

# 2021

**LECTURE NOTES**

**UNIT – I**

## CEES303-Engineering Mechanics



**Prepared by**

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## SYLLABUS

<b>ETES303</b>	<b>ENGINEERING MECHANICS</b>	<b>L      T      P      C</b>
		<b>3      0      0      3</b>

### **COURSE OBJECTIVES**

- To introduce the fundamentals of forces and their effects with their governing laws.
- To understand the definitions of particle, body forces and their equilibrium conditions.
- To understand and predict the forces and its related motions

**UNIT-I** Introduction to Engineering Mechanics-Force Systems-Basic concepts, Particle equilibrium in 2-D & 3-D; Rigid Body equilibrium; System of Forces, Coplanar Concurrent Forces, Components in Space – Resultant- Moment of Forces and its Application; Couples and Resultant of Force System, Equilibrium of System of Forces, Free body diagrams, Equations of Equilibrium of Coplanar Systems and Spatial Systems; Static Indeterminacy

**UNIT-II** Basic Structural Analysis covering, Equilibrium in three dimensions; Method of Sections; Method of Joints; How to determine if a member is in tension or compression; Simple Trusses; Zero force members; Beams & types of beams; Frames & Machines Centroid and Centre of Gravity covering, Centroid of simple figures from first principle, centroid of composite sections; Centre of Gravity and its implications; Area moment of inertia- Definition, Moment of inertia of plane sections from first principles, Theorems of moment of inertia, Moment of inertia of standard sections and composite sections; Mass moment inertia of circular plate, Cylinder, Cone, Sphere, Hook.

**UNIT-III** Friction covering, Types of friction, Limiting friction, Laws of Friction, Static and Dynamic Friction; Motion of Bodies, wedge friction, screw jack & differential screw jack.

Virtual Work and Energy Method- Virtual displacements, principle of virtual work for particle and ideal system of rigid bodies, degrees of freedom. Active force diagram, systems with friction, mechanical efficiency. Conservative forces and potential energy (elastic and gravitational), energy equation for equilibrium. Applications of energy method for equilibrium. Stability of equilibrium.

**UNIT-IV** Review of particle dynamics- Rectilinear motion; Plane curvilinear motion (rectangular, path, and polar coordinates). 3-D curvilinear motion; Relative and constrained motion; Newton's 2nd law (rectangular, path, and polar coordinates). Work-kinetic energy, power, potential energy. Impulse-momentum (linear, angular); Impact (Direct and oblique).

**UNIT-V** Introduction to Kinetics of Rigid Bodies covering, Basic terms, general principles in dynamics; Types of motion, Instantaneous centre of rotation in plane motion and simple problems; D'Alembert's principle and its applications in plane motion and connected bodies; Work energy principle and its application in plane motion of connected bodies; Kinetics of rigid body rotation

Mechanical Vibrations covering, Basic terminology, free and forced vibrations, resonance and its effects; Degree of freedom; Derivation for frequency and amplitude of free vibrations without damping and single degree of freedom system, simple problems, types of pendulum, use of simple, compound and torsion pendulums;

**Tutorials** from the above modules covering, To find the various forces and angles including resultants in various parts of wall crane, roof truss, pipes, etc.; To verify the line of polygon on various forces; To find coefficient of friction between various materials on inclined plan; Free body diagrams various systems including block-pulley; To verify the principle of moment in the disc apparatus; Helical block; To draw a load efficiency curve for a screw jack

## TEXT BOOKS

1. Irving H. Shames (2006), Engineering Mechanics, 4th Edition, Prentice Hall
2. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill

## REFERENCES

1. R. C. Hibbler (2006), Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.
2. Andy Ruina and Rudra Pratap (2011), Introduction to Statics and Dynamics, Oxford University Press
3. Shanes and Rao (2006), Engineering Mechanics, Pearson Education,
4. Hibler and Gupta (2010), Engineering Mechanics (Statics, Dynamics) by Pearson Education
5. Reddy Vijaykumar K. and K. Suresh Kumar(2010), Singer's Engineering Mechanics
6. Bansal R.K.(2010), A Text Book of Engineering Mechanics, Laxmi Publications
7. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
8. Tayal A.K. (2010), Engineering Mechanics, Umesh Publications

## COURSE OUTCOMES

At the end of this course, students will able to

1. Understand the forces system in a mechanics.
2. Analyze the structure and stability of mechanics.
3. Understand the friction and energy in rigid bodies.
4. Analyze the kinematics of motion in a particle.
5. Analyze the kinematics and kinetics in rigid bodies.

**Mapping of COs with POs**

COs	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
<b>CO1</b>	✓		✓									✓
<b>CO2</b>	✓	✓	✓	✓								
<b>CO3</b>	✓	✓	✓									
<b>CO4</b>	✓	✓	✓									
<b>CO5</b>	✓	✓	✓									

## **Unit - I**

### **Topics Covered in this notes (Part- I)**

Introduction to Engineering Mechanics-Force Systems-Basic concepts, Particle equilibrium in 2-D & 3-D; Rigid Body equilibrium; System of Forces, Coplanar Concurrent Forces, Components in Space – Resultant- Moment of Forces and its Application; Couples and Resultant of Force System, Equilibrium of System of Forces, Free body diagrams, Equations of Equilibrium of Coplanar Systems and Spatial Systems; Static Indeterminacy

### **Reference for the preparation of course material:**

1. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
3. J.L. Meriam and L. G. Kraige, Engineering Mechanics –Statics, volume I, Seventh edition, John Wiley & Sons, Inc.
4. R. C. Hibbler, Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.

**Unit-1****FORCE SYSTEM IN A MECHANICS****MECHANICS:**

It is a branch of science deals with motions, forces and time. The mechanics are grouped into three categories according the researchers as

- 1) Classical Mechanics/ Newtonian Mechanics
- 2) Relativistic Mechanics
- 3) Quantum Mechanics/ Wave Mechanics

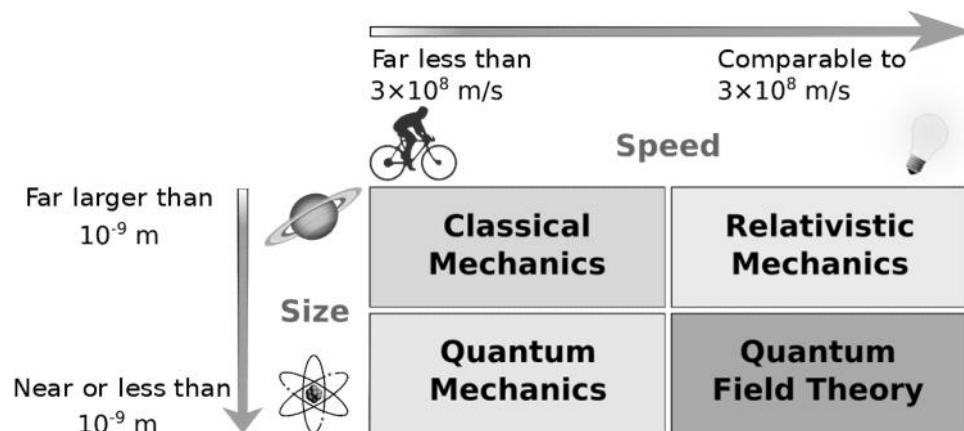


Fig. 1.1Classification of mechanics

**CLASSICAL MECHANICS/ NEWTONIAN MECHANICS:**

As per Sir Isaac Newton, the state of rest and motion of bodies undergoes/ form through three laws of motion with gravitational law is called Classical Mechanics or Newtonian Mechanics.

**Relativistic Mechanics:**

The demerits of Newtonian Mechanics to prove the behaviour of high speed bodies (Light) is studied by Albert Einstein as Relativistic Mechanics.

**Quantum Mechanics/ Wave Mechanics:**

The demerits of Newtonian Mechanics to prove the behaviour of particles at the atomic motion level is studied by Schrodinger and Broglie as Quantum Mechanics.

## **ENGINEERING MECHANICS:**

Many engineering field govern the laws of mechanics which state that the stability of work and termed as Engineering Mechanics. The engineers mostly considered classical/ Newtonian mechanics because the entire engineering problems are not dealt with in the atomic or high speed level.

## **CLASSIFICATION OF ENGINEERING MECHANICS:**

Engineering mechanics depends on the application of body (Solid or fluid) in which the mechanics is discussed. It is classified as

1. Solid Mechanics
  - a. Rigid body of mechanics
    - i. Static mechanics
    - ii. Dynamic mechanics
      1. Kinematics of mechanics
      2. Kinetics of mechanics
  - b. Deformable body of mechanics
    - i. Theory of elasticity
    - ii. Theory of plasticity
2. Fluid Mechanics
  - a. Ideal fluid
  - b. Viscous fluid
  - c. Incompressible fluid, etc.,

The solid body may be rigid and deformable. If the solid bodies are rigid and negligible deformation then the body is termed as rigid body. The study of rigid bodies at rest is named as static mechanics and at motion is dynamic mechanics. The motion of rigid bodies (Dynamic Mechanics) cause without any external force is called as Kinematics of Mechanics. Suppose the motion of rigid bodies dealt with force is named as Kinetics of Mechanics.

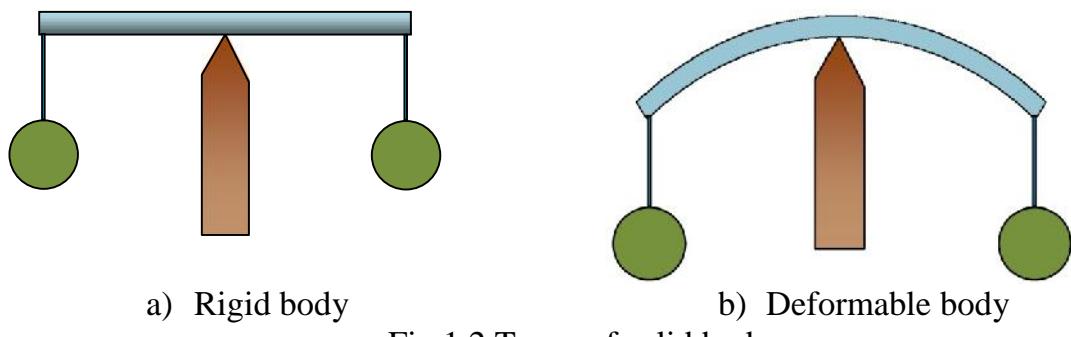


Fig.1.2 Types of solid body

If the solid bodies are deformable due to internal stress development is called as Mechanics of Deformable or Strength of Materials or Mechanics of solids. The deformable bodies undergo the limit of stress acting on the body and classified as theory of elasticity and theory of plasticity

The behaviour of material varies under applied shear force to flows or deforms. The study of fluid motion is termed as Fluid Mechanics.

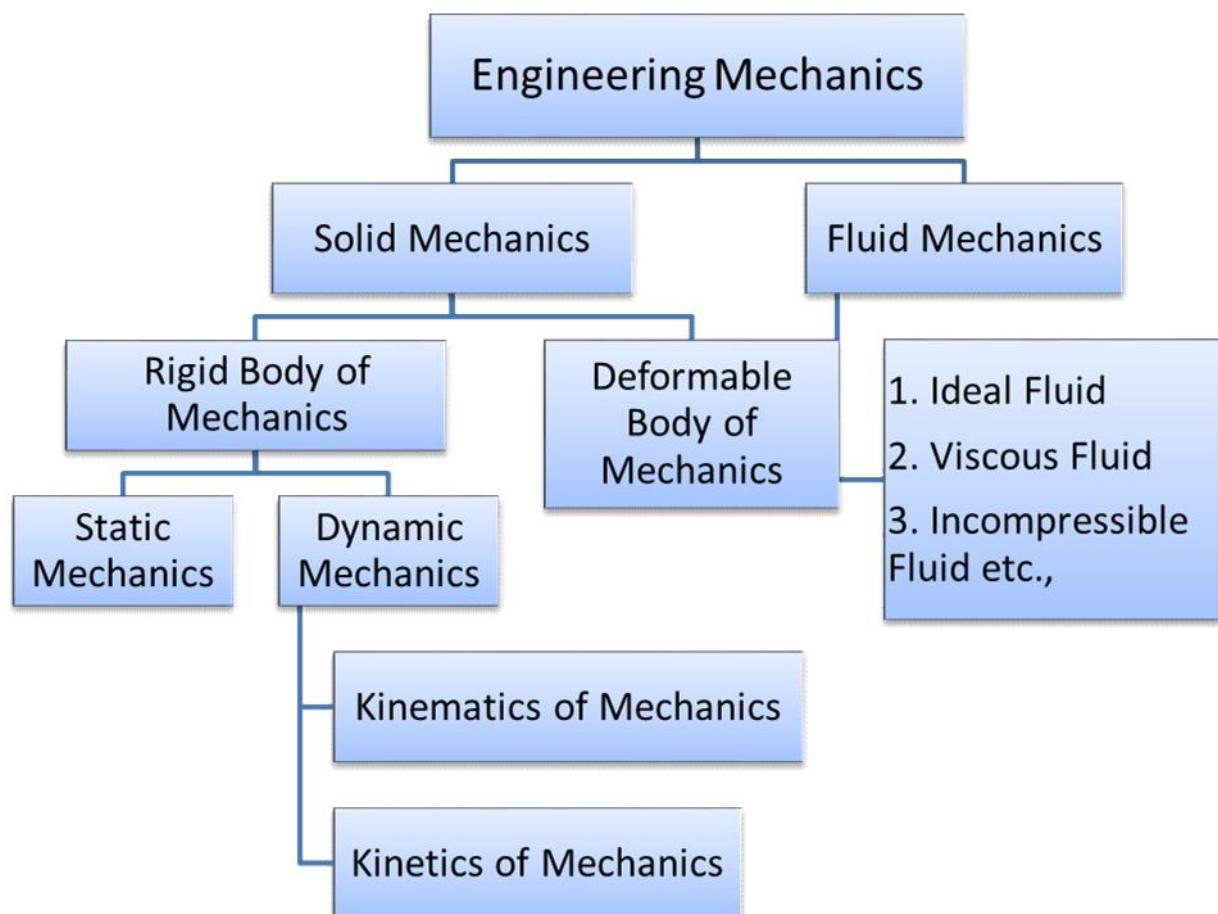


Fig. 1.3 classifications of Engineering Mechanics

**BASIC CONCEPT OF MECHANICS:**

There are many number of terms coincidence to understand the principle of mechanics and they are following as

**Space:**

A body occupied by the region with geometrical coordinates system to determines the position of linear and angular measurement.

**Mass:**

A space contains a quantity of mater in a body is known as mass. However, it helps to measure the inertia of a body. The mass of a body remain same at space but the weight of a body changes with gravitational force. Thereby the weight is the product of mass and gravitational force. Also the mass resists the change of velocity.

**Time:**

It is a parameter of dynamics of mechanics to measure the succession of events.

**Example:**

- 1) Earths rotates about its own axis and it is measured as number of days says around 365. An hour consists of a certain number of minutes as 60 seconds, a day of hours and a year of days.

**Length:**

It is a linear distance of measurement mostly the units are in metre.

**Distance or path length:**

It is total path length covered by an object during the entire journey without taking into consideration its direction it is a scalar quantity and it is always positive for an objective in motion.

**Displacement:**

A particle or body moves from one position to another, the shortest distance between the initial and final position is said to be the magnitude of the displacement and it is directed from initial to final position.

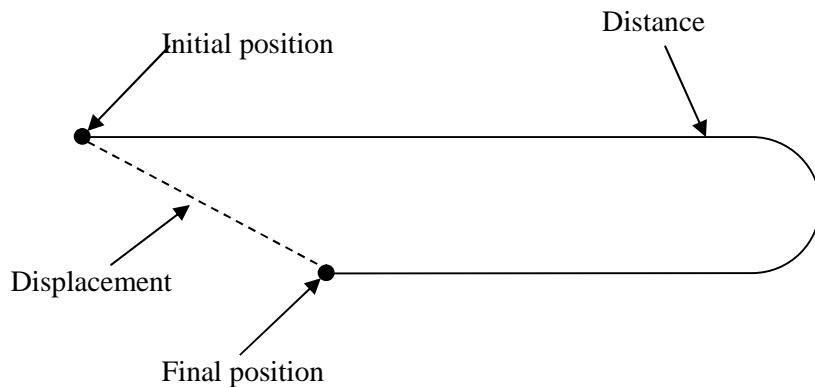


Fig. 1.4 Motion of a body

**Velocity:**

It is defined as the rate of change of displacement (i.e., the displacement changes with time)

**Acceleration:**

It is defined as the rate of change of velocity (i.e., the velocity changes with time)

**Momentum:**

It is defined as the product of mass and velocity.

**Force:**

It is an action of a body on another. The action of force is characterized by its magnitude, by the direction of its action and by its point of application. Therefore force is vector quantity. A force may produce the following effects in the body, on which its acts;

- It may change the motion of a body i.e., if a body is at rest, the force set it in motion. If the body is already in motion then the force may accelerate it.
- It may retard the motion of a body.
- It may retard the forces, already acting on the body, thus bringing it to the rest or in equilibrium.
- It may give rise to the internal stresses in the body, on which it acts.

**PARTICLE:**

A body as a particle when its dimensions are not considered to the subject of position or action of forces applied on it, particle has only mass and no size. In nature

this type of a body does not exist. However the body is treated as particle when the distance larger than the size of the body.

### Examples:

- 1) Earth as a particle when studying its orbital motion.
- 2) A ship at mid sea as a particle when its relative motion from a light house.
- 3) A fighter plane as a particle when its gunner operating from the ground.

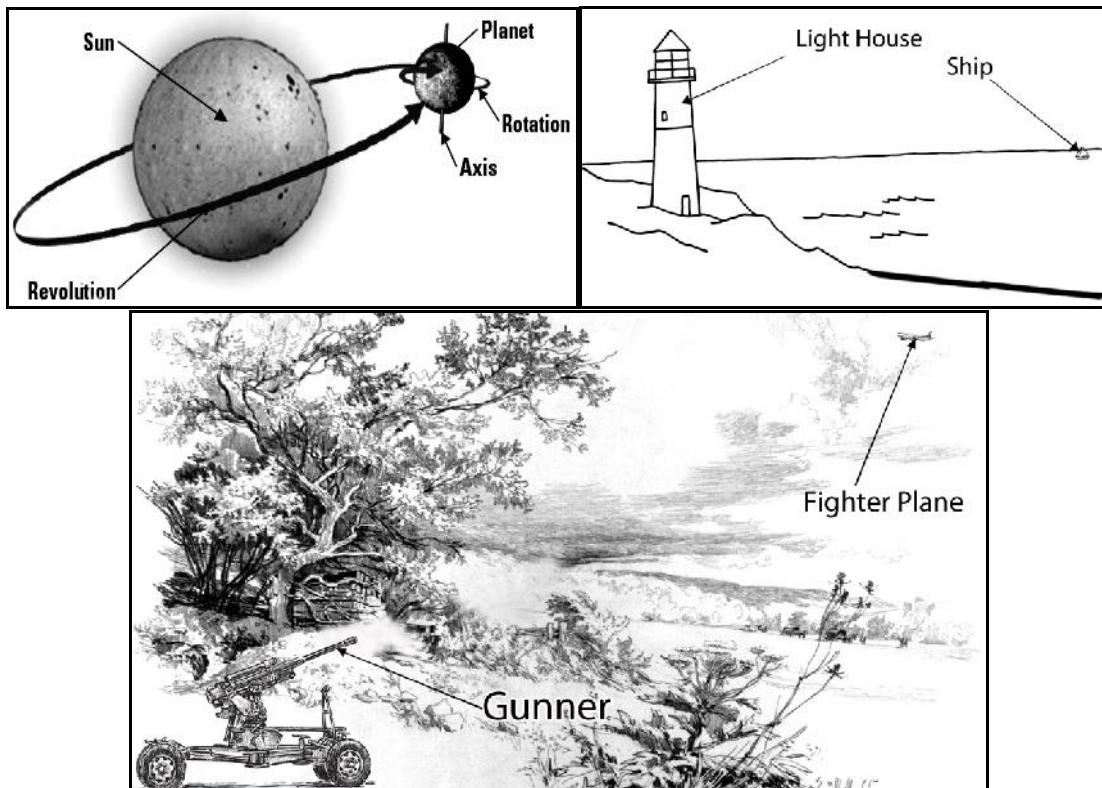


Fig. 1.5 Particles

### CONTINUUM:

A group of matters is called as a body. A body may be particle and subdivided into molecules, atoms and electrons. The discrete particles are more complex in engineering to solve by treating the body as a conglomeration. The body consists continuous distribution of matter and this is called as continuum.

### RIGID BODY:

If a body is a rigid body then the deformation over that body is negligible when there is a force. It is essential for the design and analysis of different types of members of structure, mechanical elements, shipbuilding, electrical device, etc., A rigid body mechanics deals either static or dynamics. The static of rigid body mechanics may be

either at rest or motion with a constant velocity under action of forces (equilibrium conditions). The dynamic of rigid body mechanics may be at motion or motion with a different velocity (accelerated motion) under action of force or not.

### Examples:

- 1) Static of rigid body:

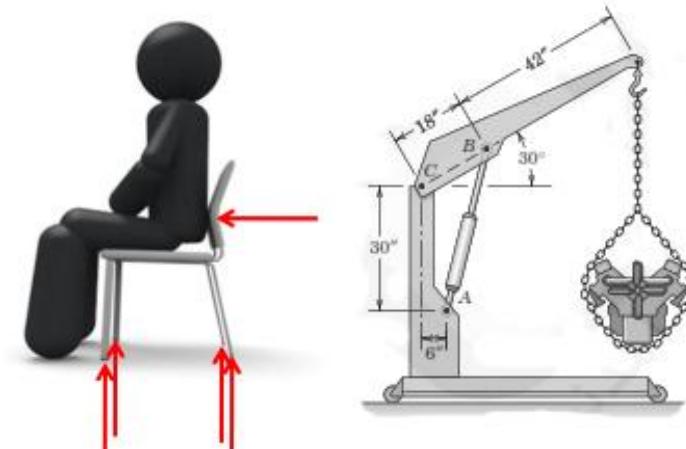


Fig. 1.6 Static of rigid body

- 2) Dynamic of rigid body:

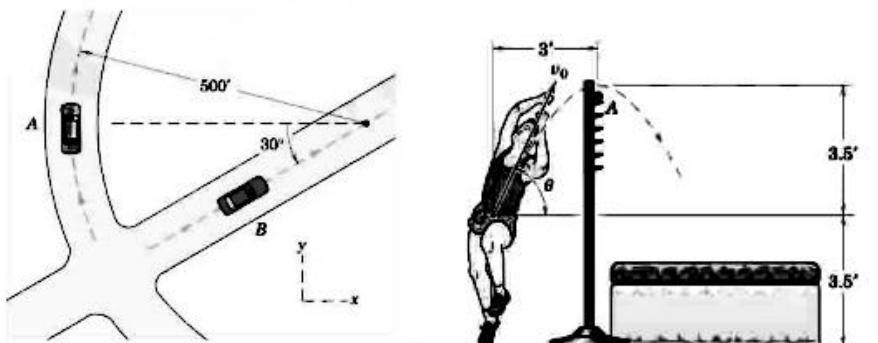


Fig. 1.7 Dynamic of rigid body

### Scalar quantity:

A scalar quantity is specified by a single value with an appropriate unit and has no direction. The rules of ordinary arithmetic are used to manipulate scalar quantities.

### Examples:

- 1) Distance
- 2) Time
- 3) Volume
- 4) Density
- 5) Speed
- 6) Temperature
- 7) Pressure

- 8) Energy and
- 9) Mass

### Vector quantity:

A vector quantity must have magnitude as well as direction and obeys the laws of vector addition.

### Examples:

- 1) Displacement
- 2) Velocity
- 3) Acceleration
- 4) Force
- 5) Torque
- 6) Moment and
- 7) Momentum

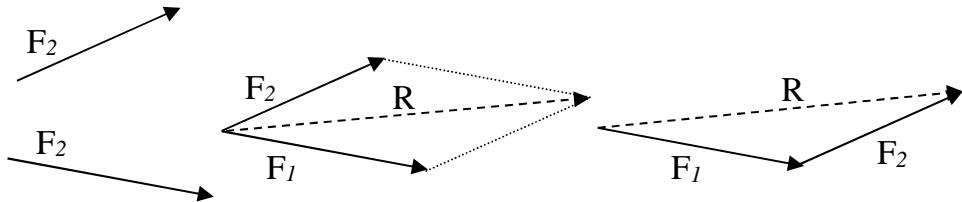


Fig.1.8 Composition of forces

In the fig.1.7 the force  $F_1$  and  $F_2$  are two force vectors acted in a direction and magnitude values. The equivalent force of vectors is  $R$  and it is represented by sum of two force vectors as  $R = F_1 + F_2$ . The force vector  $R$  can be written as

$$R = F_m \vec{n} = (F_1 + F_2)_m \vec{n} = (F_{1x} + F_{1y} + F_{2x} + F_{2y})_m (\cos \theta \cdot i + \sin \theta \cdot j)$$

$$R = (F_{1x} + F_{2x})_m \cos \theta \cdot i + (F_{1y} + F_{2y})_m \sin \theta \cdot j$$

$$R_x \cos \theta \cdot i + R_y \sin \theta \cdot j = (F_{1x} + F_{2x})_m \cos \theta \cdot i + (F_{1y} + F_{2y})_m \sin \theta \cdot j$$

$$\therefore R_x = F_{1x} + F_{2x} = \sum F_x \text{ and}$$

$$R_y = F_{1y} + F_{2y} = \sum F_y$$

Where,  $R$  – Resultant force of given two forces

$F_m$  - Magnitude of force  $F_1$  and  $F_2$ .

$\vec{n}$  - Unit vector represents the direction of force in which the magnitude of force acted.

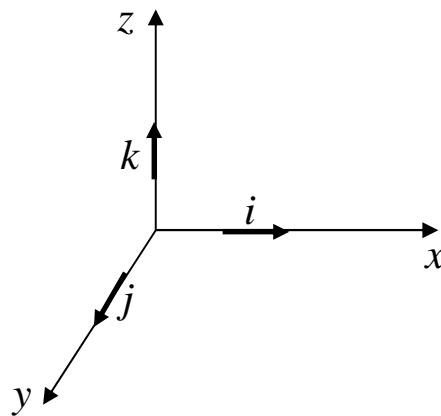
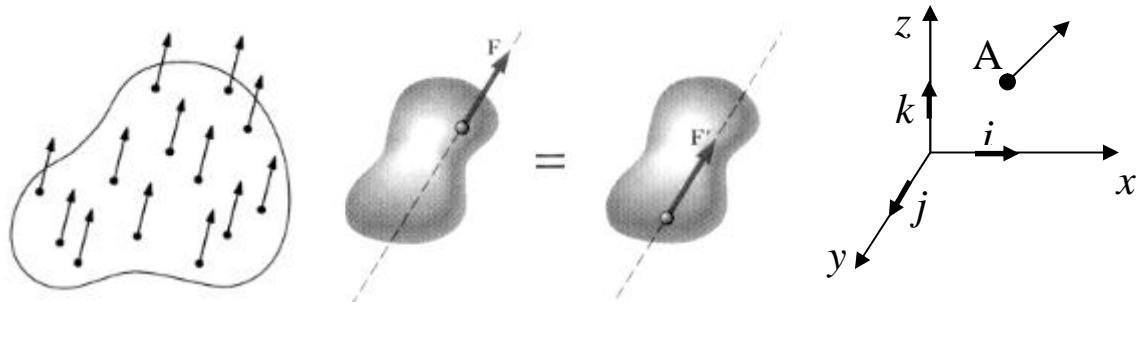


Fig.1.9 Components and unit vectors of Cartesian coordinates

Vectors representing on physical quantities can be classified as free vectors, sliding vectors or fixed vectors.

### Free vector:

The vectors are not constrained about the coordinate space. For example, the movement of a body action is without rotation.



a) Free vector

b) Sliding Vector

c) Fixed vector

Fig.1.10 Types of vectors

### Sliding Vector:

The vectors are in unidirectional movement along the line of action of force. For example, the external force on a rigid body as a principle of transmissibility.

### Fixed vector:

The vectors are constrained about the coordinate space. For example the line action of a force on deformable body is fixed.

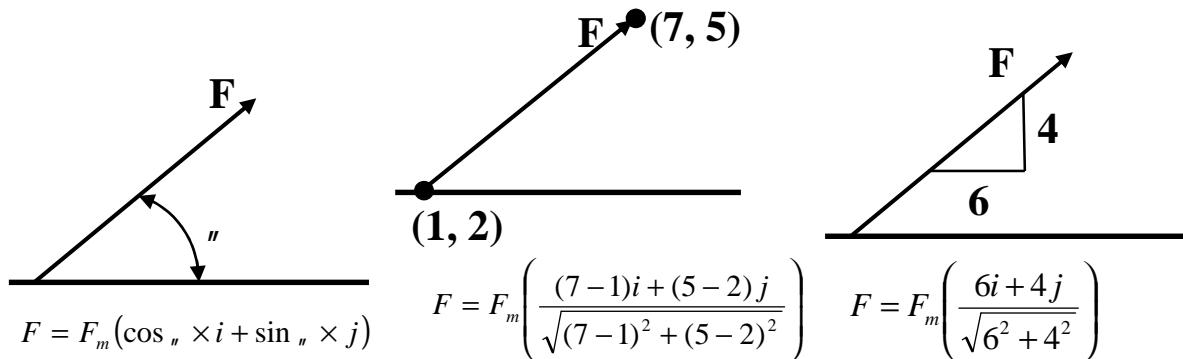


Fig.1.11 Geometry of vectors

## LAWS OF MECHANICS:

- Newton's first law
- Newton's second law
- Newton's third law
- Newton's law of gravitational
- Law of transmissibility of forces and
- Parallelogram law of forces

### Newton's first law:

A particle remains at rest or continues to move with uniform velocity by the state of equilibrium forces.

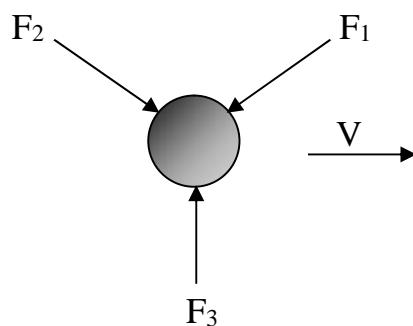


Fig. 1.12 Force systems in a particle

### Newton's second law:

The acceleration of a particle is proportional to the vector sum of forces acting on it and is in the direction of this vector sum.

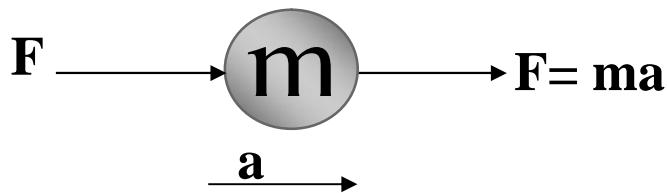


Fig. 1.13 Force systems in a body

**Newton's third law:**

For every action of force, there is an equal in magnitude, opposite in direction and collinear of reaction.

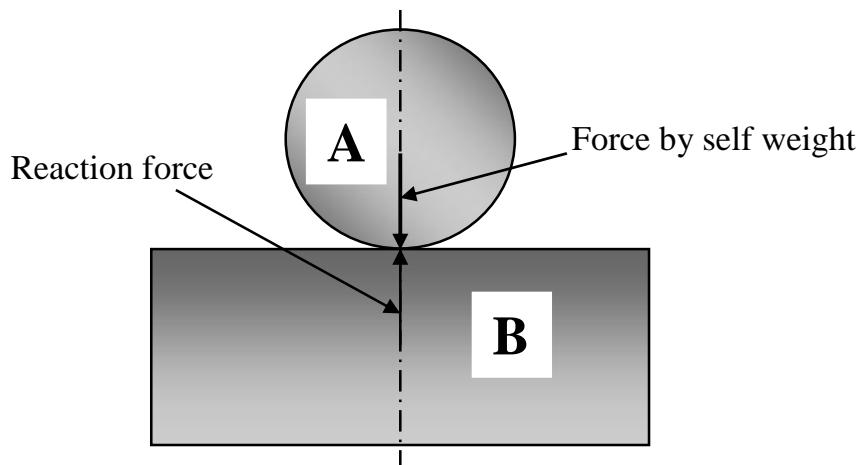


Fig. 1.14 Force systems in a two bodies

**Newton's law of gravitational force:**

Every body attracts the other body. The forces of attraction between two bodies are directly proportional to their masses and inversely proportional to the square of the distance between the two bodies. The law of gravitation is expressed by the equation

$$F \propto \frac{m_1 m_2}{d^2},$$

$$\Rightarrow F = G \frac{m_1 m_2}{d^2}$$

Where, F – Mutual force of attraction between two bodies,

G – Proportionality universal constant of gravitation,

$m_1$  &  $m_2$  – Masses of two bodies

d – Distance between two bodies

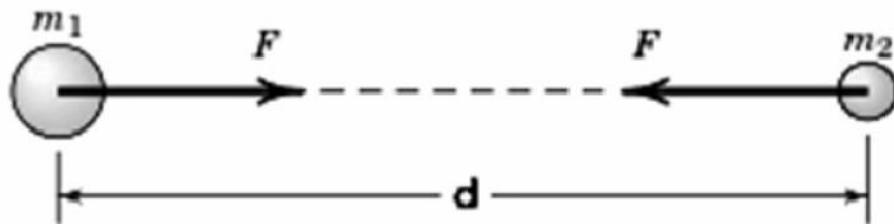


Fig.1.15 Force of attraction by gravitation

### Law of transmissibility of forces:

The force 'F' acting on the rigid plate at A or at B or at any other point on its line of action is not change by the net external effects of 'F' on the bracket. The external effects are the force exerted on the plate by the bearing support at O and the force exerted on the plate by the roller support at C.

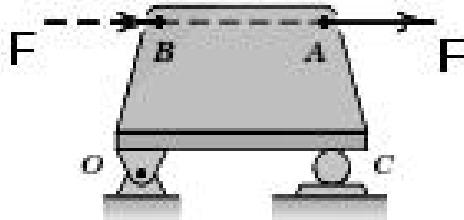


Fig.1.16 Force by transmissibility law

### Particle equilibrium in 2D & 3D:

When the resultant of all the forces acting on a particle in 2D space is zero then the particle is in equilibrium in 2D space. Therefore the equations of equilibrium are

$$\sum F_x = 0 \text{ & } \sum F_y = 0$$

$$\sum F = \sum R = 0$$

To maintain equilibrium it is necessary to satisfy Newton's first law of motion which requires the resultant force acting on a particle to be equal to zero.

If the resultant of all the forces acting on a particle in 3D space is zero then the particle is in equilibrium in 3D space. Therefore the equations of equilibrium are

$$\sum F_x = 0, \sum F_y = 0 \text{ & }$$

$$\sum F = \sum R = 0$$

### **Rigid body in equilibrium:**

A rigid body is in equilibrium when it is not undergoing a change in rotational or translational motion. This equilibrium requires that two conditions must be met. The first condition is related to the translational motion. The vector sum of the forces on the body must be zero.

$$\sum \vec{F} = 0$$

The second condition is related to the rotational motion. When the forces do not act through a common point or pivot, they may cause the body to rotate, even though the vector sum of the forces may be zero. This requires introducing the idea of moment (torque) due to a force. A net moment (torque) will cause a body initially at rest to undergo rotation. Therefore, the sum of all the moment (torque) must be zero.

$$\therefore \sum M = 0$$

### **SYSTEM OF FORCES:**

A system of forces consist number of forces act on a body simultaneously. If these forces do not lie in a single plane then it is said the system of forces in space (3D). Whereas these forces lie in a single plane (2D) then it is a coplanar force system.

The line of action of forces pass through the single point is known as concurrent force of system. The system of forces present in a line of action (1D) is termed as collinear force of system. The classifications of system of force as

- 1) One dimension (1D) or Forces lie in line
  - a) Collinear
- 2) Two dimension (2D) or Forces lie in plane
  - a) Coplanar parallel forces
  - b) Coplanar like parallel forces
  - c) Coplanar concurrent forces
  - d) Coplanar non concurrent forces
- 3) Three dimension (3D) or Forces lie in space
  - a) Non coplanar parallel forces
  - b) Non coplanar like parallel forces
  - c) Non coplanar concurrent forces
  - d) Non coplanar non concurrent forces

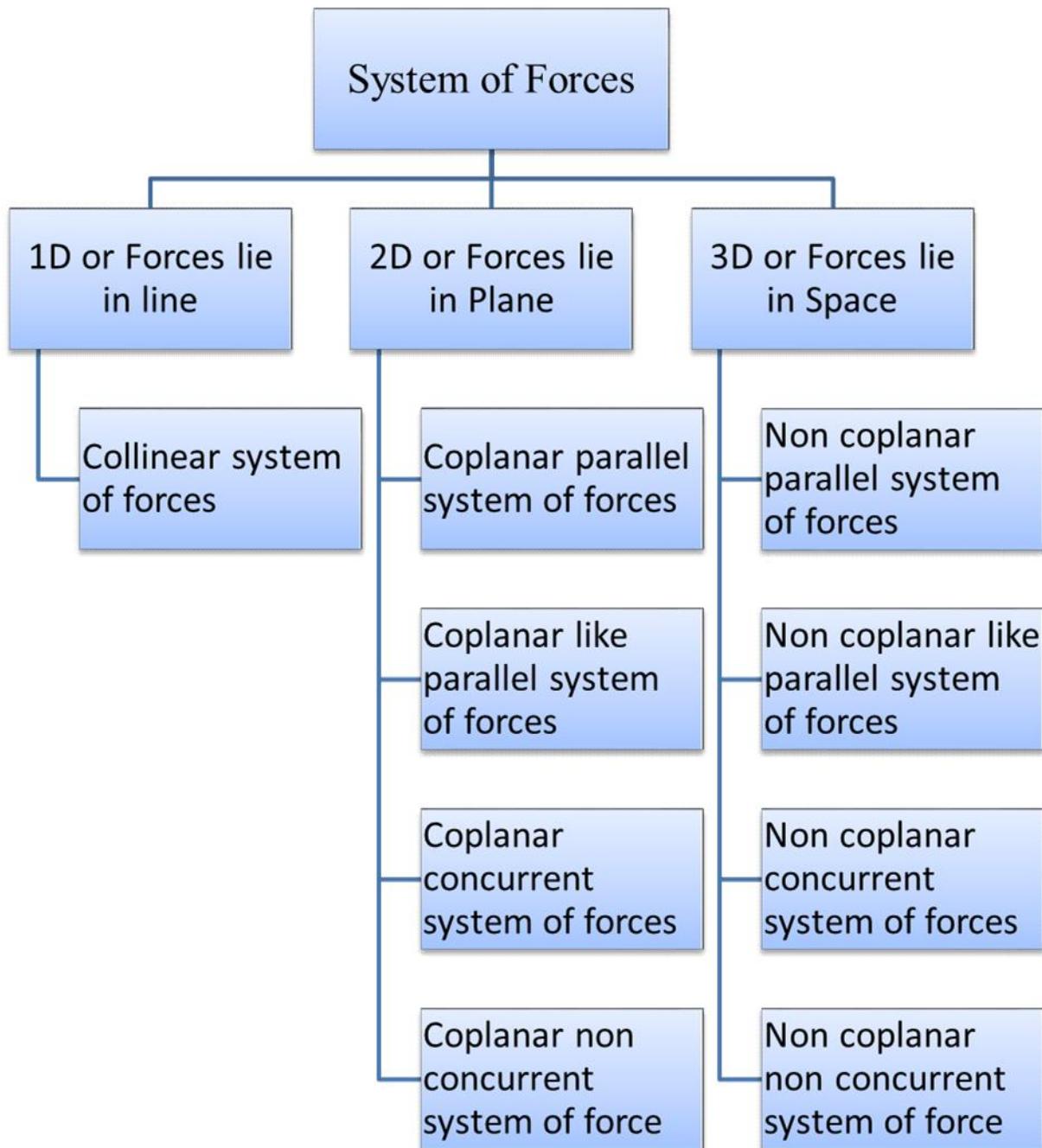


Fig.1.17 Classification of system of forces



Fig.1.18 Collinear system of forces

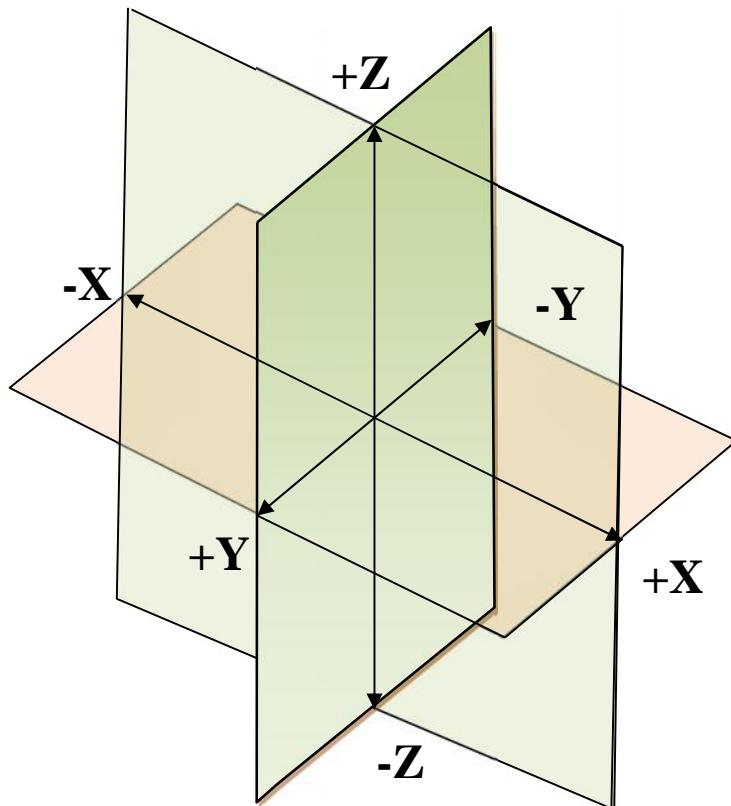
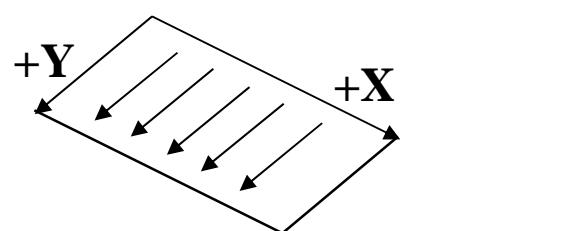
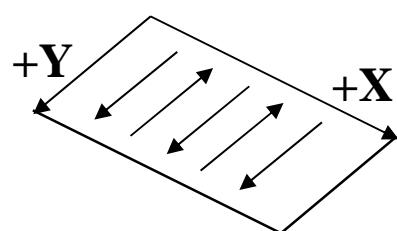


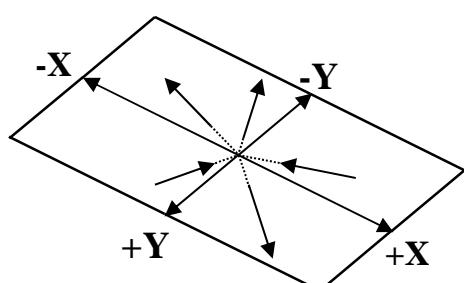
Fig.1.19 Planes of references in space coordinate



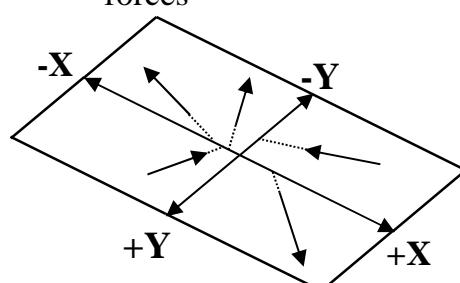
a) Coplanar parallel system of forces



b) Coplanar like parallel system of forces



c) Coplanar concurrent system of forces



d) Coplanar non concurrent system of forces

Fig.1.20 Forces lie in a plane system

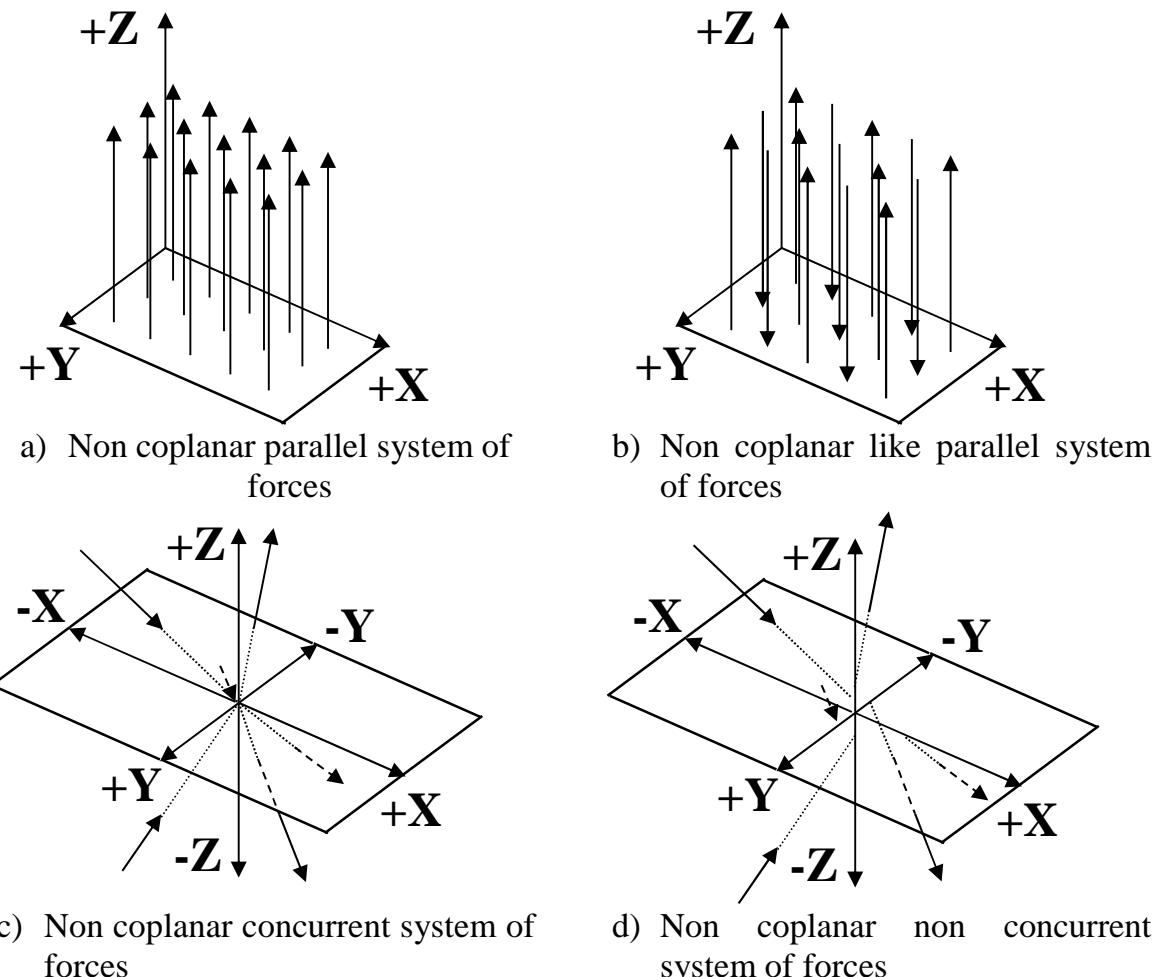


Fig.1.21 Forces lie in a space system

### Vector components of forces:

In space, a force can be resolved into mutually perpendicular components whose vector sum about the coordinates is equal to the given force. The components are parallel to the axes x, y and z with the unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . The rectangular components are considered to resolve the problem related to the system of forces. It is categorized as

#### 1. Two dimension rectangular components:

The most common two dimensional resolution of a force vector is into rectangular components. It follows from the parallelogram rule that the vector  $\mathbf{F}$  as shown in fig. is written as

$$\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j}$$

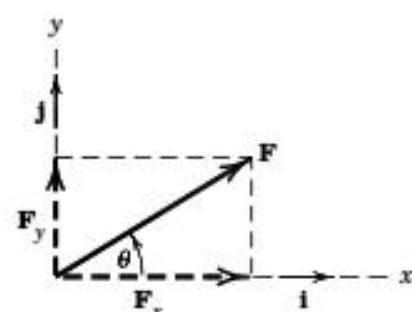


Fig.1.22 Rectangular components in two dimensions

Where  $F_x$  and  $F_y$  are vector components of  $F$  in the  $x$  and  $y$  direction. Each of the two vector components may be written as a scalar times the appropriate unit vector. In terms of the unit vectors  $i$  and  $j$ , the  $x$  component of force is  $\vec{F}_x = F_x \cdot i$  and the  $y$  component of force is  $\vec{F}_y = F_y \cdot j$  and thus the force,

Where the scalars  $F_x$  and  $F_y$  are the  $x$  and  $y$  scalar components of the vector force,  $F$ .

The scalar components can be positive or negative depending on the quadrant into which  $F$  points. For the force vector as shown in fig. The  $x$  and  $y$  scalar components are both positive and are related to the magnitude and direction of  $F$  by

$$F_x = F \cos \theta \quad \text{and} \quad F_y = F \sin \theta$$

$$\text{Therefore, } F = \sqrt{F_x^2 + F_y^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{F_y}{F_x}$$

## 2. Three dimension rectangular components:

Many problems in mechanics require analysis in three dimensions and for such problems it is often necessary to resolve a force into its three mutually perpendicular components. The force  $F$  acting at point  $O$  as shown in fig.1.22 Has the rectangular components  $F_x$ ,  $F_y$  and  $F_z$  where

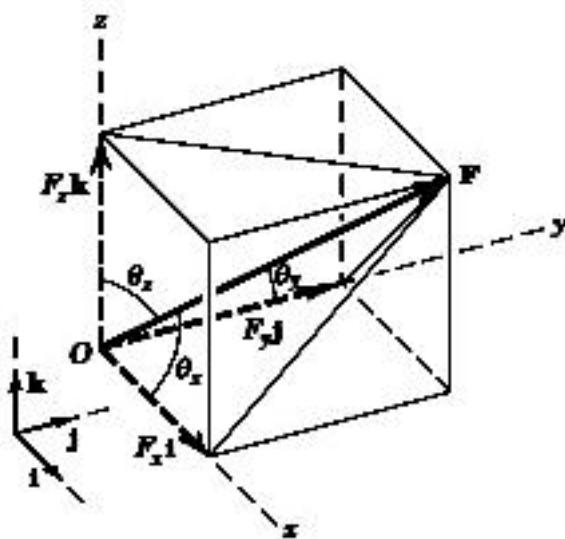


Fig.1.23 Rectangular components in three dimensions

$$F_x = F \cos \theta_x, \quad F_y = F \cos \theta_y \quad \text{and} \quad F_z = F \cos \theta_z$$

$$\text{Therefore, } F = \sqrt{F_x^2 + F_y^2 + F_z^2},$$

$$F = F_x i + F_y j + F_z k \text{ and}$$

$$F = F_m (\cos \alpha_x i + \cos \alpha_y j + \cos \alpha_z k)$$

The unit vectors  $i$ ,  $j$  and  $k$  are in the  $x$ ,  $y$  and  $z$  directions respectively. Using the direction cosines of  $F$  which are  $l = \cos \alpha_x$ ,  $m = \cos \alpha_y$  and  $n = \cos \alpha_z$ , where  $l^2 + m^2 + n^2 = 1$ , then the force as

$$F = F_m (l.i + m.j + n.k)$$

In the above equation, the right hand side of the force magnitude  $F_m$  times a unit vector  $n_F$  which characterizes the direction of  $F$ ,

$$F = F_m n_F$$

It is clear from the above equations that  $n_F = l.i + m.j + n.k$ , which shows that the scalar components of the unit vectors  $n_F$  are the direction cosines of the line of action of force,  $F$ .

In solving three dimensional problems one must usually find the  $x$ ,  $y$  and  $z$  scalar components of a force. In most cases the direction of a force is described by

- Two points on the line of the force
- Two angles which orient the line of action

#### a) The direction of a force by two point's method:

If the coordinates of points  $A$  and  $B$  as shown in fig.1.23 are known then the force  $F$  is written as

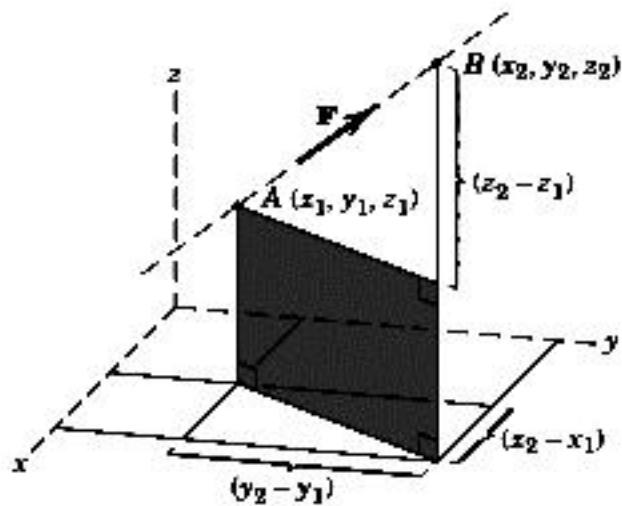


Fig.1.24 Cartesian coordinate system in a space

$$F = F_m n_F = F_m \frac{\overrightarrow{AB}}{AB} = F_m \frac{(x_2 - x_1).i + (y_2 - y_1).j + (z_2 - z_1).k}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

Thus the x, y and z scalar components of F are the scalar coefficients of the unit vectors i, j and k respectively.

**b) The direction of a force by two angles method:**

Consider the geometry as shown in fig.1.24 to resolve force F into horizontal and vertical components when angles  $\alpha$  and  $\omega$  are known.

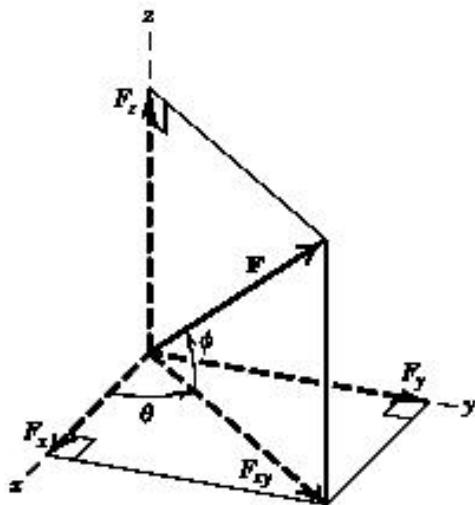


Fig.1.25 Spherical coordinate system in a space

$$F_{xy} = F \cos \omega \text{ and } F_z = F \sin \omega ,$$

Now resolve the horizontal component  $F_{xy}$  into x and y components.

$$F_x = F_{xy} \cos \alpha = F \cos \omega \cos \alpha \text{ and } F_y = F_{xy} \sin \alpha = F \cos \omega \sin \alpha$$

**Angles between two vectors:**

If the angle between the force F and the direction specified by the unit vector n is  $\theta$  then from the dot product definition, the  $F_m \cdot n_F = F n \cos \theta = F \cos \theta$ , where  $|n| = n = 1$ . Thus the angle between F and n is given by

$$\theta = \cos^{-1} \frac{F_m n_F}{F_m}$$

In general the angle between any two vectors P and Q is

$$\theta = \cos^{-1} \frac{P \cdot Q}{PQ}$$

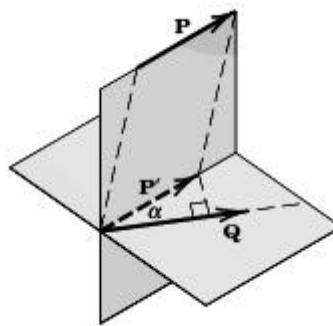


Fig.1.26 Vector angles in planes

If a force  $F$  is perpendicular to a line whose direction is specified by the unit vector  $n$ , then  $\cos \theta = 0$  and  $F_m \cdot n_F = 0$ .

Note: This relationship does not mean that either  $F_m$  or  $n_F$  is zero as would be the case with scalar multiplication where  $(A)(B) = 0$  requires that either  $A$  or  $B$  (or both) be zero.

The dot-product relationship applies to non intersecting vectors as well as to intersecting vectors. Thus the dot product of the non intersecting vectors  $P$  and  $Q$  as shown in fig. is  $Q$  times the projection of  $P'$  on  $Q$ , or  $P'Q\cos\alpha = PQ\cos\alpha$  because  $P'$  and  $P$  are the same when treated as free vectors.

### Problems:

**SP1.1:** The guy wire of the electric pole as shown in fig.1.26 makes  $60^\circ$  to the horizontal and is subjected to 20 kN force. Find horizontal and vertical components of the force.

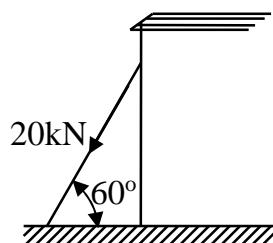


Fig.1.27 A guy wire of electric pole

### Solution:

From the figure  $F$  is 20kN and  $\theta$  is  $60^\circ$ , by the components of force method the scalar components of force  $F_m$  is 20kN.

$$F = F_m \cdot (\cos \theta \cdot i + \sin \theta \cdot j)$$

$$F = F_m \cdot \cos \theta \cdot i + F_m \cdot \sin \theta \cdot j$$

$$\therefore F_x = F_m \cdot \cos \theta \quad \text{and} \quad F_y = F_m \cdot \sin \theta$$

The horizontal component of force is,

$$F_x = 20 \cdot \cos 60^\circ = 10 \text{kN}$$

The vertical component of force is,

$$F_y = 20 \cdot \sin 60^\circ = 17.32 \text{kN}$$

**SP1.2:** A block weighing  $W=10\text{kN}$  is resting on an inclined plane as shown in fig.1.27. Determine its components normal to and parallel to the inclined plane.

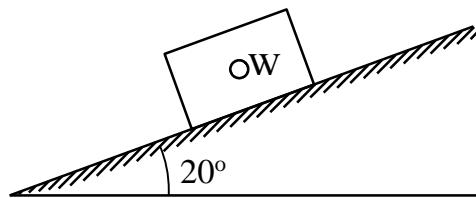
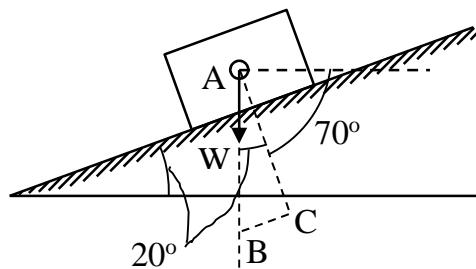


Fig.1.28 A block resting in an inclined plane

**Solution:**

The given parameter as  $W = 10\text{kN}$

From the given figure develop a space diagram to identify the forces in a block.



The plane makes an angle of  $20^\circ$  with the horizontal. Hence the normal to the plane makes an angle of  $70^\circ$  to the horizontal. That is  $20^\circ$  to the vertical. If AB represents the given force W to some scale AC represents the normal component and

CB represents component parallel to the plane. From triangle ABC, the normal component is

$$AC = W \times \cos 20^\circ = 10 \times \cos 20^\circ = 9.4 \text{ kN}$$

The parallel component is

$$BC = W \times \sin 20^\circ = 10 \times \sin 20^\circ = 3.42 \text{ kN}$$

**SP1.3:** Determine the scalar component of forces for the given parameters as shown in fig.1.28 in different ways.

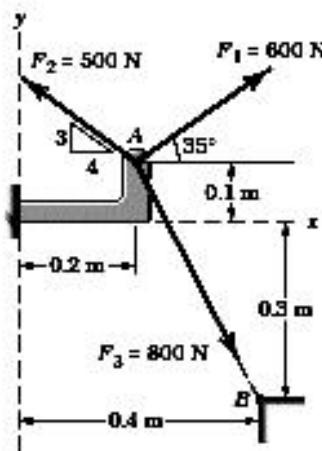
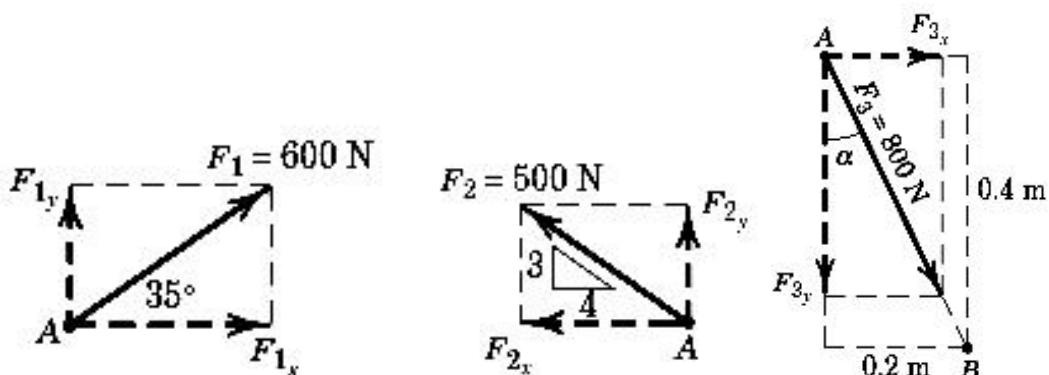


Fig. 1.29 Force systems in a bracket

**Solution:**



There are three forces acted on the bracket at point A. The force  $F_1$  has linear and angular units. Therefore, the scalar components of  $F_1$  are

$$F_1 = F_{1m} \cdot (\cos \theta \cdot i + \sin \theta \cdot j)$$

$$F_1 = F_{1m} \cdot \cos \theta \cdot i + F_{1m} \cdot \sin \theta \cdot j$$

$$\therefore F_{1x} = F_{1m} \cdot \cos \theta \quad \text{and} \quad F_{1y} = F_{1m} \cdot \sin \theta$$

From the figure,  $F_{1m}$  is 600 N and  $\theta$  is  $35^\circ$

$$F_{1x} = 600 \cdot \cos 35^\circ = 491 N$$

$$F_{1y} = 600 \cdot \sin 35^\circ = 344 N$$

$$F_1 = 491 \cdot i + 344 \cdot j$$

The scalar component of  $F_2$  as for the Dimension about x axis is 4 units and y axis is 3 units. The direction of x axis is opposite then the sign of dimension is negative.

$$F_2 = F_{2m} \left( \frac{-xi + yj}{\sqrt{(-x)^2 + y^2}} \right)$$

$$F_2 = F_{2m} \left( \frac{-4i + 3j}{\sqrt{(-4)^2 + 3^2}} \right)$$

$$F_2 = F_{2m} \left( \frac{-4i + 3j}{\sqrt{16+9}} \right) = F_{2m} \left( \frac{-4i + 3j}{\sqrt{25}} \right)$$

$$F_2 = F_{2m} \left( \frac{-4i}{5} + \frac{3j}{5} \right)$$

$$F_2 = \left( F_{2m} \cdot \frac{-4}{5} i + F_{2m} \cdot \frac{3}{5} j \right)$$

From the figure,  $F_{2m}$  is 500N

$$F_{2x} = F_{2m} \cdot \frac{-4}{5} = 500 \cdot \frac{-4}{5} = -400 N$$

$$F_{2y} = F_{2m} \cdot \frac{3}{5} = 500 \cdot \frac{3}{5} = 300N$$

$$F_2 = -400.i + 300.j$$

The scalar components of  $F_3$  as for the given dimension of two end points detail which is at point A is  $(x_A, y_A) = (0,0)$  and point B is  $(x_B, -y_B) = (0.2, -0.4)$ . The negative sign of  $y_B$  indicates the direction is opposite.

$$F_3 = F_{3m} \left( \frac{(x_B - x_A)i + (y_B - y_A)j}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}} \right)$$

$$F_3 = F_{3m} \left( \frac{(0.2 - 0)i + (-0.4 - 0)j}{\sqrt{(0.2 - 0)^2 + (-0.4 - 0)^2}} \right)$$

$$F_3 = F_{3m} \left( \frac{0.2i - 0.4j}{\sqrt{(0.2)^2 + (-0.4)^2}} \right)$$

$$F_3 = F_{3m} \left( \frac{0.2i - 0.4j}{\sqrt{(0.04) + (0.16)}} \right)$$

$$F_3 = F_{3m} \left( \frac{0.2i - 0.4j}{\sqrt{0.2}} \right) = F_{3m} \left( \frac{0.2i - 0.4j}{0.45} \right)$$

$$F_3 = F_{3m} \cdot \frac{0.2}{0.45}i + F_{3m} \cdot \frac{-0.4}{0.45}j$$

From figure  $F_3$  is 800N, then

$$F_{3x} = F_{3m} \cdot \frac{0.2}{0.45} = 800 \cdot \frac{0.2}{0.45} = 355.56N$$

$$F_{3y} = F_{3m} \cdot \frac{-0.4}{0.45} = 800 \cdot \frac{-0.4}{0.45} = -711.11N$$

$$F_3 = 355.56.i - 711.11.j$$

**SP1.4:** A machine component 1.5 m long and weight 1000 N is supported by two ropes AB and CD as shown in Fig.1.29 given below. Calculate the tensions  $T_1$  and  $T_2$  in the ropes AB and CD.

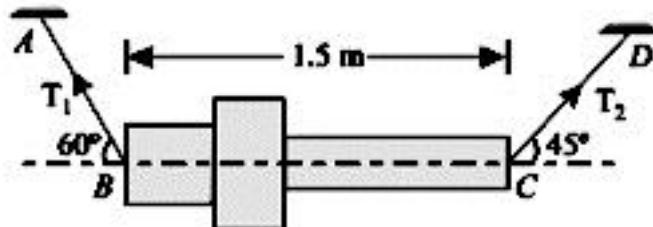
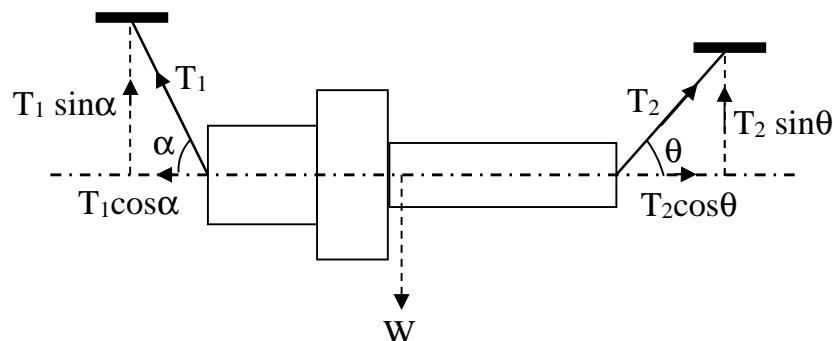


Fig. 1.30 Machine component

**Solution:**

The given data apart from dimensions of body and position is weight of a body as  $W = 1000\text{N}$ .



The sum of horizontal components of forces is equal to zero,

$\sum F_x = -T_1 \cos \alpha + T_2 \cos \theta = 0$ , the negative sign indicates the direction of component force. Therefore,

$$-T_1 \cos \alpha + T_2 \cos \theta = 0$$

$$-T_1 \cos 60^\circ + T_2 \cos 45^\circ = 0$$

$$T_1 = \frac{\cos 45^\circ}{\cos 60^\circ} T_2 = 1.414 \times T_2$$

The sum of Vertical components of forces is equal to zero,

$\sum F_y = T_1 \sin 15^\circ + T_2 \sin 45^\circ - W = 0$ , the negative sign indicates the direction of load acting. Therefore,

$$T_1 \sin 15^\circ + T_2 \sin 45^\circ - W = 0$$

$$T_1 \sin 60^\circ + T_2 \sin 45^\circ - 1000 = 0$$

Now, substitute the value of  $T_1$ , then

$$(1.414 \times T_2) \sin 60^\circ + T_2 \sin 45^\circ - 1000 = 0$$

$$1.93T_2 = 1000$$

$$T_2 = 518.1 \text{ N}$$

$$T_1 = 1.414 \times T_2 = 1.414 \times 518.1 = 732.6 \text{ N}$$

**SP1.5:** A force  $F$  with a magnitude of 100N is applied at the origin O of the axes x-y-z as shown in fig.1.30. The line of action of  $F$  passes through a point A whose coordinates are 3m, 4m and 5m. Determine i) the scalar components of  $F$ , ii) the projection  $F_{xy}$  of  $F$  on the x-y plane and iii) the projection  $F_{OB}$  of  $F$  along the line OB.

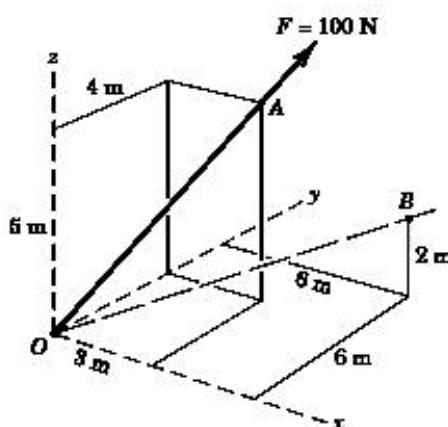
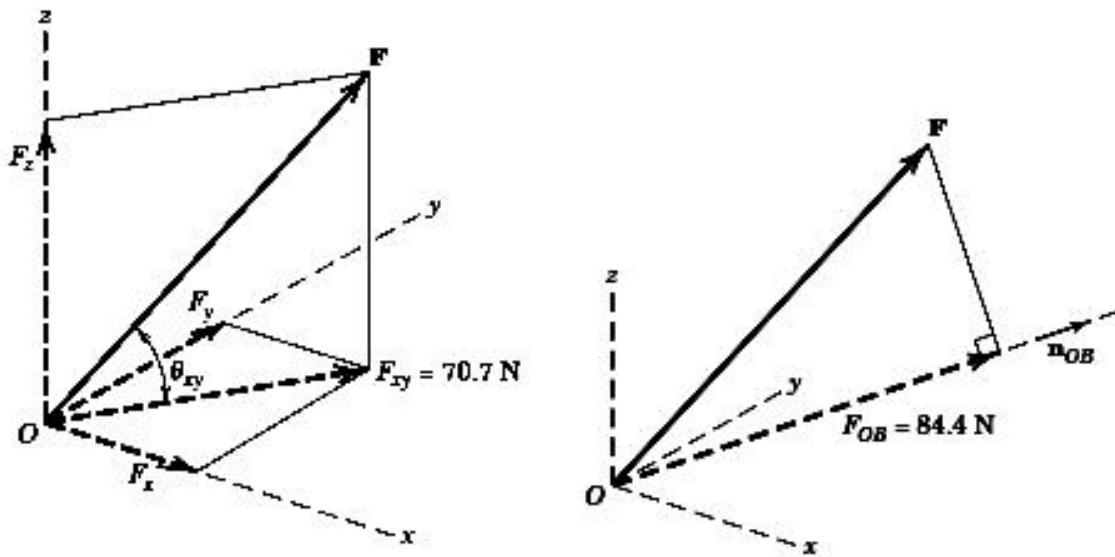


Fig.1.31 Force system in a space coordinates

**Solution:**



i) Scalar components of  $\mathbf{F}$  about the line OA,  $F_m=100\text{N}$ ,  $x=3\text{m}$ ,  $y=4\text{m}$  and  $z=5\text{m}$ .

$$F = F_m \left( \frac{x.i + y.j + z.k}{\sqrt{x^2 + y^2 + z^2}} \right) = 100 \left( \frac{3i + 4j + 5k}{\sqrt{3^2 + 4^2 + 5^2}} \right)$$

$$F = 100 \left( \frac{3i + 4j + 5k}{\sqrt{3^2 + 4^2 + 5^2}} \right) = 100 \left( \frac{3i + 4j + 5k}{\sqrt{9+16+25}} \right)$$

$$F = 100 \left( \frac{3i + 4j + 5k}{\sqrt{50}} \right) = 100 \left( \frac{3i + 4j + 5k}{7.07} \right)$$

$$F = 100 \left( \frac{3i}{7.07} + \frac{4j}{7.07} + \frac{5k}{7.07} \right) = 100(0.424i + 0.566j + 0.707k)$$

$$F = 42.4i + 56.6j + 70.7k$$

$$F_x = 42.4\text{N}$$

$$F_y = 56.6\text{N}$$

$$F_z = 70.7\text{N}$$

ii) The projection  $F_{xy}$  of  $\mathbf{F}$  on the x-y plane.

The cosine of angle between  $\mathbf{F}$  and  $xy$  plane is

$$\cos \theta_{xy} = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\cos \theta_{xy} = \frac{\sqrt{3^2 + 4^2}}{\sqrt{3^2 + 4^2 + 5^2}} = 0.707$$

Therefore,  $F_{xy} = F \cos \theta_{xy} = 100 \times 0.707 = 70.7N$

iii) The projection  $F_{OB}$  of  $\mathbf{F}$  along the line  $OB$ . The dimension of  $OB$  from origin of  $x$  is 6m,  $y$  is 6m and  $z$  is 2m.

$$\begin{aligned} F_{OB} &= F \left( \frac{x.i + y.j + z.k}{\sqrt{x^2 + y^2 + z^2}} \right) \\ &= (42.2i + 56.6j + 70.7k) \left( \frac{6i + 6j + 2k}{\sqrt{6^2 + 6^2 + 2^2}} \right) \\ F_{OB} &= (42.2i + 56.6j + 70.7k)(0.688i + 0.688j + 0.229k) \end{aligned}$$

$$F_{OB} = (42.4)(0.688) + (56.6)(0.688) + (70.7)(0.229) = 84.4N$$

If the projection is represented as vector then,

$$\begin{aligned} \overrightarrow{F_{OB}} &= F_{OB} \left( \frac{x.i + y.j + z.k}{\sqrt{x^2 + y^2 + z^2}} \right) \\ \overrightarrow{F_{OB}} &= 84.4 \left( \frac{6i + 6j + 2k}{\sqrt{6^2 + 6^2 + 2^2}} \right) = 84.4(0.688i + 0.688j + 0.229k) \\ \overrightarrow{F_{OB}} &= 58.1i + 58.1j + 19.35k \end{aligned}$$

## RESULTANT FORCE:

The resultant force,  $R$  is a force which produces the same effect of number of forces  $F_1$ ,  $F_2$ ,  $F_3$  and  $F_4$  are acting on particle as shown in fig.1.31. The different methods used to determine the resultant force of a number of given forces are as follows

- 1) Graphical Method.
- 2) Analytical Method
  - a) Geometrical resolution method
  - b) Algebraic sum of resolution method

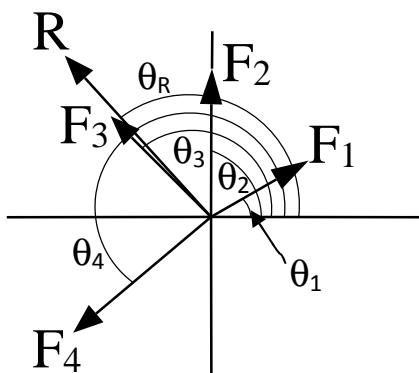


Fig.1.32 Force systems in a particle

### 1) Graphical method for the resultant force:

This method also named as polygon law of forces to find the resultant force in magnitude and direction. The graphical or vector methods started with the space drawing which shows the position of force vector acting on a particle. The graphical method of vector force is continued by addition of force one by one to the direction and magnitude of the suitable scale. The resultant force of all forces is obtained by the line joining of start point of first vector force and end point of last vector force is a magnitude value of resultant in that suitable scale. The angle of direction is measured from first vector to the position of resultant force.

Parallelogram law and triangle law of vectors can also be used to find the resultant force graphically. This method gives a clear picture of the work

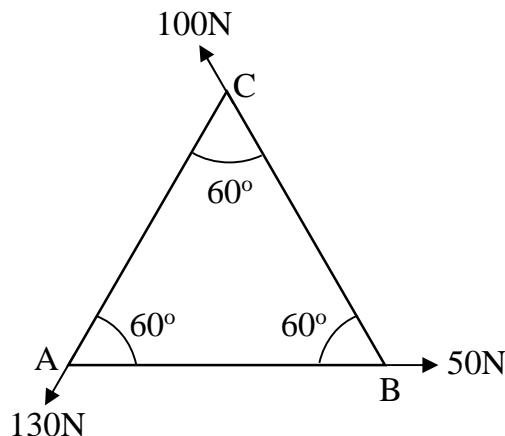
being carried out. However the main disadvantage is that it needs drawing aids like pencil, scale, drawing sheets. Hence there is need for analytical method.

### Problems:

**SP1.6:** A particle is acted upon by three forces equal to 50N, 100N and 130N along the three sides of an equilateral triangle taken in order. Find graphically the magnitude and direction of the resultant force.

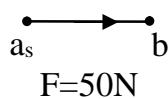
### Solution by Graphical method:

**Step 1:** Draw the space/ position diagram as per given data and denote lettering for the corners of vector force by A, B and C.

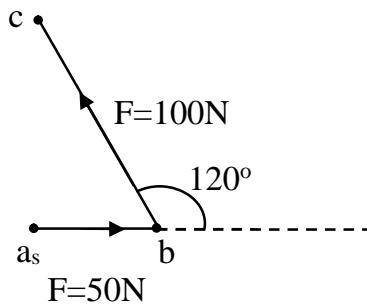


**Step 2:** Now start to draw the vector diagram for the above drawn space diagram with suitable scale. The scale is selected for the force but the angular dimension not to be changed. 100 N magnitude forces is taken as 10cm of linear scale. Only the length of the scale in cm is used to measure and marking but the denoted as original unit of N (Newton).

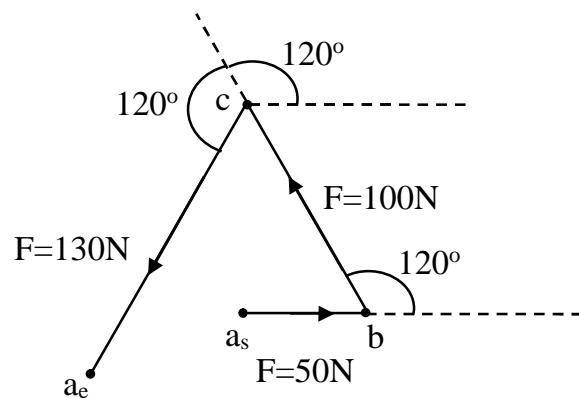
In this step draw a line from point  $a_s$  as a starting point of first force at length 5cm for the 50N ended with point b along the direction as position in space diagram.



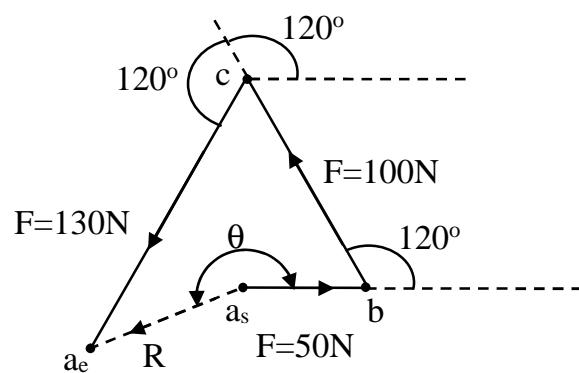
**Step 3:** The second force 100N is measured as 10cm and draw a line from b at 10cm length in the direction specified by the space diagram say  $120^\circ$  from the first vector force which is at zero degree. Now mark letter c at the end point of second vector force.



**Step 4:** The third force 130 N measured as 13cm and draw a line from point c at 13cm length in the direction specified by the space diagram as  $120^\circ$  from the second vector force which at  $120^\circ$  from horizontal. Now mark letter  $a_e$  at the end point of third vector force because this vector force ended with starting point of first vector force.



**Step 5:** The vector force by the space diagram is completed and finally joins the line between starting point of first vector force  $a_s$  and ending point of third vector force  $a_e$  is known as resultant force of given vector forces by the method graphical.



**Step 6:** Now, measure the length between point  $a_s$  and  $a_e$  in cm as 7cm and converted into the magnitude force scale as **Resultant force  $R$  , 70N**. Using protractor, measure **the angle of direction of the resultant force  $\theta$  is  $200^\circ$** . From the direction of positive x that is horizontal direction.

## 2) Analytical method for the resultant force:

### a. Geometrical resolution method for the resultant force

This method is named as parallelogram law of forces and it states that two forces acting on a particle simultaneously by in magnitude and direction which forms like a parallelogram to give the resultant force as a diagonal of parallelogram. The mathematical expression of resultant force in magnitude and direction by the parallelogram law of force as

$$R = \sqrt{F_1^2 + 2F_1F_2 \cos \alpha + F_2^2} \text{ and}$$

$$\alpha = \tan^{-1} \frac{F_2 \sin \alpha}{F_1 + F_2 \cos \alpha}$$

#### Proof:

Consider two forces  $F_1$  and  $F_2$  acting on a particle as shown in Fig.1.32 (a). Let the angle between the two forces be  $\alpha$ . If parallelogram formed as  $ABCD$  as shown in Fig.1.32 (b) with  $AB$  representing,  $F_1$  and  $AD$  representing  $F_2$  to some scale, according to parallelogram law of forces  $AC$  represents the resultant force,  $R$ . Draw a perpendicular line  $CE$  to the line  $AB$ .

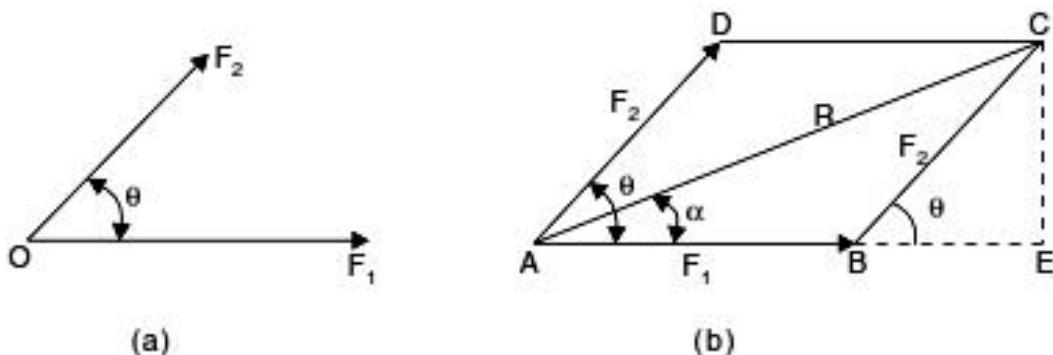


Fig.1.33 Parallelogram law of force

The resultant force,  $R$  of two forces  $F_1$  and  $F_2$  is given by,

$$R = AC = \sqrt{AE^2 + CE^2} = \sqrt{(AB + BE)^2 + CE^2}$$

But  $AB = F_1$ ,  $BE = BC \times \cos \alpha = F_2 \cos \alpha$ ,  $CE = BC \sin \alpha = F_2 \sin \alpha$

$$R = \sqrt{(F_1 + F_2 \cos \alpha)^2 + (F_2 \sin \alpha)^2} = \sqrt{F_1^2 + 2F_1F_2 \cos \alpha + F_2^2 \cos^2 \alpha + F_2^2 \sin^2 \alpha}$$

$$R = \sqrt{F_1^2 + 2F_1F_2 \cos \alpha + F_2^2} \quad [ \because \sin^2 \alpha + \cos^2 \alpha = 1 ]$$

The inclination of resultant force to the direction of  $F_1$  is given by  $\gamma$ , where

$$\tan \gamma = \frac{CE}{AE} = \frac{CE}{AB + BE} = \frac{F_2 \sin \alpha}{F_1 + F_2 \cos \alpha}$$

$$\gamma = \tan^{-1} \frac{F_2 \sin \alpha}{F_1 + F_2 \cos \alpha}$$

Particular cases:

1. When  $\alpha = 0^\circ$ ,  $R = \sqrt{F_1^2 + 2F_1F_2 + F_2^2} = F_1 + F_2$

2. When  $\alpha = 90^\circ$ ,  $R = \sqrt{F_1^2 + F_2^2}$

3. When  $\alpha = 180^\circ$ ,  $R = \sqrt{F_1^2 - 2F_1F_2 + F_2^2} = F_1 - F_2$

4. When  $F_1 = F_2 = F$ , Then

$$R = \sqrt{2F^2(1 + \cos \alpha)}$$

$$R = \sqrt{2F^2 \times 2 \cos^2 \frac{\alpha}{2}} = 2F \cos \frac{\alpha}{2}$$

### b. Algebraic sum of force resolution for the resultant force

It states that the number of force resolutions in a given direction is equal to the resultant force resolution. The force resolution or resultant force resolution is a component of forces without changing its effect on the body. The components of forces are mutually perpendicular to each other in directions. The magnitude of resultant force,

$$R = \sqrt{(\sum F_x)^2 + (\sum F_y)^2}$$

$$R_x + R_y = \sqrt{(\sum F_x)^2} + \sqrt{(\sum F_y)^2}$$

$$\sum F_x = F_{1x} + F_{2x} + \dots + F_{nx}$$

$$\sum F_y = F_{1y} + F_{2y} + \dots + F_{ny}$$

The direction of resultant force,

$$\tan \alpha = \frac{\sum F_y}{\sum F_x}$$

$$\theta = \tan^{-1} \left( \frac{\sum F_y}{\sum F_x} \right)$$

Particular cases:

1. When,  $\theta$  is  $0^\circ$  and  $90^\circ$  or  $270^\circ$  and  $360^\circ$ ,  $\sum F_x$  is positive
2. When,  $\theta$  is  $0^\circ$  and  $180^\circ$ ,  $\sum F_y$  is positive
3. When,  $\theta$  is  $90^\circ$  and  $270^\circ$ ,  $\sum F_x$  is negative
4. When,  $\theta$  is  $180^\circ$  and  $360^\circ$ ,  $\sum F_y$  is negative

### Problems:

**SP1.7:** Combine the two forces P and T which act on the fixed structure at B into a single equivalent force R.

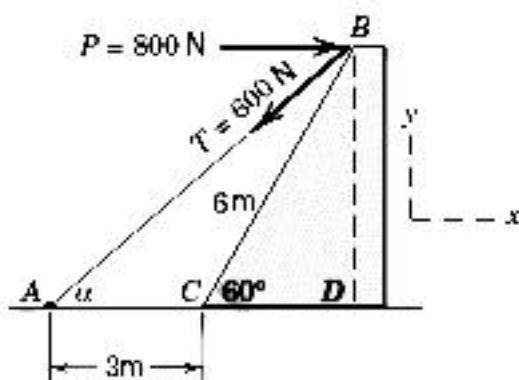
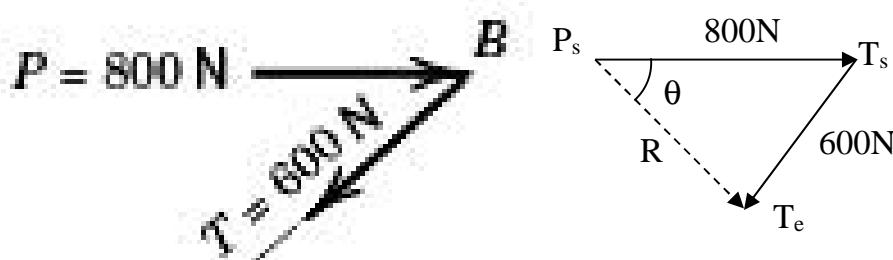


Fig.1.34 force system in a structure

### Graphical Solution:

The scale used here is 4cm = 800 N and 3cm = 600N would be more suitable for regular size paper and would give greater accuracy.

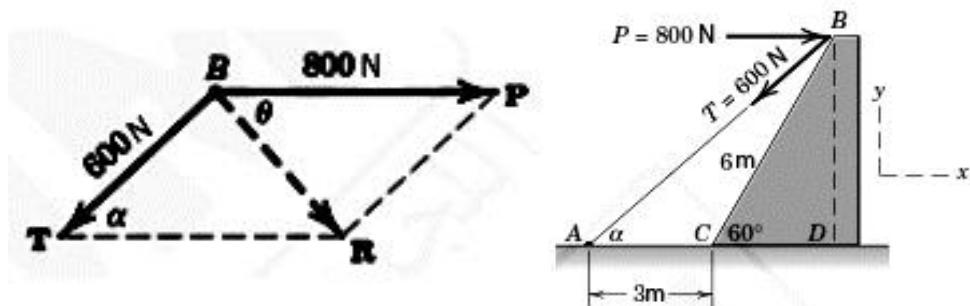


a) Space diagram

b) Vector diagram

Measurement of the length  $R$  is 2.6cm and direction  $\alpha$  of the resultant force  $R$  yields the approximate results as  $R = 520 \text{ N}$  and  $\alpha = 49^\circ$

## Geometric Solution:



From the triangle ABD, the angle  $r$  is calculated as shown in above diagram.

$$\tan r = \frac{BD}{AD} = \frac{BD}{AD + CD}$$

From triangle CBD,

$$BD = \sin 60^\circ \times 6 \text{ and } CD = \cos 60^\circ \times 6$$

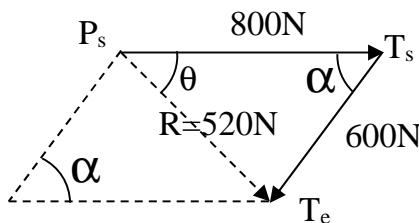
$$\text{Therefore, } \tan r = \frac{\sin 60^\circ \times 6}{3 + (\sin 60^\circ \times 6)} = 0.866$$

$$r = \tan^{-1}(0.866) = 40.9^\circ$$

The law of cosines or parallelogram law of force gives

$$R = \sqrt{P^2 + T^2 - 2PT \cos \gamma} = \sqrt{600^2 + 800^2 - 2 \times 600 \times 800 \times \cos 40.9^\circ} = 524N$$

From the law of sines, the angle „ which orients R. Thus,



$$\frac{600}{\sin 60^\circ} = \frac{524}{\sin 40.9^\circ}, \sin 60^\circ = 0.750 \text{ and } 40.9^\circ = 48.6^\circ$$

**Algebraic solution:**

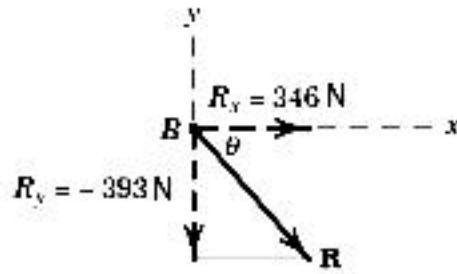
By using the x-y coordinate system on the given figure ,

$$R_x = \sum F_x = 800 - 600 \cos 40.9^\circ = 346N \text{ and } R_y = \sum F_y = -600 \sin 40.9^\circ = -393N$$

The magnitude and direction of the resultant force  $R$  as shown in figure 2.2c are then

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{346^2 + (-393)^2} = 524N \text{ and}$$

$$\theta = \tan^{-1} \frac{|R_y|}{|R_x|} = \tan^{-1} \frac{393}{346} = 48.6^\circ$$



**SP1.8:** A frame ABC is supported in part by cable DBE that passes through a frictionless ring at B. Knowing that the tension in the cable is 385 N, determine the components of the force exerted by the cable on the support at E.

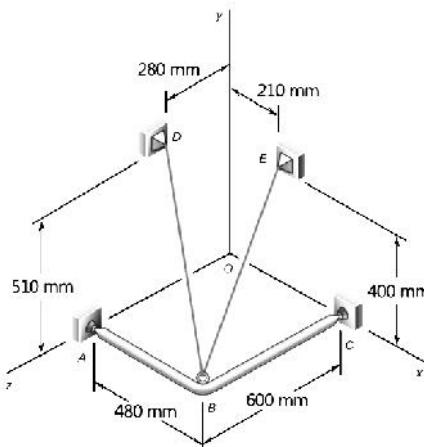


Fig.1.35 Frame supported by a cable

**Solution:**

The direction of a force by two point's method in cable from E to B, then the force  $F$  is written as

$$\vec{F} = F_m \vec{n}_F = F_m \frac{\overrightarrow{EB}}{EB} = F_m \frac{(B_x - E_x).i + (B_y - E_y).j + (B_z - E_z).k}{\sqrt{(B_x - E_x)^2 + (B_y - E_y)^2 + (B_z - E_z)^2}}$$

Where,  $F_m = 385$  N, the dimension from origin for point B is  $B_x = 480$  mm,  $B_y = 0$  mm and  $B_z = 600$  mm. At point E,  $E_x = 210$  mm,  $E_y = 400$  mm and  $E_z = 0$  mm.

Therefore,

$$\vec{F} = 385 \times \frac{(480 - 210).i + (0 - 400).j + (600 - 0).k}{\sqrt{(480 - 210)^2 + (0 - 400)^2 + (600 - 0)^2}}$$

$$\vec{F} = 385 \times \frac{(270).i + (-400).j + (600).k}{\sqrt{(270)^2 + (-400)^2 + (600)^2}}$$

$$\vec{F} = 385 \times \frac{(270).i + (-400).j + (600).k}{770}$$

$$\vec{F} = 135.i - 200.j + 300k$$

Hence,  $F_x = 135\text{ N}$ ,  $F_y = -200\text{ N}$  and  $F_z = 300\text{ N}$

**SP1.9:** A frame ABC is supported in part by cable DBE that passes through a frictionless ring at B. Knowing that the tension in the cable is 385 N, determine the magnitude and direction of the resultant of the forces exerted by the cable at B.

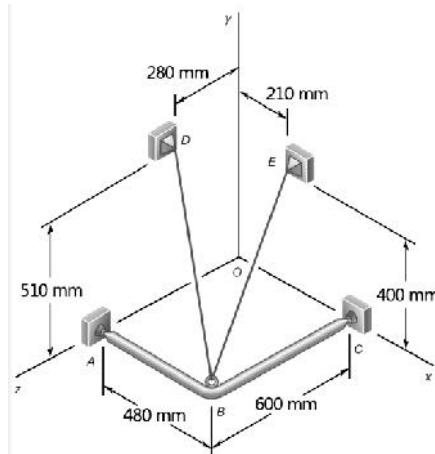


Fig.1.36 A frame support by a cable

**Solution:**

The direction of a force by two point's method in cable from B to E, then the force F is written as

$$\vec{F} = F_m \vec{n}_F = F_m \frac{\overrightarrow{BE}}{|BE|} = F_m \frac{(E_x - B_x).i + (E_y - B_y).j + (E_z - B_z).k}{\sqrt{(E_x - B_x)^2 + (E_y - B_y)^2 + (E_z - B_z)^2}}$$

Where,  $F_m = 385\text{ N}$ , the dimension from origin for point E,  $E_x = 210\text{ mm}$ ,  $E_y = 400\text{ mm}$  and  $E_z = 0\text{ mm}$ . At point B is  $B_x = 480\text{ mm}$ ,  $B_y = 0\text{ mm}$  and  $B_z = 600\text{ mm}$

Therefore,

$$\overrightarrow{F_{EB}} = 385 \times \frac{(210 - 480).i + (400 - 0).j + (0 - 600).k}{\sqrt{(210 - 480)^2 + (400 - 0)^2 + (0 - 600)^2}}$$

$$\overrightarrow{F_{EB}} = 385 \times \frac{(-270).i + (400).j + (-600).k}{\sqrt{(-270)^2 + (400)^2 + (-600)^2}}$$

$$\overrightarrow{F_{EB}} = 385 \times \frac{(-270).i + (400).j + (-600).k}{770}$$

$$\overrightarrow{F_{EB}} = -135.i + 200.j - 300k$$

The direction of a force by two point's method in cable from B to D, then the force F is written as

$$\overrightarrow{F} = F_m \overrightarrow{n_F} = F_m \frac{\overrightarrow{BD}}{BD} = F_m \frac{(D_x - B_x).i + (D_y - B_y).j + (D_z - B_z).k}{\sqrt{(D_x - B_x)^2 + (D_y - B_y)^2 + (D_z - B_z)^2}}$$

Where,  $F_m = 385$  N, the dimension from origin for point D,  $D_x = 0$ mm,  $D_y = 510$ mm and  $D_z = 280$ mm. At point B is  $B_x = 480$ mm,  $B_y = 0$ mm and  $B_z = 600$ mm.

Therefore,

$$\overrightarrow{F_{DB}} = 385 \times \frac{(0 - 480).i + (510 - 0).j + (280 - 600).k}{\sqrt{(0 - 480)^2 + (510 - 0)^2 + (280 - 600)^2}}$$

$$\overrightarrow{F_{DB}} = 385 \times \frac{(-480).i + (510).j + (-320).k}{\sqrt{(-480)^2 + (510)^2 + (-320)^2}}$$

$$\overrightarrow{F_{DB}} = 385 \times \frac{(-480).i + (510).j + (-320).k}{770}$$

$$\overrightarrow{F_{DB}} = -240.i + 255.j - 160k$$

$$\overrightarrow{R} = \overrightarrow{F_{EB}} + \overrightarrow{F_{DB}} = (-135.i + 200.j - 300k) + (-240.i + 255.j - 160k)$$

$$\overrightarrow{R} = -375.i + 455.j - 460.k$$

$$R = \sqrt{(-375)^2.i^2 + (455)^2.j^2 + (-460)^2.k^2}$$

$$R = 747.83N$$

$$\theta_x = \cos^{-1}\left(\frac{-375}{747.83}\right) = 120.1^\circ$$

$$\theta_y = \cos^{-1}\left(\frac{455}{747.83}\right) = 52.5^\circ$$

$$\theta_z = \cos^{-1}\left(\frac{-460}{747.83}\right) = 128.0^\circ$$

## MOMENT OF FORCE AND ITS APPLICATIONS:

The effect of force on a body about an axis influenced to rotate in addition to move the body in the direction of its applications. The rotation of a body about an axis is known as moment of force or torque. The axis may be any line which intersects and is parallel to the line of action of the force. In most of the engineering applications the handling of spanner is applied to tightened or loosened by the moment of force. The distance between the applied force and point of axis to rotate will either increase or decrease the effort for the applications. If the distance is high then the less effort is required whereas the more effort is required for the lesser distance.

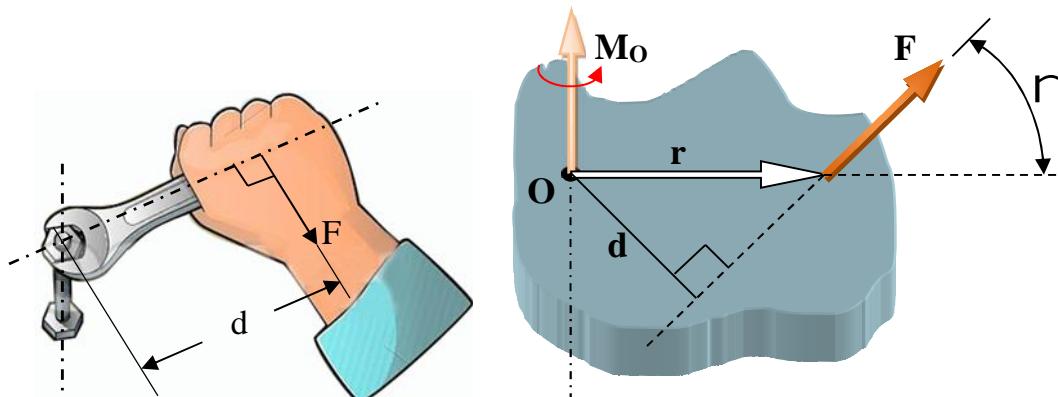


Fig.1.37 Moment of force

The magnitude of the moment or tendency of the force to rotate the body about the axis at point O perpendicular to the plane of the body is proportional to the both magnitude of the force and to the moment arm distance  $d$ , which is the perpendicular distance from the axis to the line of action of the force. Therefore the magnitude of the moment is defined as

$$M = Fd$$

The moment is a vector  $\vec{M}$  perpendicular to the plane of the body. The sense of  $\vec{M}$  depends on the direction in which  $F$  tends to rotate the body. The moment  $\vec{M}$  obeys all the rules of vector combination and may be considered a sliding vector with a line of action coinciding with the moment axis. The moments are the following two types.

1. Clockwise moments and
2. Anticlockwise moments

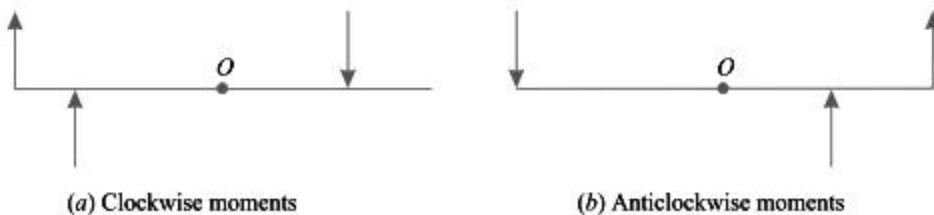


Fig.1.38 Direction of moment of force

### VARIGNON'S THEOREM

The distributive property of vector products can be used to determine the moment of the resultant of several concurrent forces. If several forces  $F_1, F_2, F_3$ , and  $F_4$  are applied at the same point A as shown in fig.1.38 and it is denoted by letter 'r' the position vector of A, it follows immediately as

$$r \times R = r \times F_1 + r \times F_2 + r \times F_3 + r \times F_4$$

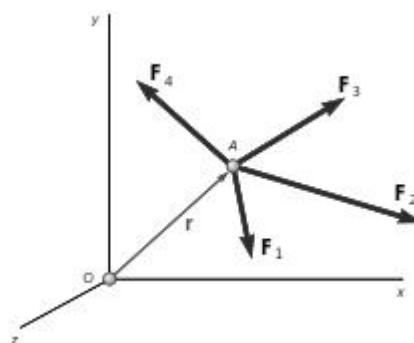


Fig.1.39 Moment of force in a space

In words, the moment about a given point O of the resultant of several concurrent forces is equal to the sum of the moments of the various forces about the same point O. This property, which was originally established by the French mathematician Varignon (1654–1722) long before the introduction of vector algebra, is known as Varignon's theorem. The relation ( $r \times R$ ) makes it possible to replace

the direct determination of the moment of a force  $F$  by the determination of the moments of two or more component forces.

## **APPLICATIONS OF MOMENTS:**

There are many applications in the field of engineering for the use of moments but frequently the moments are applied for two cases. They are

1. Position of the resultant force
2. Levers.

### **1. Position of the Resultant force by Moments:**

It is also known as analytical method for the resultant force. The position of a resultant force may be found out by moments as discussed below:

- i) First of all, find out the magnitude and direction of the resultant force by the method of algebraic sum of resolution as discussed earlier in this chapter.
- ii) Now equate the moment of the resultant force with the algebraic sum of moments of the given system of forces about any point. This may also be found out by equating the sum of clockwise moments and that of the anticlockwise moments about the point, through which the resultant force will pass.

### **2. Levers:**

A lever is a rigid bar (straight, curved or bent) and is hinged at one point. It is free to rotate about the hinged end called fulcrum. The common examples of the use of lever are crow bar, pair of scissors, fire tongs, etc. It may be noted that there is a point for effort (called effort arm) and another point for overcoming resistance or lifting load (called load arm).

## **TYPES OF LEVERS:**

Levers are natural in many variety and used for different applications but in this section only discussed about

- i) Simple levers and
- ii) Compound levers

### SIMPLE LEVERS:

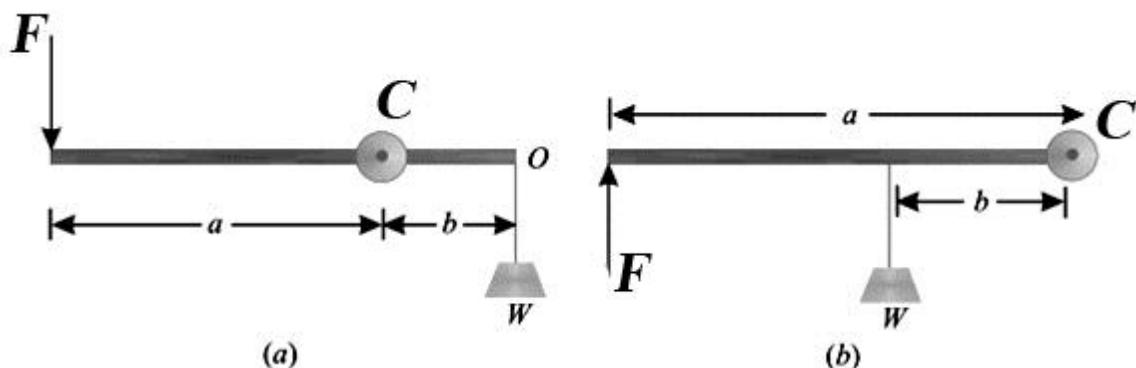


Fig.1.40 Simple levers

A lever, which consists of one bar having one fulcrum is known as simple lever as shown in Fig.1.39 (a) and (b).

Let,

$F$  = Applied effort,

$W$  = Weight,

$a$  = Length between fulcrum and effort, and

$b$  = Length between fulcrum and weight.

Now taking moments of the effort and load about the fulcrum ( $F$ ) and equating the same,

$$F \times a = W \times b \quad \text{or} \quad \frac{W}{F} = \frac{a}{b}$$

The above terms are commonly known as mechanical advantage and leverage. A little consideration will show that in order to increase the mechanical advantage, either length of the lever arm ( $a$ ) is to be increased or length of the load arm ( $b$ ) is to be reduced.

**COMPOUND LEVERS:**

A lever which consists of a number of simple levers is known as a compound lever, as shown in fig.1.40 (a) and (b).

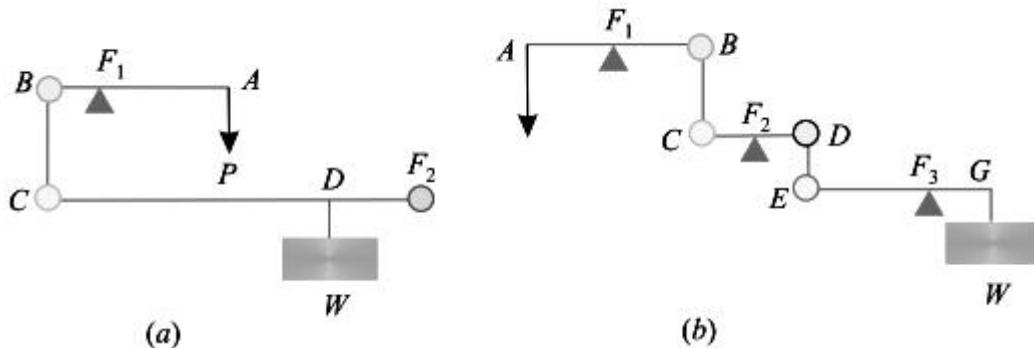


Fig.1.41 Compound levers

A little consideration will show, that in a compound lever, the mechanical advantage (or leverage) is greater than that in a simple lever. Mathematically, leverage in a compound lever = Leverage of 1st lever  $\times$  Leverage of 2nd lever  $\times$  and so on. The platform weighing machine is an important example of a compound lever. This machine is used for weighing heavy loads such as trucks, wagons along with their contents. On smaller scales, these machines are used in godowns and parcel offices of transport companies for weighing consignment goods.

**Problems:**

**SP1.9:** A force of 20N is applied at an angle  $70^\circ$  to the edge of a door 1.2m wide as shown fig. Find the moment of the force about the hinge.

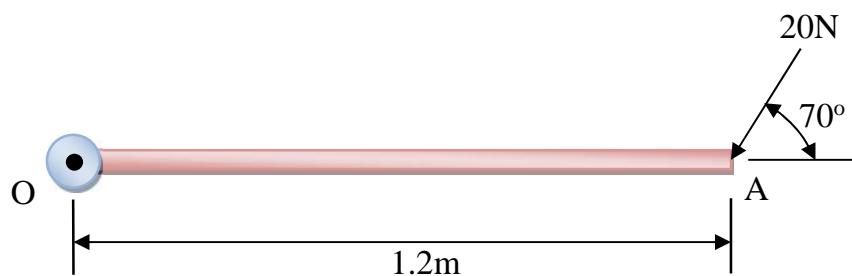
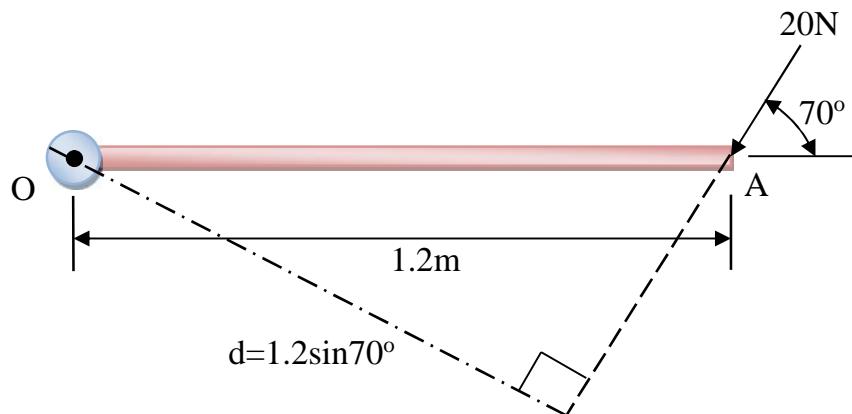


Fig.1.42 Force systems in a simple lever

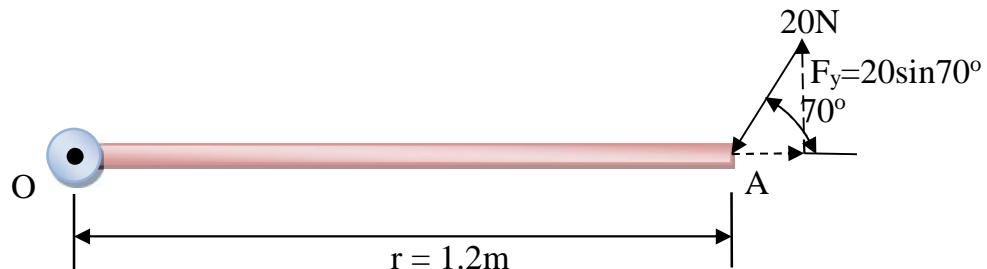
**Solution: method I,**



We know that the moment of force about hinge, O

$$M_O = d \times F = 1.2 \sin 70^\circ \times 20 = 22.55 \text{ N-m}$$

**Method II:** By component of force



We know that the moment of force about hinge, O

$$M_O = r \times (F_x + F_y) = r \times F_x + r \times F_y = 0 + 1.2 (20 \sin 70^\circ) = 22.55 \text{ N-m}$$

The perpendicular distance from point O to A for the x component of force  $F_x$  is zero.

**SP1.10:** A force of 800 N acts on a bracket as shown in fig. Determine the moment of the force about B.

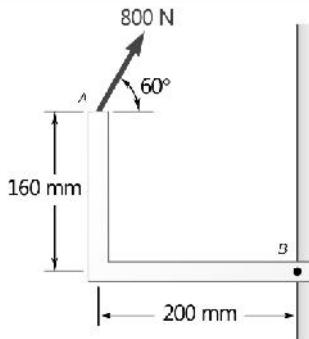


Fig.1.43 Force systems in a bracket

**Solution:**

The moment of force about B is obtained by forming the vector product

$$\overrightarrow{M}_B = \overrightarrow{r}_{AB} \times \overrightarrow{F}$$

Where,  $\overrightarrow{r}_{AB}$  is the vector drawn from B to A. The components of vector in rectangular is

$$\overrightarrow{r}_{AB} = (-0.2m).i + (0.16m).j \text{ and}$$

$$\overrightarrow{F} = (800N)\{\cos 60^\circ .i + \sin 60^\circ .j\} = 800(\cos 60^\circ .i + \sin 60^\circ .j)$$

$$\overrightarrow{F} = 400.i + 693.j$$

Therefore,

$$\overrightarrow{M}_B = \overrightarrow{r}_{AB} \times \overrightarrow{F} = (-0.2i + 0.16j) \times (400i + 693j)$$

$$\overrightarrow{M}_B = -0.2(693)(i \times j) + 0.16(400)(j \times i)$$

$$\overrightarrow{M}_B = -138.6 - 64.0 = -202.6 N \cdot m$$

**SP1.11:** Calculate the magnitude of the moment about the base point O of the 600N force in five different ways as shown in fig.1.43.

