{dx}{1+x^2} = \tan^{-1}x + C.$

18. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x} + C$; $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a} + C.$

19. $\int \dfrac{dx}{a+bx^2} = \dfrac{1}{\sqrt{ab}}\tan^{-1}x\sqrt{\dfrac{b}{a}} + C$, when $a > 0$ and $b > 0$.

20. $\displaystyle\int\frac{dx}{a^2-b^2x^2}=\frac{1}{2\,ab}\log\frac{a+bx}{a-bx}+C.$

21. $\displaystyle\int x^m(a+bx^n)^p\,dx$
$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(np+m+1)}-\frac{a(m-n+1)}{b(np+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.$$

22. $\displaystyle\int x^m(a+bx^n)^p\,dx=\frac{x^{m+1}(a+bx^n)^p}{np+m+1}+\frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx.$

23. $\displaystyle\int\frac{dx}{x^m(a+bx^n)^p}$
$$=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)^{p-1}}-\frac{(m-n+np-1)b}{(m-1)a}\int\frac{dx}{x^{m-n}(a+bx^n)^p}.$$

24. $\displaystyle\int\frac{dx}{x^m(a+bx^n)^p}$
$$=\frac{1}{an(p-1)x^{m-1}(a+bx^n)^{p-1}}+\frac{m-n+np-1}{an(p-1)}\int\frac{dx}{x^m(a+bx^n)^{p-1}}.$$

25. $\displaystyle\int\frac{(a+bx^n)^p\,dx}{x^m}=-\frac{(a+bx^n)^{p+1}}{a(m-1)x^{m-1}}-\frac{b(m-n-np-1)}{a(m-1)}\int\frac{(a+bx^n)^p\,dx}{x^{m-n}}.$

26. $\displaystyle\int\frac{(a+bx^n)^p\,dx}{x^m}=\frac{(a+bx^n)^p}{(np-m+1)x^{m-1}}+\frac{anp}{np-m+1}\int\frac{(a+bx^n)^{p-1}\,dx}{x^m}.$

27. $\displaystyle\int\frac{x^m dx}{(a+bx^n)^p}=\frac{x^{m-n+1}}{b(m-np+1)(a+bx^n)^{p-1}}-\frac{a(m-n+1)}{b(m-np+1)}\int\frac{x^{m-n}dx}{(a+bx^n)^p}.$

28. $\displaystyle\int\frac{x^m dx}{(a+bx^n)^p}=\frac{x^{m+1}}{an(p-1)(a+bx^n)^{p-1}}-\frac{m+n-np+1}{an(p-1)}\int\frac{x^m dx}{(a+bx^n)^{p-1}}.$

29. $\displaystyle\int\frac{dx}{(a^2+x^2)^n}=\frac{1}{2(n-1)a^2}\left[\frac{x}{(a^2+x^2)^{n-1}}+(2\,n-3)\int\frac{dx}{(a^2+x^2)^{n-1}}\right].$

30. $\displaystyle\int\frac{dx}{(a+bx^2)^n}=\frac{1}{2(n-1)a}\left[\frac{x}{(a+bx^2)^{n-1}}+(2\,n-3)\int\frac{dx}{(a+bx^2)^{n-1}}\right].$

31. $\displaystyle\int\frac{xdx}{(a+bx^2)^n}=\frac{1}{2}\int\frac{dz}{(a+bz)^n}$, where $z=x^2$.

32. $\displaystyle\int\frac{x^2dx}{(a+bx^2)^n}=\frac{-x}{2\,b(n-1)(a+bx^2)^{n-1}}+\frac{1}{2\,b(n-1)}\int\frac{dx}{(a+bx^2)^{n-1}}.$

33. $\displaystyle\int\frac{dx}{x(a+bx^n)}=\frac{1}{an}\log\frac{x^n}{a+bx^n}+C.$

34. $\displaystyle\int\frac{dx}{x^2(a+bx^2)^n}=\frac{1}{a}\int\frac{dx}{x^2(a+bx^2)^{n-1}}-\frac{b}{a}\int\frac{dx}{(a+bx^2)^n}.$