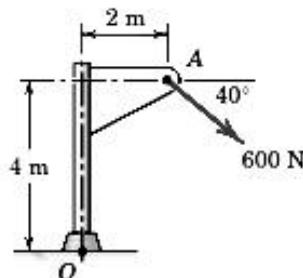
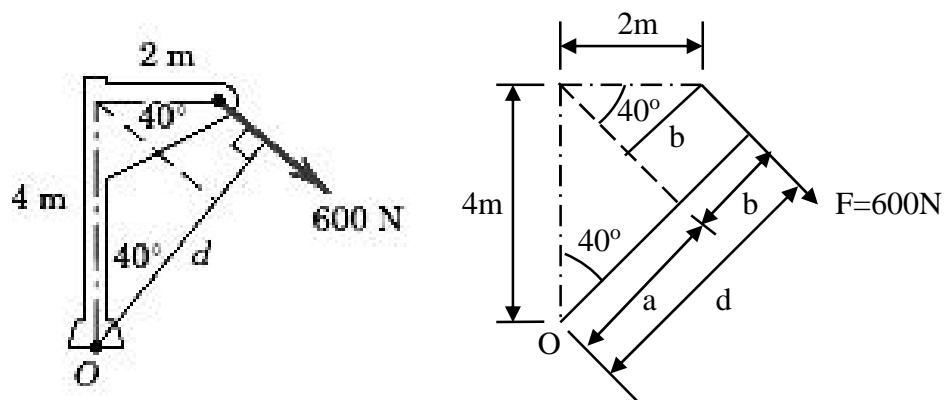


Fig.1.44 Force system in pole

**Solution:**

**Method I:** Determine the moment arm to the magnitude force F as 600N.



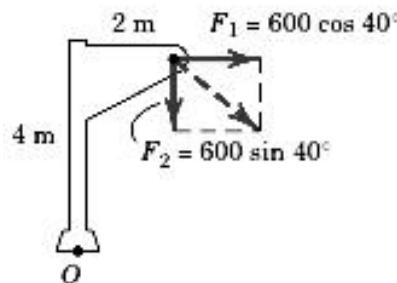
From the geometry of figure,  $d = a + b$ ,

$$a = \cos 40^\circ \times 4 \text{ and } b = \sin 40^\circ \times 2$$

$$d = 4\cos 40^\circ + 2\sin 40^\circ = 4.35 \text{ m}$$

The moment about O is clockwise and has the magnitude

$$M_O = d \times F = 4.35 \times 600 = 2610 \text{ N-m}$$

**Method II:** Components of force

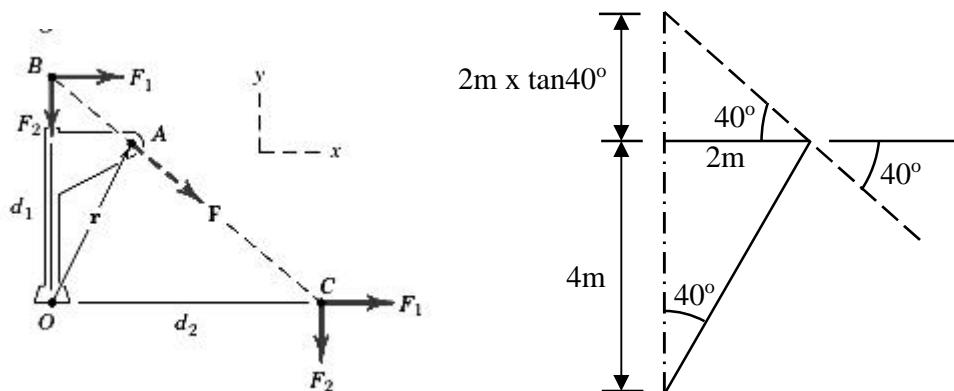
$$F_x = F \cos 40^\circ = 600 \cos 40^\circ = 460 \text{ N}$$

$$\text{and } F_y = F \sin 40^\circ = 600 \sin 40^\circ = 386 \text{ N}$$

By Varignon's theorem, the moment of force becomes

$$M_O = (\text{Vertical distance} \times F_x) + (\text{Horizontal distance} \times F_y)$$

$$M_O = 4 \times 460 + 2 \times 386 = 2610 \text{ N-m}$$

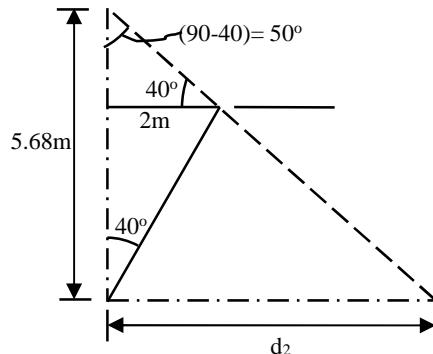
**Method III:** By Law of transmissibility

Now move the force  $F$  (600N) without change of line of action to the intersect point of vertical axis from the point O is termed as Point B. The distance between the point O and B is measured vertically for the moment of force  $F_1$  ( $F_x$ ) is  $d_1$  but the distance between the point O and B horizontally for the moment of force  $F_2$  ( $F_y$ ) is zero.

$$\text{Therefore, } d_1 = 4 + 2 \tan 40^\circ = 5.68 \text{ m and}$$

The moment of force is

$$M_O = d_1 \times F_x = 5.68 \times 460 = 2610 \text{ N-m}$$

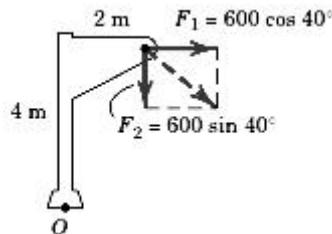
**Method IV:** By Law of transmissibility

Now move the force  $F$  (600N) without change of line of action to the intersect point of horizontal axis from the point O is termed as Point C. The distance between the point O and C is measured vertically for the moment of force  $F_1$  ( $F_x$ ) is zero but the distance between the point O and C horizontally for the moment of force  $F_2$  ( $F_y$ ) is  $d_2$ .

$$\text{Therefore, } d_2 = d_1 \times \tan 50^\circ = 5.68 \times \tan 50^\circ = 6.77\text{m}$$

The moment of force is

$$M_O = d_2 \times F_y = 6.77 \times 386 = 2610 \text{ N-m}$$

**Method V:** By vector expression

$$\text{We know that, } \overrightarrow{M_O} = \vec{r} \times \vec{F} = (r_x \cdot i + r_y \cdot j) + F_m (\cos 40^\circ \cdot i + \sin 40^\circ \cdot j)$$

$$\overrightarrow{M_O} = (2 \cdot i + 4 \cdot j) + 600(\cos 40^\circ \cdot i - \sin 40^\circ \cdot j) = -2610 \text{ N-m}$$

The negative sign indicates that the vector is in the negative z – direction. The magnitude of the vector expression is  $M_O = 2610 \text{ N-m}$ .

**SP1.12:** A rectangular plate is supported by brackets at A and B and by a wire CD as shown in fig.1.44. Knowing that the tension in the wire is 200N, determine the moment about A of the force exerted by the wire on point C.

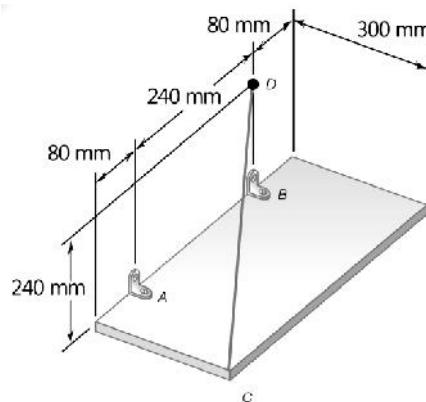
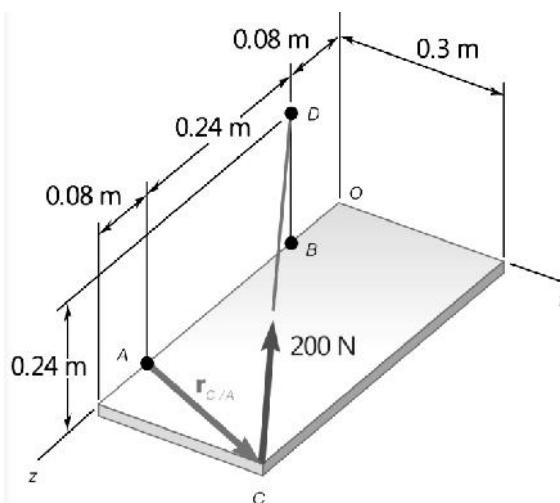


Fig.1.45 A rectangular plate supported by bracket

**Solution:**



The moment  $M_A$  about A of the force F exerted by the wire on point C is obtained by forming the vector product

$$\overrightarrow{M_A} = \overrightarrow{r_{CA}} \times \overrightarrow{F_{CD}}$$

Where,  $\overrightarrow{r_{CA}}$  is the vector drawn from A to C. The components of vector in rectangular is

$$\overrightarrow{r_{CA}} = \overrightarrow{AC} = (0.3m)\mathbf{i} + 0\mathbf{j} + (0.08m)\mathbf{k} \text{ and}$$

$$\overrightarrow{F_{CD}} = F_{CD} \frac{\overrightarrow{CD}}{CD}$$

$$\overrightarrow{F_{CD}} = 200 \frac{(-0.3m.i + 0.24m.j - 0.32m.k)}{0.5m}$$

$$\overrightarrow{F_{CD}} = -120.i + 96.j - 128.k$$

Therefore,

$$\overrightarrow{M_A} = \overrightarrow{r_{CA}} \times \overrightarrow{F_{CD}} = (0.3i + 0.j + 0.08k) \times (-120.i + 96.j - 128.k)$$

$$\overrightarrow{M_A} = 0.3(96)(i \times j) + 0.3(-128)(i \times k) + 0.08(-120)(k \times i) + 0.08(96)(k \times j)$$

$$\overrightarrow{M_A} = 0.3(96)(k) + 0.3(-128)(-j) + 0.08(-120)(j) + 0.08(96)(-i)$$

$$\overrightarrow{M_A} = 0.3(96)(k) + 0.3(128)(j) - 0.08(120)(j) - 0.08(96)(i)$$

$$\boxed{\overrightarrow{M_A} = -7.68.i + 28.8.j + 28.8.k}$$

Alternative solution:

$$M_A = \begin{vmatrix} i & j & k \\ x_C - x_A & y_C - y_A & z_C - z_A \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\overrightarrow{M_A} = -7.68.i + 28.8.j + 28.8.k}$$

## COUPLES AND RESULTANT OF FORCE SYSTEM:

The two forces having equal in magnitude and unlike parallel line of action of force is known as a couple. Consider the action of two equal and opposite forces  $F$  and  $-F$  a distance 'd' apart as shown in fig. These two forces cannot be combine into a single force due to their sum of direction in everywhere is zero. The rotation is produced by their effect. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O in their plane is the couple  $M^C$ . This couple has a magnitude (Scalar product)

$$M^C = F(a + d) - Fa \quad \text{or} \quad M^C = Fd$$

The moment of a couple is also expressed by using vector algebra. The moment of force  $F$  and  $-F$  is a cross product of distance from the point of moment (axis line about O) and the line of action of force. The couple is written as

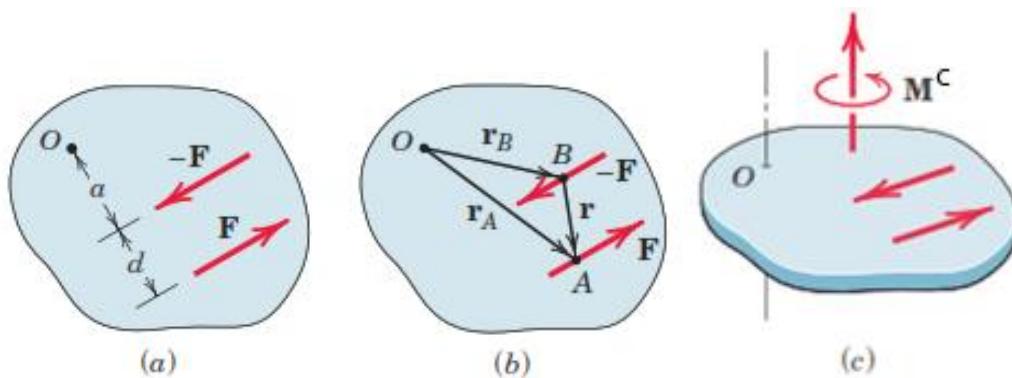


Fig.1.46 Couple

$$M^C = r_A \times F + r_B \times (-F) = (r_A - r_B) \times F$$

Where  $\mathbf{r}_A$  and  $\mathbf{r}_B$  are position vectors which run from point O to arbitrary points A and B on the lines of action of F and  $-F$  respectively. Therefore,  $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$  then the couple written as

$$M^C = \mathbf{r} \times \mathbf{F}$$

The couple of moment force not having the centre of moment about point O as reference but it is free to rotate. The distance between the two unlike parallel vectors is accounted to determine the couple of given forces. Fig. 1.45 (c) shows the couple vector is counter clockwise by the lines of action of forces.

## CLASSIFICATION OF COUPLES:

The couples are classified according to the direction of forces. They are

1. Clockwise couple and
2. Counter clockwise couple

### 1. Clockwise couple:

A body is rotated by forces in the direction of clockwise is called clockwise couple as shown in fig.1.46.

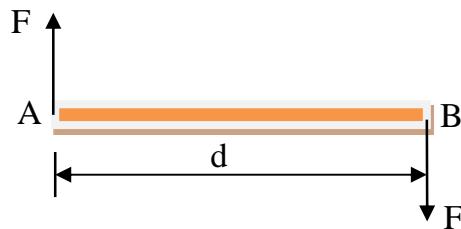


Fig.1.47 Clockwise couple

## 2. Counter clockwise couple:

A body is rotated by forces in the direction counter clockwise then it is known as counter clockwise couple as shown in fig.1.47.

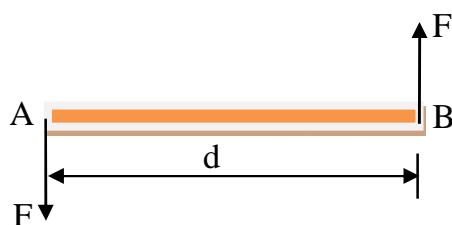


Fig.1.48 Clockwise couple

## Characteristics of a couple:

A couple (whether clockwise or anticlockwise) has the following characteristics:

1. The algebraic sum of the forces, constituting the couple, is zero.
2. The algebraic sum of the moments of the forces, constituting the couple, about any point is the same, and equal to the moment of the couple itself.
3. The couple cannot be balanced by a single force. But it can be balanced only by a couple of opposite sense.
4. Any number of coplanar couples can be reduced to a single couple, whose magnitude will be equal to the algebraic sum of the moments of all the couples.

## Resultant Couple Moment:

The sum of couple moment forces about the each couple moment force as a free vector and join together by the tail end of each couple is called as resultant couple moment force. The resultant couple moment force is written as

$$M_R^C = M_1^C + M_2^C + M_3^C + \dots + M_n^C = \sum M^C$$

**Problems:**

**SP1.13** A machine component of length 2.5 metres and height 1 metre is carried upstairs by two men, who hold it by the front and back edges of its lower face. If the machine component is inclined at  $30^\circ$  to the horizontal and weighs 100 N, find how much of the weight each man supports?

**Solution:**

Given parameter: Length of machine component = 2.5 m; Height of the component = 1 m; Inclination =  $30^\circ$  and weight of component = 100 N

Let,  $P$  = Weight supported by the man at A.

$Q$  = Weight supported by the man at B.

C = Point where the vertical line through the centre of gravity cuts the lower face.

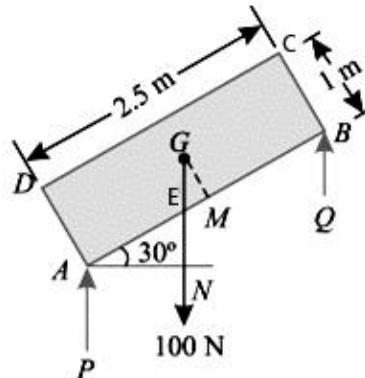


Fig.1.49

Now join G (i.e., centre of gravity) with M (i.e., mid-point of AB) as shown in Fig. 1.47.

From the geometry of the figure,

$$GM = \frac{BC}{2} = \frac{1}{2} = 0.5m \text{ and } AM = \frac{CD}{2} = \frac{2.5}{2} = 1.25m$$

Now, treat the vector force as couple moment say the force P and Weight of the body W ( $P+Q=100$ ). Therefore couple moment of force P is  $P \times AE$  which is equals to the support carrying the load at point Q, such that the couple moment of Q is  $Q \times EB$

The  $AE$  is distance between couple moment of force P and body weight 'W'.

Similarly the  $EB$  is distance between couple moment of force Q and body weight 'W'.

The weight is shared by the two couple moment of force at point A and B by the force P and Q respectively. Since the weight of the body W is P+Q.  $W = P + Q = 100$

The couple formed at both ends are equal and magnitude, therefore

$$P \times AE = Q \times EB,$$

$$\Rightarrow P \times AE = 100 - P \times EB$$

$$AE = AM - EM = 1.25 - GM \tan 30^\circ$$

$$AE = 1.25 - (0.5 \times 0.577) = 0.96m$$

$$EB = AB - AE = 2.5 - 0.96 = 1.54m$$

$$\therefore P \times AE = 100 - P \times EB$$

$$P \times 0.96 = (100 - P) \times 1.54 = 154 - 1.54P$$

$$0.96P + 1.54P = 154$$

$$2.5P = 154$$

$$P = \frac{154}{2.5} = 61.6N$$

$$Q = 100 - P = 100 - 61.6$$

$$Q = 38.4N$$

**SP1.14:** Find the resultant couple moment forces for the given figure 1.50 as below.

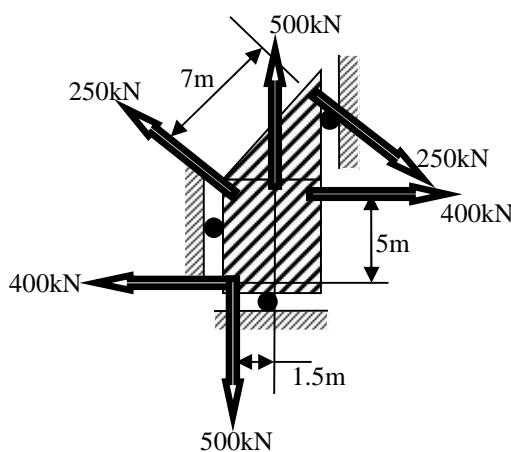
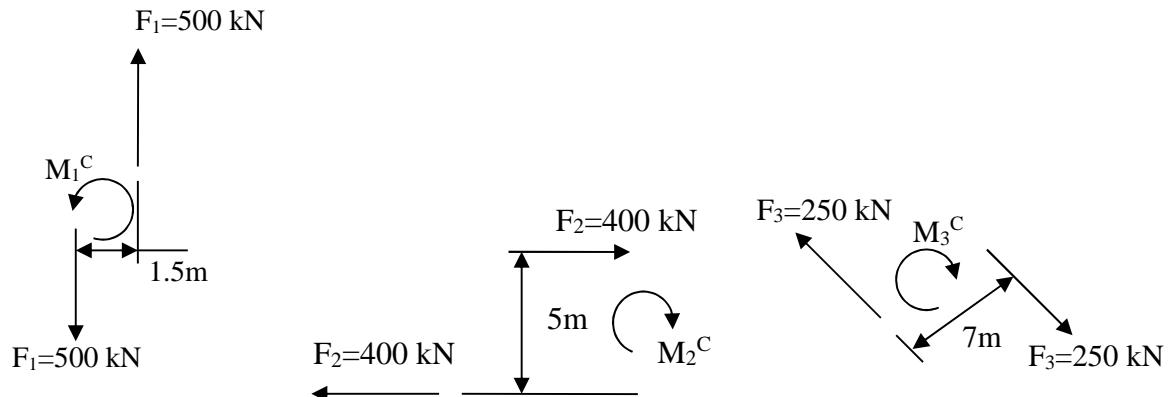


Fig.1.50

### Solution:

Given parameter; the counter clockwise direction of couple moment force is considered as positive values.



$$M_1^C = 1.5 \times 500 = 750 \text{ Nm}$$

$$M_2^c = -5 \times 400 \\ \equiv -2000 \text{ Nm}$$

$$M_3^C = -7 \times 250 \\ \equiv -1750 \text{ Nm}$$

We know that, the resultant of couple moment force is the sum of couple moment force as

$$M_R^C = M_1^C + M_2^C + M_3^C = 750 - 2000 - 1750$$

$$M_R^C = -3000 \, Nm \, ,$$

The negative sign implies the direction of couple moment force is in clockwise direction.

**SP1.15:** Determine the distance of couple moment force on the lever has 400N as shown in fig.1.51.

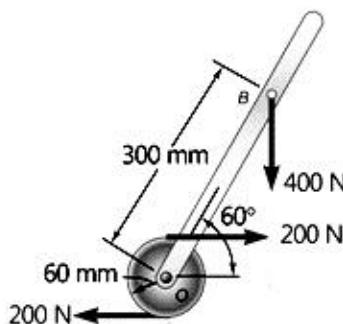


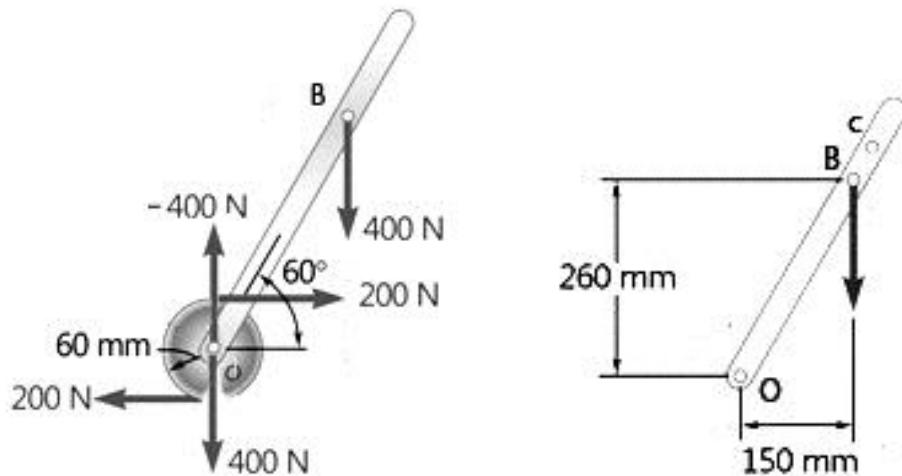
Fig.1.51

**Solution:**

First considered the force has 200N to determine the couple moment force as

$$\begin{aligned}
 M_{200N}^C &= r \times F = (0i + (60mm + 60mm)j) \times (200i + 0j) \\
 &= (0i + 0.120j) \times (200i + 0j) = (0.120)(200)(-k), \quad [i \times i = 0, j \times j = 0 \text{ and } j \times i = -k] \\
 &= -24k, \text{ The negative sign indicates the direction of couple moment force in clockwise.}
 \end{aligned}$$

Second considered the force has 400N to determine couple moment force about point O, by treating as equivalent force system.



$$\begin{aligned}
 M_{400N}^C &= r \times F = (0.150i + 0.260j) \times (0i + (-400)j) \\
 &= -400(0.150)k = -60k \text{ Nm} \quad [i \times i = 0, j \times j = 0 \text{ and } j \times i = -k]
 \end{aligned}$$

The negative sign indicates the direction of couple moment force in clockwise.

Therefore the total moment is  $M^C = M_{200N}^C + M_{400N}^C$

$$M^C = (-24 - 60)k \text{ Nm} = -84k \text{ Nm},$$

Hence the total couple moment force of 400N is

$$\begin{aligned}
 M^C &= OC \times F = (OC \cos 60^\circ i + OC \sin 60^\circ j) \times (0i - 400j) \\
 -84 &= (OC \cos 60^\circ)(-400)k = -OC(200)k
 \end{aligned}$$

$$OC = \frac{84}{200} = 0.42m = 420mm$$

## **EQUILIBRIUM OF SYSTEM OF FORCE:**

Equilibrium is a condition in static mechanics to keep the particle or rigid bodies at rest. This is termed as the sum of all forces  $\sum F = \sum(F_x + F_y + F_z)$  or resultant force  $R$  or the sum of moment force  $\sum M$  or resultant moment force  $M_R$  is zero. In particle, the moment force and couple force is not present but only the system of forces and resultant force were studied. This is due the assumption of size and mass is negligible compared to the rigid bodies. The equilibrium condition for the system of force satisfies the Newton's second law of motion by writing the condition as  $\sum F = ma = 0$ . Therefore the acceleration of particle is zero ( $a=0$ ) which means that the velocity is constant or the move is remains at rest.

### **Example:**

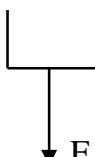
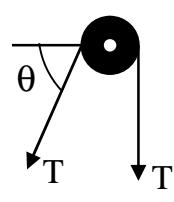
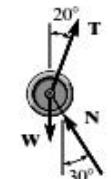
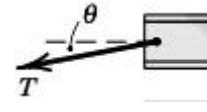
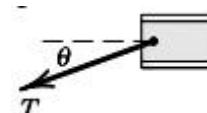
For a particle,  $\sum F = \sum(F_x + F_y + F_z) = 0$  and  $R = 0$

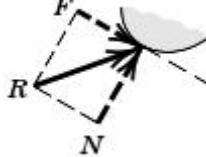
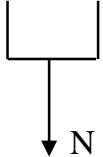
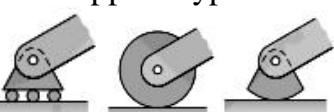
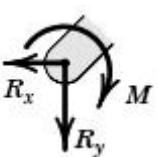
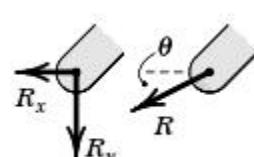
For a rigid body,  $\sum F = \sum(F_x + F_y + F_z) = 0$ ,  $R = 0$ ,  $\sum(M)_O = 0$ ,  $(M_R)_O = 0$  and  $\sum(M)_A = (M_R)_O + r \times R = 0$

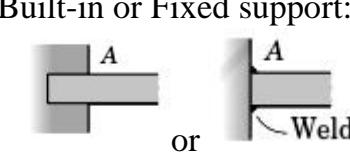
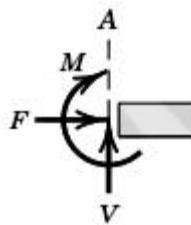
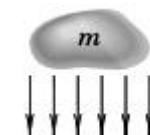
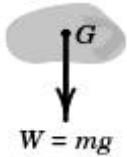
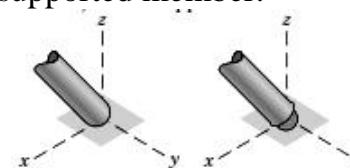
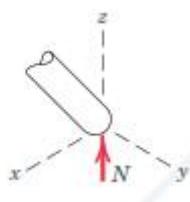
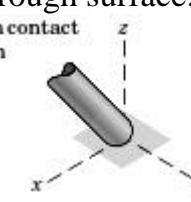
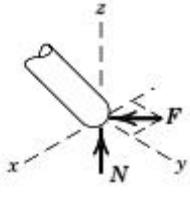
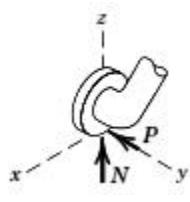
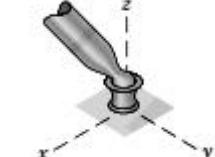
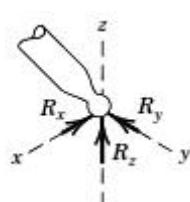
## **FREE BODY DIAGRAM:**

To form the equation of equilibrium, the various forces or moment or couple must be considered for all the known and unknown forces which act on a particle or rigid body. Without considered the real object of particle or rigid body, only the number of all forces acts on the particle or rigid body is drawn by isolation and separate from the surroundings which known as free body diagram (FBD). A system may be a single body or a combination of connected bodies. The bodies may be rigid or non-rigid. Before discuss about a free body diagram, the various types of supports involved in a particle or rigid bodies are enumerated as below.

**Table 1.1: Types of supports to describe system of force.**

Sl. No.	Types of supports	Free Body Diagram	Description
<b><u>COPLANAR SYSTEM</u></b>			
1.	Springs:		<p>*The springs considered as un-deformed at initial distance as <math>l_0</math> is used to support a particle.</p> <p>* The length of the linearly elastic spring will change in the direction of the force, <math>F</math> act on a particle. <math>F = k(l - l_0)</math>. Where <math>l</math> is a change in length by force and <math>k</math> is spring constant or stiffness.</p>
2.	Cables and Pulleys:		* A cable can support only a tension or pulling force.
3.	Smooth contact:		* An object rests on a smooth surface then the surface will exert a force on the object that is normal to the surface at the point of contact.
4.	Flexible cable, belt chain or rope: Weight of cable negligible		* Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
5.	Flexible cable, belt chain or rope: Weight of cable not negligible		* Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.

6.	Rough surface:		* Rough surfaces are capable of supporting a tangential component $F$ (Frictional force) as well as a normal component $N$ of the resultant contact force $R$ .
7.	Roller support: type1		* Roller, rocker or ball support transmits a compressive force normal to the supporting surface.
8.	Roller support: type2		* Roller, rocker or ball support transmits a compressive force normal to the supporting surface.
9.	Freely sliding guide: type1		* Collar or slider free to move along smooth guides; can support force normal to guide only.
10.	Freely sliding guide: type2		* Collar or slider free to move along smooth guides; can support force normal to guide only.
11.	Pin connection: Not free to turn		* A pin not free to turn also supports a couple $M$ with the force components $F_x$ and $F_y$ or a magnitude $F$ and direction $\theta$ .
12.	Pin connection: Free to turn		* A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. Hence the component of force along horizontal $F_x$ and vertical $F_y$ or a magnitude of force $F$ and direction $\theta$ .

13.	Built-in or Fixed support: 		* A built-in or fixed support is capable of supporting an axial force $F$ , a transverse force $V$ (Shear force) and a couple $M$ (Bending moment) to prevent rotation.
14.	Gravitational attraction: 		* The resultant of gravitational attraction on all elements of a body of mass $m$ is the weight $W=mg$ and acts towards the centre of the earth through the centre mass $G$ .
<b>SPATIAL SYSTEM</b>			
15.	Member in contact with smooth surface or ball supported member: 		* Force must be normal to the surface and directed towards the member.
16.	Member in contact with rough surface: 		* The possibility exists for a force $F$ tangent to the surface (friction force) to act on the member as well as a normal force $N$ .
17.	Roller or wheel support with lateral constraint: 		* A lateral force $P$ exerted by the guide on the wheel can exist in addition to the normal force $N$ .
18.	Ball and socket joint: 		* A ball and socket joint free to pivot about the centre of the ball can support a force $R$ with all three components of force.

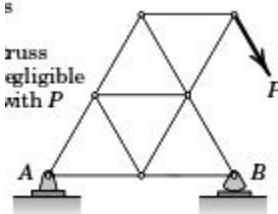
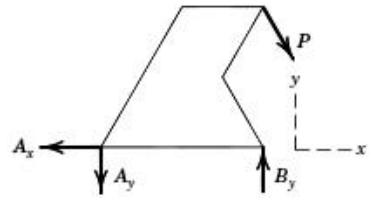
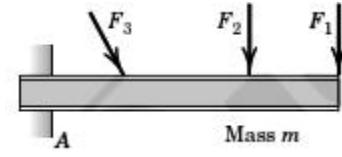
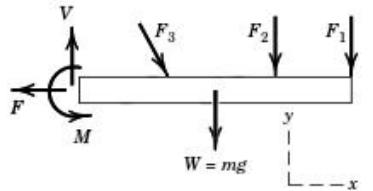
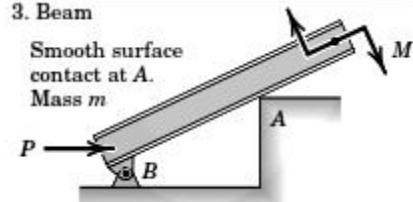
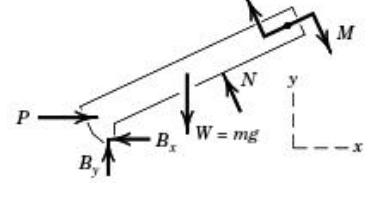
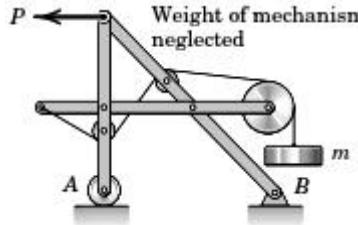
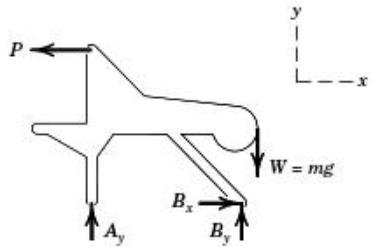
19.	Fixed connection: Embedded or welded 		* In addition to three components of force a fixed connection can support a couple moment $M$ represented by its three components.
20.	Thrust bearing support: 		* Thrust bearing is capable of supporting axial force $R_y$ as well as radial forces $R_x$ and $R_z$ . Couples $M_x$ and $M_z$ must in some cases be assumed zero in order to provide statical determinacy.

### Procedure to draw Free Body Diagram (FBD):

The isolation of system which has number of system of force in a particle or rigid bodies is discussed in different steps as follows;

1. First identify the minimum number of unknown force in a system and then isolates this system to draw a free body diagram which means the various forces acting in that systems.
2. Second the connected system to first steps is isolated and determines the unknowns and so on.
3. Similarly the remaining steps are followed to determine the number of unknowns.

**Table 1.2: Free body diagrams for mechanisms and structures**

Sl. No.	Mechanical Systems	Free Body Diagram
1.	Plane Truss: Weight of truss assumed negligible compared with $P$ 	
2.	Cantilever beam: 	
3.	Beam: Smooth surface contact at A. 3. Beam Smooth surface contact at A. Mass m 	
4.	Frames and Machines: Rigid system of interconnected bodies analysed as a single unit. 	

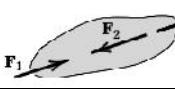
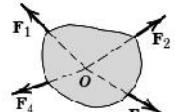
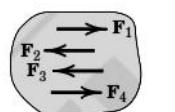
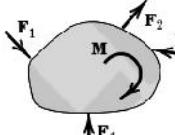
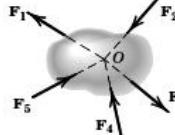
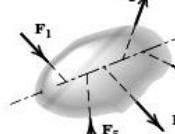
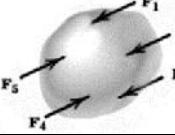
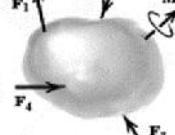
## Equations of Equilibrium of Coplanar Systems and Spatial Systems:

In statics, the most of the problems are treated in equilibrium condition to explore the known and unknown forces act on the particle or rigid body system. The sum of all forces on the particle or the rigid body termed as resultant force is to be zero which means equilibrium in state. Therefore the resultant force  $R$  and resultant moment force  $M$  are both zero for the equilibrium body. The equations of equilibrium

for the different system of force are listed in the table 1.3. The general equations of equilibrium as

$$R = \sum F = 0 \text{ and } M_R = \sum M = 0$$

**Table 1.3: Types of force system in equilibrium conditions.**

Sl. No.	Types of force system	Free Body Diagram	Equilibrium equations
<b>LINEAR SYSTEM</b>			
1.	Collinear:		$\sum F_x = 0$
<b>COPLANAR SYSTEM</b>			
2.	Concurrent at a point:		$\sum F_x = 0, \sum F_y = 0$
3.	Parallel:		$\sum F_x = 0$ and $\sum M_z = 0$
4.	Non concurrent:		$\sum F_x = 0, \sum F_y = 0$ and $\sum M_z = 0$
<b>SPATIAL SYSTEM</b>			
5.	Concurrent at a point:		$\sum F_x = 0, \sum F_y = 0$ and $\sum F_z = 0$
6.	Concurrent with a line:		$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_y = 0$ and $\sum M_z = 0$
7.	Parallel:		$\sum F_x = 0, \sum M_y = 0$ and $\sum M_z = 0$
8.	Non concurrent:		$\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_x = 0, \sum M_y = 0$ and $\sum M_z = 0$

Before trying to solve the equilibrium problem, the nature of constraints present in the body must be understood. If many unknown reaction forces are available than the independent equilibrium equations for the system of force are named as statically indeterminate. The unknown variables may be forces, couples, distances and angles.

### Lami's Theorem:

It states, "If three coplanar forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two."

Mathematically,

$$\frac{F_1}{\sin \theta_1} = \frac{F_2}{\sin \theta_2} = \frac{F_3}{\sin \theta_3}$$

Where,  $F_1$ ,  $F_2$  and  $F_3$  are three forces and  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are the angles between the forces.

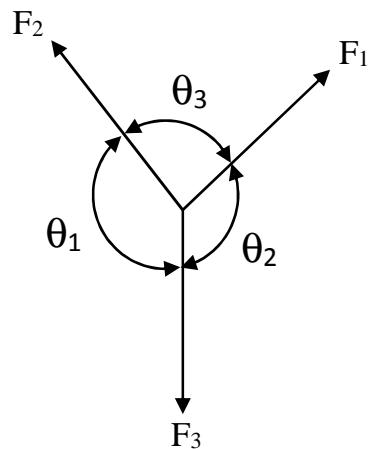


Fig.1.52 Lami's Theorem

### Proof:

Consider three coplanar forces  $F_1$ ,  $F_2$  and  $F_3$  acting at a point O. Let the opposite angles to three forces be  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  as shown in fig1.52. Now construct the parallelogram OABC with OA and OC as adjacent sides as shown in the fig1.52. We know that the resultant of two forces  $F_1$  and  $F_2$  will be given by the diagonal OB both in magnitude and direction of the parallelogram OABC. Since these forces are in equilibrium, therefore the resultant of the forces  $F_1$  and  $F_2$  must be in line with OD and equal to R, but in opposite direction.

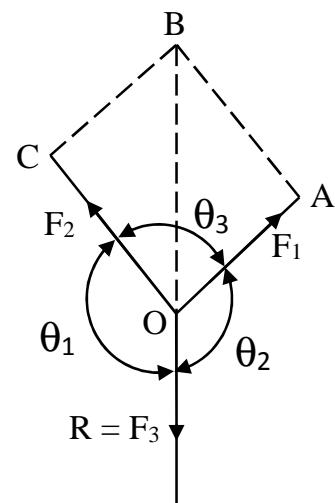


Fig.1.53 Proof of Lami's Theorem

From the geometry of the fig.1.52,

$$AB = F_2 \text{ and } BC = F_1$$

$$\text{Therefore, } \angle COB = (180^\circ - \theta_1) \text{ and } \angle CBO = \angle AOB = (180^\circ - \theta_2)$$

$$\angle BCO = 180^\circ - (\angle COB + \angle CBO)$$

$$\angle BCO = 180^\circ - \{(180^\circ - \alpha_1) + (180^\circ - \alpha_2)\} = 180^\circ - 180^\circ + \alpha_1 - 180^\circ + \alpha_2$$

$$\angle BCO = \alpha_1 + \alpha_2 - 180^\circ$$

$$\text{But, } \alpha_1 + \alpha_2 + \alpha_3 = 360^\circ$$

Now substitute  $\alpha_1 + \alpha_2 = 360^\circ - \alpha_3$  in  $\angle BCO = \alpha_1 + \alpha_2 - 180^\circ$  then

$$\angle BCO = 360^\circ - \alpha_3 - 180^\circ = 180^\circ - \alpha_3$$

Considered the triangle COB, then sine law as

$$\frac{OC}{\sin \angle CBO} = \frac{CB}{\sin \angle COB} = \frac{OB}{\sin \angle BCO}$$

$$\frac{OC}{\sin(180^\circ - \alpha_2)} = \frac{CB}{\sin(180^\circ - \alpha_1)} = \frac{OB}{\sin(180^\circ - \alpha_3)}$$

Here, CB = OA and OB = OD

$$\frac{OC}{\sin(\alpha_2)} = \frac{OA}{\sin(\alpha_1)} = \frac{OD}{\sin(\alpha_3)}$$

$$\therefore \frac{F_2}{\sin(\alpha_2)} = \frac{F_1}{\sin(\alpha_1)} = \frac{F_3}{\sin(\alpha_3)}$$

### Problems:

**SP1.16:** In a ship unloading a 3500 N automobile is supported by a cable. A rope is tied to the cable at A and pulled in order to centre the automobile over its intended position as shown in fig1.53. The angle between the cable and vertical is  $2^\circ$  while the angle between the rope and the horizontal is  $30^\circ$ . Determine the tension in the rope.

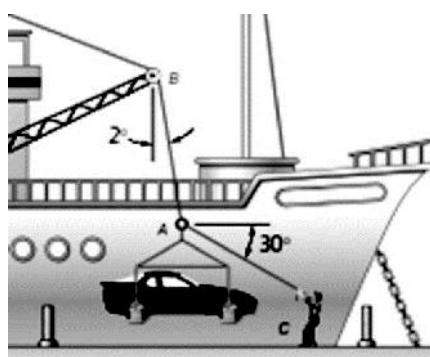


Fig.1.54

**Solution:**

As per Lami's theorem,

$$\frac{T_{AB}}{\sin(90^\circ - 30^\circ)} = \frac{T_{AC}}{\sin(180^\circ - 2^\circ)} = \frac{W}{\sin(90^\circ + 2^\circ + 30^\circ)}$$

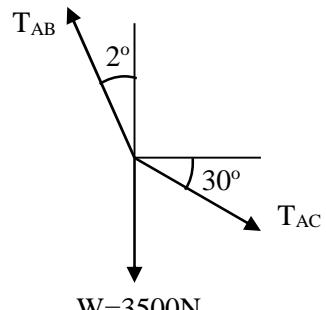
$$\frac{T_{AB}}{\sin(60^\circ)} = \frac{T_{AC}}{\sin(178^\circ)} = \frac{3500}{\sin(122^\circ)}$$

$$\frac{T_{AB}}{\sin(60^\circ)} = \frac{3500}{\sin(122^\circ)} \Rightarrow T_{AB} = \frac{3500 \times \sin(60^\circ)}{\sin(122^\circ)}$$

$$\Rightarrow T_{AB} = 3575N$$

$$\frac{T_{AC}}{\sin(178^\circ)} = \frac{3500}{\sin(122^\circ)}, \Rightarrow T_{AC} = \frac{3500 \times \sin(178^\circ)}{\sin(122^\circ)}$$

$$\Rightarrow T_{AC} = 144N$$



Free body diagram

**SP1.17:** Find the tension in cable BA and BC necessary to support the 100kg cylinder as shown in fig.1.54

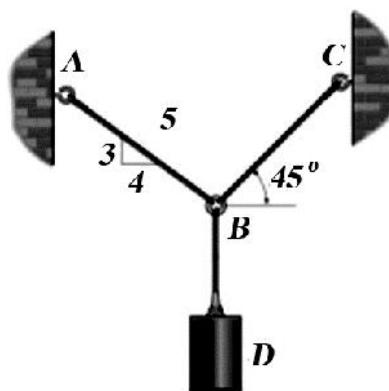


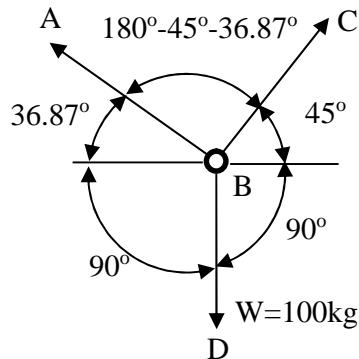
Fig.1.55

**Solution:**

The angle of force vector  $F_{AB}$  from the geometry is

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.87^\circ$$

Free body diagram;



By law of sine or Lami's theorem, we know that,

$$\frac{F_{BA}}{\sin(90 + 45)} = \frac{F_{BC}}{\sin(90 + 36.87)} = \frac{F_{BD}}{\sin(180 - 36.87 - 45)}$$

$$\frac{F_{BA}}{\sin(90 + 45)} = \frac{F_{BC}}{\sin(90 + 36.87)} = \frac{W}{\sin(180 - 36.87 - 45)}$$

$$\frac{F_{BC}}{\sin(90 + 36.87)} = \frac{100\text{kg}}{\sin(180 - 36.87 - 45)}$$

$$F_{BC} = \frac{100 \sin(90 + 36.87)}{\sin(180 - 36.87 - 45)} = \frac{79.999}{0.989} = 80.89\text{kg} = 793.53\text{N}$$

$$F_{BA} = \frac{100 \sin(90 + 45)}{\sin(180 - 36.87 - 45)} = \frac{70.71}{0.989} = 71.50\text{kg} = 701.42\text{N}$$

$$F_{BD} = W = 100\text{kg} = 981\text{N}$$

$$F_{BC} = 80.89\text{kg} = 793.53\text{N}$$

$$F_{BA} = 71.50\text{kg} = 701.42\text{N}$$

**SP1.18:** Find the required length of cord AC have the 8 kg lamp is suspended in the position as shown in fig.1.55. The un-deformed length of spring AB is 0.4m and the stiffness of spring is 300 N/m.

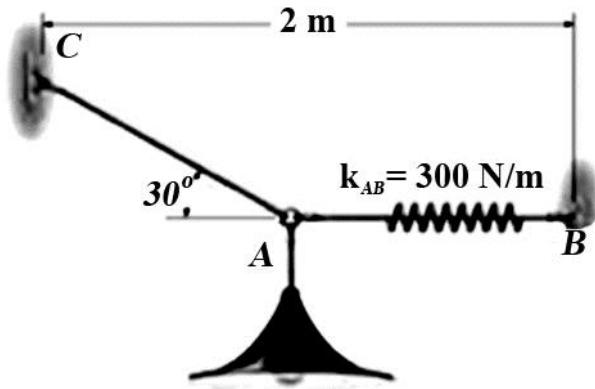
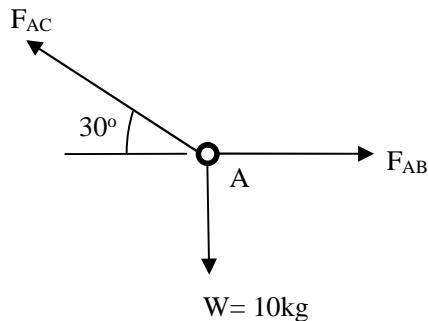


Fig.1.56

**Solution:**

Free body diagram,



By law of sine or Lami's theorem, we know that,

$$\frac{F_{AC}}{\sin(90)} = \frac{F_{AB}}{\sin(90 + 30)} = \frac{W}{\sin(180 - 30)}$$

$$\frac{F_{AC}}{\sin(90)} = \frac{F_{AB}}{\sin(90 + 30)} = \frac{8}{\sin(180 - 30)}$$

$$\frac{F_{AB}}{\sin(90 + 30)} = \frac{8}{\sin(180 - 30)},$$

$$\Rightarrow F_{AB} = \frac{8 \sin(90 + 30)}{\sin(180 - 30)} = \frac{6.93}{0.5} = 13.86 \text{ kg} = 135.97 \text{ N}$$

$$\frac{F_{AC}}{\sin(90)} = \frac{8}{\sin(180 - 30)}$$

$$\Rightarrow F_{AC} = \frac{8 \sin(90)}{\sin(180 - 30)} = \frac{8}{0.5} = 16 \text{ kg} = 156.96 \text{ N}$$

The force in the spring is find by the given data of spring constant and change in dimension, from this find the deformed position of spring as

$$F_{AB} = ku = 300 \times u$$

$$u = \frac{F_{AB}}{k} = \frac{135.97}{300} = 0.45 \text{ m}$$

Since the total deformed length of spring is

$$l'_{AB} = l_{AB} + u = 0.4 + 0.45 = 0.85 \text{ m}$$

The horizontal distance between the point C and B is 2m as given in this problem, This is the sum of the distance AB and horizontal distance AC is  $\cos\theta$  AC.

$$2m = AC \cos \theta + AB(l'_{AB}) = AC \cos 30^\circ + 0.85$$

$$AC = \frac{2 - 0.85}{\cos 30} = \frac{1.15}{0.866} = 1.328 \text{ m}$$

$$AC = 1.328 \text{ m}$$

**SP1.19:** Find the magnitude of the force C and T which along with the other three forces act on the bridge truss joint as shown in fig.1.56.

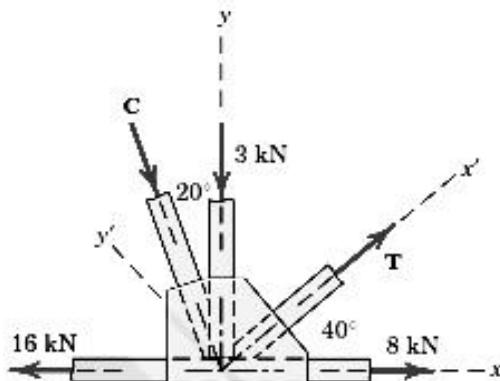
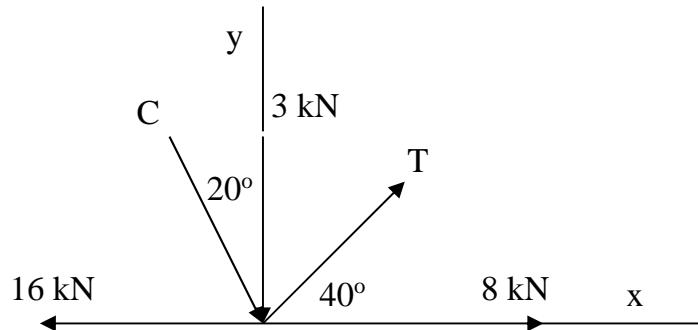


Fig.1.57

**Solution:**

Case1: about x and y axes

Free body diagram;



$$\sum F_x = 0, \quad 8 + T \cos 40^\circ + C \sin 20^\circ - 16 = 0$$

$$0.766T + 0.342C = 8 \quad \text{----} \rightarrow (1)$$

$$\sum F_y = 0, \quad T \sin 40^\circ - C \cos 20^\circ - 3 = 0$$

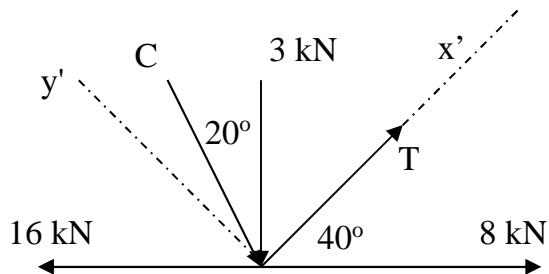
$$0.643T - 0.940C = 3 \quad \text{----} \rightarrow (2)$$

The solution of equation is

$$T = 9.09 \text{ kN} \quad \text{and} \quad C = 3.03 \text{ kN}$$

Case 2: about x' and y' axes

Free body diagram;



$$\sum F_{x'} = 0, \quad T + 8 \cos 40^\circ - 16 \cos 40^\circ - 3 \cos 50^\circ - C \cos 70^\circ = 0$$

$$T - 0.342C = 8.057 \quad \text{----} \rightarrow (3)$$

$$\sum F_{y'} = 0, \quad -C \sin 70^\circ - 3 \sin 50^\circ - 8 \sin 40^\circ + 16 \sin 40^\circ = 0$$

$$-0.939C = -2.844 \quad \text{----} \rightarrow (4)$$

$$C = 3.03 \text{ kN} \quad \text{and} \quad T = 9.1 \text{ kN}$$

Case 3: using vector force by unit vectors  $i$  and  $j$  in the  $x$  and  $y$  direction

$$\sum F = 0, 8i + T(\cos 40^\circ i + \sin 40^\circ j) - 3j + C(-\cos 70^\circ i + \sin 70^\circ j) - 16i = 0$$

Equating the coefficients of the unit vectors  $i$  and  $j$  terms to zero then

$$i \Rightarrow 8 + T \cos 40^\circ - C \cos 70^\circ - 16 = 0 \quad \rightarrow (5)$$

$$j \Rightarrow T \sin 40^\circ - 3 + C \sin 70^\circ = 0 \quad \rightarrow (6)$$

The solution of equation is

$$T = 9.09kN \text{ and } C = 3.03kN$$

**SP1.20:** Find the tension  $T$  in the cable which supports the 1000 N load with the pulley arrangement as shown in fig.1.57. Each pulley is free to rotate about its bearing and the weights of all parts are small compared with the load. Also determine the magnitude of the total force on the bearing of pulley C.

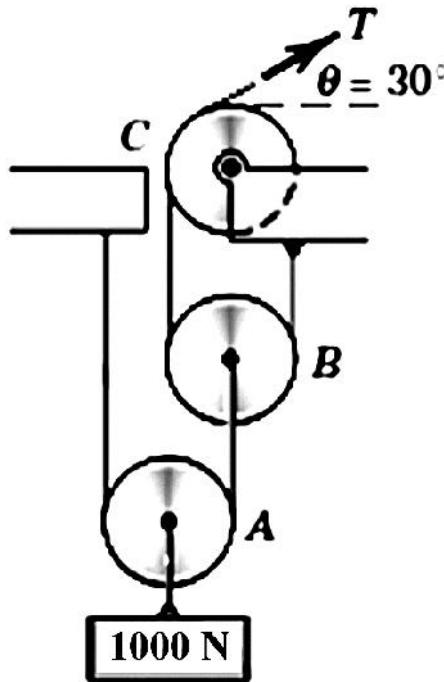
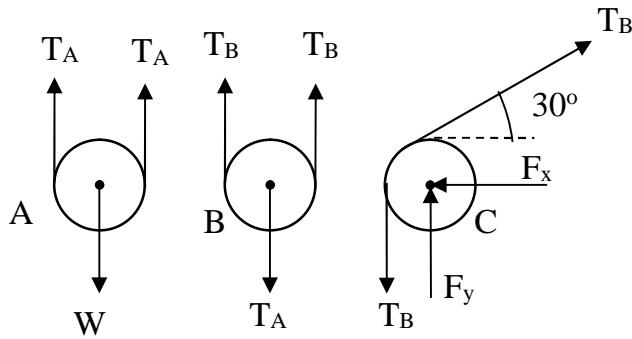


Fig.1.58

**Solution:**

Free body diagram;



Now start to solve the problem from known values that is the pulley A has carrying the load W,

$$\sum F_y^A = 0, 2T_A - 1000 = 0$$

$$T_A = 500N$$

Pulley B,

$$\sum F_y^B = 0, 2T_B - T_A = 0, 2T_B = 500$$

$$T_B = 250$$

Pulley C,

$$\sum F_x^C = 0, T_B \cos 30^\circ - F_x = 0, 250 \cos 30^\circ - F_x = 0, \Rightarrow F_x = 217N$$

$$\sum F_y^C = 0, T_B \sin 30^\circ - T_B + F_y = 0, 250 \sin 30^\circ - 250 + F_y = 0, \Rightarrow F_y = 125N$$

The magnitude of the total force on bearing of pulley C as

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{217^2 + 125^2} = 250N$$

**SP1.21:** Find the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane as shown in fig.1.58. The beam AB is a standard 0.5m I beam with a mass of 95 kg per meter of length.

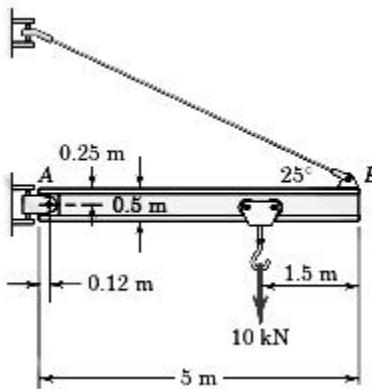
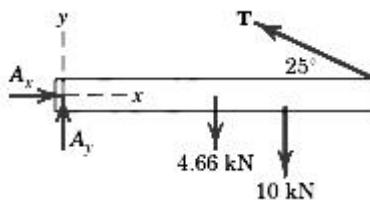


Fig.1.59

**Solution:**

Free body diagram



Taking moment about A,

$$\sum M_A = 0, \Rightarrow 0.25(T \cos 25^\circ) + (5 - 0.12)(T \sin 25^\circ) - (5 - 1.5 - 0.12)10 - (2.5 - 0.12)4.66 = 0$$

$$T = 19.61 \text{ kN}$$

Equating the sums of forces in the x and y directions to zero, then

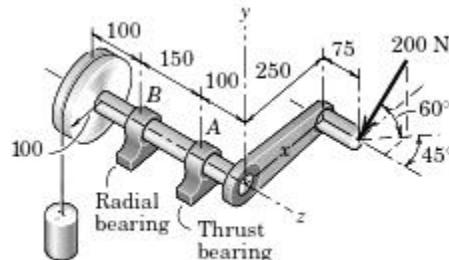
$$\sum F_x = 0, \Rightarrow A_x - 19.61 \cos 25^\circ = 0, \Rightarrow A_x = 17.77 \text{ kN}$$

$$\sum F_y = 0, \Rightarrow A_y + 19.61 \sin 25^\circ - 4.66 = 0, \Rightarrow A_y = 6.37 \text{ kN}$$

The magnitude value at end A,

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{17.77^2 + 6.37^2} = 18.88 \text{ kN}$$

**SP1.22:** A 200 N force is applied to the handle of the hoist in the direction as shown in fig 1.59. The bearing A supports the thrust (force in the direction of the shaft axis) while bearing B supports only radial load (Load normal the shaft axis). Find the mass m which can be supported and the total radial force exerted on the shaft by each bearing. Assume neither bearing to be capable of supporting a moment about a line normal to the shaft axis.

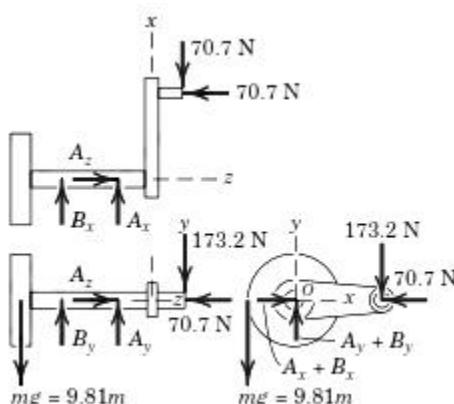


Dimensions in millimeters

Fig.1.60

### Solution:

### Free body diagram,



Taking moment about O, from the x-y projection

$$\sum M_Q = 0, \Rightarrow -100(9.81m) + 250(173.2) = 0, m = 44.1kg$$

Taking moment about A, from the x-Z projection

$$\sum M_A = 0, 150B_r + 175(70.7) - 250(70.7) = 0, B_r = 35.4N$$

$$\sum F_x = 0, \Rightarrow A_x + 35.4 - 70.7 = 0, A_x = 35.4$$

Taking moment about A, from the y-Z projection

$$\sum M_A = 0, 150B_v + 175(173.2) - 250(44.1)(9.81) = 0, B_v = 520N$$

$$\sum F_y = 0, \Rightarrow A_y + 520 - 173.2 - (44.1)(9.81) = 0, A_y = 86.8N$$

$$\sum F_z = 0, \Rightarrow A_z = 70.7N$$

The total radial forces on the bearing become

$$A_r = \sqrt{A_x^2 + A_y^2} = \sqrt{35.4^2 + 86.8^2} = 93.5N$$

$$B_r = \sqrt{B_x^2 + B_y^2} = \sqrt{35.4^2 + 520^2} = 521N$$

**SP1.23:** Find the tension in each cord used to support the 100 kg crate as shown in fig.1.60.

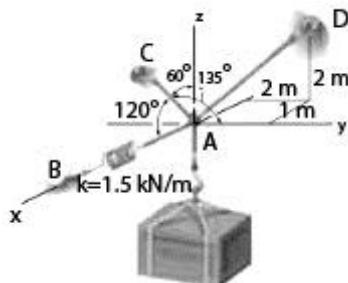
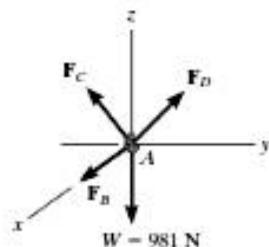


Fig. 1.61

**Solution:**

Free body diagram,



$$F_B = F_B i$$

$$F_C = F_C \cos 120^\circ i + F_C \cos 135^\circ j + F_C \cos 60^\circ k = -0.5F_C i - 0.707F_C j + 0.5F_C k$$

$$F_D = F_D \left( \frac{-1i + 2j + 2k}{\sqrt{(-1)^2 + 2^2 + 2^2}} \right) = -0.33F_D i + 0.66F_D j + 0.667F_D k$$

$$W = -100(9.81) = (-981k)N$$

$$\sum F = 0, \quad F_B + F_C + F_D + W = 0$$

$$F_B i - 0.5F_C i - 0.707F_C j + 0.5F_C k - 0.33F_D i + 0.66F_D j + 0.667F_D k - 981k = 0$$

Equating the unit vectors  $i$ ,  $j$  and  $k$  equal to zero, then

$$\sum F_x = 0, \Rightarrow F_B - 0.5F_C - 0.33F_D = 0 \quad \text{-----} \rightarrow (1)$$

$$\sum F_y = 0, \Rightarrow -0.707F_C + 0.667F_D = 0 \quad \text{-----} \rightarrow (2)$$

$$\sum F_z = 0, \Rightarrow 0.5F_C + 0.667F_D - 981 = 0 \quad \text{-----} \rightarrow (3)$$

Solving the above equations 1, 2 and 3, then

$$F_C = 813N$$

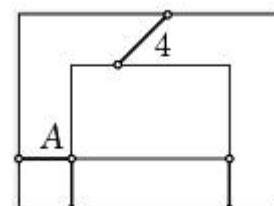
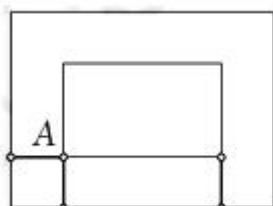
$$F_D = 862N$$

$$F_B = 694N$$

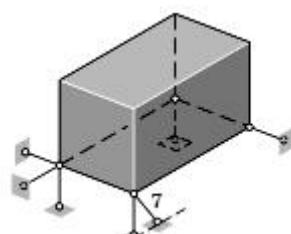
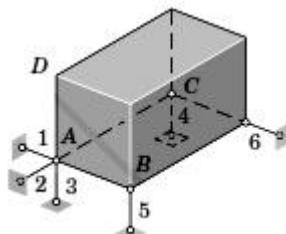
## STATIC INDETERMINACY:

A rigid body or rigid combination of elements treated as a single body, which possesses more external supports or constraints than are necessary to maintain an equilibrium position is called static indeterminacy.

Supports which can be removed without destroying the equilibrium condition of the body are said to be redundant. The number of redundant supporting elements present corresponds to the degree of statically indeterminacy and equals the total number of unknown external forces, minus the number of available independent equations of equilibrium.



Coplanar system



Spatial system

a) Statically determinate

b) Statically indeterminate

Fig.1.62 Constraints in a body

**LECTURE NOTES**

**UNIT – 2**

**2021**

**CEES303-Engineering  
Mechanics**



**Prepared by**

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**Sep-2021**

## Unit - 2

Basic Structural Analysis covering, Equilibrium in three dimensions; Method of Sections; Method of Joints; How to determine if a member is in tension or compression; Simple Trusses; Zero force members; Beams & types of beams; Frames & Machines Centroid and Centre of Gravity covering, Centroid of simple figures from first principle, centroid of composite sections; Centre of Gravity and its implications; Area moment of inertia- Definition, Moment of inertia of plane sections from first principles, Theorems of moment of inertia, Moment of inertia of standard sections and composite sections; Mass moment inertia of circular plate, Cylinder, Cone, Sphere, Hook.

### Reference for the preparation of course material:

1. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
3. J.L. Meriam and L. G. Kraige, Engineering Mechanics –Statics, volume I, Seventh edition, John Wiley & Sons, Inc.

### TEXT BOOKS

1. Irving H. Shames (2006), Engineering Mechanics, 4th Edition, Prentice Hall
2. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill

### REFERENCES

1. R. C. Hibbler (2006), Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.

**Unit - 2****STRUCTURE AND STABILITY OF MECHANICS**

The Engineering Mechanics deals with various forces like internal and external force in different bodies. In the previous section the system of forces were studied to examine the external force acting in a particle or rigid bodies on a smooth surfaces. The internal forces of structure in which the self-weight is neglected is considered to examine in the field of engineering that is forces of action and reaction between the connected members. An engineering structure is any connected system of members built to support or transfer forces and to safely withstand the loads applied to it. To determine the forces internal to an engineering structure, this must dismember the structure and analyse separate free-body diagrams of individual members or combinations of members. This analysis requires careful application of Newton's third law, which states that each action is accompanied by an equal and opposite reaction.

Statically determinate structure is considered to determine all unknown reactions with adequate number of equilibrium equations. In this section the internal forces acting in several types of structures to be analyse – namely trusses, beams and frames and machines.

**HOW TO DETERMINE IF A MEMBER IS IN TENSION OR COMPRESSION;**

The simple truss structure as shown in fig.2.1 is considered to demonstrate whether the member of truss is in tension or compression. In previous chapter, the force on various supports were studied and drawn the free body diagram to states that the system is in equilibrium condition to determine the unknown external forces. Whereas this section discuss about the unknown forces of both internal and external forces.

Fig.2.2 shows that the free body diagram of simple truss structure with unknown external force at the ends of supports. The unknown internal forces on members AB,

BC, CD, DE, EF, FG, GH, HA, BH, CG and DF are to be determined by the method of joints or the method of section. Before that the members to be specified the direction of force acted on each joint by the force of tension or compression in members.

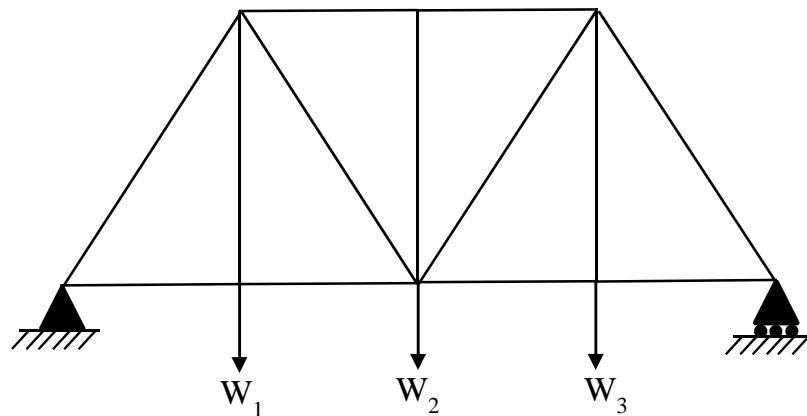


Fig.2.1 Simple truss element

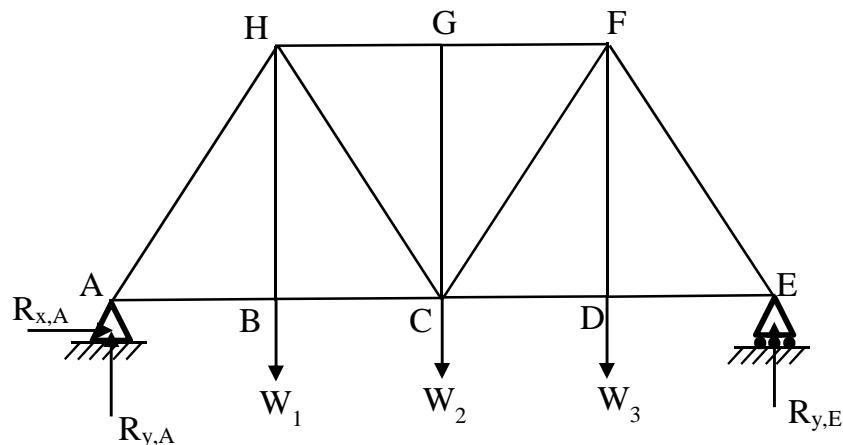
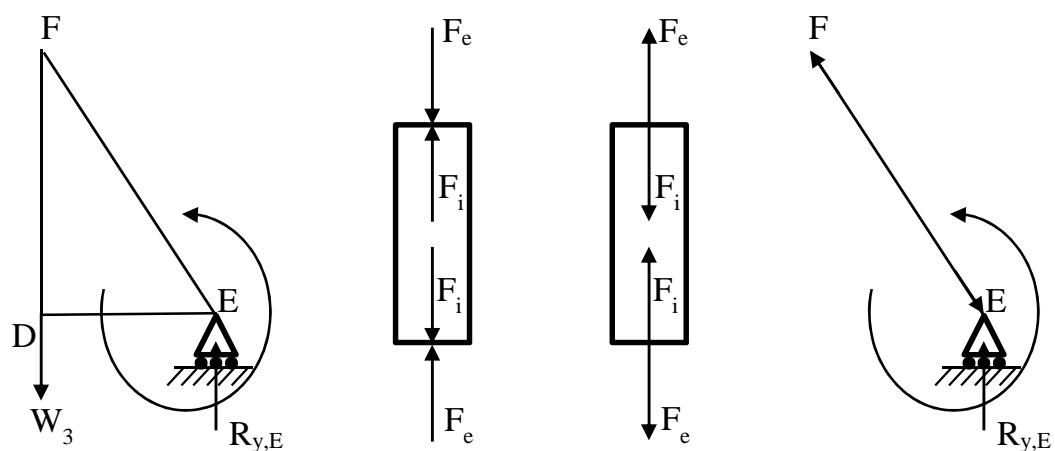


Fig.2.2 Free body diagram of simple truss



- a) Moment force at end E      b) Compressive force in a member      c) Tension in a member      d) Internal force direction

Fig.2.3 Forces at supports and members

The reaction force at end E takes moment about A and have rotation as shown in fig.2.3 (a). The moment force by reaction force at end E applies the compressive force on member EF. Therefore the compressive force by external ( $F_e$ ) creates an internal force ( $F_i$ ) which towards the joint of member as shown in fig.2.3 (b). However the member EF has compression due the moment force in counter clockwise at end E as shown in fig.2.3 (c).

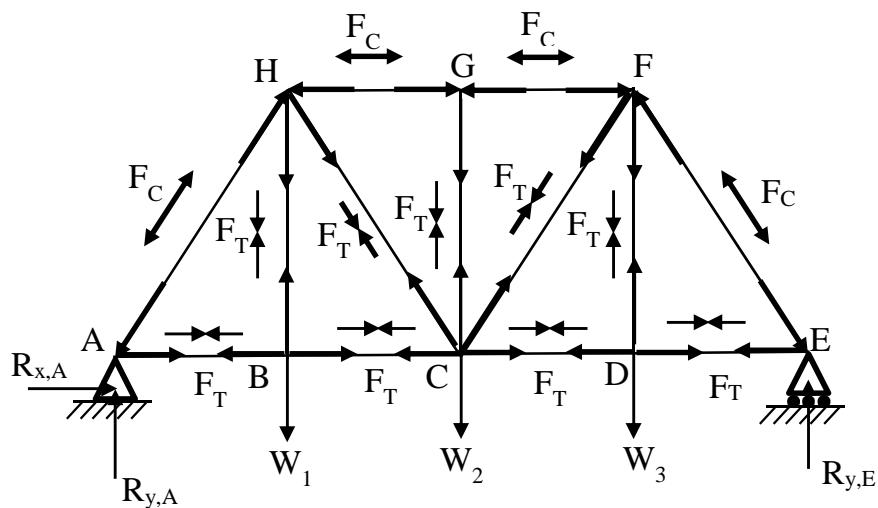


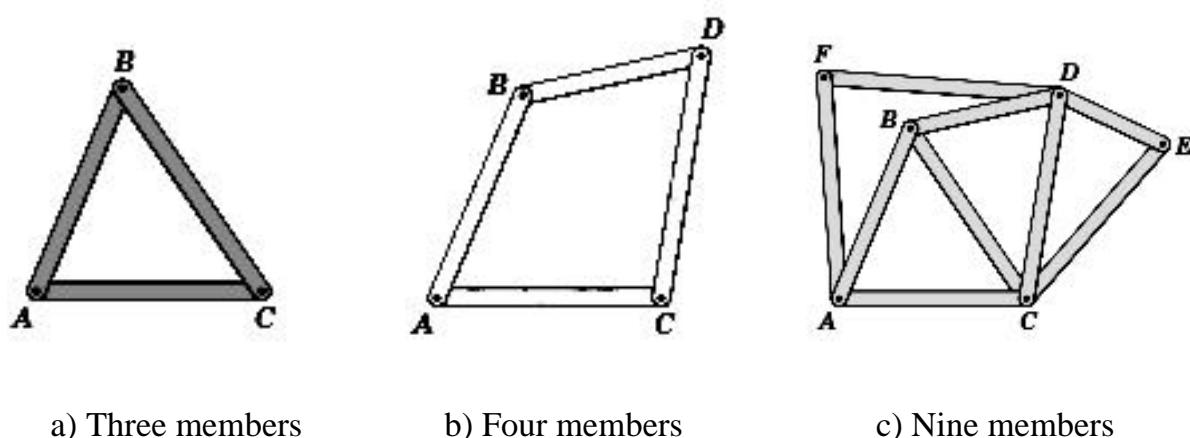
Fig.2.4 Representation of tension or compression force in a members

The moment force at end E by taking moment about A generates the compression in a member EF. This direction is noted as towards the joint E and F. Similarly the moment force at end A by taking moment about B takes the compression in a member AH and denote towards the joint A and H. The compression of member AH and EF also implies into the member HG and FG therefor it denoted towards the joint H, G and F accordingly. The member AB is pulled by the load at joint B as  $W_1$ . This will generates tension in a member AB and therefore the direction is denoted away from the joint. Similarly the member BC, CD, DE, BH, CH, CG, CF and DF have tension in a member by the load  $W_1$ ,  $W_2$  and  $W_3$  accordingly. The tension and compression in a members of simple truss structure is shown fig.2.4.

## SIMPLE TRUSSES:

The basic element of a plane truss is the triangle. Three bars joined by pins at their ends, Fig.2.5 (a), constitute a rigid frame. The term *rigid* is used to mean non collapsible and also to mean that deformation of the members due to induced internal strains is negligible. On the other hand, four or more bars pin-jointed to form a polygon of as many sides constitute a non-rigid frame. We can make the non-rigid frame in fig.2.5 (b) rigid, or stable, by adding a diagonal bar joining A and D or Band C and thereby forming two triangles. We can extend the structure by adding additional units of two end-connected bars, such as *DE* and *CE* or *AF* and *DF*, Fig.2.5(c), which are pinned to two fixed joints. In this way the entire structure will remain rigid.

Structures built from a basic triangle in the manner described are known as *simple trusses*. When more members are present than are needed to prevent collapse, the truss is statically indeterminate. A statically indeterminate truss cannot be analysed by the equations of equilibrium alone. Additional members or supports which are not necessary for maintaining the equilibrium configuration are called *redundant*.



a) Three members

b) Four members

c) Nine members

Fig.2.5 Simple truss members

To design a truss we must first determine the forces in the various members and then select appropriate sizes and structural shapes to withstand the forces. Several assumptions are made in the force analysis of simple trusses. First, we assume all members to be *two-force members*. A two-force member is one in equilibrium under the action of two forces only, as defined in general terms with Fig. 2.6. Each member

of a truss is normally a straight link joining the two points of application of force. The two forces are applied at the ends of the member and are necessarily equal, opposite, and *collinear* for equilibrium.

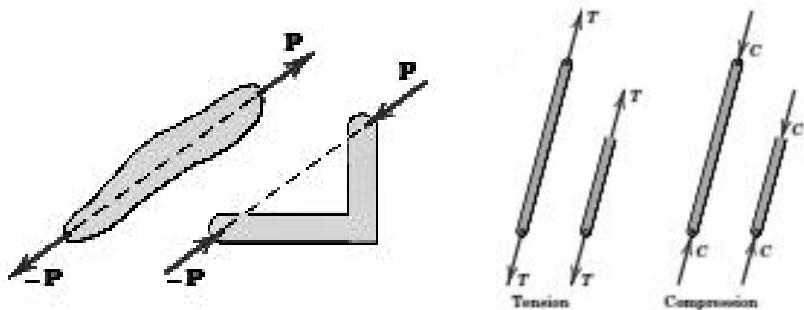


Fig.2.6 Two force members

The member may be in tension or compression, as shown in Fig.2.6. When we represent the equilibrium of a portion of a two-force member, the tension  $T$  or compression  $C$  acting on the cut section is the same for all sections. We assume here that the weight of the member is small compared with the force it supports. If it is not, or if we must account for the small effect of the weight, we can replace the weight  $W$  of the member by two forces, each  $W/2$  if the member is uniform, with one force acting at each end of the member. These forces, in effect, are treated as loads externally applied to the pin connections. Accounting for the weight of a member in this way gives the correct result for the average tension or compression along the member but will not account for the effect of bending of the member.

## **ZERO FORCE MEMBERS;**

Truss analysis using the method of joints is greatly simplified if we can first identify those members which support *no loading*. These *zero-force members* are used to increase the stability of the truss during construction and to provide added support if the loading is changed. The zero-force members of a truss can generally be found by *inspection* of each of the joints. For example, consider the truss shown in Fig.2.7 (a). If a free-body diagram of the pin at joint  $A$  is drawn, Fig.2.7 (b), it is seen that members  $AB$  and  $AF$  are zero-force members. (We could not have come to this conclusion if we had considered the free-body diagrams of joints  $F$  or  $B$  simply because there are five

unknowns at each of these joints.) In a similar manner, consider the free-body diagram of joint *D*, Fig.2.7 (c). Here again it is seen that *DC* and *DE* are zero-force members. From these observations, we can conclude that *if only two non-collinear members form a truss joint and no external load or support reaction is applied to the joint, the two members must be zero-force members*. The load on the truss in Fig.2.7 (a) is therefore supported by only five members as shown in Fig.2.7 (d).

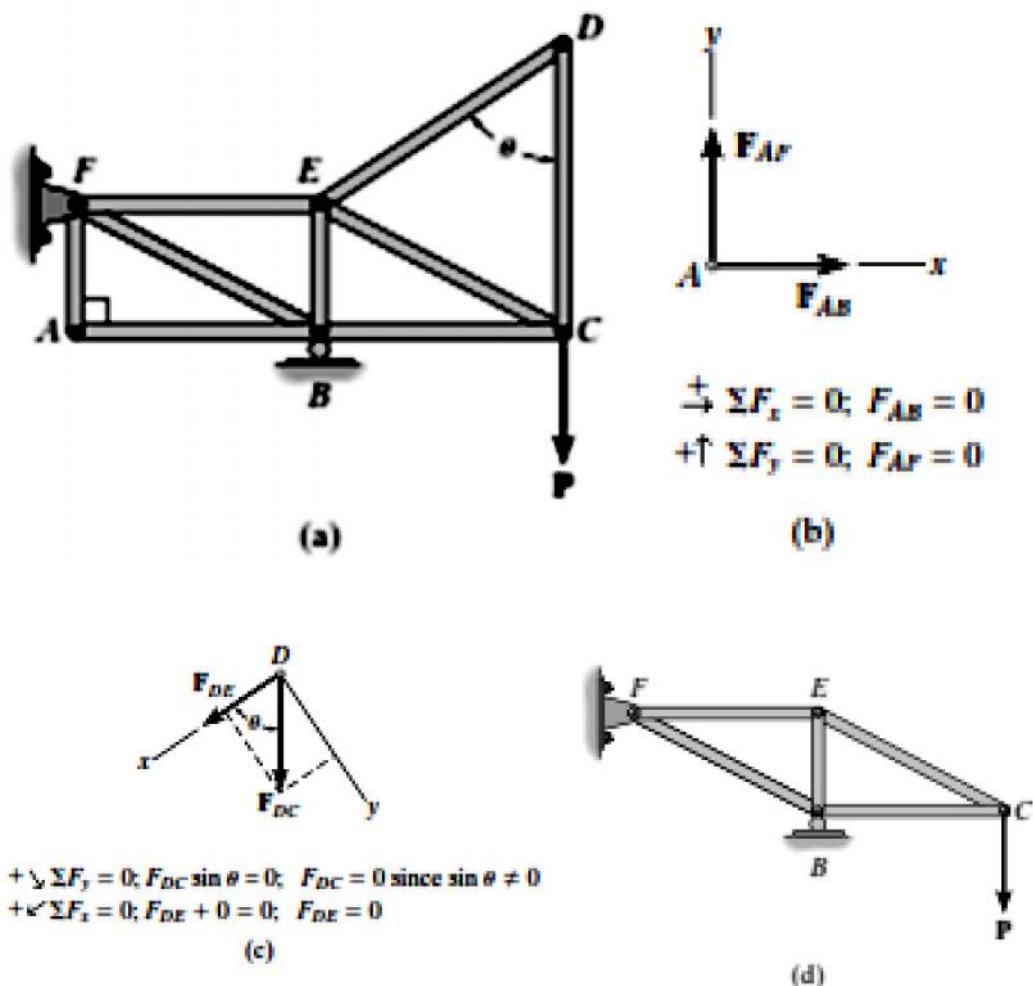


Fig. 2.7 Truss and zero force members

Now consider the truss shown in Fig.2.8 (a). The free-body diagram of the pin at joint *D* is shown in Fig.2.8 (b). By orienting the *y* axis along members *DC* and *DE* and the *x* axis along member *DA*, it is seen that *DA* is a zero-force member. Note that this is also the case for member *CA*, Fig.2.8 (c). In general then, *if three members form a truss joint for which two of the members are collinear, the third member is a zero-force member*.

force member provided no external force or support reaction has a component that acts along this member. The truss shown in Fig.2.8 (d) is therefore suitable for supporting the load  $\mathbf{P}$ .

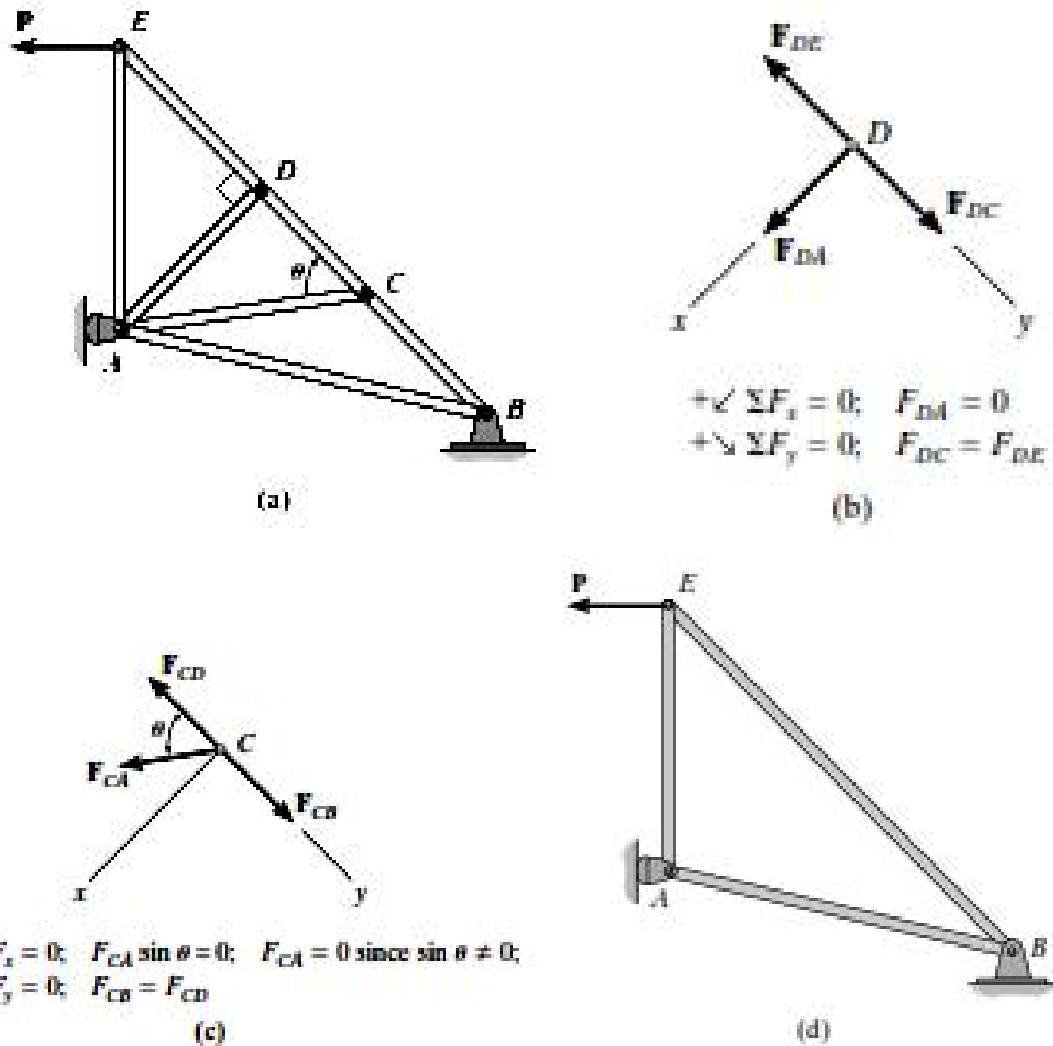


Fig. 2.8 Truss and their forces

### INTERNAL AND EXTERNAL REDUNDANCY:

If a plane truss has more external supports than are necessary to ensure a stable equilibrium conditions, then the truss is said to be statically indeterminate. The extra supports in that plane truss constitute external redundancy force.

If a truss has more internal members than are necessary to prevent collapse when the truss is removed from its supports then the extra members constitute internal redundancy and the truss is said to be statically indeterminate.

For a truss which is statically determinate externally there is a definite relation between the number of its members and the number of its joints necessary for internal stability without redundancy. The equilibrium of each truss joint is specified by two scalar force equations such that all joints have the equation as  $2J$ .

The entire truss is composed of ' $m$ ' number of two force members ( $m$  is a tension or compression forces) and having the maximum of three unknown support reaction which equals the twice of joint in a truss. Then the relation is stated as statically determinate as

$$m + 3 = 2J$$

If  $m + 3 > 2J$  then the truss is statically indeterminate which means more number of members present in that truss.

If  $m + 3 < 2J$  then the truss is not sufficient to hold the stable position thereby the truss tends to collapse under load.

## THE METHOD OF JOINTS:

In order to analyse or design a truss, it is necessary to determine the force in each of its members. One way to do this is to use the *method of joints*. This method is based on the fact that if the entire truss is in equilibrium, then each of its joints is also in equilibrium. Therefore, if the free-body diagram of each joint is drawn, the force equilibrium equations can then be used to obtain the member forces acting on each joint. Since the members of a *plane truss* are straight two-force members lying in a single plane, each joint is subjected to a force system that is *coplanar and concurrent*. As a result, only  $\sum F_x = 0$  and  $\sum F_y = 0$  need to be satisfied for equilibrium.

For example, consider the pin at joint *B* of the truss in Fig.2.9 (a). Three forces act on the pin, namely, the 500-N force and the forces exerted by members *BA* and *BC*. The free-body diagram of the pin is shown in Fig.2.9 (b). Here,  $\mathbf{F}_{BA}$  is “pulling” on the pin, which means that member *BA* is in *tension*; whereas  $\mathbf{F}_{BC}$  is “pushing” on the pin, and consequently member *BC* is in *compression*. These effects are clearly demonstrated by isolating the joint with small segments of the member connected to the pin, Fig.2.9 (c). The pushing or pulling on these small segments indicates the effect of the member being either in compression or tension.

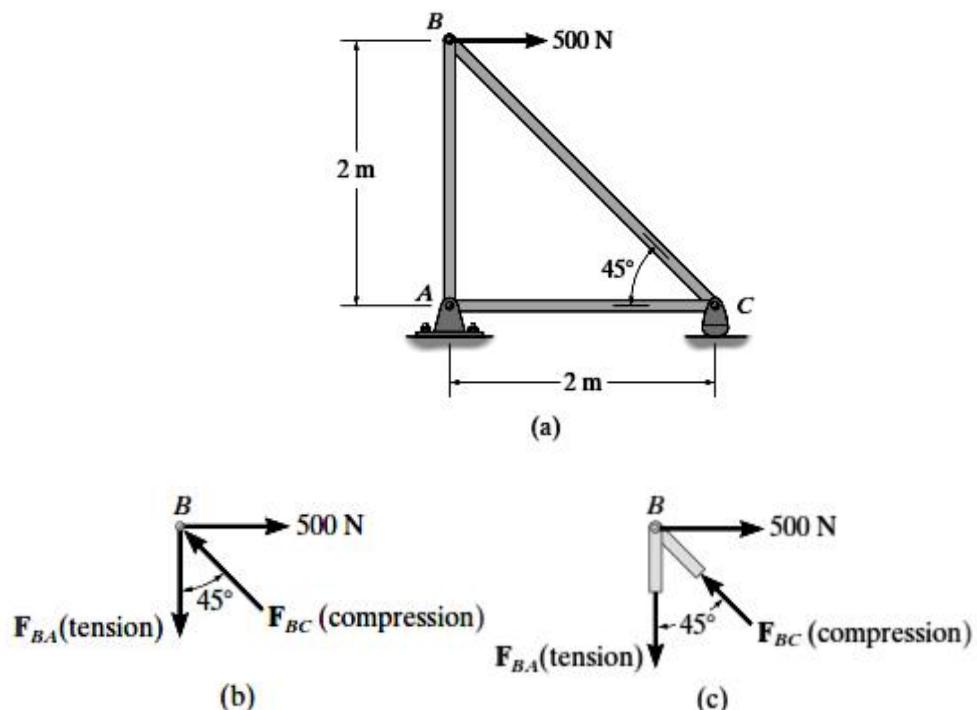


Fig.2.9 Method of joints

When using the method of joints, always start at a joint having at least one known force and at most two unknown forces, as in Fig.2.9 (b). In this way, application of  $\sum F_x = 0$  and  $\sum F_y = 0$  yields two algebraic equations which can be solved for the two unknowns. When applying these equations, the correct sense of an unknown member force can be determined using one of two possible methods.

**Method 1:** The *correct* sense of direction of an unknown member force can, in many cases, be determined “by inspection.” For example,  $\mathbf{F}_{BC}$  in fig.2.9 (b) must push on the pin (compression) since its horizontal component,  $F_{BC} \sin 45^\circ$ , must balance the 500 N

force ( $\sum F_x = 0$ ). Likewise,  $\mathbf{F}_{AB}$  is a tensile force since it balances the vertical component,  $F_{BC} \cos 45^\circ$  ( $\sum F_y = 0$ ). In more complicated cases, the sense of an unknown member force can be *assumed*; then, after applying the equilibrium equations, the assumed sense can be verified from the numerical results. A *positive* answer indicates that the sense is *correct*, whereas a *negative* answer indicates that the sense shown on the free-body diagram must be *reversed*.

**Method 2:** Always assume the *unknown member forces* acting on the joint's free-body diagram to be in *tension*; i.e., the forces "pull" on the pin. If this is done, then numerical solution of the equilibrium equations will yield *positive scalars for members in tension and negative scalars for members in compression*. Once an unknown member force is found, use its *correct* magnitude and sense (T or C) on subsequent joint free-body diagrams.

### Procedure using the method of joints:

- 1) Draw the free-body diagram of a joint having at least one known force and at most two unknown forces. (If this joint is at one of the supports, then it may be necessary first to calculate the external reactions at the support.)
- 2) Use one of the two methods described above for establishing the sense of an unknown force.
- 3) Orient the  $x$  and  $y$  axes such that the forces on the free-body diagram can be easily resolved into their  $x$  and  $y$  components and then apply the two force equilibrium equations  $\sum F_x = 0$  and  $\sum F_y = 0$ . Solve for the two unknown member forces and verify their correct sense.
- 4) Using the calculated results, continue to analyse each of the other joints. Remember that a member in *compression* "pushes" on the joint and a member in *tension* "pulls" on the joint. Also, be sure to choose a joint having at most two unknowns and at least one known force.

## Problems:

**SP2.1:** Determine the force in each member of the truss shown in Fig.2.10. Indicate whether the members are in tension or compression.

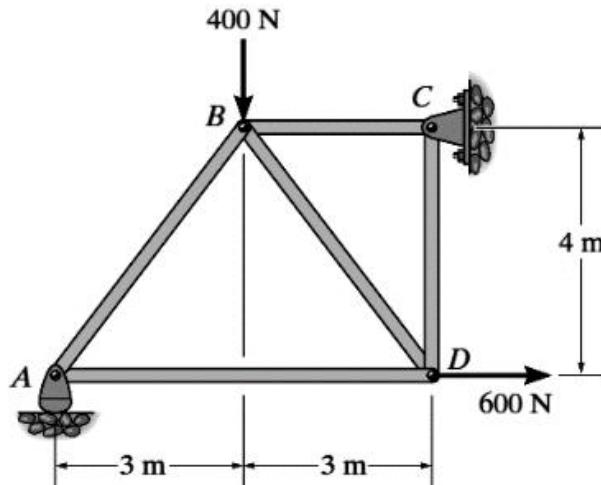
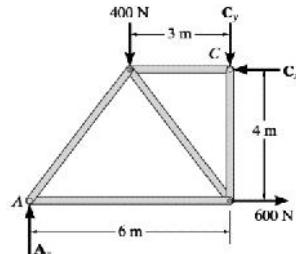


Fig.2.10 Truss

### Solution:

A free body diagram of the entire truss is draw as below



Applying the equations of equilibrium as

$$\sum F_x = o, \Rightarrow 600 - C_x = 0,$$

$$C_x = 600N$$

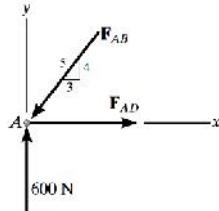
$$\sum M_C = 0, \Rightarrow -A_y(6) + 400(3) + 600(4) = 0$$

$$A_v = 600N$$

$$\sum F_y = o, \Rightarrow 600 - 400 - C_y = 0,$$

$$C_y = 200\text{N}$$

The analysis can now start at either joint *A* or *C*. The choice is arbitrary since there are one known and two unknown member forces acting on the pin at each of these joints.



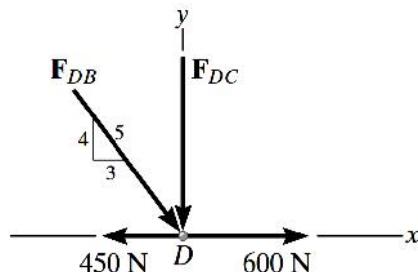
From the free-body diagram of joint *A*,  $\mathbf{F}_{AB}$  is assumed to be compressive and  $\mathbf{F}_{AD}$  is tensile. Applying the equations of equilibrium, then

$$\sum F_y = 0, \Rightarrow 600 - \frac{4}{5} F_{AB} = 0,$$

$$F_{AB} = 750\text{N (C)}$$

$$\sum F_x = 0, \Rightarrow F_{AD} - \frac{3}{5} F_{AB} = 0, \Rightarrow F_{AD} - \frac{3}{5}(750) = 0$$

$$F_{AD} = 450\text{N (T)}$$



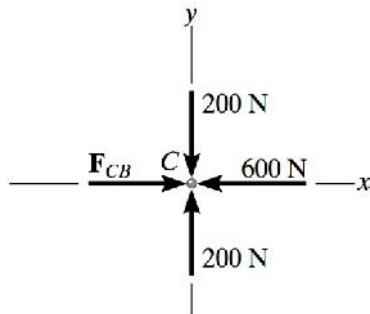
From the free-body diagram of joint *D* and the result of  $F_{AD}$ , then

$$\sum F_x = 0, \Rightarrow -F_{AD} + \frac{3}{5} F_{DB} + 600 = 0, \Rightarrow -450 + \frac{3}{5} F_{DB} + 600 = 0$$

$$F_{DB} = -250\text{N (T)}$$

$$\sum F_y = 0, \Rightarrow -F_{DC} - \frac{4}{5} F_{DB} = 0, \Rightarrow -F_{DC} - \frac{4}{5}(-250) = 0$$

$$F_{DC} = 200\text{N (c)}$$



From the free-body diagram of joint D, then

$$\sum F_x = 0, \Rightarrow F_{CB} - 600 = 0$$

$$F_{CB} = 600\text{N (C)}$$

$$\sum F_y = 0, \Rightarrow F_{DC} - C_y = 0, \Rightarrow 200 - 200 = 0$$

The above equation is used to check the solution.

**SP2.2:** Using the method of joints, determine all the zero-force members of the *Fink roof truss* shown in Fig.2.11. Assume all joints are pin connected.

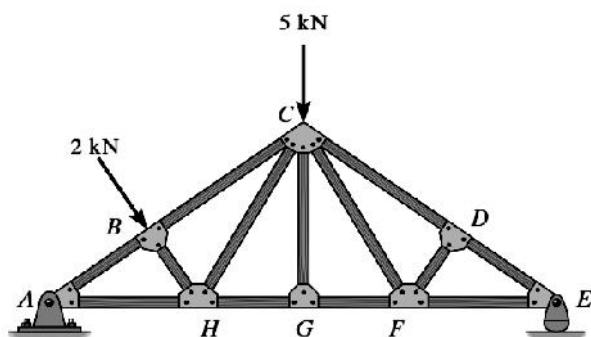
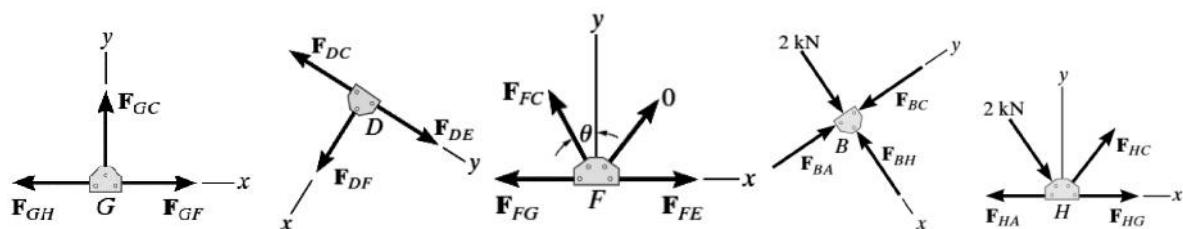


Fig.2.11 Fink roof truss

**Solution:**



Look for joint geometries that have three members for which two are collinear.

Now take joint G, then

$$\sum F_y = 0, F_{GC} = 0$$

By considered joint C, it not conclude that the member GC is a zero force member because there are five unknown force at joint C. But GC is a zero force member by means that 5 kN load at C must be supported by the members CB, CH, CF and CD.

Now take joint D, then

$$\sum R = \sum F_x + \sum F_y = 0, F_{CD} = 0$$

Joint F, then

$$\sum F_y = 0, \Rightarrow F_{FC} \cos \theta = 0, \text{ Since } \theta \neq 90^\circ, F_{FC} = 0$$

Joint B, Here the coordinates of x and y are taken in the direction of the members BH and AB respectively. Then

$$\sum F_x = 0, \Rightarrow 2 - F_{BH} = 0, F_{BH} = 2 \text{ kN}$$

Joint H, by taking vertical components

$$\sum F_y = 0, -2 \sin \theta + F_{HC} \sin \theta = 0,$$

$$\text{Therefore } F_{HC} \neq 0$$

## THE METHOD OF SECTIONS:

When we need to find the force in only a few members of a truss, we can analyse the truss using the ***method of sections***. It is based on the principle that if the truss is in equilibrium then any segment of the truss is also in equilibrium. For example, consider the two truss members shown on the left in Fig.2.12. If the forces within the members are to be determined, then an imaginary section, indicated by the blue line, can be used to cut each member into two parts and thereby “expose” each internal force as “external” to the free-body diagrams shown on the right. Clearly, it can be seen that

equilibrium requires that the member in tension (T) be subjected to a “pull,” whereas the member in compression (C) is subjected to a “push”.

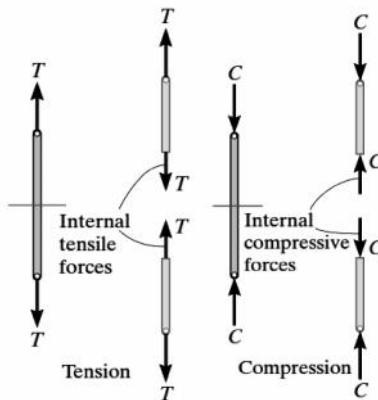


Fig.2.12 Internal forces in members

The method of sections can also be used to “cut” or section the members of an entire truss. If the section passes through the truss and the free-body diagram of either of its two parts is drawn, we can then apply the equations of equilibrium to that part to determine the member forces at the “cut section.” Since only *three* independent equilibrium equations ( $\sum F_x = 0$ ,  $\sum F_y = 0$  and  $\sum M_o = 0$ ) can be applied to the free-body diagram of any segment, then we should try to select a section that, in general, passes through not more than *three* members in which the forces are unknown.

For example, consider the truss in Fig.2.13 (a). If the forces in members *BC*, *GC*, and *GF* are to be determined, then section *aa* would be appropriate. The free-body diagrams of the two segments are shown in Figs.2.13 (b) and (c). Note that the line of action of each member force is specified from the *geometry* of the truss, since the force in a member is along its axis. Also, the member forces acting on one part of the truss are equal but opposite of those acting on the other part—Newton’s third law. Members *BC* and *GC* are assumed to be in *tension* since they are subjected to a “pull,” whereas *GF* in *compression* since it is subjected to a “push.”

The three unknown member forces  $\mathbf{F}_{BC}$ ,  $\mathbf{F}_{GC}$  and  $\mathbf{F}_{GF}$  can be obtained by applying the three equilibrium equations to the free-body diagram in Fig.2.13 (b). If, however,

the free-body diagram in Fig.2.13 (c) is considered, the three support reactions  $\mathbf{D}_x$ ,  $\mathbf{D}_y$  and  $\mathbf{E}_x$  will have to be known, because only three equations of equilibrium are available.

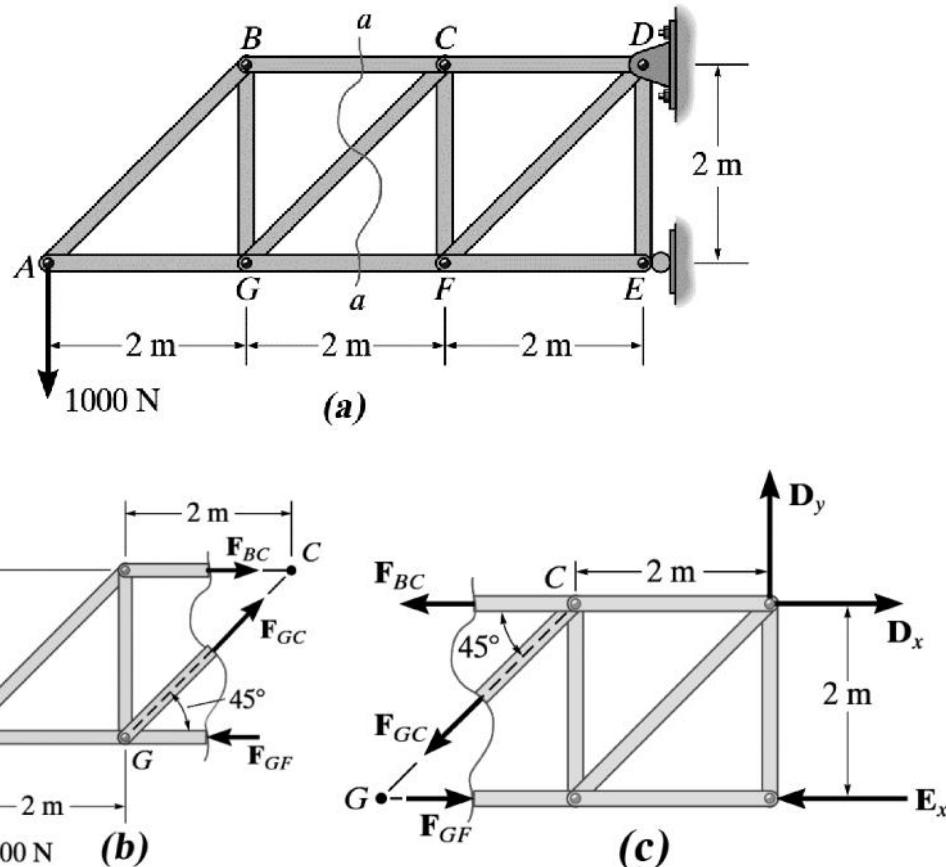


Fig. 2.13 Truss

When applying the equilibrium equations, we should carefully consider ways of writing the equations so as to yield a *direct solution* for each of the unknowns, rather than having to solve simultaneous equations. For example, using the truss segment in Fig.2.13 (b) and summing moments about C would yield a direct solution for  $\mathbf{F}_{GF}$ . Since  $\mathbf{F}_{BC}$  and  $\mathbf{F}_{GC}$  create zero moment about C. Likewise,  $\mathbf{F}_{BC}$  can be directly obtained by summing moments about G. Finally,  $\mathbf{F}_{GC}$  can be found directly from a force summation in the vertical direction since  $\mathbf{F}_{GF}$  and  $\mathbf{F}_{BC}$  have no vertical components. This ability to *determine directly* the force in a particular truss member is one of the main advantages of using the method of sections. **Suppose the method of joints were used to determine say the force in member GC it would be necessary to analyse joints A, B and G in sequence.** As in the method of joints, in this method of section also there are two ways in which we can determine the correct sense of an unknown member force:

### Procedure using the method of section:

- 1) Free-Body Diagram:
  - a) Make a decision on how to “cut” or section the truss through the members where forces are to be determined.
  - b) Before isolating the appropriate section, it may first be necessary to determine the truss’s support reactions. If this is done then the three equilibrium equations will be available to solve for member forces at the section.
  - c) Draw the free-body diagram of that segment of the sectioned truss which has the least number of forces acting on it.
  - d) Use one of the two methods described above for establishing the sense of the unknown member forces.
- 2) Equations of Equilibrium:
  - a) Moments should be summed about a point that lies at the intersection of the lines of action of two unknown forces, so that the third unknown force can be determined directly from the moment equation.
  - b) If two of the unknown forces are *parallel*, forces may be summed *perpendicular* to the direction of these unknowns to determine *directly* the third unknown force.

**SP2.3:** Find the forces included in members KL, CL and BC by the 20 kN load on the cantilever truss as shown in fig.2.14.

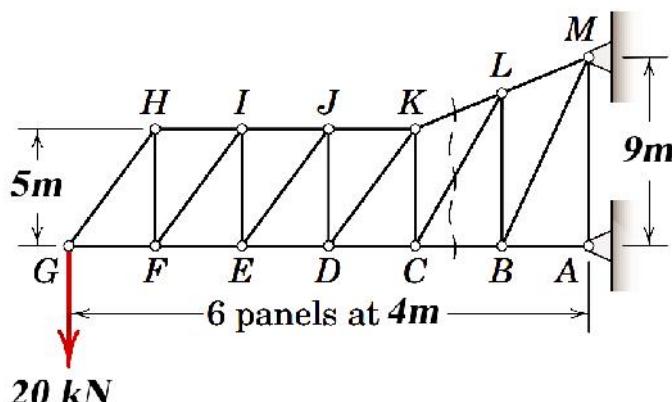
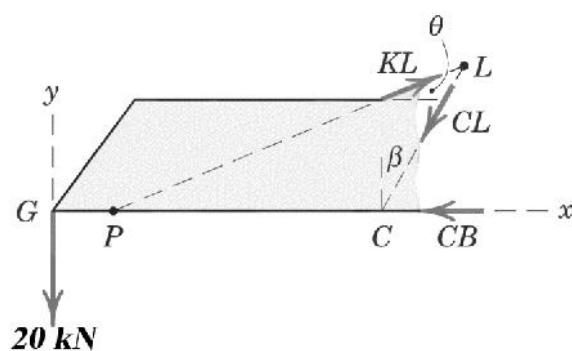


Fig. 2.14 Cantilever truss

### Solution:

The vertical components of the reaction at A and M are statically indeterminate with the two fixed supports but all members other than A and M are statically determinate.

From the given problem the force at KL, CL and CB is to find thereby the method of joint is not suitable. This is due to the method of joints need many number of joints as 13 to determine these force members. So the method of section simplifies to find the particular force of member by passing the section through KL, CL and CB. The free body diagram of left portion truss with statically determinate rigid body as below.



The sum of moment about L quickly verifies the assignment of CB as compression and the sum of moment about C quickly disclose that KL is in tension. The direction of CL is not quite so obvious until it observe that KL and CB intersect at a point P to the right of G. The sum of moment about P eliminates reference to KL and CB which shows that CL must be compressive to balance the moment of the 20 kN force about P.

Now take moment about L and summing to equilibrium condition, then

The distance from B to G is five times of 4m i.e.  $d_{BG}$  is  $5 \times 4 = 20$ m and L to B is find from the geometry as  $HF + ((MA - HF)/2)$  i.e. BL is  $5 + ((9 - 5)/2) = 7$ m.

$$\sum M_L = 0, \quad \Rightarrow 20(d_{BG}) - F_{CB}(d_{CB}) = 0, \Rightarrow 20(20) - F_{CB}(7) = 0$$

$$F_{CB} = 57.14\text{N}$$

Next, take moment about C or P and summing to equilibrium condition, then

The distance from C to G is four times of 4m i.e.  $d_{CG}$  is  $4 \times 4 = 16$ m and C to K is 5m. The horizontal force at K to L is determined as  $(F_{KL})_x = F_{KL} \cos \theta$ , therefore  $\theta$  is to determine from the geometry of KLB' (B' is a point which obtains by line intersecting from K and B). The height of B'L is 2m. Therefore  $\cos \theta$  is  $\frac{4}{\sqrt{4^2 + 2^2}} = \frac{16}{20}$

$$\sum M_C = 0, \Rightarrow 20(d_{CG}) - F_{KL} \cos \theta (d_{CK}) = 0, \Rightarrow 20(16) - F_{KL} \frac{16}{20} (5) = 0$$

$$F_{KL} = 71.1 \text{N}$$

The distance from C to P is find from the similarity of geometry as KMA' (A' is a point which obtains by the line intersecting from K and A) to PKC. Therefore

$$\frac{d_{PC}}{d_{CK}} = \frac{d_{KA'}}{d_{MA'}}, \Rightarrow \frac{d_{PC}}{5} = \frac{8}{(9-5)}, d_{PC} = 5 \left( \frac{8}{4} \right) = 10 \text{m}$$

$$\text{The distance } d_{PG} = (d_{CG} - d_{PC}) = 16 - 10 = 6 \text{m}$$

The vertical force at C to L is determined as  $(F_{CL})_y = F_{CL} \cos \beta$ , therefore  $\beta$  is to determine from the geometry of CBL. Therefore  $\beta = 90 - \angle BCL$ .

$$\tan \angle BCL = \frac{d_{BL}}{d_{CB}} = \frac{5+2}{4}$$

$$\angle BCL = \tan^{-1} \frac{7}{4} = 60.26^\circ$$

$$\beta = 90 - 60.26 = 29.74^\circ$$

Taking moment about P and summing to equilibrium conditions, then

$$\sum M_P = 0, \Rightarrow 20(d_{PG}) - (F_{CL})_y (d_{PC}) = 0, \Rightarrow 20(6) - (F_{CL} \cos \beta) (10) = 0$$

$$\Rightarrow 20(6) - (F_{CL} \cos 29.74) (10) = 0$$

$$F_{CL} = 13.82 \text{N}$$

**SP2.4:** Determine the force in members *GE*, *GC*, and *BC* of the truss shown in Fig. 2.15. Indicate whether the members are in tension or compression.

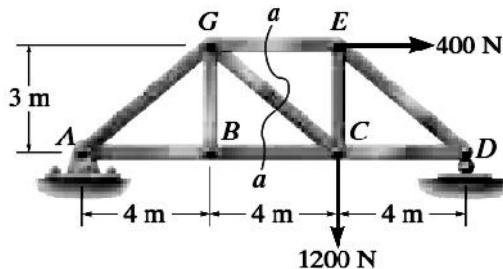
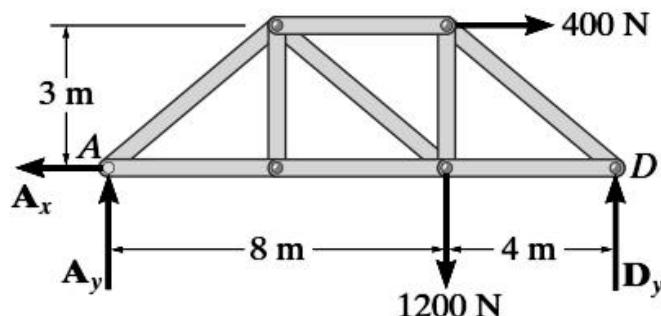


Fig.2.15 simple truss

**Solution:**



Section *aa* in Fig.2.15 has been chosen since it cuts through the *three* members whose forces are to be determined. In order to use the method of sections, however, it is *first* necessary to determine the external reactions at *A* or *D*. Why? A free-body diagram of the entire truss is shown in above figure. Applying the equations of equilibrium, we have

$$\sum F_x = 0, \Rightarrow -F_{Ax} + 400 = 0,$$

$$F_{Ax} = 400\text{N}$$

$$\sum M_A = 0, \Rightarrow -1200(d_{AC}) - 400(d_{CE}) + F_{Dy}(d_{AD}) = 0$$

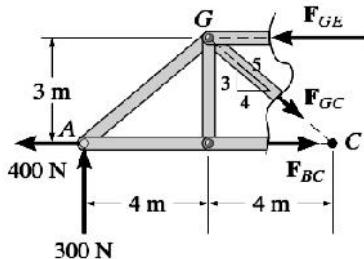
$$\Rightarrow -1200(8) - 400(3) + F_{Dy}(12) = 0$$

$$F_{Dy} = 900\text{N}$$

$$\sum F_y = 0, \Rightarrow F_{Ay} - 1200 + 900 = 0$$

$$F_{Ay} = 300\text{N}$$

Free-Body Diagram:



For the analysis the free-body diagram of the left portion of the sectioned truss will be used, since it involves the least number of forces.,.