35. $\displaystyle\int\frac{xdx}{a+bx^2}=\frac{1}{2\,b}\log\left(x^2+\frac{a}{b}\right)+C.$

36. $\displaystyle\int\frac{x^2dx}{a+bx^2}=\frac{x}{b}-\frac{a}{b}\int\frac{dx}{a+bx^2}.$

37. $\displaystyle\int\frac{dx}{x(a+bx^2)}=\frac{1}{2\,a}\log\frac{x^2}{a+bx^2}+C.$

38. $\displaystyle\int\frac{dx}{x^2(a+bx^2)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{dx}{a+bx^2}.$

39. $\displaystyle\int\frac{dx}{(a+bx^2)^2}=\frac{x}{2\,a(a+bx^2)}+\frac{1}{2\,a}\int\frac{dx}{a+bx^2}.$

FORMS CONTAINING $\sqrt{a+bx}$

40. $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C.$

41. $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C.$

42. $\int \frac{xdx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx} + C.$

43. $\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx} + C.$

44. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} + C$, for $a>0$.

45. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}} + C$, for $a<0$.

46. $\int \frac{dx}{x^2\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}.$

47. $\int \frac{\sqrt{a+bx}\,dx}{x} = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}.$

FORMS CONTAINING $\sqrt{x^2+a^2}$

48. $\int (x^2+a^2)^{\frac{1}{2}}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + C.$

49. $\int (x^2+a^2)^{\frac{3}{2}}dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^4}{8}\log(x+\sqrt{x^2+a^2}) + C.$

50. $\int (x^2+a^2)^{\frac{n}{2}}dx = \frac{x(x^2+a^2)^{\frac{n}{2}}}{n+1} + \frac{na^2}{n+1}\int(x^2+a^2)^{\frac{n}{2}-1}dx.$

51. $\int x(x^2+a^2)^{\frac{n}{2}}dx = \frac{(x^2+a^2)^{\frac{n+2}{2}}}{n+2} + C.$

52. $\int x^2(x^2+a^2)^{\frac{1}{2}}dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2+a^2}) + C.$

53. $\int \frac{dx}{(x^2+a^2)^{\frac{1}{2}}} = \log(x+\sqrt{x^2+a^2}) + C.$

54. $\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C.$

55. $\int \frac{xdx}{(x^2+a^2)^{\frac{1}{2}}} = \sqrt{x^2+a^2} + C.$

56. $\int \frac{x^2dx}{(x^2+a^2)^{\frac{1}{2}}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2+a^2}) + C.$

57. $\int \frac{x^2dx}{(x^2+a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2+a^2}} + \log(x+\sqrt{x^2+a^2}) + C.$

58. $\int \frac{dx}{x(x^2+a^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{x}{a+\sqrt{x^2+a^2}} + C.$

59. $\int \frac{dx}{x^2(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + C.$

60. $\int \frac{dx}{x^3(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\log\frac{a+\sqrt{x^2+a^2}}{x} + C.$

61. $\int \frac{(x^2+a^2)^{\frac{1}{2}}dx}{x} = \sqrt{a^2+x^2} - a\log\frac{a+\sqrt{a^2+x^2}}{x} + C.$

62. $\int \frac{(x^2+a^2)^{\frac{1}{2}}dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \log(x+\sqrt{x^2+a^2}) + C.$

Forms containing $\sqrt{x^2-a^2}$

63. $\int (x^2-a^2)^{\frac{1}{2}}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.$

64. $\int (x^2-a^2)^{\frac{3}{2}}dx = \frac{x}{8}(2\,x^2-5\,a^2)\sqrt{x^2-a^2} + \frac{3\,a^4}{8}\log(x+\sqrt{x^2-a^2}) + C.$

65. $\int (x^2-a^2)^{\frac{n}{2}}dx = \frac{x(x^2-a^2)^{\frac{n}{2}}}{n+1} - \frac{na^2}{n+1}\int (x^2+a^2)^{\frac{n}{2}-1}dx.$

66. $\int x(x^2-a^2)^{\frac{n}{2}}dx = \frac{(x^2-a^2)^{\frac{n+2}{2}}}{n+2} + C.$

67. $\int x^2(x^2-a^2)^{\frac{1}{2}}dx = \frac{x}{8}(2\,x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\log(x+\sqrt{x^2-a^2}) + C.$

68. $\int \frac{dx}{(x^2-a^2)^{\frac{1}{2}}} = \log(x+\sqrt{x^2-a^2}) + C.$

69. $\int \frac{dx}{(x^2-a^2)^{\frac{3}{2}}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C.$

70. $\int \frac{xdx}{(x^2-a^2)^{\frac{1}{2}}} = \sqrt{x^2-a^2} + C.$

71. $\int \frac{x^2dx}{(x^2-a^2)^{\frac{1}{2}}} = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2-a^2}) + C.$

72. $\int \frac{x^2dx}{(x^2-a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2-a^2}} + \log(x+\sqrt{x^2-a^2}) + C.$

73. $\int \frac{dx}{x(x^2-a^2)^{\frac{1}{2}}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C;\quad \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C.$

74. $\int \frac{dx}{x^2(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C.$

75. $\int \frac{dx}{x^3(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\sec^{-1}\frac{x}{a} + C.$

76. $\int \frac{(x^2-a^2)^{\frac{1}{2}}\,dx}{x} = \sqrt{x^2-a^2} - a\cos^{-1}\frac{a}{x} + C.$

77. $\int \frac{(x^2-a^2)^{\frac{1}{2}}\,dx}{x^2} = -\frac{\sqrt{x^2-a^2}}{x} + \log\left(x+\sqrt{x^2-a^2}\right) + C.$

FORMS CONTAINING $\sqrt{a^2-x^2}$

78. $\int (a^2-x^2)^{\frac{1}{2}}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

79. $\int (a^2-x^2)^{\frac{3}{2}}dx = \frac{x}{8}(5\,a^2-2\,x^2)\sqrt{a^2-x^2} + \frac{3\,a^4}{8}\sin^{-1}\frac{x}{a} + C.$

80. $\int (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1} + \frac{a^2n}{n+1}\int (a^2-x^2)^{\frac{n}{2}-1}\,dx.$

81. $\int x(a^2-x^2)^{\frac{n}{2}}\,dx = -\frac{(a^2-x^2)^{\frac{n+2}{2}}}{n+2} + C.$

82. $\int x^2(a^2-x^2)^{\frac{1}{2}}\,dx = \frac{x}{8}(2\,x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + C.$

83. $\int \frac{dx}{(a^2-x^2)^{\frac{1}{2}}} = \sin^{-1}\frac{x}{a}$; $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x.$

84. $\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C.$ 85. $\int \frac{xdx}{(a^2-x^2)^{\frac{1}{2}}} = -\sqrt{a^2-x^2} + C.$

86. $\int \frac{x^2dx}{(a^2-x^2)^{\frac{1}{2}}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

87. $\int \frac{x^2dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C.$

88. $\int \frac{x^mdx}{(a^2-x^2)^{\frac{1}{2}}} = -\frac{x^{m-1}}{m}\sqrt{a^2-x^2} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{(a^2-x^2)^{\frac{1}{2}}}dx.$

89. $\int \frac{dx}{x(a^2-x^2)^{\frac{1}{2}}} = \frac{1}{a}\log\frac{x}{a+\sqrt{a^2-x^2}} + C.$

90. $\int \frac{dx}{x^2(a^2-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C.$

91. $\int \frac{dx}{x^3(a^2-x^2)^{\frac{1}{2}}} = -\frac{\sqrt{a^2-x^2}}{2\,a^2x^2} + \frac{1}{2\,a^3}\log\frac{x}{a+\sqrt{a^2-x^2}} + C.$

92. $\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x} dx = \sqrt{a^2 - x^2} - a \log \frac{a + \sqrt{a^2 - x^2}}{x} + C.$

93. $\int \frac{(a^2 - x^2)^{\frac{1}{2}}}{x^2} dx = - \frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a} + C.$