Equations of Equilibrium:

Summing moments about point G eliminates  $\mathbf{F}_{GE}$  and  $\mathbf{F}_{GC}$  and yields a direct solution for  $\mathbf{F}_{BC}$ .

$$\sum M_G = 0, \Rightarrow -F_{Ay}(d_{BA}) - F_{Ax}(d_{GB}) + F_{BC}(d_{GB}) = 0$$

$$\Rightarrow -300(4) - 400(3) + F_{BC}(3) = 0$$

$$F_{BC} = 800\text{N}$$

In same manner by summing moments about point C, then

$$\sum M_C = 0, \Rightarrow -F_{Ay}(d_{BA}) + F_{GE}(d_{GB}) = 0$$

$$\Rightarrow -300(8) + F_{GE}(3) = 0$$

$$F_{GE} = 800\text{N}$$

Since  $\mathbf{F}_{BC}$  and  $\mathbf{F}_{GE}$  have no vertical components summing forces in the y direction directly yields  $\mathbf{F}_{GC}$

$$\sum F_y = 0, \Rightarrow F_{Ay} - F_{GC}\left(\frac{3}{5}\right) = 0, \Rightarrow 300 - F_{GC}\left(\frac{3}{5}\right) = 0$$

$$F_{GC} = 500\text{N}$$

**NOTE:** Here it is possible to tell, by inspection, the proper direction for each unknown member force. For example,  $\sum M_C = 0$  requires  $F_{GE}$  to be *compressive* because it must balance the moment of the 300-N force about  $C$ .

## SPACE TRUSSES (EQUILIBRIUM IN THREE DIMENSIONS):

A *space truss* consists of members joined together at their ends to form a stable three-dimensional structure. The simplest form of a space truss is a **tetrahedron**, formed by connecting six members together, as shown in Fig.2.16. Any additional members added to this basic element would be redundant in supporting the force  $\mathbf{P}$ . A *simple space truss* can be built from this basic tetrahedral element by adding three additional members and a joint, and continuing in this manner to form a system of multi-connected tetrahedrons.

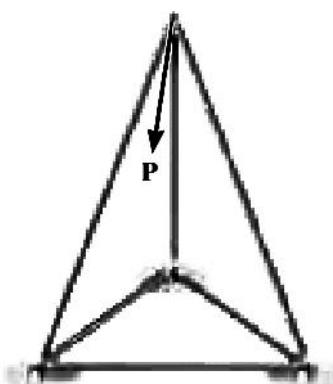


Fig.2.16 Simple space truss

If the truss is statically determinate internally then the equation for any space truss is satisfied as

$$m + 6 = 3J$$

If  $m + 6 > 3J$  then the truss is statically indeterminate which means more number of members present in that truss.

If  $m + 6 < 3J$  then the truss is not sufficient to hold the stable position thereby the truss tends to collapse under load.

### Assumptions for Design:

The members of a space truss may be treated as two-force members provided the external loading is applied at the joints and the joints consist of ball-and-socket connections. These assumptions are justified if the welded or bolted connections of the joined members intersect at a common point and the weight of the members can be neglected. In cases where the weight of a member is to be included in the analysis, it is generally satisfactory to apply it as a vertical force, half of its magnitude applied at each end of the member.

### Procedure for analysis of Space truss by:

#### Method of joints:

If the forces in *all* the members of truss are to be determined, then the method of joints is most suitable for the analysis. Here it is necessary to apply the three equilibrium equations  $\sum F_x = 0$ ,  $\sum F_y = 0$  and  $\sum F_z = 0$  to the forces acting at each joint. Remember that the solution of many simultaneous equations can be avoided if the force analysis begins at a joint having at least one known force and at most three unknown forces. Also, if the three-dimensional geometry of the force system at the joint is hard to visualize, it is recommended that a Cartesian vector analysis be used for the solution.

#### Method of Sections:

If only a *few* member forces are to be determined, the method of sections can be used. When an imaginary section is passed through a truss and the truss is separated into two parts, the force system acting on one of the segments must satisfy the *six* equilibrium equations:  $\sum F_x = 0$ ,  $\sum F_y = 0$ ,  $\sum F_z = 0$ ,  $\sum M_x = 0$ ,  $\sum M_y = 0$  and  $\sum M_z = 0$ . By proper choice of the section and axes for summing forces and moments, many of the unknown member forces in a space truss can be computed *directly*, using a single equilibrium equation.

**SP2.5:** Determine the forces acting in the members of the space truss shown in Fig.2.17. Indicate whether the members are in tension or compression.

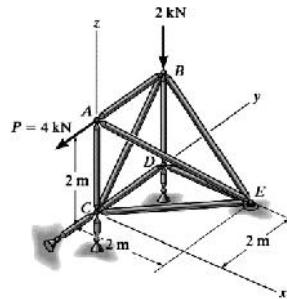
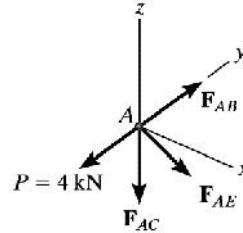


Fig.2.17 Space truss

**Solution:**

Since there are one known force and three unknown forces acting at joint A, the force analysis of the truss will begin at this joint.



Joint A: Expressing each force acting on the free-body diagram of joint A as a Cartesian vector, we have

$$P = 4(-j)$$

$$F_{AB} = (F_{AB})_m j$$

$$F_{AC} = (F_{AC})_m (-k)$$

$$F_{AE} = (F_{AE})_m \frac{\overrightarrow{r_{AE}}}{r_{AE}} = (F_{AE})_m \frac{(E_x - A_x)i + (E_y - A_y)j + (E_z - A_z)k}{\sqrt{(E_x - A_x)^2 + (E_y - A_y)^2 + (E_z - A_z)^2}}$$

$$F_{AE} = (F_{AE})_m \frac{(2 - 0)i + (2 - 0)j + (0 - 2)k}{\sqrt{(2)^2 + (2)^2 + (-2)^2}}$$

$$F_{AE} = (F_{AE})_m \frac{(2-0)i + (2-0)j + (0-2)k}{\sqrt{(2)^2 + (2)^2 + (-2)^2}} = (F_{AE})_m \frac{(2)i + (2)j + (-2)k}{3.46}$$

$$F_{AE} = (F_{AE})_m (0.578i + 0.578j - 0.578k)$$

For equilibrium,

$$\sum F = 0, \Rightarrow P + F_{AB} + F_{AC} + F_{AE} = 0$$

$$\Rightarrow -4j + (F_{AB})_m j - (F_{AC})_m k + 0.578(F_{AE})_m i + 0.578(F_{AE})_m j - 0.578(F_{AE})_m k = 0$$

By equating the unit vectors I, j and k, then

$$\sum F_x = 0, \Rightarrow 0.578(F_{AE})_m = 0$$

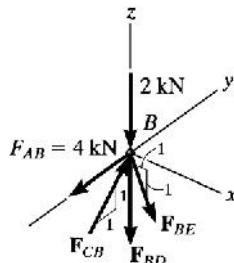
$$\sum F_y = 0, \Rightarrow -4 + (F_{AB})_m + 0.578(F_{AE})_m = 0$$

$$\sum F_z = 0, \Rightarrow -(F_{AC})_m - 0.578(F_{AE})_m = 0$$

$$F_{AC} = F_{AE} = 0$$

$$F_{AB} = 4 \text{ kN}$$

Since  $F_{AB}$  is known, then joint B can be analysed next.



Joint B:

$$\sum F_x = 0, \Rightarrow (F_{BE})_m \frac{1}{\sqrt{2}} = 0$$

$$\sum F_y = 0, \Rightarrow -4 + (F_{CB})_m \frac{1}{\sqrt{2}} = 0$$

$$\sum F_z = 0, \Rightarrow -2 + (F_{BD})_m - (F_{BE})_m \frac{1}{\sqrt{2}} + (F_{CB})_m \frac{1}{\sqrt{2}} = 0$$

$$F_{BE} = 0,$$

$$F_{CB} = 5.65 \text{ kN}$$

$$F_{BD} = 2 \text{ kN}$$

The scalar equations of equilibrium can now be applied to the forces acting on the free body diagrams of joints D and C as

$$F_{DE} = F_{DC} = F_{CE} = 0$$

## BEAMS:

*Beams* are structural members which offer resistance to bending due to applied loads. Most beams are long prismatic bars, and the loads are usually applied normal to the axes of the bars. Beams are undoubtedly the most important of all structural members, so it is important to understand the basic theory underlying their design. To analyse the load-carrying capacities of a beam we must first establish the equilibrium requirements of the beam as a whole and any portion of it considered separately. Second, we must establish the relations between the resulting forces and the accompanying internal resistance of the beam to support these forces. The first part of this analysis requires the application of the principles of statics. The second part involves the strength characteristics of the material and is usually treated in studies of the mechanics of solids or the mechanics of materials.

## Types of support:

The various types of supports and reactions developed are already discussed in previous section of unit -1. But here at glance looks about the types as

- 1) Simple support
- 2) Roller support
- 3) Hinged or pinned support
- 4) Fixed support

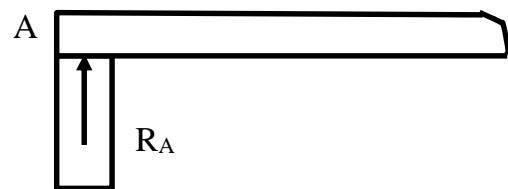


Fig.2.18 Simple support

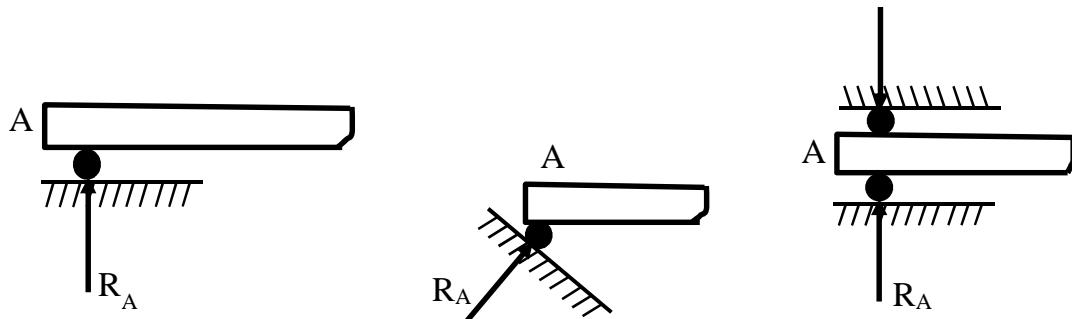


Fig.2.19 Roller support

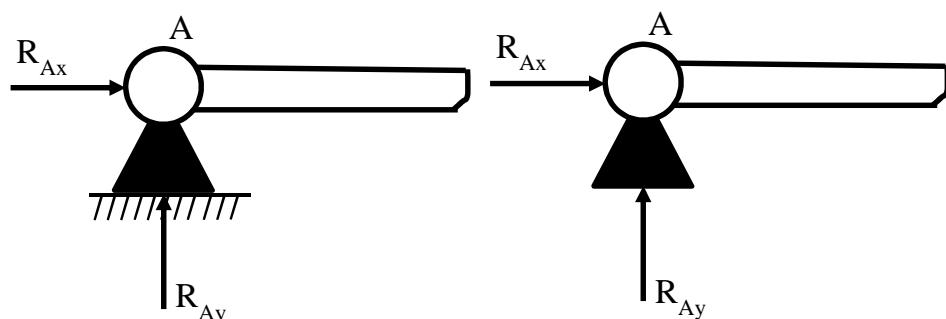


Fig.2.20 Hinged or pinned support

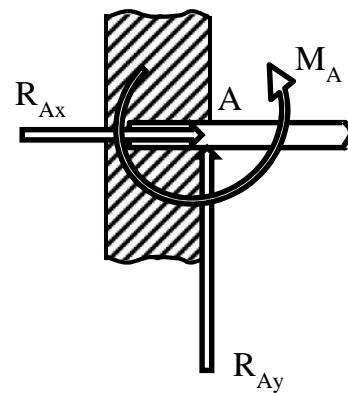


Fig.2.21 Fixed support

## Types of Beams:

The beams are classified according to the types of supports as

- 1) Cantilever beam
- 2) Simply supported beam
- 3) One end hinged and other on roller type of beam
- 4) Overhanging beam
- 5) Both ends hinged beam
- 6) Propped cantilever beam and
- 7) Continuous beam

### 1) Cantilever beam:

If a beam is fixed at one end and other end is free which is not supported by any it is called a cantilever beam. In this there are three reaction components at fixed end such as  $R_{Ax}$ ,  $R_{Ay}$  and  $M_A$  as shown in fig.2.22 and no reaction components at free end.

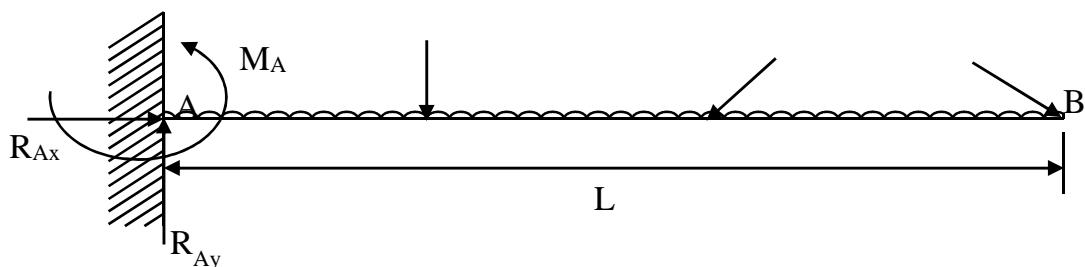


Fig.2.22 Cantilever beam

### 2) Simply supported beam:

In this type of beam both ends are simply supported as shown in fig.2.23. There is one reaction component at each end as  $R_{Ay}$  and  $R_{By}$ . They act at right angles to the support and this type of beam can resist forces normal to the beam axis. In other words the equilibrium condition that summation of forces parallel to axis equal to zero is to be satisfied automatically by the loading condition. Hence two equation of equilibrium are available.

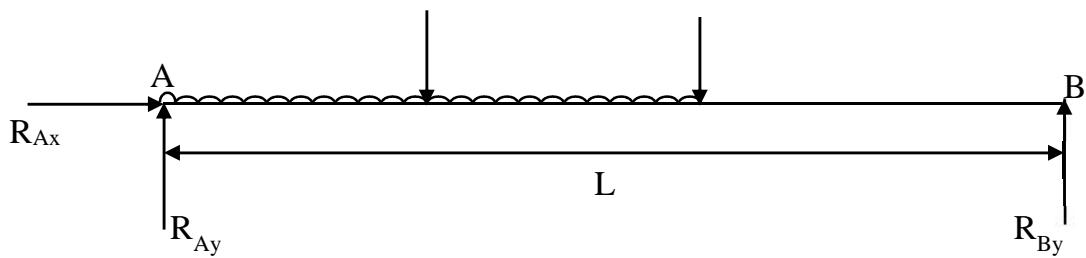


Fig.2.23 Simple supported beam

### 3) One end hinged and the other on roller:

As the name suggests one end of the beam is hinged and the other end is on roller as shown in fig.2.24. At hinge the reaction can be in both direction and at roller end it is at right angles to the roller support. The hinged end reaction in any direction can be represented by its two components perpendicular to each other. Thus its reaction components for such beam as  $R_{Ax}$ ,  $R_{Ay}$  and  $R_{By}$ .

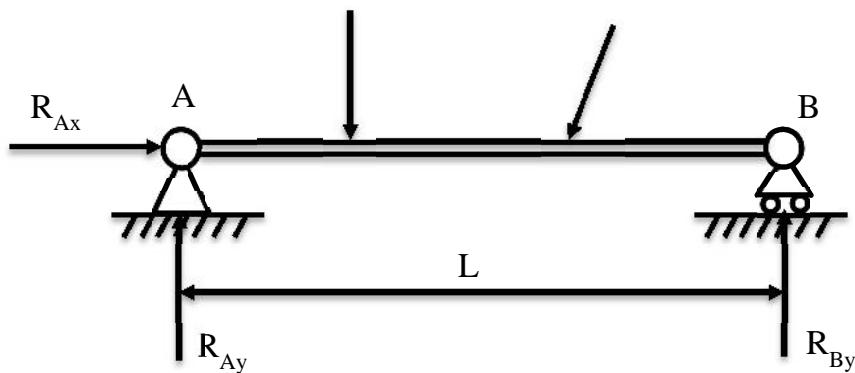
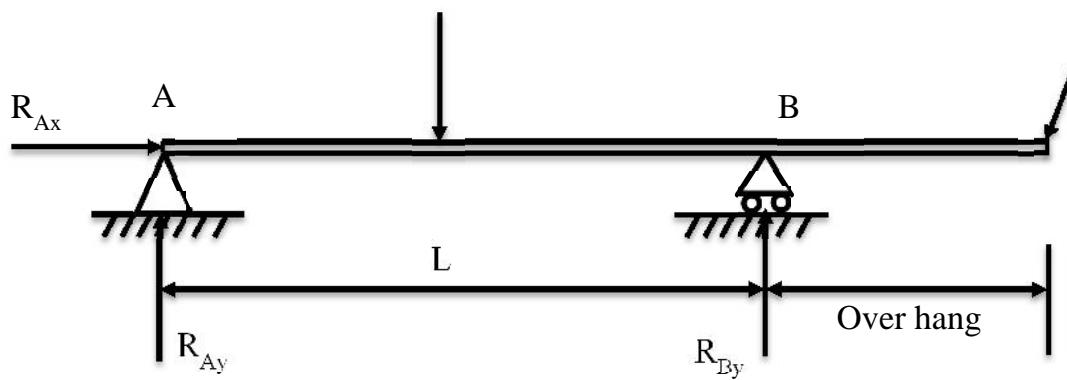


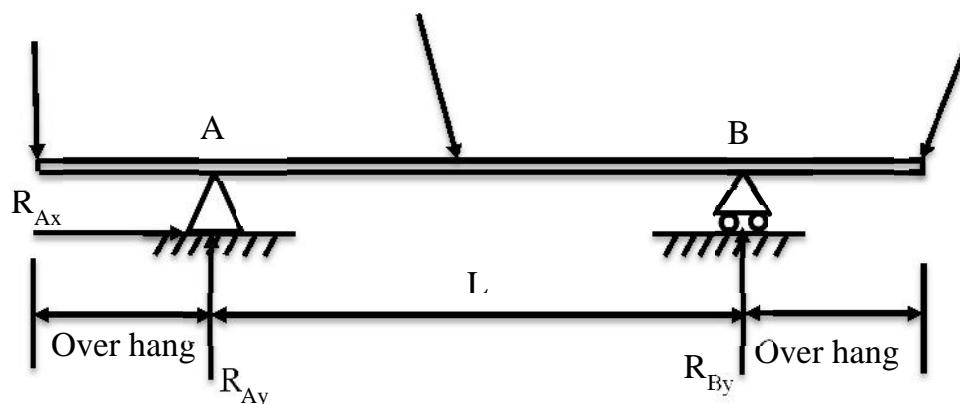
Fig.2.24 One end hinged and other end on roller

### 4) Overhanging beam:

If a beam is projecting beyond the support/ supports it is called an overhanging beam as shown in fig.2.25 . The overhang may be on only one side or may be on both sides.



a) Single overhang



b) Double overhang

Fig.2.25 Overhanging beam

### 5) Both ends hinged beam:

As the name suggested both ends of the beam are hinged. These are two reaction components at each end, then the total reaction components are four such as  $R_{Ax}$ ,  $R_{Ay}$ ,  $R_{Bx}$  and  $R_{By}$ .

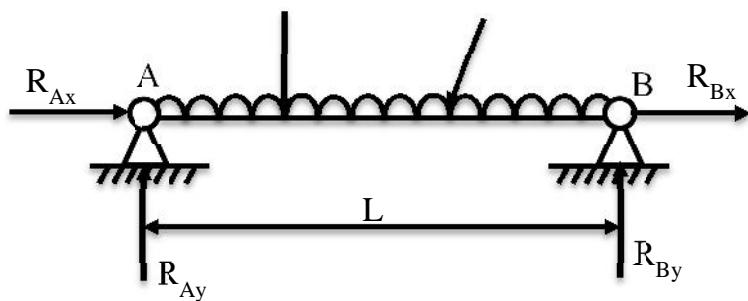


Fig. 2.26 Both ends hinged beam

## 6) Propped cantilever beam:

In this type of beam one end of the beam is fixed and the other end is simply supported or is on rollers. It has got four reaction components such as  $R_{Ax}$ ,  $R_{Ay}$ ,  $R_{By}$  and  $M_A$  and it is shown fig.2.27

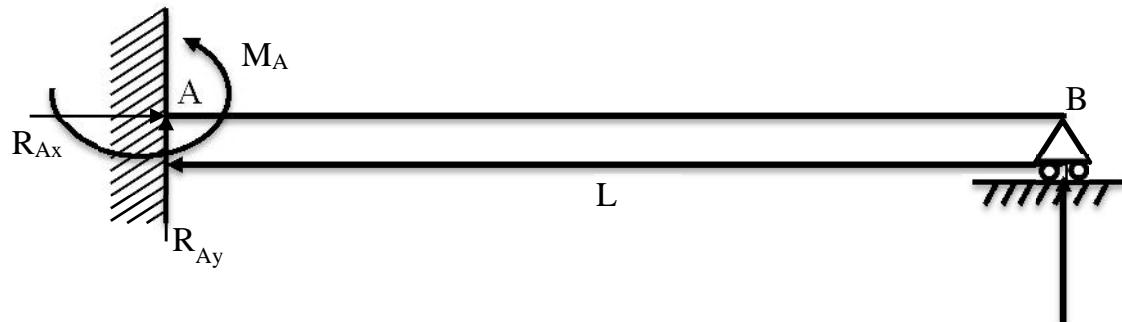
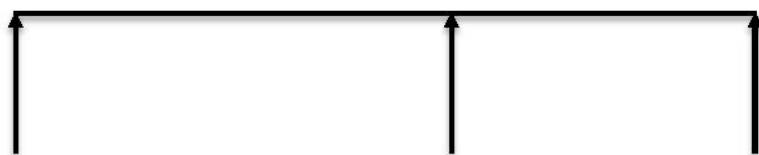


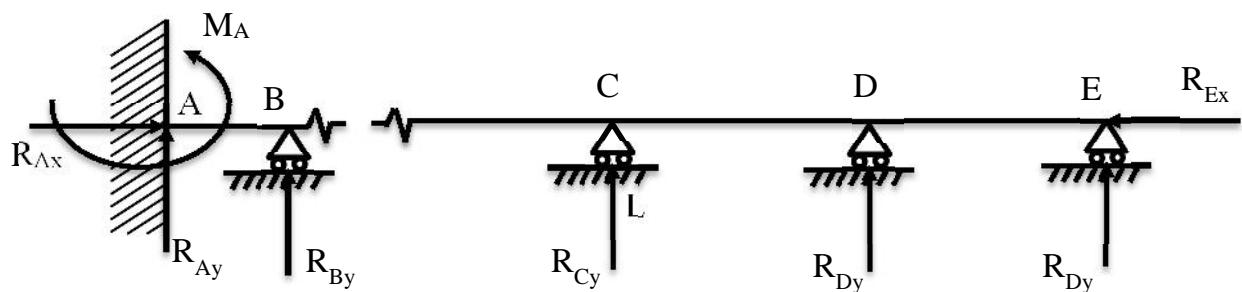
Fig.2.27 Propped cantilever beam

## 7) Continuous beam:

A beam having three or more supports is called continuous beam. In such beams three or more reaction components exist as shown in fig.2.28.



a) Two length of continuous beam



b) Multi length continuous beams

Fig.2.28 Continuous beams

## SUPPORT REDUNDANCY OF BEAMS:

The external support reactions can be calculated by the methods of statics alone are called *statically determinate beams*. Therefore the number of unknown reactions are equal to the number of independent equations of equilibrium such beams are cantilever beams, simply supported beams, one end hinged and the other end on roller support, and overhanging beams. The all reaction components of beam is determined by the equations of equilibrium only. A beam like both ends hinged, propped cantilever and continuous beams which has more supports with number of unknown reaction components are more than needed to provide the number of available equations of equilibrium is called *statically indeterminate*. To determine the support reactions for such a beam we must consider its load-deformation properties in addition to the equations of static equilibrium. Fig. 2.29 shows examples of both types of beams. In this section, the analyses are carried only by statically determinate beams.

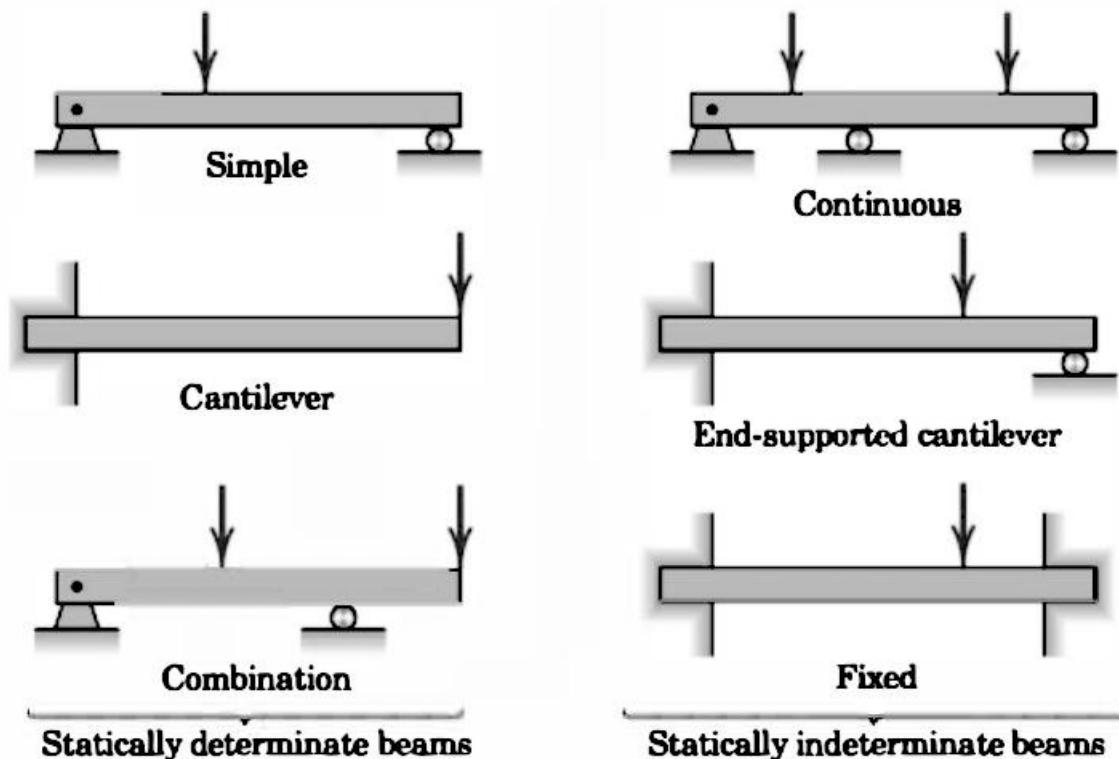


Fig.2.29 Support redundancy of beams

## Types of loading:

Beams may also be identified by the type of external loading they support as

- a) Concentrated loads
- b) Distributed loads
  - i) Uniform distributed loads
  - ii) Linearly varying distributed loads
  - iii) Composite of uniform and varying distributed loads
  - iv) Non uniform distributed loads
- c) External moment loads

### a) Concentrated loads:

The beams in Fig. 2.29 are supporting concentrated loads, in which the loads are acting over a very small length compared to of the beam and it is approximated as a point load.

### b) Distributed loads:

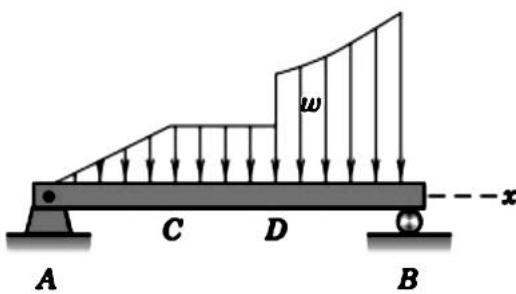


Fig.2.30 Distributed loads

The beam in Fig. 2.30 is supporting a distributed load. The intensity  $w$  of a distributed load may be expressed as force per unit length of beam. The intensity may be constant or variable, continuous or discontinuous. The intensity of the loading in Fig. 2.30 is constant from  $C$  to  $D$  and variable from  $A$  to  $C$  and from  $D$  to  $B$ . The intensity is

discontinuous at  $D$ , where it changes magnitude abruptly. Although the intensity itself is not discontinuous at  $C$ , the rate of change of intensity  $\frac{dw}{dx}$  is discontinuous.

### i) Uniform distributed loads:

Loading intensities which are constant over the length of beam is called uniform distribution loads as shown in fig.2.31 , The resultant load  $R$  is represented by the area formed by the intensity  $w$  (force per unit length of beam) and the length  $L$  over which the force is distributed. The resultant passes through the centroid of this area.

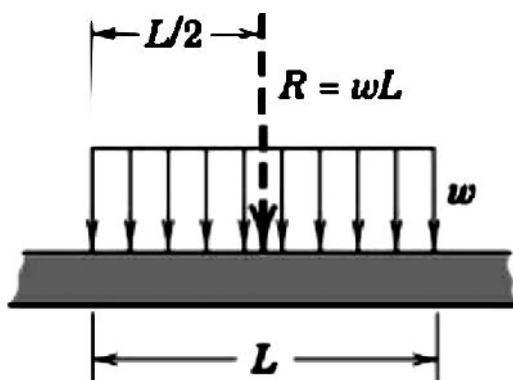


Fig.2.31 Uniform distributed loads

### ii) Linearly varying distributed loads:

Loading intensities which are vary linearly over the length of beam is called uniformly varying distributed loads as shown in fig.2.32. The resultant load  $R$  is represented by the area formed by the intensity  $w$  (force per unit length of beam) and the length  $L$  over which the force is distributed. The resultant passes through the centroid of this area. The centroid area is  $\frac{2}{3}L$

from zero intensity load and Resultant load is  $\frac{1}{2}wL$ .

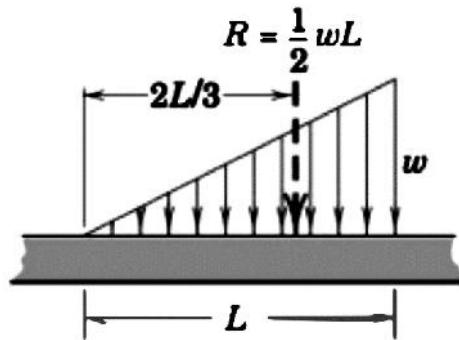


Fig.2.32 Linearly varying distributed loads

### iii) Composite of uniform and varying distributed loads:

The composite of uniform and varying distributed loads looks like a trapezoidal area which is broken into a rectangular and a triangular area, and the corresponding resultants  $R_1$  and  $R_2$  of these sub-areas are determined separately. The centroid area and resultant are considered from the rectangular and triangular as in previous sections.

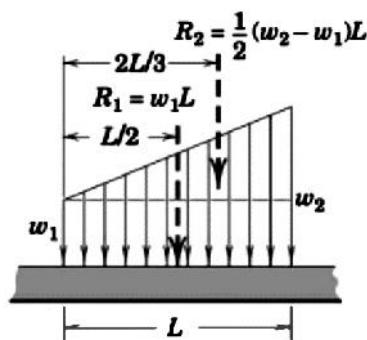


Fig.2.33 Composite uniform and varying distributed loads

### iv) Non uniform distributed loads:

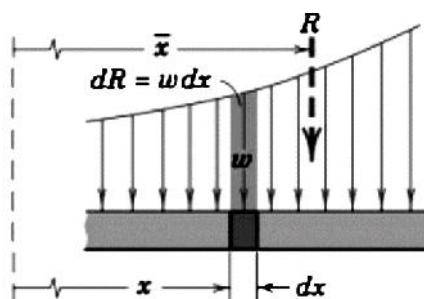


Fig.2.34 Non uniform distributed loads

Fig. 2.34, starts with a differential increment of force  $dR = w dx$ . The total load  $R$  is then the sum of the differential forces, or

$$R = \int w dx$$

As before, the resultant  $R$  is located at the centroid of the area under consideration. The x-coordinate of this centroid is found by the principle of moments  $R\bar{x} = \int x w dx$  or

$$\bar{x} = \frac{1}{R} \int x w dx$$

For the distribution as in Fig.2.34, the vertical coordinate of the centroid need not be found. Once the distributed loads have been reduced to their equivalent concentrated loads, the external reactions acting on the beam may be found by a straightforward static analysis.

v) **External moment loads:**

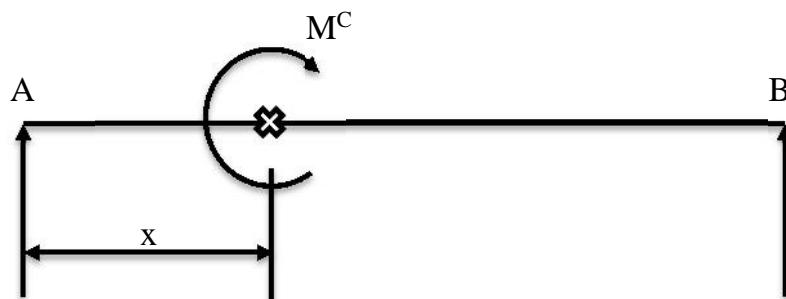


Fig. 2.35 External moment loads

A beam may be subjected to external moments at certain points. In fig.2.35 The beam subjected to a clockwise moment of  $M^C$  at a distance  $x$  from the support A.

**SP2.6:** Find the reactions developed in the cantilever beam as shown in fig.2.36.

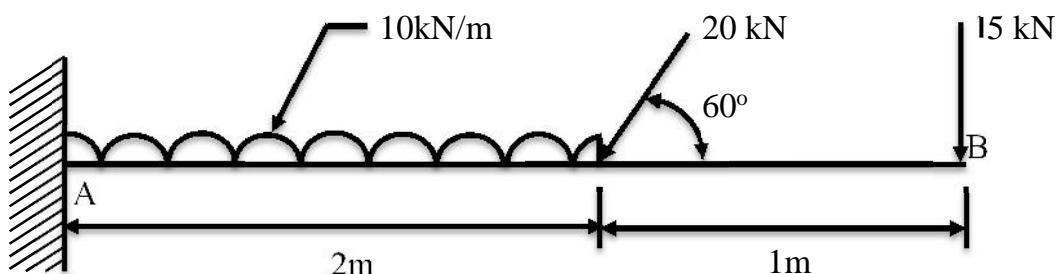
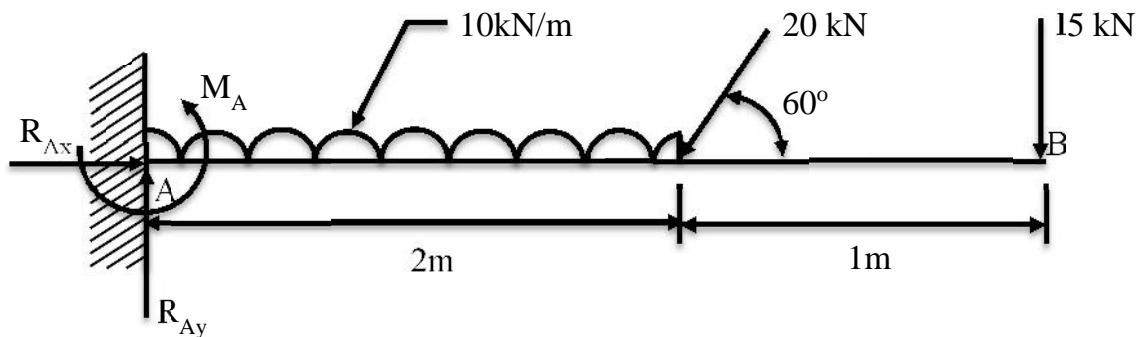


Fig.2.36

**Solution:**

First draw the free body diagram consist of reaction force developed at fixed support A as shown below.



The equilibrium conditions about vertical,

$$\sum F_y = 0, \Rightarrow R_{Ay} - (R_{10}) - 20 \sin 60^\circ - 15 = 0$$

$$\Rightarrow R_{Ay} - (10 \times 2) - 20 \sin 60^\circ - 15 = 0, R_{Ay} = 52.32 \text{ kN}$$

The equilibrium conditions about horizontal,

$$\sum F_x = 0, \Rightarrow R_{Ax} - 20 \cos 60^\circ = 0, R_{Ax} = 10 \text{ kN}$$

Taking moment about A ,

$$\sum M_A = 0, M_A - 10 \times 2 \times 1 - 20 \sin 60^\circ \times 2 - 15 \times 3 = 0$$

$$M_A = 99.64 \text{ kN-m}$$

**SP2.7:** Determine the reaction developed in the simply supported beam as shown in fig.2.37.

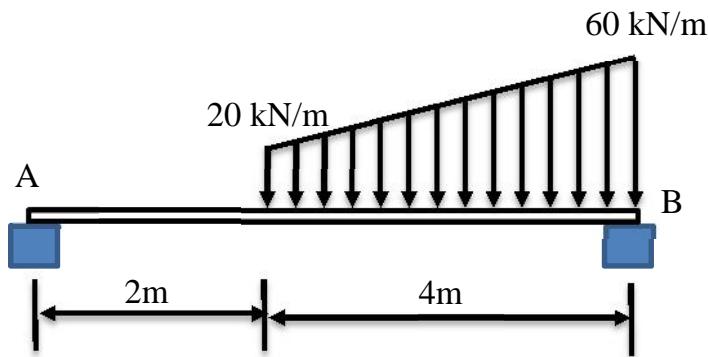
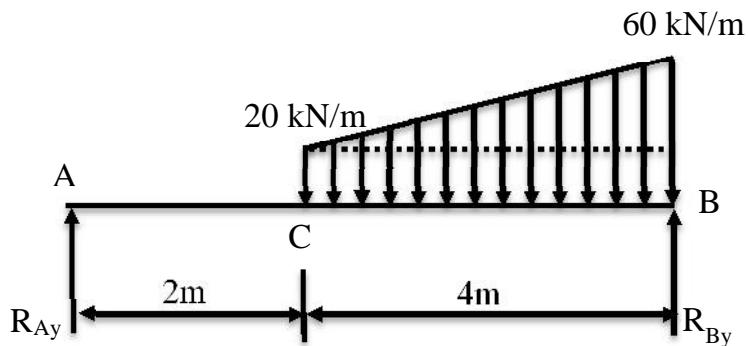


Fig.2.37

**Solution:**

The free body diagram of given space diagram as below



The varying distributed load may be split into a uniformly distributed load of 20 kN/m as in rectangular geometry and a linearly varying load from point C to point B. The linearly varying load at end C is zero and point B is 40 kN/m(60-20).

Taking moment about B, then

$$\sum M_B = 0, \Rightarrow -R_{Ay} \times 6 + 20 \times 4 \times 2 + \frac{1}{2} \times 40 \times 4 \times \frac{1}{3} = 0$$

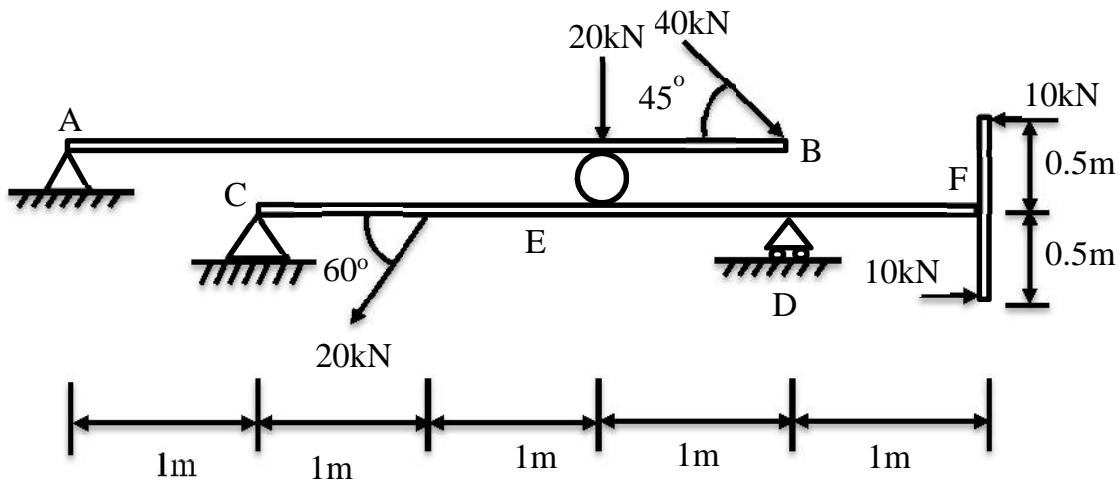
$$R_{Ay} = 44.44 \text{ kN}$$

The equilibrium equations for vertical force,

$$\sum F_y = 0, \Rightarrow R_{Ay} + R_{By} - 20 \times 4 - \frac{1}{2} \times 40 \times 4 = 0$$

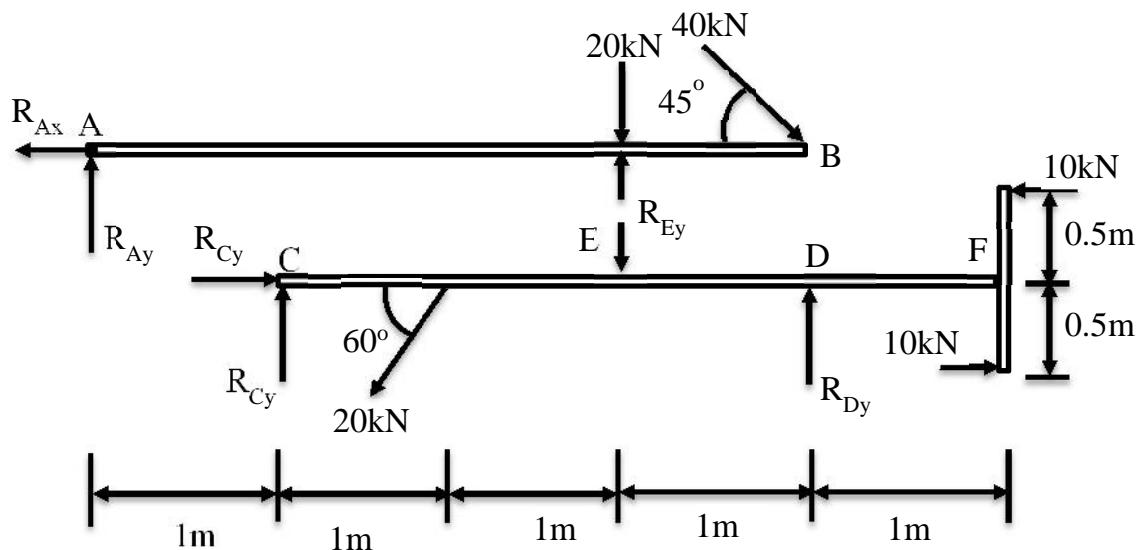
$$\Rightarrow 44.44 + R_{By} - 80 - 80 = 0, R_{By} = 115.56 \text{ kN}$$

**Sp.2.8:** The beam AB and CF are arranged as shown in fig.2.38. Find the reactions at A, C and D due to the loads acting on the beam.



**Solution:**

The free body diagrams of beams AB and CF is drawn below as



Now consider beam AB,

Taking moment about A, then

$$\sum M_A = 0, \Rightarrow R_{Ey} \times 3 - 20 \times 3 - 40 \sin 45^\circ \times 4 = 0$$

$$R_{Ey} = 57.71 \text{ kN}$$

The equilibrium equation about horizontal, then

$$\sum F_x = 0, \Rightarrow -R_{Ax} + 40 \cos 45^\circ = 0$$

$$R_{Ax} = 28.28 \text{ kN}$$

The equilibrium equations about vertical, then

$$\sum F_y = 0, \Rightarrow R_{Ay} - 20 - 40 \sin 45^\circ + R_{Ey} = 0$$

$$\Rightarrow R_{Ay} - 20 - 40 \sin 45^\circ + 57.71 = 0$$

$$R_{Ay} = -9.43 \text{ kN}$$

Here, the negative sign indicates the direction of reaction is downwards.

Consider beam CF,

Taking moment about C, then

$$\sum M_C = 0, \Rightarrow R_{Dy} \times 3 - 20 \sin 60^\circ \times 1 - R_{Ey} \times 2 + 10 \times 1 = 0$$

$$\Rightarrow R_{Dy} \times 3 - 20 \sin 60^\circ \times 1 - 57.71 \times 2 + 10 \times 1 = 0$$

$$R_{Dy} = 40.91 \text{ kN}$$

The equilibrium equations about horizontal is

$$\sum F_x = 0, \Rightarrow R_{Cx} - 20 \cos 60^\circ = 0$$

$$R_{Cx} = 10 \text{ kN}$$

The equilibrium equations about vertical is

$$\sum F_y = 0, \Rightarrow R_{Cy} - 20 \sin 60^\circ - R_{Ey} + R_{Dy} = 0$$

$$\Rightarrow R_{Cy} - 20 \sin 60^\circ - 57.71 + 40.91 = 0$$

$$R_{Cy} = 34.12 \text{ kN}$$

## FRAMES AND MACHINES:

Frames and machines are two types of structures which are often composed of pin-connected ***multi-force members***, i.e., members that are subjected to more than two forces. ***Frames*** are used to support loads, whereas ***machines*** contain moving parts and are designed to transmit and alter the effect of forces. Provided a frame or machine contains no more supports or members than are necessary to prevent its collapse, the forces acting at the joints and supports can be determined by applying the equations of equilibrium to each of its members. Once these forces are obtained, it is then possible to *design* the size of the members, connections, and supports using the theory of mechanics of materials and an appropriate engineering design code.

### Free-Body Diagrams.

In order to determine the forces acting at the joints and supports of a frame or machine, the structure must be disassembled and the free-body diagrams of its parts must be drawn. The following important points *must* be observed:

- Isolate each part by drawing its *outlined shape*. Then show all the forces and/or couple moments that act on the part. Make sure to *label* or *identify* each known and unknown force and couple moment with reference to an established  $x$ ,  $y$  coordinate system. Also, indicate any dimensions used for taking moments. Most often the equations of equilibrium are easier to apply if the forces are represented by their rectangular components. As usual, the sense of an unknown force or couple moment can be assumed.
- Identify all the two-force members in the structure and represent their free-body diagrams as having two equal but opposite collinear forces acting at their points of application. By recognizing the two-force members, we can avoid solving an unnecessary number of equilibrium equations.
- Forces common to *any two contacting members* act with equal magnitudes but opposite sense on the respective members. If the two members are treated as a “*system*” of *connected members*, then these forces are “*internal*” and are *not shown* on the *free-body diagram of the system*; however, if the free-body diagram

of *each member* is drawn, the forces are “*external*” and *must* be shown as equal in magnitude and opposite in direction on each of the two free-body diagrams.

**SP2.9:** For the frame as shown in fig.2.39, draw the free body diagram of i) each member, ii) the pins at B and A and iii) the two members connected together.

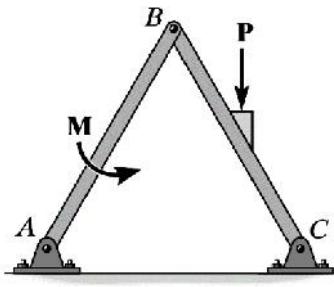


Fig.2.39

**Solution:**

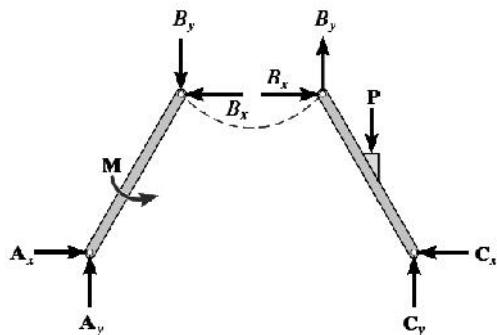


Fig.2.39 b

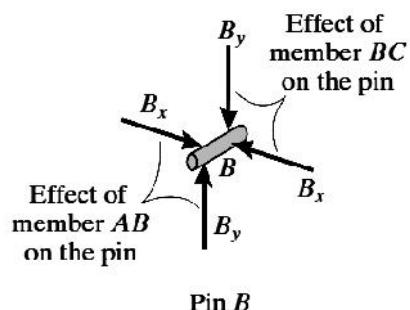


Fig.2.39 c

Part (a): By inspection, members *BA* and *BC* are *not* two-force members. Instead, as shown on the free-body diagrams, Fig. 2.39 *b*, *BC* is subjected to a force from each of the pins at *B* and *C* and the external force **P**. Likewise, *AB* is subjected to a force from each of the pins at *A* and *B* and the external couple moment **M**. The pin forces are represented by their *x* and *y* components.

Part (b). The pin at *B* is subjected to only *two forces*, i.e., the force of member *BC* and the force of member *AB*. For *equilibrium* these forces (or their respective components) must be equal but opposite, Fig. 2.39 *c*. Realize that Newton’s third law is applied between the pin and its connected members, i.e., the effect of the pin on the

two members, Fig. 2.39 *b*, and the equal but opposite effect of the two members on the pin, Fig. 2.39 *c*. In the same manner, there are three forces on pin *A*, Fig. 2.39 *d*, caused by the force components of member *AB* and each of the two pin leafs.

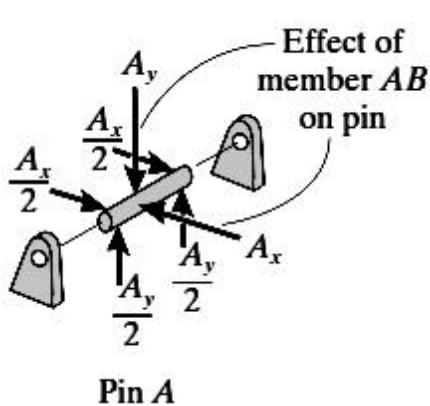


Fig.2.39 d

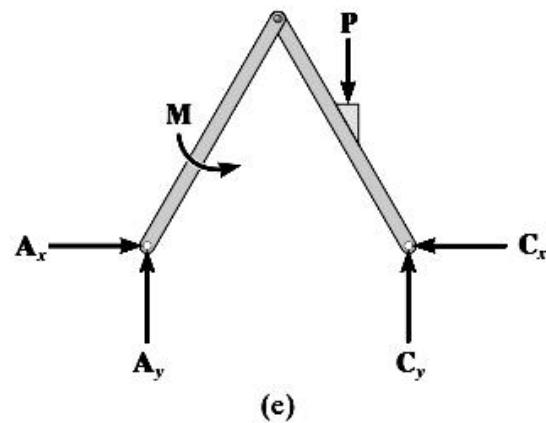


Fig.2.39 e

Part (c). The free-body diagram of both members connected together, yet removed from the supporting pins at *A* and *C*, is shown in Fig. 2.39 *e*. The force components  $\mathbf{B}_x$  and  $\mathbf{B}_y$  are *not shown* on this diagram since they are *internal* forces (Fig. 2.39 *b*) and therefore cancel out. Also, to be consistent when later applying the equilibrium equations, the unknown force components at *A* and *C* must act in the *same sense* as those shown in Fig. 2.39 *b*.

**SP2.10:** A constant tension in the conveyor belt is maintained by using the device as shown in fig.2.40. Draw the free body diagram of the frame and the cylinder that the belt surrounds. The suspended blocks has a weight of *W*.

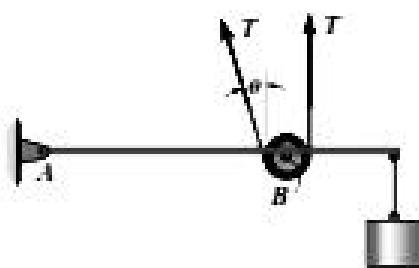
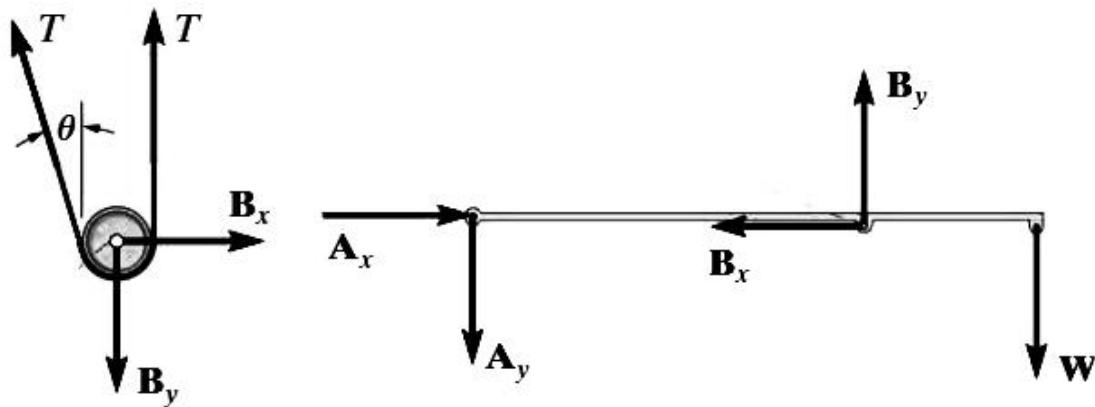


Fig.2.40

**Solution:**



**SP2.11:** For the frame as shown in fig.2.41, draw the free body diagrams of i) the entire frame including the pulleys and cords, ii) the frame without the pulleys and cords and iii) each of the pulleys.

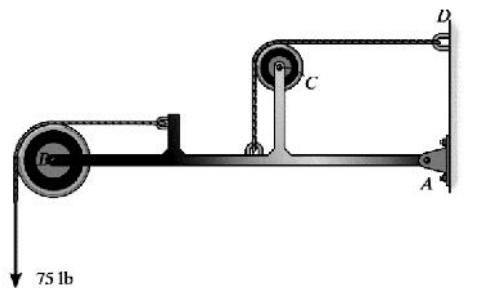


Fig.2.41

**Solution:**

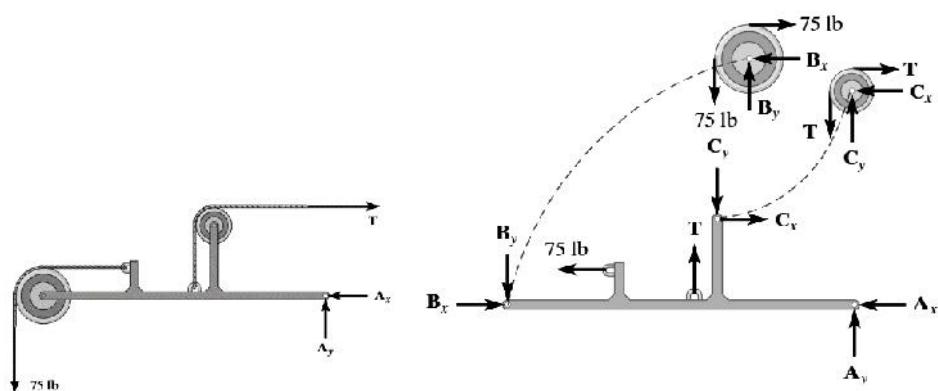


Fig.2.41 a

Fig.2.41 b

Part (a): When the entire frame including the pulleys and cords is considered, the interactions at the points where the pulleys and cords are connected to the frame become pairs of *internal* forces which cancel each other and therefore are not shown on the free-body diagram, Fig.2.41 a.

Part (b). When the cords and pulleys are removed, their effect *on the frame* must be shown, Fig.2.41 b.

Part (c). The force components  $\mathbf{B}_x$ ,  $\mathbf{B}_y$ ,  $\mathbf{C}_x$ ,  $\mathbf{C}_y$  of the pins on the pulleys, Fig. 2.41 c, are equal but opposite to the force components exerted by the pins on the frame, Fig.2.41 c.

**SP2.12:** The 75-kg man as shown in Fig.2.41 attempts to lift the 40-kg uniform beam off the roller support at *B*. Determine the tension developed in the cable attached to *B* and the normal reaction of the man on the beam when this is about to occur.

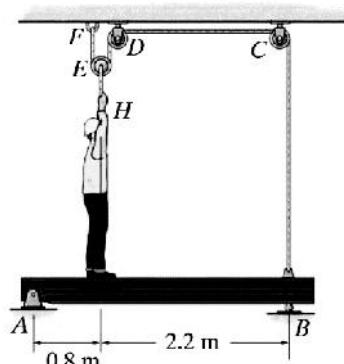


Fig.2.42

**Solution:**

The free-body diagrams of the pulley *E*, the man, and the beam are shown in Fig. 2.42 b. Since the man must lift the beam off the roller *B* then  $N_B = 0$ . When drawing each of these diagrams, it is very important to apply Newton's third law. The tensile force in the cable will be denoted as  $T_1$ .

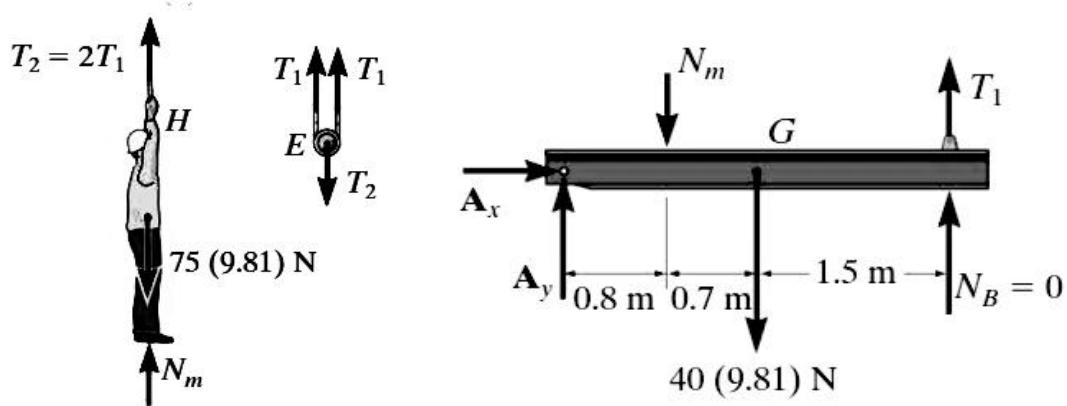


Fig.2.42 a

Equations of Equilibrium: Using the free-body diagram of pulley  $E$ , About vertical, then

$$\sum F_y = 0, \Rightarrow 2T_1 - T_2 = 0, T_2 = 2T_1 \rightarrow (1)$$

Using the free body diagram of man for vertical components, then

$$\sum F_y = 0, N_m + 2T_1 - 75(9.81) = 0 \rightarrow (2)$$

Taking moment about A on the beam, then

$$\sum M_A = 0, T_1 \times 3 - N_m \times 0.8 - (40(9.81)) \times 1.5 = 0 \rightarrow (3)$$

Solving equations 2 and 3 simultaneously for  $T_1$  and  $N_m$  then using equations (1) for  $T_2$   
 $T_1 = 256 \text{ N}$ ,  $N_m = 224 \text{ N}$  and  $T_2 = 512 \text{ N}$ .

**SP2.13:** The frame supports the 400-kg load in the manner shown in fig.2.43. Neglect the weights of the members compared with the forces induced by the load and compute the horizontal and vertical components of all forces acting on each of the members.

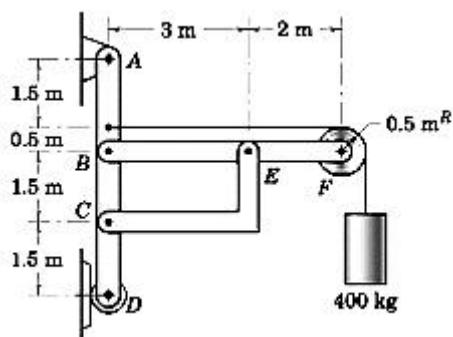
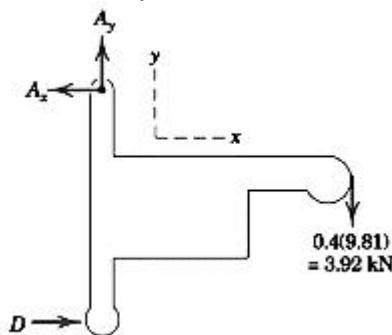


Fig.2.43

**Solution:**

First, the three supporting members which constitute the frame form a rigid assembly can be analysed as a single unit. It also observe that the arrangement of the external supports makes the frame statically determinate.



From the free-body diagram of the entire frame we determine the external reactions. Thus,

$$\sum M_A = 0, \Rightarrow -400 \times 9.81(5.5) + R_{Dx}(5) = 0,$$

$$R_{Dx} = 4.32 \text{ kN}$$

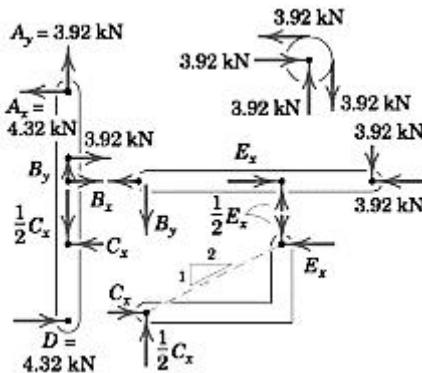
$$\sum F_x = 0, \Rightarrow R_{Ax} - R_{Dx} = 0, \Rightarrow R_{Ax} - 4.32 = 0$$

$$R_{Ax} = 4.32 \text{ kN}$$

$$\sum F_y = 0, \Rightarrow R_{Ay} - (400)(9.81) = 0$$

$$R_{Ay} = 3.92 \text{ kN}$$

Next we dismember the frame and draw a separate free-body diagram of each member. The diagrams are arranged in their approximate relative positions to aid in keeping track of the common forces of interaction. The external reactions just obtained are entered onto the diagram for AD. Other known forces are the 3.92-kN forces exerted by the shaft of the pulley on the member BF, as obtained from the free body diagram of the pulley. The cable tension of 3.92 kN is also shown acting on AD at its attachment point.



Next, the components of all unknown forces are shown on the diagrams. Here it observes that CE is a two-force member. The force components on CE have equal and opposite reactions, which are shown on BF at E and on AD at C. This may not recognize the actual sense of the components at B at first glance, so they may be arbitrarily but consistently assigned. The solution may proceed by use of a moment equation about B or E for member BF, followed by the two force equations. Thus,

$$\sum M_B = 0, \Rightarrow -400 \times 9.81(5) + \frac{1}{2} R_{Ex}(3) = 0,$$

$$R_{Ex} = 13.08 \text{ kN}$$

$$\sum F_x = 0, \Rightarrow -R_{Bx} + R_{Ex} - 400(9.81) = 0, \Rightarrow -R_{Bx} + 13.08 - 3.92 = 0$$

$$R_{Bx} = 9.15 \text{ kN}$$

$$\sum F_y = 0, \Rightarrow -R_{By} + \frac{1}{2} R_{Ex} - (400)(9.81) = 0$$

$$R_{By} = 2.62 \text{ kN}$$

Positive numerical values of the unknowns mean that we assumed their directions correctly on the free body diagrams. The value of  $C_x = E_x = 13.08 \text{ kN}$  obtained by inspection of the free-body diagram of CE is now entered onto the diagram for AD, along with the values of  $B_x$  and  $B_y$  just determined. The equations of equilibrium may now be applied to member AD as a check, since all the forces acting on it have already been computed. The equations give

$$\sum M_C = 0, \Rightarrow R_{Ax}(3.5) + R_{Dx}(1.5) - 400(9.81)(2) - R_{Bx}(1.5) = 0,$$

$$\Rightarrow 4.32(3.5) + 4.32(1.5) - 400(9.81)(2) - 9.15(1.5) = 0$$

$$\sum F_x = 0, \Rightarrow R_{Dx} - R_{Cx} + R_{Bx} + (400)(9.81) - R_{Ax} = 0,$$

$$\Rightarrow 4.32 - 13.08 + 9.15 + 3.92 - 4.32 = 0$$

$$\sum F_y = 0, \Rightarrow -\frac{1}{2}R_{Cx} + R_{By} + R_{Ay} = 0$$

$$\Rightarrow -\frac{1}{2}13.08 + 2.62 + 3.92 = 0$$

**SP2.14:** In the particular position as shown in fig.2.44, the excavator applies a 20-kN force parallel to the ground. There are two hydraulic cylinders AC to control the arm OAB and a single cylinder DE to control arm EBIF. i) Find the force in the hydraulic cylinders AC and the pressure  $P_{AC}$  against their pistons, which have an effective diameter of 95 mm, ii) Also find the force in hydraulic cylinder DE and the pressure  $P_{DE}$  against its 105 mm diameter piston. Neglect the weights of the members compared with the effects of the 20 kN force.

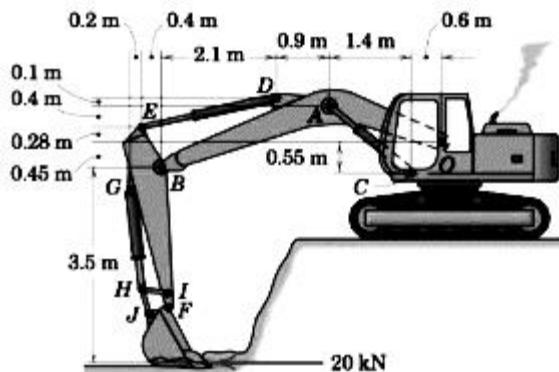
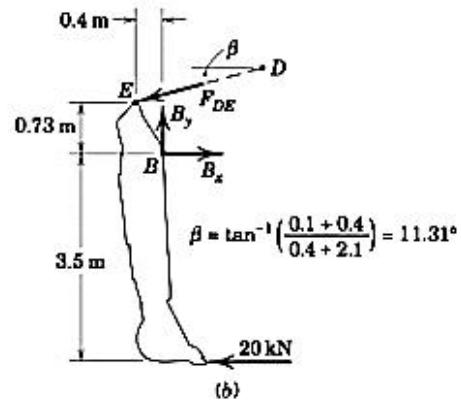
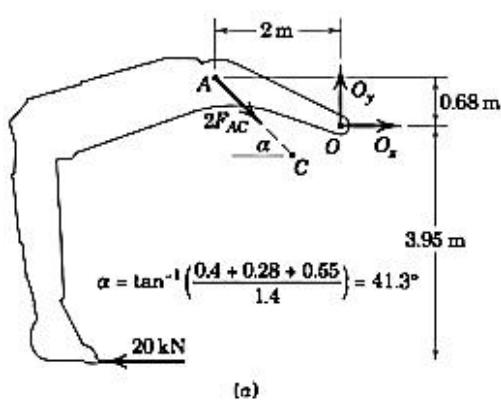


Fig.2.44

### Solution:



i) The constructing a free body diagram of the entire arm assembly is shown in above figure a. In this only dimensions are necessary for the portion of problem – details of the cylinders DE and GH are unnecessary at this time.

$$\sum M_O = 0, \Rightarrow -20(3.95) - 2F_{AC} \cos 17.3^\circ (0.68) + 2F_{AC} \sin 17.3^\circ (2) = 0$$

$$\Rightarrow -20(3.95) - 2F_{AC} \cos 41.3^\circ (0.68) + 2F_{AC} \sin 41.3^\circ (2) = 0$$

$$F_{AC} = 48.8 \text{ kN}$$

$$\text{From } F_{AC} = P_{AC}A_{AC}, P_{AC} = \frac{F_{AC}}{A_{AC}} = \frac{48800}{\left(f \frac{0.095^2}{4}\right)} = 6.89(10^6) \text{ Pa or } 6.89 \text{ MPa}$$

ii) For cylinder DF, cut the assembly at a location which makes the desired cylinder force external to our free body diagram. This means isolating the vertical arm EBIF along with the bucket and its applied force.

$$\sum M_B = 0, \Rightarrow -20(3.5) + F_{DE} \cos 53.1^\circ (0.73) + F_{DE} \sin 53.1^\circ (0.4) = 0$$

$$\Rightarrow -20(3.5) + F_{DE} \cos 11.31^\circ (0.73) + F_{DE} \sin 11.31^\circ (0.4) = 0$$

$$F_{DE} = 88.1 \text{ kN}$$

$$P_{DE} = \frac{F_{DE}}{A_{DE}} = \frac{88100}{\left(f \frac{0.105^2}{4}\right)} = 10.18(10^6) \text{ Pa or } 10.18 \text{ MPa}$$

## CENTROID AND CENTRE OF GRAVITY COVERING:

The plane geometry like triangle, quadrilateral, circle etc. or space geometry have only areas or volume, but no mass. The centre of area/ volume of such figures is known as *centroid*. The method of finding out the centroid of a figure is the same as that of finding out the centre of gravity of a body. The centre of gravity is similar to the centroid but mass is accounted.

## Difference between Centre of Gravity and Centroid:

Sl. No	Centre of Gravity	Centroid
1	The point where the total weight of the body focuses upon	It is referred to the geometrical centre of a body
2	It is the point where the gravitational force (weight) acts on the body	It is referred to the centre of gravity of uniform density objects
3	It is denoted by $g$	It is denoted by $c$
4	Centre of Gravity in a uniform gravitational field is the average of all points, weighted by local density or specific weight	The centroid is a point in a plane area in such a way that the moment of area about any axis throughout that point is 0
5	It is a physical behaviour of the object, a point where all the weight of an object is acting	It is a geometrical behaviour. It is the centre of measure of the amount of geometry.

## Methods for Centroid:

The centroid may be found out by any one of the following methods:

1. by geometrical considerations
2. by moments
3. by graphical method

As a matter of fact, the graphical method is a tedious and cumbersome method for finding out the centroid of simple figures. So this graphical method is not included here. Before to discuss the method of moments, the procedure for finding out the centroid of simple figures by geometrical considerations as follows.

## Centroid by Geometrical Considerations:

The centroid of simple figures may be found out from the geometry of the figure as given below.

1. The centroid of uniform rod/ length is at its middle point.
2. The centroid of a rectangle (or a parallelogram) is at the point, where its diagonals meet each other. It is also a middle point of the length as well as the breadth of the rectangle as shown in Fig. 2.45.

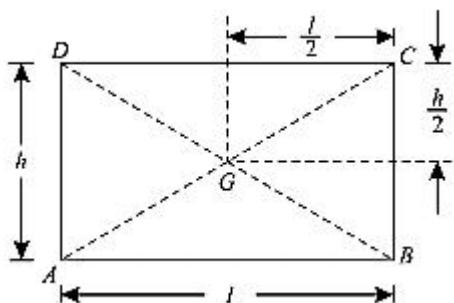


Fig.2.45 Rectangular geometry

3. The centroid of a triangle is at the point, where the three medians (a median is a line connecting the vertex and middle point of the opposite side) of the triangle meet as shown in Fig. 2.46.

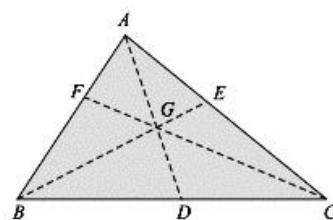


Fig.2.46

4. The centroid of a trapezium with parallel sides  $a$  and  $b$  is at a distance of  $\frac{h}{3} \times \left( \frac{b+2a}{b+a} \right)$  measured from the side  $b$  as shown in Fig.2.47.

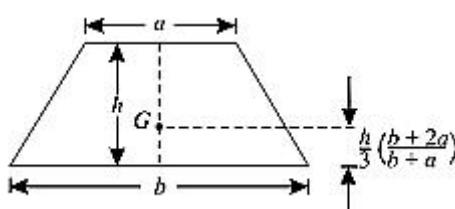


Fig.2.47 Trapezium

5. The centroid of a semicircle is at a distance of  $\frac{4r}{3\pi}$  from its base measured along the vertical radius as shown in Fig.2.48.

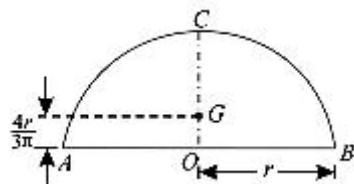


Fig.2.48 Semicircle

6. The centroid of a circular sector making semi-vertical angle  $\alpha$  is at a distance of  $\frac{2r}{3} \times \frac{\sin \alpha}{r}$  from the centre of the sector measured along the central axis as shown in Fig.2.49.

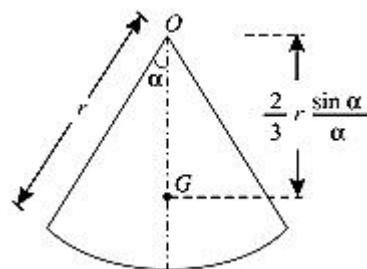


Fig.2.49 Circular sector

7. The centroid of a cube is at a distance of  $\frac{l}{2}$  from every face (where  $l$  is the length of each side).
8. The centroid of a sphere is at a distance of  $\frac{d}{2}$  from every point (where  $d$  is the diameter of the sphere).
9. The centroid of a hemisphere is at a distance of  $\frac{3r}{8}$  from its base, measured along the vertical radius as shown in Fig.2.50.

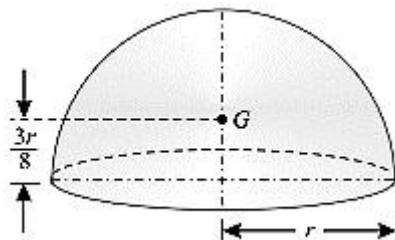


Fig.2.50 Hemisphere solid

10. The centroid of right circular solid cone is at a distance of  $\frac{h}{4}$  from its base, measured along the vertical axis as shown in Fig.2.51.

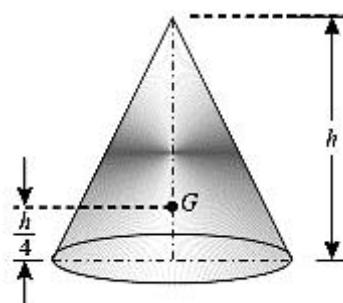


Fig.2.51 Right circular solid

11. The centroid of a segment of sphere of a height  $h$  is at a distance of  $\frac{3}{4} \times \frac{(2r-h)^2}{(3r-h)}$

from the centre of the sphere measured along the height as shown in Fig.2.52.

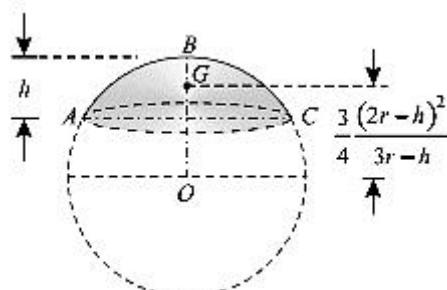


Fig.2.52 Sphere segment

#### CENTROID BY FIRST PRINCIPLE (MOMENTS):

The centroid of a body may also be found out by moments as discussed below:

Consider a body of area  $A$  whose centroid is required to be found out. Divide the body into small areas, whose centroids are known as shown in Fig.2.53. Let  $a_1, a_2, \dots, a_n$  be the areas of the particles and  $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$  be the co-ordinates of the centroids from a fixed point  $O$  as shown in Fig.2.53.

Let  $\bar{x}$  and  $\bar{y}$  be the co-ordinates of the centroid of the body. From the first principle of moments, we know that

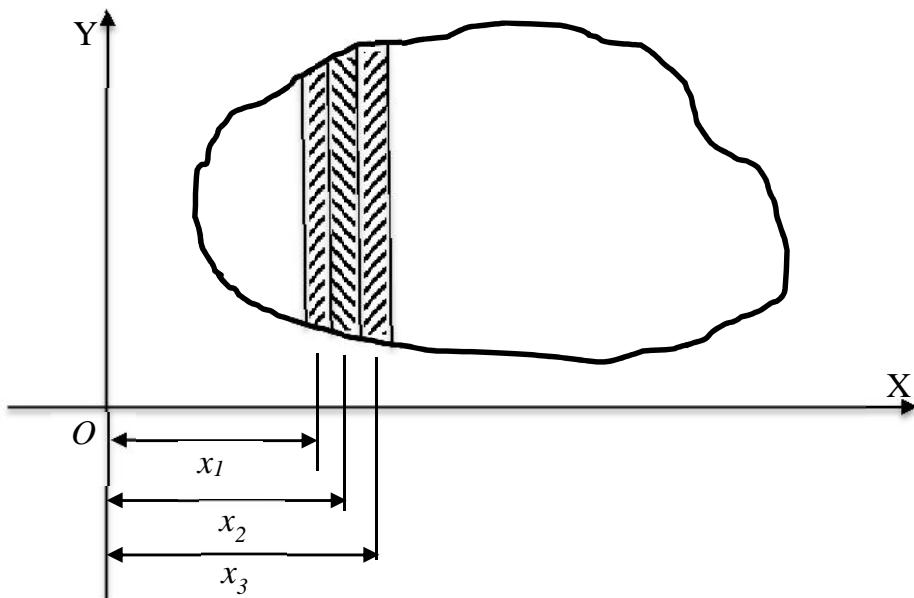


Fig.2.53 Centroid by moments

$$A\bar{x} = a_1 x_1 + a_2 x_2 + a_3 x_3 \dots \text{ or } \bar{x} = \frac{\sum ax}{A}$$

$$\text{Similarly, } \bar{y} = \frac{\sum ay}{A}$$

Where,  $A = a_1 + a_2 + a_3 + \dots$

### AXIS OF REFERENCE:

The centroid of a body is always calculated with reference to some assumed axis known as axis of reference (or sometimes with reference to some point of reference). The axis of reference, of plane figures, is generally taken as the lowest line of the figure for calculating  $\bar{y}$  and the left line of the figure for calculating  $\bar{x}$ .

### CENTROID OF SYMMETRICAL SECTIONS:

Sometimes, the given section, whose centroid is required to be found out, is symmetrical about  $X-X$  axis or  $Y-Y$  axis. In such cases, the procedure for calculating the centroid of the body is very much simplified; as it has only to calculate either  $\bar{x}$  or  $\bar{y}$ . This is due to the reason that the centre of gravity of the body will lie on the axis of symmetry.

## CENTROID OF UNSYMMETRICAL SECTIONS:

Sometimes, the given section, whose centroid is required to be found out, is not symmetrical either about  $X-X$  axis or  $Y-Y$  axis. In such cases, it has to find out both the values of  $\bar{x}$  and  $\bar{y}$ .

## CENTROID OF SIMPLE FIGURES FROM FIRST PRINCIPLE:

The calculation of centroids falls within three distinct categories, depending on whether it can model the shape of the body involved as a line, an area, or a volume.

(1) Lines: For a slender rod or wire of length  $L$ , cross-sectional area  $A$ , and density  $\rho$ , Fig.2.54, the body approximates a line segment, and  $dm = \rho A dL$ . If  $\rho$  and  $A$  are constant over the length of the rod, the coordinates of the centre of mass also become the coordinates of the centroid  $C$  of the line segment, which, may be written as

$$\bar{x} = \frac{\int x dL}{L}, \bar{y} = \frac{\int y dL}{L} \text{ and } \bar{z} = \frac{\int z dL}{L}$$

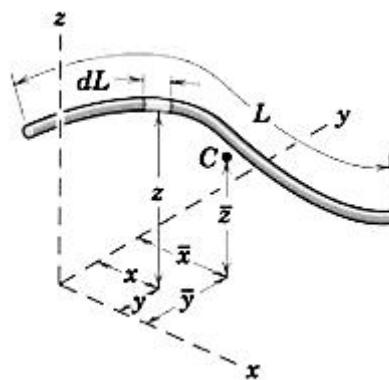


Fig.2.54 Slender rod or wire

Note that in general the centroid  $C$  will not lie on the line. If the rod lies on a single plane, such as the  $x-y$  plane only two coordinates need to be calculated.

(2) Areas: When a body of density  $\rho$  has a small but constant thickness  $t$ , it can model as a surface area  $A$ , Fig.2.55. The mass of an element becomes  $dm = \rho t dA$ . Again,

if  $p$  and  $t$  are constant over the entire area, the coordinates of the centre of mass of the body also become the coordinates of the centroid  $C$  of the surface area, and the coordinates may be written

$$\bar{x} = \frac{\int x dA}{A}, \bar{y} = \frac{\int y dA}{A} \text{ and } \bar{z} = \frac{\int z dA}{A}$$

The numerators in above equations are called the first moments of area. If the surface is curved, as illustrated in Fig.2.55 with the shell segment, all three coordinates will be involved. The centroid  $C$  for the curved surface will in general not lie on the surface. If the area is a flat surface in, say, the  $x$ - $y$  plane, only the coordinates of  $C$  in that plane need to be calculated.

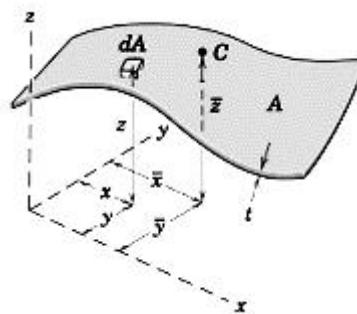


Fig.2.55 Surface area

(3) Volumes: For a general body of volume  $V$  and density  $p$ , the element has a mass  $dm = \rho dV$ . The density  $\rho$  cancels if it is constant over the entire volume, and the coordinates of the centre of mass also become the coordinates of the centroid  $C$  of the body.

$$\bar{x} = \frac{\int x dV}{V}, \bar{y} = \frac{\int y dV}{V} \text{ and } \bar{z} = \frac{\int z dV}{V}$$

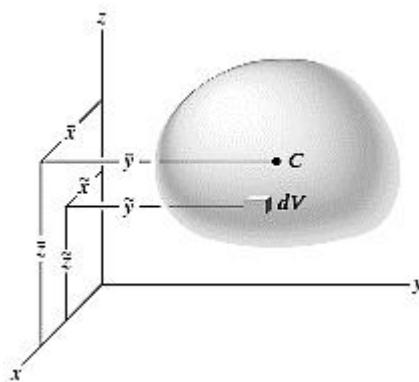


Fig.2.56

These equations represent a balance of the moments of the volume of the body. Therefore, if the volume possesses two planes of symmetry, then its centroid must lie along the line of intersection of these two planes. For example, the cone in Fig.2.57 has a centroid that lies on the  $y$  axis so that  $\bar{x} = \bar{z} = 0$ . The location  $\bar{y}$  can be found using a single integration by choosing a differential element represented by a *thin disk* having a thickness  $dy$  and radius  $r = z$ . Its volume is  $dV = \pi r^2 dy = \pi z^2 dy$  and its centroid is at  $\tilde{x} = 0$ ,  $\tilde{y} = y$  and  $\tilde{z} = 0$ .

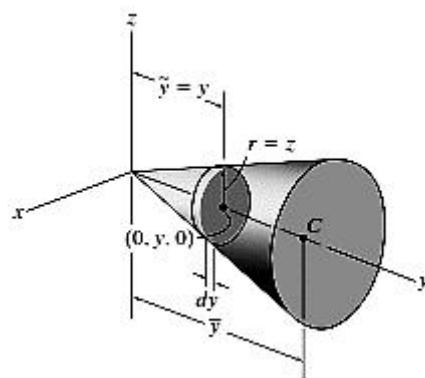


Fig.2.57 Cone solids

**SP2.15:** Locate the centroid of the rod bent into the shape of a parabolic arc as shown in fig.2.58.

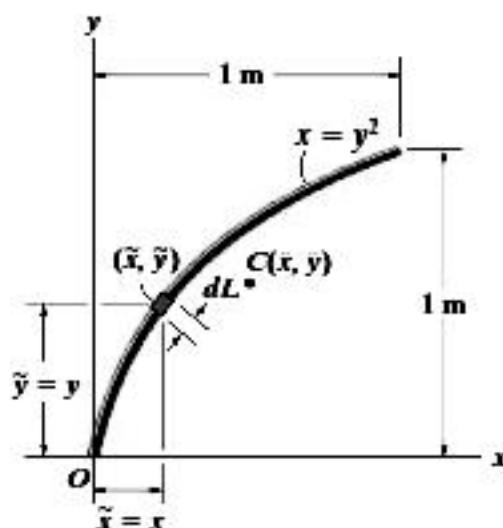


Fig.2.58 Rod bent

**Solution:**

The differential element of the length  $dL$  can be expressed in terms of the differential  $dx$  and  $dy$  using the Pythagorean theorem.

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} dy$$

Since  $x = y^2$ , then  $\frac{dx}{dy} = 2y$ . Therefore expressing  $dL$  in terms of  $y$  and  $dy$ , then

$$dL = \sqrt{(2y)^2 + 1} dy$$

Assume the centroid of the element is located at  $\bar{x} = x$  and  $\bar{y} = y$

From the equation,

$$\bar{x} = \frac{\int x dL}{L} = \frac{\int_0^1 x \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy} = \frac{\int_0^1 x \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy} = \frac{\int_0^1 y^2 \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy}$$

$$\bar{x} = \frac{0.6063}{1.479} = 0.410m$$

$$\bar{y} = \frac{\int y dL}{L} = \frac{\int_0^1 y \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy} = \frac{\int_0^1 y \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy} = \frac{\int_0^1 y \sqrt{4y^2 + 1} dy}{\int_0^1 \sqrt{4y^2 + 1} dy}$$

$$\bar{y} = \frac{0.848}{1.479} = 0.574m$$

**SP2.16:** Locate the centroid of a circular arc as shown in fig.2.59

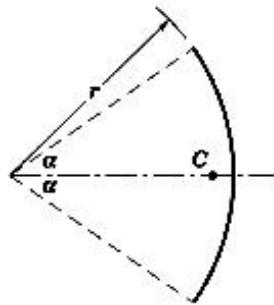


Fig.2.59 Circular arc

**Solution:**

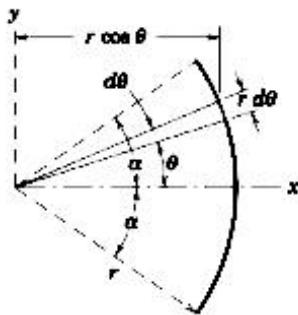


Fig.2.59a

Assume that the given figure is a symmetry about x axis and make the centroid by y axis is zero i.e  $\bar{y} = 0$ . A differential element of arc has length  $dL = rd\theta$ , expressed in polar coordinates and the x – coordinate of the element is  $r \cos \theta$ . Applying the first principle and substituting  $L = 2\pi r$  in equation

$$\bar{x} = \frac{\int x dL}{L}, \Rightarrow \bar{x} = \frac{\int (r \cos \theta)(rd\theta)}{2\pi r}$$

$$\bar{x} = \frac{[r^2 \sin \theta]_{-\pi}^{\pi}}{2\pi r} = \frac{r^2 (\sin \pi - \sin(-\pi))}{2\pi r} = \frac{r^2 (2 \sin \pi)}{2\pi r}$$

$$\bar{x} = \frac{r \sin \pi}{\pi}$$

**SP2.17:** Find the distance  $\bar{h}$  from the base of a triangle of altitude  $h$  to the centroid of its area.

**Solution:**

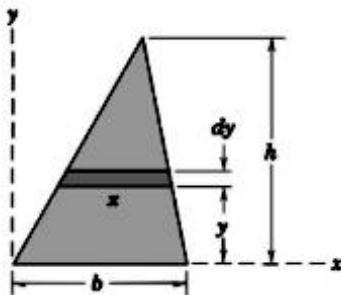


Fig.2.60 Triangle

The x axis is taken to coincide with the base. A differential strip of area  $dA = xdy$  is chosen. By similar triangles  $\frac{x}{h-y} = \frac{b}{h}$ . Applying the equations as

$$\bar{y} = \frac{\int y dA}{A}, \Rightarrow A = \frac{bh}{2} \text{ and } x = \frac{b(h-y)}{h}$$

$$\bar{y} = \frac{\int y x dy}{A}, \Rightarrow \bar{y} = \frac{\int y \left( \frac{b(h-y)}{h} \right) dy}{A}$$

$$\bar{y} = \frac{\int 2(hy - y^2) dy}{h^2} = \frac{2 \left( h \frac{y^2}{2} - \frac{y^3}{3} \right)_0^h}{h^2} = \frac{2 \left( h \frac{h^2}{2} - \frac{h^3}{3} \right)}{h^2} = \frac{2 \left( \frac{3h^3 - 2h^3}{6} \right)}{h^2}$$

$$\bar{y} = \frac{h}{3}$$

**SP2.18:** Locate the centroid of the area of a circular sector with respect to its vertex.

**Solution: method 1,**

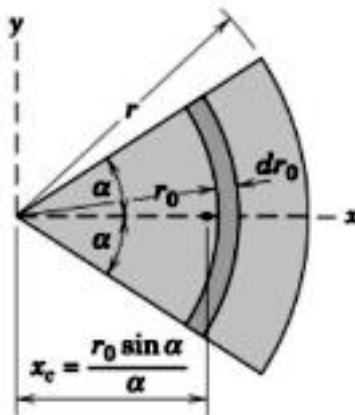


Fig.2.61 Circular sector

The x axis is chosen as the axis of symmetry and  $\bar{y} = 0$ . The area may be covered by moving an element in the form of a partial circular ring as shown in the figure from the centre to the outer periphery. The radius of the ring is  $r_o$  and its thickness is  $dr_o$ , so that its area is  $dA = 2r_o r dr_o$ .

The x coordinate to the centroid of the circular arc element is  $\bar{x}_c = \frac{r_o \sin r}{r}$ ,

where  $r_o$  replaces  $r$  in the formula. Thus, the equations as

$$\bar{x} = \frac{\int \bar{x}_c dA}{A} = \frac{\int_0^r \left( \frac{r_o \sin r}{r} \right) (2r_o r dr_o)}{\frac{2r}{2f} (fr^2)} = \frac{\frac{2}{3} r^3 \sin r}{r^2 r}$$

$$\bar{x} = \frac{2}{3} \frac{r \sin r}{r}$$

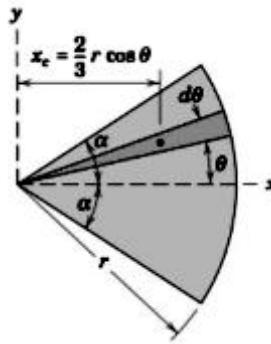
**Method 2:**

Fig.2.61a

The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle shown in the figure and it has an area  $dA = \left(\frac{r}{2}\right)(rd\alpha)$ , where higher order terms are neglected. The centroid of the triangular element of area is two thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is  $\bar{x}_c = \frac{2}{3}r \cos \alpha$ . The equation as

$$\bar{x} = \frac{\int \bar{x}_c dA}{A} = \frac{\int_{-\pi}^{\pi} \left(\frac{2}{3}r \cos \alpha\right) \left(\frac{1}{2}r^2 d\alpha\right)}{r^2 \pi} = \frac{\frac{2}{3}r^3 \sin \pi}{r^2 \pi}$$

$$\bar{x} = \frac{2}{3} \frac{r \sin \pi}{\pi}$$

$$\bar{x} = \frac{2}{3} \frac{r \sin \pi}{\pi}$$

**SP2.19:** Locate the centroid for the area of a quarter circle as shown in fig.2.62

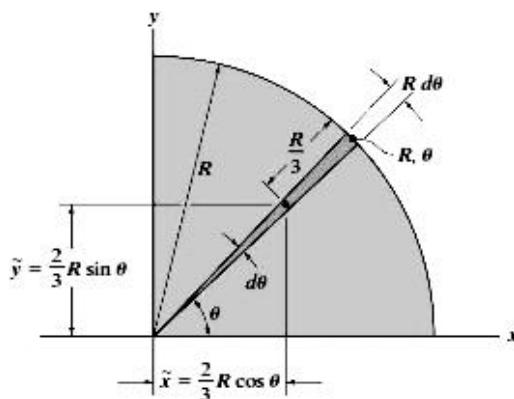


Fig.2.62 Quarter circle

**Solution:**

The area of the element is  $dA = \frac{1}{2}(R)(Rd\theta) = \frac{R^2}{2}d\theta$  and the centroid of the element (assumed as triangle) is located at  $\tilde{x} = \frac{2}{3}R \cos \theta$  and  $\tilde{y} = \frac{2}{3}R \sin \theta$ .

From the equations

$$\tilde{x} = \frac{\int x dA}{dA} = \frac{\int_0^{\pi/2} x \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\int_0^{\pi/2} \left(\frac{2}{3}R \cos \theta\right) \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right) \int_0^{\pi/2} \cos \theta d\theta}{\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi}$$

$$\tilde{y} = \frac{\int y dA}{dA} = \frac{\int_0^{\pi/2} y \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\int_0^{\pi/2} \left(\frac{2}{3}R \sin \theta\right) \frac{R^2}{2} d\theta}{\int_0^{\pi/2} \frac{R^2}{2} d\theta} = \frac{\left(\frac{2}{3}R\right) \int_0^{\pi/2} \sin \theta d\theta}{\int_0^{\pi/2} d\theta} = \frac{4R}{3\pi}$$

**SP2.20:** Locate the centroid of the area as shown in fig.2.63

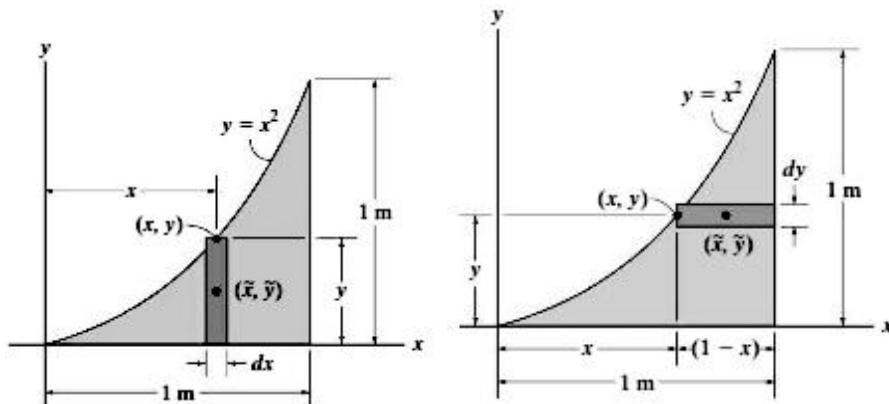


Fig.2.63

**Solution: method 1**

The area of the element is  $dA = ydx$ , and its centroid is located at  $\tilde{x} = x$  and  $\tilde{y} = \frac{y}{2}$

From the equations,

$$\bar{x} = \frac{\int x dA}{dA} = \frac{\int_0^{1m} \tilde{x} dA}{\int_0^{1m} dA} = \frac{\int_0^{1m} xy dx}{\int_0^{1m} y dx} = \frac{\int_0^{1m} x^3 dx}{\int_0^{1m} x^2 dx} = \frac{0.250}{0.333} = 0.75m$$

$$\bar{y} = \frac{\int y dA}{dA} = \frac{\int_0^{1m} \tilde{y} dA}{\int_0^{1m} dA} = \frac{\int_0^{1m} \left(\frac{y}{2}\right) y dx}{\int_0^{1m} y dx} = \frac{\int_0^{1m} \left(\frac{x^2}{2}\right) x^2 dx}{\int_0^{1m} x^2 dx} = \frac{0.100}{0.333} = 0.3m$$

### Method 2:

The area of the element is  $dA = (1-x)dy$ , and its centroid is located at

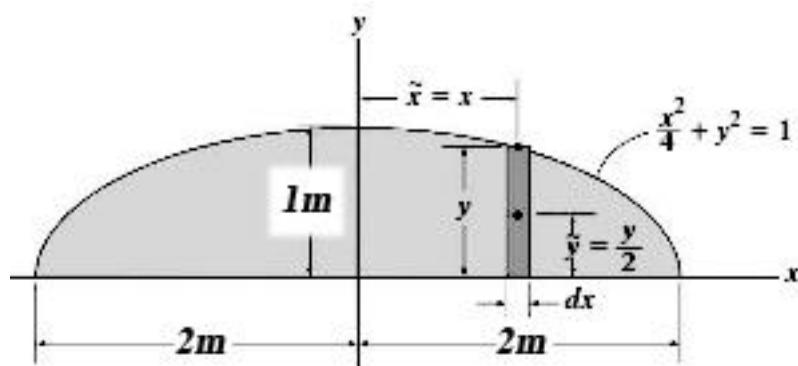
$$\tilde{x} = x + \left(\frac{1-x}{2}\right) = \frac{1+x}{2} \text{ and } \tilde{y} = y$$

From the equations,

$$\bar{x} = \frac{\int x dA}{dA} = \frac{\int_0^{1m} \tilde{x} dA}{\int_0^{1m} dA} = \frac{\int_0^{1m} \left(\frac{1+x}{2}\right) (1-x) dy}{\int_0^{1m} (1-x) dy} = \frac{\int_0^{1m} (1-y) dy}{\int_0^{1m} (1-\sqrt{y}) dy} = \frac{0.250}{0.333} = 0.75m$$

$$\bar{y} = \frac{\int y dA}{dA} = \frac{\int_0^{1m} \tilde{y} dA}{\int_0^{1m} dA} = \frac{\int_0^{1m} y (1-x) dy}{\int_0^{1m} (1-x) dy} = \frac{\int_0^{1m} (y - y^{3/2}) dy}{\int_0^{1m} (1-\sqrt{y}) dy} = \frac{0.100}{0.333} = 0.3m$$

**SP2.21:** Locate the centroid of the semi elliptical area as shown in fig.2.64



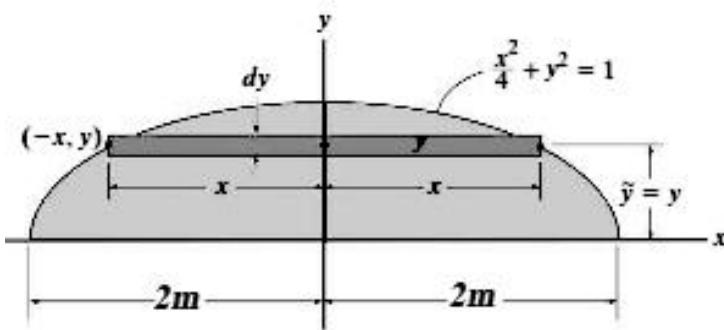


Fig.2.64

**Solution: Method 1**

The area of the element is  $dA = ydx$  and its centroid is located at  $\tilde{x} = x$  and  $\tilde{y} = \frac{y}{2}$

. Since the area is symmetrical about the y axis, then  $\bar{x} = 0$ .

From the equation,

$$\bar{y} = \frac{\int y dA}{dA} = \frac{\int_{-2m}^{2m} \tilde{y} dA}{\int_{-2m}^{2m} dA} = \frac{\int_{-2m}^{2m} \frac{y}{2} (ydx)}{\int_{-2m}^{2m} ydx} = \frac{\int_{-2m}^{2m} \left(1 - \frac{x^2}{4}\right) dx}{\int_{-2m}^{2m} \sqrt{1 - \frac{x^2}{4}} dx} = \frac{4/3}{f} = 0.424m$$

**Method 2:**

The area of the element is  $dA = xdy$  and its centroid is located at  $\tilde{x} = 0$  and  $\tilde{y} = y$

. Since the area is symmetrical about the y axis, then  $\bar{x} = 0$ . Here  $x = 2\sqrt{1 - y^2}$

From the equation,

$$\bar{y} = \frac{\int y dA}{dA} = \frac{\int_0^{1m} \tilde{y} dA}{\int_0^{1m} dA} = \frac{\int_0^{1m} y (2x dy)}{\int_0^{1m} (2x dy)} = \frac{\int_0^{1m} y 4\sqrt{1 - y^2} dy}{\int_0^{1m} 4\sqrt{1 - y^2} dy} = \frac{4/3}{f} = 0.424m$$

**SP2.22:** Locate the centroid of the volume of a hemisphere of radius with respect to its base.

**Solution:**

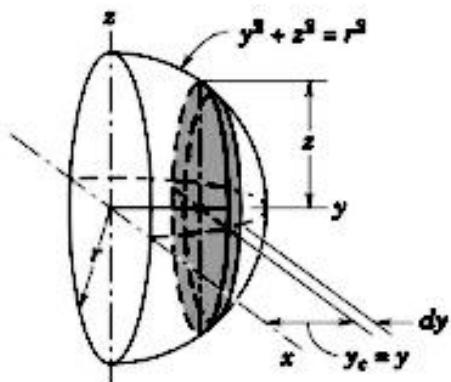


Fig.2.65 Hemisphere solids

With the axes chosen as shown in the figure,  $\bar{x} = \bar{z} = 0$  by symmetry. The most convenient is a circular slice of thickness  $dy$  parallel to the  $x - z$  plane. Since the hemisphere intersect the  $y - z$  plane in the circle  $y^2 + z^2 = r^2$ , the radius of the circular slice is  $z = \sqrt{r^2 - y^2}$ . The volume of the elemental slice becomes

$$dV = f(r^2 - y^2)dy$$

The equation becomes,

$$\bar{y} = \frac{\int \bar{y}_c dV}{V}, \Rightarrow \bar{y} \int_0^r f(r^2 - y^2)dy = \int_0^r yf(r^2 - y^2)dy$$

Where,  $\bar{y}_c = y$  and integrating the above gives

$$\frac{2}{3}fr^3\bar{y} = \frac{1}{4}fr^4, \Rightarrow \bar{y} = \frac{3}{8}r$$

## CENTROID OF COMPOSITE SECTIONS:

A **composite body** consists of a series of connected “simpler” shaped bodies, which may be rectangular, triangular, semi-circular, etc. Such a body can often be sectioned or divided into its composite parts and, provided location of the centroid of

each of these parts are known, it can then eliminate the need for integration to determine the centroid for the entire body. The centroid by moments is used as

$$\bar{x} = \frac{\sum ax}{A} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots}{a_1 + a_2 + a_3 + \dots}$$

**SP2.23:** Find the centre of gravity of a 100mm x 150mm x 30mm of T section.

**Solution:**

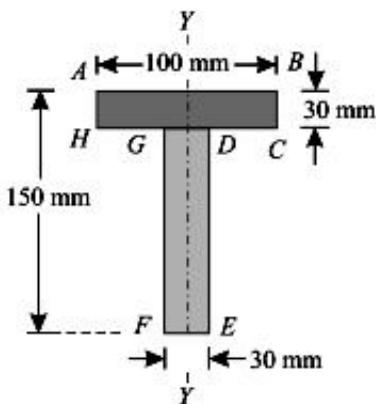


Fig.2.66 T section

As the section is symmetrical about  $Y-Y$  axis, bisecting the web, therefore its centre of gravity will lie on this axis. Split up the section into two rectangles  $ABCH$  and  $DEFG$ .

Let bottom of the web  $FE$  be the axis of reference.

1) Rectangle  $ABCH$ ,

$$a_1 = 100 \times 30 = 3000 \text{ mm}^2 \text{ and}$$

$$y_1 = \left( 150 - \frac{30}{2} \right) = 135 \text{ mm}$$

2) Rectangle  $DEFG$ ,

$$a_2 = 120 \times 30 = 3600 \text{ mm}^2 \text{ and}$$

$$y_2 = \left( \frac{120}{2} \right) = 60 \text{ mm}$$

We know that distance between centroid of the section and bottom of the flange FE.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(3000 \times 135) + (3600 \times 60)}{3000 + 3600} = 94.1 \text{ mm}$$

**SP2.24:** Find the centroid of an unequal angle section 100mm x 80mm x 20mm.

**Solution:**

As the section is not symmetrical about any axis, therefore we have to find out the values of  $x$  and  $y$  for the angle section. Split up the section into two rectangles.

Let left face of the vertical section and bottom face of the horizontal section be axes of reference.

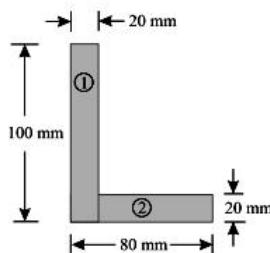


Fig.2.67 L section

1) Rectangle 1,

$$a_1 = 100 \times 20 = 2000 \text{ mm}^2$$

$$x_1 = \left( \frac{20}{2} \right) = 10 \text{ mm} \text{ and}$$

$$y_1 = \left( \frac{100}{2} \right) = 50 \text{ mm}$$

2) Rectangle 2,

$$a_2 = (80 - 20) \times 20 = 1200 \text{ mm}^2$$

$$x_2 = \left( 20 + \frac{60}{2} \right) = 50 \text{ mm} \text{ and}$$

$$y_2 = \left( \frac{20}{2} \right) = 10 \text{ mm}$$

We know that distance between centroid of the section and left face.

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2} = \frac{(2000 \times 10) + (1200 \times 50)}{2000 + 1200} = 25\text{mm}$$

Similarly, distance between centroid of the section and bottom face.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{(2000 \times 50) + (1200 \times 10)}{2000 + 1200} = 35\text{mm}$$

**SP2.25:** Find the centroid of a segment of height 30mm of a sphere of radius 60mm.

**Solution:**

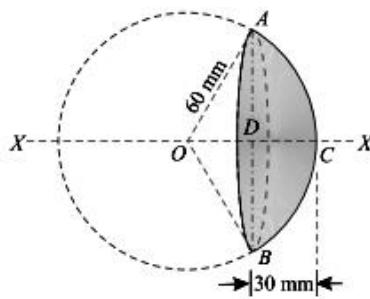


Fig.2.68 Segment of sphere

Let  $O$  be the centre of the given sphere and  $ABC$  is the segment of this sphere as shown in figure. As the section is symmetrical about  $X-X$  axis, therefore its centre of gravity lies on this axis.

Let  $O$  be the reference point. We know that centre of gravity of the segment of sphere.

$$\bar{x} = \frac{3(2r - h)^2}{4(3r - h)} = \frac{3(2 \times 60 - 30)^2}{4(3 \times 60 - 30)} = 40.5\text{mm}$$

**SP2.26:** Locate the centroid of the wire as shown in fig.2.69

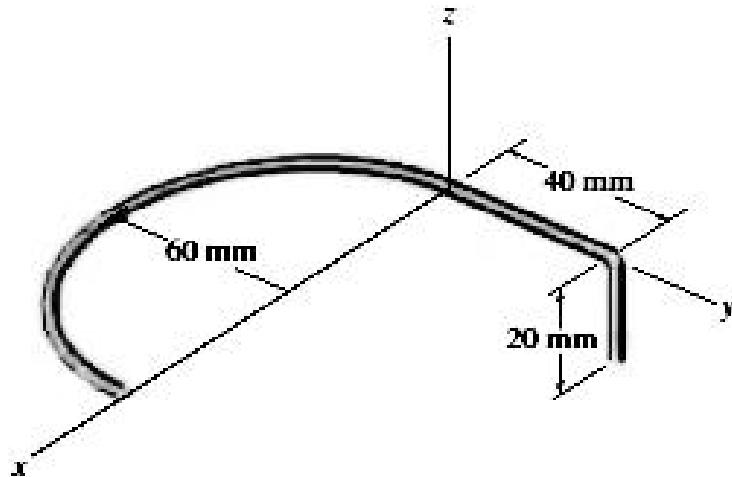


Fig.2.69 Wire geometry

**Solution:**

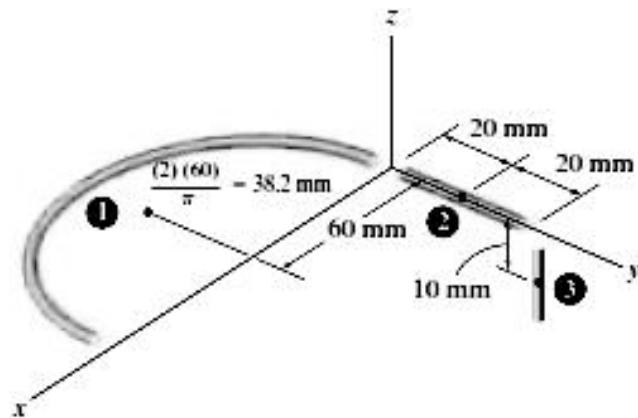


Fig.2.69

The wire is divided into three segments as shown in figure.

i) Segment 1:

$$L_1 = fr = f \times 60 = 188.5 \text{ mm}, \bar{x}_1 = r = 60 \text{ mm},$$

$$\bar{y}_1 = \frac{\int_0^L y dL}{L} = \frac{\int_0^{\pi/2} r \sin \theta (r d\theta)}{\int_0^{\pi/2} r d\theta} = \frac{\int_0^{\pi/2} r^2 \sin \theta d\theta}{\int_0^{\pi/2} r d\theta} = \frac{2r}{f} = \frac{2 \times 60}{f} = 38.2 \text{ mm}$$

$$\text{and } \bar{z}_1 = 0$$

ii) Segment :

$$L_2 = 40\text{mm}$$

$$\bar{x}_2 = 0, \bar{y}_2 = \frac{y_2}{2} = 20\text{mm} \text{ and } \bar{z}_2 = 0$$

iii) Segment:

$$L_3 = 20\text{mm}$$

$$\bar{x}_3 = 0, \bar{y}_3 = 0 \text{ and } \bar{z}_3 = \frac{z_3}{2} = 10\text{mm}$$

We know that distance between centroid of the section

$$\bar{x} = \frac{L_1 x_1 + L_2 x_2 + L_3 x_3}{L_1 + L_2 + L_3} = \frac{(188.5 \times 60) + (40 \times 0) + (20 \times 0)}{188.5 + 40 + 20} = 45.5\text{mm}$$

$$\bar{y} = \frac{L_1 y_1 + L_2 y_2 + L_3 y_3}{L_1 + L_2 + L_3} = \frac{(188.5 \times -38.2) + (40 \times 20) + (20 \times 40)}{188.5 + 40 + 20} = -22.5\text{mm}$$

$$\bar{z} = \frac{L_1 z_1 + L_2 z_2 + L_3 z_3}{L_1 + L_2 + L_3} = \frac{(188.5 \times 0) + (40 \times 0) + (20 \times -10)}{188.5 + 40 + 20} = -0.805\text{mm}$$

**SP2.27:** A semi-circular area is removed from a trapezium as shown in fig.2.70 Find the centroid of the remaining area.

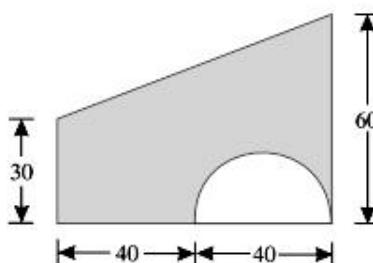


Fig.2.70 Composite section

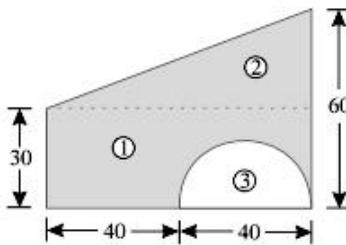
**Solution:**

Fig.2.70a

As the section is not symmetrical about any axis, therefore we have to find out the values of  $\bar{x}$  and  $\bar{y}$  for the area. Split up the area into three parts as shown in Figure. Let left face and base of the trapezium be the axes of reference.

1) Rectangle ,

$$a_1 = 80 \times 30 = 2400 \text{ mm}^2$$

$$x_1 = \left( \frac{80}{2} \right) = 40 \text{ mm and}$$

$$y_1 = \left( \frac{30}{2} \right) = 15 \text{ mm}$$

2) Triangle,

$$a_2 = \frac{80 \times 30}{2} = 1200 \text{ mm}^2$$

$$x_2 = \left( \frac{80 \times 2}{3} \right) = 53.3 \text{ mm and}$$

$$y_2 = 30 + \frac{30}{3} = 40 \text{ mm}$$

3) Semicircle,

$$a_3 = \frac{\pi}{2} \times r^2 = \frac{\pi}{2} \times (20)^2 = 628.3 \text{ mm}^2$$

$$x_3 = \left( 40 + \frac{40}{3} \right) = 60 \text{ mm and}$$

$$y_3 = \frac{4r}{3f} = \frac{4 \times 20}{3f} = 8.5 \text{ mm}$$

We know that distance between centroid of the section and left face of trapezium.

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 - a_3 x_3}{a_1 + a_2 - a_3} = \frac{(2400 \times 40) + (1200 \times 53.3) - (628.3 \times 60)}{2400 + 1200 - 628.3} = 41.1 \text{ mm}$$

Similarly, distance between centroid of the section and base of the trapezium.

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2 - a_3 y_3}{a_1 + a_2 - a_3} = \frac{(2400 \times 15) + (1200 \times 40) - (628.3 \times 8.5)}{2400 + 1200 - 628.3} = 26.5 \text{ mm}$$

**SP2.28:** A solid consists of a right circular cylinder and a hemisphere with a cone cut out from the cylinder as shown in fig.2.71 Find the centroid of the body.

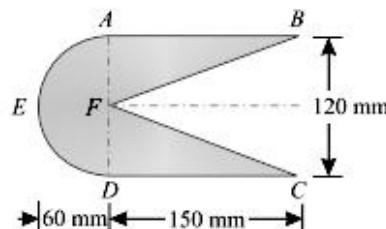


Fig.2.71 Composite solids

### Solution:

As the solid is symmetrical about horizontal axis, therefore its centroid lie on this axis.

Let the left edge of the hemispherical portion be the axis of reference.

i) Hemisphere ADE,

$$V_1 = \frac{2f}{3} \times r^3 = \frac{2f}{3} \times 60^3 = 144000f \text{ mm}^3 \text{ and}$$

$$x_1 = \frac{5r}{8} = \frac{5 \times 60}{8} = 37.5 \text{ mm}$$

ii) Right circular cylinder ABCD

$$V_2 = f \times r^2 \times h = f \times 60^2 \times 150 = 540000f \text{ mm}^3 \text{ and}$$

$$x_2 = 60 + \frac{150}{2} = \frac{5 \times 60}{8} = 135 \text{ mm}$$

iii) Cone BCF

$$v_3 = \frac{\pi}{3} \times r^2 \times h = \frac{\pi}{3} \times 60^2 \times 150 = 180000\pi \text{ mm}^3 \text{ and}$$

$$x_3 = 60 + 150 \times \frac{3}{4} = 172.5 \text{ mm}$$

We know that distance between centroid of the solid and left edge of the hemisphere, then

$$\bar{x} = \frac{v_1 x_1 + v_2 x_2 - v_3 x_3}{v_1 + v_2 - v_3}$$

$$\bar{x} = \frac{(144000\pi \times 37.5) + (540000\pi \times 135) - (180000\pi \times 172.5)}{144000\pi + 540000\pi - 180000\pi} = 93.75$$

## CENTRE OF GRAVITY AND ITS IMPLICATIONS:

The **centre of gravity** is a geometric property of any object. The centre of gravity is the average location of the weight of an object. It can completely describe the motion of any object through space in terms of the **translation** of the centre of gravity of the object from one place to another, and the **rotation** of the object about its centre of gravity if it is free to rotate. If the object is confined to rotate about some other point, like a hinge, it can still describe its motion.