FORMS CONTAINING $\sqrt{2\,ax - x^2}$, $\sqrt{2\,ax + x^2}$

94. $\int \sqrt{2\,ax - x^2}\, dx = \frac{x - a}{2} \sqrt{2\,ax - x^2} + \frac{a^2}{2} \operatorname{vers}^{-1} \frac{x}{a} + C.$

95. $\int \frac{dx}{\sqrt{2\,ax - x^2}} = \operatorname{vers}^{-1} \frac{x}{a}; \int \frac{dx}{\sqrt{2\,x - x^2}} = \operatorname{vers}^{-1} x + C.$

96. $\int x^m \sqrt{2\,ax - x^2}\, dx = - \frac{x^{m-1}(2\,ax - x^2)^{\frac{3}{2}}}{m + 2} + \frac{(2\,m + 1)a}{m + 2} \int x^{m-1} \sqrt{2\,ax - x^2}\, dx.$

97. $\int \frac{dx}{x^m \sqrt{2\,ax - x^2}} = - \frac{\sqrt{2\,ax - x^2}}{(2\,m - 1)ax^m} + \frac{m - 1}{(2\,m - 1)a} \int \frac{dx}{x^{m-1} \sqrt{2\,ax - x^2}}.$

98. $\int \frac{x^m dx}{\sqrt{2\,ax - x^2}} = - \frac{x^{m-1} \sqrt{2\,ax - x^2}}{m} + \frac{(2\,m - 1)a}{m} \int \frac{x^{m-1} dx}{\sqrt{2\,ax - x^2}}.$

99. $\int \frac{\sqrt{2\,ax - x^2}}{x^m} dx = - \frac{(2\,ax - x^2)^{\frac{3}{2}}}{(2\,m - 3)ax^m} + \frac{m - 3}{(2\,m - 3)a} \int \frac{\sqrt{2\,ax - x^2}}{x^{m-1}} dx.$

100. $\int x \sqrt{2\,ax - x^2}\, dx = - \frac{3\,a^2 + ax - 2\,x^2}{6} \sqrt{2\,ax - x^2} + \frac{a^3}{2} \operatorname{vers}^{-1} \frac{x}{a}.$

101. $\int \frac{dx}{x \sqrt{2\,ax - x^2}} = - \frac{\sqrt{2\,ax - x^2}}{ax} + C.$

102. $\int \frac{x dx}{\sqrt{2\,ax - x^2}} = - \sqrt{2\,ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a} + C.$

103. $\int \frac{x^2 dx}{\sqrt{2\,ax - x^2}} = - \frac{x + 3\,a}{2} \sqrt{2\,ax - x^2} + \frac{3}{2} a^2 \operatorname{vers}^{-1} \frac{x}{a} + C.$

104. $\int \frac{\sqrt{2\,ax - x^2}}{x} dx = \sqrt{2\,ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a} + C.$

105. $\int \frac{\sqrt{2\,ax - x^2}}{x^2} dx = - \frac{2 \sqrt{2\,ax - x^2}}{x} - \operatorname{vers}^{-1} \frac{x}{a} + C.$

106. $\int \frac{\sqrt{2\,ax - x^2}}{x^3} dx = - \frac{(2\,ax - x^2)^{\frac{3}{2}}}{3\,ax^3} + C.$

107. $\int \frac{dx}{(2\,ax - x^2)^{\frac{3}{2}}} = \frac{x - a}{a^2 \sqrt{2\,ax - x^2}} + C.$

108. $\int \frac{x dx}{(2\,ax - x^2)^{\frac{3}{2}}} = \frac{x}{a \sqrt{2\,ax - x^2}} + C.$

109. $\int F(x, \sqrt{2\,ax - x^2}) dx = \int F(z + a, \sqrt{a^2 - z^2}) dz$, where $z = x - a$.

110. $\int \frac{dx}{\sqrt{2\,ax + x^2}} = \log(x + a + \sqrt{2\,ax + x^2}) + C.$

111. $\int F(x,\ \sqrt{2\,ax + x^2})dx = \int F(z - a,\ \sqrt{z^2 - a^2})dz$, where $z = x + a$.