In flight, both airplanes and rockets rotate about their centres of gravity. A kite, on the other hand, rotates about the bridle point. But the trim of a kite still depends on the location of the centre of gravity relative to the bridle point, because for every object the weight always acts through the centre of gravity.

Determining the centre of gravity is very important for any flying object. How do engineers determine the location of the centre of gravity for an aircraft which they are designing? In general, determining the centre of gravity (cg) is a complicated

procedure because the mass (and weight) may not be uniformly distributed throughout the object. The general case requires the use of calculus. If the mass is uniformly distributed, the problem is greatly simplified. If the object has a line (or plane) of **symmetry**, the centre of gravity lies on the line of symmetry. For a solid block of uniform material, the centre of gravity is simply at the average location of the physical dimensions. (For a rectangular block, 50 X 20 X 10, the centre of gravity is at the point (25,10, 5) ). For a triangle of height  $h$ , the centre of gravity is at  $h/3$ , and for a semi-circle of radius  $r$ , the centre of gravity is at  $\frac{4r}{3\pi}$  where  $\pi$  is ratio of the circumference of the circle to the diameter.

For a general shaped object, there is a simple mechanical way to determine the centre of gravity:

1. If just balance the object using a string or an edge, the point at which the object is balanced is the centre of gravity. (Just like balancing a pencil on your finger!)
2. Another, more complicated way, is a two-step method shown on the slide.

In Step 1, you hang the object from any point and drop a weighted string from the same point. Draw a line on the object along the string.

For Step 2, repeat the procedure from another point on the object and now it has two lines drawn on the object which intersect. The centre of gravity is the point where the lines intersect. This procedure works well for irregularly shaped objects that are hard to balance.

If the mass of the object is not uniformly distributed, it must use calculus to determine centre of gravity. It will use the symbol  $\int dW$  to denote the integration of a continuous function with respect to weight. Then the centre of gravity can be determined from:

$$\bar{x}W = \int x dW$$

Where  $x$  is the distance from a reference line,  $dw$  is an increment of weight, and  $W$  is the total weight of the object. To evaluate the right side, it has to determine how the weight varies geometrically. From the weight equation, we know that:

$$W = m g$$

Where  $m$  is the mass of the object, and  $g$  is the gravitational constant. In turn, the mass  $m$  of any object is equal to the density ..., of the object times the volume,  $V$ :

$$m = \dots V$$

Now, it can combine the last two equations:

$$W = \dots g V$$

Then

$$dW = \dots g dV$$

$$dW = \dots (x, y, z) g (dx dy dz)$$

If it has a functional form for the mass distribution, then it can solve the equation for the center of gravity:

$$c..g \times W = g \times \iiint \dots (x, y, z) dx dy dz$$

where  $\iiint$  indicates a triple integral over  $dx$ ,  $dy$  and  $dz$ . If don't know the functional form of the mass distribution, it can numerically integrate the equation using a spreadsheet. Divide the distance into a number of small volume segments and determining the average value of the weight/volume (density times gravity) over that small segment. Taking the sum of the average value of the weight/volume times the distance times the volume segment divided by the weight will produce the centre of gravity.

## MOMENT OF INERTIA:

The moment of a force ( $F$ ) about a point, is the product of the force and perpendicular distance ( $d$ ) between the point and the line of action of the force (i.e.  $M = dF$ ). This moment is also called first moment of force. If this moment is again multiplied by the perpendicular distance ( $d$ ) between the point and the line of action of the force i.e.  $M = (d.d)F = d^2F$ , then this quantity is called moment of the moment of a force or second moment of force or moment of inertia (briefly written as M.I. or I). Sometimes, instead of force, area or mass of a figure or body is taken into consideration. Then the second moment is known as second moment of area or second moment of mass. But all such second moments are broadly termed as moment of inertia.

## AREA MOMENT OF INERTIA- DEFINITION:

Consider a plane area, whose moment of inertia is required to be found out. Split up the whole area into a number of small elements.

Let  $a_1, a_2, a_3, \dots$  = Areas of small elements, and

$r_1, r_2, r_3, \dots$  = Corresponding distances of the elements from the line about which the moment of inertia is required to be found out.

Now the moment of inertia of the area,

$$I = a_1r^2 + a_2r^2 + a_3r^3 + \dots = \sum ar^2$$

## MOMENT OF INERTIA OF PLANE SECTIONS FROM FIRST PRINCIPLES:

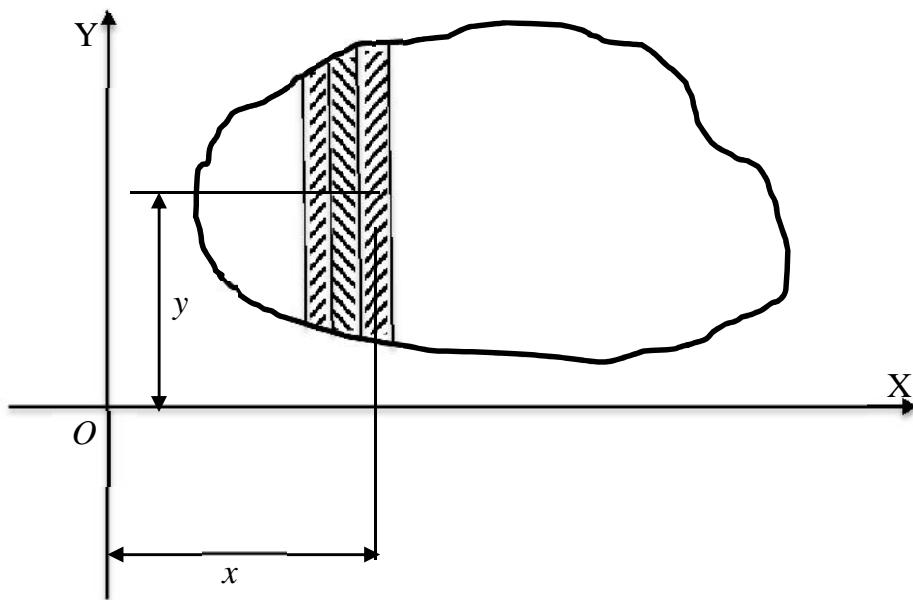


Fig.2.72 Moment of inertia of plane sections

The moment of inertia of an area may also be found out by the method of integration as discussed below:

Consider a plane figure, whose moment of inertia is required to be found out about X-X axis and Y-Y axis as shown in Fig.2.72. Let us divide the whole area into a no. of strips. Consider one of these strips.

Let  $dA$  = Area of the strip

$x$  = Distance of the centre of gravity of the strip on X-X axis and

$y$  = Distance of the centre of gravity of the strip on Y-Y axis.

We know that the moment of inertia of the strip about Y-Y axis

$$= dA \cdot x^2$$

Now the moment of inertia of the whole area may be found out by integrating above equation. *i.e.*,

$$I_{yy} = \sum dA \cdot x^2$$

Similarly  $I_{xx} = \sum dA \cdot y^2$

## THEOREMS OF MOMENT OF INERTIA:

1. Theorem of parallel axis
2. Theorem of perpendicular axis

### 1) Theorem of parallel axis:

It states, *If the moment of inertia of a plane area about an axis through its centre of gravity is denoted by  $I_G$ , then moment of inertia of the area about any other axis  $AB$ , parallel to the first, and at a distance 'h' from the centre of gravity is given by:*

$$I_{AB} = I_G + ah^2$$

where ,  $I_{AB}$  = Moment of inertia of the area about an axis  $AB$ ,

$I_G$  = Moment of Inertia of the area about its centre of gravity

$a$  = Area of the section, and

$h$  = Distance between centre of gravity of the section and axis  $AB$ .

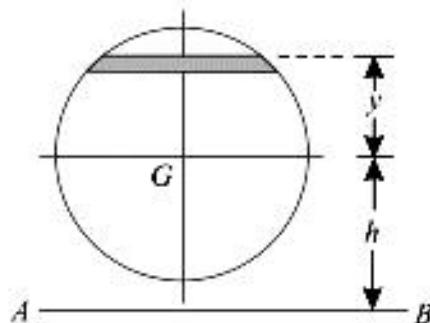


Fig.2.73 Circular lamina

### Proof:

Consider a strip of a circle, whose moment of inertia is required to be found out about a line  $AB$  as shown in Figure.

Let  $da$  = Area of the strip,

$y$  = Distance of the strip from the centre of gravity the section and

$h$  = Distance between centre of gravity of the section and the axis  $AB$ .

We know that moment of inertia of the whole section about an axis passing through the centre of gravity of the section

$$= da \cdot y^2$$

and moment of inertia of the whole section about an axis passing through its centre of gravity,

$$I_G = \sum da \cdot y^2$$

∴ Moment of inertia of the section about the axis  $AB$ ,

$$I_{AB} = \sum da(h + y)^2 = \sum da(h^2 + y^2 + 2hy)$$

$$I_{AB} = (\sum h^2 \cdot da) + (\sum y^2 \cdot da) + (\sum 2hy \cdot da)$$

$$I_{AB} = ah^2 + I_G + 0$$

It may be noted that  $h^2 \cdot da = a \cdot h^2$  and  $y^2 \cdot da = I_G$

Now the  $y \cdot da$  is the algebraic sum of moments of all the areas about an axis through centre of gravity of the section and is equal to  $a \cdot \bar{y}$ , where  $\bar{y}$  is the distance between the section and the axis passing through the centre of gravity, which obviously is zero.

## 2) Theorem of perpendicular axis

It states, If  $I_{XX}$  and  $I_{YY}$  be the moments of inertia of a plane section about two perpendicular axis meeting at  $O$ , the moment of inertia  $I_{ZZ}$  about the axis  $Z-Z$ , perpendicular to the plane and passing through the intersection of  $X-X$  and  $Y-Y$  is given by:

$$I_{ZZ} = I_{XX} + I_{YY}$$

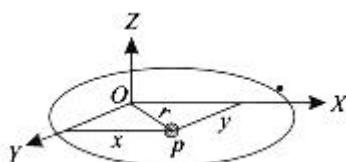


Fig.2.74 Theorem of perpendicular axis

**Proof:**

Consider a small lamina ( $P$ ) of area  $da$  having co-ordinates as  $x$  and  $y$  along  $OX$  and  $OY$  two mutually perpendicular axes on a plane section as shown in Figure.

Now consider a plane  $OZ$  perpendicular to  $OX$  and  $OY$ .

Let ( $r$ ) be the distance of the lamina ( $P$ ) from  $Z-Z$  axis such that  $OP = r$ .

From the geometry of the figure, we find that

$$r^2 = x^2 + y^2$$

We know that the moment of inertia of the lamina  $P$  about  $X-X$  axis,

$$I_{xx} = y^2.da$$

Similarly,  $I_{yy} = x^2.da$  and

$$I_{zz} = z^2.da = r^2.da = (x^2 + y^2).da$$

$$I_{zz} = x^2.da + y^2.da = I_{yy} + I_{xx}$$

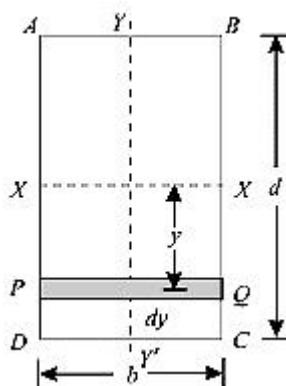
**MOMENT OF INERTIA OF STANDARD SECTIONS:****1) A rectangular section:**

Fig.2.75 Rectangular section

Consider a rectangular section  $ABCD$  as shown in Fig. whose moment of inertia is required to be found out.

Let  $b$  = Width of the section and

$d$  = Depth of the section.

Now consider a strip  $PQ$  of thickness  $dy$  parallel to  $X-X$  axis and at a distance  $y$  from it as shown in the figure

$\therefore$  Area of the strip =  $b \cdot dy$

We know that moment of inertia of the strip about  $X-X$  axis,

$$= \text{Area} \times y^2 = (b \cdot dy) y^2 = b \cdot y^2 \cdot dy$$

Now the moment of inertia of the whole section may be found out by integrating the above equation for the whole length of the lamina i.e. from  $-\frac{d}{2}$  to  $\frac{d}{2}$ .

$$I_{xx} = \int_{-\frac{d}{2}}^{\frac{d}{2}} b \cdot y^2 \cdot dy = b \int_{-\frac{d}{2}}^{\frac{d}{2}} y^2 \cdot dy$$

$$I_{xx} = b \left[ \frac{y^3}{3} \right]_{-\frac{d}{2}}^{\frac{d}{2}} = b \left[ \frac{\left(\frac{d}{2}\right)^3}{3} - \frac{\left(-\frac{d}{2}\right)^3}{3} \right] = \frac{bd^3}{12}$$

Similarly,

$$I_{xx} = \frac{db^3}{12}$$

## 2) Circular Section:

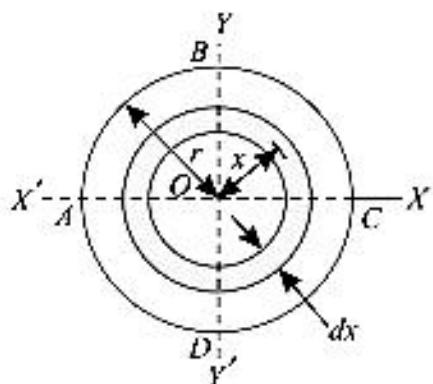


Fig.2.76 Circular section

Consider a circle  $ABCD$  of radius ( $r$ ) with centre  $O$  and  $XX'$  and  $Y-Y'$  be two axes of reference through  $O$  as shown in Figure.

Now consider an elementary ring of radius  $x$  and thickness  $dx$ . Therefore area of the ring,

$$da = 2 \pi x \cdot dx$$

and moment of inertia of ring, about  $X-X$  axis or  $Y-Y$  axis

$$\begin{aligned} &= \text{Area} \times (\text{Distance})^2 \\ &= 2 \pi x \cdot dx \times x^2 \\ &= 2 \pi x^3 \cdot dx \end{aligned}$$

Now the moment of inertia of whole section, about the central axis, can be found out by integrating the above equation for the whole radius of the circle *i.e.*, from 0 to  $r$ .

$$\begin{aligned} I_{zz} &= \int_0^r 2\pi x^3 \cdot dx = 2\pi \int_0^r x^3 \cdot dx \\ I_{zz} &= 2\pi \left[ \frac{x^4}{4} \right]_0^r = \frac{\pi}{2} (r)^4 = \frac{\pi}{32} (d)^4 \quad [\text{since } r = d/2] \end{aligned}$$

We know that from theorem of perpendicular axis that

$$I_{zz} = I_{xx} + I_{yy},$$

$$\text{If } I_{xx} = I_{yy} = \frac{I_{zz}}{2} = \frac{1}{2} \times \frac{\pi}{32} (d)^4$$

### 3) A triangular section:

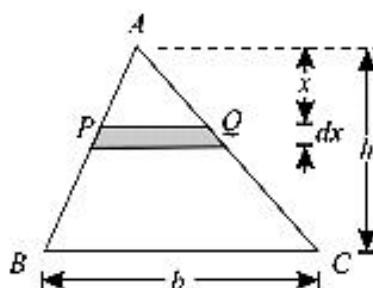


Fig.2.77 Triangular section

Consider a triangular section  $ABC$  whose moment of inertia is required to be found out.

Let,  $b$  = Base of the triangular section and

$h$  = Height of the triangular section.

Now consider a small strip  $PQ$  of thickness  $dx$  at a distance of  $x$  from the vertex  $A$  as shown in Fig. 7.8. From the geometry of the figure, we find that the two triangles  $APQ$  and  $ABC$  are similar.

Therefore

$$\frac{PQ}{BC} = \frac{x}{h} \text{ or } PQ = \frac{BC \cdot x}{h} = \frac{bx}{h} \quad [\text{since } BC = \text{base} = b]$$

We know that area of the strip  $PQ$

$$= \frac{bx}{h} \cdot dx$$

and moment of inertia of the strip about the base  $BC$

$$= \text{Area} \times (\text{Distance})^2 = \frac{bx}{h} dx (h-x)^2 = \frac{bx}{h} (h-x)^2 dx$$

Now moment of inertia of the whole triangular section may be found out by integrating the above equation for the whole height of the triangle *i.e.*, from 0 to  $h$ .

$$I_{BC} = \int_0^h \frac{bx}{h} (h-x)^2 dx$$

$$I_{BC} = \int_0^h \frac{bx}{h} (h^2 + x^2 - 2hx) dx = \frac{b}{h} \int_0^h (xh^2 + x^3 - 2hx^2) dx$$

$$I_{BC} = \frac{b}{h} \left[ \frac{x^2 h^2}{2} + \frac{x^4}{4} - \frac{2hx^3}{3} \right]_0^h = \frac{bh^3}{12}$$

We know that distance between centre of gravity of the triangular section and base  $BC$ .

$$d = \frac{h}{3}$$

Since, the moment of inertia of the triangular section about an axis through its centre of gravity and parallel to X-X axis.

$$I_G = I_{BC} - ad^2$$

$$I_G = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36}$$

Note: The moment of inertia of section about an axis through its vertex and parallel to the base

$$= I_G + ad^2 = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{9bh^3}{36} = \frac{bh^3}{4}$$

### Moment of inertia of composite sections:

The moment of inertia of a composite section may be found out by the following steps:

1. First of all, split up the given section into plane areas (*i.e.*, rectangular, triangular, circular etc., and find the centre of gravity of the section).
2. Find the moments of inertia of these areas about their respective centres of gravity.
3. Now transfer these moment of inertia about the required axis (AB) by the Theorem of Parallel Axis, *i.e.*,

$$I_{AB} = I_G + ah^2$$

where  $I_G$  = Moment of inertia of a section about its centre of gravity and parallel to the axis.

$a$  = Area of the section,

$h$  = Distance between the required axis and centre of gravity of the section.

4. The moments of inertia of the given section may now be obtained by the algebraic sum of the moment of inertia about the required axis.

**Sp2.29:** Find the moment of inertia of the area as shown in fig2.78 About the x axis.

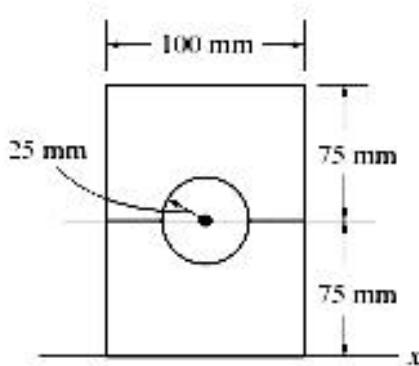


Fig.2.78 A rectangular with circular hole

**Solution:**

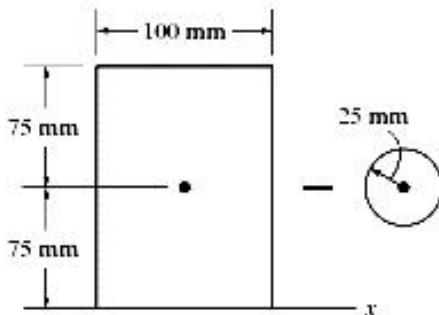


Fig.2.78a

First split the given figure into separate plane areas such as rectangular section and circular section.

From the parallel axis theorem, we know that

1. Rectangular section:

$$I_x = I_G + Ah^2$$

$$I_x^R = \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6) \text{ mm}^4$$

2. Circular section:

$$I_x^C = \frac{1}{4}f(25)^4 + f(25)^2(75)^2 = 11.4(10^6) \text{ mm}^4$$

Composite section:

$$I_x = I_x^R - I_x^C = 112.5(10^6) - 11.4(10^6) = 101.1(10^6) \text{ mm}^4$$

**SP2.30:** Find the moments of inertia for the cross sectional area of the member as shown in fig.2.79 about the x and y centroid axes.

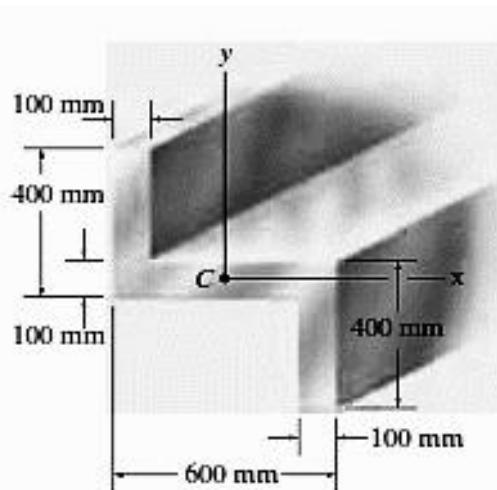


Fig.2.79

**Solution:**

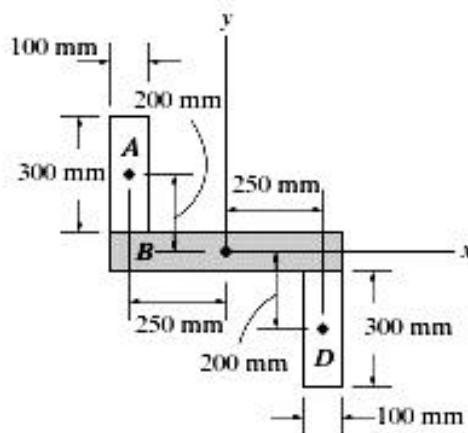


Fig.2.79a

First split the given figure into separate plane areas such as rectangular section and denoted each section as A, B and D.

From the parallel axis theorem, we know that

1. Rectangular section A and D:

$$I_x = I_G + Ah^2$$

$$I_x^A = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2 = 1425(10^6) \text{ mm}^4$$

$$I_y^C = \frac{1}{12}(300)(100)^3 + (300)(100)(250)^2 = 1900(10^6) \text{ mm}^4$$

2. Rectangular section B:

$$I_x^B = \frac{1}{12}(600)(100)^3 + 0 = 50(10^6) \text{ mm}^4$$

$$I_y^B = \frac{1}{12}(100)(600)^3 + 0 = 1800(10^6) \text{ mm}^4$$

Composite section:

$$I_x = I_x^A + I_x^C + I_x^D = 1425(10^6) + 50(10^6) + 1425(10^6) = 2900(10^6) \text{ mm}^4$$

$$I_y = I_y^A + I_y^C + I_y^D = 1900(10^6) + 1800(10^6) + 1900(10^6) = 5600(10^6) \text{ mm}^4$$

## MASS MOMENT OF INERTIA:

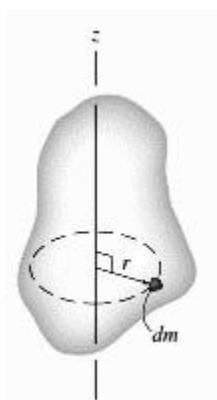


Fig.2.80 Rigid body

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration. Since it is used in dynamics to study rotational motion. Consider the rigid body as shown in Fig.2.80. It defines the *mass moment of inertia* of the body about the *z* axis as

$$I = \int_m r^2 dm$$

Here  $r$  is the perpendicular distance from the axis to the arbitrary element  $dm$ . Since the formulation involves  $r$ , the value of  $I$  is *unique* for each axis about which it is computed. The axis which is generally chosen, however, passes through the body's mass centre  $G$ . Common units used for its measurement are  $\text{kg}\cdot\text{m}^2$ .

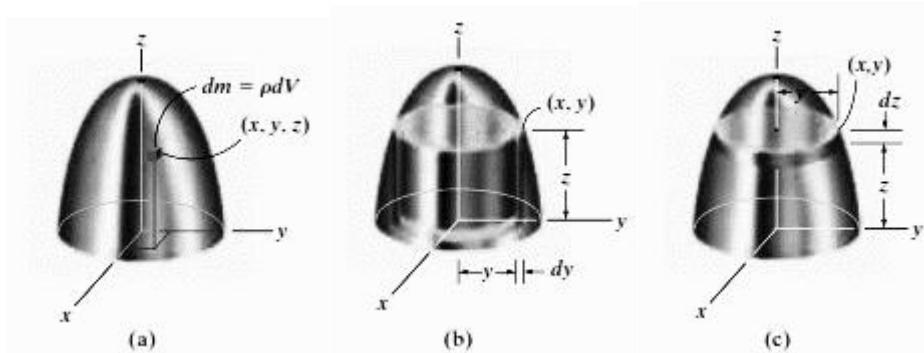


Fig.2.81 solid object

If the body consists of material having a density  $\rho$ , then  $dm = \rho dV$ , Fig.2.81. Substituting this into above equations, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 ... dV$$

For most applications,  $r$  will be a *constant*, and so this term may be factored out of the integral, and the integration is then purely a function of geometry.

$$I = ... \int_V r^2 dV$$

### Mass moment inertia of circular plate:

Integrating to find the moment of inertia of circular plate i.e., a uniform thin disk about an axis through its centre (Fig.2.82).

Since the circular plate is thin, it can take the mass as distributed entirely in the  $xy$ -plane and start with the relationship for the **surface mass density**, which is the mass per unit surface area. Since it is uniform, the surface mass density  $\rho$  is constant.

$$\dots = \frac{m}{A}$$

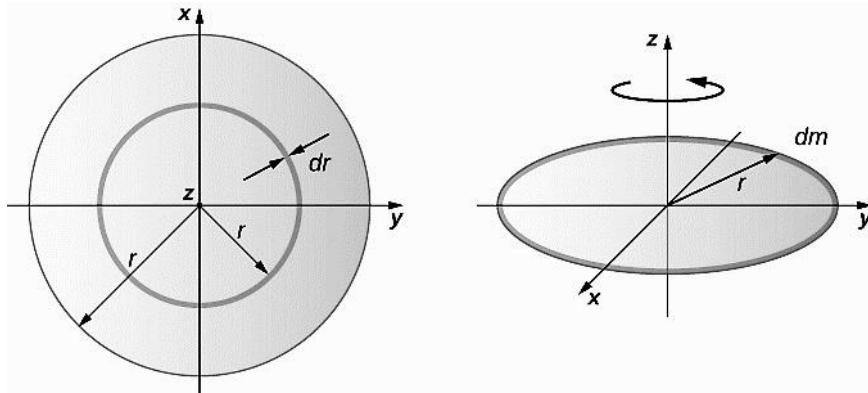


Fig.2.82 Circular plate

$$\text{So } dm = \dots (dA)$$

The area can be made up of a series of thin rings, where each ring is a mass increment  $dm$  of radius  $r$  equidistant from the axis, as shown in part (b) of the figure. The infinitesimal area of each ring  $dA$  is therefore given by the length of each ring ( $2\pi r$ ) times the infinitesimal width of each ring  $dr$ :

$$A = \pi r^2, \quad dA = d(\pi r^2) = 2\pi r dr$$

The full area of the circular plate is then made up from adding all the thin rings with a radius range from 0 to  $R$ . This radius range then becomes our limits of integration for  $dr$ , that is, it integrate from  $r=0$  to  $r=R$ . Putting this all together, we have

$$I_{zz} = \int_0^R r^2 \dots (2\pi r) dr = 2\pi \int_0^R r^3 dr = 2\pi \left[ \frac{r^4}{4} \right]_0^R$$

$$I_{zz} = 2\pi \left[ \frac{R^4}{4} - 0 \right] = 2\pi \frac{m}{A} \frac{R^4}{4} = 2\pi \frac{m}{fR^2} \frac{R^4}{4} = \frac{1}{2} mR^2$$

**SP2.31:** A 25-kg child stands at a distance  $r=1.0\text{m}$  from the axis of a rotating merry-go-round (Fig.2.83). The merry-go-round can be approximated as a uniform solid disk with a mass of 500 kg and a radius of 2.0 m. Find the moment of inertia of this system.

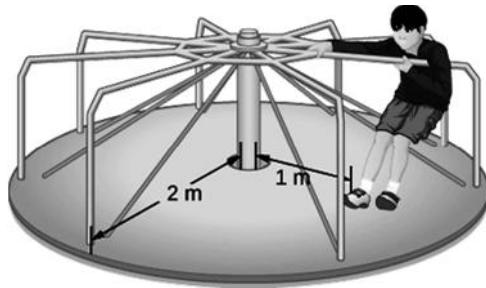


Fig.2.83 Merry – go – round with child

**Solution:**

For the child,  $I_c = m_c r^2$  and for the merry – go – round  $I_m = \frac{1}{2} m_m r^2$ . Therefore

$$I_{total} = 25(1)^2 + \frac{1}{2}(500)(2)^2 = 25 + 1000 = 1025 \text{ kg.m}^2$$

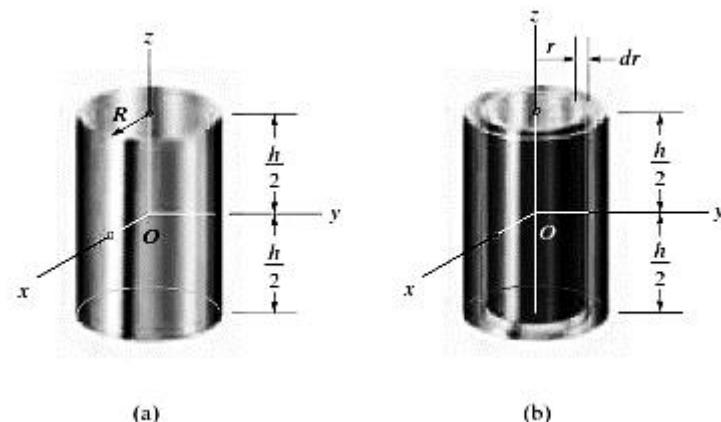
**Mass moment inertia of Cylinder:**

Fig.2.84 Cylinder solids

The shell element is considered to resolve the mass moment of inertia of cylinder as shown in fig.2.84b. Thus only a single integration is required. The volume of the element is  $dV = (2\pi r)(h)dr$  and so its mass is  $dm = \rho dV = \rho(2\pi r h r dr)$ . Since the entire element lies at the same distance  $r$  from the  $z$  axis, the moment of inertia of the element is

$$dI_Z = \int_m r^2 dm = \dots 2fh \int_0^R r^3 dr = \frac{f}{2} R^2 h$$

Since the mass of the cylinder is

$$m = \int_m dm = \dots 2\pi h \int_0^R r dr = \dots \pi h R^2$$

Then,

$$I_{zz} = \frac{1}{2} m R^2$$

### Mass moment inertia of Cone:

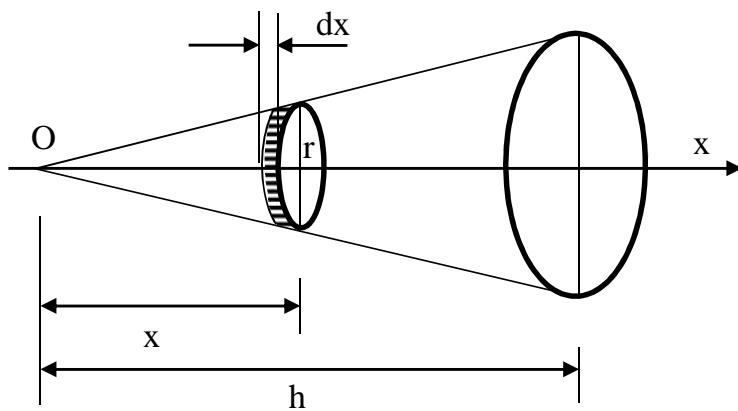


Fig.2.85 Moment of inertia of cone

A circular slice of radius  $r$  and thickness  $dx$  is chosen as the volume element. By the similarity of the triangle  $r = \frac{R}{h}x$ . The volume of element  $dV = \pi r^2 dx$  and its elemental mass  $dm = \dots dV = \dots \pi r^2 dx$ . Also to recall the moment of inertia of elemental disc which is

$$= \frac{1}{2} dm r^2 = \frac{1}{2} \dots \pi r^2 dx \times r^2 = \frac{1}{2} \dots \pi r^4 dx$$

$$\text{We know that, } r = \frac{R}{h}x$$

The moment of inertia about the X-axis of the cone is

$$I_{xx} = \int_0^h \left( \frac{1}{2} dm \right) r^2 = \int_0^h \left( \frac{1}{2} \dots \pi \left( \frac{R}{h} x \right)^4 \right) dx = \int_0^h \left( \frac{1}{2} \dots \pi \left( \frac{R^4}{h^4} x^4 \right) \right) dx = \frac{\dots \pi}{2} \frac{R^4}{h^4} \left[ \frac{x^5}{5} \right]_0^h$$

$$I_{xx} = \frac{\rho}{2} \frac{R^4}{h^4} \left[ \frac{h^5}{5} \right] = \frac{\rho R^4 h}{10}$$

But mass of the cone

$$m = \int_0^h \rho r^2 dx = \int_0^h \rho \frac{R^2}{h^2} x^2 dx = \frac{\rho R^2}{h^2} \left[ \frac{x^3}{3} \right]_0^h = \frac{\rho R^2 h}{3}$$

Therefore,

$$I_{xx} = \frac{3}{10} m R^2$$

### Mass moment inertia of Sphere:

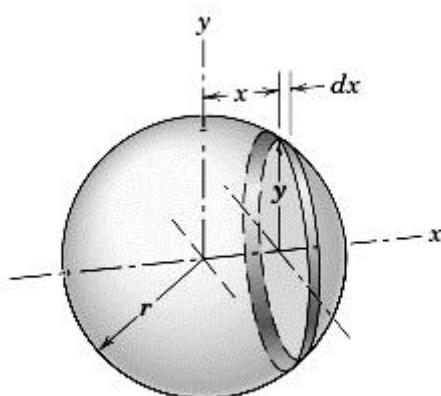


Fig.2.86 Moment of inertia Of sphere

A circular slice of radius  $y$  and thickness  $dx$  is chosen as the volume element.

The volume of element  $dV = \frac{1}{2} \rho y^2 dx$  and its mass  $dm = \rho dV = \frac{1}{2} \rho y^2 dx$ . The moment of inertia about the x-axis of the elemental cylinder is

$$dI_{xx} = (dm) y^2 = \left( \frac{1}{2} \rho y^2 dx \right) y^2 = \frac{\rho}{2} (y^2)^2 dx = \frac{\rho}{2} (r^2 - x^2)^2 dx$$

Where  $\rho$  is the constant density of the sphere. The total moment of inertia about the axis is

$$I_{xx} = \frac{\rho}{2} \int_{-r}^r (r^2 - x^2)^2 dx = \frac{8}{15} \rho r^5$$

Since the mass of the sphere is

$$m = \int dm = \dots \int_{-r}^r y^2 dx = \dots f \int_{-r}^r (r^2 - x^2) dx = \dots f \left[ (xr^2 - \frac{x^3}{3}) \right]_{-r}^r$$

$$m = \dots f \left[ (rr^2 - \frac{r^3}{3}) - (-rr^2 - \frac{-r^3}{3}) \right] = \dots f \left[ \left( \frac{2r^3}{3} + \frac{2r^3}{3} \right) \right] = \frac{4 \dots f}{3} r^3$$

Therefore,

$$I_{xx} = \frac{4}{3} \times \frac{2}{5} \dots f r^3 \times r^2 = \frac{2}{5} m r^2$$

**LECTURE NOTES**

**UNIT – 3**

**2021**

# CEES303-Engineering Mechanics



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## Unit - 3

Friction covering, Types of friction, Limiting friction, Laws of Friction, Static and Dynamic Friction; Motion of Bodies, wedge friction, screw jack & differential screw jack.

Virtual Work and Energy Method- Virtual displacements, principle of virtual work for particle and ideal system of rigid bodies, degrees of freedom. Active force diagram, systems with friction, mechanical efficiency. Conservative forces and potential energy (elastic and gravitational), energy equation for equilibrium. Applications of energy method for equilibrium. Stability of equilibrium.

### Reference for the preparation of course material:

1. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
3. J.L. Meriam and L. G. Kraige, Engineering Mechanics –Statics, volume I, Seventh edition, John Wiley & Sons, Inc.

### TEXT BOOKS

1. Irving H. Shames (2006), Engineering Mechanics, 4th Edition, Prentice Hall
2. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill

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1. R. C. Hibbler (2006), Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.

## **Unit - 3**

### **Friction and Energy of Rigid Bodies**

#### **FRICTION:**

A force that resist the movement of two contacting surfaces that slide relative to one other is known as friction force or friction. This force always acts tangent to the surface at the point of contact and is directed so as to oppose the possible or existing motion between the surfaces.

In some types of machines and processes, this want to minimize the retarding effect of friction forces. Examples are bearings of all types, power screws, gears, the flow of fluids in pipes, and the propulsion of aircraft and missiles through the atmosphere. In other situations this to maximize the effects of friction, as in brakes, clutches, belt drives, and wedges. Wheeled vehicles depend on friction for both starting and stopping, and ordinary walking depends on friction between the shoe and the ground.

Friction forces are present throughout nature and exist in all machines no matter how accurately constructed or carefully lubricated. A machine or process in which friction is small enough to be neglected is said to be ideal. When friction must be taken into account, the machine or process is termed real. In all cases where there is sliding motion between parts, the friction forces result in a loss of energy which is dissipated in the form of heat. Wear is another effect of friction.

#### **TYPES OF FRICTION;**

The types of frictional resistance encountered in mechanics are

**a) Dry Friction:**

Dry friction occurs when the unlubricated surfaces of two solids are in contact under a condition of sliding or a tendency to slide. A friction force

tangent to the surfaces of contact occurs both during the interval leading up to impending slippage and while slippage takes place. The direction of this friction force always opposes the motion or impending motion. This type of friction is also called Coulomb friction. The principles of dry or Coulomb friction were developed largely from the experiments of Coulomb in 1781 and from the work of Morin from 1831 to 1834. Although we do not yet have a comprehensive theory of dry friction. This dry friction is followed as

**i) Static friction:**

It is the friction experienced by a body when it is rest or it is the friction when the body tends to move.

**ii) Dynamic friction:**

It is the friction experience by a body when it is in motion. It is also called kinetic friction. The dynamic friction is of the following two types;

**1) Sliding friction:**

It is the friction experienced by a body when it slides over another body.

**2) Rolling friction:**

It is the friction, experienced by a body when it rolls over another.

**b) Fluid Friction:**

Fluid friction occurs when adjacent layers in a fluid (liquid or gas) are moving at different velocities. This motion causes frictional forces between fluid elements, and these forces depend on the relative velocity between layers. When there is no relative velocity, there is no fluid friction. Fluid friction depends not only on the velocity gradients within the fluid but also on the viscosity of the fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and will not be discussed here.

**c) Internal Friction:**

Internal friction occurs in all solid materials which are subjected to cyclical loading. For highly elastic materials the recovery from deformation occurs with very little loss of energy due to internal friction. For materials which have low limits of elasticity and which undergo appreciable plastic deformation during loading, a considerable amount of internal friction may accompany this deformation. The mechanism of internal friction is associated with the action of shear deformation, which is discussed in references on materials science.

**LIMITING FRICTION:**

It has been observed that when a body, lying over another body, is gently pushed, it does not move because of the frictional force, which prevents the motion. It shows that the force of the hand is being exactly balanced by the force of friction, acting in the opposite direction. If again push the body, a little harder, it is still found to be in equilibrium. It shows that the force of friction has increased itself so as to become equal and opposite to the applied force. Thus the force of friction has a remarkable property of adjusting its magnitude, so as to become exactly equal and opposite to the applied force, which tends to produce motion. There is, however, a limit beyond which the force of friction cannot increase. If the applied force exceeds this limit, the force of friction cannot balance it and the body begins to move, in the direction of the applied force. This maximum value of frictional force, which comes into play, when a body just begins to slide over the surface of the other body, is known as limiting friction. It may be noted that when the applied force is less than the limiting friction, the body remains at rest, and the friction is called static friction, which may have any value between zero and limiting friction.

**LAWS OF FRICTION**

Prof. Coulomb, after extensive experiments, gave some laws of friction, which may be grouped under the following heads :

1. Laws of static friction, and

2. Laws of kinetic or dynamic friction.

## LAWS OF STATIC FRICTION

Following are the laws of static friction as

1. The force of friction always acts in a direction, opposite to that in which the body tends to move, if the force of friction would have been absent.
2. The magnitude of the force of friction is exactly equal to the force, which tends to move the body.
3. The magnitude of the limiting friction bears a constant ratio to the normal reaction between the two surfaces. Mathematically:

$$\frac{F}{R} = \text{Cons} \tan t \quad \text{-----} \rightarrow (3.1)$$

Where,  $F$  = Limiting friction, and

$R$  = Normal reaction.

4. The force of friction is independent of the area of contact between the two surfaces.
5. The force of friction depends upon the roughness of the surfaces.

## LAWS OF KINETIC OR DYNAMIC FRICTION:

Following are the laws of kinetic or dynamic friction as

1. The force of friction always acts in a direction, opposite to that in which the body is moving.
2. The magnitude of kinetic friction bears a constant ratio to the normal reaction between the two surfaces. But this ratio is slightly less than that in case of limiting friction.
3. For moderate speeds, the force of friction remains constant. But it decreases slightly with the increase of speed.

## MECHANISM OF DRY FRICTION:

Consider a solid block of mass 'm' resting on a horizontal surface, as shown in Fig.3.1 (a). Assume that the contacting surfaces have some roughness. The experiment involves the application of a horizontal force 'P' which continuously increases from zero to a value sufficient to move the block and give it an appreciable velocity. The free-body diagram of the block for any value of 'P' is shown in Fig.3.1 (b), where the tangential friction force exerted by the plane on the block is labelled 'F'. This friction force acting on the body will always be in a direction to oppose motion or the tendency toward motion of the body. There is also a normal force N which in this case equals 'mg' and the total force 'R' exerted by the supporting surface on the block is the resultant of 'N' and 'F'.

A magnified view of the irregularities of the mating surfaces, Fig.3.2 (c), helps us to visualize the mechanical action of friction. Support is necessarily intermittent and exists at the mating humps. The direction of each of the reactions on the block,  $R_1, R_2, R_3$ , etc., depends not only on the geometric profile of the irregularities but also on the extent of local deformation at each contact point. The total normal force 'N' is the sum of the n-components of the  $R$ 's, and the total frictional force 'F' is the sum of the t-components of the  $R$ 's. When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps, and the t-components of the  $R$ 's are smaller than when the surfaces are at rest relative to one another. This observation helps to explain the well-known fact that the force 'P' necessary to maintain motion is generally less than that required to start the block when the irregularities are more nearly in mesh.

If perform the experiment and record the friction force 'F' as a function of 'P', then obtain the relation shown in Fig.3.1 (d). When 'P' is zero, equilibrium requires that there be no friction force. As 'P' is increased, the friction force must be equal and opposite of 'P' as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally, it reach a value of 'P' which causes the block to slip and to move in the direction of the applied force. At this same time the friction force decreases slightly and abruptly.

It then remains essentially constant for a time but then decreases still more as the velocity increases.

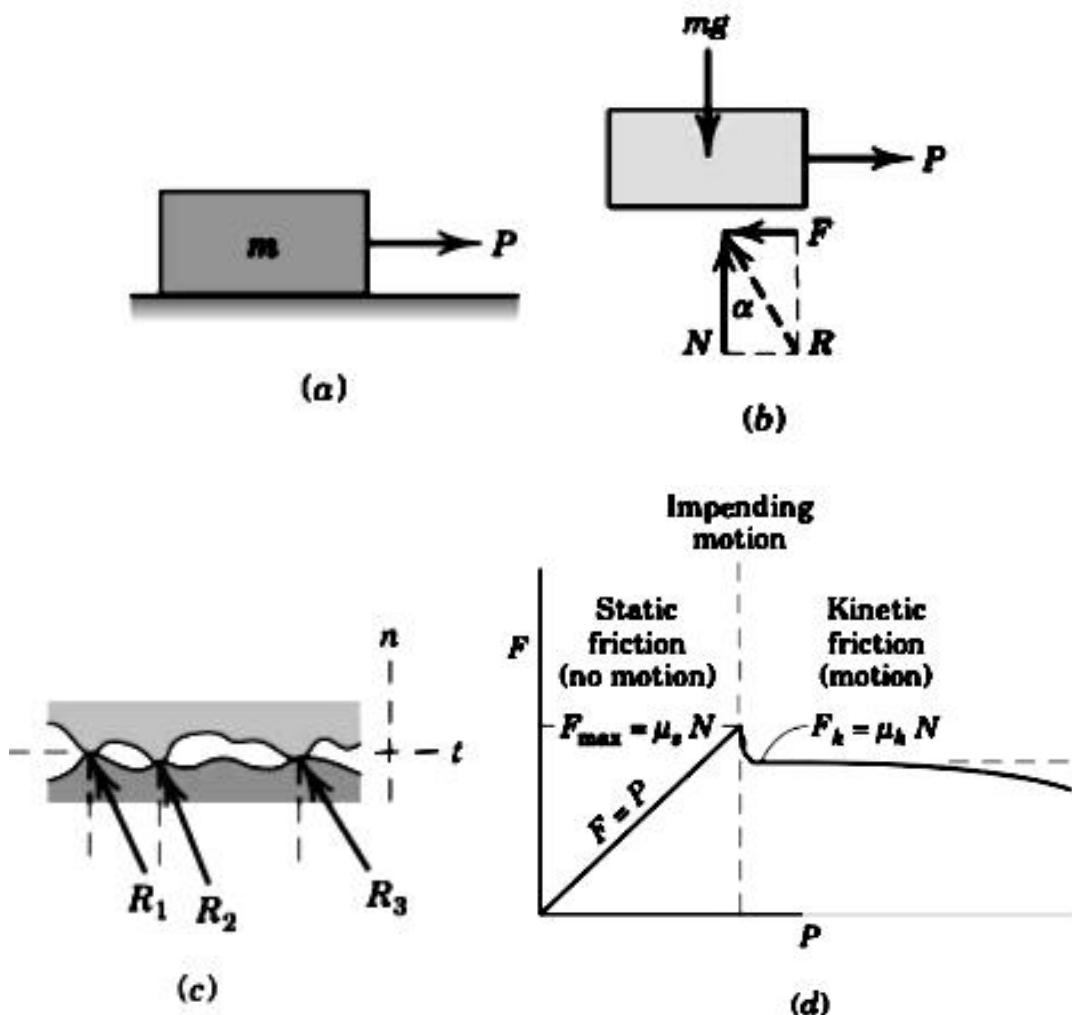


Fig.3.1 Mechanism of dry friction

### Static Friction:

The region in Fig.3.1 (d) up to the point of slippage or impending motion is called the range of static friction, and in this range the value of the friction force is determined by the equations of equilibrium. This friction force may have any value from zero up to and including the maximum value. For a given pair of mating surfaces the experiment shows that this maximum value of static friction  $F_{\max}$  is proportional to the normal force 'N'. Thus, we may write

$$F_{\max} = \mu_s N \quad \text{-----} \rightarrow (3.2)$$

Where  $\sim_s$  is the proportionality constant, called **the coefficient of static friction**.

Be aware that Eqn. 3.2 describes only the limiting or maximum value of the static friction force and not any lesser value. Thus, the equation applies only to cases where motion is impending with the friction force at its peak value. For a condition of static equilibrium when motion is not impending, the static friction force is

$$F < \sim_s N \quad \text{-----} \rightarrow (3.3)$$

## KINETIC / DYNAMIC FRICTION:

After slippage occurs, a condition of kinetic friction accompanies the ensuing motion. Kinetic friction force is usually somewhat less than the maximum static friction force. The kinetic friction force  $F_k$  is also proportional to the normal force. Thus,

$$F_k = \sim_k N \quad \text{-----} \rightarrow (3.4)$$

Where  $\sim_k$  is the **coefficient of kinetic friction**. It follows that  $\sim_k$  is generally less than  $\sim_s$ . As the velocity of the block increases, the kinetic friction decreases somewhat, and at high velocities, this decrease may be significant. Coefficients of friction depend greatly on the exact condition of the surfaces, as well as on the relative velocity, and are subject to considerable uncertainty.

Because of the variability of the conditions governing the action of friction, in engineering practice it is frequently difficult to distinguish between a static and a kinetic coefficient, especially in the region of transition between impending motion and motion. Well-greased screw threads under mild loads, for example, often exhibit comparable frictional resistance whether they are on the verge of turning or whether they are in motion.

In the engineering literature it frequently find expressions for maximum static friction and for kinetic friction written simply as  $F = \sim N$ . It is understood from the problem at hand whether maximum static friction or kinetic friction is described.

Although it will frequently distinguish between the static and kinetic coefficients, in other cases no distinction will be made, and the friction coefficient will be written simply as  $\sim$ . In those cases it must decide which of the friction conditions, maximum static friction for impending motion or kinetic friction, is involved. It emphasizes again that many problems involve a static friction force which is less than the maximum value at impending motion, and therefore under these conditions the friction relation Eqn.3.2 cannot be used.

Fig.3.1 (c) shows that rough surfaces are more likely to have larger angles between the reactions and the n-direction than are smoother surfaces. Thus, for a pair of mating surfaces, a friction coefficient reflects the roughness, which is a geometric property of the surfaces. With this geometric model of friction, it describes mating surfaces as "smooth" when the friction forces they can support are negligibly small. It is meaningless to speak of a coefficient of friction for a single surface.

## FRICTION ANGLES:

The direction of the resultant 'R' in Fig.3.1 (b) measured from the direction of 'N' is specified by  $\tan \gamma = \frac{F}{N}$ . When the friction force reaches its limiting static value  $F_{max}$ , the angle  $\alpha$  reaches a maximum value  $\phi_s$ . Thus,

$$\tan \phi_s = \sim_s \quad \text{-----} \rightarrow (3.5)$$

When slippage is occurring, the angle  $\alpha$  has a value  $\phi_k$  corresponding to the kinetic friction force. In like manner,

$$\tan \phi_k = \sim_k \quad \text{-----} \rightarrow (3.6)$$

In practice it often see the expression  $\tan \phi = \sim$ , in which the coefficient of friction may refer to either the static or the kinetic case, depending on the particular problem. The angle  $\phi_s$  is called the angle of static friction, and the angle  $\phi_k$  is called the angle of kinetic friction. The friction angle for each case clearly defines the limiting

direction of the total reaction 'R' between two contacting surfaces. If motion is impending, 'R' must be one element of a right-circular cone of vertex angle  $2\phi_s$ , as shown in Fig.3.2. If motion is not impending, 'R' is within the cone. This cone of vertex angle  $2\phi_s$  is called the cone of static friction and represents the locus of possible directions for the reaction 'R' at impending motion. If motion occurs, the angle of kinetic friction applies, and the reaction must lie on the surface of a slightly different cone of vertex angle  $2\phi_k$ . This cone is the cone of kinetic friction.

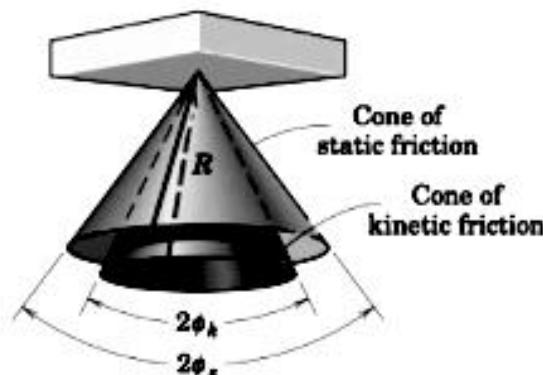


Figure 6/2

Fig.3.2 Friction angles

### Angle of limiting friction:

The angle of limiting friction may be defined as the angle between the resultant reaction and the normal to the plane on which the motion of the body is impending.

### Angle of repose:

It is well known that when grains (food grain, soil, sand etc.) are heaped, there exists a limit for the inclination of the surface. Beyond this limiting inclinations the grains start rolling down. This limiting angle up to which the grains repose (slip) is called **the angle of repose**.

Consider the block of weight 'W' resting on an inclined plane which makes an angle  $\alpha$  with the horizontal as shown in fig.3.3 when  $\alpha$  is small the block will rest on

the plane. If  $\alpha$  is increased gradually a stage is reached at which the block starts sliding down the plane. The angle  $\alpha$  for which motion is impending is called the angle of repose. Thus the maximum inclination of the plane on which a body, free from external forces can repose (sleep) is called Angle of repose.

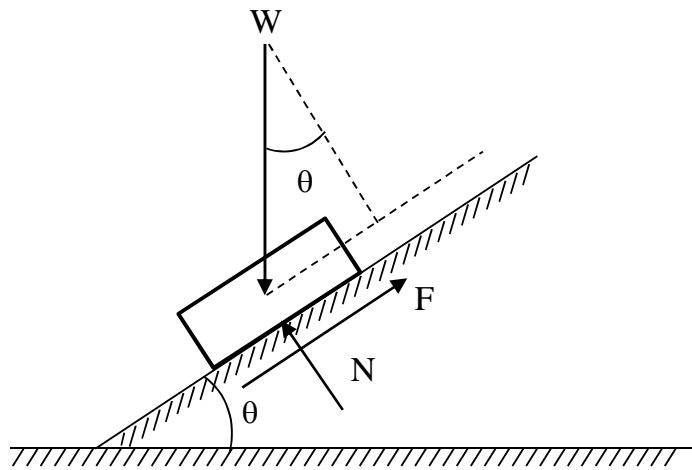


Fig.3.3 Angle of repose

Consider the equilibrium of the block shown in fig.3.3. Since the surface of contact is not smooth, not only normal reaction, but frictional force also develops. Since the body tends to slide downward, the frictional force will be up the plane.

$\Sigma$ Forces normal to the plane = 0, gives

$$N = W \cos \alpha \quad \text{-----} \rightarrow (3.7)$$

$\Sigma$ Forces parallel to the plane = 0, gives

$$F = W \sin \alpha \quad \text{-----} \rightarrow (3.8)$$

Dividing eqn. by eqn. we get,

$$\tan \alpha = \frac{F}{N} \quad \text{-----} \rightarrow (3.9)$$

If  $\phi$  is the value of  $\alpha$  when motion is impending frictional force will be limiting friction and hence

$$\tan \phi = \frac{F}{N} = \mu = \tan \alpha \quad \text{or} \quad \text{-----} \rightarrow (3.10)$$

Therefore,  $w = r$

Thus the value of angle of repose is same as the value of limiting angle of repose.

### Factors Affecting Friction:

- The friction force is essentially independent of the apparent or projected area of contact.
- The true contact area is much smaller than the projected value, since only the peaks of the contacting surface irregularities support the load.
- Even relatively small normal loads result in high stresses at these contact points.
- As the normal force increases, the true contact area also increases as the material undergoes yielding, crushing, or tearing at the points of contact.
- Dry friction are the generation of high local temperatures and adhesion at contact points, relative hardness of mating surfaces, and the presence of thin surface films of oxide, oil, dirt, or other substances.

### TYPES OF FRICTION PROBLEMS:

Now recognize the following three types of problems encountered in applications involving dry friction. The first step in solving a friction problem is to identify its type.

1. In the first type of problem, the condition of impending motion is known to exist. Here a body which is in equilibrium is on the verge of slipping, and the friction force equals the limiting static friction  $F_{\max} = \mu_s N$ . The equations of equilibrium also hold.

2. In the second type of problem, neither the condition of impending motion nor the condition of motion is known to exist. To determine the actual friction conditions, in this first assume static equilibrium and then solve for the friction force 'F' necessary for equilibrium. Three outcomes are possible:

- (a)  $F < (F_{\max} = \mu_s N)$ : Here the friction force necessary for equilibrium can be supported, and therefore the body is in static equilibrium as assumed. This emphasize that the actual friction force 'F' is less than the limiting value  $F_{\max}$  given by Eqn.3.2 and that F is determined solely by the equations of equilibrium.
- (b)  $F = (F_{\max} = \mu_s N)$ : Since the friction force 'F' is at its maximum value  $F_{\max}$  motion impends, as discussed in problem type (1). The assumption of static equilibrium is valid.
- (c)  $F > (F_{\max} = \mu_s N)$ : Clearly this condition is impossible, because the surfaces cannot support more force than the maximum  $\mu_s N$ . The assumption of equilibrium is therefore invalid, and motion occurs. The friction force F is equal to  $\mu_k N$  from Eqn. 3.4.

3. In the third type of problem, relative motion is known to exist between the contacting surfaces, and thus the kinetic coefficient of friction clearly applies. For this problem type, Eqn.3.4 always gives the kinetic friction force directly.

### Problems:

**SP3.1:** Block A weighing 1000N rests over block B which weighs 2000N as shown in fig.3.4 Block A is tied to wall with a horizontal string. If the coefficient of friction between blocks A and B is 0.25 and between B and floor is 1/3 what should be the value of P to move the block (B), if 1) P is horizontal and 2) F acts at 30° upwards to horizontal?

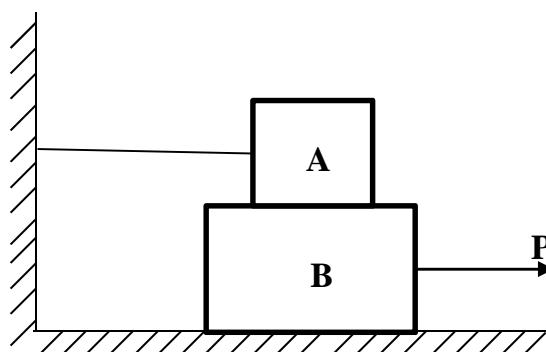


Fig.3.4

**Solution:**

- a) Find the value of P, if that force is horizontal

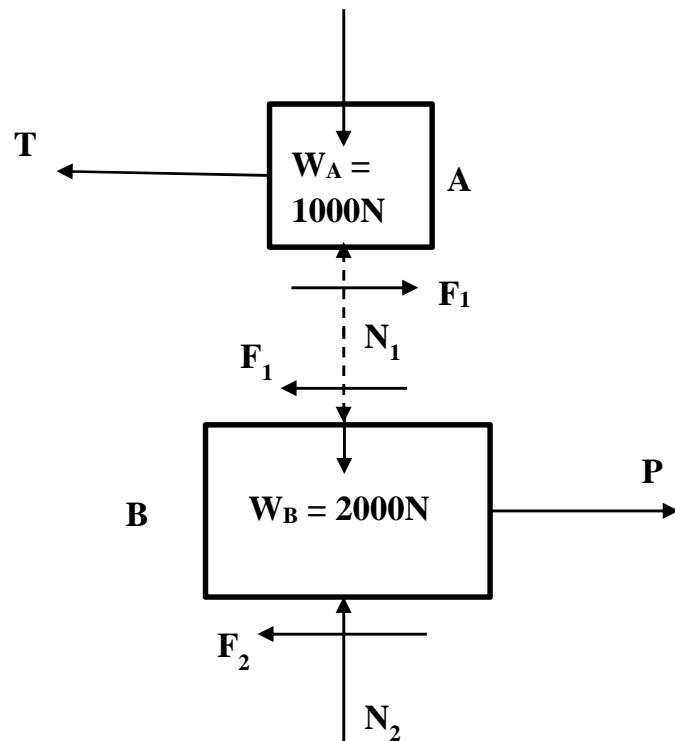


Fig.3.4a

The free body diagrams of the two blocks are shown in above figure. It may be noted that the frictional forces  $F_1$  and  $F_2$  are to be marked in the opposite directions of impending relative motion. Considering the block A, then the equilibrium equations of vertical,

$$\sum F_y = 0, N_1 - W_A = 0, \Rightarrow N_1 = 1000N$$

Since  $F_1$  is limiting friction,

$$\frac{F_1}{N_1} = \mu = 0.25, \Rightarrow \frac{F_1}{1000} = 0.25$$

$$F_1 = 250N$$

The equilibrium equations of horizontal,

$$\sum F_x = 0, \Rightarrow F_1 - T = 0, \Rightarrow T = 250N$$

Now considering the block B, then the equilibrium equations of vertical,

$$\sum F_y = 0, N_2 - W_B - N_1 = 0, \Rightarrow N_2 = 2000 + 1000 = 3000N$$

Since  $F_2$  is limiting friction,

$$\frac{F_2}{N_2} = \mu = \frac{1}{3}, \Rightarrow \frac{F_2}{3000} = \frac{1}{3}$$

$$F_2 = 1000N$$

The equilibrium equations of horizontal,

$$\sum F_x = 0, \Rightarrow P - F_1 - F_2 = 0, \Rightarrow P = 250 + 1000 = 1250N$$

$$P = 1250N$$

b) Find the value of P, if that force is inclined at  $30^\circ$  to horizontal

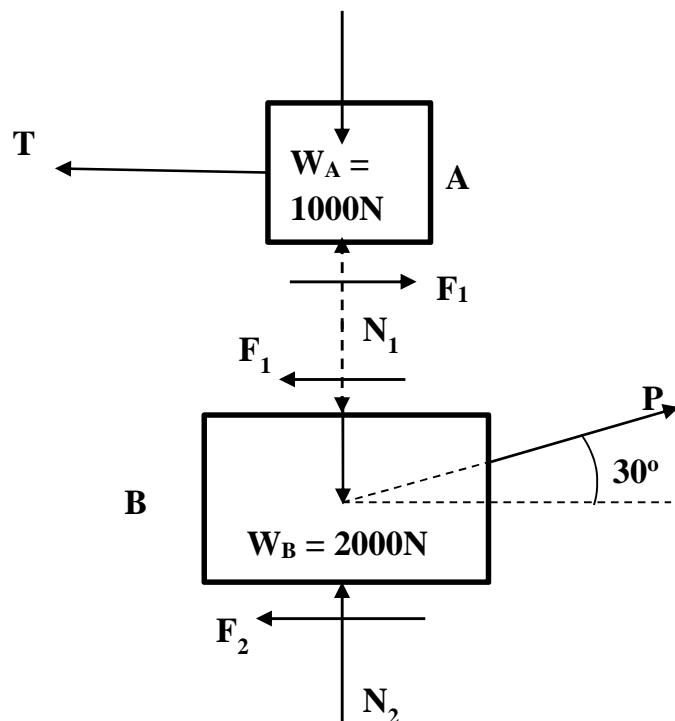


Fig.3.4b

The free body diagrams of the two blocks are shown in above figure. It may be noted that the frictional forces  $F_1$  and  $F_2$  are to be marked in the opposite

directions of impending relative motion. Considering the block A, then the equilibrium equations of vertical,

$$\sum F_y = 0, N_1 - W_A = 0, \Rightarrow N_1 = 1000N$$

Since  $F_1$  is limiting friction,

$$\frac{F_1}{N_1} = \mu = 0.25, \Rightarrow \frac{F_1}{1000} = 0.25$$

$$F_1 = 250N$$

The equilibrium equations of horizontal,

$$\sum F_x = 0, \Rightarrow F_1 - T = 0, \Rightarrow T = 250N$$

Now considering the block B, then the equilibrium equations of vertical,

$$\sum F_y = 0, N_2 - W_B - N_1 + P \sin 30^\circ = 0,$$

$$\Rightarrow N_2 + 0.5P = 2000 + 1000, \Rightarrow N_2 + 0.5P = 3000$$

$$\Rightarrow N_2 = 3000 - 0.5P$$

Since  $F_2$  is limiting friction,

$$\frac{F_2}{N_2} = \mu = \frac{1}{3}, \Rightarrow \frac{F_2}{(3000 - 0.5P)} = \frac{1}{3}$$

$$F_2 = \frac{1}{3}(3000 - 0.5P) = 1000 - \frac{0.5}{3}P$$

The equilibrium equations of horizontal,

$$\sum F_x = 0, \Rightarrow P \cos 30^\circ - F_1 - F_2 = 0,$$

$$\Rightarrow P \cos 30^\circ - 250 - \left( 1000 - \frac{0.5}{3}P \right) = 0$$

$$\Rightarrow P \left( \cos 30^\circ + \frac{0.5}{3} \right) = 1250$$

$$P = 1210.43N$$

**SP3.2:** What is the maximum load (W) which a force P equal to 5 kN will hold up if the coefficient of friction at C is 0.2 in the arrangement as shown in fig.3.5. Neglect other friction and weight of the member. If W is 3 kN and P is 4.5 kN what are the normal and tangential forces transmitted at C?

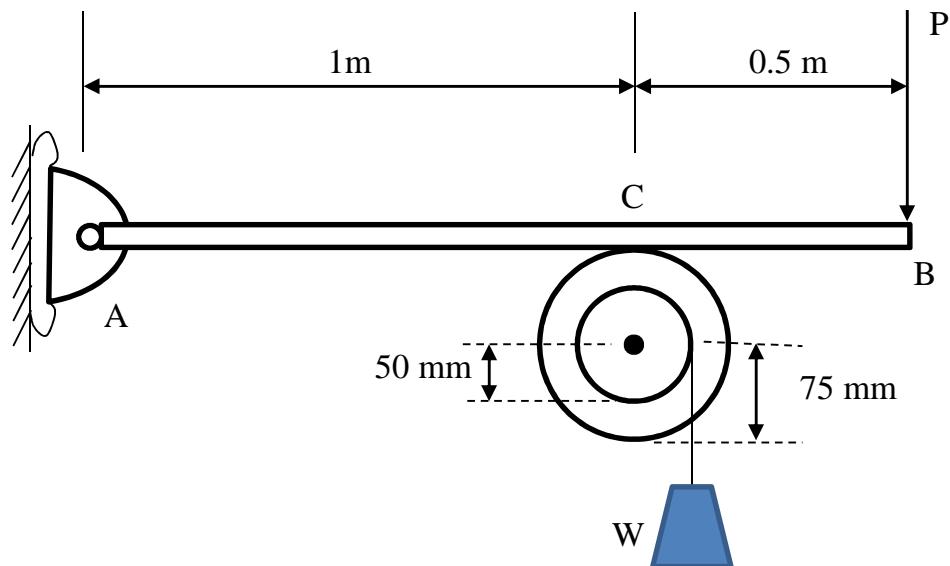


Fig.3.5

**Solution:**

The force P is 5 kN and The coefficient of friction at C is 0.2 ( $\mu$ )

a) Find maximum load W:

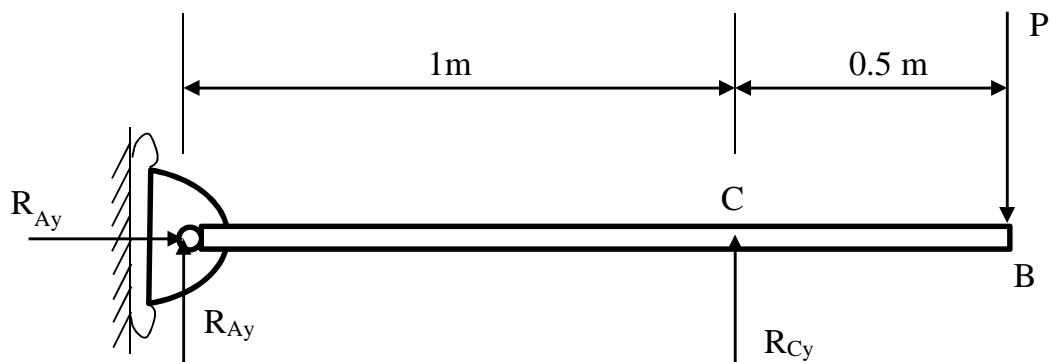


Fig.3.5a

Taking moment about A of beam AB, then

$$\sum M_A = 0, \Rightarrow R_{Cy} \times d_{AC} - P \times d_{AB} = 0$$

$$\Rightarrow R_{Cy} \times (1) - 5 \times (1.5) = 0$$

$$\Rightarrow R_{Cy} = 5 \times (1.5) = 7.5 \text{ kN}$$

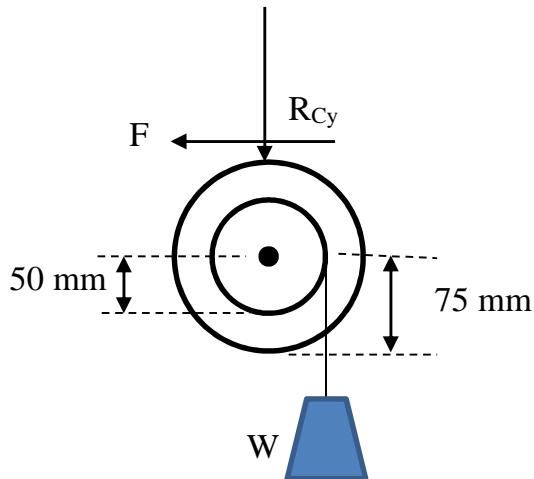


Fig.3.5b

Now consider Pulley, the limiting friction of pulley as

$$\frac{F}{R_{Cy}} = \sim = 0.2$$

$$F = 0.2 \times 7.5 = 1.5 \text{ kN}$$

Now taking moments about the centre of the pulley,

$$\sum M_{Pulley\ Centre} = 0, \Rightarrow -W \times (0.5) + F \times (0.75) = 0$$

$$\Rightarrow -W \times (0.5) + 1.5 \times (0.75) = 0$$

$$\Rightarrow W = 2.25 \text{ kN}$$

b) Normal and tangential forces transmitted at C:

Now consider a weight W equal to 3 kN suspended from the pulley and a force P equal to 4.5 kN applied at B.

Let  $R_{Cy}$  is Normal force or normal reaction at C and

F is Tangential force at C

Again consider equilibrium of the beam. Taking moments about the hinge A and equating the same

$$\sum M_A = 0, \Rightarrow R_{Cy} \times d_{AC} - P \times d_{AB} = 0$$

$$\Rightarrow R_{Cy} \times (1) - 4.5 \times (1.5) = 0$$

$$\Rightarrow R_{Cy} = 4.5 \times (1.5) = 6.75 \text{ kN}$$

We know that the tangential force at C will be the frictional force between the pulley and beam. Again taking moments about the centre of the pulley and equating the same.

$$\sum M_{\text{Pulley Centre}} = 0, \Rightarrow F \times (0.75) - W \times (0.5) = 0$$

$$\Rightarrow F = W \times \left(\frac{0.5}{0.75}\right) = 3 \times \frac{0.5}{0.75} = 2 \text{ kN}$$

**SP3.3:** What should be the value of  $\theta$  as shown in fig.3.6. If that will make the motion of 900N block the plane to impend? The coefficient of friction for all contact surfaces is 1/3.

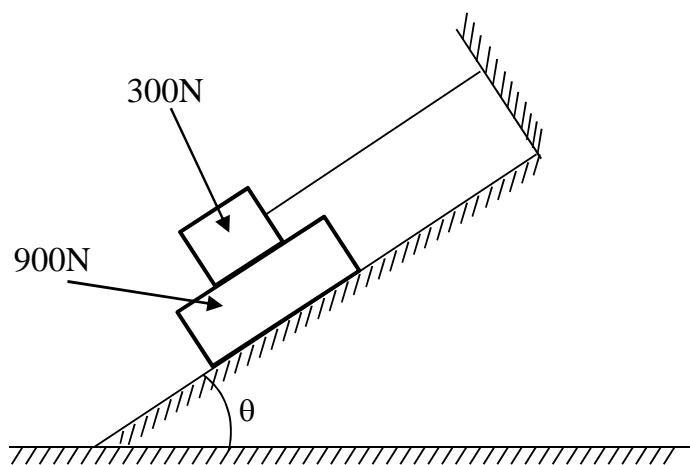


Fig.3.6

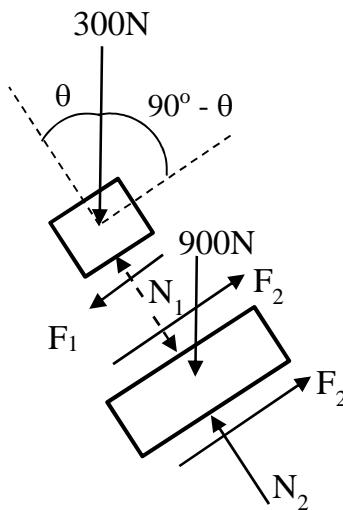
**Solution:**

Fig.3.6a

The free body diagram of the block is as shown in figure above. The block has 9000 N is on the verge of moving downward. Hence the frictional forces  $F_1$  and  $F_2$  act up on the plane on 900N block.

Considered the equilibrium equations of normal to the plane for the block has 300N as

$\sum$  Forces normal to plane = 0 , then

$$N_1 - 300 \cos \theta = 0 \quad \text{-----} \rightarrow (1)$$

From the law of friction,

$$\frac{F_1}{N_1} = \mu = \frac{1}{3}, \Rightarrow F_1 = \frac{1}{3} N_1 = \frac{1}{3} 300 \cos \theta$$

$$\Rightarrow F_1 = 100 \cos \theta \quad \text{-----} \rightarrow (2)$$

Considered the equilibrium equations of normal to the plane for the block has 900N as

$\sum$  Forces normal to plane = 0 , then

$$N_2 - N_1 - 900 \cos \theta = 0$$

$$N_2 = 300 \cos \theta + 900 \cos \theta = 1200 \cos \theta \quad \text{-----} \rightarrow (3)$$

From the law of friction,

$$\frac{F_2}{N_2} = \mu = \frac{1}{3}, \Rightarrow F_2 = \frac{1}{3} N_2 = \frac{1}{3} 1200 \cos \theta$$

$$\Rightarrow F_2 = 400 \cos \theta \quad \text{-----} \rightarrow (4)$$

The equilibrium equations of parallel to the plane for the block has 900N as

$$F_1 + F_2 - 900 \sin \theta = 0$$

$$100 \cos \theta + 400 \cos \theta = 900 \sin \theta$$

$$500 \cos \theta = 900 \sin \theta$$

$$\tan \theta = \frac{500}{900} = \frac{5}{9}$$

$$\theta = \tan^{-1} \left( \frac{5}{9} \right) = 29.05^\circ$$

**SP3.4:** What is the value of P in the system as shown in fig.3.7 to cause the motion of 500 N block to the right side? Assume the pulley is smooth and the coefficient of friction between other contact surfaces is 0.20.

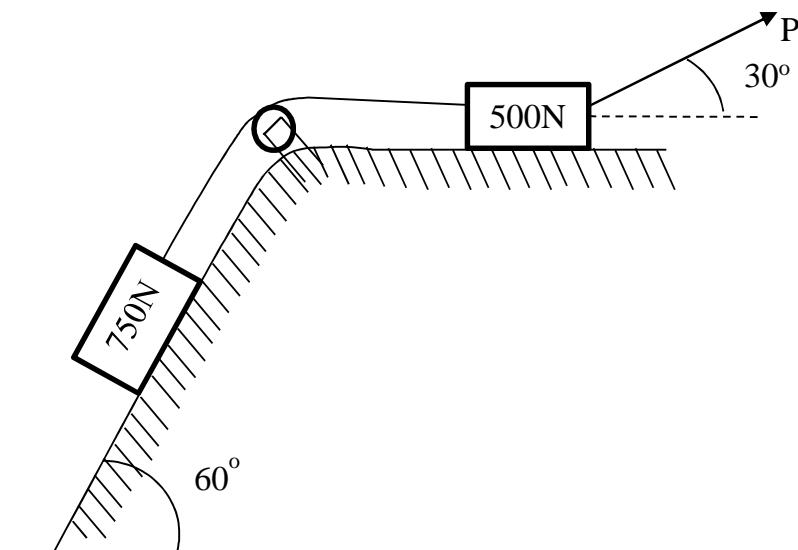


Fig.3.7

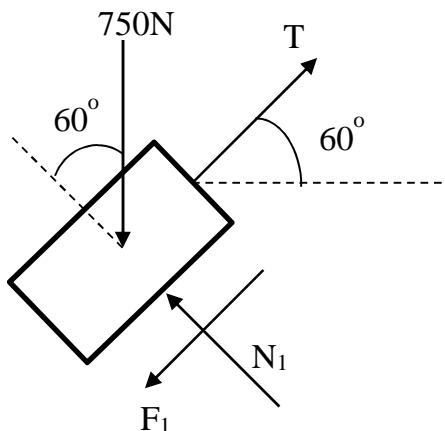
**Solution:**

Fig.3.7a

The free body diagram of the block has 750N is shown in above figure. The equilibrium equations of 750 N block is consider the force normal to the plane, then

$$\sum F_{N1} = 0, \Rightarrow N_1 - 750 \cos 60^\circ = 0$$

$$\Rightarrow N_1 = 750 \cos 60^\circ = 375N$$

Since the motion is impending,

$$\frac{F_1}{N_1} = \mu = 0.2,$$

$$F_1 = 0.2 \times N_1 = 0.2 \times 375 = 75N$$

Now consider the equilibrium equations for the force parallel to the plane, then

$$\sum F_{parallel} = 0, \Rightarrow T - F_1 - 750 \sin 60^\circ = 0$$

$$\Rightarrow T = F_1 + 750 \sin 60^\circ = 75 + 750 \sin 60^\circ = 724.52N$$

$$T = 724.52N$$

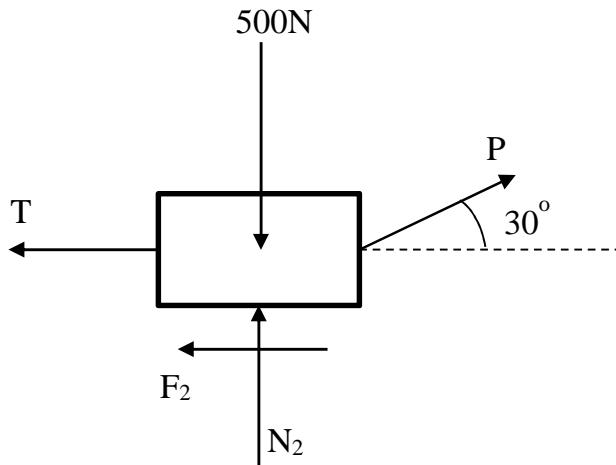


Fig.3.7b

The free body diagram of the block has 500N is shown in above figure. The equilibrium equations of 500N block is consider the vertical force, then

$$\sum F_y = 0, \Rightarrow N_2 - 500 + P \sin 30^\circ = 0$$

$$\Rightarrow N_2 = 500 + 0.5P$$

From the law of friction, then

$$\frac{F_2}{N_2} = \mu = 0.2$$

$$F_2 = 0.2 \times N_2 = 0.2 \times (500 + 0.5P) = 100 - 0.1P$$

The equilibrium equations of 500N block is consider the Horizontal force, then

$$\sum F_x = 0, \Rightarrow P \cos 30^\circ - T - F_2 = 0$$

$$\Rightarrow P \cos 30^\circ - 724.52 - (100 - 0.1P) = 0$$

$$\Rightarrow P(\cos 30^\circ + 0.1) = 724.52 + 100 = 824.52$$

$$P = 853.52 \text{ N}$$

**SP3.5:** Two blocks connected by a horizontal link AB are supported on two rough planes as shown in fig.3.8. The coefficient of friction between the block A and horizontal surface is 0.4. The limiting angle of friction between block B and inclined plane is  $20^\circ$ . What is the smallest weight  $W$  of the block A for which equilibrium of the system can exist, if the weight of block B is 5 kN?

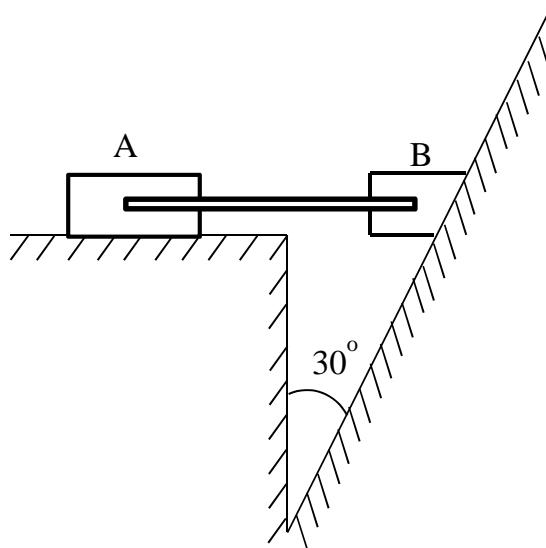


Fig.3.8

**Solution:**

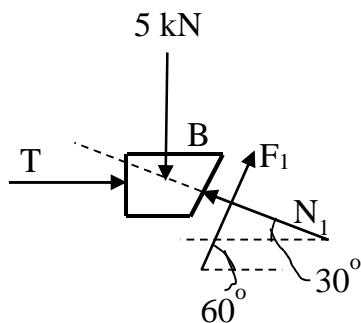


Fig.3.8a

The free body diagram of the block B is as shown in above figure and the limiting friction as

$$\frac{F_1}{N_1} = \mu = \tan 20^\circ$$

The equilibrium equations of block B for vertical force as

$$\sum F_y = 0, \Rightarrow N_1 \sin 30^\circ + F_1 \sin 60^\circ - 5 = 0$$

$$\Rightarrow 0.5N_1 + N_1 \tan 20^\circ \sin 60^\circ = 5$$

$$\Rightarrow N_1 = 6.133 \text{ kN}$$

$$\because F_1 = N_1 \times \tan 20^\circ = 6.133 \tan 20^\circ$$

$$F_1 = 2.232 \text{ kN}$$

The equilibrium equations of block B for horizontal force as

$$\sum F_x = 0, \Rightarrow T + F_1 \cos 60^\circ - N_1 \cos 30^\circ = 0$$

$$\Rightarrow T = -F_1 \cos 60^\circ + N_1 \cos 30^\circ = -2.232 \cos 60^\circ + 6.133 \cos 30^\circ$$

$$T = 4.196 \text{ kN}$$

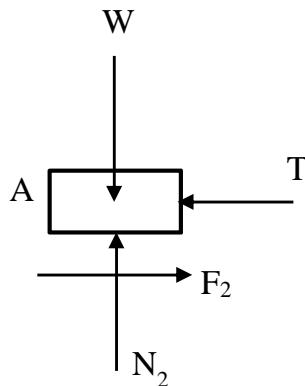


Fig.3.8b

The free body diagram of the block A is as shown in above figure and the equilibrium equations of block A for Horizontal force as

$$\sum F_x = 0, \Rightarrow F_2 - T = 0$$

$$\Rightarrow F_2 = T = 4.196 \text{ kN}$$

The limiting friction as

$$\frac{F_2}{N_2} = \sim = 0.4, \Rightarrow N_2 = \frac{F_2}{0.4} = \frac{4.196}{0.4}$$

$$N_2 = 10.49 \text{kN}$$

The equilibrium equations of block A for vertical force as

$$\sum F_y = 0, \Rightarrow W - N_2 = 0$$

$$W = 10.49 \text{kN}$$

## WEDGES:

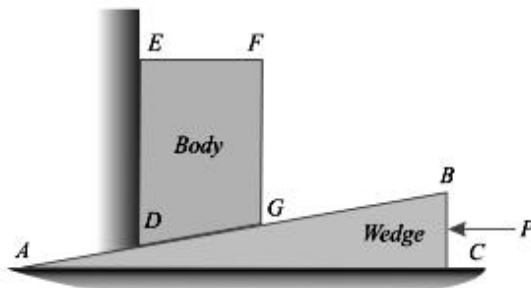


Fig.3.9 Wedge friction

A wedge is, usually, of a triangular or trapezoidal in cross-section. It is, generally, used for slight adjustments in the position of a body *i.e.* for tightening fits or keys for shafts. Sometimes, a wedge is also used for lifting heavy weights as shown in Fig.3.9. It will be interesting to know that the problems on wedges are basically the problems of equilibrium on inclined planes. Thus these problems may be solved either by the equilibrium method or by applying Lami's theorem.

Now consider a wedge *ABC*, which is used to lift the body *DEFG*.

Let  $W$  = Weight of the body *DEFG*,

$P$  = Force required to lift the body, and

$\mu$  = Coefficient of friction on the planes *AB*, *AC* and *DE*

Such that,  $\tan \theta = \sim$

A little consideration will show that when the force is sufficient to lift the body, the sliding will take place along three planes  $AB$ ,  $AC$  and  $DE$  will also occur as shown in Fig.3.10 and Fig.3.11.

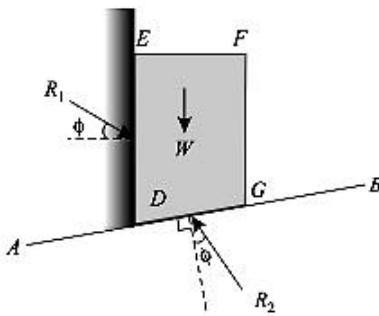


Fig.3.10 Forces on the body DEFG

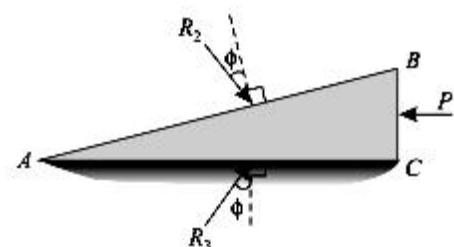


Fig.3.11 Forces on the wedge ABC

### Problems:

**SP3.6:** Find the minimum force required to move the wedge as shown in fig.3.12. The angle of friction for all contact surface is  $15^\circ$ .

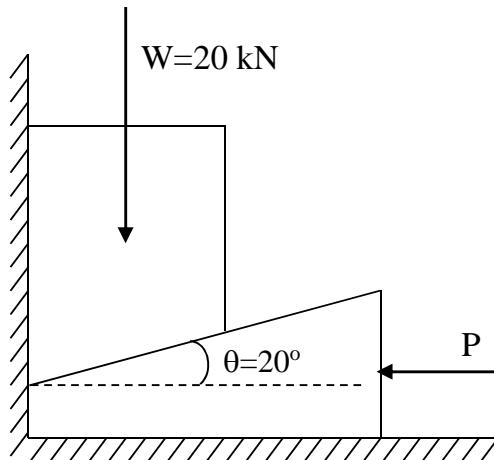


Fig.3.12

### Solution:

As wedge is driven it moves towards left and block moves upwards. Force  $P$  required to move the system is minimum when the motion is impending and hence at this stage limiting frictional force acts. Hence resultant makes limiting angle of  $15^\circ$  with normal.

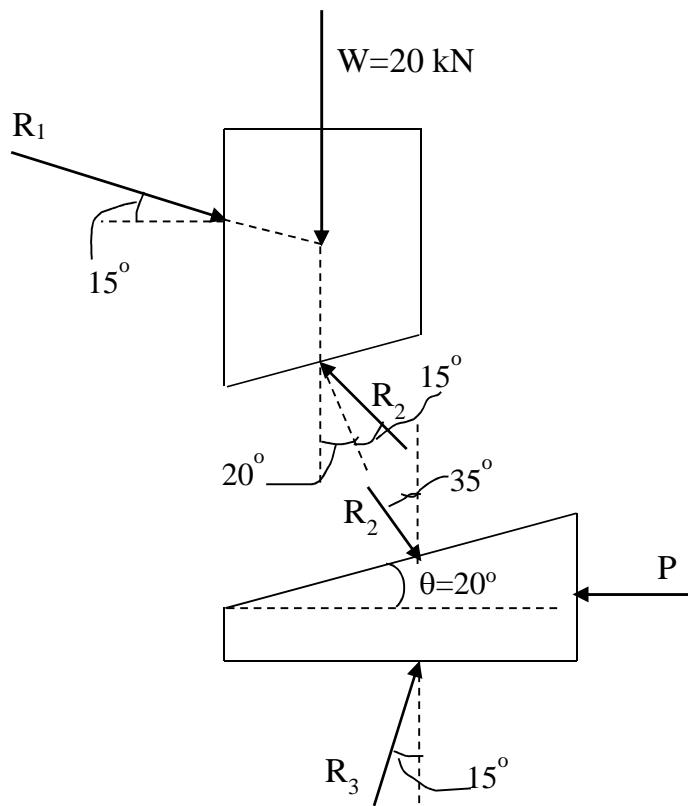


Fig.3.12a

The free body diagrams for block and wedge are shown in above figure. The force on block and wedge are redrawn in figure as given below. So that Lami's theorem can be applied conveniently.

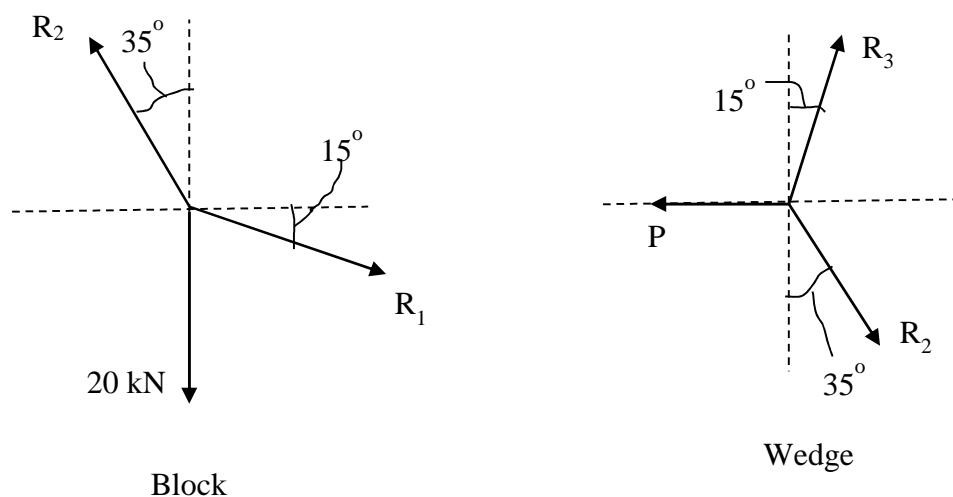


Fig.3.12b

Applying Lamis's theorem to the system of forces on block, then

$$\frac{R_1}{\sin 145^\circ} = \frac{R_2}{\sin 75^\circ} = \frac{20}{\sin 140^\circ}$$

$$\therefore \frac{R_1}{\sin 145^\circ} = \frac{20}{\sin 140^\circ}$$

$$R_1 = 17.847 \text{ kN}$$

$$\frac{R_2}{\sin 75^\circ} = \frac{20}{\sin 140^\circ}$$

$$R_2 = 30.047 \text{ kN}$$

Now applying Lamis's theorem to the system of forces on wedge, then

$$\frac{P}{\sin 130^\circ} = \frac{R_2}{\sin 105^\circ} = \frac{R_3}{\sin 125^\circ}$$

$$\therefore \frac{P}{\sin 130^\circ} = \frac{30.047}{\sin 105^\circ}$$

$$P = 23.835 \text{ kN}$$

$$\frac{30.047}{\sin 105^\circ} = \frac{R_3}{\sin 125^\circ}$$

$$R_3 = 25.48 \text{ kN}$$

**SP3.7:** A  $15^\circ$  wedge A has to be driven for tightening a body B loaded with 1000 N weight as shown in fig.3.13. If the angle of friction for all the surfaces is  $14^\circ$  find the force P which should be applied to the wedge.

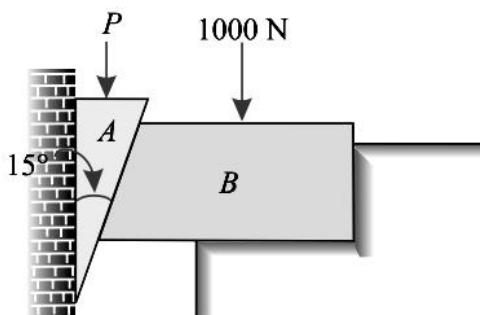
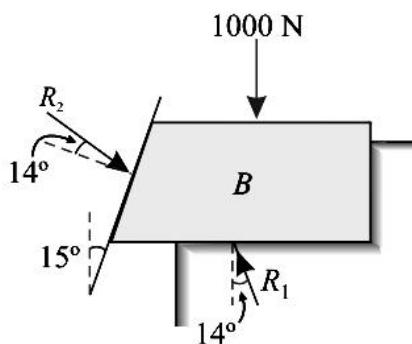
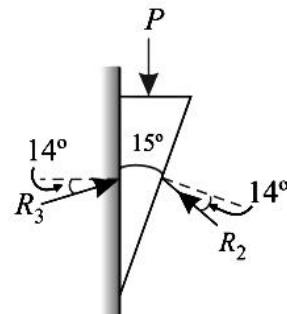


Fig.3.13

**Solution:**

(a) Body B



(b) Wedge A

Fig.3.13

The free body diagram of body B is as shown in above figure (a). The equilibrium equations of body B for horizontal force is

$$\sum F_x = 0, \Rightarrow -R_1 \sin 14^\circ + R_2 \cos(15+14)^\circ = 0$$

$$R_1 \times 0.2419 = R_2 \times 0.8746$$

$$R_1 = \frac{0.8746}{0.2419} R_2 = 3.616 R_2$$

The equilibrium equations of body B for vertical force is

$$\sum F_y = 0, \Rightarrow -R_2 \sin(15+14)^\circ + R_1 \cos 14^\circ - 1000 = 0$$

$$\Rightarrow -R_2 \times 0.4848 + 3.616 R_2 \times 0.9703 - 1000 = 0$$

$$\Rightarrow R_2(3.51 - 0.4848) = 1000$$

$$\Rightarrow R_2 = 330.6 N$$

The free body diagram of Wedge A is as shown in above figure (b). The equilibrium equations of wedge A for horizontal force is

$$\sum F_x = 0, \Rightarrow R_3 \cos 14^\circ - R_2 \cos(15+14)^\circ = 0$$

$$\Rightarrow R_3 \times 0.9703 - 330.6 \times 0.8746 = 0$$

$$\Rightarrow R_3 = 297.9N$$

The equilibrium equations of wedge A for vertical force is

$$\sum F_y = 0, \Rightarrow -P + R_3 \sin 14^\circ + R_2 \sin(15 + 14)^\circ = 0$$

$$\Rightarrow P = R_3 \sin 14^\circ + R_2 \sin(15 + 14)^\circ = 297.9 \times 0.2419 + 330.6 \times 0.4848$$

$$\Rightarrow P = 232.3N$$

## SCREW JACK:

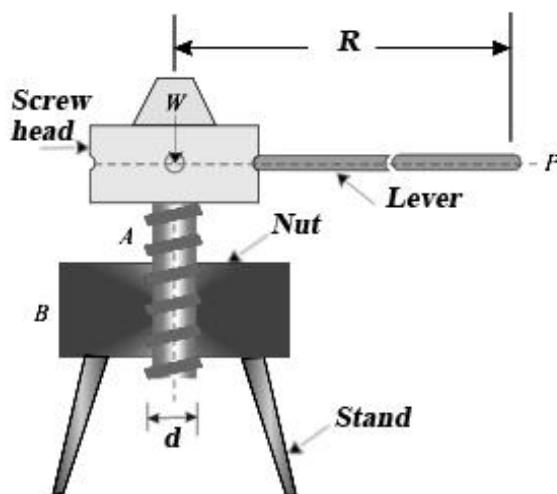


Fig.3.14 Screw jack

This is a device commonly used to lift heavy loads. Screw jack works on the principle same as that of inclined plane. A typical section of the screw jack is shown in fig.3.14.

The device consists of a nut and a screw. Monolithically cast nut and stand form the body of the jack. The load is carried by the screw head fitted onto the screw as shown in the figure. The body (consisting of nut) is fixed and the screw is rotated by means of a lever.

The axial distance moved by the nut (or by the screw, relative to each other) when it makes one complete revolution is known as **lead of the screw head**. The

distance between consecutive thread is called **pitch** (of a screw thread). If the screw is single threaded, then lead of the screw is equal to the pitch. If the screw is double threaded then lead of the screw is twice the pitch.

Let 'R' be the length of the lever and d be the mean diameter of the screw.

Let a load W be lifted using an effort P.

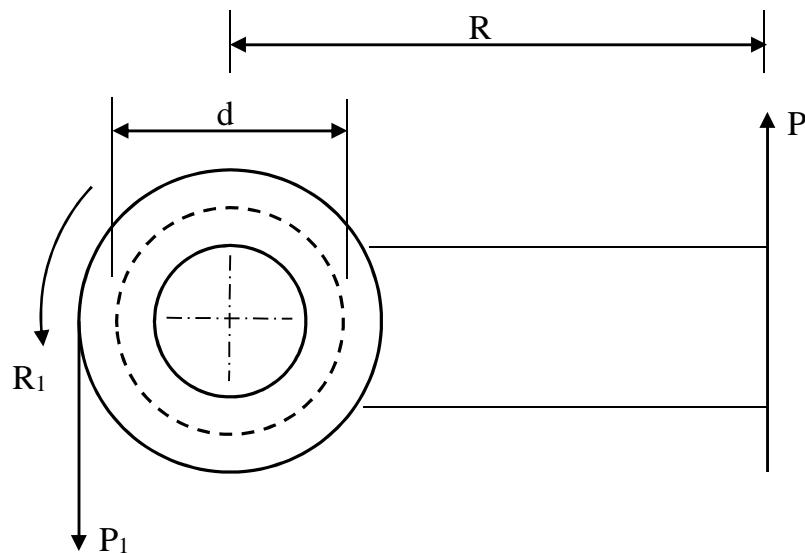


Fig.3.15 Plan view of screw jack

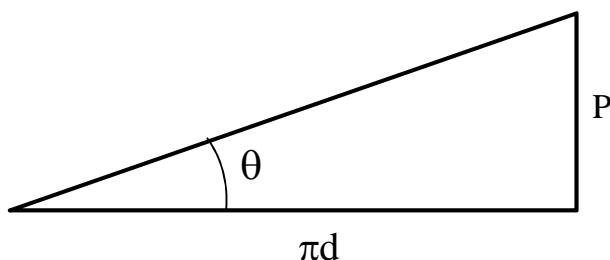


Fig.3.16 Geometry of effort and circumference of screw

If an effort P is applied at the lever end it is equivalent to an effort  $P_1$  at the screw [fig.3.15] and  $P_1$  is given by the condition:

$$P \times R = P_1 \times \frac{d}{2}$$

$$P_1 = \frac{2PR}{d}$$

Now consider one complete revolution of the lever. The load  $W$  is lifted up by a distance  $p$  equal to the lead of the screw. This can be compared with that of inclined plane having inclination is

$$\tan^{-1} \frac{p}{fd}$$

Where,  $p$  – lead of the screw

$d$  – mean diameter of screw

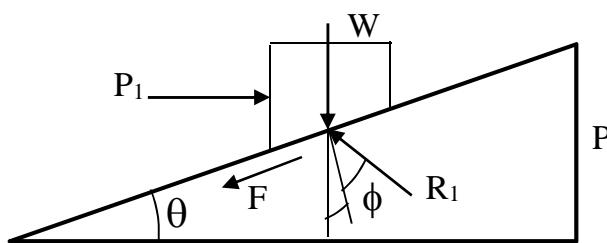


Fig.3.17 Force system of screw in upward

Applying an effort  $P$  at the end of the lever is as good as applying an effort  $P_1$  (at the screw) on this inclined plane (fig.3.17).

Resolving horizontally i.e., parallel to  $P_1$

$$P_1 = R_1 \sin(\theta + \omega)$$

Where,  $R_1$  is resultant reaction and

$\phi$  is limiting angle of friction.

Resolving vertically

$$W = R_1 \cos(\theta + \omega)$$

Dividing equation by equation

$$\frac{P_1}{W} = \frac{R_1 \sin(\alpha + \omega)}{R_1 \cos(\alpha + \omega)}$$

$$P_1 = W \tan(\alpha + \omega)$$

$$\text{But, } P_1 = \frac{2PR}{d}$$

$$\text{Therefore, } \frac{2PR}{d} = W \tan(\alpha + \omega)$$

$$P = \frac{d}{2R} W \tan(\alpha + \omega)$$

$$\text{We have, } \tan \omega = \mu$$

Where,  $\mu$  is the coefficient of friction.

Then,

$$P = \frac{d}{2R} W \frac{\tan \alpha + \tan \omega}{1 - \tan \alpha \tan \omega}$$

$$P = \frac{d}{2R} W \frac{\tan \alpha + \mu}{1 - \mu \tan \alpha}$$

$$\text{Where, } \tan \alpha = \frac{P}{fd}$$

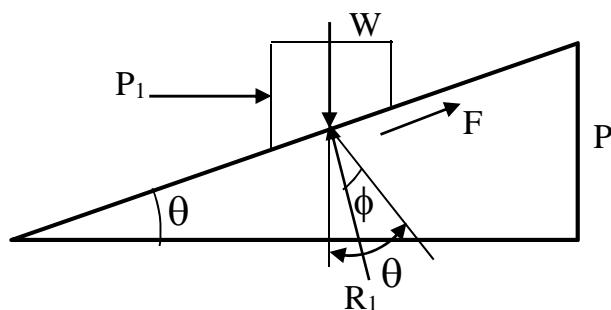


Fig.3.18 Force system of screw in downward

If the load is descending then the friction will be acting in the reverse direction so that the resultant reaction R shifts as shown in fig.3.18

Then the equation changes to

$$P = \frac{d}{2R} W \tan(\theta + \omega)$$

Torque required,  $T = PR$

$$= \frac{d}{2} W \tan(\theta + \omega)$$

Hence torque required while ascending

$$T = \frac{d}{2} W \tan(\theta + \omega)$$

And torque required while descending

$$T = \frac{d}{2} W \tan(\theta - \omega)$$

Now, the velocity ratio (VR)

$$VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{2fR}{P}$$

## Problems:

**SP3.8:** The mean radius of the screw of a square threaded screw jack is 25mm. The pitch of thread is 7.5 mm. If the coefficient of friction is 0.12. What effort applied at the end of lever 600 mm length is needed to raise a weight of 2 kN.

### Solution:

Mean diameter of the screw (d) =  $2r = 2(25) = 50\text{mm}$

Lead of the screw,  $p = 7.5\text{mm}$

$\mu = 0.12$

Length of the lever,  $R = 600\text{ mm}$

Weight to be raised,  $W = 2000\text{ N}$

We know that,

$$\tan \theta = \frac{p}{fd} = \frac{7.5}{f \times 50} = 0.048$$

$$\mu = \tan^{-1}(0.048) =$$

$$\tan \mu = \sim = 0.12$$

$$\mu = \tan^{-1}(0.12) =$$

The effort required at mean diameter of the screw to raise the weight is

$$P = W \times \frac{\sim + \tan \mu}{1 - \sim \tan \mu} = 2000 \times \frac{0.12 + 0.048}{1 - 0.12(0.048)} = 338N$$

Now, the effort applied at the end of the lever may be found out from the relation

$$P_1 \times R = P \times \frac{d}{2}$$

$$P_1 \times 600 = 338 \times \frac{50}{2}$$

$$P_1 = 14.1N$$

**SP3.9:** A screw jack raises a load 40 kN. The screw is square threaded having three threads per 20mm length and 40mm in diameter. Find the force required at the end of a lever 400mm long measured from the axis of the screw, if the coefficient of friction between screw and nut is 0.12.

### Solution:

Screw diameter,  $d=40mm$ ,

$$\text{Lead of the screw, } p = \frac{20}{3} = 6.667mm$$

Load,  $W = 40 \text{ kN}$

Lever length,  $R = 400\text{mm}$

$$\mu = 0.12$$

We have,

$$P = \frac{d}{2R} W \frac{\sim + \tan \mu}{1 - \sim \tan \mu}$$

$$\text{and } \tan \phi = \frac{\text{lead}}{fd} = \frac{6.667}{f \times 40} = 0.05305$$

$$P = \frac{40}{2 \times 400} \times 40000 \frac{0.12 + 0.05305}{1 - (0.12 \times 0.05305)}$$

$$P = 348.32 \text{ kN}$$

**SP3.10:** A screw jack has square threads 50mm mean diameter and 10mm pitch. The load on the jack revolves with the screw. The coefficient of friction at the screw thread is 0.05. a) Determine the tangential force required at the end of 300 mm lever to lift a load of 6000 N. and a) State whether the jack is self-locking. If not find the torque which must be applied to keep the load from descending.

Solution:

$$\tan \phi = \frac{\text{lead}}{fd} = \frac{p}{fd} = \frac{10}{f \times 50} = 0.0637$$

$$\phi = \tan^{-1} 0.0637 = 3.6426^\circ$$

$$\tan \omega = \sim = 0.05$$

$$\omega = \tan^{-1} 0.05 = 2.8624^\circ$$

a) Determine the tangential force,

$$P = \frac{d}{2R} \times W \tan(\phi + \omega) = \frac{50}{2 \times 300} \times 6000 \tan(3.6426^\circ + 2.8624^\circ) = 57.01 \text{ N}$$

b) State whether the jack is self-locking,

$$VR = \frac{2fR}{p} = \frac{2f \times 300}{10} = 188.496$$

$$MA = \frac{W}{P} = \frac{6000}{57.01} = 105.245$$

Therefore, the efficiency is

$$y = \frac{MA}{VR} = \frac{105.245}{188.496} = 0.5583 = 55.83\%$$

Hence the efficiency of screw jack is greater than 50%, then the screw jack is not self-locking.

The torque required to keep the load from descending is

$$T = \frac{d}{2} W \tan(\mu - \phi) = \frac{50}{2} \times 600 \times \tan(3.6426^\circ - 2.8624^\circ) = 204.3 \text{ N-mm}$$

## DIFFERENTIAL SCREW JACK:

Differential screw jack is an improvement over simple screw jack. A typical differential screw jack is shown in fig.3. It consists of two threaded elements A and B. Both A and B have threads in the same direction (right handed). The element A is a cylinder which has threads on both its outer and inner surfaces. The threads on the outer surface of the element A fits into the nut C which also functions as the base of the whole mechanism. The threads on the element B fit into the threads cut on the inner surface of A. Thus the element A acts as a screw for the nut C and also as a nut for the element B. With the help of a lever inserted in the holes made on the top of the block D, which is attached to the element B, block D can be rotated. When D is rotated, A rotates with it. Rotation of B is prevented by suitable arrangement.

Let  $D_A$  and  $d_B$  be the mean diameters of the screws A and B respectively.

Let  $P_A$  and  $P_B$  be the pitch of the screws A and B respectively and  $P_A$  be greater than  $P_B$ .

If the lever is rotated through one complete revolution then the height through which the element A moves up is equals to  $P_A$

In the meantime the element B moves down with respect to C.

The distance through which B comes down is equals to  $P_B$ .

Therefore, net height through which load is lifted is equal to  $(P_A - P_B)$ .

Let  $R$  be the radial distance (from the centre line of A and B) at which an effort  $P$  is applied.

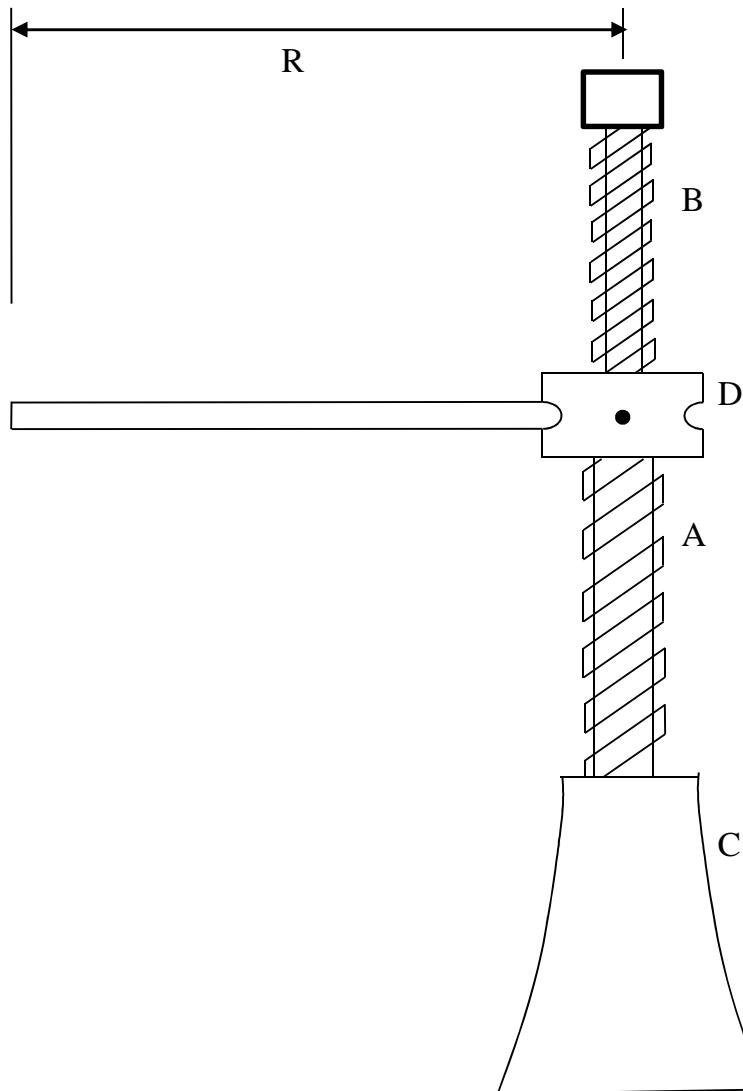


Fig.3.19 Differential screw jack

Now, the velocity ratio (VR) is

$$VR = \frac{\text{Distance moved by the effort}}{\text{Distance moved by the load}} = \frac{2fR}{P_A - P_B}$$

It can be seen from the above equations that the velocity ratio in the differential screw jack is increased as compared to that of simple screw jack.

**Problems:**

**SP3.11:** The following are the specifications for a differential screw jack;

- a) Pitch of smaller screw is 5mm,
- b) Pitch of larger screw is 10mm and
- c) Lever arm length from the centre of screw is 500mm

The screw jack raises a load of 15 kN with an effort of 185N. Determine the efficiency of the differential screw jack at this load. If the above jack can raise a load of 40 kN with an effort of 585 N, determine the law of machine.

**Solution:**

$$\text{Now } P_A = 10\text{mm}$$

$$P_B = 5\text{mm}$$

$$\text{Lever arm length, } R = 500\text{mm}$$

We know that. The velocity ratio

$$VR = \frac{2fR}{P_A - P_B} = \frac{2f \times 500}{10 - 5} = 628.32$$

The mechanical advantage,

$$MA = \frac{W}{P} = \frac{15000}{185} = 81.08$$

The efficiency of machine

$$\gamma = \frac{MA}{VR} = \frac{81.08}{628.32} = 0.129 = 12.9\%$$

To find law of machine:

$$\text{Let law of machine be } P = mW + C$$

$$\text{From first case: } 185 = m \times 15000 + C \rightarrow (1)$$

$$\text{From second case: } 585 = m \times 50000 + C \rightarrow (2)$$

Subtracting eqn. (1) from eqn. (2) then

$$400 = 35000 \times m, \Rightarrow m = \frac{4}{350} \longrightarrow (3)$$

Substituting the eqn. (3) in eqn. (1) then we get

$$185 = \frac{4}{350} \times 15000 + C$$

$$\Rightarrow C = 13.57N$$

Therefore, the law of machine is

$$P = \frac{4}{350} W + 13.57$$

## WORK OF A FORCE:

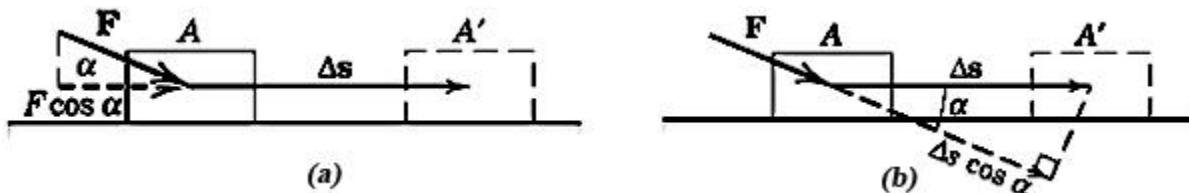


Fig.3.20 Work of a force

Consider the constant force  $F$  acting on the body shown in fig3.20 (a), whose movement along the plane from  $A$  to  $A'$  is represented by the vector  $\Delta s$ , called the displacement of the body. By definition the work  $U$  done by the force  $F$  on the body during this displacement is the component of the force in the direction of the displacement times the displacement, or

$$U = (F \cos \alpha) \Delta s$$

From fig.3.20 (b) it seems that the same result is obtained if multiply the magnitude of the force by the component of the displacement in the direction of the force. This gives

$$U = F(\Delta s \cos \alpha)$$

Because this obtains the same result regardless of the direction in which it resolves the vectors, it concludes that work  $U$  is a scalar quantity.

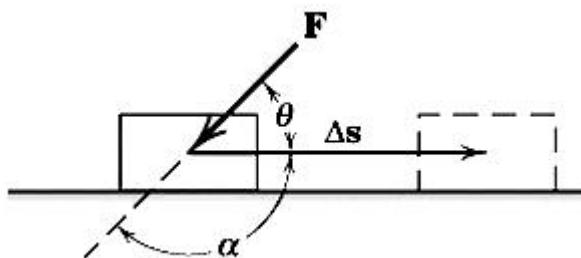


Fig.3.21 Work of a force act opposite of the displacement

Work is positive when the working component of the force is in the same direction as the displacement. When the working component is in the direction opposite to the displacement, fig.3.21, the work done is negative. Thus,

$$U = (F \cos \Gamma) \Delta s = -(F \cos \Gamma) \Delta s$$

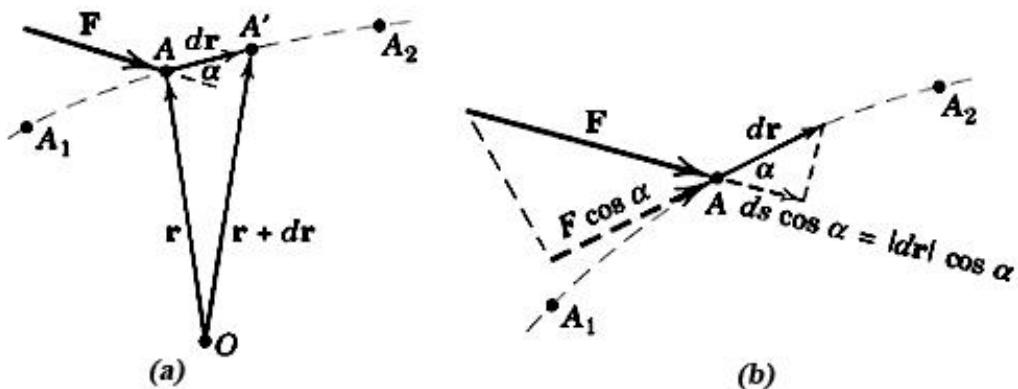


Fig.3.22 Force moves along the path

Now generalize the definition of work to account for conditions under which the direction of the displacement and the magnitude and direction of the force are variable.

Fig.3.22 (a) shows a force  $F$  acting on a body at a point  $A$  which moves along the path shown from  $A_1$  to  $A_2$ . Point  $A$  is located by its position vector  $r$  measured from some arbitrary but convenient origin  $O$ . The infinitesimal displacement in the motion from  $A$  to  $A'$  is given by the differential change  $dr$  of the position vector. The work done by the force  $F$  during the displacement  $dr$  is defined as

$$dU = F \cdot dr$$

If  $F$  denotes the magnitude of the force  $F$  and  $ds$  denotes the magnitude of the differential displacement  $dr$ , the definition of the dot product to obtain.

$$dU = F ds \cos \gamma$$

It may again interpret this expression as the force component  $F \cos \gamma$  in the direction of the displacement times the displacement, or as the displacement component  $ds \cos \alpha$  in the direction of the force times the force, as represented in fig.3.22 (b). If it expresses  $F$  and  $dr$  in terms of their rectangular components, then

$$\begin{aligned} dU &= (iF_x + jF_y + kF_z)(idx + jdy + kdz) \\ &= F_x dx + F_y dy + F_z dz \end{aligned}$$

To obtain the total work  $U$  done by  $F$  during a finite movement of point A from A1 to A2, fig.3.22 (a), It integrates  $dU$  between these positions. Thus,

$$\begin{aligned} U &= \int F \cdot dr = \int (F_x dx + F_y dy + F_z dz) \text{ or} \\ U &= \int F \cos \gamma ds \end{aligned}$$

To carry out this integration, it must know the relation between the force components and their respective coordinates, or the relations between  $F$  and  $s$  and between  $\cos \alpha$  and  $s$ .

In the case of concurrent forces which are applied at any particular point on a body, the work done by their resultant equals the total work done by the several forces. This is because the component of the resultant in the direction of the displacement equals the sum of the components of the several forces in the same direction.

## WORK OF A COUPLE:

In addition to the work done by forces, couples also can do work. In fig.3.23 (a) the couple  $M$  acts on the body and changes its angular position by an amount  $d\theta$ . The

work done by the couple is easily determined from the combined work of the two forces which constitute the couple. In part b of the figure it represents the couple by two equal and opposite forces  $F$  and  $-F$  acting at two arbitrary points  $A$  and  $B$  such that  $F = M/b$ . During the infinitesimal movement in the plane of the figure, line  $AB$  moves to  $A'B'$ . Now take the displacement of  $A$  in two steps, first, a displacement  $dr_B$  equal to that of  $B$  and, second, a displacement  $dr_{A/B}$  (read as the displacement of  $A$  with respect to  $B$ ) due to the rotation about  $B$ . Thus the work done by  $F$  during the displacement from  $A$  to  $A'$  is equal and opposite in sign to that due to  $-F$  acting through the equal displacement from  $B$  to  $B'$ . Therefore conclude that no work is done by a couple during a translation (movement without rotation).

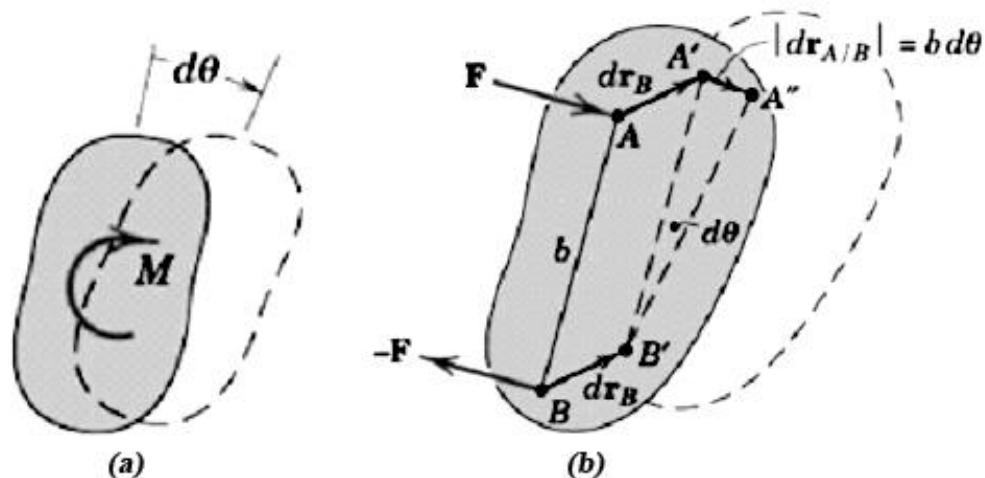


Fig.3.23 Work of a couple

During the rotation, however,  $F$  does work equal to  $F \cdot dr_{A/B} = Fbd_{\theta}$ , where  $dr_{A/B} = bd_{\theta}$  and where  $d\theta$  is the infinitesimal angle of rotation in radians. Since  $M = Fb$ , then

$$dU = Md_{\theta}$$

The work of the couple is positive if  $M$  has the same sense as  $d\theta$  (clockwise in this illustration), and negative if  $M$  has a sense opposite to that of the rotation. The total work of a couple during a finite rotation in its plane becomes.

$$U = \int M d_{\theta}$$

## VIRTUAL DISPLACEMENTS:

A virtual displacement is a first-order differential change in a linear or angular position. This change is fictitious in that it is an assumed movement which need not take place in reality. Mathematically, a virtual displacement is treated the same as a differential change in an actual movement. The symbol  $\delta$  is used for the differential virtual change and the usual symbol  $d$  for the differential change in a real movement.

Relating the linear and angular virtual displacements of the parts of a mechanical system during a virtual movement consistent with the constraints is often the most difficult part of the analysis.

To do this,

1. Write the geometric relationships which describe the configuration of the system.
2. Establish the differential changes in the positions of parts of the system by differentiating the geometric relationship to obtain expressions for the differential virtual movements.

## VIRTUAL WORK:

The definitions of the work of a force and a couple have been presented in terms of actual movements expressed by differential displacements having magnitudes of  $dr$  and  $d\theta$ . Consider now an imaginary or **virtual displacement** of a body in static equilibrium, which indicates a displacement or rotation that is assumed and does not actually exist. These movements are first-order differential quantities and will be denoted by the symbols  $\delta r$  and  $\delta\theta$  ( $\delta r$  and  $\delta\theta$ ), respectively. The virtual work done by a force having a virtual displacement  $\delta r$  is

$$uU = F \cos_{\alpha} \delta r$$

Similarly when a couple undergoes a virtual rotation  $\delta\theta$  in the plane of the couple forces, the virtual work is

$$uU = Mu_{\alpha}$$

## PRINCIPLE OF VIRTUAL WORK:

The **principle of virtual work** states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

$$uU = 0$$

## PRINCIPLE OF VIRTUAL WORK FOR A PARTICLE:

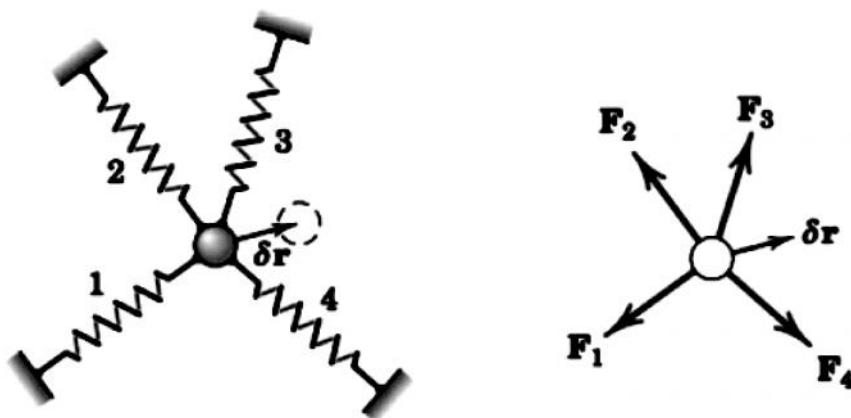


Fig.3.24 System of forces on particle

Consider the particle or small body in Fig.3.24 which attains an equilibrium position as a result of the forces in the attached springs. If the mass of the particle were significant, then the weight  $mg$  would also be included as one of the forces. For an assumed virtual displacement  $\delta r$  of the particle away from its equilibrium position, the total virtual work done on the particle is

$$uU = F_1 \cdot \delta r + F_2 \cdot \delta r + F_3 \cdot \delta r + \dots = \sum F \cdot \delta r$$

Now express  $\sum F$  in terms of its scalar sums and  $\delta r$  in terms of its component virtual displacements in the coordinate directions, as follows:

$$\begin{aligned} uU &= \sum F \cdot \delta r = (i \sum F_x + j \sum F_y + k \sum F_z)(i \delta x + j \delta y + k \delta z) \\ &= \sum F_x \delta x + \sum F_y \delta y + \sum F_z \delta z = 0 \end{aligned}$$

The sum is zero, since  $\Sigma F = 0$ , which gives  $\Sigma F_x = 0$ ,  $\Sigma F_y = 0$  and  $\Sigma F_z = 0$ . The equation  $uU = 0$  is therefore an alternative statement of the equilibrium conditions for a particle. This condition of zero virtual work for equilibrium is both necessary and sufficient, since it may be applied to virtual displacements taken one at a time in each of the three mutually perpendicular directions, in which case it becomes equivalent to the three known scalar requirements for equilibrium.

The principle of zero virtual work for the equilibrium of a single particle usually does not simplify this already simple problem because  $uU = 0$  and  $\Sigma F = 0$  provide the same information. However, it introduces the concept of virtual work for a particle so that it can later apply it to systems of particles.

## **PRINCIPLE OF VIRTUAL WORK FOR A RIGID BODY:**

It can easily extend the principle of virtual work for a single particle to a rigid body treated as a system of small elements or particles rigidly attached to one another. Because the virtual work done on each particle of the body in equilibrium is zero, it follows that the virtual work done on the entire rigid body is zero. Only the virtual work done by external forces appears in the evaluation of  $uU = 0$  for the entire body, since all internal forces occur in pairs of equal, opposite, and collinear forces, and the net work done by these forces during any movement is zero.

As in the case of a particle, we again find that the principle of virtual work offers no particular advantage to the solution for a single rigid body in equilibrium. Any assumed virtual displacement defined by a linear or angular movement will appear in each term in  $uU = 0$  and when cancelled will leave us with the same expression we would have obtained by using one of the force or moment equations of equilibrium directly.

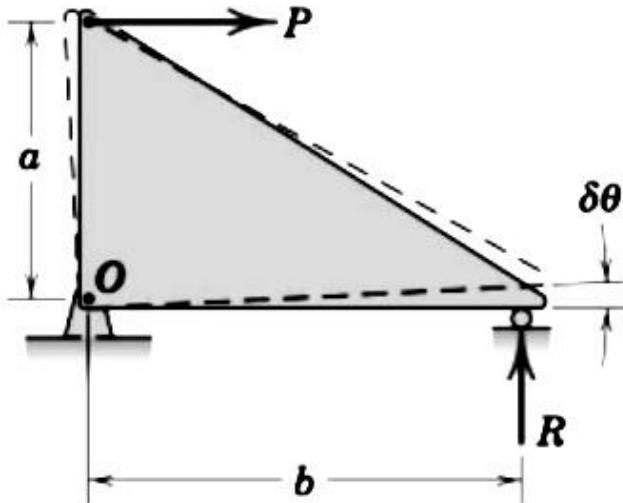


Fig.3.25 Rigid body

This condition is illustrated in Fig.3.25, where it needs to determine the reaction  $R$  under the roller for the hinged plate of negligible weight under the action of a given force  $P$ . A small assumed rotation  $\delta\theta$  of the plate about O is consistent with the hinge constraint at O and is taken as the virtual displacement. The work done by  $P$  is  $-Pa\delta\theta$ , and the work done by  $R$  is  $+Rb\delta\theta$ . Therefore, the principle  $\sum U = 0$  gives

$$-Pa\delta\theta + Rb\delta\theta = 0$$

Cancelling  $\delta\theta$  leaves

$$Pa - Rb = 0$$

which is simply the equation of moment equilibrium about O. Therefore, nothing is gained by using the virtual-work principle for a single rigid body. The principle is, however, decidedly advantageous for inter-connected bodies, as discussed next.

## PRINCIPLE OF VIRTUAL WORK FOR IDEAL SYSTEMS OF RIGID BODIES:

Now extend the principle of virtual work to the equilibrium of an interconnected system of rigid bodies. This treatment will be limited to so-called ideal systems. These are systems composed of two or more rigid members linked together by mechanical

connections which are incapable of absorbing energy through elongation or compression, and in which friction is small enough to be neglected.

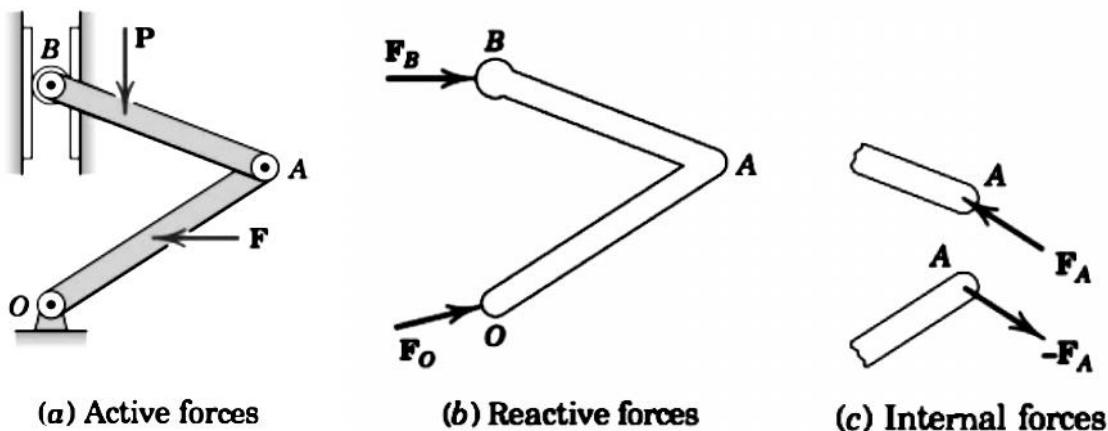


Fig.3.26 Ideal systems of rigid bodies

Fig.3.26 (a) shows a simple example of an ideal system where relative motion between its two parts is possible and where the equilibrium position is determined by the applied external forces  $P$  and  $F$ . It can identify three types of forces which act in such an interconnected system. They are as follows:

(1) **Active forces:** These are external forces capable of doing virtual work during possible virtual displacements. In fig.3.26 (a) forces  $P$  and  $F$  are active forces because they would do work as the links move.

(2) **Reactive forces:** These are forces which act at fixed support positions where no virtual displacement takes place in the direction of the force. Reactive forces do no work during a virtual displacement. In fig.3.26 (b) the horizontal force  $F_B$  exerted on the roller end of the member by the vertical guide can do no work because there can be no horizontal displacement of the roller. The reactive force  $F_O$  exerted on the system by the fixed support at  $O$  also does no work because  $O$  cannot move.

(3) **Internal forces:** These are forces in the connections between members. During any possible movement of the system or its parts, the net work done by the internal forces at the connections is zero. This is so because the internal forces always exist in pairs of equal and opposite forces, as indicated for the internal forces  $F_A$  and

- $F_A$  at joint A in fig.3.26 (c). The work of one force therefore necessarily cancels the work of the other force during their identical displacements.

### DEGREES OF FREEDOM:

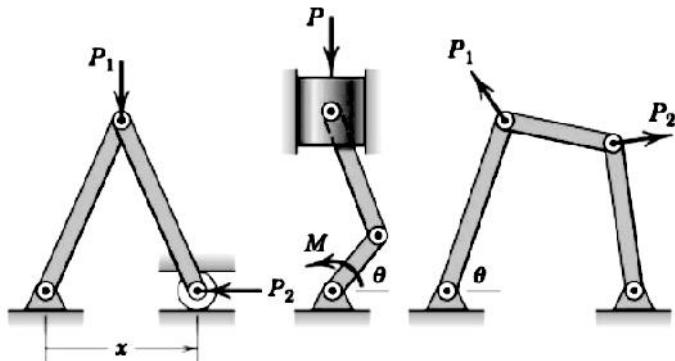


Fig.3.27 One degree of freedom systems

The number of degrees of freedom of a mechanical system is the number of independent coordinates needed to specify completely the configuration of the system. Fig.3.27 shows three examples of one degree-of-freedom systems. Only one coordinate is needed to establish the position of every part of the system. The coordinate can be a distance or an angle.

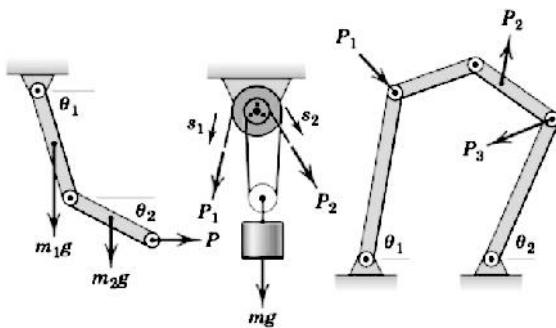


Fig.3.28 Two degree of freedom systems

Fig.3.28 shows three examples of two – degree – of - freedom systems where two independent coordinates are needed to determine the configuration of the system.

By the addition of more links to the mechanism in the right-hand figure, there is no limit to the number of degrees of freedom which can be introduced.

The principle of virtual work  $\delta U = 0$  may be applied as many times as there are degrees of freedom. With each application, it allows only one independent coordinate to change at a time while holding the others constant. In this treatment of virtual work in this section, consider only one-degree-of-freedom systems.

## SYSTEMS WITH FRICTION:

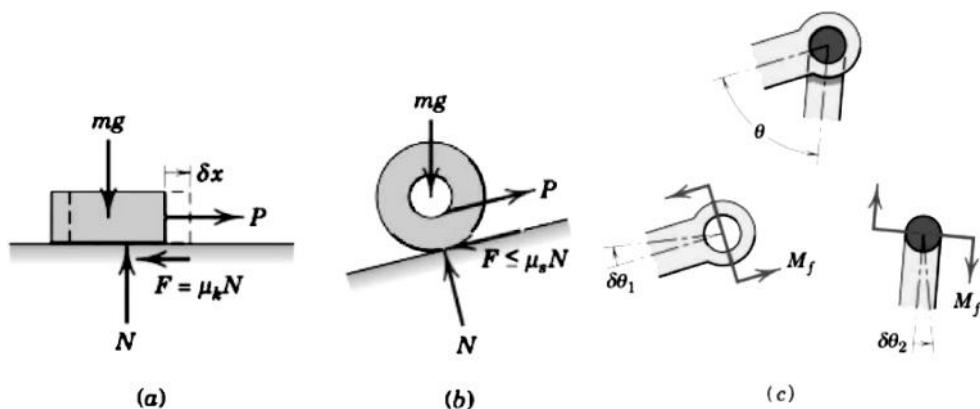


Fig.3.29 Friction in a system

When sliding friction is present to any appreciable degree in a mechanical system, the system is said to be "real." In real systems some of the positive work done on the system by external active forces (input work) is dissipated in the form of heat generated by the kinetic friction forces during movement of the system. When there is sliding between contacting surfaces, the friction force does negative work because its direction is always opposite to the movement of the body on which it acts. This negative work cannot be regained.

Thus, the kinetic friction force  $\sim_k N$  acting on the sliding block in fig.3.29 (a) does work on the block during the displacement  $x$  in the amount of  $-\sim_k N_x$ . During a virtual displacement  $\delta x$ , the friction force does work equal to  $-\sim_k N_x \delta x$ . The static

friction force acting on the rolling wheel in Fig.3.29 (b), on the other hand, does no work if the wheel does not slip as it rolls.

In Fig.3.29 (c) the moment  $M_f$  about the centre of the pinned joint due to the friction forces which act at the contacting surfaces does negative work during any relative angular movement between the two parts. Thus, for a virtual displacement  $\delta\theta$  between the two parts, which have the separate virtual displacements  $\delta\theta_1$  and  $\delta\theta_2$  as shown, the negative work done is  $-M_f u_{\theta_1} - M_f u_{\theta_2} = -M_f (u_{\theta_1} + u_{\theta_2})$ , or simply  $-M_f u_{\theta}$ . For each part,  $M_f$  is in the sense to oppose the relative motion of rotation.

A major advantage of the method of virtual work is in the analysis of an entire system of connected members without taking them apart. If there is appreciable kinetic friction internal to the system, it becomes necessary to dismember the system to determine the friction forces. In such cases the method of virtual work finds only limited use.

## MECHANICAL EFFICIENCY:

Because of energy loss due to friction, the output work of a machine is always less than the input work. The ratio of the two amounts of work is the mechanical efficiency  $\gamma$ . Thus,

$$\gamma = \frac{\text{output work}}{\text{input work}}$$

The mechanical efficiency of simple machines which have a single degree of freedom and which operate in a uniform manner may be determined by the method of work by evaluating the numerator and denominator of the expression fore during a virtual displacement.

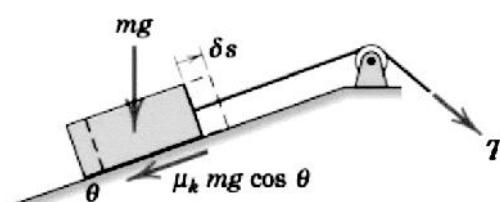


Fig.3.30 Connected body

As an example, consider the block being moved up the inclined plane in fig.3.30. For the virtual displacement  $\delta s$  shown, the output work is that necessary to elevate the block, or  $mg \delta s \sin\theta$ . The input work is  $T_{us} = (mgs \sin\theta + \gamma_k mg \cos\theta) \delta s$ . The efficiency of the inclined plane is, therefore,

$$\eta = \frac{mg \delta s \sin\theta}{mg(\sin\theta + \gamma_k \cos\theta) \delta s} = \frac{1}{1 + \gamma_k \cot\theta}$$

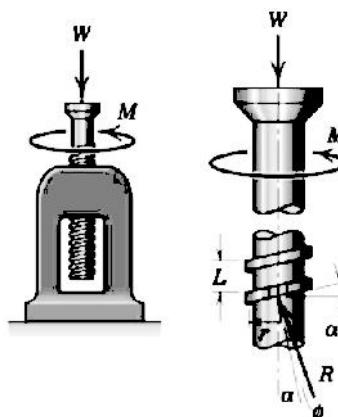


Fig.3.31 Simple screw jack

As a second example, consider the screw jack described in previous and shown in fig.3.31.  $M = Wr \tan(r + w)$  gives the moment  $M$  required to raise the load  $W$ , where the screw has a mean radius  $r$  and a helix angle  $\alpha$ , and where the friction angle is  $w = \tan^{-1} \gamma_k$ . During a small rotation  $\delta\theta$  of the screw, the input work  $M u_s = Wr u_s \tan(r + w)$ . The output work is that required to elevate the load, or  $Wr u_s \tan r$ . Thus the efficiency of the jack can be expressed as

$$\eta = \frac{Wr u_s \tan r}{Wr u_s \tan(r + w)} = \frac{\tan r}{\tan(r + w)}$$

As friction is decreased,  $c/J$  becomes smaller, and the efficiency approaches unity.

**Problems:**

**SP3.12:** Determine the force exerted by the vice on the block when a given force  $P$  is applied at  $C$  as shown in fig.3.32. Assume that there is no friction.

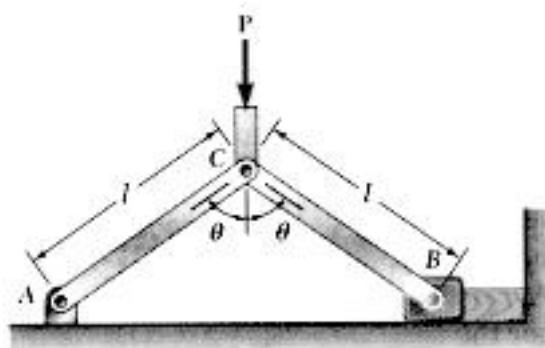
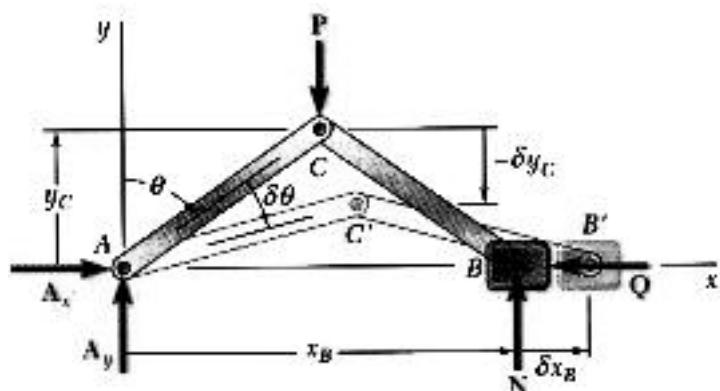


Fig.3.32

**Solution:**

The above free body diagram shows that the reactive and active forces in a system. Consider the work done by the external forces for a virtual displacement  $\delta\theta$ . Now  $\delta\theta$  is a positive increment to  $\theta$  as shown in free body diagram. As per the virtual work method, the only forces  $P$  and  $Q$  produces non-zero work.

$x_B$  increases while  $y_C$  decreases then

The positive increment for  $x_B$  is  $Ux_B \rightarrow UU_Q = -QUx_B$ . The negative sign indicates opposite sense of movement.

The negative increment for  $y_C$  is  $-u y_C \rightarrow u U_P = +P(-u y_C)$ . The negative sign indicates the same sense of above.

The principle of virtual work is

$$uU = 0, \Rightarrow uU_Q + uU_P = 0$$

$$\Rightarrow uU_Q + uU_P = -Qux_B - Pu y_C = 0$$

The terms  $x_B$  and  $y_C$  are expressed as  $\theta$ , then

$$x_B = 2l \sin \theta \text{ and } y_C = l \cos \theta$$

Now differentiate w.r.t  $\theta$ , then

$$ux_B = 2l \cos \theta u \text{ and } uy_C = -l \sin \theta u$$

Therefore,

$$\Rightarrow -Qux_B - Pu y_C = -Q2l \cos \theta u - P(-l \sin \theta u) = 0$$

$$\Rightarrow -2Ql \cos \theta u + Pl \sin \theta u = 0$$

$$\Rightarrow Q = \frac{1}{2} P \tan \theta$$

**SP3.13:** Find the magnitude of the couple  $M$  required to maintain the equilibrium of the mechanism.

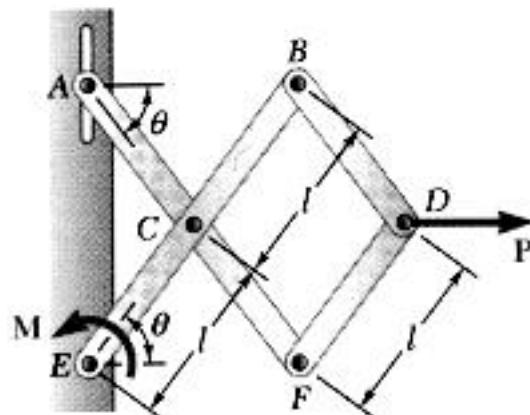


Fig.3.33

**Solution:**

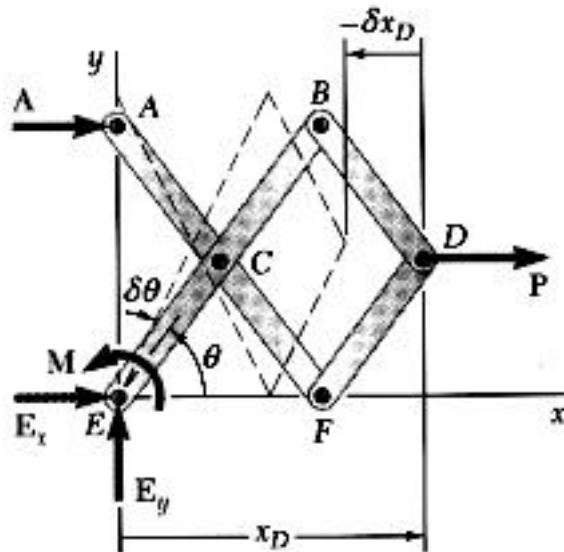


Fig.3.33a

Now apply the principle of virtual work, then

$$\mathbf{u}U = 0, \Rightarrow \mathbf{u}U_M + \mathbf{u}U_P = 0$$

$$\Rightarrow Mu_{\theta} + Pux_D = 0$$

We know that,

$$x_D = 3l \cos \theta$$

Differentiate w.r.t.  $\theta$ ,

$$ux_D = -3l \sin \theta u_{\theta}$$

$$\text{Therefore, } Mu_{\theta} + P(-3l \sin \theta u_{\theta}) = 0$$

$$M = 3Pl \sin \theta$$

**SP3.14:** Determine the angle  $\theta$  for equilibrium of the two member linkage as shown in fig.3.34a. Each member has a mass of 10kg.

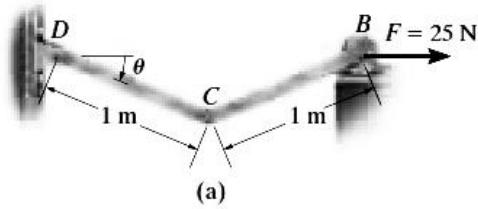


Fig.3.34 a

**Solution:**

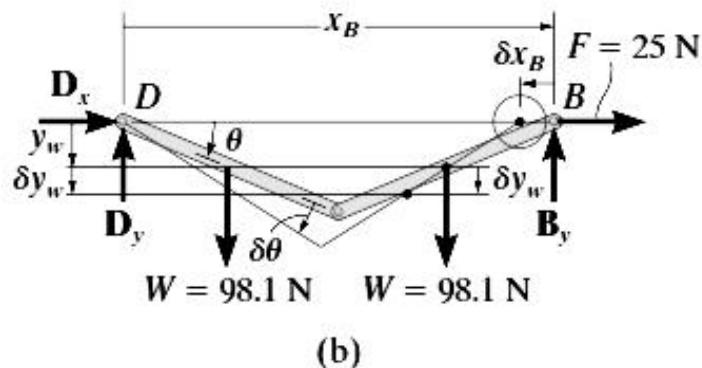


Fig.3.34b

The position of F and W can be specified by the position coordinates  $x_B$  and  $y_B$  as

$$x_B = 2(1 \cos \theta), \Rightarrow u x_B = -2 \sin \theta u \theta$$

$$y_B = \frac{1}{2}(1 \sin \theta), \Rightarrow u y_B = 0.5 \cos \theta u \theta$$

The principle of virtual work is

$$uU = 0, \Rightarrow W u y_B + W u y_W + F u x_B = 0$$

$$\Rightarrow 9.81(0.5 \cos \theta u \theta) + 98.1(0.5 \cos \theta u \theta) + 25(-2 \sin \theta u \theta) = 0$$

Since  $\delta\theta \neq 0$ , then

$$98.1 \cos \theta - 50 \sin \theta = 0$$

$$\theta = \tan^{-1} \frac{98.1}{50} = 63.0^\circ$$

**SP3.15:** Determine the required force P in fig.3.35a needed to maintain equilibrium of the scissors linkage when  $\theta=60^\circ$ . The spring is un-stretched when  $\theta=30^\circ$ . Neglect the mass of the links.

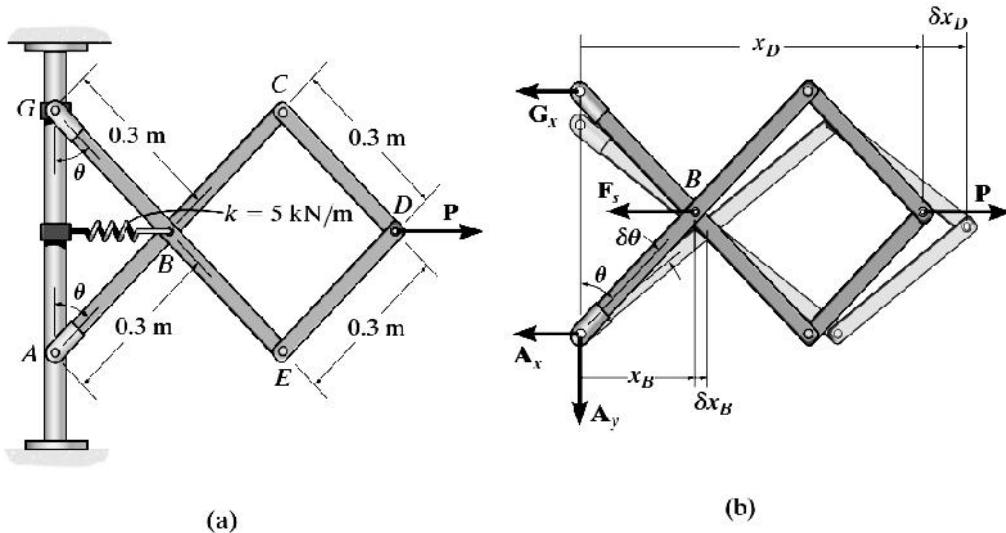


Fig.3.35

**Solution:**

The position of  $F_s$  and  $P$  can be specified by the position coordinates  $x_B$  and  $x_D$  as

$$x_B = 0.3 \sin \theta, \Rightarrow ux_B = 0.3 \cos \theta u$$

$$x_D = 3(0.3 \sin \theta), \Rightarrow ux_D = 0.9 \cos \theta u$$

The principle of virtual work is

$$uU = 0, \Rightarrow -F_s ux_B + P ux_D = 0$$

$$\Rightarrow F_s = ks,$$

Where,  $k$  is the stiffness of spring constant and  $s$  is the change in position that is change in length minus un-stretched length. If  $k$  is 5000N/m, un-stretched length is  $0.3\sin 30^\circ$  and Change in length is  $0.3\sin \theta$ , Then

$$\Rightarrow F_s = 5000(0.3\sin \theta - 0.3\sin 30^\circ) = (1500 \sin \theta - 750)N$$

Hence,

$$\Rightarrow -(1500 \sin \theta - 750)u_B + P u_D = 0$$

$$\Rightarrow -(1500 \sin \theta - 750)(0.3 \cos \theta u_D) + P(0.9 \cos \theta u_D) = 0$$

$$\Rightarrow (0.9P + 225 - 450 \sin \theta) \cos \theta u_D = 0$$

Since  $\cos \theta \neq 0$ , then

$$P = 500 \sin \theta - 250$$

When  $\theta = 60^\circ$ , then

$$P = 500 \sin 60^\circ - 250$$

$$P = 183 \text{ N}$$

**SP3.16:** If the box as shown in fig.3.36a has a mass 10kg determine the couple moment  $M$  needed to maintain equilibrium when  $\theta = 60^\circ$ . Neglect the mass of the members.

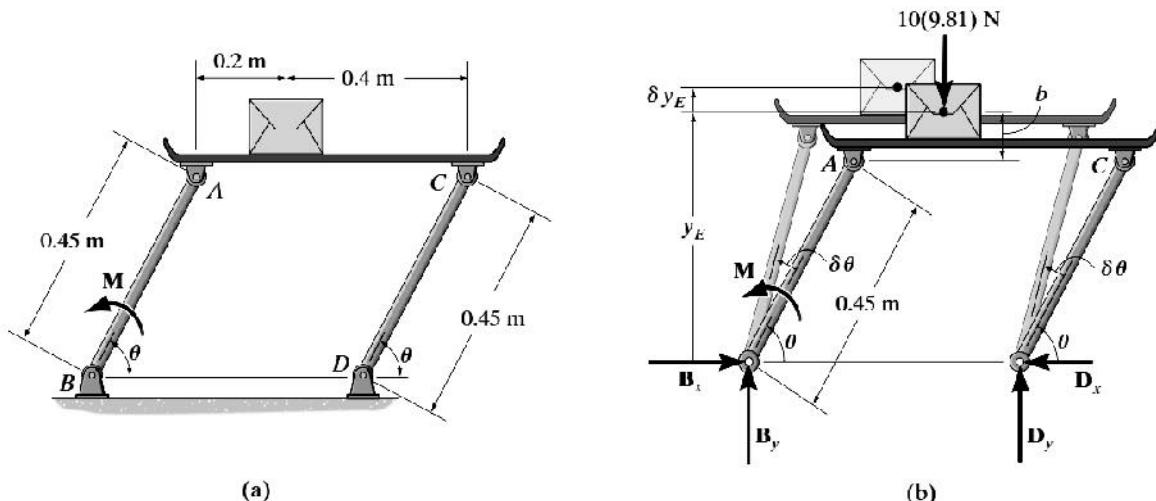


Fig.3.36

**Solution:**

The position of  $W$  and  $M$  can be specified by the position coordinates  $y_E$  and  $\theta$  as

$$y_E = 0.45 \sin \theta + b, \Rightarrow u y_E = 0.45 \cos \theta u_\theta$$

The principle of virtual work is

$$uU = 0, \Rightarrow -10(9.81)uy_E + Mu_{\theta} = 0$$

$$\Rightarrow -10(9.81)(0.45 \cos \theta) + Mu_{\theta} = 0$$

Since  $\delta\theta \neq 0$ , then

$$M - 44.145 \cos \theta = 0$$

If  $\theta = 60^\circ$ , then

$$M = 44.145 \cos 60^\circ$$

**SP3.17:** The mass  $m$  is brought to an equilibrium position by the application of the couple  $M$  to the end of one of the two parallel links which are hinged as shown in fig.3.37. The links have negligible mass and all friction is assumed to be absent. Determine the expression for the equilibrium angle  $\theta$  assumed by the links with the vertical for a given value of  $M$ . Consider the alternative of a solution by force and moment equilibrium.

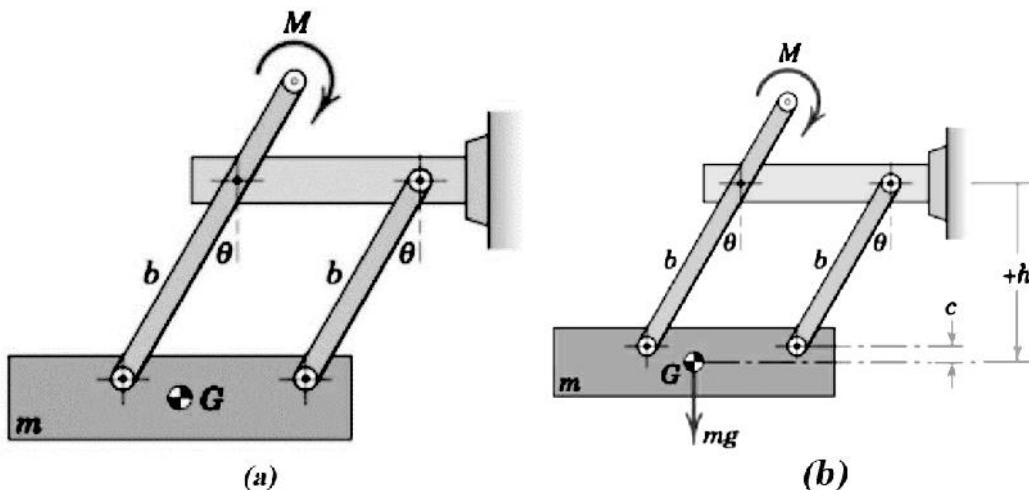


Fig.3.37

### Solution:

The position of  $W$  and  $M$  can be specified by the position coordinates  $h$  and  $\theta$  as

$$h = b \cos \theta + c, \Rightarrow uh = -b \sin \theta u_{\theta} + 0$$

The principle of virtual work is

$$uU = 0, \Rightarrow Mu_{\theta} + Wh = 0$$

$$\Rightarrow Mu_{\theta} + mg(-b \sin \theta) = 0$$

Since  $\delta\theta \neq 0$ , then

$$M = mgb \sin \theta$$

$$\theta = \sin^{-1} \frac{M}{mgb}$$

**SP3.18:** For link OA in the horizontal position as shown in fig.3.38 Determine the force P on the sliding collar which will prevent OA from rotating under the action of the couple M. Neglect the mass of the moving parts.

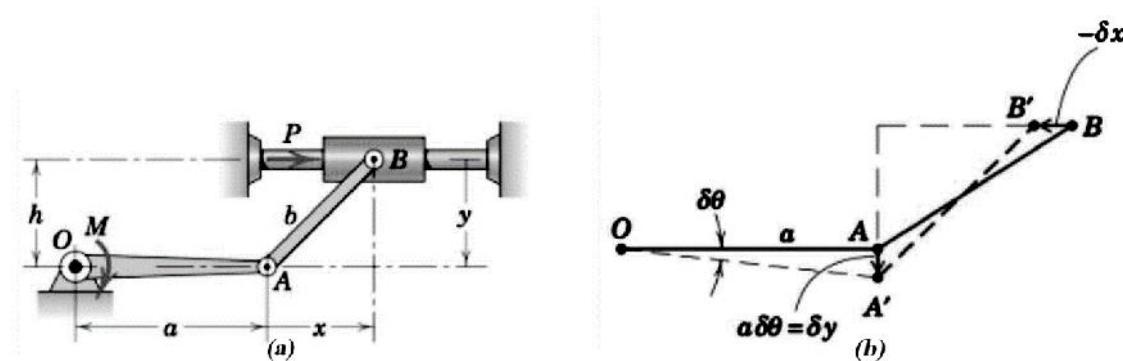


Fig.3.38

### Solution:

The position of P and M can be specified by the position coordinates x and  $\theta$  about the point O. From the right triangle for which link AB has length as

$$b^2 = x^2 + y^2$$

Differentiate the above equation to obtain displacement along x and y, then

$$2xu_x + 2yu_y = 0$$

$$u_x = -\frac{y}{x}u_y$$

The vertical displacement at point A is  $u_y = au_{\theta}$

The principle of virtual work is

$$uU = 0, \Rightarrow Mu_{\theta} + Pux = 0$$

$$\Rightarrow Mu'' + P \left( -\frac{y}{x} a u'' \right) = 0$$

Since  $\delta\theta \neq 0$ , then

$$P = \frac{Mx}{ya} = \frac{Mx}{ha}$$

**SP3.19:** Determine the reaction  $R_A$  and  $R_B$  developed in the simply supported beam shown in fig.3.39 using virtual work method.

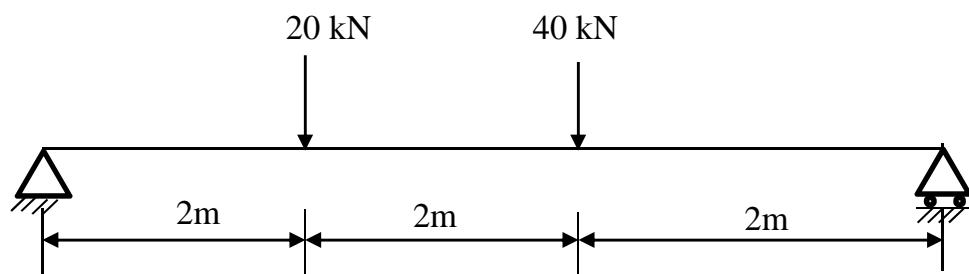


Fig.3.39

**Solution:**

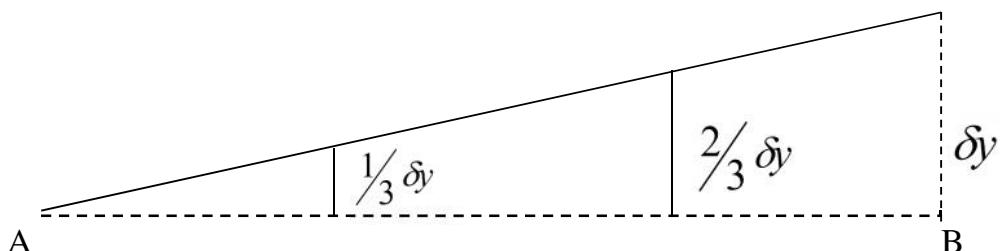


Fig.3.39a

Let  $R_A$  and  $R_B$  be the reactions at supports A and B. let  $\delta y$  displacement be given to beam at B without giving any displacement at A as shown in above free body diagram.

The corresponding displacement for the beam at 20 kN load and 40 kN load at points are  $\frac{1}{3}u_y$  and  $\frac{2}{3}u_y$  respectively.

By the principle of virtual work,

$$uU = 0, \Rightarrow R_A \times 0 - 20 \times \frac{1}{3}uy - 40 \times \frac{2}{3}uy + R_B \times uy = 0$$

$$\Rightarrow \left( \frac{-20}{3} - \frac{80}{3} + R_B \right)uy = 0$$

Since  $\delta y \neq 0$ , then

$$\Rightarrow \left( \frac{-20}{3} - \frac{80}{3} + R_B \right) = 0$$

Therefore,

$$R = 33.33 \text{ kN}$$

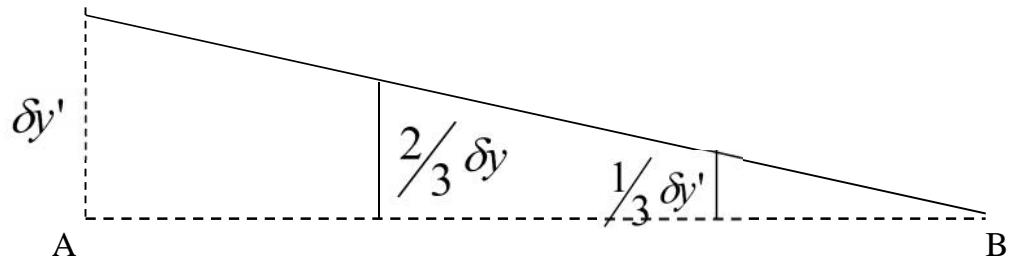


Fig.3.39b

Now consider the virtual displacement from the support at B, then the corresponding displacement for the beam at 20 kN load and 40 kN load at points are  $\frac{2}{3}uy'$  and  $\frac{1}{3}uy'$  respectively. The principle of virtual work is

$$uU' = 0, \Rightarrow R_A \times uy' - 20 \times \frac{2}{3}uy' - 40 \times \frac{1}{3}uy' + R_B \times 0 = 0$$

$$\Rightarrow \left( R_A - \frac{40}{3} - \frac{40}{3} \right)uy' = 0$$

Since  $\delta y \neq 0$ , then

$$R_A - \frac{40}{3} - \frac{40}{3} = 0$$

$$R_A = 26.67 \text{ kN}$$

**SP3.20:** Determine the reactions in the overhanging beam as shown in fig.3.40 using virtual work method.

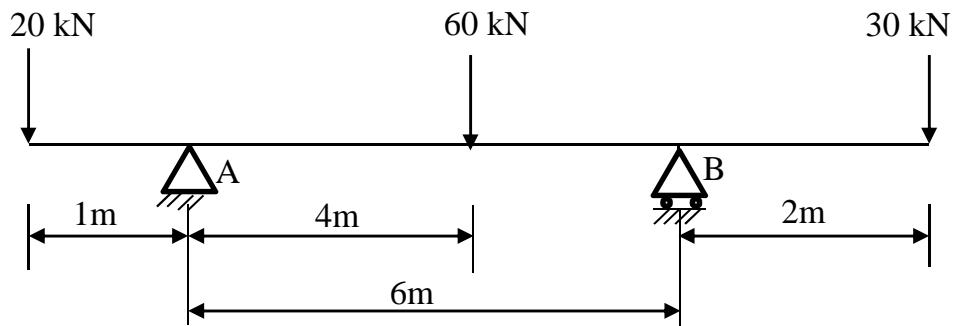


Fig.3.40

**Solution:**

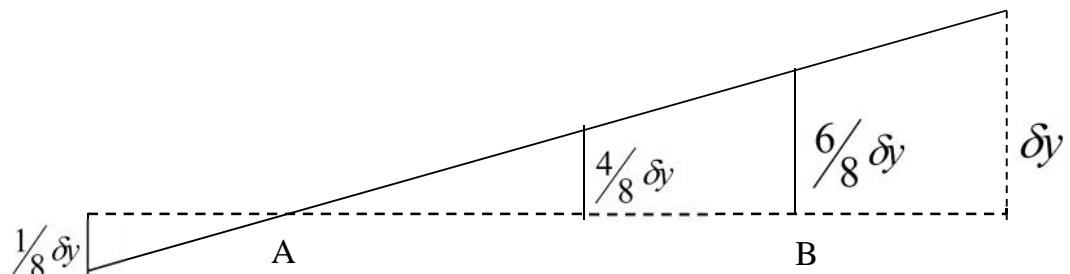


Fig.3.40a

Let  $R_A$  and  $R_B$  be the reactions at supports A and B. let  $\frac{6}{8}u_y$  displacement be given to beam at B without giving any displacement at A as shown in above free body diagram.

The corresponding displacement for the beam at 20 kN load , 60 kN and 30 kN load at points are  $\frac{1}{8}u_y$  ,  $\frac{4}{8}u_y$  and  $u_y$  respectively.

By the principle of virtual work,

$$uU = 0, \Rightarrow R_A \times 0 + 20 \times \frac{1}{8}uy - 60 \times \frac{4}{8}uy + R_B \times \frac{6}{8}uy - 30 \times uy = 0$$

$$\Rightarrow \left( \frac{20}{8} - \frac{240}{8} + \frac{6R_B}{8} - 30 \right)uy = 0$$

$$\text{Since } \delta y \neq 0, \text{ then } R_B = \frac{8}{6} \left( 30 + \frac{240}{8} - \frac{20}{8} \right)$$

$$R_B = 76.67 \text{ kN}$$

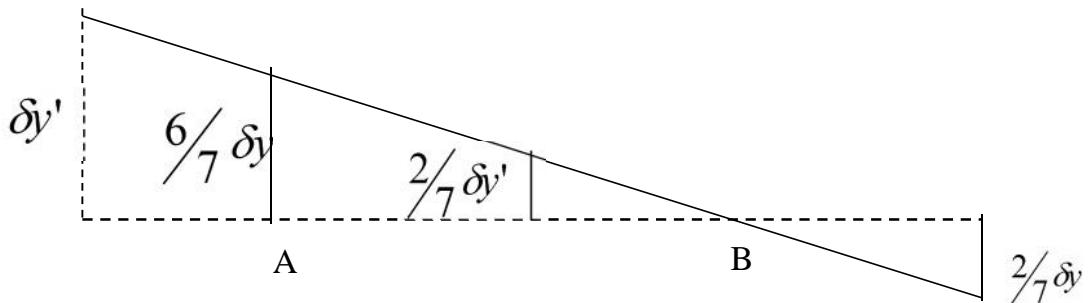


Fig.3.40b

Now consider the virtual displacement from the support at B, then the corresponding displacement for the beam at 20 kN load, 60 kN and 30 kN load at points are  $uy'$ ,  $\frac{2}{7}uy'$  and  $\frac{2}{7}uy'$  respectively.

The principle of virtual work is

$$uU' = 0, \Rightarrow -R_A \times \frac{6}{7}uy' + 20 \times uy' + 60 \times \frac{2}{7}uy' + R_B \times 0 - 30 \times \frac{2}{7}uy' = 0$$

$$\Rightarrow \left( -\frac{6}{7}R_A + 20 + \frac{120}{7} - \frac{60}{7} \right)uy' = 0$$

Since  $\delta y \neq 0$ , then

$$R_A = \frac{7}{6} \left( 20 + \frac{120}{7} - \frac{60}{7} \right)$$

$$R_A = 33.33 \text{ kN}$$

**SP3.21:** Determine the reaction  $R_A$  in the overhanging beam as shown in fig.3.41 by virtual work method.

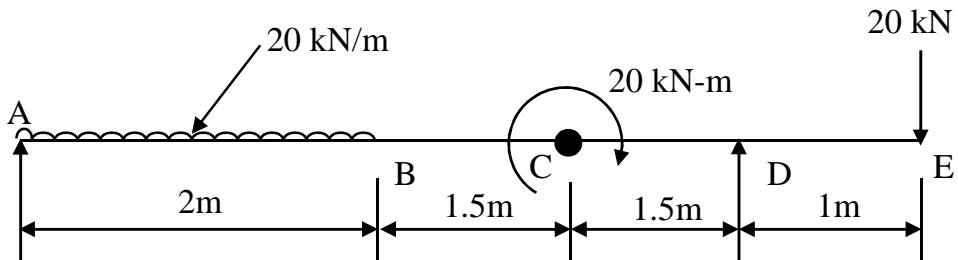


Fig.3.41

**Solution:**

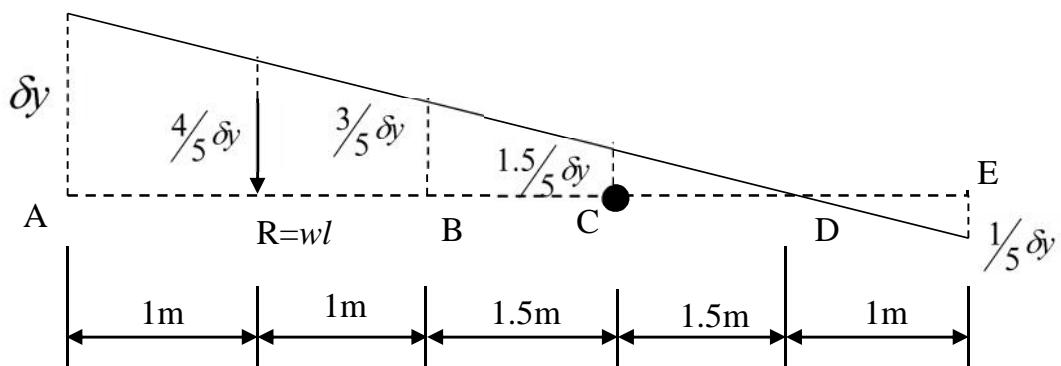


Fig.3.41a

Now consider the virtual displacement from the support at B, then the corresponding displacement for the beam

At point A is  $u_y$  ,

At point B is  $\frac{3}{5}u_y$  ,

At point R is  $\frac{4}{5}u_y$

At point C is  $\frac{u_y}{5}$

At point D is 0 and

At point E is  $\frac{1}{5}u_y$

The principle of virtual work is

$$uU' = 0, \Rightarrow -R_A \times u_y + 20 \times 2 \times \frac{4}{5}u_y - 40 \times \frac{1}{5}u_y + R_B \times 0 - 20 \times \frac{1}{5}u_y = 0$$

$$\Rightarrow (-R_A + 32 - 8 - 4)u_y = 0$$

Since  $\delta y \neq 0$ , then

$$R_A = 32 - 8 - 4$$

$$R_A = 20 \text{ kN}$$

**SP3.22:** What is the value of P in the system as shown in fig.3.42 to cause the motion of 500 N block to the right side? Assume the pulley is smooth and the coefficient of friction between other contact surfaces is 0.20.

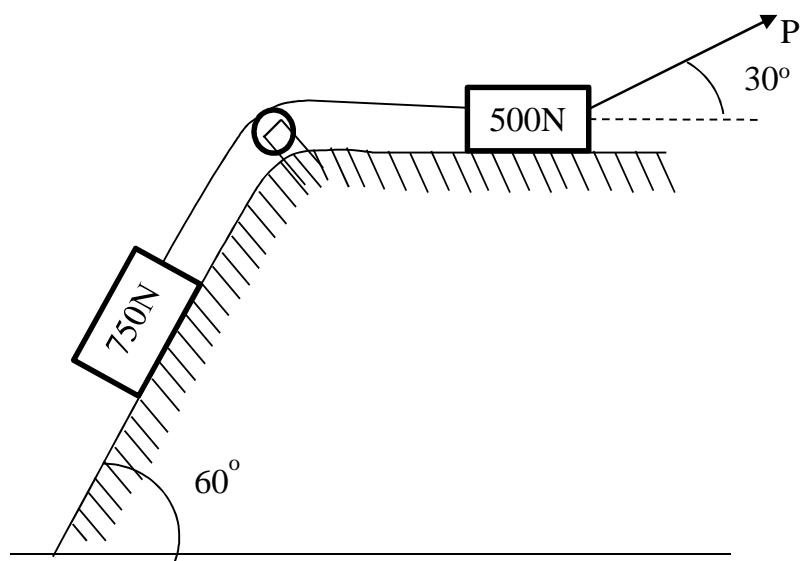


Fig.3.42

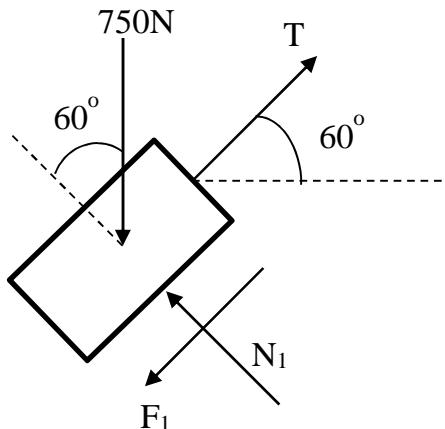
**Solution:**

Fig.3.42a

The free body diagram of the block has 750N is shown in above figure. When motion is impending up the plane. Then friction force acts down the plane as

$$\frac{F_1}{N_1} = \sim = 0.2 ,$$

$$F_1 = 0.2 \times N_1 = 0.2 \times 375 = 75N$$

Let us give virtual displacement  $\delta s$  up the plane. Let the direction up the plane be taken positive.

Then from principle of virtual work, we get

$$- 750 \sin 60^\circ \times (\delta s) - F_1 \times (\delta s) + T(\delta s) = 0$$

$$- 750 \sin 60^\circ \times (\delta s) - 75 \times (\delta s) + T(\delta s) = 0$$

$$(- 750 \sin 60^\circ - 75 + T)(\delta s) = 0$$

Since  $\delta s \neq 0$ , then

$$(- 750 \sin 60^\circ - 75 + T) = 0$$

$$T = (750 \sin 60^\circ + 75)$$

$$T = 724.52N$$

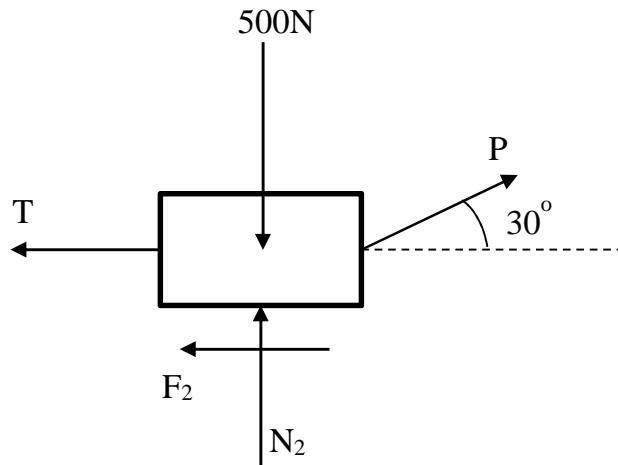


Fig.3.42b

The free body diagram of the block has 500N is shown in above figure. When motion is impending up the plane. Then friction force acts down the plane as

$$\frac{F_2}{N_2} = \mu = 0.2,$$

$$F_2 = 0.2 \times N_2 = 0.2 \times (500 - P \sin 30^\circ) = 100 - 0.1PN$$

Let us give virtual displacement  $\delta s$  up the plane. Let the direction up the plane be taken positive.

Then from principle of virtual work, we get

$$P \cos 30^\circ \times (\delta s) - F_2 \times (\delta s) - T(\delta s) = 0$$

$$0.87P \times (\delta s) - (100 - 0.1P) \times (\delta s) - 724.52(\delta s) = 0$$

$$(0.87P - (100 - 0.1P) - 724.52)(\delta s) = 0$$

Since  $\delta s \neq 0$ , then

$$(0.97P - 824.52) = 0$$

$$P = \left( \frac{824.52}{0.97} \right)$$

$$P = 850.02N$$

**SP3.23:** Determine the force P required to make the motion impending to the left for the system of connected bodies as shown in fig.3.43a. Coefficient of friction for all contacting surface is 0.25. Pulleys are smooth.

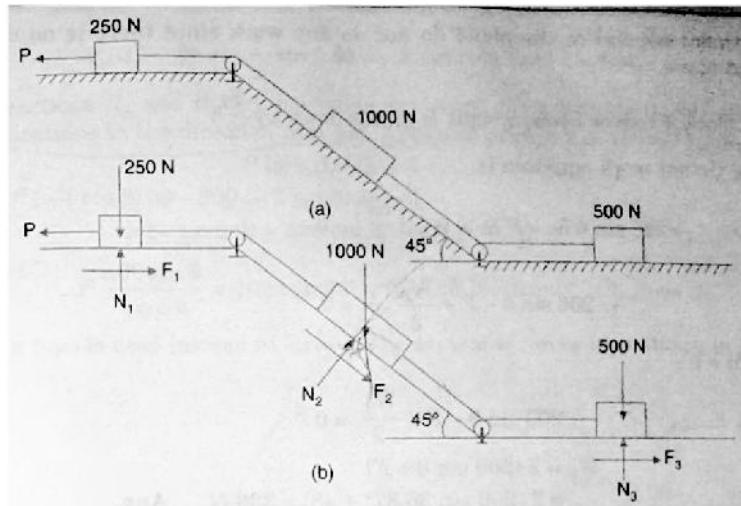


Fig.3.43

**Solution:**

Since motion is impending to the left, frictional forces are acting in the directions shown in fig.3.43b.

$$F_1 = 0.25N_1 = 0.25 \times 250 = 62.5N$$

$$F_2 = 0.25N_2 = 0.25 \times 1000 \cos 45^\circ = 176.78N$$

$$F_3 = 0.25N_3 = 0.25 \times 500 = 12.5N$$

Let us give a virtual displacement towards left. Taking the displacement of the system to left as positive, then the virtual work equation is

$$P(u_s) - F_1(u_s) - 1000 \sin 45^\circ(u_s) - F_2(u_s) - F_3(u_s) = 0$$

$$\therefore (P - F_1 - 1000 \sin 45^\circ - F_2 - F_3)(u_s) = 0$$

Since  $\delta s \neq 0$ , then

$$P = F_1 + 1000 \sin 45^\circ + F_2 + F_3, \Rightarrow P = 62.5 + 707 + 176.78 + 12.5$$

$$P = 1071.38N$$

**SP3.24:** Using the principle of virtual work find the weight  $W_B$  for which the motion of block A is impending up the plane. The coefficient of friction between block A and the plane is 0.3 and pulleys are smooth as shown in fig.3.44

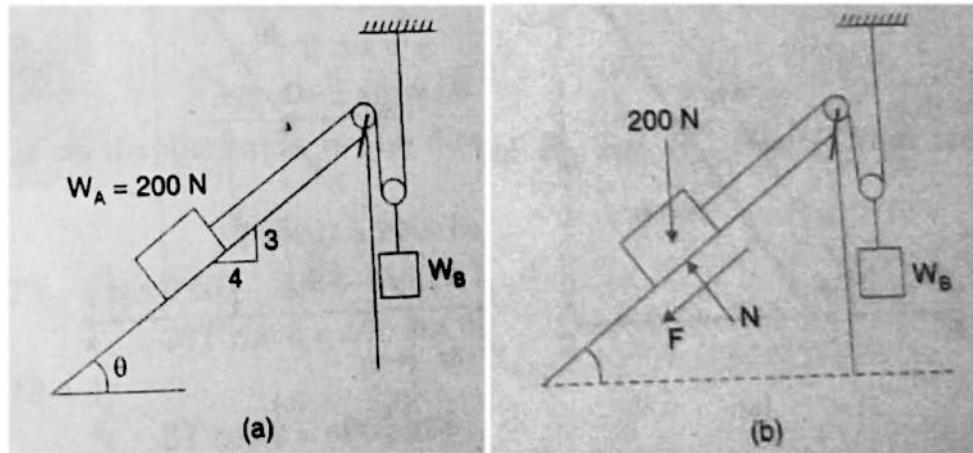


Fig.3.44

**Solution:**

Now determine  $\theta$  from the geometry of plane as

$$\tan \theta = \frac{3}{4}$$

$$\theta = \tan^{-1} \frac{3}{4} = 36.87^\circ$$

The motion is impending, then

$$\frac{F}{N} = \mu$$

$$F = \mu N = \mu W \cos \theta$$

$$F = 0.3 \times 200 \cos 36.87^\circ$$

$$F = 48 \text{ N}$$

Let us give virtual displacement of  $\delta s$  up the plane to block A. Component of block A down the plane is  $W_A \sin \theta$

The component normal to the plane do not do any work since there is no displacement normal to the plane.

When block A moves by  $\delta s$ , weight B moves down by  $\frac{us}{2}$

Hence the virtual work equation is

$$-200 \sin \theta (us) - F(us) + W_B \frac{(us)}{2} = 0$$

$$\left( -200 \sin \theta - F + \frac{W_B}{2} \right) (us) = 0$$

Since  $\delta s \neq 0$ , then

$$\left( -200 \sin \theta - F + \frac{W_B}{2} \right) = 0$$

$$\left( -200 \sin 36.87 - 48 + \frac{W_B}{2} \right) = 0$$

$$W_B = 2(200 \sin 36.87^\circ + 48)$$

$$W_B = 336 \text{ N}$$

**SP3.25:** Ladder AB of length 4m and weight 200 N is held in position by applying force P as shown in fig.3.45. Assuming smooth wall and floor determine the force P. If instead of force P for preventing from slipping, a horizontal rope is tied at a distance of 1m from A, what is the tension in the rope?

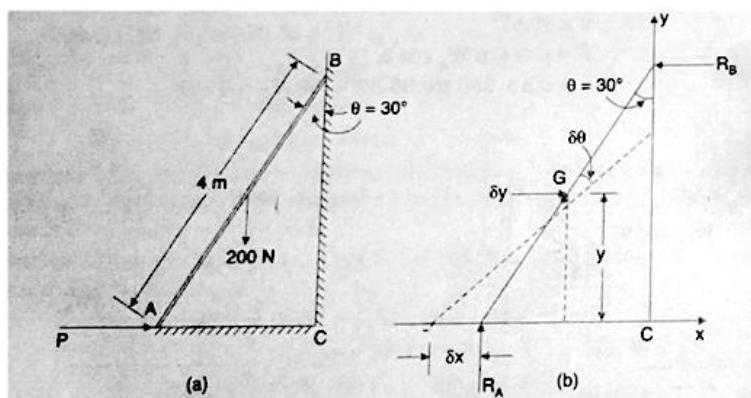


Fig.3.45

**Solution:**

a) P at floor level

$$AB = l = 4m$$

Let ladder rotate by angle  $\delta\theta$  as shown in figure 3.45b

Now taking C as origin then the coordinate of A is x

$$x = -4\sin\theta$$

Differentiate the above equation as

$$ux = -4\cos\theta \delta\theta$$

P moves by  $-4\cos\theta\delta\theta$

$$\text{Now, } y = \frac{l}{2}\cos\theta$$

$$uy = -\frac{l}{2}\sin\theta \delta\theta = -2\sin\theta \delta\theta, \quad \text{since } l = 4m$$

The reactions  $R_A$  and  $R_B$  are not doing any work since there is no movement of their point of applications in the direction they act. Applying principle of virtual work we get

$$Pux - 200uy = 0$$

$$P(-4\cos\theta)u - 200(-2\sin\theta)u = 0$$

$$P(-4\cos\theta) + 400\sin\theta = 0$$

$$P = \frac{400\sin\theta}{4\cos\theta} = 100\tan\theta = 100\tan 30^\circ$$

$$P = 57.735N$$

b) If a rope is used instead of force P, the system of forces is as shown in fig.3.46 a

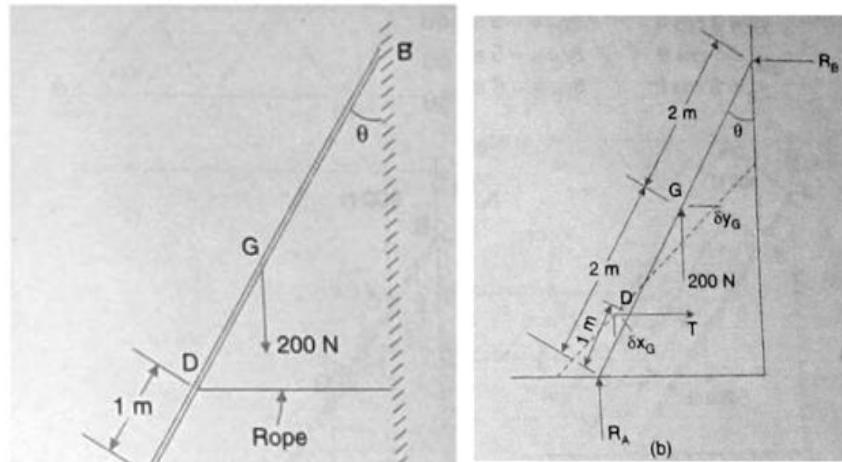


Fig.3.46

The system of forces in this case are as shown in fig.3.46b Taking C as origin.

$$x_D = -3 \sin \theta$$

Differentiate the above equation as

$$ux_D = -3 \cos \theta u_\theta$$

$$y_G = 2 \cos "$$

Differentiate the above equation as

$$uy_G = -2 \sin "u "$$

There is no displacement in the direction of  $R_A$  and  $R_B$ . Hence from the principle of virtual work,

$$T(\mathbf{u}x_D) + 200(\mathbf{u}y_G) = 0$$

$$T(-3 \cos \theta, \sin \theta) + 200(-2 \sin \theta, \cos \theta) = 0$$

$$(-3T \cos \theta + 400 \sin \theta) u_{\theta} = 0$$

Since  $\delta\theta \neq 0$ , then

$$3T \cos \theta = 400 \sin \theta, \quad T = \frac{400}{3} \tan \theta = \frac{400}{3} \tan 30^\circ$$

$$T=76.98N$$

**SP3.26:** The ladder as shown in fig.3.47a is 6m long and is supported by a horizontal floor and vertical wall. The coefficient of friction between the floor and the ladder is 0.4 and between the wall and the ladder is 0.25. The weight of the ladder is 200N. The ladder also supports a vertical load of 900N at C which is at a distance of 1m from B. Determine the least value of  $\theta$  for which the ladder may be placed without slipping. Also determine the reaction at the stage.

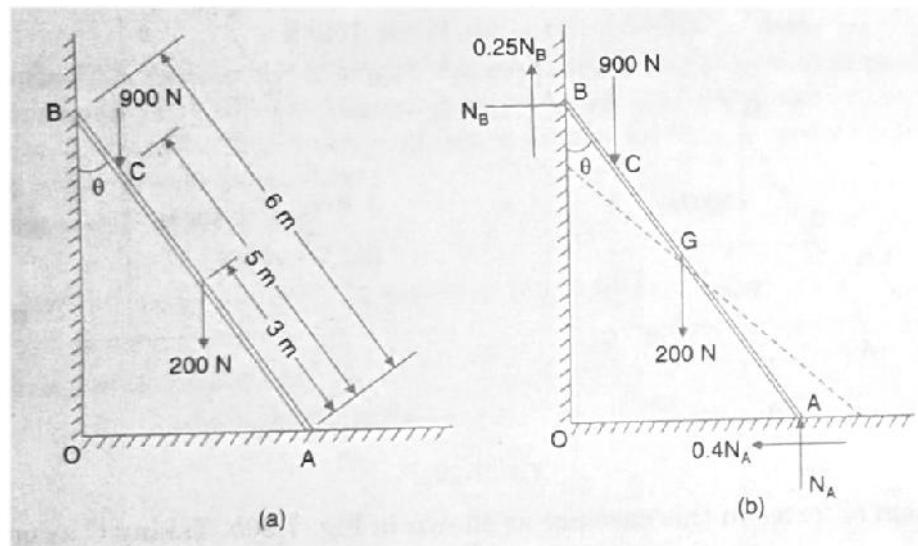


Fig.3.47

### Solution:

The various forces acting on the ladder when it is on the verge of slipping are as shown in fig.3.47b. note frictional forces always oppose impending motions and they are equal to  $\mu \times$  normal reactions where  $\mu$  is coefficient of friction. To calculate virtual works we need the displacement in the direction of forces. Normal reactions  $N_A$  and  $N_B$  are not going to do any work since displacements in direction of those forces is always zero. Taking O as origin now,

$$x_A = 6 \sin \theta, \Rightarrow \text{displacement } u_x = 6 \cos \theta u$$

$$y_G = 3 \cos \theta, \Rightarrow \text{displacement } u_y = -3 \sin \theta u$$

$$y_C = 5 \cos \theta, \Rightarrow \text{displacement } u_y = -5 \sin \theta u \text{ and}$$

$$y_B = 6 \cos \theta, \Rightarrow \text{displacement } u_y = -6 \sin \theta u$$

Applying principles of virtual work we get,

$$-0.4N_A 6 \cos \theta - 200(-3 \sin \theta) - 900(-5 \sin \theta) + 0.25N_B(-6 \sin \theta) = 0$$

Since  $\delta\theta \neq 0$ , then

$$-2.4N_A \cos \theta + 600 \sin \theta + 4500 \sin \theta - 1.5N_B \sin \theta = 0$$

From the equilibrium conditions,

$$\sum F_x = 0, \Rightarrow 0.4N_A = N_B \text{ and}$$

$$\sum F_y = 0, \Rightarrow N_A + 0.25N_B - 900 - 200 = 0$$

$$N_A + 0.25(0.4N_A) = 1100$$

$$N_A = 1000 \text{ N}$$

$$\text{Hence, } N_B = 0.4 \times 1000 = 400 \text{ N}$$

Substituting these value in above equations then we get,

$$-2.4 \times 1000 \cos \theta + 600 \sin \theta + 4500 \sin \theta - 1.5 \times 400 \sin \theta = 0$$

$$4500 \sin \theta = 2400 \cos \theta, \tan \theta = \frac{2400}{4500}$$

$$\theta = 28.073^\circ$$

**SP3.27:** Determine the force in the member FH of the truss shown in fig.3.48 Each load is 10 kN and all triangles are equilaterals with sides 4m.

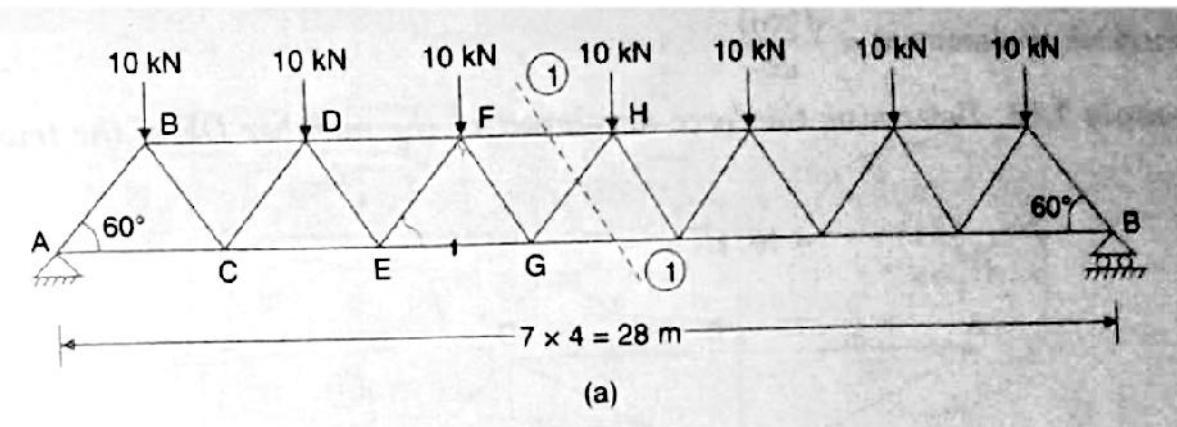


Fig.3.48

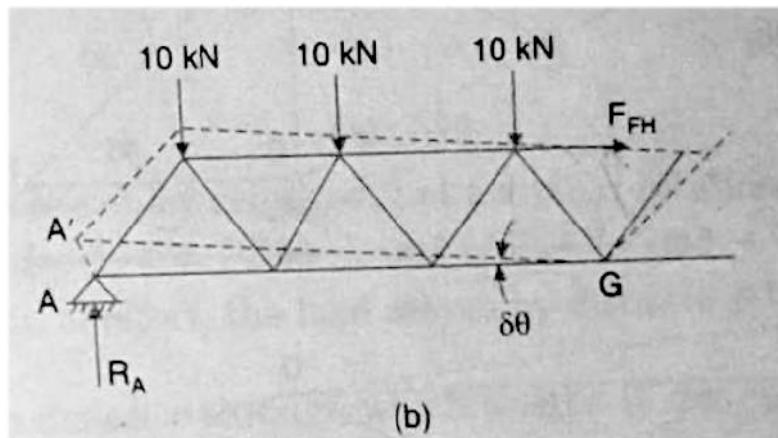
**Solution:**

Fig.3.48

$$\text{Due to symmetry, } R_A = R_B = \frac{1}{2} \text{ total load} = \frac{1}{2} \times 10 \times 7 = 35 \text{ kN}$$

Take section 1-1 as shown in fig.3.a and consider left side portion as shown in fig.3.48b.

Imagine a rotation of  $\delta\theta$  about point G.

Vertical displacement of A =  $12\delta\theta$

Vertical displacement of B =  $10\delta\theta$

Vertical displacement of D =  $6\delta\theta$  and

Vertical displacement of F =  $2\delta\theta$

Horizontal displacement of  $F_{FH} = 2 \tan 60^\circ$  towards right

Therefore from the principle of virtual displacement, we have

$$R_A 12 u_{\text{v}} - 10 \times 10 u_{\text{v}} - 10 \times 6 u_{\text{v}} - 10 \times 2 u_{\text{v}} + F_{FH} 2 \tan 60^\circ u_{\text{v}} = 0$$

$$2F_{FH} \tan 60^\circ = -R_A 12 + 100 + 60 = 20, \quad F_{FH} = \frac{240}{2 \tan 60^\circ}$$

$$F_{FH} = -69.28 \text{ kN}$$

The negative sign indicates the direction of  $F_{FH}$  is to be reverted and is compressive force.

**SP3.28:** Determine the force developed in the member DF of the truss shown in fig.3.49

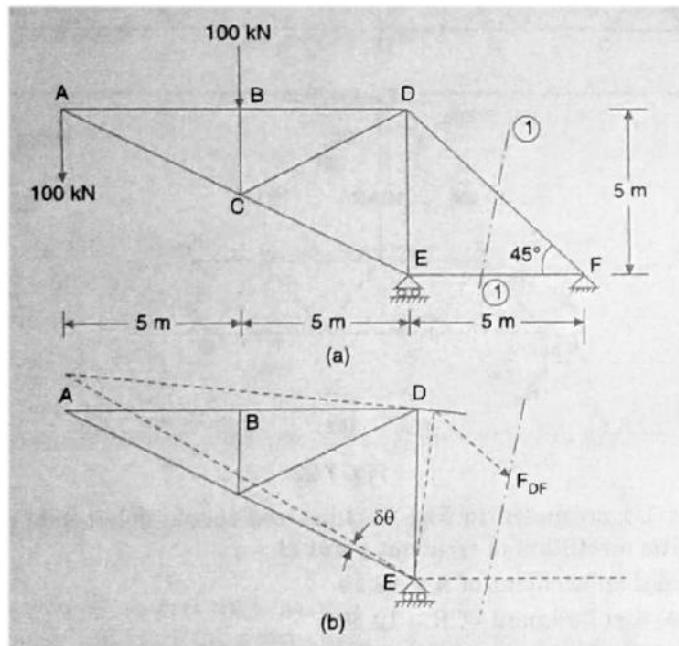


Fig.3.49

**Solution:**

Take the section 1-1 as shown in fig.3.49a and consider left side portion. Imagine a virtual rotation  $\delta\theta$  at support E. Then

Vertical displacement of point A =  $10\delta\theta$

Vertical displacement of point B =  $5\delta\theta$

Vertical displacement of point D = 0 and

Horizontal displacement of point D =  $5\delta\theta$

Therefore, from virtual work principle,

$$-100 \times 10u_u - 100 \times 5u_u + F_{DF} \cos 45^\circ \times 5u_u = 0$$

$$F_{DF} 5 \cos 45^\circ = 100 \times 10 + 100 \times 5 = 1500$$

$$F_{DF} = \frac{1500}{5 \cos 45^\circ} = 300\sqrt{2} = 424.26 \text{ kN (tensile)}$$

**SP3.29:** Neglecting the friction determine the effort required to lift a load by the screw jack as shown in fig.3.50 Take pitch of the jack = $p$ .

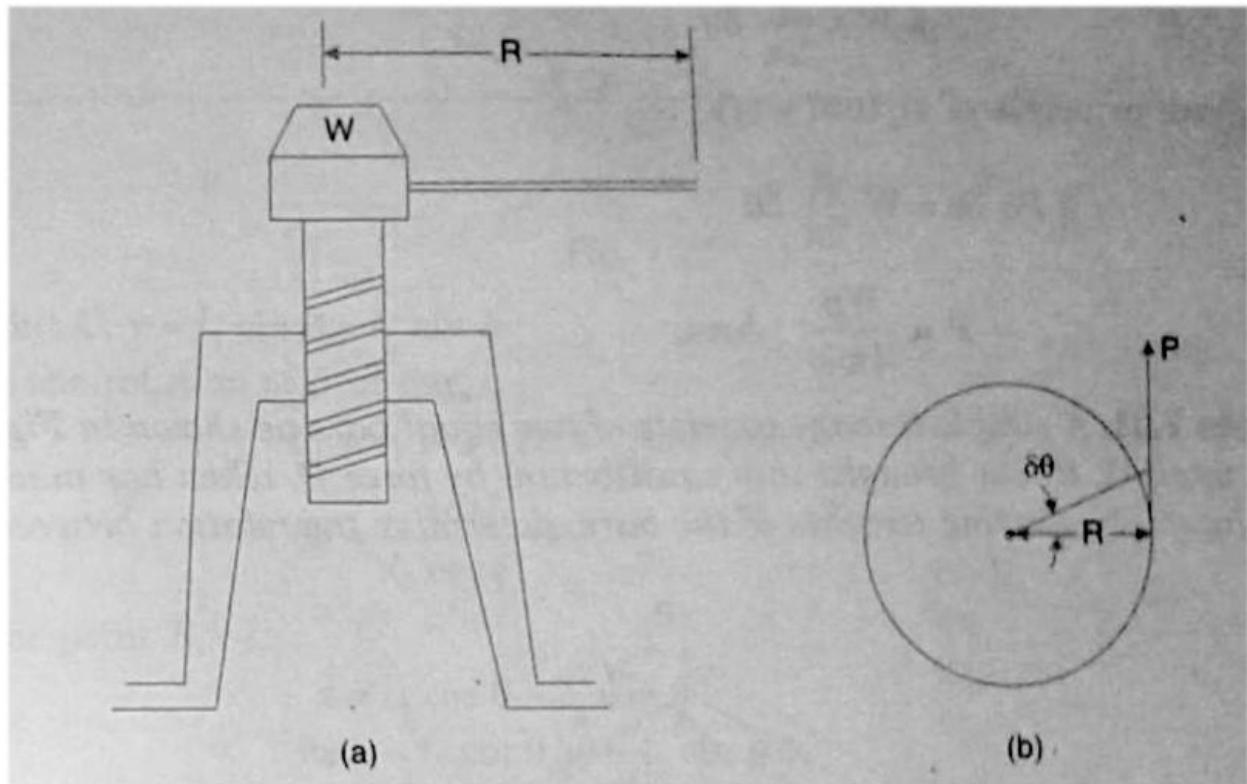


Fig.3.50

**Solution:**

Let  $P$  be the effort required. Let a virtual rotation  $\delta\theta$  be given to effort.

Therefore virtual work done by  $P = PR\delta\theta$

For one full rotation of effort, the load moves by distance  $p$  (definition of pitch)

Therefore for  $\delta\theta$  rotation distance through which weight  $W$  moves =  $\frac{P}{2f}u_{\text{u}}$

From virtual work principle total work done is zero.

$$PRu_{\text{u}} = W \frac{p}{2f}u_{\text{u}}, \Rightarrow P = \frac{Wp}{2fR}$$

**SP3.30:** Neglecting friction determine the pressure applied by screw press when effort  $P$  is applied as shown in fig.3.51.

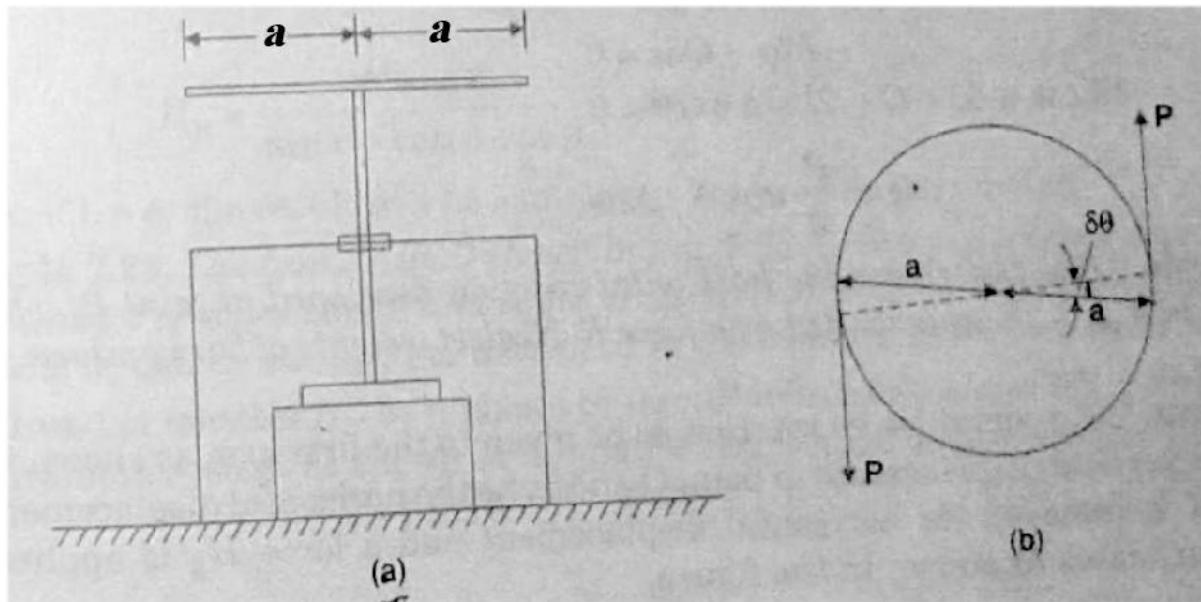


Fig.3.51

**Solution:**

Let a rotation  $\delta\theta$  be given to the screw press by applying efforts  $P$  as shown in fig.3.51b Work done by effort.

$$= 2Pau_{\text{u}}$$

If  $p$  is the pitch of the screw, work done by the load  $W$  is

$$= W \times \frac{p}{2f} u_{\text{u}}$$

Hence from principle of virtual work

$$2Pau_{\text{u}} = W \frac{p}{2f} u_{\text{u}}$$

$$P = \frac{Wp}{4fa}$$

**SP3.31:** Determine the horizontal and vertical component of reactions developed at end B of the pin jointed plane frame loaded as shown in fig.3.52a.

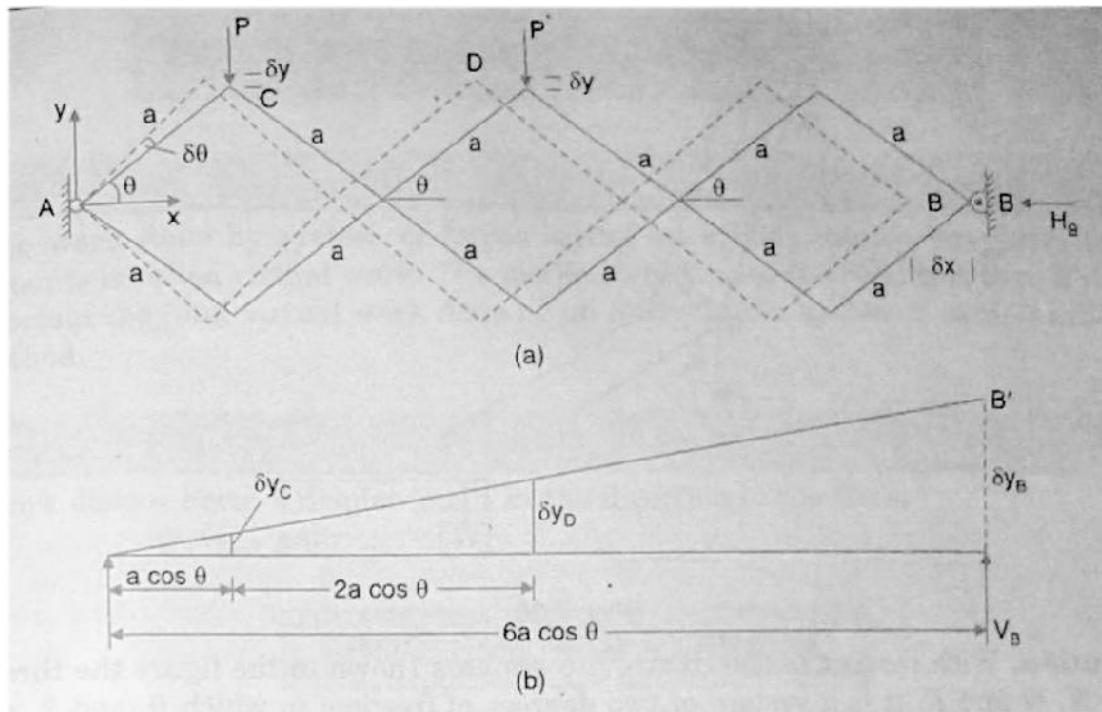


Fig.3.52

**Solution:**

## i) Horizontal reaction:

Removing the restraint to support B and allowing it to roll in x direction with a horizontal force  $H_B$ , let the angle of rotation of the members of frame be  $\delta\theta$ .

$$\text{Now, } y_C = a \sin \theta, \Rightarrow u y_C = a \cos \theta u$$

$$\text{Similarly, } y_D = a \sin \theta \Rightarrow u y_D = a \cos \theta u$$

$$x = 6a \cos \theta u \Rightarrow u x = -6a \sin \theta u$$

Considering work done by all the active forces, virtual work equation is

$$(-P)u y_C + (-P)u y_D + (-H_B)u x = 0$$

$$-P a \cos \theta u - P a \cos \theta u - H_B (-6a \sin \theta u) = 0$$

$$H_B = \frac{2P \cos \theta}{6 \sin \theta} = \frac{P}{3} \cot \theta$$

## ii) Vertical reaction:

Let  $\delta y$  be vertical displacement given to joint B after removing restraint in vertical direction. The corresponding vertical force at B be  $V_B$ .

Vertical displacement of C;

$$u_{y_C} = \frac{a \sin \alpha}{6a \sin \alpha} u_{y_B} = \frac{1}{6} u_{y_B}$$

Vertical displacement of D;

$$u_{y_D} = \frac{3a \sin \alpha}{6a \sin \alpha} u_{y_B} = \frac{1}{2} u_{y_B}$$

Therefore the virtual work equation is

$$(-P)u_{y_C} + (-P)u_{y_D} + V_B u_{y_B} = 0$$

$$-P \frac{1}{6} u_{y_B} - P \frac{1}{2} u_{y_B} + V_B u_{y_B} = 0$$

Since  $\delta y_B \neq 0$ , then

$$V_B = \frac{4}{6} P = \frac{2}{3} P$$

**SP3.32:** An inextensible string is subjected to equal loads W at distance L and 2L from end A. If a horizontal force is applied at other end, the equilibrium position is as shown in fig.3.53 Express the angles  $\theta_1$  and  $\theta_2$  with the vertical in terms of F and W.

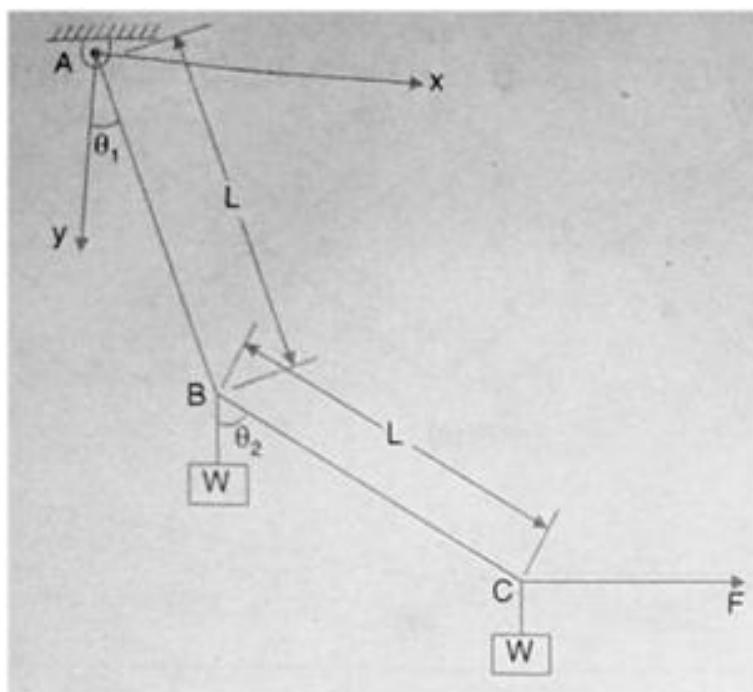


Fig.3.53

**Solution:**

With respect to the coordinate system as shown in fig.3.53 the three active force are W, W and F. It is a system of two degrees of freedom in which  $\theta_1$  and  $\theta_2$  can vary independently.

Therefore the principle of virtual work is

$$\mathbf{u}U = \frac{\partial U}{\partial \theta_1} \mathbf{u}_{\theta_1} + \frac{\partial U}{\partial \theta_2} \mathbf{u}_{\theta_2} = 0$$

Keeping  $\theta_1$  constant and varying  $\theta_2$  the principle of virtual equation is

$$W\mathbf{u}y_C + F\mathbf{u}x = 0$$

$$W \frac{\partial}{\partial \theta_2} (L \cos \theta_2) \mathbf{u}_{\theta_2} + F \frac{\partial}{\partial \theta_2} (L \sin \theta_2) \mathbf{u}_{\theta_2} = 0$$

$$-WL \sin \theta_2 + FL \cos \theta_2 = 0$$

$$\tan \theta_2 = \frac{F}{W}$$

Now, keeping  $\theta_2$  constant and varying  $\theta_1$ ,

$$y_B = L \cos \theta_1$$

$$\mathbf{u}y_B = -L \sin \theta_1 \mathbf{u}_{\theta_1}$$

$$y_C = L \cos \theta_1 + L \cos \theta_2$$

Since  $\theta_2$  is constant,

$$\mathbf{u}y_C = -L \sin \theta_1 \mathbf{u}_{\theta_1}$$

$$x_C = L \sin \theta_1 + L \sin \theta_2$$

Since  $\theta_2$  is constant,

$$\mathbf{u}x_C = L \cos \theta_1 \mathbf{u}_{\theta_1}$$

The principle of virtual work is

$$Wu_{y_B} + Wu_{y_C} + Fu_{x_C} = 0$$

$$W(-L\sin \alpha_1)u_{x_1} + W(-L\sin \alpha_1)u_{x_1} + F(L\cos \alpha_1)u_{x_1} = 0$$

$$\tan \alpha_1 = \frac{F}{2W}$$

## CONSERVATIVE FORCES:

When a force does work that depends only upon the initial and final positions of the force, and it is *independent* of the path it travels, then the force is referred to as a **conservative force**. The weight of a body and the force of a spring are two examples of conservative forces.

### a) Weight:

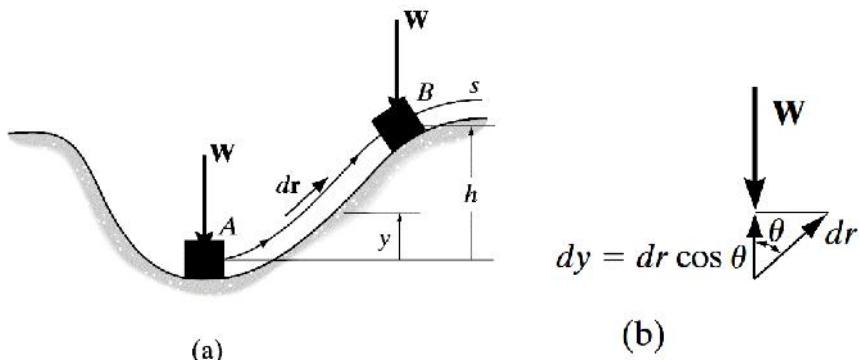


Fig.3.54 Displacement of rigid body

Consider a block of weight **W** that travels along the path as shown in fig.3.54 (a). When it is displaced up the path by an amount  $dr$ , then the work is  $dU = \mathbf{W} \cdot dr$  or  $dU = -W(dr \cos \alpha) = -Wdy$ , as shown in fig.3.54 (b). In this case, the work is *negative* since **W** acts in the opposite sense of  $dy$ . Thus, if the block moves from *A* to *B*, through the vertical displacement *h*, the work is

$$U = - \int_0^h Wdy = -Wh$$

The weight of a body is therefore a conservative force, since the work done by the weight depends only on the *vertical displacement* of the body, and is independent of the path along which the body travels.

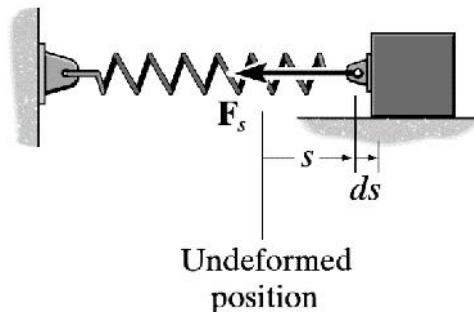
**b) Spring Force:**

Fig.3.55 Deformation of elastic body

Now consider the linearly elastic spring as shown in fig.3.55, which undergoes a displacement  $ds$ . The work done by the spring force on the block is  $dU = -F_s ds = -ksds$ . The work is *negative* because  $\mathbf{F}_s$  acts in the opposite sense to that of  $ds$ . Thus, the work of  $\mathbf{F}$  when the block is displaced from  $s = s_1$  to  $s = s_2$  is

$$U = - \int_{s_1}^{s_2} ksds = - \left( \frac{1}{2} ks_2^2 - \frac{1}{2} ks_1^2 \right)$$

Here the work depends only on the spring's initial and final positions  $s_1$  and  $s_2$ , measured from the spring's un-stretched position. Since this result is independent of the path taken by the block as it moves, then a spring force is also a *conservative force*.

**Friction:** In contrast to a conservative force, consider the force of *friction* exerted on a sliding body by a fixed surface. The work done by the frictional force depends on the path; the longer the path, the greater the work. Consequently, frictional forces are *non-conservative*, and most of the work done by them is dissipated from the body in the form of heat.

## POTENTIAL ENERGY:

A conservative force can give the body the capacity to do work. This capacity measured as potential energy, depends on the location or “position” of the body measured relative to a fixed reference position or datum. The concept of potential energy is useful for determining the stability of equilibrium by two methods. They are

- 1) Elastic potential Energy
- 2) Gravitational potential energy

### 1) Elastic Potential Energy:

The work done on an elastic member is stored in the member in the form of elastic potential energy  $V_e$ . This energy is potentially available to do work on some other body during the relief of its compression or extension.

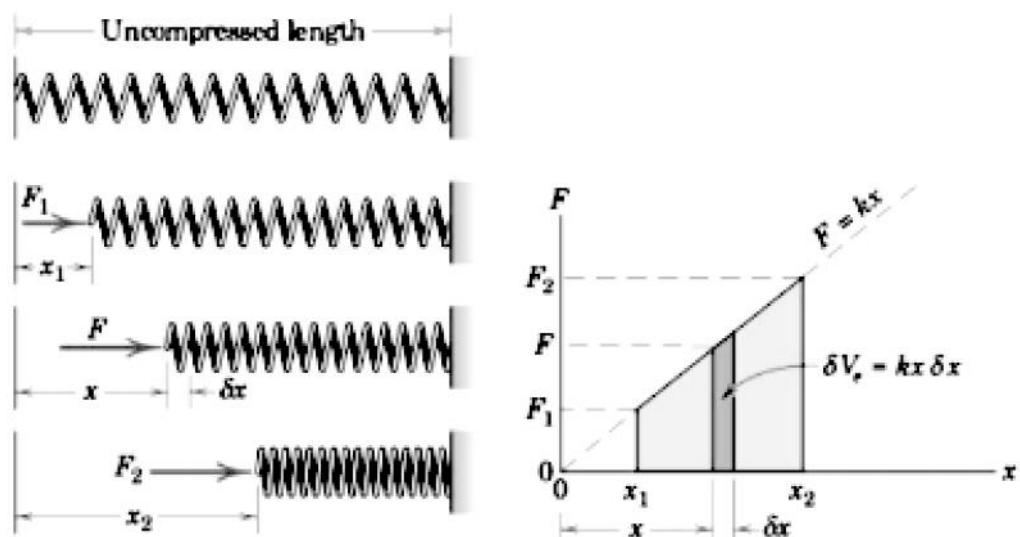


Fig.3.56 Elastic potential energy

$$V_e = \int_0^x F dx = \int_0^x kx dx, \text{ or}$$

$$V_e = \frac{1}{2} kx^2$$

Thus, the potential energy of the spring equals the triangular area in the diagram of  $F$  versus  $x$  from 0 to  $x$ .

During an increase in the compression of the spring from  $x_1$  to  $x_2$ , the work done on the spring equals its change in elastic potential energy or

$$\Delta V_e = \int_{x_1}^{x_2} kx dx = \frac{1}{2}k(x_2^2 - x_1^2)$$

which equals the trapezoidal area from  $x_1$  to  $x_2$ .

During a virtual displacement  $\delta x$  of the spring, the virtual work done on the spring is the virtual change in elastic potential energy.

$$\delta V_e = F\delta x = kx\delta x$$

During a decrease in the compression of the spring as it is relaxed from  $x = x_2$  to  $x = x_1$ , the change (final minus initial) in the potential energy of the spring is negative. Consequently, if  $\delta x$  is negative,  $\delta V_e$  is also negative.

When we have a spring in tension rather than compression, the work and energy relations are the same as those for compression, where  $x$  now represents the elongation of the spring rather than its compression. While the spring is being stretched, the force again acts in the direction of the displacement, doing positive work on the spring and increasing its potential energy.

Because the force acting on the movable end of a spring is the negative of the force exerted by the spring on the body to which its movable end is attached, the work done on the body is the negative of the potential energy change of the spring.

A torsional spring, which resists the rotation of a shaft or another element, can also store and release potential energy. If the torsional stiffness, expressed as torque per radian of twist, is a constant  $k_T$ , and if  $\theta$  is the angle of twist in radians, then the resisting torque is  $M = k_T \theta$ . The potential energy becomes  $V_e = \frac{1}{2} k_T \theta^2$  or

$$V_e = \frac{1}{2} k_T \theta^2$$

which is analogous to the expression for the linear extension spring.

The units of elastic potential energy are the same as those of work and are expressed in joules (J) in SI units and in foot-pounds (ft-lb) in U.S. customary units.

## 2) Gravitational Potential Energy:

In the previous section the work of a gravitational force or weight acting on a body treated in the same way as the work of any other active force. Thus, for an upward displacement  $\delta h$  of the body in fig.3.57 the weight  $W = mg$  does negative work  $U = -mgh$ . If, on the other hand, the body has a downward displacement  $\delta h$ , with  $h$  measured positive downward, the weight does positive work  $U = +mgh$ .

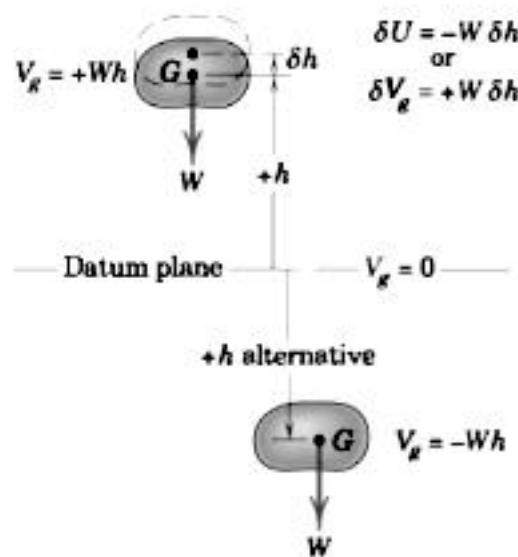


Fig.3.57 Gravitational forces

An alternative to the foregoing treatment expresses the work done by gravity in terms of a change in potential energy of the body. This alternative treatment is a useful representation when we describe a mechanical system in terms of its total energy. The gravitational potential energy  $V_g$  of a body is defined as the work done on the body by a force equal and opposite to the weight in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to

be zero. The potential energy, then, is the negative of the work done by the weight. When the body is raised, for example, the work done is converted into energy which is potentially available, since the body can do work on some other body as it returns to its original lower position. If consider  $V_g$  to be zero at  $h = 0$ , fig.3.57, then at a height  $h$  above the datum plane, the gravitational potential energy of the body is

$$V_g = mgh$$

If the body is a distance  $h$  below the datum plane, its gravitational potential energy is  $-mgh$ .

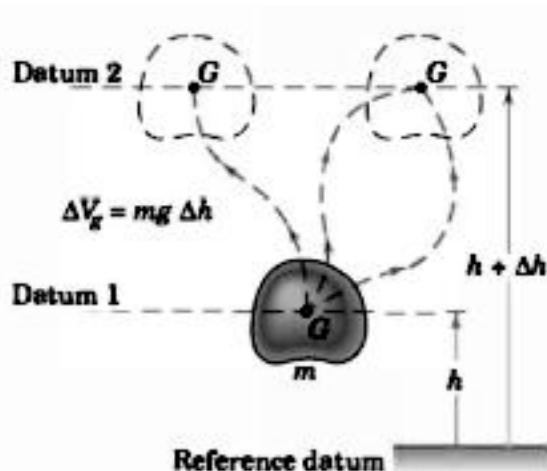


Fig.3.58 Gravitation forces with reference datum

Note that the datum plane for zero potential energy is arbitrary because only the change in potential energy matters, and this change is the same no matter where it places the datum plane. Note also that the gravitational potential energy is independent of the path followed in arriving at a particular level  $h$ . Thus, the body of mass  $m$  as shown in fig.3.58 has the same potential-energy change no matter which path it follows in going from datum plane 1 to datum plane 2 because  $\Delta h$  is the same for all three paths.

The virtual change in gravitational potential energy is simply

$$\Delta V_g = mg \Delta h$$

where  $\Delta h$  is the upward virtual displacement of the mass centre of the body. If the mass centre has a downward virtual displacement, then  $\Delta V_g$  is negative.

The units of gravitational potential energy are the same as those for work and elastic potential energy, joules (J) in SI units and foot-pounds (ft-lb) in U.S. customary units.

## ENERGY EQUATION:

The work done by a linear spring on the body to which its movable end is attached is the negative of the change in the elastic potential energy of the spring. Also, the work done by the gravitational force or weight  $mg$  is the negative of the change in gravitational potential energy. Therefore, when apply the virtual-work equation to a system with springs and with changes in the vertical position of its members, it may replace the work of the springs and the work of the weights by the negative of the respective potential energy changes.

It can use these substitutions to write the total virtual work  $\delta U$  in equation  $\delta U = 0$  as the sum of the work  $\delta U'$  done by all active forces, other than spring forces and weight forces, and the work  $-(\delta V_e + \delta V_g)$  done by the spring and weight forces. Equation  $\delta U = 0$  then becomes

$$\delta U' - (\delta V_e + \delta V_g) = 0 \text{ or } \delta U' = \delta V$$

Where  $V = V_e + V_g$  stands for the total potential energy of the system. With this formulation a spring becomes internal to the system, and the work of spring and gravitational forces is accounted for in the  $\delta V$  term.

## ACTIVE-FORCE DIAGRAMS:

With the method of virtual work it is useful to construct the active force diagram of the system these are analysing. The boundary of the system must clearly distinguish those members which are part of the system from other bodies which are not part of the system. When includes an elastic member within the boundary of system, the forces of interaction between it and the movable members to which it is attached are internal to

the system. Thus these forces need not be shown because their effects are accounted for in the  $V_e$  term. Similarly, weight forces are not shown because their work is accounted for in the  $V_g$  term.

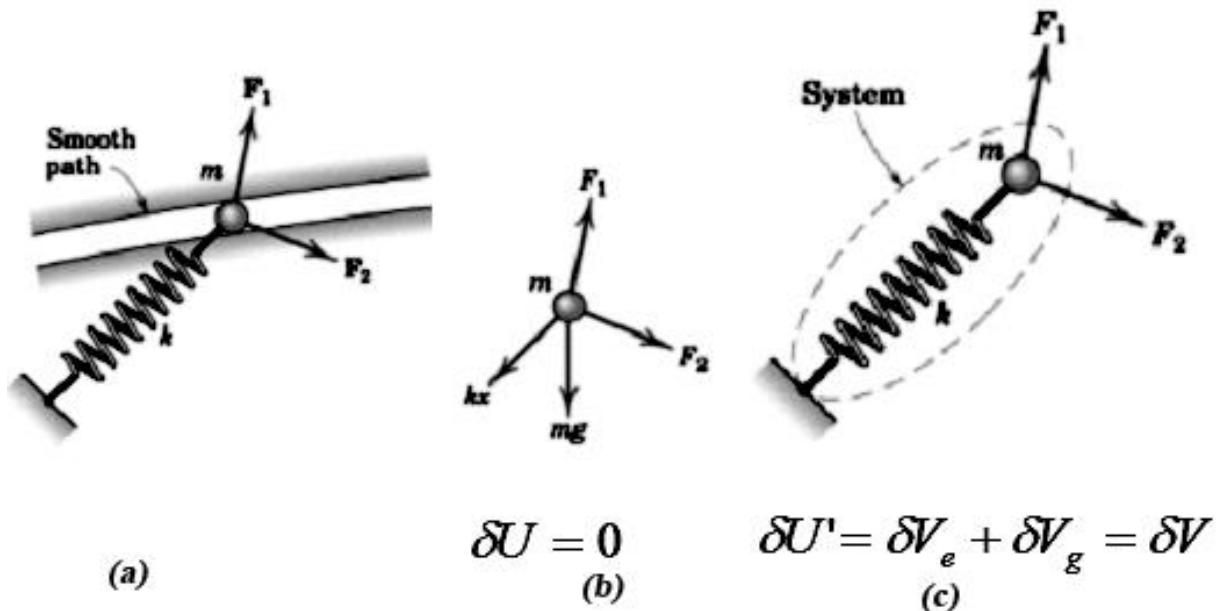


Fig.3.59 Active force diagrams

Fig.3.59 illustrates the difference between the use of equation  $\delta U = 0$  and  $\delta U' - (\delta V_e + \delta V_g) = 0$  or  $\delta U' = \delta V$ . It consider the body in part (a) of the figure to be a particle for simplicity, and it assumes that the virtual displacement is along the fixed path. The particle is in equilibrium under the action of the applied forces  $F_1$  and  $F_2$ , the gravitational force  $mg$ , the spring force  $kx$ , and a normal reaction force. In fig.3.59 (b), where the particle alone is isolated,  $\delta U$  includes the virtual work of all forces shown on the active-force diagram of the particle. (The normal reaction exerted on the particle by the smooth guide does no work and is omitted.). In Fig.3.59 (c) the spring is included in the system, and  $\delta U'$  is the virtual work of only  $F_1$  and  $F_2$ , which are the only external forces whose work is not accounted for in the potential energy terms. The work of the weight  $mg$  is accounted for in the  $\delta V_g$  term, and the work of the spring force is included in the  $\delta V_e$  term.