Forms containing $a + bx \pm cx^2$

112. $\int \frac{dx}{a + bx + cx^2} = \frac{2}{\sqrt{4\,ac - b^2}} \tan^{-1} \frac{2\,cx + b}{\sqrt{4\,ac - b^2}} + C$, when $b^2 < 4\,ac$.

113. $\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4\,ac}} \log \frac{2\,cx + b - \sqrt{b^2 - 4\,ac}}{2\,cx + b + \sqrt{b^2 - 4\,ac}} + C$, when $b^2 > 4\,ac$.

114. $\int \frac{dx}{a + bx - cx^2} = \frac{1}{\sqrt{b^2 + 4\,ac}} \log \frac{\sqrt{b^2 + 4\,ac} + 2\,cx - b}{\sqrt{b^2 + 4\,ac} - 2\,cx + b} + C.$

115. $\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$

116. $\int \sqrt{a + bx + cx^2}dx$

$$= \frac{2\,cx + b}{4\,c}\sqrt{a + bx + cx^2} - \frac{b^2 - 4\,ac}{8\,c^{\frac{3}{2}}} \log(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$$

117. $\int \frac{dx}{\sqrt{a + bx - cx^2}} = \frac{1}{\sqrt{c}} \sin^{-1} \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$

118. $\int \sqrt{a + bx - cx^2}dx = \frac{2\,cx - b}{4\,c}\sqrt{a + bx - cx^2} + \frac{b^2 + 4\,ac}{8\,c^{\frac{3}{2}}} \sin^{-1} \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$

119. $\int \frac{xdx}{\sqrt{a + bx + cx^2}} = \frac{\sqrt{a + bx + cx^2}}{c} - \frac{b}{2\,c^{\frac{3}{2}}} \log(2\,cx + b + 2\sqrt{c}\sqrt{a + bx + cx^2}) + C.$

120. $\int \frac{xdx}{\sqrt{a + bx - cx^2}} = -\frac{\sqrt{a + bx - cx^2}}{c} + \frac{b}{2\,c^{\frac{3}{2}}} \sin^{-1} \frac{2\,cx - b}{\sqrt{b^2 + 4\,ac}} + C.$

Other Algebraic Forms

121. $\int \sqrt{\frac{a + x}{b + x}}\,dx = \sqrt{(a + x)(b + x)} + (a - b)\log(\sqrt{a + x} + \sqrt{b + x}) + C.$

122. $\int \sqrt{\frac{a - x}{b + x}}\,dx = \sqrt{(a - x)(b + x)} + (a + b)\sin^{-1}\sqrt{\frac{x + b}{a + b}} + C.$

123. $\int \sqrt{\frac{a + x}{b - x}}\,dx = -\sqrt{(a + x)(b - x)} - (a + b)\sin^{-1}\sqrt{\frac{b - x}{a + b}} + C.$

124. $\int \sqrt{\frac{1 + x}{1 - x}}\,dx = -\sqrt{1 - x^2} + \sin^{-1}x + C.$

125. $\int \frac{dx}{\sqrt{(x - \alpha)(\beta - x)}} = 2\sin^{-1}\sqrt{\frac{x - \alpha}{\beta - \alpha}} + C.$

Exponential and Trigonometric Forms

126. $\int a^x dx = \frac{a^x}{\log a} + C.$

127. $\int e^x dx = e^x + C.$

128. $\int e^{ax} dx = \frac{e^{ax}}{a} + C.$

129. $\int \sin x dx = -\cos x + C.$

130. $\int \cos x dx = \sin x + C.$

131. $\int \tan x dx = \log \sec x = -\log \cos x + C.$

132. $\int \cot x dx = \log \sin x + C.$

133. $\int \sec x dx = \int \frac{dx}{\cos x} = \log(\sec x + \tan x) = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C.$

134. $\int \operatorname{cosec} x dx = \int \frac{dx}{\sin x} = \log(\operatorname{cosec} x - \cot x) = \log \tan \frac{x}{2} + C.$

135. $\int \sec^2 x dx = \tan x + C.$

136. $\int \operatorname{cosec}^2 x dx = -\cot x + C.$

137. $\int \sec x \tan x dx = \sec x + C.$

138. $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C.$

139. $\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C.$

140. $\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin 2x + C.$

141. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx.$

142. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx.$

143. $\int \frac{dx}{\sin^n x} = -\frac{1}{n-1} \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}.$