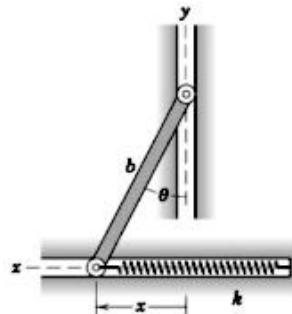
**STABILITY OF EQUILIBRIUM:**

Fig.3.60 Mechanical system changes with gravitational and elastic potential energy

Consider now the case of a mechanical system where movement is accompanied by changes in gravitational and elastic potential energies and where no work is done on the system by non-potential forces. The mechanism as shown in fig3.60 is an example of such a system. With  $\delta U' = 0$  the virtual-work relation, equation  $\delta U' - (\delta V_e + \delta V_g) = 0$  or  $\delta U' = \delta V$ , becomes

$$(\delta V_e + \delta V_g) = 0 \text{ or } \delta V = 0$$

The above equation expresses the requirement that the equilibrium configuration of a mechanical system is one for which the total potential energy  $V$  of the system has a stationary value. For a system of one degree of freedom where the potential energy and its derivatives are continuous functions of the single variable, say,  $x$ , which describes the configuration, the equilibrium condition  $\delta V = 0$  is equivalent mathematically to the requirement

$$\frac{dV}{dx} = 0$$

The above equation states that a mechanical system is in equilibrium when the derivative of its total potential energy is zero. For systems with several degrees of freedom the partial derivative of  $V$  with respect to each coordinate in turn must be zero for equilibrium.



Fig.3.61 Stability of equilibrium

There are three conditions under which equation  $\frac{dV}{dx} = 0$  applies, namely, when the total potential energy is a minimum (stable equilibrium), a maximum (unstable equilibrium), or a constant (neutral equilibrium). Fig.3.39 shows a simple example of these three conditions. The potential energy of the roller is clearly a minimum in the stable position, a maximum in the unstable position, and a constant in the neutral position.

It may also characterize the stability of a mechanical system by noting that a small displacement away from the stable position results in an increase in potential energy and a tendency to return to the position of lower energy. On the other hand, a small displacement away from the unstable position results in a decrease in potential energy and a tendency to move farther away from the equilibrium position to one of still lower energy. For the neutral position a small displacement one way or the other results in no change in potential energy and no tendency to move either way.

When a function and its derivatives are continuous, the second derivative is positive at a point of minimum value of the function and negative at a point of maximum value of the function. Thus, the mathematical conditions for equilibrium and stability of a system with a single degree of freedom  $x$  are:

$$\text{Equilibrium } \frac{dV}{dx} = 0$$

$$\text{Stable } \frac{d^2V}{dx^2} > 0$$

$$\text{Unstable} \quad \frac{d^2V}{dx^2} < 0$$

The second derivative of  $V$  may also be zero at the equilibrium position, in which case we must examine the sign of a higher derivative to ascertain the type of equilibrium. When the order of the lowest remaining nonzero derivative is even, the equilibrium will be stable or unstable according to whether the sign of this derivative is positive or negative. If the order of the derivative is odd, the equilibrium is classified as unstable, and the plot of  $V$  versus  $x$  for this case appears as an inflection point in the curve with zero slope at the equilibrium value.

Stability criteria for multiple degrees of freedom require more advanced treatment. For two degrees of freedom, for example, it uses a Taylor- series expansion for two variables.

### Problems:

**SP3.33:** The 10 kg cylinder is suspended by the spring which has a stiffness of 2kN/m. Plot the potential energy  $V$  of the system and show that it is minimum at the equilibrium position.

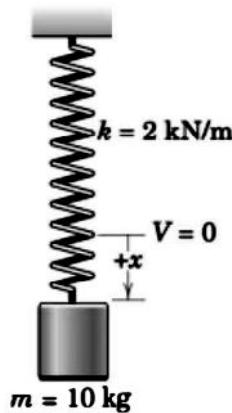


Fig.3.62

### Solution:

The elastic potential energy for an arbitrary position  $x$  is

$$V_e = \frac{1}{2}kx^2$$

The gravitational potential energy for an arbitrary position x is

$$V_g = -mgx$$

The total potential energy is

$$V = V_e + V_g = \frac{1}{2}kx^2 - mgx$$

The equilibrium of potential energy is obtained by

$$\frac{dV}{dx} = 0, \Rightarrow \frac{dV}{dx} = kx - mg = 0$$

$$\Rightarrow x = \frac{mg}{k}$$

$$x = \frac{10 \times 9.81}{2000} = 0.049m$$

X=49mm

Substituting numerical values gives

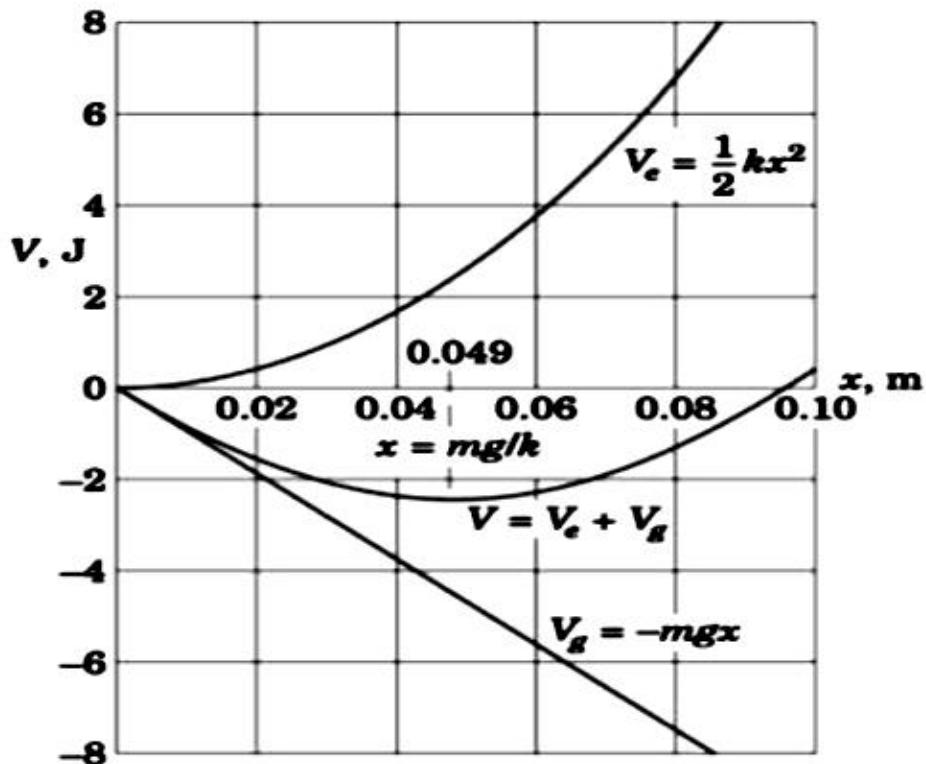
$$V = \frac{1}{2}(2000)(0.049)^2 - 10 \times 9.81 \times 0.049$$

The minimum value at equilibrium position of potential energy is

$$V = -2.4N - m$$

The below plot shows the different forms of potential energies. The minimum value

of V occurs at x = 0.049m where  $\frac{dV}{dx} = 0$  and  $\frac{d^2V}{dx^2}$  is positive.



**SP3.34:** The two uniform links, each of mass  $m$ , are in the vertical plane and are connected and constrained as shown in fig.3.63. As the angle  $\theta$  between the links increases with the application of the horizontal force  $P$ , the light rod, which is connected at A and passes through a pivoted collar at B, compresses the spring of stiffness  $k$ . If the spring is uncompressed in the position where  $\theta = 0$ , determine the force  $P$  which will produce equilibrium at the angle  $\theta$ .

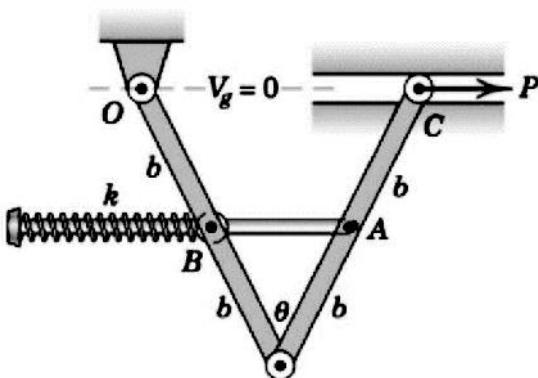


Fig.3.63

**Solution:**

The compression  $x$  of the spring is the distance which A has moved away from B, which is  $x = 2b \sin \frac{\theta}{2}$

The elastic potential energy of the spring is

$$V_e = \frac{1}{2} kx^2, \Rightarrow V_e = \frac{1}{2} k \left( 2b \sin \frac{\theta}{2} \right)^2 = 2kb^2 \sin^2 \frac{\theta}{2}$$

With the datum for zero gravitational potential energy taken through the support at O for convenience, the expression is

$$V_g = -2mgx = -2mg \left( b \cos \frac{\theta}{2} \right)$$

The distance between O and C is  $4b \sin \frac{\theta}{2}$ , so that the virtual work done by P is

$$uU' = Pu \left( 4b \sin \frac{\theta}{2} \right)$$

The principle of virtual work is

$$\begin{aligned} uU' - uV &= 0, \Rightarrow uU' - uV_e - uV_g = 0 \\ \Rightarrow Pu \left( 4b \sin \frac{\theta}{2} \right) - u \left( 2kb^2 \sin^2 \frac{\theta}{2} \right) - u \left( -2mgb \cos \frac{\theta}{2} \right) &= 0 \\ \Rightarrow 2Pb \cos \frac{\theta}{2} u - 2kb^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} u - mgb \sin \frac{\theta}{2} u &= 0 \end{aligned}$$

Since  $\delta\theta \neq 0$ , then

$$P = kb \sin \frac{\theta}{2} + \frac{1}{2} mg \tan \frac{\theta}{2}$$

**SP3.35:** The ends of the uniform bar of mass m slide freely in the horizontal and vertical guides. Examine the stability conditions for the positions of equilibrium. The spring of stiffness k is un-deformed when  $x = 0$ .

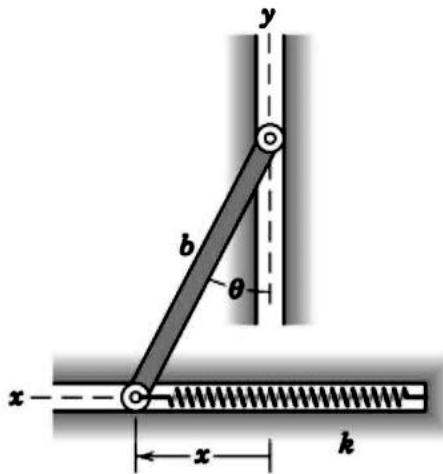


Fig.3.64

**Solution:**

The system consists of the spring and the bar. Since there are no external active forces the given sketch serves as the active force diagram. It will take the x axis as the datum for zero gravitational potential energy. In the displaced position the elastic and gravitational potential energies are

$$V_e = \frac{1}{2}kx^2 = \frac{1}{2}kb^2 \sin^2 \theta \quad \text{and} \quad V_g = mg \frac{b}{2} \cos \theta$$

The total potential energy is then,

$$V = V_e + V_g = \frac{1}{2}kb^2 \sin^2 \theta + \frac{1}{2}mgb \cos \theta$$

The equilibrium occurs for  $\frac{dV}{d\theta} = 0$ , so that

$$\frac{dV}{d\theta} = kb^2 \sin \theta \cos \theta - \frac{1}{2}mgb \sin \theta = \left( kb^2 \cos \theta - \frac{1}{2}mgb \right) \sin \theta = 0$$

The two solutions to this equation are given by

$$\sin \theta = 0 \quad \text{and} \quad \cos \theta = \frac{mg}{2kb}$$

Now determine the stability by examining the sign of the second derivative of V for each of the two equilibrium positions. The second derivative is

$$\frac{d^2V}{d\theta^2} = kb^2(\cos^2 \theta - \sin^2 \theta) - \frac{1}{2}mgb\cos \theta$$

$$\frac{d^2V}{d\theta^2} = kb^2(2\cos^2 \theta - 1) - \frac{1}{2}mgb\cos \theta$$

Solution 1:  $\sin \theta = 0, \theta = 0$

$$\frac{d^2V}{d\theta^2} = kb^2(2 - 1) - \frac{1}{2}mgb = kb^2\left(1 - \frac{mg}{2kb}\right)$$

If  $k > \frac{mg}{2b}$ , Then  $\Rightarrow \frac{d^2V}{d\theta^2} = \text{positive value (stable)}$

If  $k < \frac{mg}{2b}$ , Then  $\Rightarrow \frac{d^2V}{d\theta^2} = \text{negative value (unstable)}$

Thus, if the spring is sufficiently stiff, the bar will return to the vertical position even though there is no force in the spring at that position.

Solution 2:

$$\cos \theta = \frac{mg}{2kb}, \theta = \cos^{-1} \frac{mg}{2kb}$$

$$\frac{d^2V}{d\theta^2} = kb^2\left(2\left(\frac{mg}{2kb}\right)^2 - 1\right) - \frac{1}{2}mgb\left(\frac{mg}{2kb}\right) = kb^2\left(\left(\frac{mg}{2kb}\right)^2 - 1\right)$$

Since the cosine must be less than unity, then this solution is limited to the case where  $k > \frac{mg}{2b}$ , which makes the second derivative of V negative. Thus equilibrium for solution 2 is never stable. If  $k < \frac{mg}{2b}$ , then no longer have solution 2 since the spring will be too weak to maintain equilibrium at a value of  $\theta$  between 0 and 90°.

**SP3.36:** The uniform link as shown in fig.3.65a has a mass of 10kg. If the spring is unstretched when  $\theta=0^\circ$ , determine the angle  $\theta$  for equilibrium and investigate the stability at the equilibrium position.

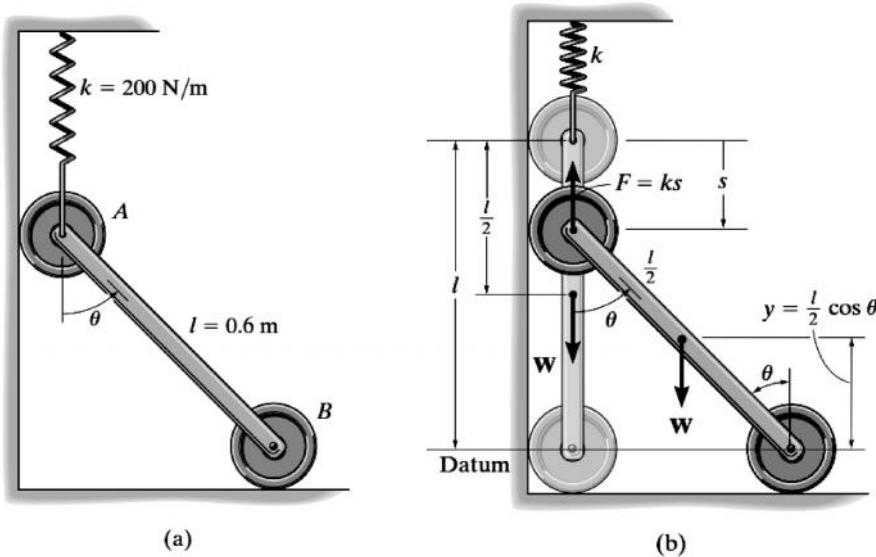


Fig.3.65

**Solution:**

The datum is established at the bottom of the link as shown in fig.3.65b. When the link is located in the arbitrary position  $\theta$ , the spring increases its potential energy by stretching and the weight decreases its potential energy. Hence

$$V = V_e + V_g = \frac{1}{2}ks^2 + Wy$$

Since  $l = s + l \cos \theta$  or  $s = l(1 - \cos \theta)$  and  $y = \frac{l}{2} \cos \theta$ , then

$$V = \frac{1}{2}kl^2(1 - \cos \theta)^2 + W\left(\frac{l}{2} \cos \theta\right)$$

The equilibrium condition,

$$\frac{dV}{d\theta} = kl^2(1 - \cos \theta)\sin \theta - \frac{Wl}{2}\sin \theta = 0 \text{ or}$$

$$l\left(kl(1 - \cos \theta) - \frac{W}{2}\right)\sin \theta = 0$$

This equation is satisfied by  $\sin \theta = 0$ ,  $\theta = 0^\circ$ . Therefore,

$$\theta = \cos^{-1}\left(1 - \frac{W}{2kl}\right) = \cos^{-1}\left(1 - \frac{10 \times 9.81}{2 \times 200 \times 0.6}\right) = 53.8^\circ$$

The second derivative of V is

$$\frac{d^2V}{d\theta^2} = kl^2(l - \cos \theta) \cos \theta + kl^2 \sin \theta \sin \theta - \frac{Wl}{2} \cos \theta$$

$$\frac{d^2V}{d\theta^2} = kl^2(\cos \theta - \cos 2\theta) - \frac{Wl}{2} \cos \theta$$

Substituting values for the constant with  $\theta=0^\circ$  and  $\theta=53.8^\circ$ , Then

$$\left. \frac{d^2V}{d\theta^2} \right|_{\theta=0^\circ} = 200(0.6)^2(\cos 0^\circ - \cos 0^\circ) - \frac{10 \times 9.81 \times 0.6}{2} \cos 0^\circ$$

$$\left. \frac{d^2V}{d\theta^2} \right|_{\theta=0^\circ} = -29.4 < 0 \text{ (Unstable equilibrium at } \theta=0^\circ\text{)}$$

$$\left. \frac{d^2V}{d\theta^2} \right|_{\theta=53.8^\circ} = 200(0.6)^2(\cos 53.8^\circ - \cos 107.6^\circ) - \frac{10 \times 9.81 \times 0.6}{2} \cos 53.8^\circ$$

$$\left. \frac{d^2V}{d\theta^2} \right|_{\theta=53.8^\circ} = 46.9 > 0 \text{ (Stable equilibrium at } \theta=53.8^\circ\text{)}$$

**SP3.37:** If the spring *AD* as shown in fig.3.66a has a stiffness of 18 kN/m and is unstretched when  $\theta = 60^\circ$ , determine the angle  $\theta$  for equilibrium. The load has a mass of 1.5 Mg. Investigate the stability at the equilibrium position.

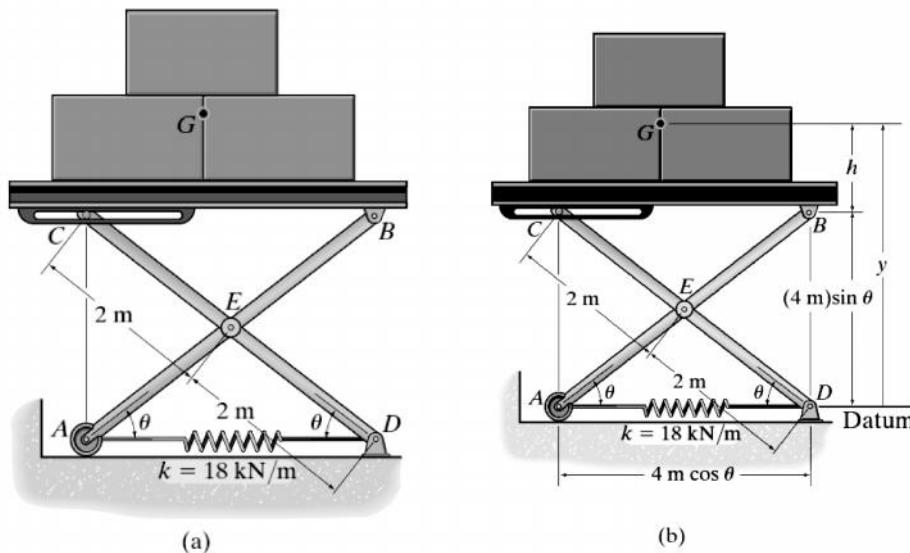


Fig.3.66

**Solution:**

The gravitational potential energy for the load with respect to the fixed datum as shown in fig.3.66b is

$$V_g = mgy = 1500(9.81)[4\sin \theta + h] = 58860 \sin \theta + 14715h$$

Where  $h$  is a constant distance. From the geometry of the system, the elongation of the spring when the load is on the platform is

$$s = 4\cos \theta - 4\cos 60^\circ = 4\cos \theta - 2$$

Thus, the elastic potential energy of the system is

$$V_e = \frac{1}{2}ks^2 = \frac{1}{2}(18000)(4\cos \theta - 2)^2 = 9000(4\cos \theta - 2)^2$$

The potential energy function for the system is therefore

$$V = V_e + V_g = 58860 \sin \theta + 14715h + 9000(4\cos \theta - 2)^2$$

When the system is in equilibrium, then

$$\frac{dV}{d\theta} = 58860 \sin \theta + 18000(4\cos \theta - 2)(-4\sin \theta) = 0$$

$$58860\cos \theta - 288000\sin \theta \cos \theta + 144000\sin \theta = 0$$

Since  $\sin 2\theta = 2\sin \theta \cos \theta$ , then

$$58860\cos \theta - 144000\sin 2\theta + 144000\sin \theta = 0$$

Solving by trial and error, then

$$\theta = 28.18^\circ \text{ and } \theta = 45.51^\circ$$

Taking the second derivative of equation, then

$$\frac{d^2V}{d\theta^2} = -58860 \sin \theta - 288000 \cos 2\theta + 144000 \cos \theta$$

Substituting  $\theta = 28.18^\circ$ , then

$$\frac{d^2V}{d\theta^2} = -60402 < 0 \quad (\text{Unstable})$$

Now,  $\mu = 45.51^\circ$

$$\frac{d^2V}{d\mu^2} = 64073 > 0 \quad (\text{Stable})$$

**SP3.38:** The uniform block having a mass  $m$  rests on the top surface of the half cylinder, fig.3.67a. Show that this is a condition of unstable equilibrium if  $h > 2R$ .

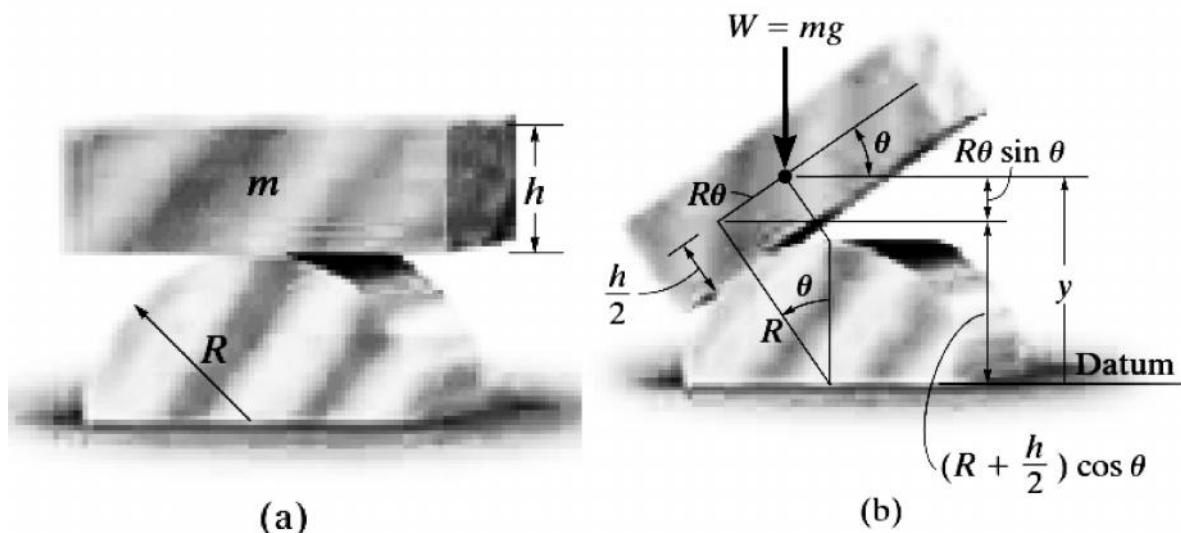


Fig.3.67

**Solution:**

The datum is established at the base of the cylinder, fig.3.67b. If the block is displaced by an amount  $u$  from the equilibrium position, the potential function is

$$V = V_e + V_g = 0 + mg y$$

From the geometry of fig.3.b,

$$y = \left( R + \frac{h}{2} \right) \cos \mu + R \mu \sin \mu$$

Thus,

$$V = mg \left[ \left( R + \frac{h}{2} \right) \cos \mu + R \mu \sin \mu \right]$$

The equilibrium conditions, then

$$\frac{dV}{d\theta} = mg \left[ -\left( R + \frac{h}{2} \right) \sin \theta + R \sin \theta + R \cos \theta \right] = 0$$

$$mg \left( -\frac{h}{2} \sin \theta + R \cos \theta \right) = 0$$

Note that  $\theta = 0^\circ$  satisfies this equation.

Taking the second derivative of V for the stability, then

$$\frac{d^2V}{d\theta^2} = mg \left( -\frac{h}{2} \cos \theta + R \cos \theta - R \sin \theta \right)$$

At  $\theta = 0^\circ$ ,

$$\left. \frac{d^2V}{d\theta^2} \right|_{\theta=0^\circ} = -mg \left( \frac{h}{2} - R \right)$$

Since all the constants are positive, the block is in unstable equilibrium provided

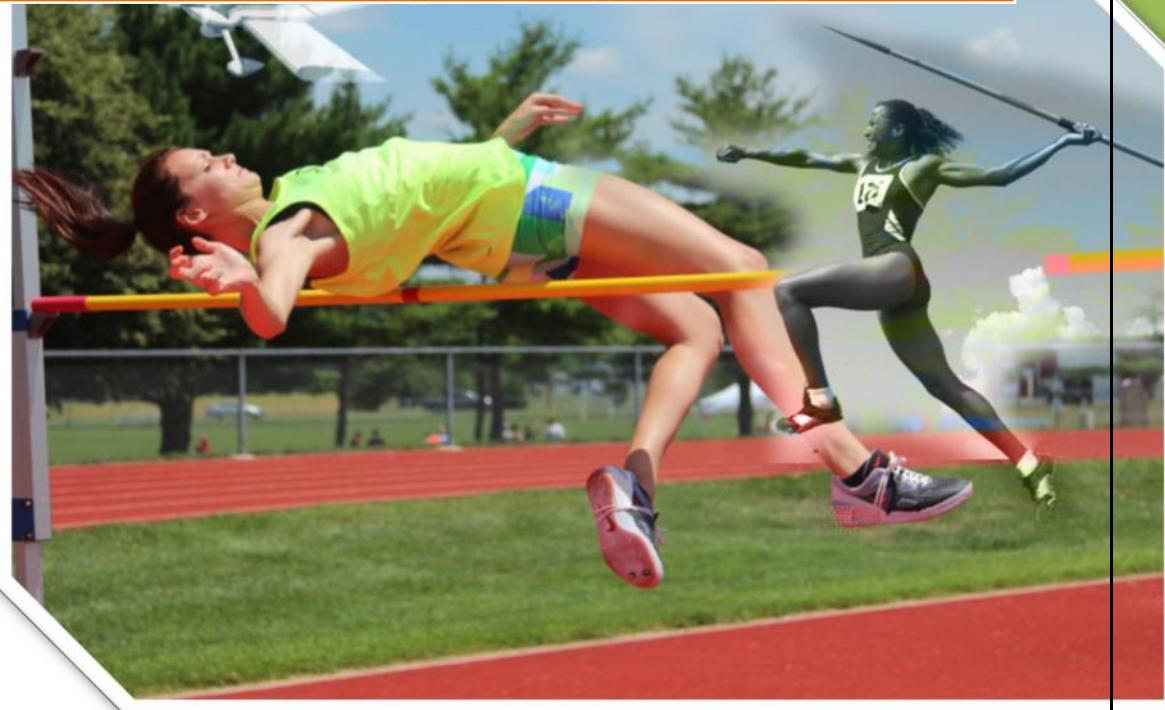
$$h > 2R, \text{ because then } \frac{d^2V}{d\theta^2} < 0.$$

**LECTURE NOTES**

**UNIT – 4**

**2021**

**CEES303-Engineering  
Mechanics**



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## Unit - 4

Review of particle dynamics- Rectilinear motion; Plane curvilinear motion (rectangular, path, and polar coordinates). 3-D curvilinear motion; Relative and constrained motion; Newton's 2nd law (rectangular, path, and polar coordinates). Work-kinetic energy, power, potential energy. Impulse-momentum (linear, angular); Impact (Direct and oblique).

### Reference for the preparation of course material:

1. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
3. J.L. Meriam and L. G. Kraige, Engineering Mechanics –Statics, volume I, Seventh edition, John Wiley & Sons, Inc.

### TEXT BOOKS

1. Irving H. Shames (2006), Engineering Mechanics, 4th Edition, Prentice Hall
2. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill

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1. R. C. Hibbler (2006), Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.

**Unit - 4****Review of particle Dynamics**

Kinematics is the branch of dynamics which describes the motion of bodies without reference to the forces which either cause the motion or are generated as a result of the motion. Kinematics is often described as the “geometry of motion.” Some engineering applications of kinematics include the design of cams, gears, linkages, and other machine elements to control or produce certain desired motions, and the calculation of flight trajectories for aircraft, rockets, and spacecraft. A thorough working knowledge of kinematics is a prerequisite to kinetics, which is the study of the relationships between motion and the corresponding forces which cause or accompany the motion.

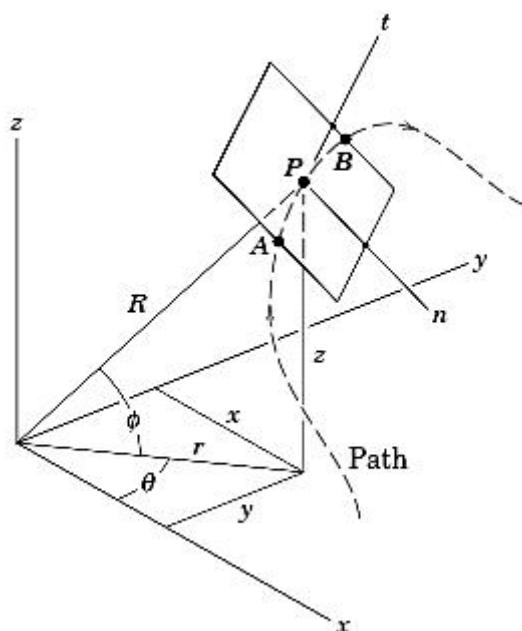
**PARTICLE MOTION:**

Fig.4.1 Particle motion

A particle is a body whose physical dimensions are so small compared with the radius of curvature of its path that treats the motion of the particle as that of a point. For example, the wingspan of a jet transport flying between Los Angeles and New York is of no consequence compared with the radius of curvature of its flight path, and thus the treatment of the airplane as a particle or point is an acceptable approximation.

It describes the motion of a particle in a number of ways, and the choice of the most convenient or appropriate way depends a great deal on experience and on how the data are given. Let us obtain an overview of the several methods developed in this section by referring to fig.4.1, which shows a particle P moving along some general path in space. If the particle is confined to a specified path, as with a bead sliding along a fixed wire, its motion is said to be **constrained**. If there are no physical guides, the motion is said to be **unconstrained**. A small rock tied to the end of a string and whirled in a circle undergoes constrained motion until the string breaks, after which instant its motion is unconstrained.

## RECTILINEAR MOTION:

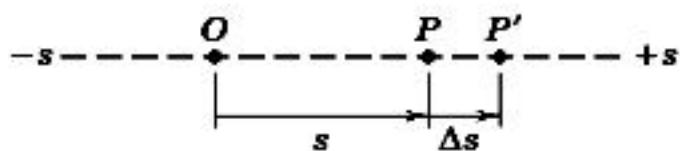


Fig.4.2 Rectilinear motion

Consider a particle P moving along a straight line, fig.4.2 shows that the position of 'P' at any instant of time  $t$  can be specified by its distance ' $s$ ' measured from some convenient reference point O fixed on the line. At time  $t + \Delta t$  the particle has moved to  $P'$  and its coordinate becomes  $s + \Delta s$ . The change in the position coordinate during the interval  $\Delta t$  is called the **displacement**,  $\Delta s$  of the particle. The displacement would be negative if the particle moved in the negative  $s$ -direction.

**a) Velocity and Acceleration:**

The average velocity of the particle during the interval  $\Delta t$  is the displacement divided by the time interval or  $v_{av} \rightarrow \frac{\Delta s}{\Delta t}$ . As  $\Delta t$  becomes smaller and approaches zero in the limit, the average velocity approaches the instantaneous velocity of the particle, which is  $v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t}$  or

$$V = \frac{ds}{dt} = \dot{s} \quad \text{-----} \rightarrow (4.1)$$

Thus, the velocity is the time rate of change of the position coordinate  $s$ . The velocity is positive or negative depending on whether the corresponding displacement is positive or negative.

The average acceleration of the particle during the interval  $\Delta t$  is the change in its velocity divided by the time interval or  $a_{av} = \frac{\Delta v}{\Delta t}$ . As  $\Delta t$  becomes smaller and approaches zero in the limit, the average acceleration approaches the instantaneous acceleration of the particle, which is  $a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$  or

$$a = \frac{dv}{dt} = \ddot{v} \quad \text{or} \quad a = \frac{d^2s}{dt^2} = \ddot{s} \quad \text{-----} \rightarrow (4.2)$$

The acceleration is positive or negative depending on whether the velocity is increasing or decreasing. Note that the acceleration would be positive if the particle had a negative velocity which was becoming less negative. If the particle is slowing down, the particle is said to be decelerating.

Velocity and acceleration are actually vector quantities, as it will see for curvilinear motion in forthcoming sections. For rectilinear motion, where the direction of the motion is that of the given straight-line path, the sense of the vector along the path is described by a plus or minus sign. In curvilinear motion, it will account for the changes in direction of the velocity and acceleration vectors as well as their changes in magnitude.

By eliminating the time  $dt$  between equation 4.1 and 4.2, then the differential equation relating displacement, velocity, and acceleration as

$$v \, dv = a \, ds \text{ or } \ddot{s} \, d \dot{s} = \ddot{s} \, ds \quad \text{-----} \rightarrow (4.3)$$

### b) Uniform acceleration motion:

When the velocity of the body changes by equal amounts in equal intervals of time, then the body is said to have uniform acceleration motion. Consider equation 4.2 as  $a = \frac{dv}{dt}$ . Here the acceleration 'a' is treated as constant then the velocity is changes from initial velocity to final velocity as  $u \rightarrow v$  and time changes from  $t=0$  to  $t=t$ .

**To find**

- **velocity as a function of time,**
- **velocity as a function of position and**
- **position as a function of time**

while keeping the acceleration as constant.

Now the integration of equation 4.2 gives the velocity as a function of time, then

$$\int_u^v dv = \int_{t=0}^{t=t} a \, dt = a \int_{t=0}^{t=t} dt$$

$$[v]_u^v = a[t]_{t=0}^{t=t} + C_1$$

$$v = at + C_1 \quad \text{-----} \rightarrow (4.4)$$

To eliminate the constant  $C_1$ , Substitute the initial condition when  $t=0$  then  $v=u$ , therefore, the equation 4.4 as

$$u = a(0) + C_1$$

$$C_1 = u \quad \text{-----} \rightarrow (4.5)$$

Now substitute the equation 4.5 into the equation 4.4 then

$$V = u + at \quad \longrightarrow \rightarrow (4.6) \quad [\text{Velocity as a function of time}]$$

The equation 4.1 is taken to find the position as a function of time, then

$$v = \frac{ds}{dt}$$

$$ds = v dt \quad \longrightarrow \rightarrow (4.7)$$

Now substitute the equation 4.6 into the equation 4.7 to get

$$ds = (u + at)dt \quad \longrightarrow \rightarrow (4.8)$$

The equation 4.8 is now integrate to obtain position as a function of time, then

$$\int_0^s ds = \int_0^t (u + at)dt$$

$$s = ut + \frac{at^2}{2} + C_2$$

When,  $t=0$ ,  $s=0$ , then

$$C_2 = 0$$

Therefore,

$$s = ut + \frac{1}{2}at^2 \quad \longrightarrow \rightarrow (4.9) \quad [\text{Position as a function of time}]$$

From equation 4.6 obtain time and substitute in equation 4.9 to get the velocity as a function of position. Therefore

$$v = u + at$$

$$t = \frac{v - u}{a}$$

The equation 4.9 as,

$$s = ut + \frac{1}{2}at^2$$

$$s = u \left( \frac{v-u}{a} \right) + \frac{1}{2} a \left( \frac{v-u}{a} \right)^2$$

$$s = \frac{uv - u^2}{a} + \frac{1}{2} \frac{a}{a^2} (v^2 - 2vu + u^2)$$

$$s = \frac{uv}{a} - \frac{u^2}{a} + \frac{1}{2a} v^2 - \frac{vu}{a} + \frac{1}{2a} u^2$$

$$s = \frac{1}{2a} v^2 - \frac{1}{2a} u^2, \quad s = \frac{v^2 - u^2}{2a}$$

$$v^2 = u^2 + 2a s \longrightarrow (4.10) \text{ [Velocity as a function of position]}$$

**The equation of motion with uniform acceleration,**

**The velocity as a function of time is  $v = u + at$ ,**

**The position as a function of time is  $s = ut + \frac{1}{2} at^2$  and**

**The velocity as a function of position is  $v^2 = u^2 + 2a s$**

### c) Uniform velocity motion:

When the velocity of the body remains constant throughout the period of time then the body is said to be uniform velocity motion. Therefore when the body moves with uniform velocity, the rate of change of velocity is zero and hence the acceleration is also zero.

The equation 4.6 and 4.10 becomes to get the initial velocity is equals to final velocity. The only one equation is suite to get function from equation 4.9.

Therefore, the equation of motion with uniform velocity from equation 4.9 as

$$s = ut \text{ or } s = vt \longrightarrow (4.11)$$

**Problems: Uniform acceleration (Motion in straight)**

**SP4.1:** If a body moves at the rate of 45 kmph for 6 minutes then 90 kmph for 10 minutes and finally 8 kmph for 3 minutes. What is the average velocity in the total interval?

**Solution:**

Given data;  $v_1 = 45 \text{ kmph} = 45(5/18) \text{ m/s} = 12.5 \text{ m/s}$  and  $t_1 = 6 \text{ min} = 360 \text{ s}$

$v_2 = 90 \text{ kmph} = 90(5/18) \text{ m/s} = 25 \text{ m/s}$  and  $t_2 = 10 \text{ min} = 600 \text{ s}$

$v_3 = 8 \text{ kmph} = 8(5/18) \text{ m/s} = 2.22 \text{ m/s}$  and  $t_3 = 3 \text{ min} = 180 \text{ s}$

By the uniform velocity of motion, the distance travelled by

$$S_1 = v_1 \times t_1 = 12.5 \times 360 = 4500 \text{ m}$$

$$S_2 = v_2 \times t_2 = 25 \times 600 = 15000 \text{ m}$$

$$S_3 = v_3 \times t_3 = 2.22 \times 180 = 400 \text{ m}$$

The total distance  $S$  travelled by body is  $S_1 + S_2 + S_3$ ,

$$S = S_1 + S_2 + S_3 = 4500 + 15000 + 400 = 19900 \text{ m}$$

The total time to reach the total distance is  $t = 360 + 600 + 180$

$$t = 1140 \text{ s}$$

We know that,

$$\text{The average velocity} = \frac{\text{Total distance travelled, } S}{\text{Total time taken, } t}$$

$$\text{The average velocity} = \frac{19900}{1140} = 17.456 \text{ m/s} = 62.84 \text{ kmph}$$

**SP4.2:** A car travels along a straight track between two points at a mean speed of 60 m/s. It returns at a mean speed of 40 m/s. What is the mean speed for the round trip?

**Solution:**

Given data; Mean speed of forward travels  $v_1$  is 60 m/s, and the mean speed of return travels  $v_2$  is 40m/s.

$$\text{The time taken for forward travel is } \frac{\text{Distance, } S}{\text{Mean Speed, } v_1} = \frac{s}{60}$$

$$\text{The time taken for return travel is } \frac{\text{Distance, } S}{\text{Mean Speed, } v_2} = \frac{s}{40}$$

$$\text{The time taken to cover total distance is } \frac{s}{60} + \frac{s}{40} = \frac{2s + 3s}{120}$$

The mean speed for the round trip is

$$\frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{s + s}{\frac{5s}{120}} = \frac{240}{5} = 48 \text{ m/s}$$

**SP4.3:** A bullet is fired with a muzzle velocity of 600m/s. If the length of the barrel is 750mm, what is the average acceleration?

**Solution:**

Given data; final velocity  $v$  is 600m/s and Distance to be travel is 750mm

From the velocity as a function of position, find the acceleration of a bullet fired with a muzzle velocity.

$$v^2 = u^2 + 2as$$

$v=600\text{m/s}$ ,  $u=0$  and  $s=0.75\text{m}$ , then

$$a = \frac{v^2 - u^2}{2s} = \frac{600^2 - 0^2}{2(0.75)} = 240 \text{ km/s}^2$$

**SP4.4:** A car is moving with a velocity of 20m/s. The car is brought to rest by applying brakes in 4 secs. Determine a) the retardation and b) distance travelled by the car after applying brakes.

**Solution:**

Given data;  $u=20\text{m/s}$ ,  $v=0\text{m/s}$ ,  $t=4\text{sec}$

By the equation of motion  $v = u + at$  find the acceleration as

$$a = \frac{v - u}{t} = \frac{0 - 20}{4} = -5\text{m/s}^2$$

The negative sign indicates the retardation.

Distance travelled by the car after applying brakes is obtain from the equation of motion as

$$s = ut + \frac{1}{2}at^2$$

$$s = 20 \times 4 + \frac{1}{2}(-5)(4)^2 = 40\text{m}$$

**SP4.5:** on turning a corner a motorist rushing at 20m/s, find a child on the road 50m ahead. He instantly stops the engine and applies brakes so as to stop the car with in 10 m of the child. Calculate a) retardation and b) time required to stop the car.

**Solution:**

Given data;  $u=20\text{m/s}$ ,  $S_{MC} = 50\text{m}$ ,  $v=0\text{m/s}$  and  $S_{MS} = 50-10=40\text{m}$

From the relation,  $v^2 = u^2 + 2aS$

$$0^2 = 20^2 + 2a40$$

$$a = -5\text{m/s}^2$$

From the relation,  $v = u + at$

$$0 = 20 + (-5)t$$

$$t = 4\text{sec.}$$

**SP4.6:** Three marks A, B and C at distance of 100m each are made along a straight road. A car starting from rest and with uniform acceleration passes the mark A and takes 10 sec to reach B and further 8 seconds to reach the mark C. Calculate a) acceleration of the car, b) the velocity of car at A, c) the velocity of car at B and d) the distance of the mark A from starting point.

**Solution:**

Given data;  $t_{AB} = 10\text{sec}$ ,  $t_{BC} = 8\text{sec}$ ,  $S_{AB} = 100\text{m}$ , and  $S_{BC} = 100\text{m}$

Consider the motion from A to B, by the relation of position as a function of time is

$$S = ut + \frac{1}{2}at^2$$

$$S = S_{AB} = 100\text{m}, u = v_A, v = v_B, t_{AB} = 10\text{sec}$$

$$100 = v_A \times 10 + \frac{1}{2}a(10)^2$$

$$50a + 10v_A = 100 \quad \text{-----} \rightarrow (i)$$

Consider the motion from A to C, by the relation of position as a function of time is

$$S = ut + \frac{1}{2}at^2$$

$$S = S_{AC} = 1200\text{m}, u = v_A, v = v_C, t_{AC} = 18\text{sec}$$

$$200 = v_A \times 18 + \frac{1}{2}a(18)^2$$

$$162a + 18v_A = 200 \quad \text{-----} \rightarrow (ii)$$

Solve equation i and ii then

$$a = 0.28\text{m/s}^2 \text{ and } v_A = 8.61 \text{ m/s}$$

Consider the motion from starting point O to A, by the relation of velocity as a function of position is

$$v^2 = u^2 + 2as$$

$u = 0$ ,  $a = 0.28 \text{m/s}^2$ , and  $v_A = 8.61 \text{m/s}$

$$8.61^2 = 0^2 + 2(0.28)(S)$$

$$S = 133.4 \text{m}$$

Consider the motion from A to B, by the relation of velocity as a function of position is

$$v^2 = u^2 + 2as$$

$$v_B^2 = v_A^2 + 2a \times 100$$

$$v_B^2 = 8.61^2 + 2 \times 0.28 \times 100$$

$$V_B = 11.39 \text{m/s}$$

**SP4.7:** A burglar's car had a start with an acceleration of  $2 \text{m/s}^2$ . A police vigilant party came after 5 seconds and continued to chase the burglar's car with a uniform velocity of  $20 \text{m/s}$ . Find the time taken in which the police van will overtake the car.

### Solution:

Consider the uniform acceleration motion of the burglar's car and let 't' be the time taken by the burglar's car while overtaking by the police van.

$$S = ut + \frac{1}{2}at^2$$

$$u = 0 \text{m/s}, a = 2 \text{m/s}^2$$

$$\text{Therefore, } S = t^2 \quad \text{-----} \rightarrow \text{(i)}$$

Consider the uniform velocity motion of police van, then

$$S = vt = 20(t-5)$$

$$S = 20t - 100 \quad \text{-----} \rightarrow \text{(ii)}$$

Substitute the equation i in ii gives,

$$t^2 = 20t - 100$$

$$t^2 - 20t + 100 = 0$$

$$(t - 10)^2 = 0$$

$$t = 10 \text{ sec.}$$

Therefore, the time for police van to overtake is  $t - 5 = 10 - 5 = 5 \text{ sec}$

**SP4.8:** A train is uniformly accelerated and passes successive kilometre stones with velocities at 18 kmph and 36 kmph respectively. Calculate the velocity when it passes the third kilometre stone. Also find time taken for each of these two intervals of one kilometre.

**Solution:**

Consider the motion of the train between A and B, where  $u = v_A = 18 \text{ kmph} = 18(5/18) \text{ m/s} = 5 \text{ m/s}$ , and  $v = v_B = 36 \text{ kmph} = 36(5/18) \text{ m/s} = 10 \text{ m/s}$  and  $S = 1000 \text{ m}$ . By the relation,  $v^2 = u^2 + 2aS$ ,

$$10^2 = 5^2 + 2 \times a \times 1000$$

$$a = 0.038 \text{ m/s}^2$$

Consider the motion of the train between B and C, where  $u = v_B = 10 \text{ m/s}$ ,  $v = v_C$ ,  $a = 0.038 \text{ m/s}^2$  and  $S = 1000 \text{ m}$ . By the relation  $v^2 = u^2 + 2aS$ ,

$$v_C^2 = 10^2 + 2 \times 0.038 \times 1000$$

$$v_C = 13.23 \text{ m/s}$$

To find the time interval 't' when motion between A and B, where  $u = 5 \text{ m/s}$ ,  $v = 10 \text{ m/s}$  and  $a = 0.038 \text{ m/s}^2$ . By the relation,  $v = u + at$

$$10 = 5 + 0.038t$$

$$t = 133.3 \text{ sec.}$$

To find the time interval 't' when motion between B and C, where  $u = 10 \text{ m/s}$ ,  $v = 13.23 \text{ m/s}$  and  $a = 0.038 \text{ m/s}^2$ . By the relation,  $v = u + at$

$$13.23 = 10 + 0.038t$$

$$t = 86.08 \text{ sec.}$$

**SP4.9:** A body is moving with uniform acceleration and covers 20m in 4<sup>th</sup> second and 30m in 8<sup>th</sup> second. Determine a) the initial velocity of the body and b) acceleration of the body.

**Solution:**

Let  $S_4$  is a distance travelled at time  $t = 4$  seconds and  $S_3$  is a distance travelled at time  $t = 3$  seconds. Therefore the distance travelled during 4<sup>th</sup> seconds is  $S_4 - S_3$ .

By the relation of motion,  $S = ut + \frac{1}{2}at^2$ , where  $t = 4$  second, then

$$S_4 = u \times 4 + \frac{1}{2} \times a \times 4^2 = 4u + 8a$$

$$\text{Similarly, } S_3 = u \times 3 + \frac{1}{2} \times a \times 3^2 = 3u + 4.5a$$

$$\text{Hence, } S_4 - S_3 = (4u + 8a) - (3u + 4.5a) = 20$$

$$u + 3.5a = 20 \quad \text{-----} \rightarrow (\text{i})$$

Similarly for 8<sup>th</sup> seconds

$$S_8 = u \times 8 + \frac{1}{2} \times a \times 8^2 = 8u + 32a$$

$$\text{Similarly, } S_7 = u \times 7 + \frac{1}{2} \times a \times 7^2 = 7u + 24.5a$$

$$\text{Hence, } S_8 - S_7 = (8u + 32a) - (7u + 24.5a) = 30$$

$$u + 7.5a = 30 \quad \text{-----} \rightarrow (\text{ii})$$

Now subtract equation 1 in 2, then

$$u + 7.5a - u - 3.5a = 30 - 20$$

$$4a = 10$$

$$a = 2.5 \text{ m/s}^2$$

Substitute the above in equation 1 then

$$u + 3.5(2.5) = 20$$

$$u = 11.25 \text{ m/s.}$$

**SP4.10:** A train travelling at 60kmph has to slow down on account of work being done on the line. Instead of continuing at a constant speed, it therefore moves with a constant deceleration of 1 kmph per second until its reduced to 15 kmph. It then travels at constant speed for 0.25 km and then accelerates at 0.5 kmph per second until its speed is once again 60 kmph. Find the delay caused.

**Solution:**

Now consider the motion of the train is to slow down from 60kmph to 15kmph, maintain the speed 15kmph upto 0.25 km and again increasing from 15kmph to 60kmph.

This motion can be divided into three parts of motion as;

- a) Part-1: Uniform deceleration or retardation
- b) Part-2: Uniform velocity and
- c) Part-3: Uniform acceleration

a) Part-1: Uniform deceleration or retardation

Given data;  $u = 60 \text{ kmph} = 60(5/18) \text{ m/s} = 16.66 \text{ m/s}$ ,  $v = 15 \text{ kmph} = 15(5/18) \text{ m/s} = 4.17 \text{ m/s}$  and  $a = -1 \text{ kmph/s} = -(5/18) \text{ m/s}^2 = -0.28 \text{ m/s}^2$ .

By the relation of equation of motion,  $v^2 = u^2 + 2aS$

$$4.17^2 = 16.66^2 + 2 \times (-0.28) \times S_1$$

$$S_1 = 468.73 \text{ m}$$

By the velocity as a function of time,  $v = u + at$

$$4.17 = 16.66 + (-0.28) \times t_1$$

$$t_1 = 45 \text{ sec.}$$

b) Part-2: Uniform velocity

Data given;  $S=S_2 = 0.25\text{km} = 250\text{m}$ ,  $v = 4.17 \text{ m/s}$ ,

By the uniform velocity of motion,  $S = vt$

$$S_2 = vt_2$$

$$250 = (4.17) \times t_2$$

$$t_2 = 60\text{sec.}$$

c) Part-3: Uniform acceleration

Data given;  $u = 4.17\text{m/s}$ ,  $v = 16.67\text{m/s}$ ,  $a = 0.5\text{kmph/s} = 0.14\text{m/s}^2$

By the equation of motion of relation,  $v^2 = u^2 + 2as$

$$16.67^2 = 4.17^2 + 2 \times (0.14) \times S_3$$

$$S_3 = 937.5\text{m}$$

By the relation,  $v = u + at$

$$16.67 = 4.17 + (0.14) \times t_3$$

$$t_3 = 90\text{sec.}$$

Total distance travelled by train is  $S_1 + S_2 + S_3$ ,

$$S = 468.76 + 250 + 937.5 = 1656.26\text{m}$$

Total time taken to reach the initial speed is  $t_1 + t_2 + t_3$

$$t = 45 + 60 + 90 = 195\text{sec.}$$

When the train moves this distance 1656.26m with the same speed of 60kmph the time taken to cover this distance is total distance / velocity, then

$$t_T = \frac{S}{v} = \frac{1656.26}{16.67} = 99.37\text{s}$$

Hence, the delay cause due to work done on the line is

$$t_{\text{delay}} = t - t_T = 195 - 99.37 = 95.62\text{s}$$

**SP4.11:** In a 100m race an athlete accelerates uniformly from the start to his maximum velocity in a distance of 4m and runs the remaining distance with that velocity. If he finishes the race in 10.4 secs, Find a) his initial acceleration and b) his maximum velocity.

**Solution:**

The motion of the athlete can be divided into two parts;

Part-1: uniform acceleration

Data given;  $u_1 = 0$ ,  $v_1 = v_{\max}$ ,  $S_1 = 4\text{m}$ ,  $t = t_1$

By the relation,  $v_1 = u_1 + a_1 t_1$

$$v_{\max} = 0 + a_1 t_1$$

$$t_1 = \frac{v_{\max}}{a_1} \quad \text{-----} \rightarrow (1)$$

By the relation,  $v_1^2 = u_1^2 + 2a_1 S_1$

$$v_{\max}^2 = 0^2 + 2a_1 4$$

$$a_1 = \frac{v_{\max}^2}{8} \quad \text{-----} \rightarrow (2)$$

Substitute, the equation 2 in equation 1, then

$$t_1 = \frac{v_{\max}}{\cancel{v_{\max}^2} / 8} = \frac{8}{v_{\max}}$$

Part-2: Uniform velocity

Total time,  $t = 10.4 \text{ sec.}$

The time taken for uniform velocity is total time minus the time taken to reach the uniform velocity from start.

Therefore,  $t_2 = t - t_1$

$$t_2 = 10.4 - \frac{8}{v_{\max}}$$

The remaining distance from the reach of uniform velocity  $S_2$  is  $S - S_1$

$$S_2 = 100 - 4 = 96 \text{m}$$

By the relation,  $S_2 = v_2 t_2$ , where  $v_2$  is  $v_{\max}$

$$96 = v_{\max} \left[ 10.4 - \frac{8}{v_{\max}} \right]$$

$$96 = 10.4v_{\max} - 8$$

$$v_{\max} = 10 \text{m/s}$$

$$\text{Hence, } a_1 = \frac{v_{\max}^2}{8} = \frac{10^2}{8} = 12.5 \text{m/s}^2$$

**SP4.12:** A bus is beginning to move with an acceleration of  $0.3 \text{ m/s}^2$ . A man who is  $12 \text{ m}$  behind the bus starts running at  $3 \text{ m/s}$  to catch the bus. After how many seconds will the man be able to catch the bus?

### Solution:

The motion of bus is uniform acceleration and the motion of the man is uniform velocity. The time for catching the bus by the man is equal to the time for travelling by the bus from start. Hence the time 't' is same for both the motion's

#### Part-1: Bus (Uniform acceleration)

Data given,  $u=0$ ,  $a_B = 0.3 \text{ m/s}^2$ ,

By the relation,  $S = ut + \frac{1}{2}at^2$

$$S_B = 0 + \frac{1}{2}(0.3)t_B^2$$

$$S_B = 0.15t_B^2 \quad \text{-----} \rightarrow (i)$$

#### Part-2: Man (Uniform velocity)

Data given,  $S_M = 12 + S_B$ ,

By the relation,  $S_M = v_M t_M$

$$12 + S_B = 3 \times t_M$$

$$S_B = 3t_M - 12 \quad \text{-----} \rightarrow \text{(ii)}$$

Now substitute the eqn. ii into the eqn. i then

$$3t_M - 12 = 0.15t_B^2$$

Here,  $t_M = t_B$ , then

$$0.15t_M^2 - 3t_M + 12 = 0$$

By solving, it gives

$$t = 14.47 \text{ s or } 5.52 \text{ s}$$

**SP4.13:** Two trains A and B leave the same station in parallel lines. Trains A starts with a uniform acceleration of  $0.15 \text{ m/s}^2$  and attains a speed of  $27 \text{ kmph}$ . Then the steam is reduced to keep speed constant. Train B leaves  $40 \text{ secs}$  later with uniform acceleration of  $0.3 \text{ m/s}^2$  to attain maximum speed of  $54 \text{ kmph}$ . When and where will B overtake A?

**Solution:**

Since the both train starts from same station while overtaking the distance travelled by the both the trains remain same. i.e.,  $S_A = S_B$ .

Part-1: uniform acceleration

Train-A:

Data given,  $u = 0$ ,  $v = 27 \text{ kmph} = 7.5 \text{ m/s}$ ,  $a_A = 0.15 \text{ m/s}^2$ .

By the relation,  $v = u + at$

$$7.5 = 0 + 0.15t$$

$$t = 50 \text{ sec.}$$

By the relation,  $S_{1A} = ut + \frac{1}{2}at^2$

$$S_{1A} = 0 + \frac{1}{2}(0.15)(50)^2$$

$$S_{1A} = 187.5m$$

Train-B:

Data given,  $u = 0$ ,  $v = 54\text{kmph} = 15\text{m/s}$ ,  $a_B = 0.3 \text{ m/s}^2$ .

By the relation,  $v = u + at$

$$15 = 0 + 0.3t$$

$$t = 50\text{sec.}$$

By the relation,  $S_{1B} = ut + \frac{1}{2}at^2$

$$S_{1B} = 0 + \frac{1}{2}(0.3)(50)^2$$

$$S_{1B} = 375m$$

### Part-2: Uniform velocity

Let  $t_A$  be the time taken by the train A while overtaking by the train B, Then the distance travelled by the train A

$$S_A = S_{1A} + S_{2A}, \text{ where, } S_{2A} = vt_A, t_A = t - t_{1A} = t - 50$$

$$\text{Therefor, } S_A = 187.5 + 7.5(t - 50)$$

$$S_A = 7.5t - 187.5 \quad \text{-----} \rightarrow (i)$$

Let  $t_B$  be the time taken by the train B while it overtaking the train A, Then the distance travelled by the train B

$$S_B = S_{1B} + S_{2B}, \text{ where, } S_{2B} = vt_B, t_B = t - t_{1B} - \text{late by } 40\text{sec} = t - 50 - 40, \text{ and}$$

$$v_B = 15\text{m/s}$$

$$\text{Therefor, } S_B = 375 + 15(t - 90)$$

$$S_B = 15t - 975 \quad \text{-----} \rightarrow (ii)$$

Now equating the eqn. i and ii by keeping  $S_A$  is equal to  $S_B$  gives

$$S_B = 600\text{m} \text{ and } t = 65\text{s.}$$

**SP4.14:** Two cars are travelling towards each other on a single lane road at velocities 12 m/s and 9 m/s respectively. When 100m apart both drivers realise the situation and apply their brakes. They succeed in stopping simultaneously and just short of colliding. Assume constant deceleration for each case, find a) time required for cars to stop, b) deceleration of each car and c) the distance travelled by each car while slowing down.

**Solution:**

Both the cars stop simultaneously. Hence times remain same. Let  $S_A$  and  $S_B$  be the distance travelled by the two cars.

$$S_A + S_B = 100\text{m}$$

Considering the motion of first car A, where  $u = 12\text{m/s}$ ,  $v = 0\text{m/s}$ ,  $S = S_A$

By the relation,  $v = u + at$

$$0 = 12 + a_A \times t$$

$$a_A = -\frac{12}{t}$$

By the relation,  $S_A = ut + \frac{1}{2}a_A t^2$

$$S_A = 12t + \frac{1}{2}a_A t^2$$

$$S_A = 12t + \frac{1}{2}\left(-\frac{12}{t}\right)t^2$$

$$S_A = 6t \quad \text{-----} \rightarrow (\text{i})$$

Now considering the motion of car B, where  $u = 9\text{m/s}$ ,  $v = 0\text{m/s}$ ,  $S = 100 - S_A$

By the relation,  $v = u + at$

$$0 = 9 + a_B \times t$$

$$a_B = -\frac{9}{t}$$

By the relation,  $S_B = ut + \frac{1}{2}a_B t^2$

$$S_B = 9t + \frac{1}{2} \left( -\frac{9}{t} \right) t^2$$

$$S_B = 4.5t \quad \text{-----} \rightarrow \text{(ii)}$$

We know that,  $S_B = 100 - S_A$

$$4.5t = 100 - 6t$$

$$10.5t = 100$$

$$t = 9.52s$$

Therefore, the eqn. i and ii, becomes

$$S_A = 6(9.52) = 57.12m \text{ and}$$

$$S_B = 4.5(9.52) = 42.8m$$

$$a_A = -\frac{12}{t} = -\frac{12}{9.52} = -1.26m/s^2 \text{ and}$$

$$a_B = -\frac{9}{t} = -\frac{9}{9.52} = -0.95m/s^2$$

### **Problems: Uniform acceleration (Motion under gravity)**

**SP4.15:** A boy throws a ball from the ground vertically up with a speed of 20m/s. Neglecting the resistance of air determine a) how high the ball will rise, b) time taken to reach the maximum heights and c) the total time taken by the ball to come back to the boy.

**Solution;**

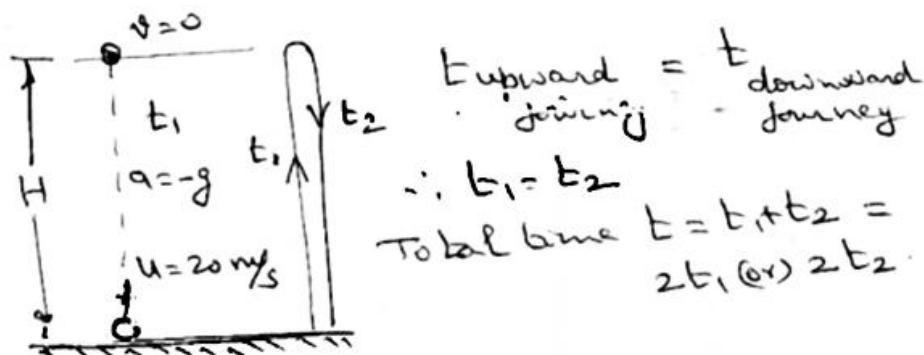


Fig.4.3

Considering the upward motion of the ball

$$u = 20 \text{ m/s}$$

$$v = 0$$

$$a = -g$$

Using the relations

$$v = u + at$$

$$0 = 20 + (-9.81)t$$

$$t = t_1 = 20/9.81 = 2.038 \text{ s}$$

$$s^2 = u^2 + 2as$$

$$0^2 = 20^2 + 2 \times (-9.81)s$$

$$s = 400/19.62 = 20.38 \text{ m}$$

Considering the downward motion of the ball

$$u = 0 \text{ m/s}$$

$$a = g$$

$$s = 20.38 \text{ m}$$

Using the relations

$$s = ut + \frac{1}{2}at^2$$

$$20.38 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$20.38 = 4.905t^2$$

$$t^2 = 20.38/4.905 = 4.1549$$

$$t = t_2 = 2.038 \text{ s}$$

$$\text{Total time taken} = 2.038 + 2.038 = 4.076 \text{ s}$$

**SP4.16:** A particle is dropped from the top of a tower 200m high and another particle is projected at the same time vertically upwards from the foot of the tower so as to meet the first particle at a height of 50m. Find the velocity of projection of the second particle.

**Solution:**

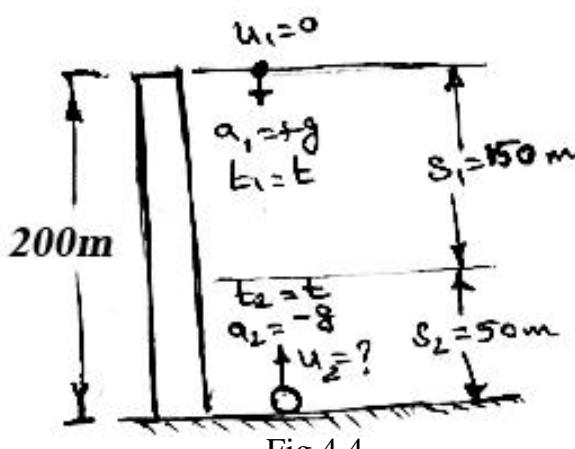


Fig.4.4

Let both the particles meet each other after time 't' secs from the start.

Considering the downward motion of the first particle:

$$u = 0; s = 200 - 50 = 150 \text{ m};$$

$$a = g$$

Using the relations

$$s = ut + \frac{1}{2} at^2$$

$$150 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$t = \sqrt{150/4.905} = 5.53 \text{ s}$$

Considering the upward motion of the second particle:

Let 'u' be the initial velocity

$a = -g; s = 50 \text{ m}; t = 5.53 \text{ s}$  (∴ the time for both the motions are same)

Using the relations

$$s = ut + \frac{1}{2} at^2$$

$$50 = u \times 5.53 + \frac{1}{2} (-9.81) (5.53)^2$$

$$u = 36.17 \text{ m/s}$$

Velocity of projection of the second particle = 36.17 m/s

**SP4.17:** A stone is thrown vertically up from the top of tower with a certain velocity. It reaches ground in 5.64 sec. A second stone thrown down from the same tower with the same initial velocity reaches the ground in 3.6 sec. Determine a) the height of the tower and b) the initial velocity of the stones.

**Solution:**

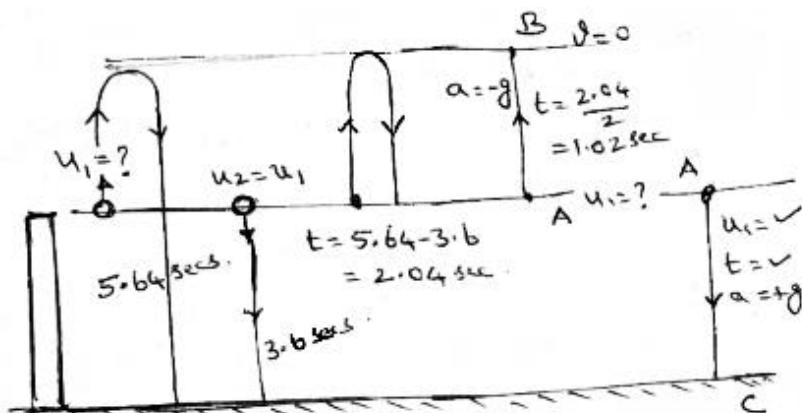


Fig.4.5

We know that when the stone is thrown upwards from the top of the tower, the time taken for upward motion and downward motion is same. Also the velocity of projection and velocity with which it cross the top level during downward motion remains same.

∴ Time taken by the stone for the upward and downward motion till it reaches the top of the tower is  $= 5.64 - 3.6 = 2.04 \text{ s}$

Therefore time for upward motion  $= 2.04/2 = 1.02 \text{ s}$

Considering upward motion

$$u = u; v = 0; a = -g; t = 1.02 \text{ s}$$

$$v = u + a t$$

$$0 = u + (-9.81) 1.02$$

$$u = 10 \text{ m/s}$$

Considering downward motion of the stone from the top of the building

$$u = 10 \text{ m/s}, t = 3.6 \text{ s}$$

$$s = h = ut + \frac{1}{2} a t^2$$

$$= 10 \times 3.6 + \frac{1}{2} \times 9.81 \times (3.6)^2$$

$$= 99.5 \text{ m}$$

Therefore the height of the tower = 99.5 m

The initial velocity of the stone = 10 m/s

**SP4.18:** A stone was thrown vertically from the ground with a velocity of 49m/s. After 2 sec. another stone was thrown vertically upwards from the same place. If both the stones strike the ground at the same time, find the velocity with which the second stone was thrown. Take  $g = 9.8 \text{ m/s}^2$ .

**Solution:**

Considering the motion of the first stone:

Upward motion:

$$u = 49 \text{ m/s}; v = 0; a = -g$$

Using the relations

$$v = u + a t$$

$$0 = 49 + (-9.8) t$$

$$t = t_1 = 49/9.8 = 5 \text{ s}$$

$$s = ut + \frac{1}{2} a t^2$$

$$s = 49 \times 5 + \frac{1}{2} (-9.8) 5^2$$

$$s = 122.5 \text{ m}$$

Downward motion:

$$u = 0 \text{ m/s}$$

$$a = g$$

$$s = 122.5 \text{ m}$$

$$s = ut + \frac{1}{2} a t^2$$

$$122.5 = 0 + \frac{1}{2} (9.8) t^2$$

$$122.5 = 4.9 t^2$$

$$t = 5 \text{ s}$$

Total time for first stone to move up and down =  $5 + 5 = 10 \text{ s}$

Considering the motion of the second stone:

Time available for second stone

$$= 10 - 2 = 8 \text{ s}$$

Time available for upward motion

$$= 8/2 = 4 \text{ s}$$

Upward motion:

$$u = u; v = 0; t = 4 \text{ s}; a = -g$$

$$v = u + a t$$

$$0 = u - 9.8 \times 4$$

$$u = 39.2 \text{ m/s}$$

Velocity of the second stone = 39.2 m/s

**SP4.19:** A body falling freely under the action of gravity passes two points 10m apart vertically in 0.2sec. From what height above the higher point did it start to fall?

**Solution:**

Considering downward motion between points A and B

$$a = 9.81 \text{ m/s}^2; s = 10 \text{ m}; t = 0.2 \text{ s}$$

Using the relations

$$s = ut + \frac{1}{2} at^2$$

$$10 = u \times 0.2 + \frac{1}{2} \times 9.81 \times (0.2)^2$$

$$u = 49.05 \text{ m/s}$$

Considering the motion from 'O' to 'A'

$$u = 0; a = g; v = 49.05 \text{ m/s}$$

Using the relation

$$v^2 = u^2 + 2 as$$

$$49.05^2 = 0 + 2 \times 9.81 \times s$$

$$s = 122.47 \text{ m}$$

The distance between O to A = 122.47 m

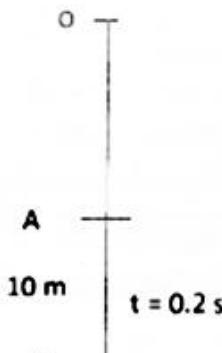


Fig.4.6

**SP4.20:** A stone dropped into well is heard to strike the water after 4sec. find the depth of the well if the velocity of the sound is 350m/s.

**Solution:**

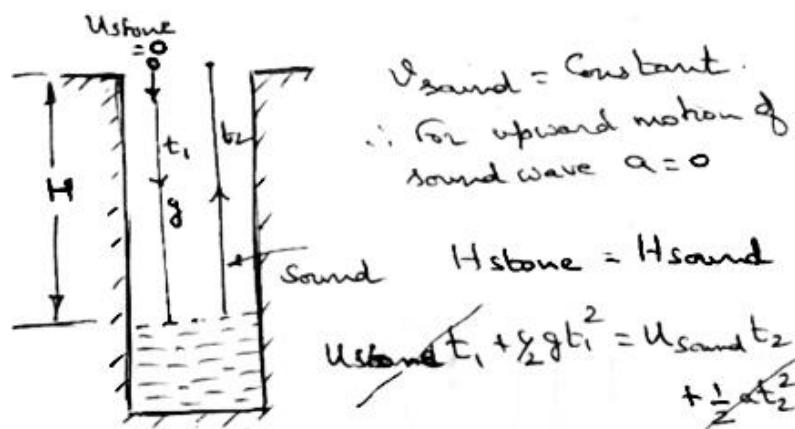


Fig.4.7

$$\text{Also } (t_1 + t_2) = 4$$

$$\therefore t_2 = (4 - t_1)$$

When the stone is dropped into well the downward motion of the stone is uniform acceleration motion (g) while the sound of splash is heard with uniform velocity. Let 't' be time taken by the stone for downward motion. The time for hearing the sound of splash will be equal to (4-t). Let the depth of well is equal to 'h'

Considering the downward motion of the stone:

$$u = 0; t = t; a = g$$

Using the relations

$$s = ut + \frac{1}{2}at^2$$

$$h = 0 + \frac{1}{2} \times 9.81 t^2$$

$$h = 4.905 t^2 \quad (1)$$

Considering the motion of the sound (upward)

Using the relation

$$s = vt$$

$$h = 350 \times (4-t)$$

$$h = 1400 - 350t \quad \dots (2)$$

Equating (1) and (2)

$$4.905 t^2 = 1400 - 350t$$

$$4.905 t^2 + 350t - 1400 = 0$$

Using the relation

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 4.905; b = 350; c = -1400$$

$$t = \frac{-350 \pm \sqrt{350^2 - 4 \times 4.905 \times (-1400)}}{2 \times 4.905}$$

$$= 3.8 \text{ s}$$

$$h = 4.905 t^2 = 4.905 \times (3.8)^2 = 70.8 \text{ m}$$

**SP4.21:** A particle falling under gravity falls 20m in a certain second. Find the time required to cover next 20m.

**Solution:**

Considering the downward motion of the particle from A to B

$$s = 20 \text{ m}; a = g; t = 1 \text{ s}$$

Using the relations

$$s = ut + \frac{1}{2}at^2$$

$$20 = u \times 1 + \frac{1}{2} \times 9.81 \times 1^2$$

$$u = 15.095 \text{ m/s}$$

$$v = u + at$$

$$v = 15.095 + 9.81 \times 1$$

$$v = 24.905 \text{ m/s}$$

Considering the downward motion of the particle from B to C

$$s = 20 \text{ m}; a = g; u = 24.905 \text{ m/s}$$

$$s = ut + \frac{1}{2}at^2$$

$$20 = 24.905 \times t + \frac{1}{2} \times 9.81 \times t^2$$

$$4.905 t^2 + 24.905 t - 20 = 0$$

Using the relation

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 4.905; b = 24.905; c = -20$$

$$t = \frac{-24.905 \pm \sqrt{24.905^2 - 4 \times 4.905 \times (-20)}}{2 \times 4.905}$$

$$= 0.71 \text{ s}$$

Time required cover next 20m = 0.71 s

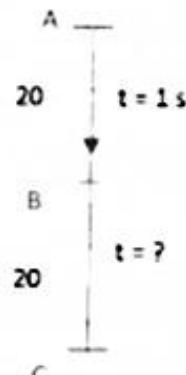


Fig.4.8

**SP4.22:** A cage descends in a mine shaft with an acceleration of  $0.5\text{m/s}^2$ . After the cage has travelled 25m, a stone is dropped from the top of the shaft. Determine a) the time taken by the stone to hit the cage and b) the distance travelled by the cage before impact.

**Solution:**

**Considering the downward motion of the cage for 1" 25 m**

$$u = 0; s = 25 \text{ m}; a = 0.5 \text{ m/s}^2$$

**Using the relations**

$$s = ut + \frac{1}{2}at^2$$

$$25 = 0 + \frac{1}{2} \times 0.5 \times t^2$$

$$t = t_1 = 10 \text{ s}$$

∴ after 10 secs the stone was dropped

Let  $t_2$  be the time taken by the stone to hit the cage

∴ Total time taken for the cage from the start till it hits by stone is =  $t_2 + 10 \text{ s}$

**Considering the motion of the cage;**

$$u = 0; a = 0.5 \text{ m/s}^2; t = 10 + t_2$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2} \times 0.5 \times (10+t_2)^2$$

$$s = 0.25 (10+t_2)^2 = 0.25 (10^2 + t_2^2 + 20t_2)$$

$$s = 25 + 0.25t_2^2 + 5t_2 \quad (1)$$

**Considering the motion of the stone:**

$$u = 0; a = g; t = t_2$$

$$s = ut + \frac{1}{2}at^2$$

$$s = 0 + \frac{1}{2} \times 9.81 \times t_2^2 = 4.905 t_2^2 \quad (2)$$

**Equating (1) and (2)**

$$4.905 t_2^2 = 25 + 0.25t_2^2 + 5t_2$$

**Solving the quadratic equation**

$$t_2 = 2.92 \text{ s}$$

$$s = 4.905t_2^2 = 4.905 \times (2.92)^2$$

Distance traveled by the stone = Distance traveled by the cage = **41.77 m**

**SP4.23:** A body was thrown vertically down from a tower and traverses a distance of 40m during its fourth second of its fall. Find the initial velocity of the body.

**Solution:**

**Distance traveled in 4 secs**

$$s = ut + \frac{1}{2}at^2$$

$$s_4 = u \times 4 + \frac{1}{2} a \times 4^2$$

$$s_4 = 4u + \frac{1}{2} \times 9.81 \times 4^2$$

$$s_4 = 4u + 78.4 \quad (1)$$

**Distance traveled in 3 secs**

$$s = ut + \frac{1}{2}at^2$$

$$s_3 = u \times 3 + \frac{1}{2} a \times 3^2$$

$$s_3 = 3u + \frac{1}{2} \times 9.81 \times 3^2$$

$$s_3 = 3u + 44.1 \quad (2)$$

**Distance traveled in 4<sup>th</sup> secs**

$$(1) - (2) = s_4 - s_3$$

$$= (4u + 78.4) - (3u + 44.1)$$

$$40 = u + 34.3$$

$$u = 5.7 \text{ m/s}$$

Initial velocity of the body = **5.7 m/s**

**SP4.24:** A particle starting from rest falls 70m in the last second of its motion. Determine the total time taken by the particle to fall and the height from which it fell.

**Solution:**

Let 'n' be the last second of the motion

$$\therefore s_n = u n + \frac{1}{2} a n^2$$

$$s_{n-1} = u (n-1) + \frac{1}{2} a (n-1)^2$$

Distance traveled in  $n^{\text{th}}$  second =  $s_n - s_{n-1}$

$$s_n - s_{n-1} = [u n + \frac{1}{2} a n^2] - [u (n-1) + \frac{1}{2} a (n-1)^2]$$

$$= u n + \frac{1}{2} a n^2 - u (n-1) + u - \frac{1}{2} a n^2 + a n - a/2$$

$$= u + a n - a/2$$

$$s = u + a (n - \frac{1}{2})$$

$$70 = 0 + g (n - \frac{1}{2})$$

$$70 = 9.81(n - \frac{1}{2})$$

$$70 = 9.81 n - 4.905$$

$$74.905 = 9.81 n$$

$$n = 74.905/9.81 = 7.635$$

$$\text{Total time} = 7.635 \text{ s}$$

$$\text{Total height} = s_n = u n + \frac{1}{2} a n^2$$

$$s_n = 0 + \frac{1}{2} \times 9.81 \times 7.635^2$$

$$s_n = 286 \text{ m}$$

**SP4.25:** Sounding rockets are used for conducting experiments in the upper atmosphere. One such rocket is fired vertically upwards with a constant acceleration of  $40 \text{ m/s}^2$ . All the fuel in the rocket is consumed in 25 sec and the rocket then ascends like a free particle decelerating under the influence of gravity. A) Determine the velocity of the rocket at the instant when all the fuel is consumed, B) What is the maximum altitude reached? C) Determine the time taken to reach the maximum altitude and D) Find the total time taken to reach the ground from the start.

**Solution:**

**Stage-I upward motion of the rocket till all the fuel is burnt:**

$$u = 0; a = 40 \text{ m/s}^2; t = t_1 = 25 \text{ s}$$

Using the relations

$$v = u + a t$$

$$v = 0 + 40 \times 25 = 1000 \text{ m/s}$$

$$s = ut + \frac{1}{2} a t^2$$

$$= 0 + \frac{1}{2} \times 400 \times 25^2$$

$$s = s_1 = 12500 \text{ m}$$

Stage II upward motion of the rocket after all the fuel is burnt (as a free particle)

$$u = 1000 \text{ m/s}; a = -g; v = 0$$

Using the relations

$$v = u + at$$

$$0 = 1000 + (-9.81)t$$

$$t = t_2 = 1000/9.81 = 101.93 \text{ s}$$

$$v^2 = u^2 + 2as$$

$$0 = 1000^2 + 2 \times (-9.81)s$$

$$s = s_2 = 50968.39 \text{ m}$$

Total distance traveled upwards by the rocket

$$= s_1 + s_2 = 12500 + 50968.39$$

$$= 63468.39 \text{ m}$$

Maximum altitude reached by the rocket = 63.468 km

$$\text{Total time taken} = t_1 + t_2 = 101.93 + 25$$

$$= 126.93 \text{ s}$$

Time taken to reach the maximum altitude = 126.93 s

Considering the downward motion of the rocket from the maximum altitude

$$u = 0 \text{ m/s}; a = g; s = 63468.39 \text{ m}$$

Using the relations

$$s = ut + \frac{1}{2}at^2$$

$$63468.39 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$t = t_3 = 113.75 \text{ s}$$

Total time for upward and downward motion = 126.93 + 113.75 = 240.68 s

**SP4.26:** A man in a balloon rising with a constant velocity of 4m/s propels a ball upward with a velocity of 1.2m/s relative to the balloon. After what time interval will the ball return to the balloon?

**Solution:**

The absolute velocity of the ball moving upwards =  $4 + 1.2 = 5.2 \text{ m/s}$

$$u = 5.2 \text{ m/s}; a = -g; v = 0$$

Using the relations

$$v = u + at$$

$$0 = 5.2 + (-9.81)t$$

$$t = 5.2/9.81 = 0.53 \text{ s}$$

$$v^2 = u^2 + 2as$$

$$0 = 5.2^2 - 2 \times 9.81 \times s$$

$$s = 1.378 \text{ m}$$

During this time 0.53 s, the balloon can travel a distance

$$= vt = 4 \times 0.53 = 2.12 \text{ m}$$

From the above calculations it is inferred that before reaching the maximum height by the stone the balloon hits the stone while moving upwards.

Let 't' be the time to hit

Considering the motion of the ball

$$u = 5.2 \text{ m/s}$$

$$a = -g$$

$$s = ut + \frac{1}{2}at^2$$

$$= 5.2t + \frac{1}{2} \times (-9.81)t^2$$

$$s_{\text{ball}} = 5.2t - 4.905t^2$$

Considering the motion of the balloon

$$s = vt$$

$$s_{\text{balloon}} = 4t$$

equating

$$s_{\text{ball}} = s_{\text{balloon}}$$

$$4t = 5.2t - 4.905t^2$$

$$4.905t^2 = 1.2t$$

$$t = 1.2/4.905 = 0.2446 \text{ s}$$

The time interval for hitting the balloon by ball is 0.2446 s

**SP4.27:** A Juggler keeps three balls going with one hand, so that at any instant two are in the air and one in the hand. If each ball rises to a height of 'h' show that each ball stays in the hand for  $0.4515\sqrt{h}$  sec.

**Solution:**

Let the ball is thrown from hand with a velocity 'u' to a height of 'h'

$$u = u; v = 0; a = -g; s = h$$

Using the relation

$$v^2 = u^2 + 2as$$

$$0 = u^2 + 2(-g)h$$

$$u = \sqrt{2gh}$$

$$v = u + at$$

$$0 = u - gt$$

$$t = u/g$$

$$\text{The time in which the ball is in air} = 2t = 2u/g \\ = 2 \times \sqrt{2gh}/g$$

Since there are two balls in air,  
The time stayed on the hand = time of flight / no. of balls in air

$$= 2 \times \sqrt{2gh}/2g$$

$$= 0.4515\sqrt{h}$$

**SP4.29:** Water drips from a faucet at the rate of 6 drops per second. The faucet is 200mm above the sink. When one drop strikes the sink how far is the next drop above the sink?

**Solution:**

Consider the downward motion of the first drop

till it reaches the sink

$$u = 0; s_1 = 200 \text{ mm} = 0.2 \text{ m}; a = g$$

$$s = ut + \frac{1}{2}at^2$$

$$0.2 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$t = 0.2019275 \text{ s}$$

Consider the downward motion of the second drop

Time gap between each drop =  $1/6 \text{ s}$

$$\text{The time for 2nd drop} = 0.2019275 - 1/6 = 0.03526 \text{ s}$$

$$u = 0; t = 0.03526 \text{ s}; a = g$$

$$s = ut + \frac{1}{2}at^2$$

$$s_2 = 0 + \frac{1}{2} \times 9.81 \times 0.03526^2 = 6.098 \text{ mm}$$

$$\text{Distance between first and second drop} = 200 - 6.098 \\ = 193.9 \text{ mm} = 0.1939 \text{ m}$$

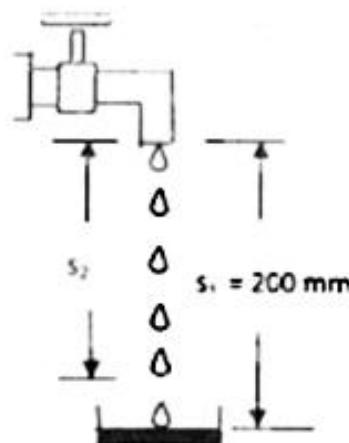


Fig.4.9

**SP4.29:** Two balls are projected simultaneously with the same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times  $t_1$  and  $t_2$  respectively show that  $\sqrt{t_1 t_2}$  is the time which each will take to reach the ground if simply let drop from the top of the tower.

**Solution:**

*Let 't' be the time taken by the ball (3) when simply drop from the height 'h'*

Using the relation

$$s = ut + \frac{1}{2} at^2$$

$$h = 0 + \frac{1}{2} gt^2$$

$$h = \frac{1}{2} g t^2$$

$t_1$  be the total time taken by the ball (1) projected vertically upwards

$t_2$  be the time taken by the ball (2) thrown vertically downwards

Considering the motion of the ball which is moving up and down above the top of the building

Time available for the above said motion  $= t_1 - t_2$

Time available for upward motion only

$$= (t_1 - t_2)/2$$

To find  $h_1$

Consider the downward motion of the 1<sup>st</sup> ball till it reaches the top of the building

$$u = 0$$

$$a = g$$

$$s = h_1$$

$$s = ut + \frac{1}{2} at^2$$

$$h_1 = 0 + \frac{1}{2} g [(t_1 - t_2)/2]^2$$

$$h_1 = \frac{1}{2} g [(t_1 - t_2)/2]^2$$

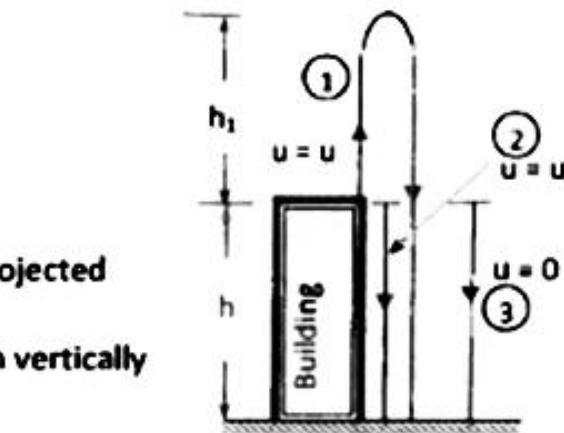


Fig.4.10

Considering the downward motion of the ball from its max. height till it reaches the ground

Time available for the above said motion

$$= [(t_1 - t_2)/2] + t_2 = (t_1 + t_2)/2$$

$$u = 0$$

$$s = h + h_1$$

$$t = (t_1 + t_2)/2$$

$$s = ut + \frac{1}{2} at^2$$

$$h + h_1 = 0 + \frac{1}{2} g [(t_1 + t_2)/2]^2$$

$$h + h_1 = (g/2) [(t_1 + t_2)^2/2^2]$$

$$h = (g/8) (t_1 + t_2)^2 - h_1$$

$$= (g/8) (t_1 + t_2)^2 - g/2 [(t_1 - t_2)/2]^2$$

$$t^2 = t_1 t_2$$

$$t = \sqrt{t_1 t_2}$$

**Problems: Motion under variable acceleration**

**SP4.30:** A car moves along a straight line whose equation of motion is given by  $S = 12t + 3t^2 - 2t^3$ , where S is in metre and t in seconds. Calculate a) the velocity and acceleration at start and b) the acceleration when velocity is zero.

**Solution:**

Equation of motion is

$$S = 12t + 3t^2 - 2t^3 \dots (1)$$

Differentiating the equation (1) with respect to 't'

$$v = \frac{ds}{dt} = 12 + 6t - 6t^2 \dots (2)$$

at start  $t = 0$

$$v = 12 \text{ m/s}$$

Differentiating the equation (2) with respect to 't'

$$a = \frac{dv}{dt} = 6 - 12t \dots (3)$$

$$\text{at } t = 0; a = 6 \text{ m/s}^2$$

To find 'a' when  $v = 0$

$$v = 12 + 6t - 6t^2 = 0$$

$$(t - 2)(t + 1) = 0$$

$$t = 2 \text{ or } -1$$

Neglecting the value obtained with - sign for time Substituting the value of t in equation

$$a = 6 - 12t = 6 - 12 \times 2 = -18 \text{ m/s}^2$$

**SP4.31:** The equation of motion of a particle moving in a straight line is given by  $S = 18t + 3t^2 - 2t^3$ , where S in metre and t in seconds. Find the velocity and acceleration at start, the time when the particle reaches its maximum velocity and the maximum velocity of the particle.

**Solution:**

Equation of motion is

$$S = 18t + 3t^2 - 2t^3$$

Differentiating the equation (1) with respect to 't'

$$v = \frac{ds}{dt} = 18 + 6t - 6t^2$$

at start  $t = 0; v = 18 \text{ m/s}$

$$a = \frac{dv}{dt} = 6 - 12t$$

at  $t = 0; a = 6 \text{ m/s}^2$

When velocity is maximum  $\frac{dv}{dt} = 0$

$$i.e., 6 - 12t = 0$$

$$12t = 6; t = 0.5$$

to get max. velocity substitute  $t = 0.5 \text{ s}$

$$v_{\max} = 18 + 6 \times 0.5 - 6 \times 0.5^2 = 18 + 3 - 1.5 = 19.5 \text{ m/s}$$

$$v_{\max} = 19.5 \text{ m/s}$$

**SP4.32:** The motion of a particle moving in a straight line is given by the expression  $S = t^3 - 3t^2 + 2t + 5$ , where  $S$  is the displacement in metre and  $t$  is the time in seconds. Determine a) the velocity and acceleration after 4 seconds, b) maximum or minimum velocity and time at which velocity is zero.

**Solution:**

$$S = t^3 - 3t^2 + 2t + 5$$

$$v = \frac{ds}{dt} = 3t^2 - 6t + 2$$

$$a = \frac{dv}{dt} = 6t - 6$$

To find max. or min.

Differentiating  $\frac{dv}{dt}$  once again

If the value obtained is +ve ,it is min

If the value obtained is -ve ,it is max

$$\frac{d^2v}{dt^2} = 6t - 6$$

$$\frac{d^2v}{dt^2} = 6 \text{ (+ve sign)}$$

Hence the value obtained is minimum value

When  $t = 4$ ;

$$v = 3 \times 4^2 - 6 \times 4 + 2 = 48 - 24 + 2 = 26 \text{ m/s}$$

$$a = 6 \times 4 - 6 = 18 \text{ m/s}^2$$

Velocity is maximum or minimum when  $\frac{dv}{dt} = 0$

$$6t - 6 = 0$$

$$6t = 6; t = 1$$

$$\text{Substituting } t = 1 \text{ s in } v = 3t^2 - 6t + 2$$

$$v = 3 \times 1^2 - 6 \times 1 + 2 = -1 \text{ m/s (min)}$$

To find the time at which the velocity is zero

$$v = 3t^2 - 6t + 2 = 0$$

$$a = 3, b = -6, c = 2$$

Using the relation

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= 1.577 \text{ s (or) } 0.423 \text{ s}$$

**SP4.33:** The acceleration of a point moving on a vertical line is given by the equation  $a = 12t - 20$ . It is known that its displacement  $S = -10 \text{ m}$  at time  $t = 0$  and that its displacement  $S = +10 \text{ m}$  at time  $t = 5 \text{ sec}$ . Derive the equation of its motion.

**Solution:**

$$a = 12t - 20$$

$$a = \frac{dv}{dt} = 12t - 20$$

$$dv = (12t - 20) dt$$

Integrating with respect to 't'

$$v = \int (12t - 20) dt$$

$$v = 12t^2/2 - 20t + C_1$$

$$v = 6t^2 - 20t + C_1$$

$$v = \frac{ds}{dt} = 6t^2 - 20t + C_1$$

$$ds = (6t^2 - 20t + C_1) dt$$

Integrating further to get s

$$s = \int (6t^2 - 20t + C_1) dt$$

$$s = \frac{6t^3}{3} - \frac{20t^2}{2} + C_1 t + C_2$$

$$s = 2t^3 - 10t^2 + C_1 t + C_2$$

$$\text{When } t = 0; s = -10$$

Substituting the above condition

$$-10 = C_2$$

$$s = 2t^3 - 10t^2 + C_1 t - 10$$

$$\text{When } t = 5; s = -10$$

$$10 = 2 \times 5^3 - 10 \times 5^2 + C_1 \times 5 - 10$$

$$5C_1 = 20; C_1 = 4$$

$$s = 2t^3 - 10t^2 + 4t - 10$$

**SP4.34:** The acceleration of a point in rectilinear motion is given by the equation,  $a = -9.8$ . It is known that the velocity  $v$  is zero and the displacement  $S$  is +25m when  $t=0$ . Determine the equation of the displacement.

**Solution;**

$$a = -9.8; \frac{dv}{dt} = -9.8$$

$$dv = -9.8 dt$$

Integrating both sides

$$dv = -9.8 dt$$

$$v = -9.8t + C_1$$

$$\text{When } v = 0 \text{ & } t = 0$$

$$0 = C_1$$

$$v = -9.8t$$

$$\frac{ds}{dt} = -9.8t$$

$$ds = -9.8t dt$$

Integrating both sides

$$\int ds = \int -9.8t dt$$

$$s = -9.8t^2/2 + C_2$$

$$s = -4.9t^2 + C_2$$

$$\text{When } t = 0; s = 25$$

$$25 = C_2$$

$$s = -4.9t^2 + 25$$

**SP4.35:** A particle moves along a horizontal straight line with an acceleration  $a = 6S^{1/3}$  when  $t = 2$ sec, its displacement  $S = +27$ m and its velocity  $v = +27$ m/s. Calculate the velocity and acceleration of the point when  $t = 4$ sec.

**Solution:**

$$a = 6s^{1/3},$$

$$\frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = 6s^{1/3}$$

$$v \cdot \frac{dv}{ds} = 6s^{1/3}$$

$$v \cdot dv = 6s^{1/3} \cdot ds$$

Integrating both sides

$$\int v \cdot dv = \int 6s^{1/3} \cdot ds$$

$$\frac{v^2}{2} - 6 \cdot \frac{s^{1/3+1}}{(1/3)+1} + c_1 = 6 \cdot \frac{s^{4/3}}{(4/3)} + c_1$$

$$\frac{v^2}{2} = \frac{9}{2} s^{4/3} + c_1$$

$$3s^{1/3} = 3t + c_2$$

When  $t = 2$ ;  $s = 27$ 

$$3 \times (27)^{1/3} = 3 \times 2 + c_2$$

$$9 = 6 + c_2$$

$$c_2 = 3$$

$$\therefore 3s^{1/3} = 3t + 3$$

$$v^2 = 9s^{4/3} + c_1$$

When  $s = 27$  m,  $v = 27$  m/s

$$27^2 = 9(27)^{4/3} + c_1$$

$$27^2 = 9 \times 81 + c_1$$

$$c_1 = 0$$

$$v^2 = 9s^{4/3}$$

$$v = 3s^{2/3}$$

$$\frac{ds}{dt} = 3s^{2/3}$$

$$s^{-2/3} \cdot ds = 3 dt$$

Integrating both sides

$$\int s^{-2/3} \cdot ds = \int 3 dt$$

$$\frac{s^{-2/3+1}}{-2/3+1} = -3t + C$$

$$\frac{s^{-1/3}}{-1/3+1} = -3t + C$$

$$s^{1/3} = t + 1$$

$$s = (t+1)^3$$

When  $t = 4$  s

$$s = 125$$
 m

$$v = 3(125)^{2/3}$$

$$v = 75$$
 m/s

$$a = 6(125)^{1/3} = 30 \text{ m/s}^2$$

**SP4.36:** A particle moves on a vertical line with an acceleration  $a = 2\sqrt{v}$  when  $t = 2$  sec,its displacement  $S = \frac{64}{3} m$  and its velocity  $v = 16$  m/s. Determine the displacement,velocity and acceleration of the particle when  $t = 3$  sec.**Solution:**

$$a = 2\sqrt{v}$$

$$\frac{dv}{dt} = 2\sqrt{v}$$

$$v^{1/2} \cdot dv = 2 dt$$

Integrating both sides

$$\frac{\sqrt{v}}{2} + 1 = 2t + c_1$$

$$2\sqrt{v} = 2t + c_1$$

When  $t = 2$ ;  $v = 16 \text{ m/s}$

$$2 \times (16)^{1/2} = 2 \times 2 + c_1$$

$$c_1 = 4$$

$$2\sqrt{v} = 2t + 4$$

$$\frac{\sqrt{v}}{2} = t + 2$$

$$v = (t + 2)^2$$

$$ds = (t + 2)^2 dt$$

Integrating both sides

$$s = [(t + 2)^3 / 3] + c_2$$

When  $t = 2$ ;  $s = 64/3$

$$64/3 = (2 + 2)^3 / 3 + c_2$$

$$64/3 = 64/3 + c_2$$

$$c_2 = 0$$

$$s = (t + 2)^3 / 3$$

When  $t = 3$

$$s = (3 + 2)^3 / 3 = 125/3 = 41.66 \text{ m}$$

$$v = (t + 2)^2 = (3 + 2)^2 = 25 \text{ m/s}$$

$$a = 2\sqrt{v}$$

$$= 2\sqrt{25} = 10 \text{ m/s}^2$$

**SP4.37:** A train starting from rest is uniformly accelerated. The acceleration at any instant is  $\frac{10}{(v+1)}$  m/s<sup>2</sup>, where v is the velocity of the train, in m/s at that instant. Find the distance in which the train will attain a velocity of 36kmph.

**Solution:**

$$a = \frac{10}{v+1}$$

$$\frac{dv}{dt} = \frac{10}{v+1}$$

$$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt} = \frac{10}{v+1}$$

$$\frac{dv}{ds} \cdot v = \frac{10}{v+1}$$

$$v dv (v+1) = 10 ds$$

$$(v^2 + v) dv = 10 ds$$

Integrating both sides

$$\frac{v^3}{3} + \frac{v^2}{2} = 10s + c_1$$

When  $v = 0, s = 0$

$$c_1 = 0$$

$$\therefore \frac{v^3}{3} + \frac{v^2}{2} = 10s$$

When  $v = 36 \text{ kmph} = 10 \text{ m/s}$

$$\frac{10^3}{3} + \frac{10^2}{2} = 10s$$

$$\frac{1000}{3} + \frac{100}{2} = 10s$$

$$s = 38.33 \text{ m}$$

**SP4.38:** A small projectile is fired vertically downwards into fluid medium with an initial velocity of 60 m/s. If the projectile experiences a deceleration which is equal to  $a = -(0.4v^3) \text{ m/s}^2$ . Where  $v$  is measured in m/s, determine the velocity  $v$  and position  $S$  in 4 seconds after the projectile is fired.

**Solution:**

$$a = -(0.4v^3)$$

$$\frac{dv}{dt} = -0.4v^3$$

$$v^{-3} \cdot dv = -0.4 dt$$

Integrating both sides

$$\frac{v^{-2+1}}{-3+1} = -0.4t + c_1$$

$$\frac{v^{-2}}{-2} = -0.4t + c_1$$

When  $t = 0, v = 60 \text{ m/s}$

$$\frac{60^{-2}}{-2} = 0 + c_1$$

$$v = (0.8t - 2.77 \times 10^{-4})^{-1/2}$$

$$ds/dt = (0.8t - 2.77 \times 10^{-4})^{-1/2}$$

$$ds = (0.8t - 2.77 \times 10^{-4})^{-1/2} dt$$

Integrating both sides

$$s = \frac{(0.8t - 2.77 \times 10^{-4})^{-\frac{1}{2}+1}}{(-\frac{1}{2}+1) \times 0.8} + c_2$$

$$s = \frac{(0.8t - 2.77 \times 10^{-4})^{-\frac{1}{2}+1}}{0.4} + c_2$$

$$c_1 = -\frac{1}{7200}$$

$$\frac{v^{-2}}{-2} = -0.4t - \frac{1}{7200}$$

Multiplying by -2

$$\frac{1}{v^2} = 0.8t + \frac{1}{3600}$$

$$\frac{1}{v} = \sqrt{0.8t + \frac{1}{3600}}$$

When  $t = 0, s = 0$

$$0 = \frac{(-2.77 \times 10^{-4})^{\frac{1}{2}}}{0.4} + c_2$$

$$c_2 = 0.04166$$

$$\therefore s = \frac{(0.8t - 2.77 \times 10^{-4})^{\frac{1}{2}}}{0.4} + 0.04166$$

When  $t = 4$

$$s = \frac{(0.8 \times 4 - 2.77 \times 10^{-4})^{\frac{1}{2}}}{0.4} + 0.04166$$

$$s = 4.51 \text{ m}$$

$$v = (0.8 \times 4 - 2.77 \times 10^{-4})^{-1/2}$$

$$v = 0.559 \text{ m/s}$$

**SP4.39:** A car is moving with a velocity of 72kmph. After seeing a child on the road brakes are applied and the vehicle is stopped in a distance of 15m. If the retardation produced is proportional to the point where brakes are applied, find the expression of retardation.

**Solution:**

- $a$  is prop to  $s$
- $a = k.s$
- $\frac{dv}{dt} = -k.s$
- $\frac{dv}{ds} \times \frac{ds}{dt} = -k.s$
- $\frac{dv}{ds} \cdot v = -k.s$
- $v \cdot dv = -k.s \cdot ds$

**Integrating both sides**

$$\frac{v^2}{2} = -k \cdot \frac{s^2}{2} + c_1$$

When  $s = 0, v = 20 \text{ m/s}$   
 $20^2/2 = c_1 = 200$

$$\frac{v^2}{2} = -k \cdot \frac{s^2}{2} + 200$$

When  $s = 15 \text{ m}, v = 0$   
 $0 = -k \cdot (15^2/2) + 200$   
 $K = 1.778$   
 $a = -1.778s$

**SP4.40:** A particle starts with velocity  $v_0$ , its acceleration and velocity are related by the equation  $a = -kv$ , where  $k$  is a constant  $v$  is the velocity of the particle  $a$  is the acceleration of the particle. Find the displacement time relation.

**Solution:**

$$a = -kv$$

$$\frac{dv}{dt} = -kv$$

$$\frac{dv}{v} = -k \cdot dt$$

**Integrating both sides!**

$$\log v = -kt + C_1$$

$$\text{when } t = 0, v = v_0$$

$$\log v_0 = C_1$$

$$\log v = -kt + \log v_0$$

$$\log v - \log v_0 = -kt$$

$$\frac{v}{v_0} = e^{-kt}$$

$$v_0$$

$$v = v_0 e^{-kt}$$

$$\frac{ds}{dt} = v_0 e^{-kt}$$

$$ds = v_0 e^{-kt} \cdot dt$$

**Integrating**

$$s = \frac{v_0 e^{-kt}}{-k} + C_2$$

$$\text{When } t = 0, s = 0$$

$$0 = \frac{v_0 e^0}{-k} + C_2$$

$$C_2 = v_0/k$$

$$s = \frac{v_0 e^{-kt}}{-k} + \frac{v_0}{k}$$

$$s = \frac{v_0}{k} [1 - e^{-kt}]$$

**SP4.41:** A particle moves along a horizontal straight line such that its velocity is given by  $v(m/s) = 3t^2 - 6t$ , where  $t$  is the time in second. If it is initially located at the origin O, determine the distance travelled during the time interval  $t = 0$  to  $t = 3.5$  sec., the average velocity and the average speed of the particle during this time interval.

**Solution:**

$$v = (3t^2 - 6t)$$

$$\frac{ds}{dt} = (3t^2 - 6t)$$

$$ds = (3t^2 - 6t)dt$$

**Integrating both sides**

$$s = \frac{3t^3}{3} - \frac{6t^2}{2} + C_1$$

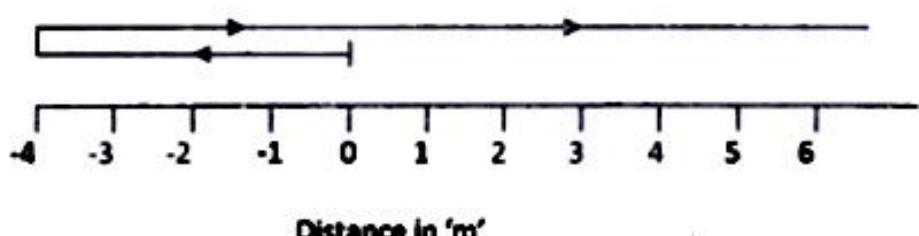


Fig.4.11

$$s = t^3 - 3t^2 + c_1$$

When  $t = 0, s = 0$

$$c_1 = 0$$

$$s = t^3 - 3t^2$$

when  $t = 0, s = 0$

when  $t = 1, s = 1 - 3 = -2 \text{ m}$

when  $t = 2, s = 8 - 12 = -4 \text{ m}$

when  $t = 3, s = 27 - 27 = 0 \text{ m}$

when  $t = 3.5,$

$$s = 3.5^3 - 3 \cdot 3.5^2 = 6.125 \text{ m}$$

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time}}$$

$$= \frac{0 - 6.125}{3.5} = 1.75 \text{ m/s}$$

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time}}$$

$$= \frac{4 + 4 + 6.125}{3.5} = 4.035 \text{ m/s}$$

## PLANE CURVILINEAR MOTION:

The motion of a particle along a curved path which lies in a single plane. This motion is a special case of the more general three dimensional motion as discussed above and illustrated in fig.4.1. If let the plane of motion be the x-y plane, for instance, then the coordinates  $z$  and  $\phi$  of fig.4.1 are both zero and  $R$  becomes the same as  $r$ . As mentioned previously, the vast majority of the motions of points or particles encountered in engineering practice can be represented as plane motion.

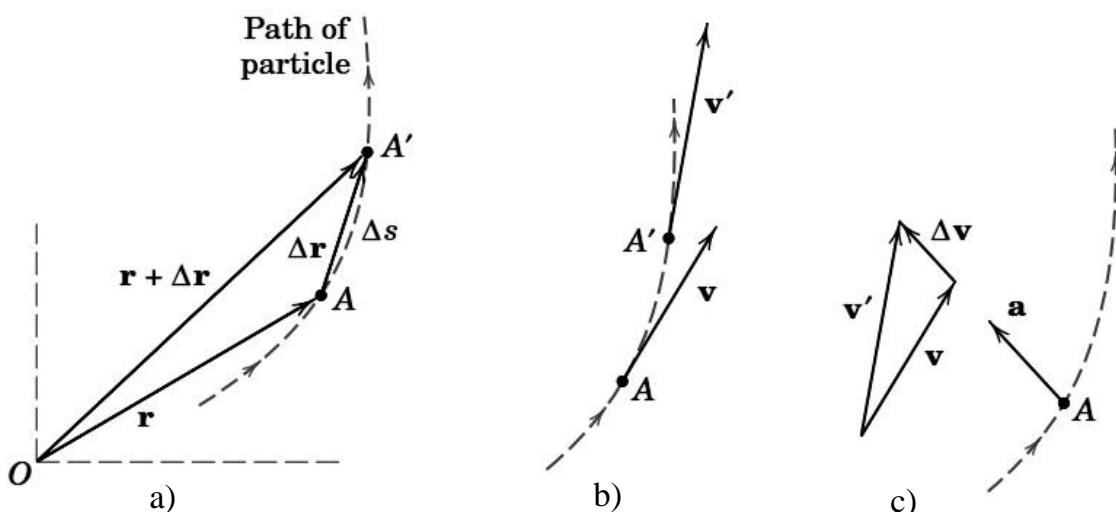


Fig.4.12 Plane curve linear motion

Consider now the continuous motion of a particle along a plane curve as represented in fig.4.12. At time 't' the particle is at position A, which is located by the position vector ' $\mathbf{r}$ ' measured from some convenient fixed origin O. If both the

magnitude and direction of 'r' are known at time 't', then the position of the particle is completely specified. At time  $t + \Delta t$ , the particle is at A', located by the position vector  $r + \Delta r$ . This combination is vector addition and not scalar addition.

### a) Displacement

The displacement of the particle during time  $\Delta t$  is the vector  $\Delta r$  which represents the vector change of position and is clearly independent of the choice of origin. If an origin were chosen at some different location, the position vector 'r' would be changed, but  $\Delta r$  would be unchanged. The distance actually travelled by the particle as it moves along the path from A to A' is the scalar length  $\Delta s$  measured along the path. Thus, it distinguishes between the vector displacement  $\Delta r$  and the scalar distance  $\Delta s$ .

### b) Velocity

The average velocity of the particle between A and A' is defined as  $v_{av} = \frac{\Delta r}{\Delta t}$ , which is a vector whose direction is that of  $\Delta r$  and whose magnitude is the magnitude of  $\Delta r$  divided by  $\Delta t$ . The average speed of the particle between A and A' is the scalar quotient  $\frac{\Delta s}{\Delta t}$ . Clearly, the magnitude of the average velocity and the speed approach one another as the interval  $\Delta t$  decreases and A and A' become closer together. The instantaneous velocity 'v' of the particle is defined as the limiting value of the average velocity as the time interval approaches zero. Thus,

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t} \quad \longrightarrow (4.12)$$

It is observed that the direction of  $\Delta r$  approaches that of the tangent to the path as  $\Delta t$  approaches zero and, thus, the velocity  $v$  is always a vector tangent to the path.

Now extend the basic definition of the derivative of a scalar quantity to include a vector quantity and write

$$v = \frac{dr}{dt} = \dot{r} \quad \longrightarrow (4.13)$$

The derivative of a vector is itself a vector having both a magnitude and a direction. The magnitude of  $v$  is called the speed and is the scalar

$$v = |v| = \frac{ds}{dt} = \dot{s} \quad \longrightarrow \quad (4.14)$$

At this point it makes a careful distinction between the magnitude of the derivative and the derivative of the magnitude.

The magnitude of the derivative can be written in any one of the several ways

$\left| \frac{dr}{dt} \right| = \left| \dot{r} \right| = \dot{s} = |v| = v$  and represents the magnitude of the velocity, or the speed, of the particle.

On the other hand, the derivative of the magnitude is written  $\frac{d|r|}{dt} = \frac{dr}{dt} = \dot{r}$ , and represents the rate at which the length of the position vector  $r$  is changing.

Thus, these two derivatives have two entirely different meanings, and it must be extremely careful to distinguish between them in thinking and in notation. For this and other reasons, these are urged to adopt a consistent notation for handwritten work for all vector quantities to distinguish them from scalar quantities. For simplicity the underline  $\underline{v}$  is recommended. Other handwritten symbols such as  $\vec{v}$ ,  $\underline{v}$  and  $\hat{v}$  are sometimes used.

With the concept of velocity as a vector established, by referring fig.4.12 and denote the velocity of the particle at A by the tangent vector  $v$  and the velocity at A' by the tangent  $v'$ . Clearly, there is a vector change in the velocity during the time  $t$ . The velocity  $v$  at A plus (vectorially) the change  $\Delta v$  must equal the velocity at A', so we can write  $v' - v = \Delta v$ . Inspection of the vector diagram shows that  $\Delta v$  depends both on the change in magnitude (length) of  $v$  and on the change in direction of  $v$ . These two changes are fundamental characteristics of the derivative of a vector.

### c) Acceleration

The average acceleration of the particle between A and A' is defined as  $\frac{\Delta v}{\Delta t}$ ,

which is a vector whose direction is that of  $\dot{v}$ . The magnitude of this average acceleration is the magnitude of  $\dot{v}$  divided by  $\Delta t$ . The instantaneous acceleration  $a$  of the particle is defined as the limiting value of the average acceleration as the time interval approaches zero. Thus,

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$$

By definition of the derivative, then

$$a = \frac{dv}{dt} = \dot{v} \quad \text{-----} \rightarrow (4.15)$$

As the interval  $\Delta t$  becomes smaller and approaches zero, the direction of the change  $\Delta v$  approaches that of the differential change  $dv$  and, thus, of  $a$ . The acceleration  $a$ , then, includes the effects of both the change in magnitude of  $v$  and the change of direction of  $v$ . It is apparent, in general, that the direction of the acceleration of a particle in curvilinear motion is neither tangent to the path nor normal to the path. It observes that the acceleration component which is normal to the path points toward the centre of curvature of the path.

## 1) RECTANGULAR COORDINATES (X- Y):

This system of coordinates is particularly useful for describing motions where the x- and y-components of acceleration are independently generated or determined. The resulting curvilinear motion is then obtained by a vector combination of the x- and y-components of the position vector, the velocity, and the acceleration.

### a) Vector Representation

The particle path of fig.4.12 is shown again in fig.4.13 along with x- and y-axes. The position vector  $r$ , the velocity  $v$ , and the acceleration 'a' of the particle as developed

in previous are represented in fig.4.13 together with their x- and y-components. With the aid of the unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , it can write the vectors  $\mathbf{r}$ ,  $\mathbf{v}$ , and  $\mathbf{a}$  in terms of their x- and y-components. Thus,

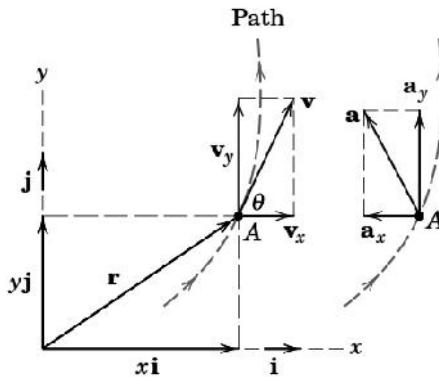


Fig.4.13 Particle motion in x and y plane

$$\left. \begin{array}{l} \mathbf{r} = x\mathbf{i} + y\mathbf{j} \\ \mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} \\ \mathbf{a} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} \end{array} \right\} \longrightarrow (4.16)$$

As differentiate with respect to time, it observes that the time derivatives of the unit vectors are zero because their magnitudes and directions remain constant. The scalar values of the components of  $\mathbf{v}$  and  $\mathbf{a}$  are merely  $v_x = \dot{x}$ ,  $v_y = \dot{y}$ ,  $a_x = \ddot{v}_x = \ddot{x}$  and  $a_y = \ddot{v}_y = \ddot{y}$ . (As drawn in fig.4.13,  $\mathbf{a}$  is in the negative x-direction, so that  $\ddot{x}$  would be a negative number). As observed previously, the direction of the velocity is always tangent to the path, and from the figure it is clear that

$$v^2 = v_x^2 + v_y^2, \quad v = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan \alpha = \frac{v_y}{v_x}$$

$$a^2 = a_x^2 + a_y^2 \quad \text{and} \quad a = \sqrt{a_x^2 + a_y^2}$$

If the angle  $\alpha$  is measured counter clockwise from the x-axis to  $\mathbf{v}$  for the configuration of axes shown, then it can also observe that  $\frac{dy}{dx} = \tan \alpha = \frac{v_y}{v_x}$ . If the coordinates  $x$  and  $y$  are known independently as functions of time,  $x = f_1(t)$  and  $y =$

$f_2(t)$ , then for any value of the time it can combine them to obtain  $r$ . Similarly, combine their first derivatives  $\dot{x}$  and  $\dot{y}$  to obtain  $v$  and their second derivatives  $\ddot{x}$  and  $\ddot{y}$  to obtain  $a$ . On the other hand, if the acceleration components  $a_x$  and  $a_y$  are given as functions of the time, it can integrate each one separately with respect to time, once to obtain  $v_x$  and  $v_y$  and again to obtain  $x = f_1(t)$  and  $y = f_2(t)$ . Elimination of the time  $t$  between these last two parametric equations gives the equation of the curved path  $y = f(x)$ .

From the foregoing discussion it can see that the rectangular coordinate representation of curvilinear motion is merely the superposition of the components of two simultaneous rectilinear motions in the  $x$ - and  $y$ -directions. Therefore, everything covered in rectilinear motion and it can be applied separately to the  $x$ -motion and to the  $y$ -motion.

### b) Projectile Motion

An important application of two-dimensional kinematic theory is the problem of projectile motion. For a first treatment of the subject, it neglects aerodynamic drag and the curvature and rotation of the earth, and assume that the altitude change is small enough so that the acceleration due to gravity can be considered constant. With these assumptions, rectangular coordinates are useful for the trajectory analysis.

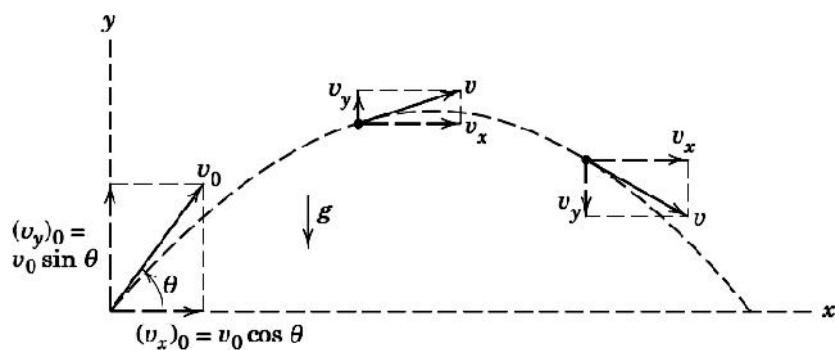


Fig.4.14 Projectile motion

For the axes shown in fig.4.14, the acceleration components are

$$a_x = 0 \text{ and } a_y = -g$$

Integration of these accelerations follows the results obtained previously for uniform/ constant acceleration motion and yields

$$v_x = (v_x)_0 \text{ and } v_y = (v_y)_0 - gt$$

$$x = x_0 + (v_x)_0 t \text{ and } y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2$$

$$v_y^2 = (v_y)_0^2 - 2g(y - y_0)$$

In all these expressions, the subscript zero denotes initial conditions, frequently taken as those at launch where, for the case illustrated,  $x_0 = y_0 = 0$ . Note that the quantity  $g$  is taken to be positive throughout this text.

### Equation of the projectile:

Consider a particle projected from a point 'O' at an angle of  $\alpha$  from the horizontal. Let 'u' be the velocity of projection. Consider any point  $P(x,y)$  as the position of particle after time 't' sec with x and y co-ordinates.

Vertical distance travelled after time  $t = y$

Using the relation  $S = ut + at^2/2$

$$\text{We can get } y = (u \sin \alpha) t + (-g) t^2/2$$

Horizontal distance moved after time  $t = x$

### Using the relation

We can get

$$S = \sqrt{t}$$

$$t = x / u \cos \alpha \quad (ii)$$

substituting eqn (ii) in (i)

$$v = (u \sin \alpha / u \cos \alpha) x - g x^2 / 2(u \cos \alpha)^2$$

$$y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$$

Since the above equation is similar to the equation of parabola (i.e.,  $y = ax + bx^2$ ), the path of the projectile is a parabola.

### Time of flight of a projectile on a horizontal plane :-

It is the time for which the projectile has remained in the air.

We know that at the end of a projectile motion (i.e., after reaching the ground) the displacement is zero.

$$\begin{aligned} y &= u \sin \alpha \cdot t - gt^2/2 \quad \text{from eqn.(i)} \\ 0 &= u \sin \alpha \cdot t - gt^2/2 \\ u \sin \alpha \cdot t &= gt^2/2 \\ t &= \frac{2u \sin \alpha}{g} \end{aligned}$$

### Horizontal range of a projectile :

Horizontal component of the velocity of projection

$$u_x = u \cos \alpha.$$

$$\text{We know that } t = 2 u \sin \alpha / g$$

$$\text{Range (R)} = \text{Horizontal velocity} \times \text{time} = u \cos \alpha \times 2 u \sin \alpha / g$$

$$R = \frac{u^2 \sin 2\alpha}{g} \quad (\text{since } \sin 2\alpha = 2 \sin \alpha \cos \alpha)$$

### Maximum range:

In the above equation R is maximum only when  $\sin 2\alpha$  is maximum

$$\text{i.e., } \sin 2\alpha = 1 \quad 2\alpha = 90^\circ; \quad \alpha = 45^\circ$$

Therefore R is maximum when  $\alpha = 45^\circ$

$$R = u^2 \sin 2\alpha / g$$

$$R_{\max} = u^2 \sin 90^\circ / g$$

$$R_{\max} = \frac{u^2}{g}$$

**Maximum height of a projectile on a horizontal plane:****In the vertical direction**the initial velocity  $u_y = u \sin \alpha$ the final velocity  $v_y = 0$ from the equation  $v^2 = u^2 - 2gs$ ,we can write  $v_y^2 = u_y^2 - 2gH$ where  $H$  = Max. height reached by the projectile

substituting the values for the initial and final velocities in the above equation

we get  $0 = (u \sin \alpha)^2 - 2gH$ Therefore  $2gH = u^2 \sin^2 \alpha$ 

$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

**Motion of the projectile on an inclined plane:**

Consider a projectile projected from O on an upward inclined plane OA. Let the projectile strike the plane at B as

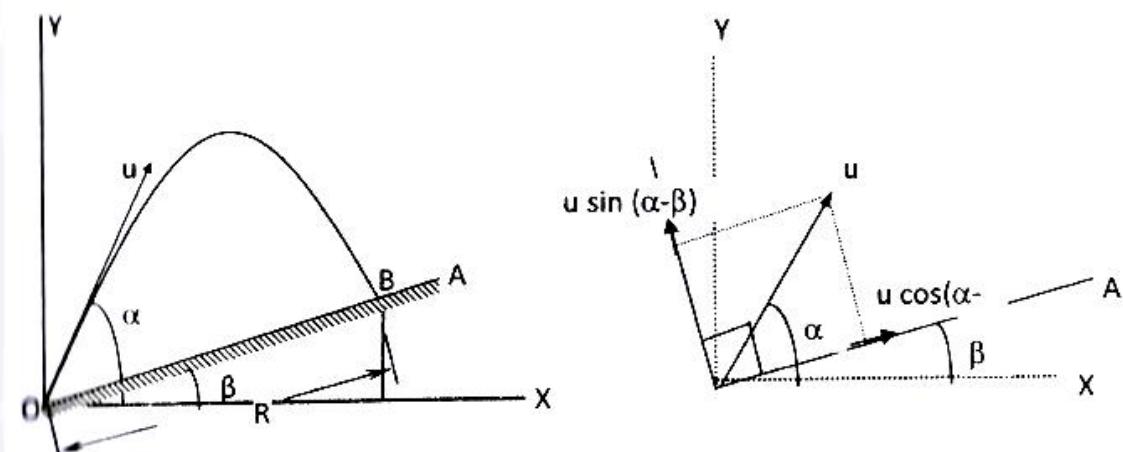


Fig.4.15 Motion of projectile

shown in fig.

Let  $u$  – velocity of projection;

- $\alpha$  - the angle of projection;
- $\beta$  - inclination of the plane OA with the horizontal;
- R - range of flight from O to B and
- t - time of flight from O to B.

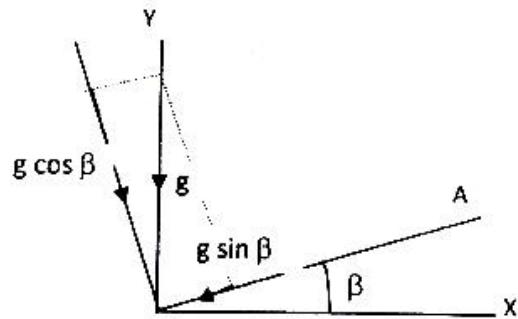


Fig.4.16

Component of velocity of projection normal to the inclined plane  $= u \sin(\alpha - \beta)$

Acceleration due to gravity normal to the plane OA

$$= g \cos \beta$$

Consider the motion of the projectile normal to the plane. We know that, the distance covered by the projectile after time 't' normal to the plane OA is zero.

$$\begin{aligned} \text{i.e., } S &= ut - gt^2/2 \\ 0 &= u \sin(\alpha - \beta)t - g \cos \beta t^2/2 \\ g \cos \beta t/2 &= u \sin(\alpha - \beta) \quad t = \frac{2u \sin(\alpha - \beta)}{g \cos \beta} \end{aligned}$$

To find the range along the inclined plane::

$$\begin{aligned} \text{The horizontal range } OC &= u \cos \alpha \cdot t \\ &= u \cos \alpha \cdot 2u \sin(\alpha - \beta) / (g \cos \beta) \\ &= 2u^2 \cdot \sin(\alpha - \beta) \cos \alpha / (g \cos \beta) \end{aligned}$$

$$\begin{aligned} \text{Range along OA} &= OB = OC / \cos \beta \\ &= 2u^2 \cdot \sin(\alpha - \beta) \cos \alpha / (g \cos^2 \beta) \end{aligned}$$

$$\begin{aligned} \text{Range } R &= [u^2 / g \cos^2 \beta] [2 \sin(\alpha - \beta) \cos \alpha] \\ \text{since } 2 \cos A \sin B &= \sin(A+B) - \sin(A-B) \end{aligned}$$

$$R = \frac{u^2}{g \cos^2 \beta} [\sin(2\alpha - \beta) - \sin \beta]$$

To get max. Range,

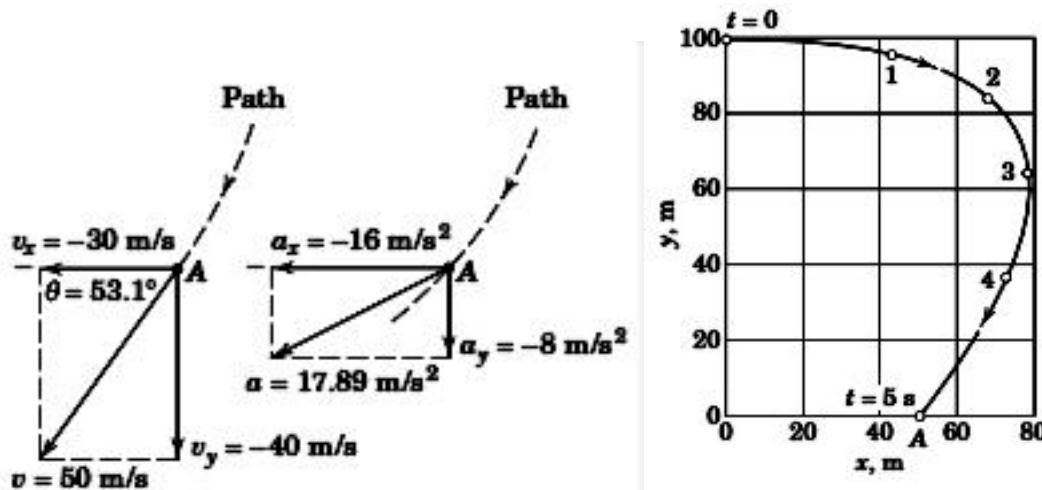
$$\begin{aligned} \sin(2\alpha - \beta) &= 1 \\ (2\alpha - \beta) &= 90^\circ = \pi/2 \\ \alpha &= \pi/4 + \beta/2 \end{aligned}$$

$$\begin{aligned} \text{Therefore } R_{\max} &= [u^2 / g \cos^2 \beta] [\sin(2(\pi/4 + \beta/2) - \beta) - \sin \beta] \\ &= [u^2 / g \cos^2 \beta] [\sin(\pi/2) - \sin \beta] \\ &= [u^2 / g \cos^2 \beta] [1 - \sin \beta] \\ &= u^2 [1 - \sin \beta] / g [1 - \sin^2 \beta] \end{aligned}$$

$$R_{\max} = \frac{u^2}{g(1 + \sin \beta)}$$

**Problems:**

**SP.4.42:** The curvilinear motion of a particle is defined by  $v_x = 50 - 16t$  and  $y = 100 - 4t^2$ , where  $v_x$  is in metres per second,  $y$  is in metres and  $t$  is in seconds. It is also known that  $x = 0$  when  $t = 0$ . Plot the path of the particle and determine its velocity and acceleration when the position  $y = 0$  is reached.

**Solution:**

The  $x$ -coordinate is obtained by integrating the expression for  $v_x$ , and the  $x$ -component of the acceleration is obtained by differentiating  $v_x$ . Thus,

$$\left[ \int dx = \int v_x dt \right] \quad \int_0^x dx = \int_0^t (50 - 16t) dt \quad x = 50t - 8t^2 \text{ m}$$

$$[a_x = \dot{v}_x] \quad a_x = \frac{d}{dt} (50 - 16t) \quad a_x = -16 \text{ m/s}^2$$

The  $y$ -components of velocity and acceleration are

$$[v_y = \dot{y}] \quad v_y = \frac{d}{dt} (100 - 4t^2) \quad v_y = -8t \text{ m/s}$$

$$[a_y = \dot{v}_y] \quad a_y = \frac{d}{dt} (-8t) \quad a_y = -8 \text{ m/s}^2$$

We now calculate corresponding values of  $x$  and  $y$  for various values of  $t$  and plot  $x$  against  $y$  to obtain the path as shown.

When  $y = 0$ ,  $0 = 100 - 4t^2$ , so  $t = 5 \text{ s}$ . For this value of the time, we have

$$v_x = 50 - 16(5) = -30 \text{ m/s}$$

$$v_y = -8(5) = -40 \text{ m/s}$$

$$v = \sqrt{(-30)^2 + (-40)^2} = 50 \text{ m/s}$$

$$a = \sqrt{(-16)^2 + (-8)^2} = 17.89 \text{ m/s}^2$$

The velocity and acceleration components and their resultants are shown on the separate diagrams for point A, where  $y = 0$ . Thus, for this condition we may write

$$\mathbf{v} = -30\mathbf{i} - 40\mathbf{j} \text{ m/s} \quad \text{Ans.}$$

$$\mathbf{a} = -16\mathbf{i} - 8\mathbf{j} \text{ m/s}^2 \quad \text{Ans.}$$

**SP4.43:** A team of engineering students designs a medium size catapult which launches 3.6kg steel spheres. The launch speed is  $v_0 = 24.4 \text{ m/s}$ , the launch angle is  $\theta = 35^\circ$  above the horizontal, and the launch position is 1.8m above ground level. The students use an athletic field with an adjoining slope topped by an 2.4m fence as shown in fig4.17. Determine:

- (a) the time duration  $t_f$  of the flight ,
- (b) the x-y coordinates of the point of first impact,
- (c) the maximum height  $h$  above the horizontal field attained by the ball and
- (d) the velocity (expressed as a vector) with which the projectile strikes the ground (or the fence)

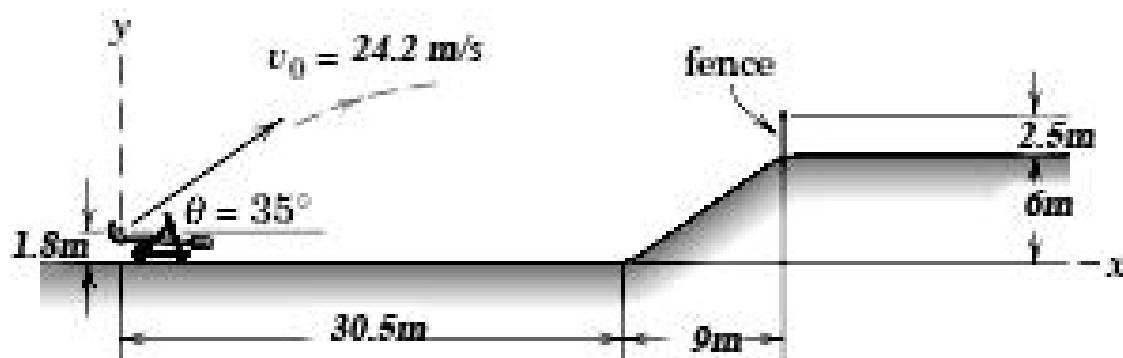


Fig.4.17

**Solution:**

Now make the assumption of constant gravitational acceleration and no aerodynamic drag. With the latter assumption, the 3.6 kg weight of the projectile is irrelevant. Using the given x-y coordinate system, this is being by checking the y-displacement at the horizontal position of the fence.

$$x = x_0 + (v_x)_0 t, \Rightarrow 30.5 + 9 = 0 + 24.2 \cos 35^\circ t, \Rightarrow t = 1.994$$

$$y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2, \Rightarrow y = 1.8 + 24.2 \sin 35^\circ (1.994) - \frac{1}{2} (9.81) (1.994)^2,$$

$$y = 9.976m$$

- a) Because the y – coordinate of the top of the fence is  $6 + 2.5 = 8.5m$ , the projectile clears the fence. Now find the flight time by setting  $y = 6m$ ,

$$y = y_0 + (v_y)_0 t_f - \frac{1}{2} g t_f^2, \Rightarrow 6 = 1.8 + 24.2 \sin 35^\circ (t_f) - \frac{1}{2} (9.81) t_f^2, t_f = 2.485s$$

$$x = x_0 + (v_x)_0 t_f, \Rightarrow x = 0 + 24.2 \cos 35^\circ (2.485), \Rightarrow x = 49.26m$$

- b) Thus the point of first impact is  $(x, y) = (49.26, 6)$   
 c) For the maximum height:

$$v_y^2 = (v_y)_0^2 - 2g(y - y_0), \Rightarrow 0^2 = (24.2 \sin 35^\circ)^2 - 2(9.81)(h - 1.8), h = 11.489m$$

- d) For the impact velocity:

$$v_x = (v_x)_0, \Rightarrow v_x = 24.2 \cos 35^\circ = 19.823m/s$$

$$v_y = (v_y)_0 - gt, v_y = 24.2 \sin 35^\circ - (9.81)(2.485) = -10.498m/s$$

So the impact velocity is  $v(m/s) = 19.823i - 10.498j$

**SP4.44:** An aircraft moving horizontal at 108 kmph at an altitude of 1000m towards a target on the ground, releases a bomb which hits it. Estimate the horizontal distance of the aircraft from the target, when it released the bomb. Calculate also the direction and magnitude of the velocity with which the bomb hits the target.

**Solution:**

Horizontal velocity of bomb = velocity of aircraft

$$= 108 \text{ kmph} = 30 \text{ m/sec}$$

Vertical component to velocity of the bomb = 0

Considering downward motion of the bomb

$$u = 0; a = g; s = 1000 \text{ m}$$

$$s = ut + \frac{1}{2} at^2$$

$$1000 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

$$t = \sqrt{\frac{1000}{4.905}}$$

$$t = 14.278 \text{ s}$$

$$v = u + at$$

$$= 0 + 9.8 \times 14.278$$

$$v = 140.06 \text{ m/sec}$$

Considering the horizontal motion of the bomb

$$v = 30 \text{ m/sec}; t = 14.27 \text{ sec}$$

$$s = v \times t = 30 \times 14.278 = 428.35 \text{ m}$$

To find the bomb velocity:

$$v_y = 140.06 \text{ m/sec}$$

$$v_x = 30 \text{ m/sec}$$

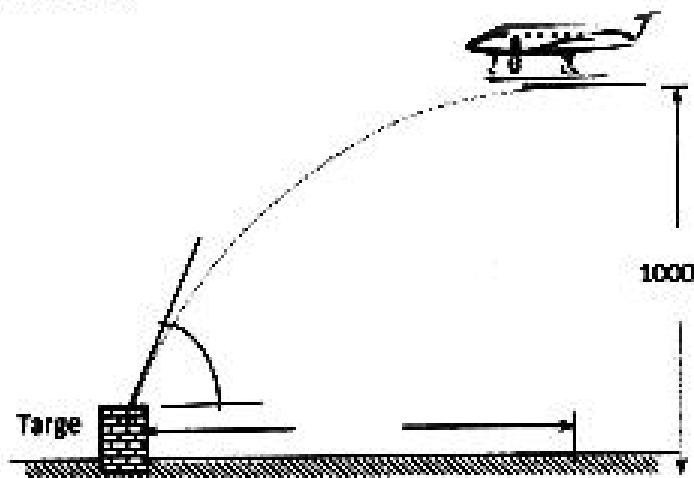


Fig.4.18

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{140.06^2 + 30^2}$$

$$v = 143.24 \text{ m/sec}$$

$$\tan \alpha = \frac{v_y}{v_x} = \frac{140.06}{30}; \alpha = 77.9^\circ$$

**SP4.45:** A motor cyclist wants to jump over a ditch as shown in figure. Find the necessary minimum velocity 'u' at 'A' in kmph of the motor cycle. Also find the inclination and the magnitude of the velocity of the motor cycle just after clearing the ditch.

**Solution:**

Considering the downward motion of the motor cycle:

Since the motor cycle is moving in the horizontal direction, the vertical component of its velocity is zero.

$$u = 0$$

$$a = 9.81 \text{ m/sec}^2$$

$$s = 2 \text{ m}$$

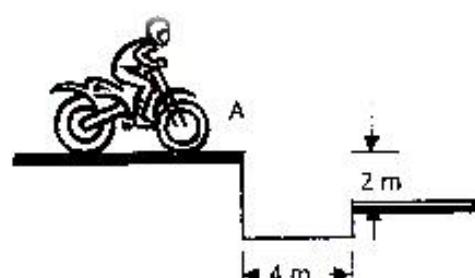


Fig.4.19

$$t = \sqrt{\frac{2}{4.905}}$$

$$t = 0.6385 \text{ sec}$$

$$s = ut + \frac{1}{2} at^2$$

$$2 = 0 + \frac{1}{2} \times 9.81 \times t^2$$

**Vertical component of the velocity of the motor cycle after clearing the ditch (i.e., after traveling the vertical distance of 2 m),**

$$v_y = u + at$$

$$= 0 + 9.81 \times 0.6385$$

$$v_y = 6.264 \text{ m/sec}$$

**Considering the horizontal motion of the motor cycle**

$$s = v \times t$$

$$4 = v \times 0.638$$

$$v = 4/0.638 = 6.264 \text{ m/s} = 22.55 \text{ kmph}$$

**The minimum velocity at A = 22.55 kmph**

**Velocity after clearing the ditch**

$$v = \sqrt{6.264^2 + 6.264^2} = 8.858 \text{ m/sec} \\ = 31.89 \text{ kmph}$$

$$\theta = \tan^{-1} \frac{6.264}{6.264}$$

**$\theta = 45^\circ$ ; The inclination of the motorbike while hitting the road =  $45^\circ$**

**SP4.46:** A shot is fired horizontally from the top of a tower with velocity of 100 m/s. If the shot hits the ground after 2 sec, find the height of the tower and the distance from the foot of the tower, where the strikes the ground.

**Solution:**

**Considering vertical motion of the shot**

$$u = 0, a = 9.81 \text{ m/s}^2; t = 2 \text{ sec}$$

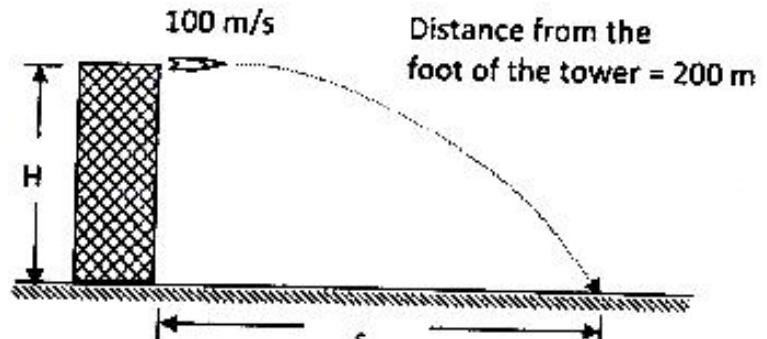
$$s = ut + \frac{1}{2} at^2$$

$$H = 0 + \frac{1}{2} \times 9.81 \times 2^2$$

$$H = 19.62 \text{ m}$$

$$\text{Height of the tower} = 19.62 \text{ m}$$

$$s = vt = 100 \times 2 = 200 \text{ m};$$



**SP4.47:** An airplane is flying on a straight level course at 200 kmph at a height of 1000m above the ground. An anti-craft gun located on the ground fires a shell with an initial velocity of 300m/s at the instant when the plane is vertically above it. At what inclination, to the horizontal should the gun be fired to hit the plane? What time after firing the gun shell will hit the plane? What will then be the horizontal distance of the plane from the gun?

**Solution:**

Plane horizontal velocity = 55.55m/sec  
 Velocity of the shell = 300m/sec  
Considering the vertical motion of the shell

Vertical component of the shell

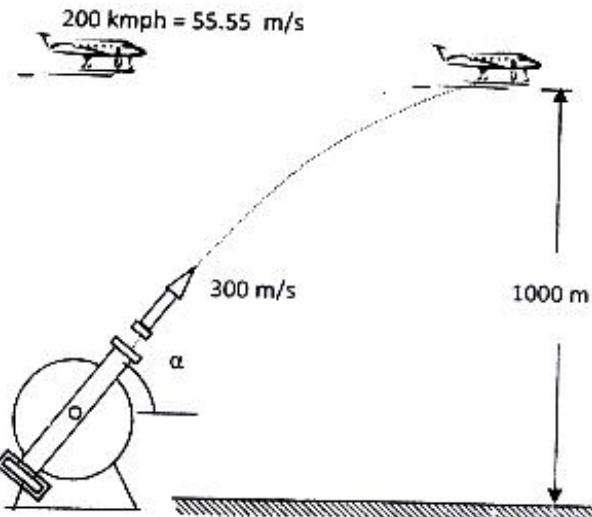
$$u_y = 300 \sin \alpha$$

$$a = -g$$

$$t = t$$

$$s = u t + \frac{1}{2} a t^2$$

$$1000 = 300 \sin \alpha t - \frac{1}{2} \times 9.81 t^2 \dots (1)$$



Considering the horizontal motion of the shell

Horizontal component of the shell =  $u \cos \alpha$

Horizontal distance moved by the shell =  $u \cos \alpha \times t = 300 \cos \alpha \times t$

Horizontal distance moved by the plane in time 't' =  $55.55 t$

Distance moved by the plane = distance moved by the shell

$$55.55 t = 300 \cos \alpha \times t$$

$$\cos \alpha = 55.55 / 300 ; \alpha = 79.33^\circ$$

Substituting the value of 'α' in equation (1) We get

$$1000 = 300 \sin 79.33 t - 4.905 t^2$$

$$4.905 t^2 - 294.81 t + 1000 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-294.81 \pm \sqrt{294.81^2 - 4 \times 4.905 \times 1000}}{2 \times 4.905}$$

$$t = 3.608 \text{ sec}$$

$$s = vt$$

$$s = 55.55 \times 3.608 = 200 \text{ m}$$

**SP4.48:** If a particle is projected inside a horizontal tunnel which is 5m high with a velocity of 60m/s., find the greatest possible range and the corresponding angle of projection.

**Solution:**

$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

$$S = \frac{(60)^2 \sin^2 \alpha}{2(9.81)}$$

$$\sin^2 \alpha = \frac{98.1}{3600} = 0.02725$$

$$\sin \alpha = 0.16507$$

$$\alpha = 9.50^\circ$$

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{60^2 \sin 2 \times 9.5}{9.81}$$

$$R = 119.5 \text{m}$$

**SP4.49:** A particle is projected from the base of a hill whose shape is that of a right circular cone with axis vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If  $\theta$  be the semi-cone angle,  $h$  its height,  $u$  the initial velocity of the projectile and  $\alpha$  the angle of projection measured from the horizontal show that  $\tan r = 2 \cot \alpha$  and  $u^2 = gh \left[ 2 + \frac{\tan^2 \alpha}{2} \right]$ .

**Solution:**

we know that,

$$h = \frac{u^2 \sin^2 \alpha}{2g}; R = \frac{u^2 \sin 2\alpha}{g}$$

$$\frac{h}{R} = \frac{\sin^2 \alpha}{2 \sin 2\alpha} = \frac{\sin^2 \alpha}{4 \sin \alpha \cos \alpha} = \frac{\tan \alpha}{4} \quad \dots(1)$$

$$\text{But } \cot \theta = \frac{h}{\left(\frac{R}{2}\right)} = \frac{2h}{R} \quad \dots(2)$$

$$\text{Substitute (2) in (1)} \frac{h}{R} = \frac{\cot \theta}{2}$$

$$\frac{\cot \theta}{2} = \frac{\tan \alpha}{4}$$

$$\tan \alpha = \frac{4}{2} \cot \theta = 2 \cot \theta$$

$$\cot \alpha = \frac{1}{2 \cot \theta} = \frac{\tan \theta}{2}$$

$$h = \frac{u^2 \sin^2 \alpha}{2g}; \quad 2gh = u^2 \sin^2 \alpha$$

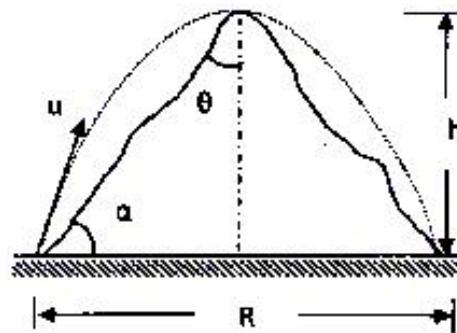


Fig.4.22

$$\begin{aligned} \frac{u^2}{gh} &= \frac{2}{\sin^2 \alpha} = 2 \cosec^2 \alpha \\ &= 2(1 + \cot^2 \alpha) = 2(1 + \frac{\tan^2 \theta}{4}) \\ &= 2 + \frac{\tan^2 \theta}{2} \end{aligned}$$

$$u^2 = gh \left[ 2 + \frac{\tan^2 \theta}{2} \right]$$

**SP4.50:** A cricket ball shot by a batsman from a height of 1.8m at an angle of  $30^\circ$  with horizontal with a velocity of 18m/s is caught by a field man at a height of 0.6m from the ground. How far apart were the two players?

**Solution:**

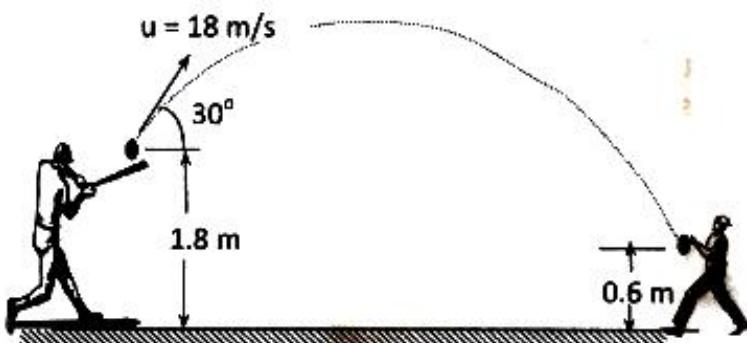
Considering vertical motion of the ball

$$u_y = u \sin \alpha = 18 \sin 30^\circ$$

$$u_y = 9 \text{ m/s}$$

Distance moved by the ball in vertical direction =  $1.8 - 0.6$

= 1.2 m. Here  $s = -1.2 \text{ m}$



$$s = u t + \frac{1}{2} a t^2$$

$$-1.2 = 9t - 4.905t^2$$

$$4.905t^2 - 9t - 1.2 = 0$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{9 \pm \sqrt{9^2 + 4 \times 4.905 \times 1.2}}{2 \times 4.905}$$

$$t = 1.9597 \text{ sec}$$

$$\text{The horizontal distance} = u \cos \alpha t = 18 \times \cos 30^\circ \times 1.9597 = 30.54 \text{ m}$$

**SP4.51:** A projectile is aimed at a mark on a horizontal plane through the point of projection. It falls 12m short when the angle of projection is  $15^\circ$  while it overshoots the mark by 24m when the angle is  $45^\circ$ . Find the angle of projection to hit the mark. Assume no air resistance.

**Solution:**

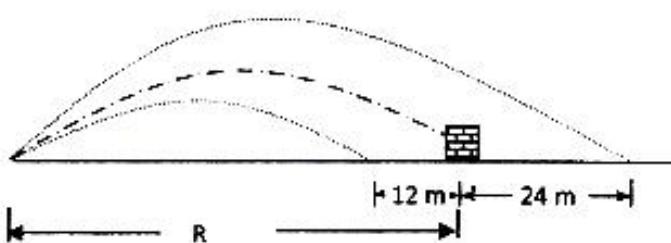
$$R = \frac{u^2 \sin 2\alpha}{g}$$

when  $\alpha = 15^\circ$

$$R-12 = \frac{u^2 \sin 2(15)}{g}$$

$$R-12 = \frac{u^2 \sin 30}{g}$$

$$R-12 = \frac{u^2}{2g} \quad \text{--- (1)}$$



when  $\alpha = 45^\circ$

$$R+24 = \frac{u^2 \sin 90}{g}$$

$$R+24 = \frac{u^2}{g} \quad \text{--- (2)}$$

(2) / (1) gives

$$\frac{R+24}{R-12} = \frac{u^2}{g} \times \frac{2g}{u^2}$$

$$R+24 = 2(R-12)$$

$$R+24 = 2R - 24$$

$$R = 48 \text{ m}$$

$$R+24 = \frac{u^2}{g}$$

$$72 = \frac{u^2}{g}$$

Substituting in general equation

$$R = \frac{u^2}{g} \sin 2\alpha$$

$$48 = 72 \sin 2\alpha$$

$$\sin 2\alpha = \frac{48}{72} = \frac{2}{3}$$

$$2\alpha = 41.76^\circ ; \alpha = 20.88^\circ$$

**SP4.52:** A projectile is fired from the edge of a 150m high cliff with the initial velocity of 180m/s at an angle of elevation of  $30^0$  with the horizontal. Neglecting air resistances find a) the greatest elevation above the ground reached by the projectile, b) time required to strike the ground, c) the horizontal distance from the gun to the point where the projectile strikes the ground and d) the velocity with which the projectile strikes the ground.

**Solution:**

**Considering vertical motion of the projectile till it reaches the maximum height**

$$u_g = 180 \sin 30 = 90 \text{ m/sec}$$

$$v = 0$$

$$a = -9.8 \text{ m/s}^2$$

$$v^2 = u^2 + 2as$$

$$0 = 90^2 - 2 \times 9.81 \times s$$

$$s = \frac{90^2}{2 \times 9.81} = 412.84 \text{ m}$$

$$\text{Max. height} = 150 + 412.84 = 562.84 \text{ m}$$

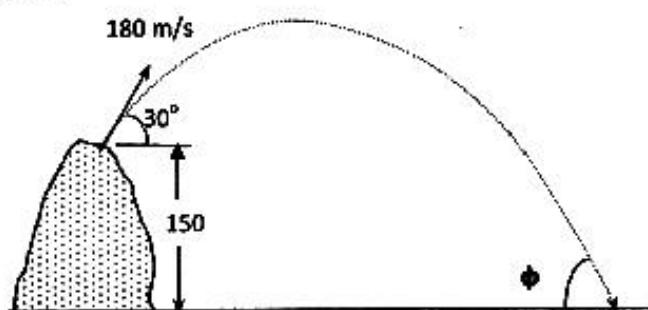


Fig.4.25

**Considering the motion of the projectile from the starting point till it reaches the ground**

$$u_g = 180 \sin 30^0 = 90 \text{ m/sec}$$

$$a = -9.81 \text{ m/s}^2$$

$$s = -150 \text{ m}$$

$$s = ut + \frac{1}{2} at^2$$

$$-150 = 90t - 4.905t^2$$

$$4.905t^2 - 90t - 150 = 0$$

$$a = 4.905; b = -90; c = -150$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{90 \pm \sqrt{90^2 + 4 \times 4.905 \times 150}}{2 \times 4.905}$$

$$t = \frac{90 \pm \sqrt{11643}}{9.81} = \frac{90 \pm 105.08}{9.81}$$

$$= 19.886 \text{ sec}$$

**Horizontal distance**

$$s = u \cos \alpha t = 180 \cos 30 \times 19.886$$

$$= 3100.2 \text{ m} = 3.1 \text{ km}$$

$$v^2 = u^2 + 2as$$

$$a = 9.81 \text{ m/s}^2$$

$$v^2 = 90^2 + 2 \times (-9.81) \times (-150) = 11004$$

$$v_y = 105.08 \text{ m/s}$$

$$v_x = 180 \cos 30 = 155.88 \text{ m/s}$$

$$v = \sqrt{(v_y)^2 + (v_x)^2}$$

$$v = \sqrt{(105.08)^2 + (155.88)^2}$$

$$v = 187.97 \text{ m/s}$$

$$\phi = \tan^{-1} \frac{165.08}{155.88} = 33.98^0$$

**SP4.53:** A rocket is fired at an angle of  $80^0$  to the horizontal and moves with constant acceleration of  $50 \text{ m/s}^2$ . All the fuel in the rocket is burnt out in 20sec. and the rocket continues as a free particle under the influence of gravity. Assume that the rocket flies

on a straight path till all the fuel is burnt out. Determine a) the maximum altitude reached by the rocket, b) the horizontal distance of the point where it strikes back on the earth from the launching point and c) the total time taken till it strike back on the earth.

**Solution:**

**Considering the straight line motion with uniform acceleration (O to A)**

$$u = 0; a = 50 \text{ m/s}^2; t = 20 \text{ sec}$$

$$v = u + at = 0 + 50 \times 20 = 1000 \text{ m/sec}$$

$$s = ut + \frac{1}{2} at^2$$

$$s = 0 + \frac{1}{2} \times 50 \times 20^2$$

$$s = 10,000 \text{ m along } 80^\circ \text{ inclined line.}$$

$$\text{Vertical distance} = 10000 \times \sin 80^\circ$$

$$= 9848.07 \text{ m} = 9.848 \text{ km}$$

$$\text{Horizontal distance} = 10000 \times \cos 80^\circ$$

$$= 1736.48 \text{ m} = 1.736 \text{ km}$$

**Considering the vertical motion from "A" upto "C"**

$$u = u_y = 1000 \sin 80^\circ = 984.8 \text{ m/s}$$

$$s = -9848 \text{ m}$$

$$a = -9.81 \text{ m/s}^2$$

$$s = ut + \frac{1}{2} at^2$$

$$-9848 = 984.8t - 4.905t^2$$

$$4.905t^2 - 984.8t - 9848 = 0$$

$$a = 4.905; b = -984.8; c = -9848$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(+984.8) \pm \sqrt{(-984.8)^2 - 4 \times 4.905 \times (-9848)}}{2 \times 4.905}$$

$$t = \frac{984.8 \pm 1078.4}{9.81} = 210.3 \text{ sec}$$

$$\text{Horizontal distance} = u \cos \alpha \cdot t$$

$$= 1000 \cos 80^\circ \times 210.3$$

$$s = \frac{984.8^2}{2 \times 9.81} = 49.43 \text{ km}$$

$$\text{Maximum height} = 49.43 + 9.848 = 59.278 \text{ km}$$

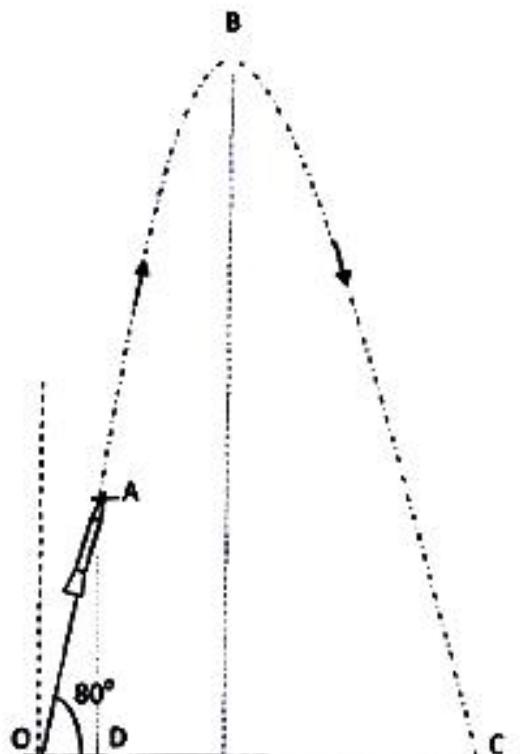


Fig.4.26

$$= 36518.2 = 36.518 \text{ km}$$

$$\text{Total distance} = 36.518 + 1.736$$

$$= 38.254 \text{ km}$$

$$\text{Total time} = 210 + 20 = 230.3 \text{ sec}$$

**To find maximum height:**

$$u = 1000 \sin 80^\circ$$

$$v = 0; a = -9.81 \text{ m/s}^2$$

$$v^2 = u^2 + 2as$$

$$0 = (984.8)^2 + 2 \times (-9.81) \times (s)$$

**SP4.54:** Find the least initial velocity which a projectile may have so that it may clear a wall 3.6m high and 4.8m distance from the point of projection and strike the horizontal plane through the foot of the wall at distance 3.6m beyond the wall. The point of projection is at the same level as the foot of the wall.

**Solution:**

$$\text{Range} = 4.8 + 3.6 = 8.4 \text{ m}$$

$$R = \frac{u^2}{g} \sin 2\alpha$$

$$8.4 = \frac{u^2}{g} \sin 2\alpha = \frac{u^2}{g} 2 \sin \alpha \cos \alpha$$

$$u^2 = 4.2 g / \sin \alpha \cos \alpha \dots (1)$$

Considering the vertical motion of projectile

$$u = u \sin \alpha; a = -g; s = 3.6 \text{ m}$$

$$s = ut + \frac{1}{2} at^2$$

$$3.6 = u \sin \alpha t - \frac{1}{2} gt^2$$

$$u \cos \alpha t = 4.8 \text{ m}$$

$$t = \frac{4.8}{u \cos \alpha}$$

$$3.6 = u \sin \alpha \frac{4.8}{u \cos \alpha} - \frac{1}{2} (g) \frac{4.8^2}{u^2 \cos^2 \alpha}$$

$$3.6 = \tan \alpha \times 4.8 - \frac{1}{2} \times g \times \frac{4.8^2}{u^2 \cos^2 \alpha}$$

$$= 4.8 \tan \alpha - \frac{g \times 4.8^2 \times \sin \alpha \cos \alpha}{2 \times \cos^2 \alpha \times 4.2g}$$

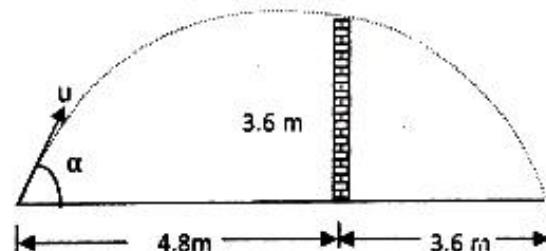


Fig.4.27

$$3.6 = 4.8 \tan \alpha - \frac{11.52}{4.2} \tan \alpha = 2.06 \tan \alpha$$

$$\tan \alpha = \frac{3.6}{2.06} = 1.748$$

$$\alpha = 60^\circ 14'$$

From eqn (1)

$$u^2 = 4.2 g / \sin \alpha \cos \alpha$$

$$= 4.2 \times 9.81 / \sin 60.14 \cdot \cos 60.14$$

$$u^2 = 95.42$$

$$u = 9.76 \text{ m}$$

**SP4.55:** Two guns are pointed at each other, one upward at an angle of  $30^\circ$  and the other at the same angle of depression, the nozzles being 30m apart. If the guns are shot simultaneously with velocities of 350m/s upwards and 300m/s, downwards respectively, find when and where they will meet.

**Solution:**

Horizontal distance between A & B =  $30\cos 30^\circ = 25.98\text{m}$

The horizontal distance covered by the shot from "B" after "t" secs

$$= 300 \cos 30^\circ t = 259.80 t \text{ m}$$

Similarly, the horizontal distance covered by the shot from "A" in time t

$$= 350 \cos 30^\circ t = 303.08 t \text{ m}$$

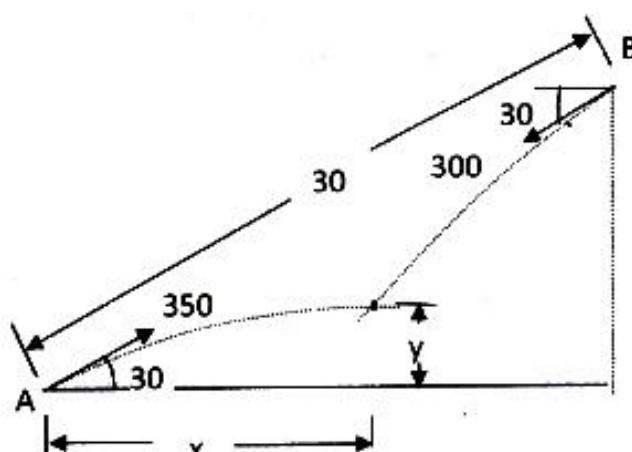


Fig.4.28

Total horizontal distance

$$= (259.8 + 303.08)t$$

$$25.98 = 562.908 t$$

$$t = \frac{25.98}{562.908} = 0.04615 \text{ sec}$$

Horizontal distance from A

$$= 350 \cos 30^\circ \times 0.04615$$

$$x = 303.1 \times 0.04615$$

$$x = 13.989\text{m}$$

$$s = ut + \frac{1}{2} at^2$$

$$y = 350 \sin 30^\circ \times 0.04615 - 4.905 \times (0.04615)^2$$

$$= 8.05 - 0.0103 = 8.04\text{m}$$

**SP4.56:** A particle is projected upwards with a velocity of 100m/s at an angle of  $30^\circ$  with the horizontal. Find the time, when the particle will move perpendicular to its initial direction.

**Solution:**

Let the tangent on the curve at the point 'P' makes  $90^\circ$  with the initial direction. Then the angle of projection at P =  $60^\circ$  downwards

We know that  $\tan \theta = v_y / v_x$

$$\tan (-60) = v_y / v_x ; \text{ but } v_x = 100 \cos 30^\circ$$

$$-1.732 = v_y / (86.6)$$

$$v_y = (86.6)(1.732) = -150\text{m/s}$$

i.e vertical component of velocity at 'P' =  $-150\text{m/s}$

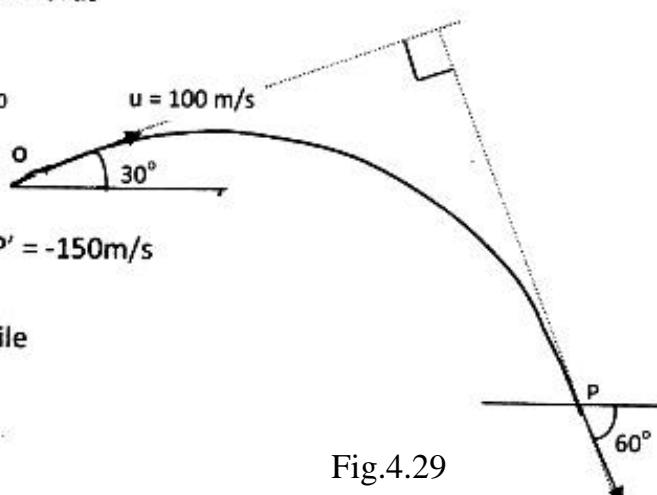


Fig.4.29

Considering the motion of the projectile in the vertical direction

$$v_y = u_y - gt$$

$$-150 = 100 \sin 30^\circ - 9.81 \times t$$

$$t = 20.38 \text{ sec}$$

**SP5.57:** A projectile is fired with a velocity of 500m/s at an elevation of 30°. find the velocity and direction of the projectile after 30sec. of its firing.

**Solution:**

$$u_x = u \cos \alpha = 500 \cos 30 = 433 \text{ m/sec}$$

$$u_y = u \sin \alpha = 500 \sin 30 = 250 \text{ m/sec}$$

Considering the projectile moving vertically

$$u = u_y = 250 \text{ m/sec}$$

$$a = -g$$

$$t = 30 \text{ sec}$$

$$v_y = u - gt$$

$$= 250 - 9.81 \times 30$$

$$v_y = -44.3 \text{ m/sec}$$

-ve sign indicate the direction is downward

Horizontal velocity remain constant



Fig.4.30

$$\text{i.e } u_x = v_x = 433.0 \text{ m/sec}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{433^2 + (-44.3)^2}$$

$$v = 435.2 \text{ m/sec}$$

$$\tan \theta = v_y/v_x = -44.3 / 433$$

$$\theta = -5.84^\circ$$

**SP4.58:** Two adjacent guns having the same muzzle velocity of 350m/s fire simultaneously at angles of  $\alpha_1$  and  $\alpha_2$  for the same target situated at the range of 4200m. Find the time difference between the hits.

**Solution:**

$$\text{The equation for } R = \frac{u^2 \sin 2\alpha}{g}$$

$$4200 = \frac{350^2 \sin 2\alpha}{9.81}$$

$$\sin 2\alpha = \frac{4200 \times 9.81}{350^2}$$

$$= 0.3363$$

$$2\alpha = 19.654^\circ \text{ or } 180 - 19.654 = 160.35^\circ$$

$$\alpha = 9.82 \text{ or } 80.17^\circ$$

$$\alpha_1 = 9.82^\circ; \alpha_2 = 80.17^\circ$$

Horizontal distance covered

$$u \cos \alpha t = 4200$$

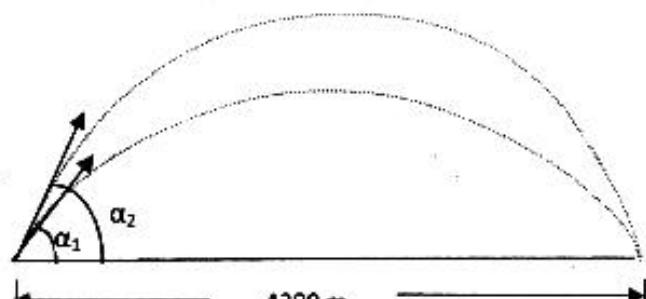


Fig.4.31

$$350 \cos 9.82 t = 4200$$

$$t = 12.17 \text{ sec}$$

$$\text{Similarly } 350 \cos 80.17 t = 4200$$

$$t = 70.339 \text{ sec}$$

$$\text{Time difference} = 70.339 - 12.17 = 58.16 \text{ sec}$$

## 2) NORMAL AND TANGENTIAL COORDINATES (n- t ) or Path:

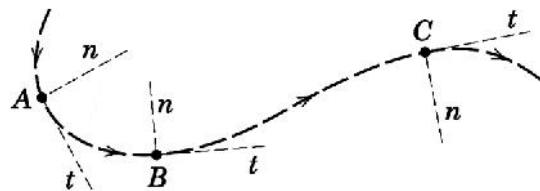


Fig.4.32 The particle moves along the path

One of the common descriptions of curvilinear motion uses path variables, which are measurements made along the tangent  $t$  and normal  $n$  to the path of the particle. These coordinates provide a very natural description for curvilinear motion and are frequently the most direct and convenient coordinates to use. The  $n$ - and  $t$ -coordinates are considered to move along the path with the particle, as seen in fig.4.32 where the particle advances from A to B to C. The positive direction for  $n$  at any position is always taken toward the centre of curvature of the path. As seen from fig.4.32, the positive  $n$ -direction will shift from one side of the curve to the other side if the curvature changes direction.

## a) Velocity and Acceleration

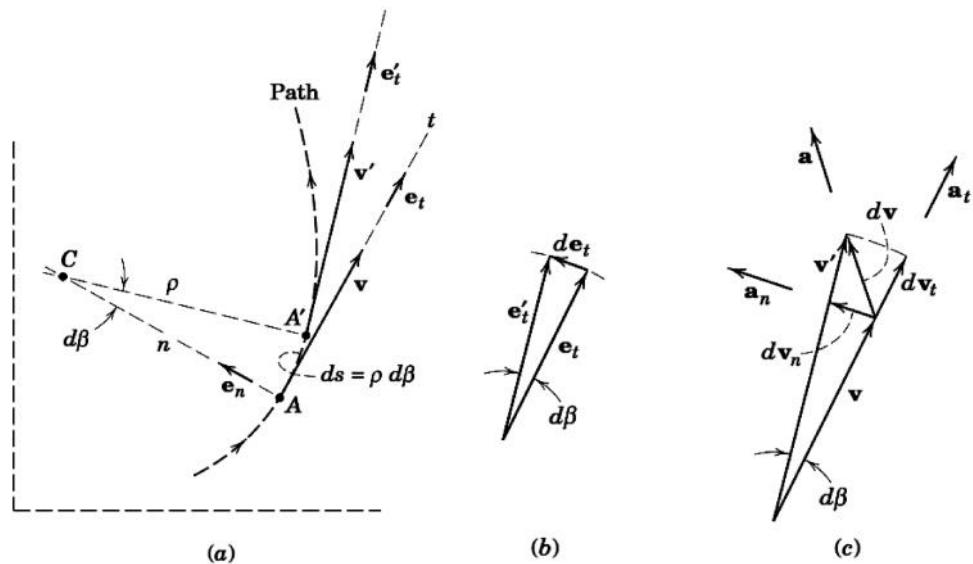


Fig.4.33 Curvilinear motion of a particle

Now use the coordinates  $n$  and  $t$  to describe the velocity 'v' and acceleration 'a' for the curvilinear motion of a particle. For this purpose, now introduce unit vectors  $e_n$  in the  $n$ -direction and  $e_t$  in the  $t$ -direction, as shown in fig.4.33a for the position of the particle at point A on its path. During a differential increment of time  $dt$ , the particle moves a differential distance  $ds$  along the curve from A to A'. With the radius of curvature of the path at this position designated by ... to obtain the differential distance as

$$ds = \dots dS, \text{ where } \beta \text{ is in radians.}$$

It is unnecessary to consider the differential change in ... between A and A' because a higher-order term would be introduced which disappears in the limit. Thus, the magnitude of the velocity can be written  $v = \frac{ds}{dt} = \dots \frac{dS}{dt}$ , and it can write the velocity as the vector

$$\mathbf{v} = v e_t = \dots \overset{\bullet}{S} e_t \quad \longrightarrow \quad (4.17)$$

The acceleration  $a$  of the particle was defined in earlier as  $a = \frac{dv}{dt}$ , and it observed from fig.4.12 that the acceleration is a vector which reflects both the change in magnitude and the change in direction of  $v$ . Now differentiate  $v$  in equation 4.17 by applying the ordinary rule for the differentiation of the product of a scalar and a vector and get

$$a = \frac{dv}{dt} = \frac{d(v e_t)}{dt} = v \dot{e}_t + v e_t \quad \longrightarrow \quad (4.18)$$

where the unit vector  $e_t$  now has a nonzero derivative because its direction changes.

To find  $\dot{e}_t$  now analyse the change in  $e_t$  during a differential increment of motion as the particle moves from A to A' in fig.4.33a. The unit vector  $e_t$  correspondingly changes to  $e'_t$ , and the vector difference  $de_t$  is shown in part b of the fig4.33. The vector

$de_t$  in the limit has a magnitude equal to the length of the arc  $|e_t|ds = ds$  obtained by swinging the unit vector  $e_t$  through the angle  $d\beta$  expressed in radians.

The direction of  $de_t$  is given by  $e_n$ . Thus it can write  $de_t = e_n ds$ . Dividing by  $d\beta$  gives

$$\frac{de_t}{ds} = e_n \quad \longrightarrow \rightarrow (4.19)$$

Dividing by  $dt$  gives  $\frac{de_t}{dt} = \left( \frac{ds}{dt} \right) e_n$ , which can be written as

$$\dot{e}_t = \dot{s} e_n \quad \longrightarrow \rightarrow (4.20)$$

With the substitution of equation 4.20 and  $\dot{s}$  from the relation  $v = \dots \dot{s}$ , then the equation 4.18 for the acceleration becomes

$$a = \frac{v^2}{\dots} e_n + \dot{v} e_t \quad \longrightarrow \rightarrow (4.21)$$

Where,

$$a_n = \frac{v^2}{\dots} = \dots \dot{s}^2 = v \dot{s}$$

$$a_t = \dot{v} = s^{\bullet \bullet}$$

$$a = \sqrt{a_n^2 + a_t^2} \quad \longrightarrow \rightarrow (4.22)$$

Now stress that  $a_t = \dot{v}$  is the time rate of change of the speed  $v$ . Finally noted that

$a_t = \dot{v} = \frac{d(\dots \dot{s})}{dt} = \dots \ddot{s} + \dots \dot{s}^2$ . This relation however finds little use because this have some reason to compute  $\dots$ .

**b) Circular motion:**

Circular motion is an important special case of plane curvilinear motion where the radius of curvature  $\rho$  becomes the constant radius  $r$  of the circle and the angle  $\beta$  is replaced by the angle  $\theta$  measured from any convenient radial reference to OP as shown in fig.4.34

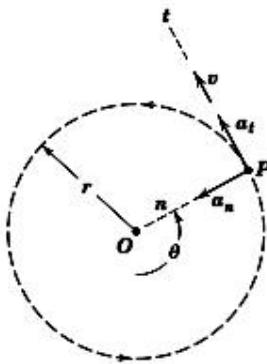


Fig.4.34 Circular motion

The velocity and the acceleration components for the motion of the particle P become

$$a_n = \frac{v^2}{r} = \left( \frac{r \dot{\theta}^2}{r} \right) = r \dot{\theta}^2 = v \ddot{\theta}$$

$$a_t = v = r \dot{\theta}$$

**Problems:**

**SP4.59:** To anticipate the dip and hump in the road the driver of a car applies her brakes to produce a uniform deceleration. Her speed is 100kmph at the bottom A of the dip and 50kmph at the top C of the hump which is 120m along the road from A. If the passengers experience a total acceleration of  $3\text{m/s}^2$  at A and if the radius of curvature of the hump at C is 150m, calculate a) the radius of curvature  $\rho$  at A, b) the acceleration at the inflection point B and c) the total acceleration at C.

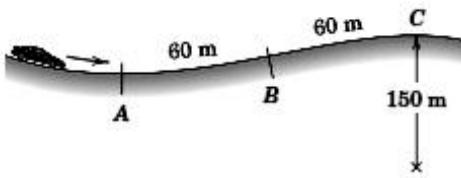
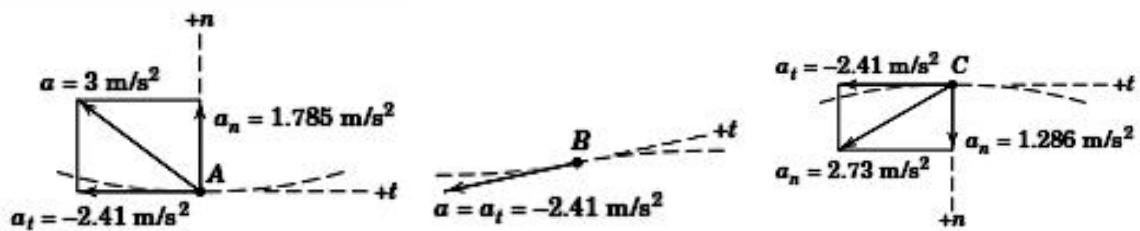


Fig.4.35

**Solution:**

The dimensions of the car are small compared with those of the path, so we will treat the car as a particle. The velocities are

$$v_A = \left( 100 \frac{\text{km}}{\text{h}} \right) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) \left( 1000 \frac{\text{m}}{\text{km}} \right) = 27.8 \text{ m/s}$$

$$v_C = 50 \frac{1000}{3600} = 13.89 \text{ m/s}$$

We find the constant deceleration along the path from

$$\left[ \int v \, dv = \int a_t \, ds \right] \quad \int_{v_A}^{v_C} v \, dv = a_t \int_0^s ds$$

$$a_t = \frac{1}{2s} (v_C^2 - v_A^2) = \frac{(13.89)^2 - (27.8)^2}{2(120)} = -2.41 \text{ m/s}^2$$

(a) Condition at A. With the total acceleration given and  $a_t$  determined, we can easily compute  $a_n$  and hence  $\rho$  from

$$[a^2 = a_t^2 + a_n^2] \quad a_n^2 = 3^2 - (2.41)^2 = 3.19 \quad a_n = 1.785 \text{ m/s}^2$$

$$[a_n = v^2/\rho] \quad \rho = v^2/a_n = (27.8)^2/1.785 = 432 \text{ m} \quad \text{Ans.}$$

(b) Condition at B. Since the radius of curvature is infinite at the inflection point,  $a_n = 0$  and

$$a = a_t = -2.41 \text{ m/s}^2 \quad \text{Ans.}$$

(c) Condition at C. The normal acceleration becomes

$$[a_n = v^2/\rho] \quad a_n = (13.89)^2/150 = 1.286 \text{ m/s}^2$$

With unit vectors  $\mathbf{e}_n$  and  $\mathbf{e}_t$  in the n- and t-directions, the acceleration may be written

$$\mathbf{a} = 1.286\mathbf{e}_n - 2.41\mathbf{e}_t \text{ m/s}^2$$

where the magnitude of  $\mathbf{a}$  is

$$|\mathbf{a}| = \sqrt{a_n^2 + a_t^2} \quad a = \sqrt{(1.286)^2 + (-2.41)^2} = 2.73 \text{ m/s}^2 \quad \text{Ans.}$$

The acceleration vectors representing the conditions at each of the three points are shown for clarification.

**SP4.60:** A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of  $6 \text{ m/s}^2$ , and the downward acceleration component is the acceleration due to gravity at that altitude, which is  $g = 9 \text{ m/s}^2$ . At the instant represented, the velocity of the mass centre G of the rocket along the  $15^\circ$  direction of its trajectory is  $20(10^3) \text{ kmph}$ . For this position determine (a) the radius of curvature of the flight trajectory, (b) the rate at which the speed  $v$  is increasing, (c) the angular rate  $\dot{\gamma}$  of the radial line from G to the centre of curvature C, and (d) the vector expression for the total acceleration  $\mathbf{a}$  of the rocket.

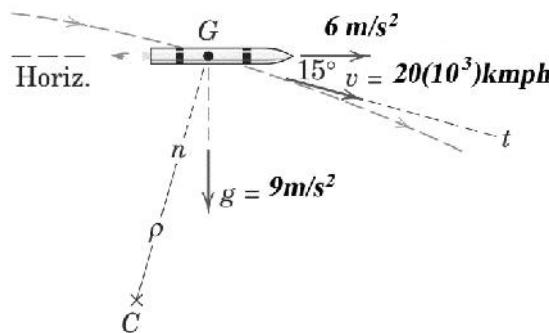


Fig.4.36

### Solution:

Now observe that the radius of curvature appears in the expression for the normal component of acceleration, so use n- and t-coordinates to describe the motion of G. The n- and t-components of the total acceleration are obtained by resolving the given horizontal and vertical accelerations into their n- and t-components and then combining. From the figure we get

$$a_n = 9 \cos 15^\circ - 6 \sin 15^\circ = 7.14 \text{ m/s}^2$$

$$a_t = 9 \sin 15^\circ + 6 \cos 15^\circ = 8.12 \text{ m/s}^2$$

- a) The radius of curvature of the flight trajectory from

$$a_n = \frac{v^2}{r}, \Rightarrow r = \frac{v^2}{a_n} = \frac{(20 \times 10^3)^2}{3.6} = 4.32 \times 10^6 \text{ m}$$

The radius of curvature of the flight trajectory is  $4.32 \times 10^6 \text{ m}$

- b) The rate at which the speed  $v$  is increasing

$$\dot{v} = a_t, \Rightarrow \dot{v} = 8.12 \text{ m/s}^2$$

- c) The angular rate  $\dot{\theta}$  of the radial line from G to the centre of the curvature C,

$$\dot{v} = \dot{\theta} r, \Rightarrow \dot{\theta} = \frac{v}{r} = \frac{(20 \times 10^3)}{4.32 \times 10^6} = 12.85 \times 10^{-4} \text{ rad/s}$$

- d) The vector expression for the total acceleration  $a$  of the rocket,

$$a = a_n e_n + a_t e_t, \Rightarrow a(\text{m/s}^2) = 7.14 e_n + 8.12 e_t$$

### 3) POLAR COORDINATES ( $r - \theta$ ):

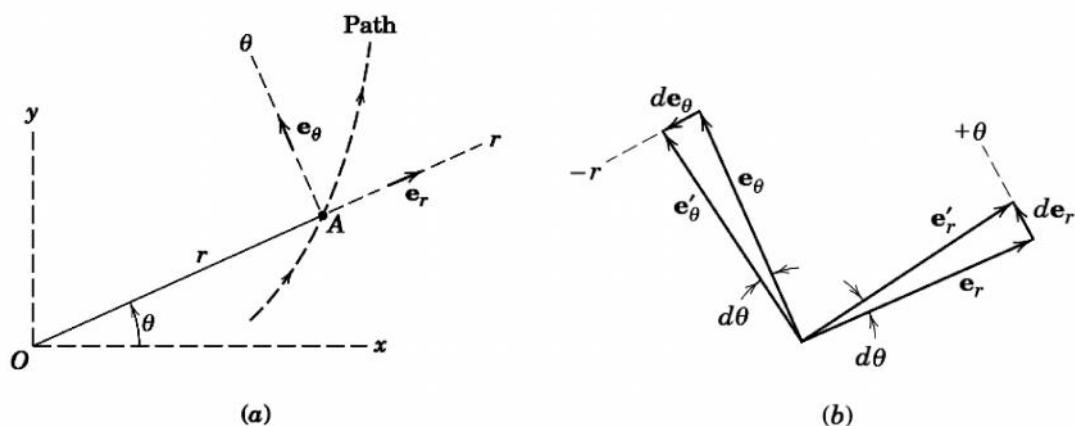


Fig.4.37 Polar coordinates motion of a particle

Now consider the third description of plane curvilinear motion, namely, polar coordinates where the particle is located by the radial distance  $r$  from a fixed point and

by an angular measurement  $\theta$  to the radial line. Polar coordinates are particularly useful when a motion is constrained through the control of a radial distance and an angular position or when an unconstrained motion is observed by measurements of a radial distance and an angular position. Fig.4.37 (a) shows the polar coordinates  $r$  and  $\theta$  which locate a particle traveling on a curved path. An arbitrary fixed line, such as the x-axis, is used as a reference for the measurement of  $\theta$ . Unit vectors  $e_r$  and  $e_\theta$  are established in the positive  $r$ - and  $\theta$ -directions, respectively. The position vector  $r$  to the particle at A has a magnitude equal to the radial distance  $r$  and a direction specified by the unit vector  $e_r$ . Thus, express the location of the particle at A by the vector

$$r = r e_r \quad \longrightarrow \rightarrow (4.24)$$

### a) Time Derivatives of the Unit Vectors

To differentiate this relation with respect to time to obtain  $v = \dot{r}$  and  $a = \dot{v}$ , now need expressions for the time derivatives of both unit vectors  $e_r$  and  $e_\theta$ . It obtains  $\dot{e}_r$  and  $\dot{e}_\theta$  in exactly the same way it derived  $\dot{e}_r$  in the preceding article. During time  $dt$  the coordinate directions rotate through the angle  $d_\theta$ , and the unit vectors also rotate through the same angle from  $e_r$  and  $e_\theta$  to  $e_r'$  and  $e_\theta'$ , as shown in fig.4.37 (b). It noted that the vector change  $de_r$  is in the plus  $\theta$ -direction and that  $de_\theta$  is in the minus  $r$ -direction. Because their magnitudes in the limit are equal to the unit vector as radius times the angle  $d_\theta$  in radians, it can write them as  $de_r = e_\theta d_\theta$  and  $de_\theta = -e_r d_\theta$ . If divide these equations by  $d_\theta$ , Then

$$\frac{de_r}{d_\theta} = e_\theta \quad \text{and} \quad \frac{de_\theta}{d_\theta} = -e_r$$

If, on the other hand, it divides that by  $dt$ , then have  $\frac{de_r}{dt} = \left( \frac{d_\theta}{dt} \right) e_\theta$  and

$$\frac{de_\theta}{dt} = -\left( \frac{d_\theta}{dt} \right) e_r \quad \text{or simply}$$

$$\dot{e}_r = \dot{\theta} e_\theta \quad \text{and} \quad \dot{e}_\theta = -\dot{\theta} e_r \quad \longrightarrow \rightarrow (4.25)$$

**b) Velocity**

Now ready to differentiate  $\mathbf{r} = r\mathbf{e}_r$  with respect to time. Using the rule for differentiating the product of a scalar and a vector gives

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

With the substitution of  $\dot{\mathbf{e}}_r$  from equation 4.25, the vector expression for the velocity becomes

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \quad \xrightarrow{\text{---}} (4.26)$$

$$\text{Where, } v_r = \dot{r}, \quad v_\theta = r\dot{\theta} \quad \text{and} \quad v = \sqrt{v_r^2 + v_\theta^2}$$

The  $r$  - component of  $\mathbf{v}$  is merely the rate at which the vector  $\mathbf{r}$  stretches. The  $\theta$ - component of  $\mathbf{v}$  is due to the rotation of  $\mathbf{r}$ .

**c) Acceleration**

Now differentiate the expression for  $\mathbf{v}$  to obtain the acceleration  $\mathbf{a} = \dot{\mathbf{v}}$ . Note that the derivative of  $r\dot{\theta}\mathbf{e}_\theta$  will produce three terms, since all three factors are variable. Thus,

$$\mathbf{a} = \dot{\mathbf{v}} = \left( \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta \right) + \left( \dot{r}\ddot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\theta}\mathbf{e}_r \right)$$

Substitution of  $\dot{\mathbf{e}}_r$  and  $\dot{\mathbf{e}}_\theta$  from equation 4.e and collecting terms give

$$\mathbf{a} = \left( \ddot{r} - r\dot{\theta}^2 \right) \mathbf{e}_r + \left( r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) \mathbf{e}_\theta \quad \xrightarrow{\text{---}} (4.27)$$

$$\text{Where, } a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad \text{and} \quad a = \sqrt{a_r^2 + a_\theta^2}$$

Now it can write the  $\theta$ - component alternatively as

$$a_\theta = \frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\theta} \right)$$

Which can be verified easily by carrying out the differentiation. This form of  $a_\theta$  will be useful when it treats the angular momentum of particles.

#### d) Circular motion;

For motion in a circular path with  $r$  constant, the components of equation

$v = \dot{r} e_r + r \dot{\theta} e_\theta$  and equation  $a = \left( \ddot{r} - r \dot{\theta}^2 \right) e_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) e_\theta$  become simply

$$v_r = 0, v_\theta = r \dot{\theta}, a_r = -r \dot{\theta}^2 \text{ and } a_\theta = r \ddot{\theta}$$

This description is the same as that obtained n- and t- components, where the  $\theta$ - and t - directions coincide but the positive r-direction is in the negative n-direction. Thus  $a_r = -a_\theta$  for circular motion centred at the origin of the polar coordinates.

The expressions for  $a_r$  and  $a_\theta$  in scalar form can also be obtained by direct differentiation of the coordinate relations  $x = r \cos \theta$  and  $y = r \sin \theta$  to obtain  $\ddot{a}_x = \ddot{x}$  and

$\ddot{a}_y = \ddot{y}$ . Each of these rectangular components of acceleration can then be resolved r- and  $\theta$ - components which when combined will yield the expressions of equation

$$a = \left( \ddot{r} - r \dot{\theta}^2 \right) e_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) e_\theta$$

#### Problems:

**SP4.61:** Rotation of the radially slotted arm is governed by  $\theta = 0.2t + 0.02t^3$ , where  $\theta$  is in radians and  $t$  is in seconds. Simultaneously the power screw in the arm engages the slider B and controls its distance from O according to  $r = 0.2 + 0.04t^2$ , where  $r$  is in metres and  $t$  is in second. Calculate the magnitude of the velocity and acceleration of the slider for the instant when  $t = 3s$ .

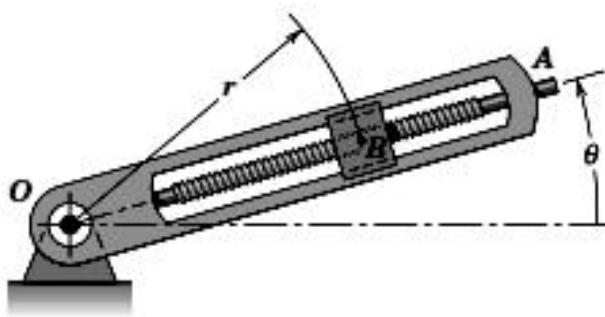
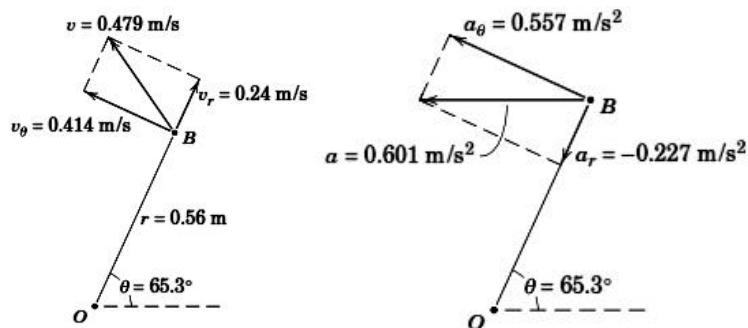


Fig.4.38

**Solution:**

The coordinates and their time derivatives which appear in the expressions for velocity and acceleration in polar coordinates are obtained first and evaluated for  $t = 3$  s.

$$\begin{aligned}
 r &= 0.2 + 0.04t^2 & r_3 &= 0.2 + 0.04(3^2) = 0.56 \text{ m} \\
 \dot{r} &= 0.08t & \dot{r}_3 &= 0.08(3) = 0.24 \text{ m/s} \\
 \ddot{r} &= 0.08 & \ddot{r}_3 &= 0.08 \text{ m/s}^2 \\
 \theta &= 0.2t + 0.02t^3 & \theta_3 &= 0.2(3) + 0.02(3^3) = 1.14 \text{ rad} \\
 && \text{or } \theta_3 = 1.14(180/\pi) = 65.3^\circ \\
 \dot{\theta} &= 0.2 + 0.06t^2 & \dot{\theta}_3 &= 0.2 + 0.06(3^2) = 0.74 \text{ rad/s} \\
 \ddot{\theta} &= 0.12t & \ddot{\theta}_3 &= 0.12(3) = 0.36 \text{ rad/s}^2
 \end{aligned}$$

The velocity components are obtained from Eq. 2/13 and for  $t = 3$  s are

$$\begin{aligned}
 [v_\theta = r\dot{\theta}] & \quad v_r = 0.24 \text{ m/s} \\
 [v_\theta = r\dot{\theta}] & \quad v_\theta = 0.56(0.74) = 0.414 \text{ m/s} \\
 [v = \sqrt{v_r^2 + v_\theta^2}] & \quad v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s}
 \end{aligned}$$

The velocity and its components are shown for the specified position of the arm.

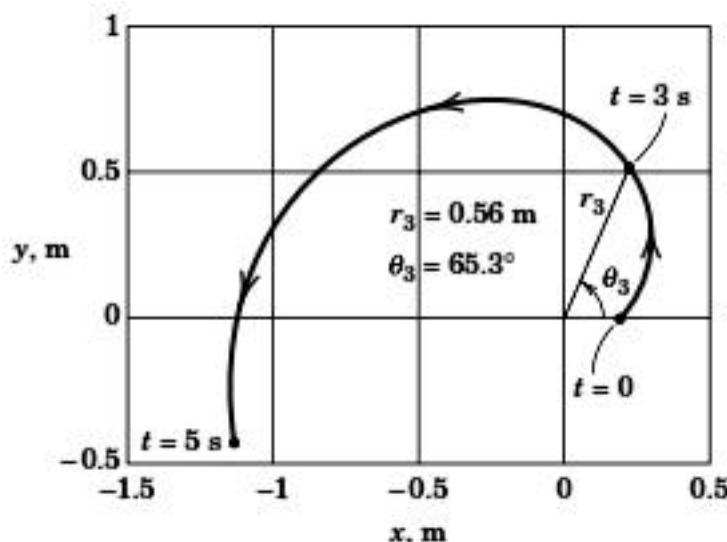
The acceleration components are obtained from Eq. 2/14 and for  $t = 3$  s are

$$\begin{aligned}
 [a_r = \ddot{r} - r\dot{\theta}^2] \quad a_r &= 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2 \\
 [a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}] \quad a_\theta &= 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2 \\
 [a = \sqrt{a_r^2 + a_\theta^2}] \quad a &= \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2
 \end{aligned}$$

The acceleration and its components are also shown for the  $65.3^\circ$  position of the arm.

Plotted in the final figure is the path of the slider *B* over the time interval  $0 \leq t \leq 5$  s. This plot is generated by varying *t* in the given expressions for *r* and  $\theta$ . Conversion from polar to rectangular coordinates is given by

$$x = r \cos \theta \quad y = r \sin \theta$$



**SP4.62:** A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when  $\theta=30^\circ$ , the tracking data given  $r = 8(10^4)$  m,  $\dot{r} = 1200 \text{ m/s}$  and  $\ddot{r} = 0.80 \text{ deg/s}$ . The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is  $9.20 \text{ m/s}^2$  vertically down. For these conditions determine the velocity *v* of the rocket and the values of  $\ddot{r}$  and  $\ddot{\theta}$ .

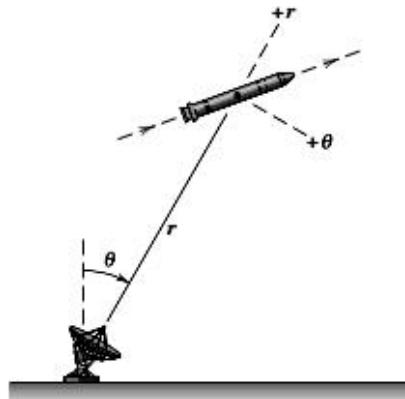


Fig.4.39

**Solution:**

The components of velocity from equation  $v = \dot{r} e_r + r \dot{\theta} e_\theta$  are

$$v_r = \dot{r}, \Rightarrow v_r = 1200 \text{ m/s}$$

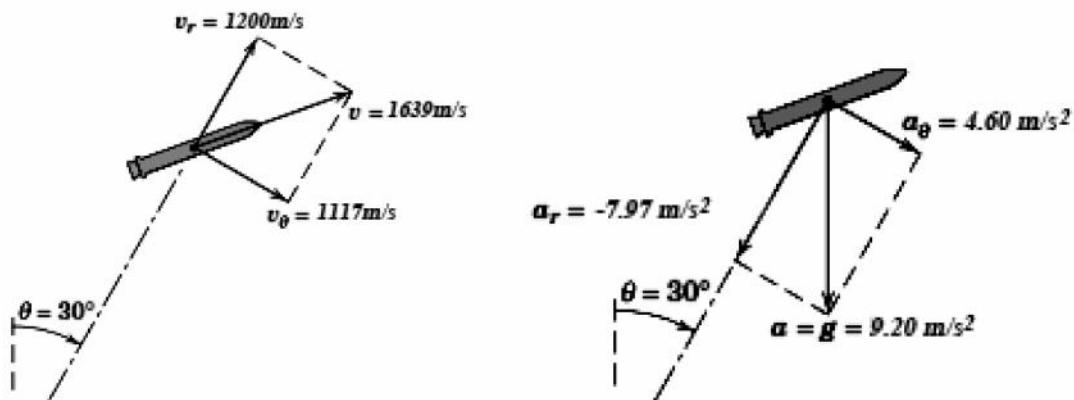
$$v_\theta = r \dot{\theta}, \Rightarrow v_\theta = 8(10^4)(0.08) \left( \frac{f}{180} \right) = 1117 \text{ m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2}, \Rightarrow v = \sqrt{(1200)^2 + (1117)^2} = 1639 \text{ m/s}$$

Since the total acceleration of the rocket is  $g = 9.20 \text{ m/s}^2$  down, it can easily find its  $r$  - and  $\theta$  - components for the given position. As shown in the figure, they are

$$a_r = -9.20 \cos 30^\circ = -7.97 \text{ m/s}^2$$

$$a_\theta = 9.20 \sin 30^\circ = 4.60 \text{ m/s}^2$$



Now equate these values to the polar coordinate expressions for  $a_r$  and  $a_\theta$  which

contain the unknowns  $\ddot{r}$  and  $\ddot{\theta}$ . Thus from equation,  $a = \left( \ddot{r} - r \dot{\theta}^2 \right) e_r + \left( r \ddot{\theta} + 2r \dot{\theta} \dot{r} \right) e_\theta$

$$a_r = \ddot{r} - r \dot{\theta}^2, \Rightarrow -7.97 = \ddot{r} - 8(10^4) \left( 0.08 \left( \frac{f}{180} \right) \right)^2, \ddot{r} = 7.63 \text{ m/s}^2$$

$$a_\theta = r \ddot{\theta} + 2r \dot{\theta} \dot{r}, \Rightarrow 4.60 = 8(10^4) \ddot{\theta} + 2(1200) \left( 0.08 \left( \frac{f}{180} \right) \right), \ddot{\theta} = -3.61 \times 10^{-4} \text{ rad/s}^2$$

**SP4.63:** A flywheel increases its speed from 30rpm to 60rpm in 10sec. Calculate a) the angular acceleration and b) number of revolutions made by the wheel in these 10 seconds.

**Solution:**

$$N_0 = 30 \text{ rpm}$$

$$\omega_0 = 2\pi N / 60 = 2\pi \cdot 30 / 60 = \pi \text{ radians}$$

$$\omega = 2\pi N / 60 = 2\pi \cdot 60 / 60 = 2\pi \text{ radians}$$

$$t = 10 \text{ sec}$$

$$\omega = \omega_0 + \alpha t$$

$$2\pi = \pi + \alpha \cdot 10$$

$$10\alpha = \pi; \alpha = \pi / 10 = 0.314 \text{ rad/s}^2$$

$$\text{Angular acceleration} = 0.314 \text{ rad/s}^2$$

To find the no of rev. made:

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$(2\pi)^2 = \pi^2 + 2 \times \frac{\pi}{10} \times \theta$$

$$\theta = (4\pi^2 - \pi^2)5 / \pi = 15\pi$$

$$\text{Number of revolution made} = \frac{15\pi}{2\pi} = 7.5 \text{ rev.}$$

**SP4.64:** A pulley starting from rest is given an acceleration of  $0.5 \text{ rad/s}^2$ . What will be its speed in rpm at the end of 2min? If then it is uniformly retarded at the rate of  $0.3 \text{ rad/s}^2$ , in how many minutes the pulley will come to rest.

**Solution:**

Acceleration:

$$\omega_0 = 0$$

$$\alpha = 0.5 \text{ rad/s}^2$$

$$t = 2 \times 60 = 120 \text{ secs.}$$

$$\omega = \omega_0 + \alpha t$$

$$= 0 + 0.5 \times 120$$

Retardation:

$$\omega_0 = 60 \text{ rad/s}$$

$$\omega = 0$$

$$\alpha = -0.3 \text{ rad/s}^2$$

$$\omega = \omega_0 + \alpha t$$

$$0 = 60 - 0.3 \cdot t$$

$$\omega = 60 \text{ rads/sec}$$

$$2\pi N/60 = 60$$

$$N = \frac{60 \times 60}{2\pi} = 573.2 \text{ rpm}$$

Speed at the end of 2 min. = 573.2 rpm

$$t = \frac{60}{0.3}$$

$$t = 200 \text{ secs}$$

$$t = 3.33 \text{ mins}$$

The pulley comes to rest after 3.33 min.

**SP4.65:** A wheel rotates for 5 sec. with a constant angular acceleration and describes during this time 100 radians. It then rotates with a constant angular velocity during the next five sec. and describes 80 radians. Find the initial angular velocity and the angular acceleration.

**Solution:**

Uniform acceleration:

$$\theta = 100 \text{ radians}$$

$$t = 5 \text{ sec}$$

$$\omega_0 = \omega_0$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$100 = \omega_0 \times 5 + \frac{1}{2} \alpha (5)^2$$

$$100 = 5\omega_0 + 12.5\alpha \dots\dots (1)$$

$$\omega = \omega_0 + \alpha t$$

$$\omega = \omega_0 + 5\alpha$$

Uniform angular velocity:

$$\theta = 80 \text{ rads}; t = 5 \text{ sec};$$

$$\theta = \omega t$$

$$80 = (\omega_0 + 5\alpha) \times 5$$

$$80 = 5\omega_0 + 25\alpha \dots\dots (2)$$

$$5\omega_0 + 25\alpha = 80 \dots (2)$$

$$5\omega_0 + 12.5\alpha = 100 \dots (1)$$

$$(2) - (1)$$

$$12.5\alpha = -20$$

$$\alpha = \frac{-20}{12.5} = -1.6 \text{ rad / s}^2$$

$$\omega_0 + 5\alpha = 16$$

$$\omega_0 + 5(-1.6) = 16$$

$$\omega_0 = 24 \text{ rads / sec.}$$

**SP4.66:** A swing bridge turns through  $90^\circ$  in 2min. The bridge is uniformly accelerated from rest for the first 40 sec. Subsequently it turns with a uniform angular velocity for the next 60 sec. Then the motion of the bridge is uniformly retarded for the last 20 sec. Find a) angular acceleration, b) maximum angular velocity and c) angular retardation of the bridge.

**Solution:**

Part I: Uniform acceleration

$$\omega_0 = 0$$

$$t_1 = 40 \text{ sec}; \alpha = \alpha_1$$

$$\omega = \omega_0 + \alpha_1 t$$

$$\omega = 0 + \alpha_1 40$$

$$\omega = 40 \alpha_1 \quad \dots \dots (1)$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$= 0 + \frac{1}{2} \alpha_1 (40)^2$$

$$\theta_1 = 800 \alpha_1 \quad \dots \dots (2)$$

Part II: Uniform angular velocity

$$\theta = \omega t$$

$$\omega_2 = \omega = 40 \alpha_1$$

$$t_2 = 60 \text{ sec}$$

$$\theta_2 = \omega_2 \times t_2 = 40 \alpha_1 \times 60$$

$$\theta_2 = 2400 \alpha_1 \quad \dots \dots (3)$$

Part III: Uniform angular retardation

$$\omega_0 = \omega = 40 \alpha_1$$

$$\omega = 0$$

$$t_3 = 20 \text{ sec}$$

$$\omega_3 = \omega_0 + \alpha_3 t$$

$$0 = 40 \alpha_1 + \alpha_3 \times 20$$

$$20 \alpha_3 = -40 \alpha_1$$

$$\alpha_3 = -2 \alpha_1 \quad \dots \dots (4)$$

$$\theta_3 = \omega_0 t_3 + \frac{1}{2} \alpha_3 t_3^2$$

$$\theta_3 = 40 \alpha_1 \times 20 + \frac{1}{2} (-2 \alpha_1) 20^2$$

$$= 800 \alpha_1 - 400 \alpha_1$$

$$\theta_3 = 400 \alpha_1 \quad \dots \dots (5)$$

$$\theta_1 + \theta_2 + \theta_3 = 800 \alpha_1 + 2400 \alpha_1 + 400 \alpha_1$$

$$= 3600 \alpha_1 = 90 = \frac{\pi}{2} \text{ radians}$$

$$\alpha_1 = \frac{\pi}{(3600 \times 2)}$$

$$\alpha_1 = 4.36 \times 10^{-4} \text{ rads/sec}^2$$

$$\alpha_3 = -2 \alpha_1, \alpha_3 = -2(4.36 \times 10^{-4})$$

$$\alpha_3 = -8.72 \times 10^{-4} \text{ rad/sec}^2$$

$$\text{Max. angular velocity} = 40 \alpha_1$$

$$= 40 \times 4.36 \times 10^{-4}$$

$$\text{Max. angular velocity} = 0.0174 \text{ rad/sec.}$$

**SP4.67:** A rotor of an electric motor is uniformly accelerated to a speed of 1800 rpm from rest for 5 sec and then immediately power is switched off and the rotor decelerates uniformly. If the total time elapsed from start to stop is 12.5 sec, determine the number of revolutions made in a) acceleration and b) deceleration. Also determine the value of deceleration.

**Solution:**

Acceleration:

$$\omega_0 = 0; t = 5 \text{ sec}; N = 1800 \text{ rpm}$$

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 1800}{60} = 60\pi \text{ rad/sec}$$

$$\omega = \omega_0 + \alpha t$$

$$60\pi = 0 + \alpha \times 5$$

$$\alpha = 60\pi / 5 = 12\pi = 37.6 \text{ rad/sec}^2$$

To find 'θ'

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$(60\pi)^2 = 0 + 2 \times 12\pi \times \theta$$

$$\theta = \frac{3600\pi^2}{2 \times 12\pi} = 150\pi$$

$$\text{No. of revolutions made} = \frac{150\pi}{2\pi} = 75$$

Retardation:

$$\omega_0 = 60\pi; \omega = 0$$

$$t = 12.5 - 5 = 7.5 \text{ sec}$$

$$\omega = \omega_0 + \alpha t$$

$$0 = 60\pi + \alpha \times 7.5$$

$$\alpha = \frac{-60\pi}{7.5} = -8\pi = -25.132 \text{ rad/sec}^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$0 = (60\pi)^2 - (2) \times (-8\pi) \times \theta$$

$$\theta = \frac{3600\pi^2}{2 \times 8\pi} = 225\pi$$

$$\text{No. of revolutions made} = \frac{225\pi}{2\pi} = 112.5 \text{ rev.}$$

**SP4.68:** A car is moving at 72kmph, if the wheels are 75cm diameter, find the angular velocity of wheel about its axis. If the car comes to rest in a distance of 20m under a uniform retardation, find the angular retardation of the wheels.

**Solution:**

$$\begin{aligned}
 v_{car} &= 72 \text{ kmph} = 20 \text{ m/sec;} \\
 \text{wheel dia} &= 75\text{cm} = 0.75\text{m}; r = 0.375 \\
 \omega &= v/r = 20/0.375 = 53.32\text{rad/sec} \\
 s &= 20 \text{ m} = r\theta \\
 20 &= 0.375 \times \theta \\
 \theta &= \frac{20}{0.375} = 53.33 \text{ radians}
 \end{aligned}$$

$$\begin{aligned}
 \omega &= 0; \omega_0 = 53.32\text{rad} ; \theta = 53.33\text{rad} \\
 \omega^2 &= \omega_0^2 + 2\alpha\theta \\
 0 &= (53.32)^2 + 2 \times \alpha \times 53.33 \\
 \alpha &= \frac{-(53.32)^2}{2 \times 53.33} = -26.66 \text{ rad/sec}^2 \\
 \omega &= 53.32 \text{ rad/sec}; \alpha = -26.67 \text{ rad/sec}^2
 \end{aligned}$$

**SP4.69:** A step pulley shown in fig.4. starts from rest and accelerates at  $2\text{rad/s}^2$ . What time is required for block A to move 20m? Find also the velocity of A and B at that time.

**Solution:**

$$\begin{aligned}
 s &= 20 \text{ m}, s = r\theta \\
 \theta &= \frac{s}{r} = \frac{20}{1} = 20 \text{ radians} \\
 \omega_0 &= 0 \\
 \alpha &= 2\text{rad/sec}^2; \\
 \theta &= 20 \text{ radians} \\
 \theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\
 20 &= 0 + \frac{1}{2} \times 2 \times t^2 \\
 20 &= t^2 \\
 t &= 4.472 \text{ sec} \\
 \omega &= \omega_0 + \alpha t \\
 &= 0 + 2 \times 4.472 = 8.944 \text{ rad/sec} \\
 v_A &= r_A \times \omega = 1 \times 8.944 = 8.944 \text{ m/sec} \\
 v_B &= r_B \times \omega = 0.75 \times 8.944 = 6.7 \text{ m/sec.}
 \end{aligned}$$

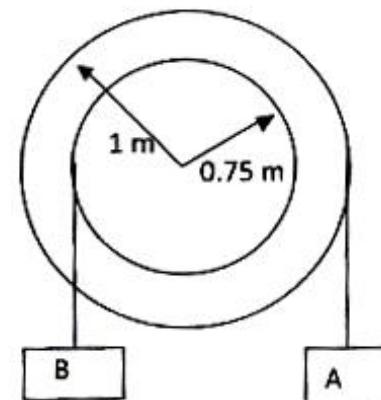


Fig.4.40

**SP4.70:** One end of a cord is wrapped around a pulley of 1.5m diameter while a block is attached to the other end. The pulley is given an angular acceleration of 2 rpm per second. Find the displacement, velocity and acceleration of the block 5 sec. after the system was at rest.

**Solution:**

$$\alpha = 2 \text{ rpm/sec} = \frac{2}{60} \text{ rps}^2$$

$$\alpha = \frac{2 \times 2\pi}{60} \text{ rad/sec}^2 = 0.2094 \text{ rad/sec}^2$$

$$a = r\alpha = 0.75 \times 0.2094 = 0.157079 \text{ m/sec}^2$$

$$v = u + at$$

$$v = 0 + (0.157079 \times 5) = 0.78535 \text{ m/sec}$$

$$s = ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2} \times 0.157079 \times 5^2$$

$$s = 1.9635 \text{ m}$$

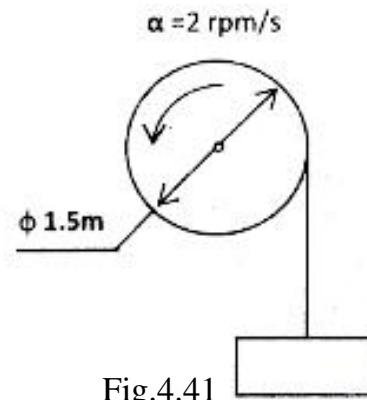


Fig.4.41

**SP4.71:** A horizontal bar of length 1.5m and of small cross-section rotates about the vertical axis through one end. It accelerates uniformly from 1200 rpm to 1500 rpm in an interval of 5 sec. What is the linear velocity at the beginning and at the end of the interval? What are the normal and tangential components of the acceleration of the mid-point of the bar after 5 sec. after the acceleration begins?

**Solution:**

$$\omega_0 = \frac{2\pi N}{60} = \frac{2\pi \times 1200}{60} = 125.66 \text{ rad/sec}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1500}{60} = 157.07 \text{ rad/sec}$$

$$t = 5 \text{ sec}$$

$$\omega = \omega_0 + \alpha t$$

$$157.07 = 125.66 + \alpha \times 5$$

$$\alpha = \frac{(157.07 - 125.66)}{5} = 6.2839 \text{ rad/sec}^2$$

Initial velocity of the rod end =  $\omega_0 \times r = 125.66 \times 1.5 = 188.5 \text{ m/s}$

Final velocity of the rod end =  $\omega \times r = 157.07 \times 1.5 = 235.6 \text{ m/s}$

after 5 sec

$$a_t = \alpha r = 0.75 \times 6.2839 = 4.712 \text{ rad/s}^2$$

$$a_n = \omega^2 r = (157.07)^2 \times 0.75 = 18.503 \text{ rad/s}^2$$

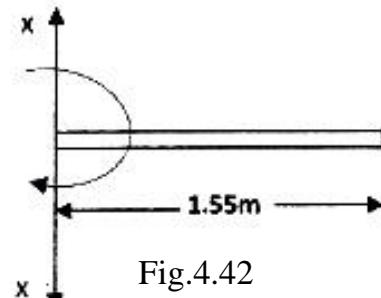


Fig.4.42

**SP4.72:** What is the smallest radius which should be used for a high way curve if the normal component of the accelerations of a car travelling at 72kmph is not to exceed  $0.72 \text{ m/s}^2$ ?

**Solution:**

$$a_n = 0.72 \text{ m/sec}^2 \quad v = 72 \text{ kmph} = 20 \text{ m/sec}$$

$$a_n = \omega^2 \times r = v^2 / r = 20^2 / r = 0.72$$

$$0.72 = \frac{(20)^2}{r}; \quad 0.72 r = 20^2$$

$$r = \frac{400}{0.72}; \quad r = 555.55 \text{ m}$$

**SP4.73:** A point describes a circular path of radius 1.2m. The angular velocity and angular acceleration of a radius to the point at a certain instant are 4 rad/sec, and 2 rad/s<sup>2</sup>, both clockwise. If the angular acceleration is constant, determine for the given instant and at 4 sec. later the normal and tangential components of acceleration of the point and the angle turned through in 4 sec.

**Solution:**

$$r = 1.2 \text{ m}$$

$$\omega = 4 \text{ rad/sec}$$

$$\alpha = 2 \text{ rad/sec}^2$$

$$a_n = \omega^2 r = 4^2 \times 1.2 = 19.2 \text{ m/sec}^2$$

$$a_t = \alpha r = 1.2 \times 2 = 2.4 \text{ m/sec}^2$$

after '4'sec

$$\omega = \omega_0 + \alpha t$$

$$= 4 + (2 \times 4)$$

$$\omega = 12 \text{ rad/sec}$$

$$a_n = \omega^2 r = 12^2 \times 1.2 = 144 \times 2$$

$$= 172.32 \text{ m/sec}^2$$

$$a_t = \alpha r = 1.2 \times 2 = 2.4 \text{ m/sec}^2$$

to find  $\theta$

$$\omega^2 = \omega_0^2 + 2 \times \alpha \theta$$

$$(1.2)^2 = 4^2 + 2 \times 2 \times \theta$$

$$\theta = \frac{144 - 16}{4} = \frac{128}{4}$$

$$\theta = 32 \text{ radians}$$

**SP4.74:** A motorist is travelling on a curved sections of high way of radius 750m at the speed of 100 kmph. The motorist suddenly applies brakes, causing the automobile to slow down at a constant rate. Knowing that after 8sec. the speed has been reduced to 75kmph, determine the acceleration of the automobile immediately after the brakes have been applied.

**Solution:**

$$u = 100 \text{ kmph} = \frac{100}{3.6} = 27.77 \text{ m/sec}$$

$$v = 75 \text{ kmph} = \frac{75}{3.6} = 20.83 \text{ m/sec}$$

$$t = 8 \text{ sec}$$

$$\omega_0 = \frac{u}{r} = \frac{27.77}{750} = 0.03702 \text{ rad/sec}$$

$$\omega = \frac{\theta}{t} = \frac{20.83}{750} = 0.02777 \text{ rad/sec}$$

$$\omega = \omega_0 + \alpha t$$

$$0.0277 = 0.03702 + \alpha \times 8$$

$$\alpha = \frac{0.0277 - 0.03702}{8} = -0.001153 \text{ rad/sec}^2$$

$$a_t = r \alpha = 750 \times 0.001153 = 0.86535 \text{ m/sec}^2$$

$$a_n = \omega^2 r = (0.02777)^2 \times 750 = 0.5783 \text{ m/sec}^2$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(0.5783)^2 + (0.86535)^2}$$

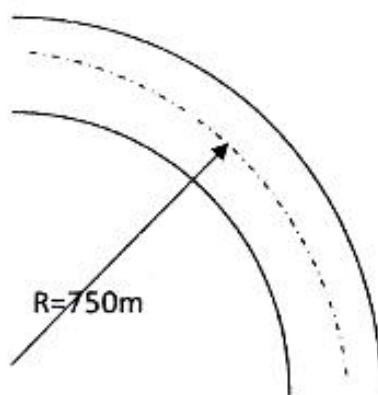


Fig.4.43

$$a = 1.0407 \text{ m/sec}^2$$

$$\tan \alpha = \frac{a_n}{a_t}$$

$$\alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{(0.5783)}{(0.86535)}$$

$$\alpha = 33.75^\circ$$

**SP4.75:** A bus starts from rest on a curve of 300m radius and accelerates at the constant tangential acceleration of 0.75m/s<sup>2</sup>. Determine the distance and time that the bus will travel before the magnitude of its total acceleration is 0.9m/s<sup>2</sup>.

**Solution:**

$$r = 300 \text{ m}; a_t = 0.75 \text{ m/sec}^2; a = 0.9 \text{ m/sec}^2$$

$$a = \sqrt{a_n^2 + a_t^2}$$

$$a^2 = a_n^2 + a_t^2$$

$$(0.9)^2 = a_n^2 + (0.75)^2$$

$$a_n^2 = (0.9)^2 - (0.75)^2$$

$$a_n = \sqrt{(0.9)^2 - (0.75)^2}$$

$$a_n = 0.4974 \text{ m/sec}^2$$

$$\omega_0 = 0$$

$$a_n = 0.4974 \text{ m/sec}^2$$

$$\omega^2 r = 0.4974$$

$$\omega^2 \times 300 = 0.4974$$

$$r \alpha = 0.75$$

$$\alpha = \frac{0.75}{300} = 0.0025 \text{ rad/sec}^2$$

$$\omega = \omega_0 + \alpha t$$

$$0.0407 = 0 + (0.0025) \times t$$

$$t = \frac{0.0407}{0.0025} = 16.2884 \text{ sec}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\theta = \frac{\omega^2 - \omega_0^2}{2\alpha} = \frac{(0.0407)^2 - 0^2}{2(0.0025)}$$

$$= 0.3312 \text{ rad}$$

$$\omega = \sqrt{\frac{(0.4974)}{300}} = 0.0407 \text{ rad/s}$$

$$a_t = 0.75 \text{ m/sec}^2$$

$$s = r\theta = 300 \times 0.3312 = 99.4 \text{ m}$$

**SP4.76:** A car enters a curved section of the road of length equal to the quarter of a circle of radius 100m at 18kmph and leaves at 36kmph. If the car is travelling with a constant tangential acceleration, find the magnitude and direction of acceleration a) when it enters the curve and b) when it leaves the curve.

**Solution:**

$$r = 100 \text{ m}$$

$$\theta = \frac{\pi}{2} \text{ rad}$$

$$s = r\theta = 100 \times \frac{\pi}{2} = 157.08 \text{ m}$$

$$u = 18 \text{ kmph} = 5 \text{ m/s}$$

$$v = 36 \text{ kmph} = 10 \text{ m/s}$$

$$\omega_0 = \frac{u}{r} = \frac{5}{100} = 0.05 \text{ rad/sec}$$

$$\omega = \frac{\theta}{t} = \frac{10}{100} = 0.1 \text{ rad/sec}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$0 = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

$$\frac{\pi}{2} = \frac{(0.1)^2 - (0.05)^2}{2\alpha}$$

$$\alpha = \frac{(0.1)^2 - (0.05)^2}{\pi} = 2.387 \times 10^{-3} \text{ rad/sec}^2$$

When entering the curve

$$a_t = r\alpha = (2.387 \times 10^{-3}) \times (100) = 0.2387 \text{ m/sec}^2$$

$$a_n = \omega^2 r = (0.05)^2 \times 100 = 0.25 \text{ m/sec}^2$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(0.25)^2 + (0.2387)^2}$$

$$a = 0.345 \text{ m/sec}^2$$

$$\tan \alpha = \frac{a_n}{a_t}$$

$$\alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{(0.25)}{(0.2387)}$$

$$\tan \alpha = \frac{a_n}{a_t}; \alpha = 46.32^\circ$$

$$\alpha = \tan^{-1} \frac{a_n}{a_t} = \tan^{-1} \frac{(1)}{(0.2387)}$$

$$\alpha = 76.57^\circ$$

$$v_{car} = 72 \text{ kmph} = 20 \text{ m/sec}$$

$$\text{wheel dia} = 75 \text{ cm} = 0.75 \text{ m}; r = 0.375$$

36km

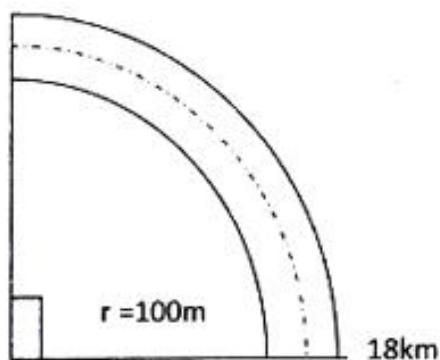


Fig.4.44

When leaving the curve

$$a_t = r\alpha = (2.387 \times 10^{-3}) \times (100) = 0.2387 \text{ m/sec}^2$$

$$a_n = \omega^2 r = (0.1)^2 \times 100 = 1 \text{ m/sec}^2$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(1)^2 + (0.2387)^2}$$

$$a = 1.028 \text{ m/sec}^2$$

$$\omega = 0; \omega_0 = 53.32 \text{ rad}; \theta = 53.33 \text{ rad}$$

$$\omega = v/r = 20/0.375 = 53.32 \text{ rad/sec}$$

$$s = 20 \text{ m} = r\theta$$

$$20 = 0.375 \times \theta$$

$$\theta = \frac{20}{0.375} = 53.33 \text{ radians}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$0 = (53.32)^2 + 2 \times \alpha \times 53.33$$

$$\alpha = \frac{-(53.32)^2}{2 \times 53.33} = -26.66 \text{ rad/sec}^2$$

$$\omega = 53.32 \text{ rad/sec}; \alpha = -26.67 \text{ rad/s}^2$$

### 3-D CURVILINEAR MOTION:

The general case of three-dimensional motion of a particle along a space curve was illustrated in fig.4.1. Three coordinate systems, rectangular (x-y-z), cylindrical (r-θ-z), and spherical (R-<sub>ϕ</sub> - <sub>θ</sub>), are commonly used to describe this motion. These systems are indicated in fig.4.45, which also shows the unit vectors for the three coordinate systems.

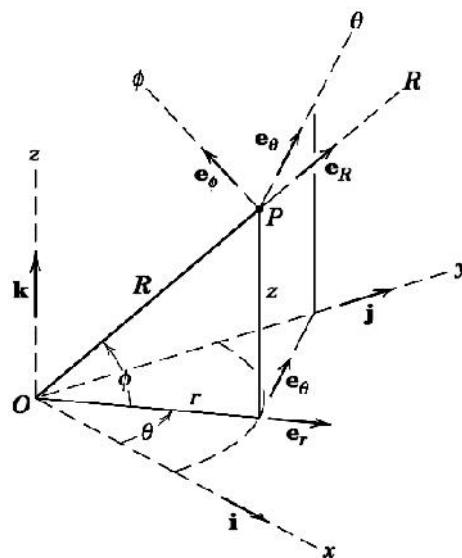


Fig.4.45

#### 1) Rectangular Coordinates (x-y-z)

The extension from two to three dimensions offers no particular difficulty. It merely add the z-coordinate and its two time derivatives to the two-dimensional expressions of Eqs.4.16, so that the position vector R, the velocity v, and the acceleration a become

$$\left. \begin{aligned} R &= xi + yj + zk \\ v &= \dot{R} = \dot{x}i + \dot{y}j + \dot{z}k \\ a &= \ddot{v} = \ddot{R} = \ddot{x}i + \ddot{y}j + \ddot{z}k \end{aligned} \right\} \quad \text{-----} \rightarrow (4.28)$$

Note that in three dimensions we are using R in place of r for the position vector.

## 2) Cylindrical Coordinates (r-, $\theta$ -z)

If we understand the polar-coordinate description of plane motion, then there should be no difficulty with cylindrical coordinates because all that is required is the addition of the z-coordinate and its two time derivatives. The position vector R to the particle for cylindrical coordinates is simply

$$R = re_r + zk$$

In place of equation  $v = \dot{r}e_r + r\dot{\theta}e_\theta$  for plane motion, it can write the velocity as

$$v = \dot{r}e_r + r\dot{\theta}e_\theta + \dot{z}k \quad \text{-----} \rightarrow (4.29)$$

Where,  $v_r = \dot{r}$ ,  $v_\theta = r\dot{\theta}$ ,  $v_z = \dot{z}$  and  $v = \sqrt{v_r^2 + v_\theta^2 + v_z^2}$

Similarly the acceleration is written by the adding the z – component to equation

$$a = \left( \ddot{r} - r\dot{\theta}^2 \right) e_r + \left( r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) e_\theta \text{ which gives us}$$

$$a = \left( \ddot{r} - r\dot{\theta}^2 \right) e_r + \left( r\ddot{\theta} + 2\dot{r}\dot{\theta} \right) e_\theta + \ddot{z}k \quad \text{-----} \rightarrow (4.30)$$

Where,  $a_r = \ddot{r} - r\dot{\theta}^2$ ,  $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \frac{d}{dt} \left( r^2 \dot{\theta} \right)$ ,  $a_z = \ddot{z}$  and

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$

Whereas the unit vectors  $e_r$  and  $e_\theta$  have non zero time derivatives due to the changes in their directions, it notes at that the unit vector k in the z-direction remains fixed in direction and therefore has a zero time derivative.

### 3) Spherical Coordinates (R, $\theta$ , $\phi$ ):

Spherical coordinates  $R, \theta, \phi$  are utilized when a radial distance and two angles are utilized to specify the position of a particle, as in the case of radar measurements, for example. Derivation of the expression for the velocity  $v$  is easily obtained, but the expression for the acceleration 'a' is more complex because of the added geometry. Consequently, only the results will be cited here. First it designates the unit vectors  $e_R, e_\theta, e_\phi$  as shown in fig.4.45. Note that the unit vector  $e_R$  is in the direction in which the particle P would move if  $R$  increases but  $\theta$  and  $\phi$  are held constant. The unit vector  $e_\theta$  is in the direction in which P would move if  $\theta$  increases while  $R$  and  $\phi$  are held constant. Finally, the unit vector  $e_\phi$  is in the direction in which P would move if  $\phi$  increases while  $R$  and  $\theta$  are held constant. The resulting expressions for  $v$  and  $a$  are

$$v = v_r e_r + v_\theta e_\theta + v_\phi e_\phi \quad \rightarrow (4.31)$$

Where,  $v_R = \dot{R}$ ,  $v_\theta = R \dot{\theta} \cos\phi$  and  $v_\phi = R \dot{\phi}$

And

$$a = a_R e_R + a_\theta e_\theta + a_\phi e_\phi \quad \rightarrow (4.32)$$

Where,  $a_R = \ddot{R} - R \dot{\theta}^2 - R \dot{\phi}^2 \cos^2\phi$ ,  $a_\theta = \frac{\cos\phi}{R} \frac{d}{dt} \left( R^2 \dot{\theta} \right) - 2R \dot{\theta} \dot{\phi} \sin\phi$  and

$$a_\phi = \frac{1}{R} \frac{d}{dt} \left( R^2 \dot{\phi} \right) + R \dot{\phi}^2 \sin\phi \cos\phi$$

Linear algebraic transformations between any two of the three coordinate-system expressions for velocity or acceleration can be developed. These transformations make it possible to express the motion component in rectangular coordinates, for example, if the components are known in spherical coordinates, or vice versa. These transformations are easily handled with the aid of matrix algebra and a simple computer program.

## Problems

**SP4.77:** The power screw starts from rest and is given a rotational speed  $\dot{\theta}$  which increases uniformly with time  $t$  according to  $\dot{\theta} = kt$ , where  $k$  is a constant. Determine the expressions for the velocity  $v$  and acceleration  $a$  of the centre of ball A when the screw has turned through one complete revolution from rest. The lead of the screw (advancement per revolution) is  $L$ .

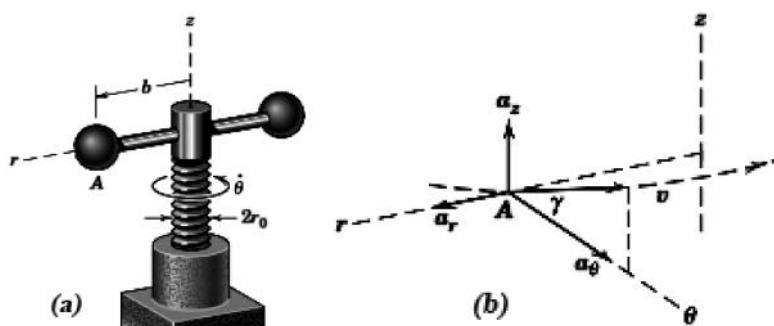


Fig.4.46

### Solution:

The center of ball A moves in a helix on the cylindrical surface of radius  $b$ , and the cylindrical coordinates  $r, \theta, z$  are clearly indicated.