144. $\int \frac{dx}{\cos^n x} = \frac{1}{n-1} \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$

145. $\int \cos^m x \sin^n x dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x dx.$

146. $\int \cos^m x \sin^n x dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x dx.$

147. $\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{n-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2} x}.$

148. $\int \frac{dx}{\sin^m x \cos^n x} = -\frac{1}{m-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cos^n x}.$

149. $\int \frac{\cos^m x dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\cos^m x dx}{\sin^{n-2} x}.$

150. $\int \frac{\cos^m x dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} x dx}{\sin^n x}.$

151. $\int \sin x \cos^n x dx = -\frac{\cos^{n+1} x}{n+1} + C.$

152. $\int \sin^n x \cos x dx = \frac{\sin^{n+1} x}{n+1} + C.$

153. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx + C.$

154. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx + C.$

155. $\int \sin mx \sin nx dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$

156. $\int \cos mx \cos nx dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$

157. $\int \sin mx \cos nx dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C.$

158. $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right) + C,$ when $a > b$.

159. $\int \frac{dx}{a + b\cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{\sqrt{b-a}\tan\frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a}\tan\frac{x}{2} - \sqrt{b+a}} + C,$ when $a < b$.

160. $\int \frac{dx}{a + b\sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}} + C,$ when $a > b$.

161. $\int \frac{dx}{a + b\sin x} = \frac{1}{\sqrt{b^2 - a^2}} \log \frac{a\tan\frac{x}{2} + b - \sqrt{b^2 - a^2}}{a\tan\frac{x}{2} + b + \sqrt{b^2 - a^2}} + C,$ when $a < b$.

162. $\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab} \tan^{-1}\left(\frac{b\tan x}{a}\right) + C.$

163. $\int e^{ax} \sin nx dx = \frac{e^{ax}(a\sin nx - n\cos nx)}{a^2 + n^2} + C;$

$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2} + C.$

164. $\int e^{ax} \cos nx dx = \frac{e^{ax}(n\sin nx + a\cos nx)}{a^2 + n^2} + C;$

$\int e^x \cos x dx = \frac{e^x(\sin x + \cos x)}{2} + C.$

165. $\int xe^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1) + C.$

166. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx.$

167. $\int a^{mx} x^n dx = \frac{a^{mx} x^n}{m\log a} - \frac{n}{m\log a}\int a^{mx} x^{n-1} dx.$

168. $\int \frac{a^x dx}{x^m} = -\frac{a^x}{(m-1)x^{m-1}} + \frac{\log a}{m-1} \int \frac{a^x dx}{x^{m-1}}.$

169. $\int e^{ax} \cos^n x dx = \frac{e^{ax} \cos^{n-1} x (a \cos x + n \sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2+n^2} \int e^{ax} \cos^{n-2} x dx.$

170. $\int x^m \cos ax dx = \frac{x^{m-1}}{a^2}(ax \sin ax + m \cos ax) - \frac{m(m-1)}{a^2} \int x^{m-2} \cos ax dx.$

Logarithmic Forms

171. $\int \log x dx = x \log x - x + C.$

172. $\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{1}{2^2} \log^2 x + \cdots.$

173. $\int \frac{dx}{x \log x} = \log(\log x) + C.$

174. $\int x^n \log x dx = x^{n+1}\left[\frac{\log x}{n+1} - \frac{1}{(n+1)^2}\right] + C.$

175. $\int e^{ax} \log x dx = \frac{e^{ax} \log x}{a} - \frac{1}{a} \int \frac{e^{ax}}{x} dx.$

176. $\int x^m \log^n x dx = \frac{x^{m+1}}{m+1} \log^n x - \frac{n}{m+1} \int x^m \log^{n-1} x dx.$

177. $\int \frac{x^m dx}{\log^n x} = -\frac{x^{m+1}}{(n-1)\log^{n-1} x} + \frac{m+1}{n-1} \int \frac{x^m dx}{\log^{n-1} x}.$

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