Integrating the given relation for  $\dot{\theta}$  gives  $\theta = \Delta\theta = \int \dot{\theta} dt = \frac{1}{2}kt^2$ . For one revolution from rest we have

$$2\pi = \frac{1}{2}kt^2$$

giving

$$t = 2\sqrt{\pi/k}$$

Thus, the angular rate at one revolution is

$$\dot{\theta} = kt = k(2\sqrt{\pi/k}) = 2\sqrt{\pi k}$$

The helix angle  $\gamma$  of the path followed by the center of the ball governs the relation between the  $\theta$ - and  $z$ -components of velocity and is given by  $\tan \gamma = L/(2\pi b)$ . Now from the figure we see that  $v_\theta = v \cos \gamma$ . Substituting  $v_\theta = r\dot{\theta} = b\dot{\theta}$  from Eq. 2/16 gives  $v = v_\theta/\cos \gamma = b\dot{\theta}/\cos \gamma$ . With  $\cos \gamma$  obtained from  $\tan \gamma$  and with  $\dot{\theta} = 2\sqrt{\pi k}$ , we have for the one-revolution position

$$v = 2b\sqrt{\pi k} \frac{\sqrt{L^2 + 4\pi^2 b^2}}{2\pi b} = \sqrt{\frac{k}{\pi}} \sqrt{L^2 + 4\pi^2 b^2} \quad \text{Ans.}$$

The acceleration components from Eq. 2/17 become

$$[a_r = \ddot{r} - r\dot{\theta}^2] \quad a_r = 0 - b(2\sqrt{\pi k})^2 = -4b\pi k$$

$$[a_\theta = r\ddot{\theta} + 2r\dot{\theta}] \quad a_\theta = bk + 2(0)(2\sqrt{\pi k}) = bk$$

$$[a_z = \ddot{z} = \dot{v}_z] \quad a_z = \frac{d}{dt}(v_z) = \frac{d}{dt}(v_\theta \tan \gamma) = \frac{d}{dt}(b\dot{\theta} \tan \gamma)$$

$$= (b \tan \gamma)\ddot{\theta} = b \frac{L}{2\pi b} k = \frac{kL}{2\pi}$$

Now we combine the components to give the magnitude of the total acceleration, which becomes

$$a = \sqrt{(-4b\pi k)^2 + (bk)^2 + \left(\frac{kL}{2\pi}\right)^2}$$

$$= bk \sqrt{1 + 16\pi^2} + L^2/(4\pi^2 b^2) \quad \text{Ans.}$$

**SP4.78:** An aircraft P takes off at A with a velocity  $v_0$  of 250kmph and climbs in the vertical  $y'$ - $z'$  plane at the constant  $15^\circ$  angle with an acceleration along its flight path of  $0.8 \text{ m/s}^2$ . Flight progress is monitored by radar at point O. (a) Resolve the velocity of P into cylindrical coordinate components 60 seconds after take-off and find  $\dot{r}$ ,  $\dot{\theta}$  and  $\dot{z}$  for that instant. (b) Resolve the velocity of the aircraft P into spherical coordinate components 60 seconds after take-off and find  $\dot{R}$ ,  $\dot{\theta}$  and  $\dot{\phi}$  for that instant.

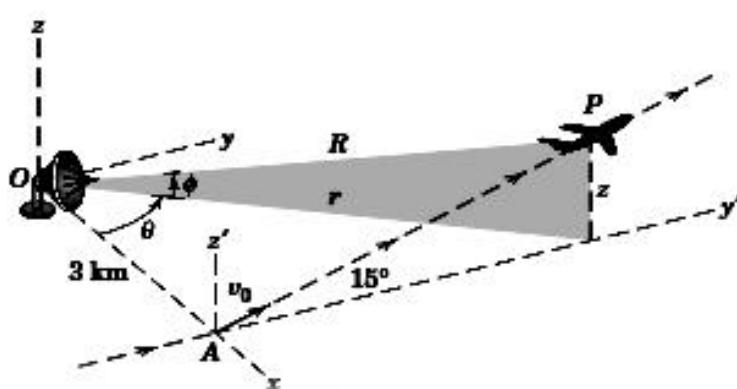
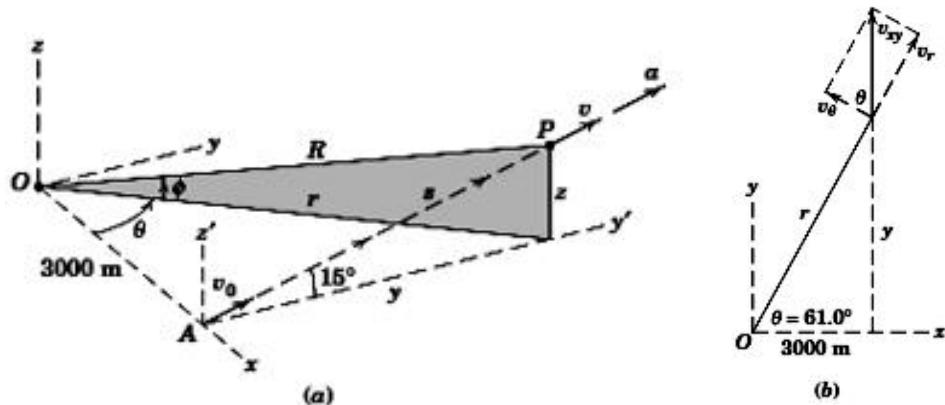


Fig.4.47

**Solution:**



(a) The accompanying figure shows the velocity and acceleration vectors in the  $y'$ - $z'$  plane. The takeoff speed is

$$v_0 = \frac{250}{3.6} = 69.4 \text{ m/s}$$

and the speed after 60 seconds is

$$v = v_0 + at = 69.4 + 0.8(60) = 117.4 \text{ m/s}$$

The distance  $s$  traveled after takeoff is

$$s = s_0 + v_0 t + \frac{1}{2} a t^2 = 0 + 69.4(60) + \frac{1}{2} (0.8)(60)^2 = 5610 \text{ m}$$

The  $y$ -coordinate and associated angle  $\theta$  are

$$y = 5610 \cos 15^\circ = 5420 \text{ m}$$

$$\theta = \tan^{-1} \frac{5420}{3000} = 61.0^\circ$$

From the figure (b) of  $x$ - $y$  projections, we have

$$r = \sqrt{3000^2 + 5420^2} = 6190 \text{ m}$$

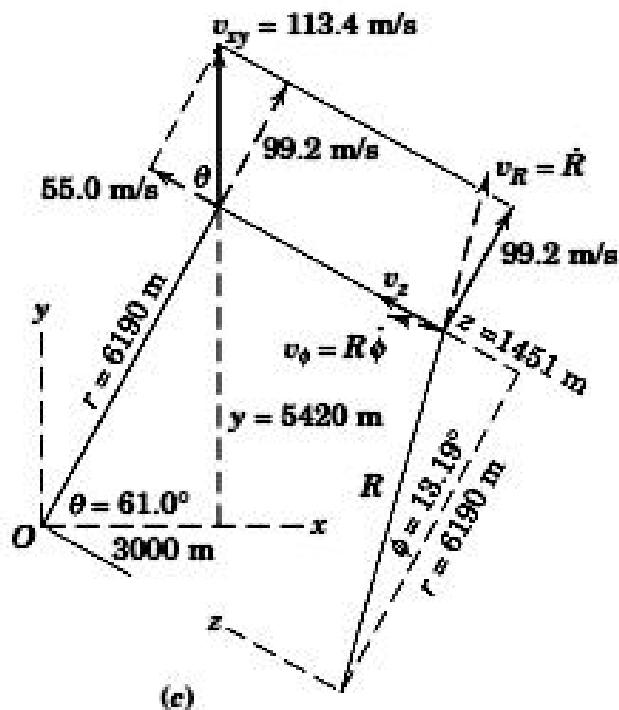
$$v_{xy} = v \cos 15^\circ = 117.4 \cos 15^\circ = 113.4 \text{ m/s}$$

$$v_r = \dot{r} = v_{xy} \sin \theta = 113.4 \sin 61.0^\circ = 99.2 \text{ m/s} \quad \text{Ans.}$$

$$v_\theta = r \dot{\theta} = v_{xy} \cos \theta = 113.4 \cos 61.0^\circ = 55.0 \text{ m/s}$$

$$\text{So} \quad \dot{\theta} = \frac{55.0}{6190} = 8.88(10^{-3}) \text{ rad/s} \quad \text{Ans.}$$

Finally  $z = v_z = v \sin 15^\circ = 117.4 \sin 15^\circ = 30.4 \text{ m/s}$  **Ans.**



(b) Refer to the accompanying figure (c), which shows the  $x$ - $y$  plane and various velocity components projected into the vertical plane containing  $r$  and  $R$ . Note that

$$z = r \tan 15^\circ = 5420 \tan 15^\circ = 1451 \text{ m}$$

$$\phi = \tan^{-1} \frac{z}{r} = \tan^{-1} \frac{1451}{6190} = 13.19^\circ$$

$$R = \sqrt{r^2 + z^2} = \sqrt{6190^2 + 1451^2} = 6360 \text{ m}$$

**From the figure,**

$$v_B = \dot{R} = 99.2 \cos 13.19^\circ + 30.4 \sin 13.19^\circ = 103.6 \text{ m/s} \quad \text{Ans.}$$

$$\dot{\theta} = 8.88(10^{-3}) \text{ rad/s, as in part (a)} \quad \text{Ans.}$$

$$v_A = R\dot{\phi} = 30.4 \cos 13.19^\circ - 99.2 \sin 13.19^\circ = 6.95 \text{ m/s}$$

$$\dot{\phi} = \frac{6.95}{6360} = 1.093(10^{-3}) \text{ rad/s} \quad \text{Ans.}$$

## RELATIVE MOTION (TRANSLATING AXES):

In the previous sections, we have described particle motion using coordinates referred to fixed reference axes. The displacements, velocities and accelerations so determined are termed absolute. It is not always possible or convenient, however, to use a fixed set of axes to describe or to measure motion. In addition, there are many engineering problems for which the analysis of motion is simplified by using measurements made with respect to a moving reference system. These measurements, when combined with the absolute motion of the moving coordinate system, enable us to determine the absolute motion in question. This approach is called a relative-motion analysis.

Now consider two particles A and B which may have separate curvilinear motions in a given plane or in parallel planes, fig.4.48. We will arbitrarily attach the origin of a set of translating (nonrotating) axes x-y to particle B and observe the motion of A from our moving position on B.

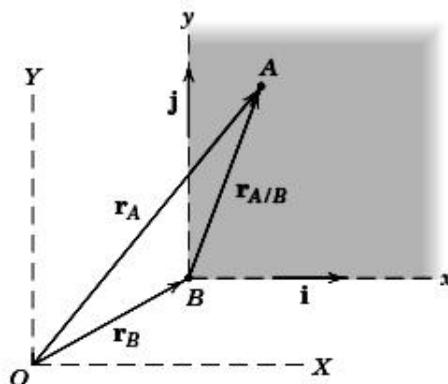


Fig.4.48

The position vector of A as measured relative to the frame x-y is  $r_{A/B} = xi + yj$  r , where the subscript notation “A / B” means “A relative to B” or “A with respect to B.” The unit vectors along the x- and y-axes are i and j, and x and y are the coordinates of ‘A’ measured in the x-y frame. The absolute position of B is defined by the vector  $r_B$  measured from the origin of the fixed axes X-Y. The absolute position of A is seen, therefore, to be determined by the vector

$$\dot{r}_A = \dot{r}_B + \dot{r}_{A/B}$$

Now differentiate this vector equation once with respect to time to obtain velocities and twice to obtain accelerations. Thus,

$$\dot{\dot{r}}_A = \dot{\dot{r}}_B + \dot{\dot{r}}_{A/B} \quad \text{or} \quad \dot{v}_A = \dot{v}_B + \dot{v}_{A/B} \quad \rightarrow (4.33)$$

$$\ddot{\dot{r}}_A = \ddot{\dot{r}}_B + \ddot{\dot{r}}_{A/B} \quad \text{or} \quad \ddot{a}_A = \ddot{a}_B + \ddot{a}_{A/B} \quad \rightarrow (4.34)$$

In Eq.4.33 the velocity which we observe A to have from our position at B attached to the moving axes x-y is  $\dot{v}_{A/B} = \dot{x}i + \dot{y}j$ . This term is the velocity of A with respect to B. Similarly, in Eq.4.34 the acceleration which we observe A to have from our nonrotating position on B is  $\ddot{v}_{A/B} = \ddot{x}i + \ddot{y}j$ . This term is the acceleration of A with respect to B. We note that the unit vectors i and j have zero derivatives because their directions as well as their magnitudes remain unchanged.

Equation 4.33 (or 4.34) states that the absolute velocity (or acceleration) of A equals the absolute velocity (or acceleration) of B plus, vectorially, the velocity (or acceleration) of A relative to B. The relative term is the velocity (or acceleration) measurement which an observer attached to the moving coordinate system x-y would make. We can express the relative-motion terms in whatever coordinate system is convenient—rectangular, normal and tangential, or polar—and the formulations in the preceding articles can be used for this purpose.

### Problems:

**SP4.79:** Passengers in the jet transport A flying east at a speed of 800kmph observe a second jet plane B that passes under the transport in horizontal flight. Although the nose of B is pointed in the  $45^\circ$  northeast direction plane B appears to the passengers in A to be moving away from the transport at the  $60^\circ$  angle as shown in fig.4.. Determine the true velocity of B.

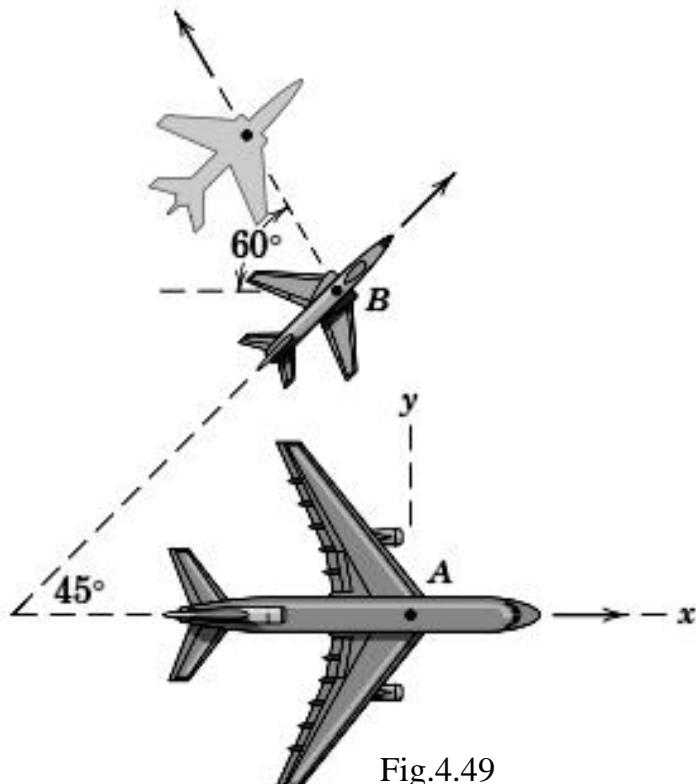


Fig.4.49

**Solution:**

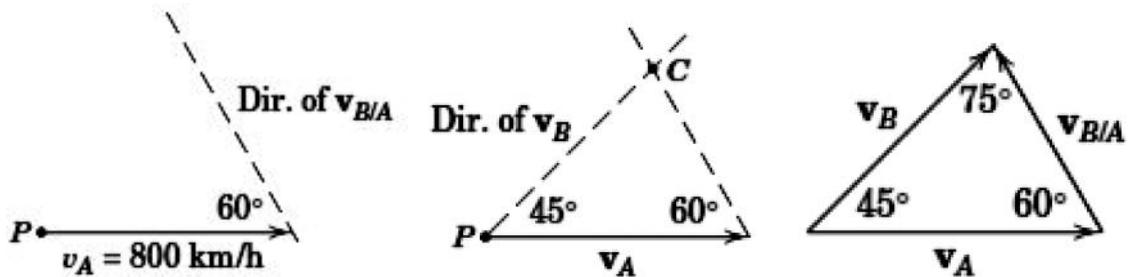
The moving reference axes  $x$ - $y$  are attached to  $A$ , from which the relative observations are made. We write, therefore,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

Next we identify the knowns and unknowns. The velocity  $\mathbf{v}_A$  is given in both magnitude and direction. The  $60^\circ$  direction of  $\mathbf{v}_{B/A}$ , the velocity which  $B$  appears to have to the moving observers in  $A$ , is known, and the true velocity of  $B$  is in the  $45^\circ$  direction in which it is heading. The two remaining unknowns are the magnitudes of  $\mathbf{v}_B$  and  $\mathbf{v}_{B/A}$ . We may solve the vector equation in any one of three ways.

**(I) Graphical.** We start the vector sum at some point  $P$  by drawing  $\mathbf{v}_A$  to a convenient scale and then construct a line through the tip of  $\mathbf{v}_A$  with the known direction of  $\mathbf{v}_{B/A}$ . The known direction of  $\mathbf{v}_B$  is then drawn through  $P$ , and the intersection  $C$  yields the unique solution enabling us to complete the vector triangle and scale off the unknown magnitudes, which are found to be

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$



**(II) Trigonometric.** A sketch of the vector triangle is made to reveal the trigonometry, which gives

$$\frac{v_B}{\sin 60^\circ} = \frac{v_A}{\sin 75^\circ} \quad v_B = 800 \frac{\sin 60^\circ}{\sin 75^\circ} = 717 \text{ km/h} \quad \text{Ans.}$$

**(III) Vector Algebra.** Using unit vectors  $\mathbf{i}$  and  $\mathbf{j}$ , we express the velocities in vector form as

$$\begin{aligned} \mathbf{v}_A &= 800\mathbf{i} \text{ km/h} & \mathbf{v}_B &= (v_B \cos 45^\circ)\mathbf{i} + (v_B \sin 45^\circ)\mathbf{j} \\ \mathbf{v}_{B/A} &= (v_{B/A} \cos 60^\circ)(-\mathbf{i}) + (v_{B/A} \sin 60^\circ)\mathbf{j} \end{aligned}$$

Substituting these relations into the relative-velocity equation and solving separately for the  $\mathbf{i}$  and  $\mathbf{j}$  terms give

$$\begin{aligned} (\mathbf{i}\text{-terms}) \quad v_B \cos 45^\circ &= 800 - v_{B/A} \cos 60^\circ \\ (\mathbf{j}\text{-terms}) \quad v_B \sin 45^\circ &= v_{B/A} \sin 60^\circ \end{aligned}$$

Solving simultaneously yields the unknown velocity magnitudes

$$v_{B/A} = 586 \text{ km/h} \quad \text{and} \quad v_B = 717 \text{ km/h} \quad \text{Ans.}$$

It is worth noting the solution of this problem from the viewpoint of an observer in  $B$ . With reference axes attached to  $B$ , we would write  $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$ . The apparent velocity of  $A$  as observed by  $B$  is then  $\mathbf{v}_{A/B}$ , which is the negative of  $\mathbf{v}_{B/A}$ .

**SP4.80:** Car A is accelerating in the direction of its motion at the rate of  $1.2 \text{ m/s}^2$ . Car B is rounding a curve of  $150 \text{ m}$  radius at a constant speed of  $54 \text{ kmph}$ . Determine the velocity and acceleration which car B appears to have to an observer in car A if car A has reached a speed of  $72 \text{ kmph}$  for the positions represented.

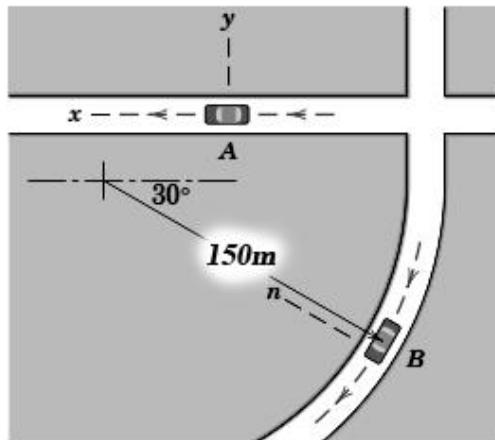


Fig.4.50

**Solution:**

Now choose non rotating reference axes attached to car A since the motion of B with respect to A is desired.

Velocity; The relative velocity equation is  $v_B = v_A + v_{A/B}$  and the velocities of A and B for the position considered have the magnitudes

$$v_A = \frac{72}{3.6} = 20 \text{ m/s} \text{ and } v_B = \frac{54}{3.6} = 15 \text{ m/s}$$

The triangle of velocity vectors is drawn in the sequence required by the equation, and application of the law of cosines and the law of sines gives

$$v_{A/B} = \sqrt{v_B^2 + v_A^2 - 2v_A v_B \cos 60} = \sqrt{20^2 + 15^2 - 2 \times 20 \times 15 \times \cos 60} = 18.03 \text{ m/s}$$

$$\frac{v_{A/B}}{\sin 60} = \frac{v_B}{\sin \theta}, \Rightarrow \theta = \sin^{-1} \left( \frac{v_B}{v_{A/B}} \sin 60 \right) = \sin^{-1} \left( \frac{15}{18.03} \sin 60 \right) = 46.1^\circ$$

Acceleration; The relative acceleration equation is  $a_B = a_A + a_{A/B}$ . The acceleration of A is given and the acceleration of B is normal to the curve in the n-direction and has the magnitude

$$a_n = \frac{v^2}{r}, \Rightarrow a_B = \frac{15^2}{150} = 1.5 \text{ m/s}^2$$

The triangle of acceleration vectors is drawn in the sequence required by the equation as illustrated. Solving for the x and y components of  $a_{A/B}$  gives us

$$(a_{B/A})_x = 1.5 \cos 30^\circ - 1.2 = 0.0990 \text{ m/s}^2$$

$$(a_{B/A})_y = 1.5 \sin 30^\circ = 0.750 \text{ m/s}^2$$

$$\text{From which } a_{B/A} = \sqrt{0.0990^2 + 0.750^2} = 0.757 \text{ m/s}^2$$

The direction of  $a_{B/A}$  may be specified by the angle  $s$  which, by the law of sines becomes

$$\frac{1.5}{\sin s} = \frac{0.757}{\sin 30^\circ}, \Rightarrow s = \sin^{-1} \left( \frac{1.5}{0.757} \sin 30^\circ \right) = 97.5^\circ$$

**SP.4.81:** A passenger train 250m long moving with a velocity of 72 kmph overtakes a goods train moving on a parallel path in the same direction completely in 45 sec. if the length of the goods train is 200m determine the speed of goods train.

**Solution:**

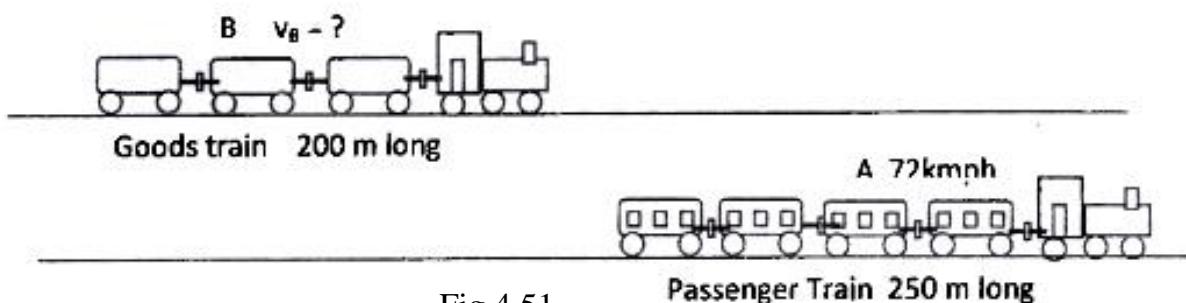


Fig.4.51

The relative distance to be covered to overtake the train B by A =  $250 + 200 = 450 \text{ m}$

Relative velocity  $V_{A/B} = V_A - V_B = 20 - V_B$

Time to overtake = 45 s

We know that

Relative distance = Relative velocity x time

$$450 = (20 - V_B) \times 45$$

$$20 - V_B = 450 / 45 = 10$$

$$V_B = 10 \text{ m/s} = 36 \text{ kmph}, \text{ The speed of goods train} = 36 \text{ kmph}$$

**SP4.82:** Two trains A and B are moving on parallel tracks with velocity 72 kmph and 36 kmph in the same direction. The driver of train A finds that it took 42 sec. to overtake train B. While driver of train B find that train A took 30 sec. to overtake him. Determine a) Length of each train and b) time taken for complete overtaking. If the two trains move in opposite direction with the same velocities, how much time is taken for complete crossing.

**Solution:**

$v_A = 72 \text{ kmph} = 20 \text{ m/s}$ ,  $v_B = 36 \text{ kmph} = 10 \text{ m/s}$ ,  $L_A$  = length of train A, and  $L_B$  = length of train B.

To overtake the train B by train A, time taken is 42 seconds, then

$$L_B = (v_A - v_B)t = (20 - 10) \times 42 = 420 \text{ m}$$

When the trains moving in the same direction, for complete overtaking the distance to be travelled as

$$= 420 + 300 = 720 \text{ m}$$

$$\text{Therefore, } 720 = (v_A - v_B)t = (20 - 10)t, \Rightarrow t = \frac{720}{30} = 24 \text{ s}$$

**SP4.83:** A railway coach having an ordinary cross seat arrangement is travelling at 14.4 kmph. A person runs at 5 m/s on the platform. In what direction he must run so that he may enter parallel to the seats? Also find the velocity with which he entered.

**Solution:**

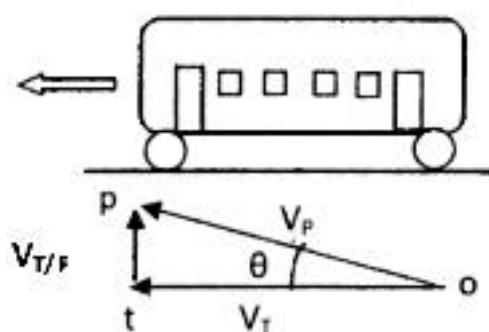


Fig.4.52

$$v_T = 14.4 \text{ kmph} = 4 \text{ m/s},$$

The relative velocity of the passenger with respect to the train

$$v_{P/T} = \sqrt{v_P^2 - v_T^2} = \sqrt{5^2 - 4^2} = 3 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.86^\circ$$

**SP4.84:** A cyclist travelling east with a speed of 2m/s feels the wind to be blowing directly from north. On doubling his speed he feels the wind to blow from north-east. Find the direction and the speed of the wind.

**Solution:**

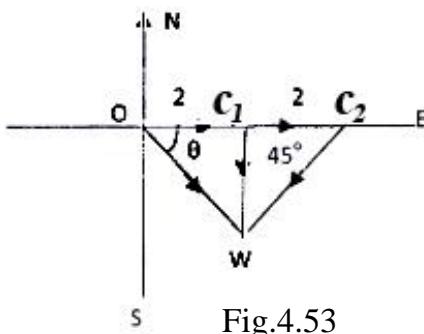


Fig.4.53

Let  $OC_1$  – Cycle velocity = 2m/s

$OC_2$  – Cycle velocity = 4m/s

$C_1W$  – Feels wind is blowing from North

$C_2W$  – Feels wind is blowing from North East

$OW$  – Wind velocity

From the right angled triangle  $C_1C_2W$ ,

$$C_1C_2 = C_1W = 2 \text{ m/s},$$

$$\text{Hence, } OW = \sqrt{2^2 + 2^2} = 2.828 \text{ m/s}$$

$$\text{To find direction, } \theta = \tan^{-1}\left(\frac{2}{2}\right) = 45^\circ$$

Therefore, the Wind is blowing from North West.

**SP4.85:** Two ships A and B leave a port at the same time. The ship A is travelling north – west at 32kmph and ship B 40° south of west at 24kmph. Determine a) the speed of the ship B relative to ship A and b) after what time they will be 150km apart?

**Solution:**

$$V_{B/A} = \sqrt{V_A^2 + V_B^2 - 2V_A V_B \cos(45 + 40)^\circ}$$

$$V_{B/A} = \sqrt{32^2 + 24^2 - 2 \times 32 \times 24 \cos(85)^\circ}$$

$$V_{B/A} = \sqrt{1461.12} = 38.29 \text{ kmph}$$

Time at which the two ships are 150 km away

Relative distance = Relative velocity x time

$$150 = V_{B/A} \times t$$

$$t = 150/38.29 = 3.91 \text{ hrs}$$

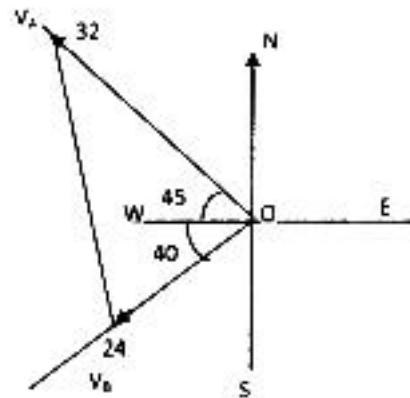


Fig.4.54

**SP4.86:** A steam ship is travelling north at the rate of 20kmph and there is a wind blowing from north –east at 30kmph. Find the direction in which the smoke from the chimney will appear to an observer sitting on the ship.

**Solution:**

OS – Ship velocity = 20kmph

OW – Wind velocity = 30kmph

From the triangle SOW, the angle SOW is 135°

SW – Relative velocity of the ship with respect to wind

$$v_{S/W} = \sqrt{30^2 + 20^2 - 2 \times 30 \times 20 \cos 135^\circ}$$

$$v_{S/W} = 46.35 \text{ kmph}$$

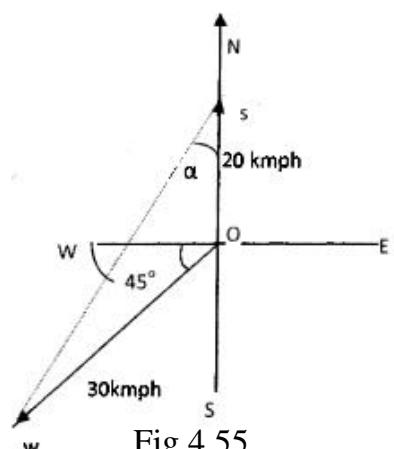


Fig.4.55

Using law of sine,

$$\frac{v_{S/W}}{\sin 135^\circ} = \frac{30}{\sin \alpha}, \Rightarrow \alpha = \sin^{-1} \left( \frac{\sin 135^\circ}{46.35} 30 \right) = 27.22^\circ$$

The direction of smoke from the chimney is 27.22°

**SP4.87:** When a cyclist is riding at 20kmph he finds the rain meeting him at an angle of  $45^\circ$  with vertical. When he rides at 12kmph, he meets the rain at an angle of  $30^\circ$  with the vertical. Find the magnitude and direction of the absolute velocity of the rain.

**Solution:**

Let the distance  $OD = x$

$$\therefore RD = \tan 45^\circ (20 - x) \dots (1)$$

$OC_1$  – velocity of cycle = 20 kmph

From triangle  $DC_2R$

$OC_2$  – velocity of cycle = 12 kmph

$$\tan 60^\circ = \frac{RD}{C_2D} = \frac{RD}{12 - x}$$

From the triangle  $DC_3R$

$$\therefore RD = \tan 60^\circ (12 - x) \dots (2)$$

$$\tan 45^\circ = \frac{RD}{C_1D} = \frac{RD}{20 - x}$$

From Eqn. (1) and (2)

$$\tan 45^\circ (20 - x) = \tan 60^\circ (12 - x)$$

$$(20 - x) = (12 - x) \tan 60^\circ / \tan 45^\circ$$

$$x = 1.072 \text{ kmph}$$

$$RD = 18.92$$

Therefore velocity of rain

$$V_R = \sqrt{x^2 + RD^2} = \sqrt{1.072^2 + 18.92^2}$$

$$V_R = 18.95 \text{ kmph}$$

$$\alpha = \tan^{-1} \frac{RD}{x} = \tan^{-1} \frac{18.92}{1.072} = 86.75^\circ$$

$$\text{Direction of velocity of rain with respect to vertical} = 90 - 86.75 = 3.24^\circ$$

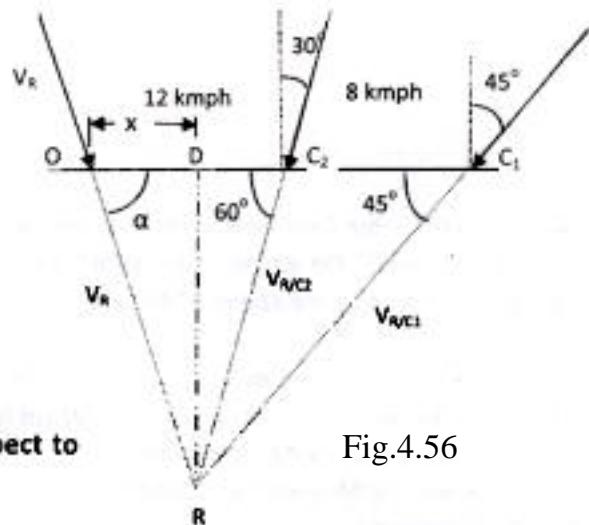


Fig.4.56

**SP4.88:** Car A when at a distance of 32.5m north of a crossing was observed to start from rest and travelling towards the crossing with a constant acceleration of  $1\text{m/s}^2$ . At the same instant another car B was observed to be just at the crossing and travelling east at a constant speed of 36kmph. Find the position, velocity and acceleration of car A relative to car B 5 seconds after the observation.

**Solution:**

Car A is moving with uniform acceleration towards crossing;

$$u = 0, a = 1\text{m/s}^2, t = 5\text{s}$$

Using the relation,

$$S = ut + \frac{1}{2}at^2$$

$$S = (0)(5) + \frac{1}{2}(1)(5)^2$$

$$S = 12.5 \text{ m}$$

$$v = u + at ,$$

$$\Rightarrow v = (0)(5) + (1)(5) = 5 \text{ m/s}$$

Car B is moving with uniform velocity;  $S = vt$ ,

$$S = 10 \times 5 = 50 \text{ m}$$

Relative position of A with respect to B is  $= \sqrt{20^2 + 50^2} = 53.85 \text{ m}$

$$\theta = \tan^{-1} \left( \frac{20}{50} \right) = 21.8^\circ$$

Relative acceleration,  $a_A = 1 \text{ m/s}^2$ ,  $a_B = 0 \text{ m/s}^2$  (velocity is uniform) and  $a_{A/B} = \sqrt{1^2 + 0^2 - 2 \times 1 \times 0 \times \cos 90^\circ} = 1 \text{ m/s}^2$

Relative velocity of A with respect to B after 5 second is  $v_A = 5 \text{ m/s}$ ,  $v_B = 10 \text{ m/s}$  and

$$v_{A/B} = \sqrt{5^2 + 10^2 - 2 \times 5 \times 10 \times \cos 90^\circ} = 11.2 \text{ m/s}$$

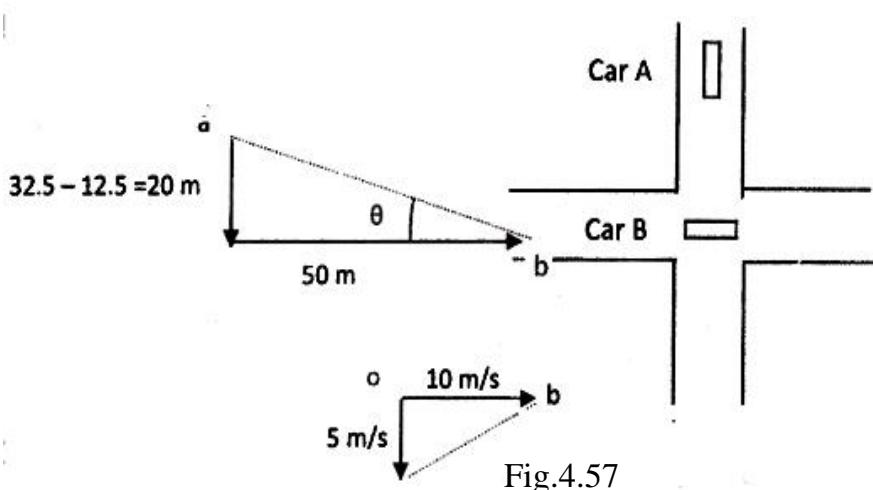


Fig.4.57

**SP4.89:** A submarine is travelling on a course of bearing  $80^\circ$  East of North with a speed of 21 knots reports the presence of an enemy ship travelling on a course of bearing  $135^\circ$  West of North with a speed of 15 knots with respect to the submarine. Assuming that the submarine and the enemy ship are at the same place at the time of sighting. A) find the true direction and speed of the enemy ship, B) if a warship stationed 150 nautical miles South of the submarine immediately starts at 18 knots to intercept the enemy ship ( $34.66^\circ$  South of East,  $12.24$  knots,  $34^\circ$  East of North, 6.85 hour). 1 nautical mile = 6080 feet. A ship travelling 6080 ft per hour is said to be travelling at a speed of 1 knot.

**Solution:**

From the triangle ose

$$\text{Angle ose} = 45 - 10 = 35^\circ$$

$$Oe = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \cos 35}$$

$$Oe = 12.24 \text{ knots}$$

Using Sine Law

$$\frac{15}{\sin \alpha} = \frac{12.24}{\sin 35}$$

$$\alpha = 44.66^\circ$$

where  $\alpha$  is the angle eOs

Hence the direction of Enemy ship

$$= 44.66 - 10 = 34.66^\circ \text{ South of East}$$

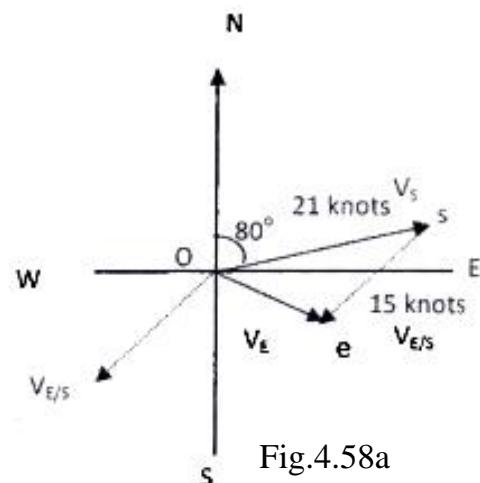


Fig.4.58a

To find the time and direction of warship:

Velocity of enemy ship = 12.24 knots

Let 't' be the time to catch the enemy ship by the war ship

During this time the enemy ship travels a distance of  $12.24 \cdot t$

OE – the distance traveled by the enemy ship

OW- distance of war ship at the beginning  
= 150 nautical miles

WE – distance traveled by the war ship in time 't'  
=  $18 \cdot t$

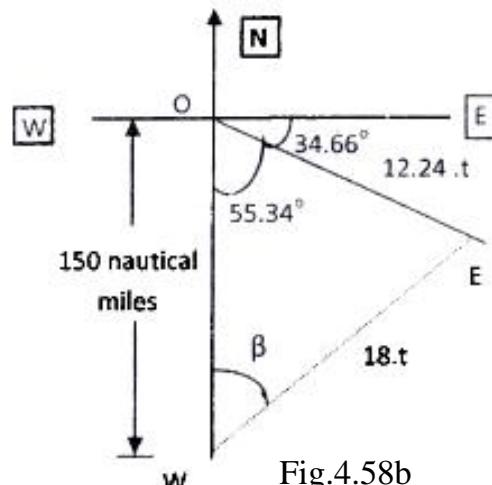


Fig.4.58b

Consider the triangle OEW

Let the angle OWE =  $\beta$

$$\text{Angle OEW} = 180 - 55.34 - 34 = 90.66$$

Using Sine Law

$$\frac{12.24 \cdot t}{\sin \beta} = \frac{18 \cdot t}{\sin 55.34}$$

$$\beta = 34^\circ$$

Using Sin law

$$\frac{150}{\sin 90.66} = \frac{18 \cdot t}{\sin 55.34}$$

$$t = 6.85 \text{ hrs; 6 hrs; 51 min.}$$

War ship travels  $34^\circ$  East of North

## CONSTRAINED MOTION:

Sometimes the motions of particles are interrelated because of the constraints imposed by interconnecting members. In such cases it is necessary to account for these constraints in order to determine the respective motions of the particles.

### 1) One Degree of Freedom

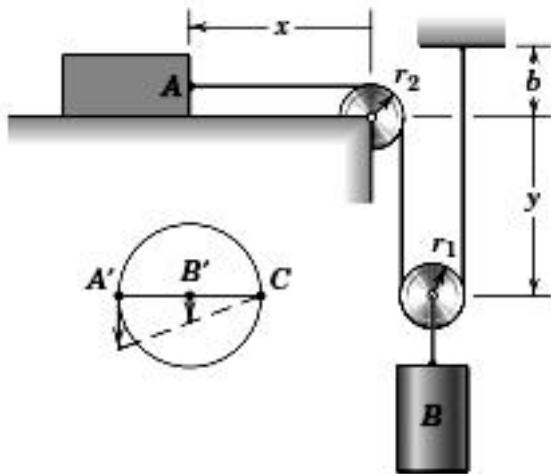


Fig.4.59

Consider first the very simple system of two interconnected particles A and B shown in fig.4.59. It should be quite evident by inspection that the horizontal motion of A is twice the vertical motion of B. Nevertheless we will use this example to illustrate the method of analysis which applies to more complex situations where the results cannot be easily obtained by inspection. The motion of B is clearly the same as that of the centre of its pulley, so we establish position coordinates  $y$  and  $x$  measured from a convenient fixed datum. The total length of the cable is

$$L = x + \frac{fr_2}{2} + 2y + fr_1 + b \quad \longrightarrow \quad (4.36)$$

With  $L$ ,  $r_2$ ,  $r_1$  and  $b$  all constant, the first and second time derivatives of the equation give

$$0 = \dot{x} + 2\dot{y} \text{ or } 0 = v_A + 2v_B \quad \longrightarrow \quad (4.37)$$

$$0 = \ddot{x} + 2\ddot{y} \text{ or } 0 = a_A + 2a_B \quad \longrightarrow \quad (4.38)$$

The velocity and acceleration constraint equations indicate that, for the coordinates selected, the velocity of A must have a sign which is opposite to that of the velocity of B, and similarly for the accelerations. The constraint equations are valid for

the motion of the system in either direction. We emphasize that  $v_A = \dot{x}$  is positive to the left and that  $v_B = \dot{y}$  is positive down.

Because the results do not depend on the lengths or pulley radii, we should be able to analyze the motion without considering them. In the lower-left portion of fig.4.59 is shown an enlarged view of the horizontal diameter A'B'C' of the lower pulley at an instant of time. Clearly, A' and A have the same motion magnitudes, as do B and B'. During an infinitesimal motion of A', it is easy to see from the triangle that B' moves half as far as A' because point C as a point on the fixed portion of the cable momentarily has no motion. Thus, with differentiation by time in mind, we can obtain the velocity and acceleration magnitude relationships by inspection. The pulley, in effect, is a wheel which rolls on the fixed vertical cable. The system of fig.4.59 is said to have one degree of freedom since only one variable, either x or y, is needed to specify the positions of all parts of the system.

## 2) Two Degrees of Freedom

The system with two degrees of freedom is shown in fig.4.60. Here the positions of the lower cylinder and pulley C depend on the separate specifications of the two coordinates  $y_A$  and  $y_B$ . The lengths of the cables attached to cylinders A and B can be written, respectively, as

$$L_A = y_A + 2y_D + \text{constant} \rightarrow (4.39)$$

$$L_B = y_B + 2y_C + (y_C - y_D) + \text{constant} \rightarrow (4.40)$$

and their time derivatives are

$$0 = \dot{y}_A + 2\dot{y}_D \text{ and } 0 = \dot{y}_B + 2\dot{y}_C - \dot{y}_D \rightarrow (4.41)$$

$$0 = \ddot{y}_A + 2\ddot{y}_D \text{ and } 0 = \ddot{y}_B + 2\ddot{y}_C - \ddot{y}_D \rightarrow (4.42)$$

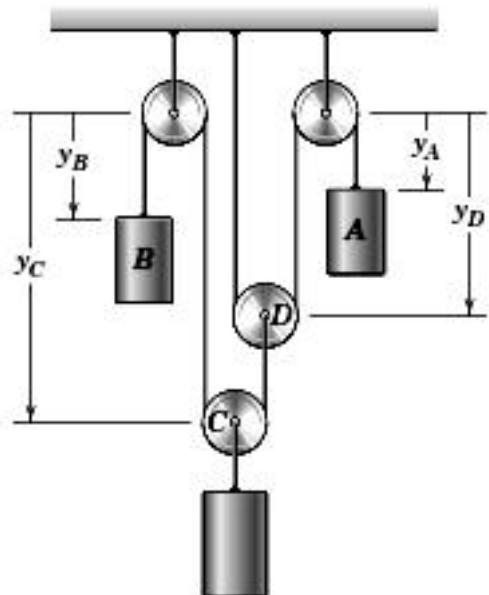


Fig.4.60

Eliminating the terms in  $\dot{y}_D$  and  $\ddot{y}_D$  gives

$$\dot{y}_A + 2\dot{y}_B + 4\dot{y}_C = 0 \text{ or } v_A + 2v_B + 4v_C = 0 \longrightarrow (4.43)$$

$$\ddot{y}_A + 2\ddot{y}_B + 4\ddot{y}_C = 0 \text{ or } a_A + 2a_B + 4a_C = 0 \longrightarrow (4.44)$$

It is clearly impossible for the signs of all three terms to be positive simultaneously. So, for example, if both A and B have downward (positive) velocities, then C will have an upward (negative) velocity.

These results can also be found by inspection of the motions of the two pulleys at C and D. For an increment  $dy_A$  (with  $y_B$  held fixed), the centre of D moves up an amount  $\frac{dy_A}{2}$ , which causes an upward movement  $\frac{dy_A}{4}$  of the centre of C. For an increment  $dy_B$  (with  $y$  held fixed), the center of C moves up a distance  $\frac{dy_B}{2}$ . A combination of the two movements gives an upward movement

$$-dy_C = \frac{dy_A}{4} + \frac{dy_B}{2} \longrightarrow (4.45)$$

so that  $-v_C = \frac{v_A}{4} + \frac{v_B}{2}$  as before. Visualization of the actual geometry of the motion is an important ability.

A second type of constraint where the direction of the connecting member changes with the motion is illustrated in the second of the two sample problems which follow.

### Problems:

**SP4.90:** In the pulley configuration as shown in fig.4.61 cylinder A has a downward velocity of 0.3m/s. Determine the velocity of B. Solve in two ways.

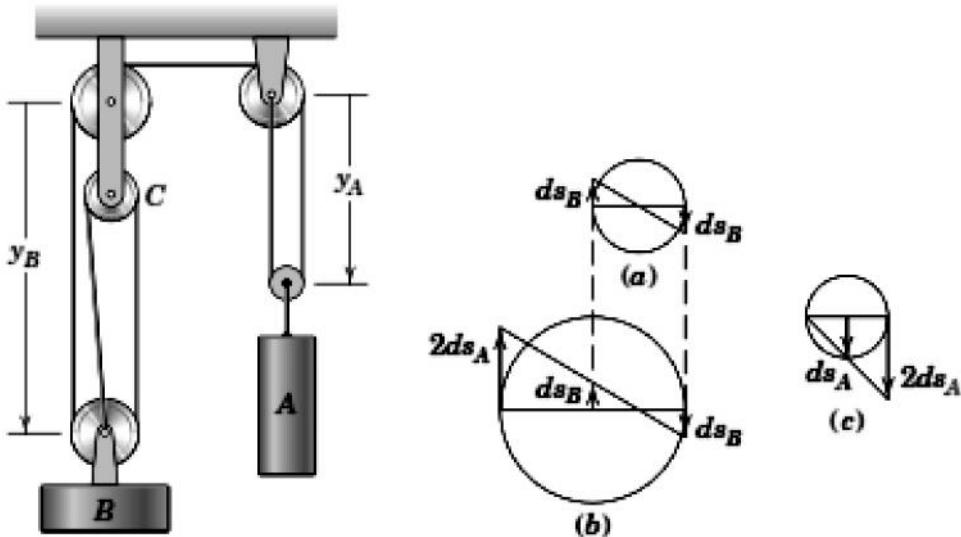


Fig.4.61

**Solution (I).** The centers of the pulleys at *A* and *B* are located by the coordinates  $y_A$  and  $y_B$  measured from fixed positions. The total constant length of cable in the pulley system is

$$L = 3y_B + 2y_A + \text{constants}$$

where the constants account for the fixed lengths of cable in contact with the circumferences of the pulleys and the constant vertical separation between the two upper left-hand pulleys. Differentiation with time gives

$$0 = 3\dot{y}_B + 2\dot{y}_A$$

Substitution of  $v_A = \dot{y}_A = 0.3 \text{ m/s}$  and  $v_B = \dot{y}_B$  gives

$$0 = 3(v_B) + 2(0.3) \quad \text{or} \quad v_B = -0.2 \text{ m/s} \quad \text{Ans.}$$

**Solution (II).** An enlarged diagram of the pulleys at *A*, *B*, and *C* is shown. During a differential movement  $ds_A$  of the center of pulley *A*, the left end of its horizontal diameter has no motion since it is attached to the fixed part of the cable. Therefore, the right-hand end has a movement of  $2ds_A$  as shown. This movement is transmitted to the left-hand end of the horizontal diameter of the pulley at *B*. Further, from pulley *C* with its fixed center, we see that the displacements on each side are equal and opposite. Thus, for pulley *B*, the right-hand end of the diameter has a downward displacement equal to the upward displacement  $ds_B$  of its center. By inspection of the geometry, we conclude that

$$2ds_A = 3ds_B \quad \text{or} \quad ds_B = \frac{2}{3}ds_A$$

Dividing by  $dt$  gives

$$|v_B| = \frac{2}{3}v_A = \frac{2}{3}(0.3) = 0.2 \text{ m/s (upward)} \quad \text{Ans.}$$

**SP4.91:** The tractor A is used to hoist the bale B with the pulley arrangement as shown in fig.4.62. If A has a forward velocity  $v_A$  determine an expression for the upward velocity  $v_B$  of the bale in terms of  $x$ .

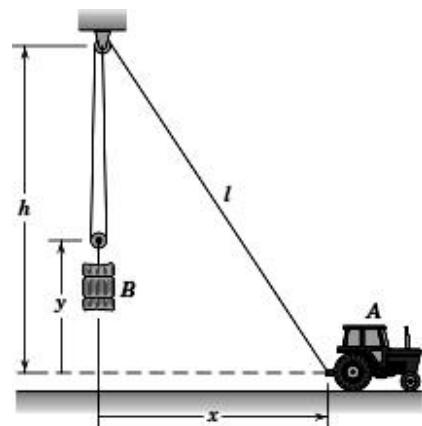


Fig.4.62

**Solution.** We designate the position of the tractor by the coordinate  $x$  and the position of the bale by the coordinate  $y$ , both measured from a fixed reference. The total constant length of the cable is

$$L = 2(h - y) + l = 2(h - y) + \sqrt{h^2 + x^2}$$

Differentiation with time yields

$$0 = -2\dot{y} + \frac{x\dot{x}}{\sqrt{h^2 + x^2}}$$

Substituting  $v_A = \dot{x}$  and  $v_B = \dot{y}$  gives

$$v_B = \frac{1}{2} \frac{xv_A}{\sqrt{h^2 + x^2}}$$

Ans.

## NEWTON'S SECOND LAW OF MOTION:

*Kinetics* is a branch of dynamics that deals with the relationship between the change in motion of a body and the forces that cause this change. The basis for kinetics is Newton's second law, which states that when an *unbalanced force* acts on a particle, the particle will *accelerate* in the direction of the force with a magnitude that is proportional to the force. This law can be verified experimentally by applying a known unbalanced force  $\mathbf{F}$  to a particle, and then measuring the acceleration  $\mathbf{a}$ . Since the force and acceleration are directly proportional, the constant of proportionality,  $m$ , may be determined from the ratio  $m = F/a$ . This positive scalar  $m$  is called the *mass* of the particle. Being constant during any acceleration,  $m$  provides a quantitative measure of the resistance of the particle to a change in its velocity that is its *inertia*.

If the mass of the particle is  $m$ , Newton's second law of motion may be written in mathematical form as

$$\mathbf{F} = m\mathbf{a} \quad \text{-----} \rightarrow (4.46)$$

The above equation, which is referred to as the *equation of motion*, is one of the most important formulations in mechanics. As previously stated, its validity is based solely on *experimental evidence*. In 1905, however, Albert Einstein developed the theory of relativity and placed limitations on the use of Newton's second law for describing general particle motion. Through experiments it was proven that *time* is not an absolute quantity as assumed by Newton; and as a result, the equation of motion fails to predict the exact behaviour of a particle, especially when the particle's speed approaches the speed of light (0.3 Gm/s).

### 1) Rectangular coordinates:

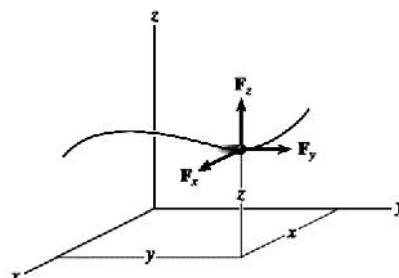


Fig.4.63

When a particle moves relative to an inertial x,y,z frame of reference, the forces acting on the particle as well as its acceleration, can be expressed in terms of their i,j and k components as shown in fig.4.63. Applying the equation of motion, we have

$$\sum F = m\mathbf{a}, \Rightarrow \sum F_x i + \sum F_y j + \sum F_z k = m(a_x i + a_y j + a_z k)$$

For this equation to be satisfied, the respective i, j and components on the left side must equal the corresponding components on the right side. Consequently it may write the following three scalar equations:

$$\left. \begin{array}{l} \sum F_x = ma_x \\ \sum F_y = ma_y \\ \sum F_z = ma_z \end{array} \right\} \quad \longrightarrow \rightarrow (4.47)$$

Where, the acceleration and resultant force are given by,

$$a = a_x i + a_y j + a_z k, \quad a = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad \longrightarrow \rightarrow (4.48)$$

$$\sum F = \sum F_x i + \sum F_y j + \sum F_z k, \quad |\sum F| = \sqrt{(\sum F_x)^2 + (\sum F_y)^2 + (\sum F_z)^2} \quad \rightarrow (4.49)$$

In particular, if the particle is constrained to move only in the x-y plane then the first two of these equations are used to specify the motion.

### Problems:

**SP4.92:** The 50kg crate as shown in fig.4.64 rests on a horizontal surface for which the coefficient of kinetic friction is  $\mu_k = 0.3$ . If the crate is subjected to a 400-N towing force, determine the velocity of the crate in 3seconds starting from rest.

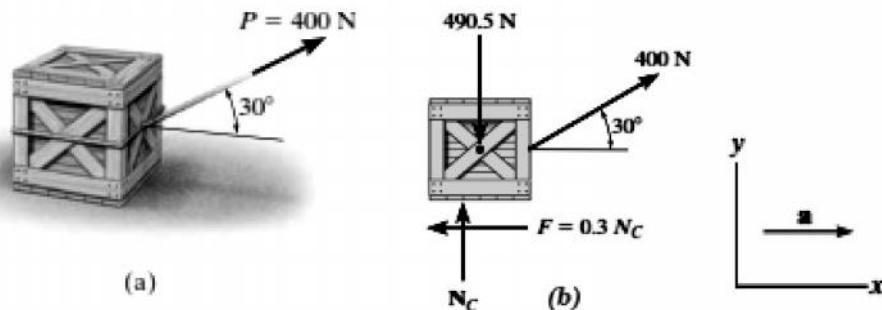


Fig.4.64

### Solution:

Using the equations of motion, we can relate the crate's acceleration to the force causing the motion. The crate's velocity can then be determined using kinematics.

**Free-Body Diagram.** The weight of the crate is  $W = mg = 50 \text{ kg} (9.81 \text{ m/s}^2) = 490.5 \text{ N}$ . As shown in Fig. 13-6b, the frictional force has a magnitude  $F = \mu_k N_C$  and acts to the left, since it opposes the motion of the crate. The acceleration  $a$  is assumed to act horizontally, in the positive  $x$  direction. There are two unknowns, namely  $N_C$  and  $a$ .

in the positive  $x$  direction. There are two unknowns, namely  $N_C$  and  $a$ .

**Equations of Motion.** Using the data shown on the free-body diagram, we have

$$\xrightarrow{\text{+}} \sum F_x = ma_x; \quad 400 \cos 30^\circ - 0.3N_C = 50a \quad (1)$$

$$+\uparrow \sum F_y = ma_y; \quad N_C - 490.5 + 400 \sin 30^\circ = 0 \quad (2)$$

Solving Eq. 2 for  $N_C$ , substituting the result into Eq. 1, and solving for  $a$  yields

$$N_C = 290.5 \text{ N}$$

$$a = 5.185 \text{ m/s}^2$$

**Kinematics.** Notice that the acceleration is *constant*, since the applied force  $\mathbf{P}$  is constant. Since the initial velocity is zero, the velocity of the crate in 3 s is

$$(\xrightarrow{\text{+}}) \quad v = v_0 + a_t t = 0 + 5.185(3) \\ = 15.6 \text{ m/s} \rightarrow \quad \text{Ans.}$$

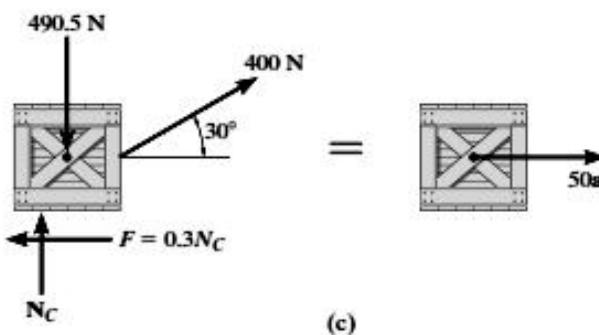


Fig.4.64

**SP4.93:** A 10 kg projectile is fired vertically upward from the ground with an initial velocity of 50m/s as shown in fig.4. Determine the maximum height to which it will travel if (a) atmospheric resistance is neglected; and b) atmospheric resistance is measured as  $F_D = (0.01v^2)N$ , where  $v$  is the speed of the projectile at any instant measured in m/s.

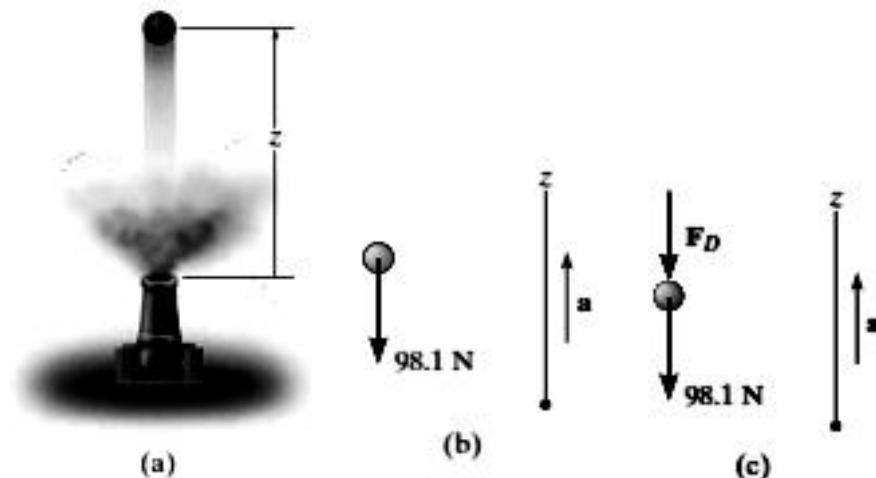


Fig.4.65

**Solution:**

In both cases the known force on the projectile can be related to its acceleration using the equation of motion. Kinematics can then be used to relate the projectile's acceleration to its position.

**Part (a) Free-Body Diagram.** As shown in Fig. 13-7b, the projectile's weight is  $W = mg = 10(9.81) = 98.1 \text{ N}$ . We will assume the unknown acceleration  $a$  acts upward in the *positive z* direction.

**Equation of Motion.**

$$+\uparrow \sum F_z = ma_z; \quad -98.1 = 10a, \quad a = -9.81 \text{ m/s}^2$$

The result indicates that the projectile, like every object having free-flight motion near the earth's surface, is subjected to a *constant* downward acceleration of  $9.81 \text{ m/s}^2$ .

**Kinematics.** Initially,  $z_0 = 0$  and  $v_0 = 50 \text{ m/s}$ , and at the maximum height  $z = h$ ,  $v = 0$ . Since the acceleration is *constant*, then

$$\begin{aligned}
 (+\uparrow) \quad v^2 &= v_0^2 + 2a_c(z - z_0) \\
 0 &= (50)^2 + 2(-9.81)(h - 0) \\
 h &= 127 \text{ m} \quad \text{Ans.}
 \end{aligned}$$

**Part (b) Free-Body Diagram.** Since the force  $F_D = (0.01v^2)$  N tends to retard the upward motion of the projectile, it acts downward as shown on the free-body diagram, Fig. 13-7c.

**Equation of Motion.**

$$+\uparrow \sum F_z = ma_z; \quad -0.01v^2 - 98.1 = 10a, \quad a = -(0.001v^2 + 9.81)$$

**Kinematics.** Here the acceleration is *not constant* since  $F_D$  depends on the velocity. Since  $a = f(v)$ , we can relate  $a$  to position using

$$(+\uparrow) a dz = v dv; \quad -(0.001v^2 + 9.81) dz = v dv$$

Separating the variables and integrating, realizing that initially  $z_0 = 0$ ,  $v_0 = 50$  m/s (positive upward), and at  $z = h$ ,  $v = 0$ , we have

$$\int_0^h dz = - \int_{50 \text{ m/s}}^0 \frac{v dv}{0.001v^2 + 9.81} = -500 \ln(v^2 + 9810) \Big|_{50 \text{ m/s}}^0$$

$$h = 114 \text{ m}$$

*Ans.*

**SP4.94:** A smooth 2 kg collar as shown in fig.4.66 is attached to a spring having a stiffness  $k = 3$  N/m and an un-stretched length of 0.75m. If the collar is released from rest at A, determine its acceleration and the normal force of the rod on the collar at the instant  $y=1$ m.

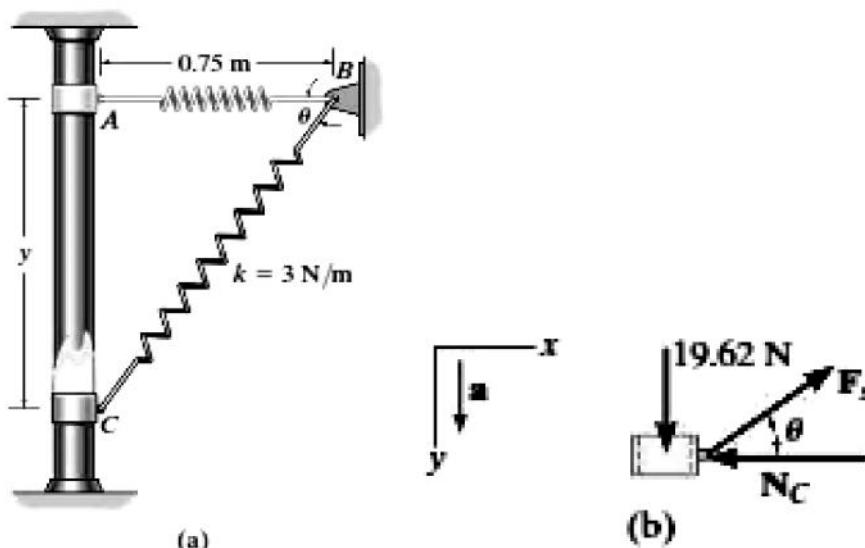


Fig.4.66

**Solution:**

**Free-Body Diagram.** The free-body diagram of the collar when it is located at the arbitrary position  $y$  is shown in Fig. 13-9b. Furthermore, the collar is *assumed* to be accelerating so that “ $a$ ” acts downward in the *positive*  $y$  direction. There are four unknowns, namely,  $N_C$ ,  $F_s$ ,  $a$ , and  $\theta$ .

**Equations of Motion.**

$$\xrightarrow{+} \sum F_x = ma_x; \quad -N_C + F_s \cos \theta = 0 \quad (1)$$

$$+\downarrow \sum F_y = ma_y; \quad 19.62 - F_s \sin \theta = 2a \quad (2)$$

From Eq. 2 it is seen that the acceleration depends on the magnitude and direction of the spring force. Solution for  $N_C$  and  $a$  is possible once  $F_s$  and  $\theta$  are known.

The magnitude of the spring force is a function of the stretch  $s$  of the spring; i.e.,  $F_s = ks$ . Here the unstretched length is  $AB = 0.75$  m. Fig. 13-9a; therefore,  $s = CB - AB = \sqrt{y^2 + (0.75)^2} - 0.75$ . Since  $k = 3$  N/m, then

$$F_s = ks = 3 \left( \sqrt{y^2 + (0.75)^2} - 0.75 \right) \quad (3)$$

From Fig. 13-9a, the angle  $\theta$  is related to  $y$  by trigonometry.

$$\tan \theta = \frac{y}{0.75}$$

Substituting  $y = 1$  m into Eqs. 3 and 4 yields  $F_s = 1.50$  N and  $\theta = 53.1^\circ$ . Substituting these results into Eqs. 1 and 2, we obtain

$$N_C = 0.900 \text{ N} \quad \text{Ans.}$$

$$a = 9.21 \text{ m/s}^2 \downarrow \quad \text{Ans.}$$

**SP4.95:** The 100 kg block A as shown in fig.4.67a is released from rest. If the masses of the pulleys and the cord are neglected, determine the velocity of the 20kg block B in 2seconds.

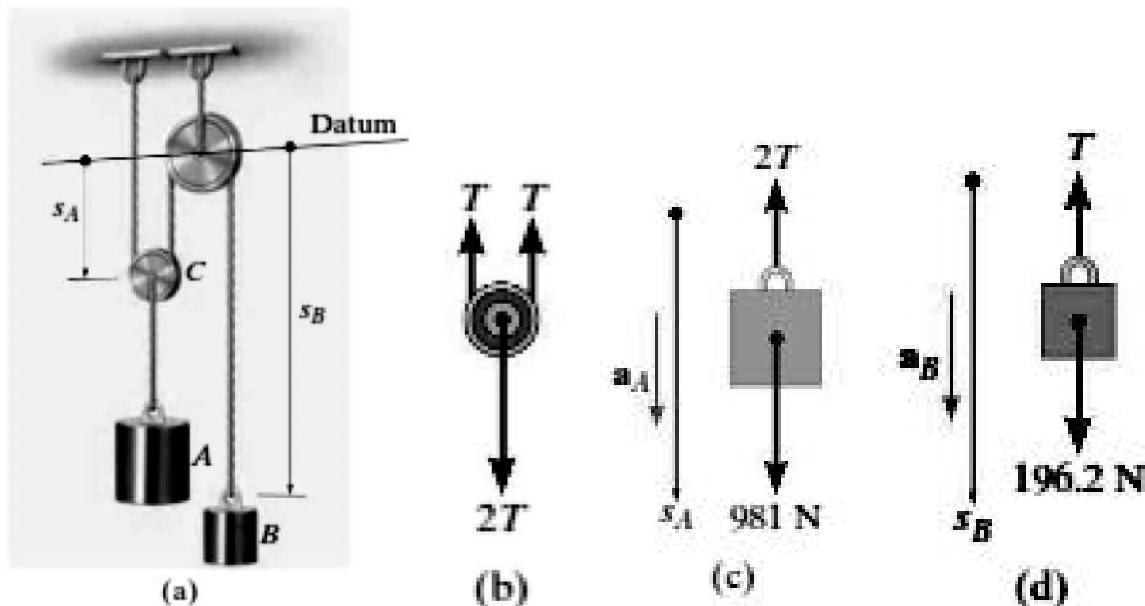


Fig.4.67

**Solution:**

**Free-Body Diagrams.** Since the mass of the pulleys is neglected, then for pulley C,  $ma = 0$  and we can apply  $\sum F_y = 0$ , as shown in Fig. 13-10b. The free-body diagrams for blocks A and B are shown in Fig. 13-10c and d, respectively. Notice that for A to remain stationary  $T = 490.5$  N, whereas for B to remain static  $T = 196.2$  N. Hence A will move down while B moves up. Although this is the case, we will assume both blocks accelerate downward, in the direction of  $+s_A$  and  $+s_B$ . The three unknowns are  $T$ ,  $a_A$ , and  $a_B$ .

**Equations of Motion. Block A.**

$$+\downarrow \sum F_y = ma_A; \quad 981 - 2T = 100a_A \quad (1)$$

**Block B.**

$$+\downarrow \sum F_y = ma_B; \quad 196.2 - T = 20a_B \quad (2)$$

**Kinematics.** The necessary third equation is obtained by relating  $a_A$  to  $a_B$  using a dependent motion analysis, discussed in Sec. 12.9. The coordinates  $s_A$  and  $s_B$  in Fig. 13-10a measure the positions of A and B from the fixed datum. It is seen that

$$2s_A + s_B = l$$

where  $l$  is constant and represents the total vertical length of cord. Differentiating this expression twice with respect to time yields

$$2a_A = -a_B \quad (3)$$

Notice that when writing Eqs. 1 to 3, the positive direction was always assumed downward. It is very important to be consistent in this assumption since we are seeking a simultaneous solution of equations. The results are

$$T = 327.0 \text{ N}$$

$$a_A = 3.27 \text{ m/s}^2$$

$$a_B = -6.54 \text{ m/s}^2$$

Hence when block  $A$  accelerates downward, block  $B$  accelerates upward as expected. Since  $a_B$  is constant, the velocity of block  $B$  in 2 s is thus

(+↓)

$$\begin{aligned} v &= v_0 + a_B t \\ &= 0 + (-6.54)(2) \\ &= -13.1 \text{ m/s} \end{aligned}$$

Ans.

The negative sign indicates that block  $B$  is moving upward.

**SP4.96:** The baggage truck A has a weight of 4kN as shown in fig.4.68 and tows a 2.5 kN cart B and a 1.5kN cart C. for a short time the driving frictional force developed at the wheels of the truck is  $F_A = (180t)N$ , where  $t$  is in seconds. If the truck starts from rest, determine its speed in 2 second. Also what is the horizontal force acting on the coupling between the truck and cart B at this instant? Neglect the size of the truck and carts.

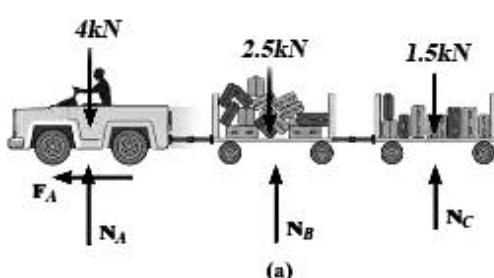


Fig.4.68

**Solution:**

Free body diagram: As shown in fig.4.68a. it is the frictional driving force that gives both the truck and carts an acceleration. Here we have considered all three vehicles as a single system.

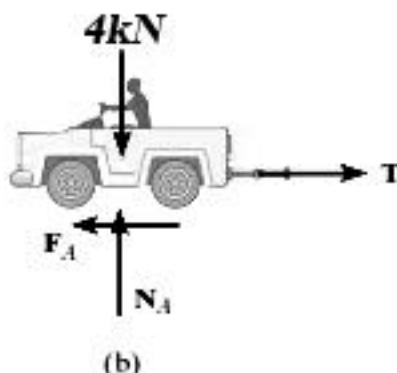
Equation of motion: Only motion in the horizontal direction has to be considered.

$$\sum F_x = ma_x, \Rightarrow 0.18t = \left( \frac{4 + 2.5 + 1.5}{9.81} \right) a$$

$$a = 0.225t$$

Kinematics: Since the acceleration is a function of time, the velocity of the truck is obtained using  $a = \frac{dv}{dt}$  with the initial condition that  $v = 0$  at  $t = 0$ , we have

$$\int_0^v dv = \int_0^2 0.225t dt, \Rightarrow v = \left[ \frac{0.225}{2} t^2 \right]_0^2 = 0.45 \text{ m/s}$$



Free body diagram: In order to determine the force between the truck and cart B we will consider a free body diagram of the truck so that we can “expose” the coupling force T as external to the free body diagram as shown in fig.4.68b

Equation of motion: When  $t = 2$  seconds, then

$$\sum F_x = ma_x, \Rightarrow 0.18(2) - T = \left( \frac{4}{9.81} \right) (0.225 \times 2)$$

$$T = 0.177 \text{ kN}$$

**SP4.97:** A body weighs 490.5 N on the earth. Find its weight , a) on the moon where the gravitational acceleration is  $1.4 \text{ m/s}^2$  and b) on the Sun, where the gravitational acceleration is  $270 \text{ m/s}^2$ .

**Solution:**

We know that weight = mass  $\times$  acceleration due to gravity

$$\begin{aligned} \text{Therefore Mass on the Earth} &= \text{Weight on the earth } /'g' \text{ for earth} \\ &= 490.5 / 9.81 = 50 \text{ kg} \end{aligned}$$

$$\begin{aligned} \text{Weight on the Moon} &= \text{mass} \times 'g' \text{ for moon} \\ &= 50 \times 1.4 = 70 \text{ N} \end{aligned}$$

$$\begin{aligned} \text{Weight on the Sun} &= \text{mass} \times 'g' \text{ for Sun} \\ &= 50 \times 270 = 13,500 \text{ N} \end{aligned}$$

**SP4.98:** A vehicle of mass 500kg is moving with a velocity of 25 m/s. A force of 200N acts on it for 2 minutes. Find the distance travelled and velocity of the vehicle. A) When the force acts in the direction of motion and B) when the force acts in the opposite direction of motion.

**Solution:**

We know that,  $\sum F = ma$ ,  $\Rightarrow 200 = 500 \times a$

$$a = 0.4 \text{ m/s}^2$$

Case A: When the force is acting in the same direction:

$$u = 25 \text{ m/s}, a = 0.4 \text{ m/s}^2 \text{ and } t = 2 \times 60 = 120 \text{ s}$$

$$\text{We know that, } S = ut + \frac{1}{2}at^2, \Rightarrow S = 25 \times 120 + \frac{1}{2} \times 0.4 \times (120)^2, S = 5880 \text{ m}$$

$$v = u + at, \Rightarrow v = 25 + 0.4 \times 120, v = 73 \text{ m/s}$$

Case B: When the force is acting in the opposite direction:

$$u = 25 \text{ m/s}, a = -0.4 \text{ m/s}^2 \text{ and } t = 2 \times 60 = 120 \text{ s}$$

$$\text{We know that, } S = ut + \frac{1}{2}(-a)t^2, \Rightarrow S = 25 \times 120 + \frac{1}{2} \times -0.4 \times (120)^2, S = 120 \text{ m}$$

$$v = u + (-a)t, \Rightarrow v = 25 + (-0.4) \times 120, v = -23 \text{ m/s}$$

The vehicle will move in the opposite direction with a velocity of 23m/s.

**SP4.99:** A man of mass 65kg dives vertically downwards into a swimming pool from a tower of height 19.62m. He was found to go down in water by 2m and then started rising. Find the average resistance of the water.

**Solution:**

Considering the downward motion of the man till it reaches the top surface of the water in the swimming pool

$$\begin{aligned} u &= 0; s = 19.62 \text{ m}; a = 9.81 \text{ m/s}^2 \\ v^2 &= u^2 + 2as \\ v^2 &= 0 + 2 \times 9.81 \times 19.62 \\ v &= 19.62 \text{ m/s} \end{aligned}$$

Considering the motion of the man inside the water

$$\begin{aligned} u &= 19.62 \text{ m/s}; v = 0; s = 2 \text{ m} \\ v^2 &= u^2 + 2as \\ 0 &= (19.62)^2 + 2a \times 2 \\ a &= -96.236 \text{ m/s}^2 \end{aligned}$$

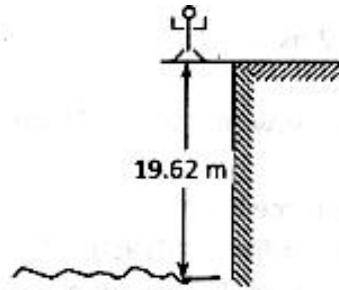


Fig.4.69

Applying D'Alembert's principle

$$\text{Net force } F = m \cdot a$$

$$\text{Net force } F = w - Fr_w = m \cdot a$$

$$Fr_w = w - m \cdot a$$

$$\begin{aligned} \text{Self weight of the man 'w' which acts} \\ \text{downwards} &= 65 \times 9.81 = 637.65 \text{ N} \\ m \cdot a &= 65 \times -96.236 \text{ m/s}^2 = -6255.34 \text{ N} \end{aligned}$$

$$\begin{aligned} \text{Resisting force acting upwards on the man} \\ \text{by the water} \\ Fr_w &= 637.65 \text{ N} - (-6255.34 \text{ N}) \\ Fr_w &= 6892.99 \text{ N} \end{aligned}$$

**SP4.100:** At a certain instant a body of mass 10kg falling freely under the force of gravity was found to be falling at the rate of 20m/s. What force will stop the body in a) 2 seconds and b) 2m.

**Solution:**

$$\text{Net force while falling down is } mg - F = ma \quad \rightarrow (1)$$

$$\text{Case a) } u = 20 \text{ m/s}, v = 0, t = 2 \text{ s}, \Rightarrow v = u + at, 0 = 20 + a \times 2, a = -10 \text{ m/s}^2$$

Therefore substituting the value of a in eqn.1, then

$$10 \times 9.81 - F = 10 \times (-10)$$

$$F = 198.1N$$

Case a)  $u = 20 \text{ m/s}$ ,  $v = 0$ ,  $S = 2 \text{ m}$ ,  $\Rightarrow v^2 = u^2 + 2aS$ ,  $0^2 = 20^2 + 2 \times a \times 2$ ,  $a = -100 \text{ m/s}^2$

Therefore substituting the value of a in eqn.1, then

$$10 \times 9.81 - F = 10 \times (-100)$$

$$F = 1098.1N$$

**SP4.101:** A 1 Mg car has to be accelerated from rest to 72kmph in a distance of 100m on a level road. The resistance to motion is 150N. Determine the tractive force needed by the car.

**Solution:**

$$1 \text{ Mg} = 1 \times 10^6 \text{ gm} = 1 \times 10^3 \text{ kg} = 1000 \text{ kg}$$

Considering the uniform acceleration motion of the car  
 $u = 0$  ;  $v = 72 \text{ kmph} / 3.6 = 20 \text{ m/s}$

$$\begin{aligned} s &= 100 \text{ m} \\ v^2 &= u^2 + 2as \\ 20^2 &= 0 + 2 \times a \times 100 \\ a &= 2 \text{ m/s}^2 \end{aligned}$$

We know that as per D'Alembert's principle

Net force  $F = m \cdot a$

But net force = Tractive force ( $F_t$ ) – resistance force ( $F_r$ )

Therefore  $F_t - F_r = m \cdot a$

$$\begin{aligned} F_t &= m \cdot a + F_r \\ &= 1000 \times 2 + 150 \text{ N} \\ F_t &= 2150 \text{ N} \end{aligned}$$

**SP4.102:** A 100kg rocket is fired vertically, if the thrust of the rocket is 5000N and the burn time is 10 seconds. Determine the maximum altitude attained by the rocket.

**Solution:**

**Upward thrust on the rocket ( $F_t$ )**

$$= 5000 \text{ N}$$

**Self weight acting downward**

$$W = 100 \times 9.81 = 981 \text{ N}$$

$$\text{Net force } F = F_t - W$$

$$= 5000 - 981 = 4019 \text{ N}$$

$$\text{Net force } F = m \cdot a$$

$$4019 = 100 \times a$$

$$a = 40.19 \text{ m/s}^2$$

**Considering the upward motion of the rocket till all the fuel is burnt**

$$u = 0 \text{ m/s}; a = 40.19 \text{ m/s}^2; t = 10 \text{ s}$$

$$s = ut + \frac{1}{2} a t^2$$

$$s = 0 \times 10 + \frac{1}{2} \times 40.19 \times (10)^2$$

$$s = 2009.5 \text{ m}$$

$$v = u + at$$

$$v = 0 + 40.19 \times 10 = 401.9 \text{ m/s}$$

**The distance moved by the rocket till all the fuel is burnt is 2.0095 km. Further the rocket continues to move as free particle till it reaches its maximum height.**

**Considering the motion of the rocket further**

$$u = 401.9 \text{ m/s}; v = 0 \text{ m/s}$$

$$a = -g = -9.81 \text{ m/s}^2$$

$$v^2 = u^2 + 2as$$

$$0^2 = (401.9)^2 - 2 \times 9.81 \times s$$

$$s = 8232.59 \text{ m}$$

**Maximum height reached by the rocket**

$$= 2009.5 + 8232.59$$

$$= 10242.09 \text{ m}$$

$$= 10.242 \text{ km}$$

**SP4.103:** A body of mass 10kg is moving over a smooth surface, whose equation of motion is given by the relation  $S = 5t + 2t^2$  where S in meter and t in seconds. Find the magnitude of force responsible for the motion.

**Solution:**

$$S = 5t + 2t^2$$

$$ds/dt = v = 5 + 4t$$

$$dv/dt = a = 4 \text{ m/s}^2$$

$$F = m \cdot a$$

$$F = 10 \times 4 = 40 \text{ N}$$

**SP4.104:** An elevator having mass 2500kg and is moving vertically downwards with a constant acceleration starting from rest, It travels a distance of 35m during an interval of 10seconds. Find the cable tension during this time. Neglecting all other resistance to motion, what are the limits of cable tension?

**Solution:****Considering the downward motion of the lift**

$$u = 0; s = 35 \text{ m}; t = 10 \text{ s}$$

$$s = ut + \frac{1}{2} a t^2$$

$$35 = 0 + \frac{1}{2} a (10)^2$$

$$a = 0.7 \text{ m/s}^2$$

**Applying D Alembert's principle**

$$\sum F - ma = 0$$

$$mg - T = ma$$

$$m(g - a) = T$$

$$T = 2500 (9.81 - 0.7) = 22.775 \text{ kN}$$

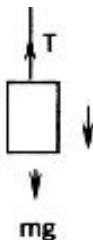
**Tension in the cable = 22.775 kN**

Fig.4.70

**Limits of cable tension**

$$\text{When } a = 0; T = mg = 2500 \times 9.81 \\ = 24.525 \text{ kN}$$

$$a = g; T = m(g - g) = 0$$

**Limits = (0, 24.525 kN)**

**SP4.105:** An elevator of mass 500kg is ascending with an acceleration of  $3 \text{ m/s}^2$ . During this ascension an operator whose mass is 70kg is standing on the scale placed on the floor. What is the scale reading? What will be the total tension in the cables of the elevator during this motion?

**Solution:**

$$\text{mass of elevator}(m_1) = 500 \text{ kg}$$

$$\text{mass of the man } (m_2) = 70 \text{ kg}$$

**Consider the motion of the operator only****Let R be the reaction or pressure acting on the platform of the scale****Then applying D Alembert's principle**

$$R - m_2 g = m_2 a$$

$$R = m_2 (g + a)$$

**The pressure exerted on the scale**

$$= m_2 (g + a)$$

$$= 70(9.81 + 3)$$

$$= 896 \text{ N} = 91.36 \text{ kg}$$

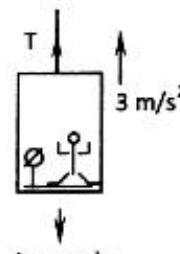
**The scale reading is = 91.36 kg**

Fig.4.71

**Consider the motion of the elevator and operator**

$$T - (m_1 + m_2)g = (m_1 + m_2) \cdot a$$

$$T = (m_1 + m_2) (g + a)$$

**Tension on the cable T =  $(m_1 + m_2) (g + a)$** 

$$= (500 + 70)(9.81 + 3)$$

$$= 7296 \text{ N}$$

**SP4.106:** A nail is to be pulled out of a wall. The radial pressure between the wall and the nail is  $4000 \text{ kN/m}^2$  and the coefficient of friction between them is 0.4. Determine the force required to pull the nail out of the wall.

**Solution:**

Surface area of the nail =  $\pi d l$

$$= \pi \times (3 \times 10^{-3}) \times (30 \times 10^{-3}) = 2.827 \times 10^{-4} \text{ m}^2$$

Radial force acting on the nail = radial pressure x area

$$\text{Radial force} = 4000 \times 2.827 \times 10^{-4} = 1130.7 \text{ N}$$

Frictional force = Coefficient of friction x normal reaction

Here normal reaction is the radial force

$$\text{Hence frictional force } (F_r) = 0.4 \times 1130.7 = 452.3 \text{ N}$$

Force needed to pull the nail is the force needed to overcome the frictional force.

$$\text{Hence the required force needed to pull} = 452.3 \text{ N}$$

**SP4.107:** 1 kN block rests on a horizontal plane as shown in fig.4.72 Find the magnitude of the force P required to give the block an acceleration of 3 m/s<sup>2</sup> to the right. The coefficient of the friction between the block and the plane is 0.25.

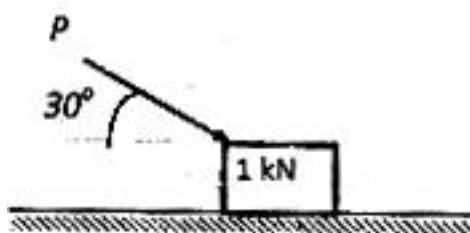


Fig.4.72

**Solution:**

$$\begin{aligned} \text{Weight of the block } (mg) &= 1000 \text{ N} \\ \text{mass} &= 1000/9.81 \\ &= 101.936 \text{ kg} \end{aligned}$$

Applying Newton's second law of motion

$$\text{Net force } F = m \cdot a$$

$$P \cos 30 - F_r = m \cdot a$$

$$\begin{aligned} \text{Frictional force } F_r &= \mu R_n \\ \text{But } R_n &= P \sin 30 + 1000 \text{ N} \\ \text{Therefore } F_r &= 0.25 (0.5 P + 1000) \\ &= 0.125 P + 250 \end{aligned}$$

$$\begin{aligned} 0.866 P - 0.125 P + 250 &= 101.936 \times 3 \\ P &= 750 \text{ N} \end{aligned}$$

**SP4.108:** A vehicle of mass 200 tonnes has a frictional resistance of 50N per tonne. Speed of the train at the top of an incline of 1 in 80, is 45 kmph. Find the speed of the train after running down the incline for 1km.

**Solution:**

$$m = 200,000 \text{ kg}$$

$$\sum F = m \cdot a$$

$$(m \cdot g \cdot \sin \alpha - F_r) = m \cdot a$$

$$200000 \times 9.81 \times (1/80) - 50 \times 200$$

$$= 200,000 \cdot a$$

$$a = 0.072 \text{ m/s}^2$$

Considering the downward motion of the train

$$u = 45 \text{ kmph} = 12.5 \text{ m/s}$$

$$a = 0.072 \text{ m/s}^2$$

$$s = 1000 \text{ m}$$

$$v^2 = u^2 + 2 a s$$

$$= (12.5)^2 + 2 \times 0.072 \times 1000$$

$$v = 17.32 \text{ m/s (or) } 62.37 \text{ kmph}$$

**SP4.109:** A body of mass 500kg initially at rest at A 50m from B on an 15% upgrade is allowed to slide down as shown in fig.4.73 If the coefficient of friction between the body and the plane is 0.1, find the velocity of the body at B. Also find the distance through which the body will travel beyond B on the level plane.

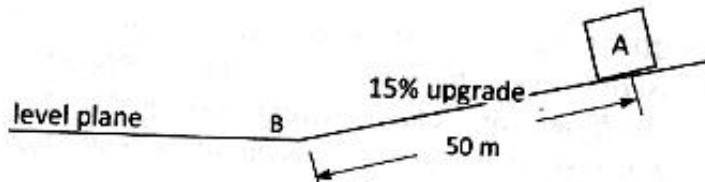


Fig.4.73

**Solution:**

Considering downward motion of the block between A & B

$$mg \sin \alpha - \mu mg \cos \alpha = ma$$

$$g \sin \alpha - \mu g \cos \alpha = a$$

$$9.81 \times 0.15 - 0.1 \times 9.81 \times 0.9889 = a$$

$$a = 0.5015 \text{ m/s}^2$$

$$v^2 = u^2 + 2as$$

$$v^2 = 0^2 + 2 \times 0.5015 \times 50$$

$$v = 7.0816 \text{ m/s}$$

Considering motion of the block beyond B

$$-\mu mg = ma$$

$$-\mu g = a$$

$$a = -0.1 \times 9.81 = -0.981 \text{ m/s}^2$$

$$v^2 = u^2 + 2as$$

$$0 = 7.0815^2 + 2 \times 0.981 \times s$$

$$s = 25.56 \text{ m}$$

**SP4.110:** A truck is moving down a  $10^\circ$  incline. When the driver applies brakes with the result that the truck decelerates at a steady rate of  $1 \text{ m/s}^2$ . Investigate whether a mass

500kg placed on the truck will slide or remain stationary relative to the truck. Assume the coefficient of friction between the truck surface and the load as 0.3. What will be the factor of safety against slipping for this load?

**Solution:**

$$\begin{aligned} F_r &= \mu R_n = \mu mg \cos \alpha \\ &= 0.3 \times 500 \times 9.81 \times \cos 10^\circ \end{aligned}$$

$$F_r = 1449.14 \text{ N}$$

$$\begin{aligned} \text{Self weight component} &= mg \sin \alpha \\ &= 500 \times 9.81 \times \sin 10^\circ = 851.74 \text{ N} \end{aligned}$$

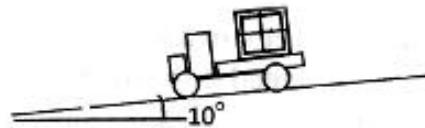


Fig.4.74

$$\text{Inertia force acting down the plane} = 500 \times 1 = 500 \text{ N}$$

$$\text{Total force acting down the plane} = 500 + 851.74 = 1351.74 \text{ N}$$

$$\text{But the frictional force} = 1449.14 \text{ N}$$

The force exerted on the body is less than the frictional force.

Hence the body will not move.

$$\text{The factor of safety against slipping} = 1449.14 / 1351.74 = 1.072$$

**SP4.111:** A locomotive of mass 200 tonnes draws a train of mass 450 tonnes. The frictional resistance is constant and equal to 75 N/tonne. Find the tractive force which will be required for the train to reach a speed of 72 kmph in a distance of 2km from the starting point: a) on a level track, b) for going upward on an inclined plane of 1 in 240 and c) for moving downwards on the same inclined plane.

**Solution:**

Considering the motion of the train

$$= 48.750 \text{ kN}$$

$$u = 0$$

$$v = 72 \text{ kmph} = 20 \text{ m/s}$$

$$s = 2000 \text{ m}$$

$$v^2 = u^2 + 2 a s$$

$$20^2 = (0)^2 + 2 \times a \times 2000$$

$$400 = 4000 a$$

$$a = 400 / 4000 = 0.1 \text{ m/s}^2$$

Case(i) on the level track

$$\sum F = m \cdot a$$

$$F_t - F_r = m \cdot a$$

$$F_r = 75 \text{ N/tonne} = 75 \times 650$$

$$\begin{aligned} F_t &= m \cdot a + F_r \\ &= 650,000 \times 0.1 + 48750 \\ &= 113750 \text{ N} = 113.75 \text{ kN} \end{aligned}$$

Case(ii) on the upward plane

$$\sum F = m \cdot a$$

$$F_t - F_r - mg \sin \alpha = m \cdot a$$

$$\begin{aligned} F_t &= F_r + mg \sin \alpha + m \cdot a \\ &= 48750 + 650,000 \times 9.81 \times (1/240) + \\ &\quad 650000 \times 0.1 \\ &= 140.318 \text{ kN} \end{aligned}$$

**Case(iii) On the downward plane**

$$\sum F = m \cdot a$$

$$F_t - F_r + mg \sin \alpha = m \cdot a$$

$$F_t = F_r - mg \sin \alpha + m \cdot a$$

$$= 48750 - 650,000 \times 9.81 \times (1/240) + 650000 \times 0.1$$

$$= 87.18 \text{ kN}$$

**SP4.112:** A truck weighing 6kN just moves freely (engine is not running at 36kmph down a slope of 1 in 40, the road resistance at this speed being sufficient to prevent only acceleration. Find the road resistance per kN weight of the truck. What power will the engine have to exert to run up the same slope at double the speed when the road resistance remains the same?

**Solution:****Down the plane**

$$mg \sin \alpha - Fr = m a = 0$$

Since the speed is constant,  $a = 0$

$$mg \sin \alpha = Fr = 6000 \times 1/40$$

$$= 150 \text{ N}$$

for 6 kN weight the frictional force = 150 N

$$\text{for } 1 \text{ kN} = 1 \times 150 / 6 = 25 \text{ N/kN}$$

**Up the plane**

$$F_t - mg \sin \alpha - Fr = m a = 0$$

$$F_t = mg \sin \alpha + Fr$$

$$= 150 + 150 = 300 \text{ N}$$

$$\text{Power} = F_t \times v = 300 \times 2 \times 10 = 6000 \text{ W}$$

$$= 6 \text{ kW}$$

**SP4.113:** An engine of weight 500kN pulls a train weighing 1500kN up an incline of 1 in 100. The train starts from rest and moves with a constant acceleration against a resistance of 5 N/kN. It attains a maximum speed of 36kmph in 1km distance. Determine the tension in the coupling between train and the engine and the tractive force developed by the engine.

**Solution:****Moving up the plane:**

$$u = 0$$

$$v = 36 \text{ kmph} = 10 \text{ m/s}$$

$$s = 100 \text{ m}$$

$$v^2 = u^2 + 2 a s$$

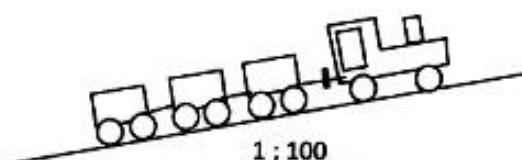


Fig.4.75

$$10^2 = 0^2 + 2 \times a \times 100$$

$$a = 0.05 \text{ m/s}^2$$

$$F_r = 5 \times (500 + 1500) = 10,000 \text{ N}$$

$$F_t - mg \sin \alpha - F_r = m \cdot a$$

$$F_t = 2000 \times 1000 \times (1/100) + 10,000 + (2000000/9.81) 0.05$$

$$F_t = 40.1937 \text{ kN}$$

### To find the tension in the coupling

Considering the engine only

$$F_t - mg \sin \alpha - F_r - T = m \cdot a$$

$$F_t = mg \sin \alpha + F_r + T + m \cdot a$$

$$40,1937 = (500000 \times (1/100) + 500 \times 5 + T + (500000/9.81) 0.05$$

$$T = 30.145 \text{ kN}$$

**SP4.114:** A train weighing 200kN moves down a slope of 1 in 150 at 18 kmph and engine develops 3.5kW. What power will be required to pull the train up the slope at the same speed?

**Solution:**

$$18 \text{ kmph} = 5 \text{ m/s}$$

We know that

Power = Speed x tractive force

$$F_t \times 5 = 3500 \text{ W}$$

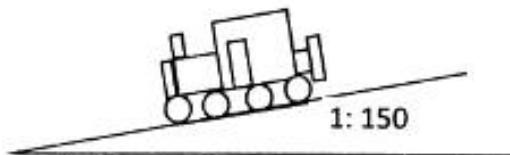


Fig.4.76

$$F_t = 700 \text{ N}$$

$$\sum F = m \cdot a$$

$$F_t - F_r + m \cdot g \cdot \sin \alpha = m \cdot a$$

Since speed is maintained constant  $a = 0$

Therefore

$$700 - F_r + 200,000 \cdot (1/150) = (200000/9.81) \times 0 = 0$$

$$F_r = 2033.3 \text{ N}$$

When the train moves upwards  $\sum F = m \cdot a$

$$F_t - F_r - m \cdot g \cdot \sin \alpha = m \cdot a$$

Since speed is constant  $a = 0$

$$F_t = 2033.3 + 200,000 \cdot (1/150)$$

$$\text{Therefore } F_t = 3366.6 \text{ N}$$

Hence Power required =  $F_t \times \text{Speed}$

$$P = 3366.6 \times 5 = 16.83 \text{ kW}$$

**SP4.115:** Two blocks A and B are held on an inclined plane 5m apart as shown in fig.4. The coefficient of friction between the block A and B and the inclined plane are 0.2 and 0.1 respectively. If the block begin to slide down the plane simultaneously calculate the time and distance travelled by each block before collision.

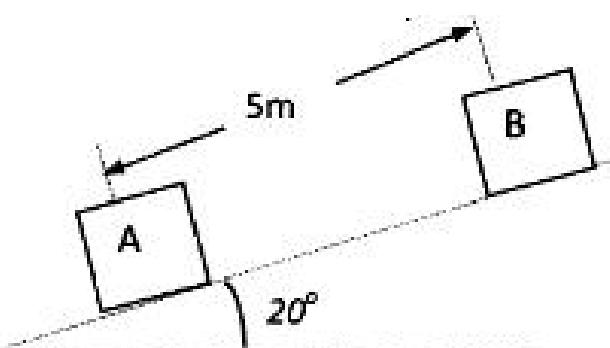


Fig.4.77

**Solution:**

Time for each block is same

$$\text{To find } a_A : \sum F = m \cdot a$$

$$m \cdot g \cdot \sin \alpha - F_f = m \cdot a_A$$

$$m \cdot g \cdot \sin \alpha - \mu_A m \cdot g \cdot \cos \alpha = m \cdot a_A$$

Canceling 'm' throughout

$$g \cdot \sin \alpha - \mu_A g \cdot \cos \alpha = a_A$$

$$9.81 \times \sin 20 - 0.2 \times 9.81 \times \cos 20 = a_A$$

$$a_A = 1.51 \text{ m/s}^2$$

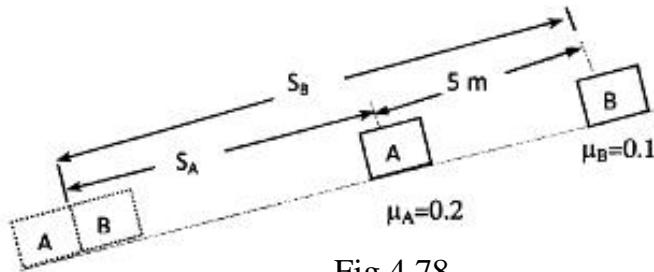


Fig.4.78

$$= 0 + \frac{1}{2} \times 1.51 \times t^2 = 0.755 t^2$$

$$S_A = 0.755 t^2$$

Similarly

$$S_B = 0 + \frac{1}{2} \times 2.43 \times t^2 = 1.217 t^2$$

$$S_B = 1.217 t^2$$

$$S_B - S_A = 5 \text{ m}$$

$$1.217 t^2 - 0.755 t^2 = 5$$

$$0.462 t^2 = 5$$

$$t = 3.29 \text{ s}$$

$$\therefore S_A = 0.755 t^2 = 8.17 \text{ m}$$

$$S_B = 1.217 t^2 = 13.17 \text{ m}$$

$$\text{To find } a_B : \sum F = m \cdot a$$

$$m \cdot g \cdot \sin \alpha - F_f = m \cdot a_B$$

$$m \cdot g \cdot \sin \alpha - \mu_B m \cdot g \cdot \cos \alpha = m \cdot a_B$$

Canceling 'm' throughout

$$g \cdot \sin \alpha - \mu_B g \cdot \cos \alpha = a_B$$

$$9.81 \times \sin 20 - 0.1 \times 9.81 \times \cos 20 = a_B$$

$$a_B = 2.43 \text{ m/s}^2$$

Considering the motion of block 'A'

$$S_A = ut + \frac{1}{2} a t^2$$

**SP4.116:** A horizontal force  $P = 70\text{N}$  is exerted on mass  $A = 16\text{kg}$  as shown in fig.4.

The coefficient of friction between A and the horizontal plane is 0.25. B has a mass of 4kg and the coefficient of friction between it and the plane is 0.5. The cord between the two masses makes an angle of  $10^\circ$  with horizontal. What is the tension in the cord?

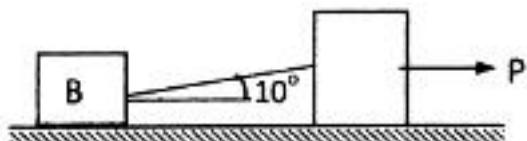
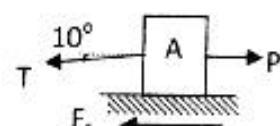
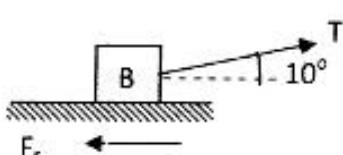


Fig.4.79

**Solution:**

Considering the motion of block B

$$\sum F = m \cdot a$$

$$T \cos 10 - F_r = m \cdot a$$

$$T \cos 10 - \mu(mg - T \sin 10) = m \cdot a$$

$$T(0.9848) - 0.5 (4 \times 9.81) - 0.5 T (0.1736)$$

$$= 4 \cdot a$$

$$0.9849 T - 19.62 + .0868 T = 4 \cdot a$$

$$\underline{1.0716 T - 19.62 = 4 \cdot a} \quad \dots \dots (1)$$

Considering the motion of the block A

$$\sum F = m \cdot a$$

$$P - T \cos 10 - F_r = m \cdot a$$

$$P - T \cos 10 - \mu(mg + T \sin 10) = m \cdot a$$

$$70 - 1.0274 T - 39.24 = 16 \cdot a$$

$$\underline{30.76 - 1.027 T = 16 \cdot a} \quad \dots \dots (2)$$

Solving the equations 1 & 2

$$T = 20.5 \text{ N} \quad \text{& } a = 0.6 \text{ m/s}^2$$

**SP4.117:** Determine the tension in the string and acceleration of two blocks of mass 150kg and 50kg connected by a string and a frictionless and weightless pulley as shown in fig.4.80

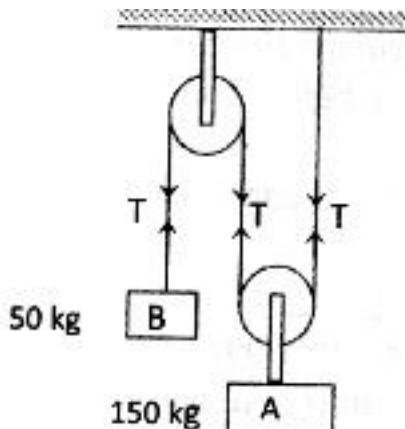


Fig.4.80

**Solution:**

Tension 'T' remains same throughout the string .

Acceleration of the block A(150 kg) is half of the acceleration of the block B(50 kg)

$$\text{i.e. } a_A = \frac{1}{2} a_B$$

$$\text{or } 2 a_A = a_B$$

Considering the downward motion of the block A(150 kg)

$$\sum F = m \cdot a$$

$$mg - 2T = m \cdot a$$

$$150 \times 9.81 - 2T = 150 \cdot a_A$$

Considering the upward motion of the block B(50 kg)

$$\sum F = m \cdot a$$

$$T - mg = m \cdot a$$

$$T - 50 \times 9.81 = 50 \cdot a_B$$

$$T - 490.5 = 50 \times (2 \cdot a_A)$$

$$T - 490.5 = 100 \cdot a_A \quad \dots \dots (2)$$

Solving equation 1 & 2

$$1471.5 - 2T = 150 \cdot a_A \quad \dots \dots \dots (1)$$

$$T = 630.6 \text{ N}, a_A = 1.4 \text{ m/s}^2 \text{ & } a_B = 2.8 \text{ m/s}^2$$

**SP4.118:** Determine the resulting motion of the body 'A' assuming the pulleys to be smooth and weightless as shown in fig.4. If the system starts from rest determine the velocity of the body 'A' after 10 seconds.

**Solution:**

Tension T in the string remains same throughout

Acceleration of block B is half of the acceleration of block A

$$\text{i.e., } a_B = \frac{1}{2} a_A \text{ or } 2 a_B = a_A$$

Considering the downward motion of the block B

$$\sum F = m \cdot a$$

$$mg - 2T = m \cdot a$$

$$15 \times 9.81 - 2T = 15 \cdot a_B$$

$$147.15 - 2T = 15 a_B \quad \dots \dots \dots (1)$$

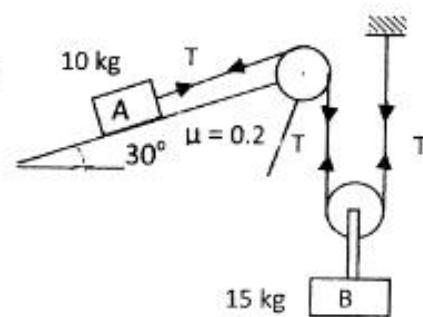


Fig.4.81

Considering the up ward motion of the block A

$$\sum F = m \cdot a$$

$$T - mg \sin \alpha - Fr = m \cdot a$$

$$T - 10 \times 9.81 \times \sin 30^\circ - 0.2 \times 10 \times 9.81 \times$$

$$\cos 30^\circ = 10 \cdot a_A = 10 (2 a_B) = 20 a_B$$

$$T - 66.04 = 20 a_B \quad \dots \dots \dots (2)$$

Solving the equation from 1 & 2

$$a_B = 0.2739 \text{ m/s}^2, a_A = 0.5478 \text{ m/s}^2$$

Considering the upward motion of the block A

$$u = 0 \text{ m/s}$$

$$a_A = 0.5478 \text{ m/s}^2$$

$$t = 10 \text{ s}$$

$$v = u + a t$$

$$v = 0 + 0.5478 \times 10 = 5.478 \text{ m/s}$$

**SP4.119:** Two rough planes inclined at  $30^\circ$  and  $15^\circ$  to the horizontal and of the same height are placed back to back. Masses of 15kg is placed on  $30^\circ$  plane and 5kg is placed on the  $15^\circ$  plane and are connected by a string over the top of the planes. If the coefficient of friction be 0.3, find from fundamentals the resulting acceleration.

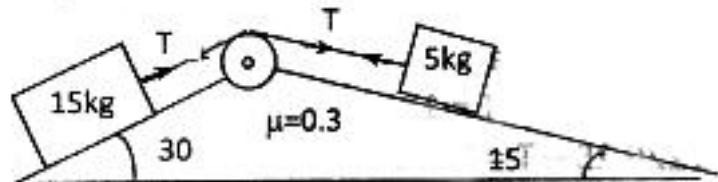


Fig.4.82

**Solution:**

Considering the downward motion of the block 15 kg,

$$\sum F = m.a$$

$$mg \sin \alpha - T - F_f = m.a$$

$$15 \times 9.81 \times \sin 30^\circ - T - \mu mg \cos \alpha = 15.a$$

$$73.575 - T - 38.23 = 15.a$$

$$35.345 - T = 15.a \quad \dots \dots \dots (1)$$

Considering the upward motion of the block 5 kg,

$$\sum F = m.a$$

$$T - mg \sin \alpha - F_f = m.a$$

$$T - 5 \times 9.81 \times \sin 15^\circ - \mu mg \cos \alpha = 5.a$$

$$T - 12.69 - 14.21 = 5.a \quad \dots \dots \dots (2)$$

Solving the equations 1 & 2

$$a = 0.422 \text{ m/s}^2$$

**SP4.120:** A block of wood 'A' of mass 5kg is resting on a table. A string attached to 'A' runs over a pulley and carries a mass of 5kg at the other end. A wooden block is of mass 3kg is placed on the top of block 'A' and is held fixed by means of a string. The coefficient of kinetic friction between A and the table is 0.25 and between B and A is 0.4. Determine acceleration of block A.

**Solution:**

Let the tension between the masses A & C is  $T_1$  and the tension on the string of mass B is  $T_2$ . Since masses A & C are connected by string the acceleration remains same for both the masses.

Considering the downward motion of the mass C

$$\sum F = m.a$$

$$mg - T_1 = m.a$$

$$5 \times 9.81 - T_1 = 5.a$$

$$49.05 - T_1 = 5.a \quad \dots \dots \dots (1)$$

Considering the frictional force between masses

A & B and A and the table

Due to the motion the block A towards left

$$T_1 - \mu_{AT}(5 + 3)g - \mu_{AB}(3)g = 5.a$$

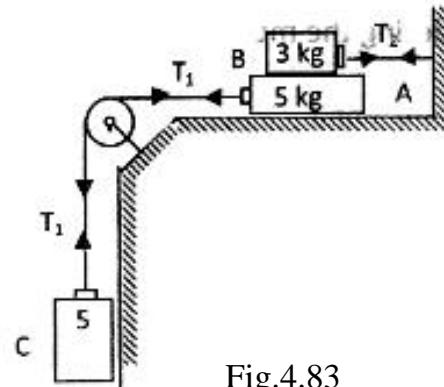


Fig.4.83

$$T_1 - 0.25(5 + 3)9.81 - 0.4(3)9.81 = 5.a$$

$$T_1 - 31.392 = 5.a \quad \dots \dots \dots (2)$$

Solving the equations 1 & 2

$$a = 1.765 \text{ m/s}^2$$

**SP4.121:** The 125kg mass concrete block 'A' is released from rest in the position as shown in fig.4. and pulls the 200kg mass block up the  $30^\circ$  ramp. If the coefficient of kinetic friction between the log and the ramp is 0.5, determine the velocity of the log when the block hits the ground at 'B'.

**Solution:**

$$a_A = 2 a_C$$

Considering the mass A

$$mg - T = m a$$

$$125 \times 9.81 - T = 125 a_A$$

$$1226.25 - T = 125 a_A$$

$$2452.5 - 2T = 250 a_A \dots\dots(1)$$

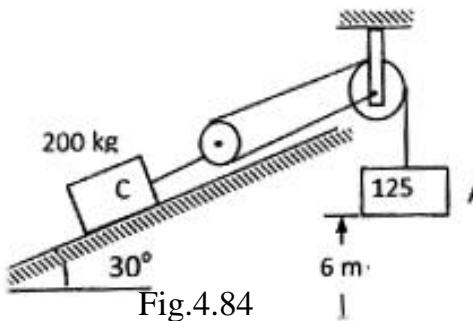


Fig.4.84

Considering mass C

$$2T - mg \sin \alpha - \mu mg \cos \alpha = m a_C$$

$$2T - 200 \times 9.81 \times \sin 30^\circ - 0.5 \times 200 \times 9.81 \times \cos 30^\circ = 200 a_C$$

$$2T - 1830.57 = 100 a_A \dots\dots(2)$$

Solving 1 & 2

$$a_A = 1.7769 \text{ m/s}^2$$

To find the velocity

$$u = 0$$

$$a = a_C = a_A/2 = 1.7769/2 = 0.88845 \text{ m/s}^2$$

$$s = 6/2 = 3 \text{ m}$$

$$v^2 = u^2 + 2 a s$$

$$v^2 = 0^2 + 2 \times 0.88845 \times 3$$

$$v = 2.309 \text{ m/s}$$

**SP4.122:** Determine the acceleration of the 5kg cylinder A. Neglect the mass of the pulleys and cords. The block at B has a mass of 10kg. The coefficient of kinetic friction between the block B and the surface is 0.1.

**Solution:**

$$\text{We know that } a_B = 4 a_A$$

Considering the motion of mass A

$$m_A g - 2T_2 = m_A \cdot a_A$$

$$5 \times 9.81 - 2T_2 = 5 a_A$$

$$49.05 - 2T_2 = 5 a_A$$

$$\text{But } T_2 = 2T_1$$

$$49.05 - 4T_1 = 5 a_A \dots\dots(1)$$

Considering the motion of mass B

$$\begin{aligned} T_1 - \mu m_B g &= m_B \cdot a_B \\ T_1 - 0.1 \times 10 \times 9.81 &= 10 a_B \\ T_1 - 9.81 &= 40 a_A \\ 4T_1 - 39.24 &= 160 a_A \dots\dots(2) \\ \text{Solving 1 \& 2} \\ a_A &= 0.05945 \text{ m/s}^2 \end{aligned}$$

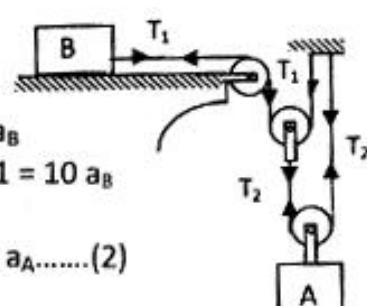


Fig.4.85

## 2) Normal and Tangential Coordinates or Path variable:

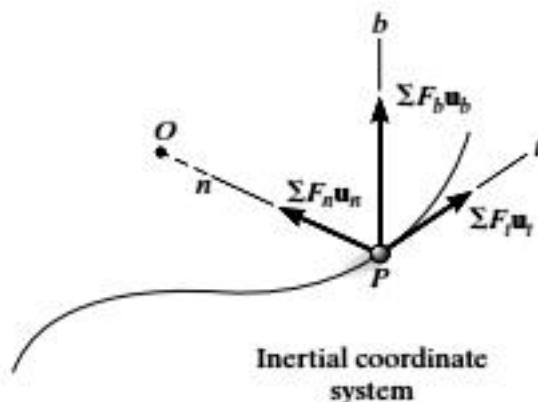


Fig.4.86

When a particle moves along a curved path which is known, the equation of motion for the particle may be written in the tangential, normal, and bi-normal directions, as shown in fig.4.86. Note that there is no motion of the particle in the bi-normal direction, since the particle is constrained to move along the path. We have

$$\sum F = ma$$

$$\sum F_t e_t + \sum F_n e_n + \sum F_b e_b = ma_t + ma_n$$

This equation is satisfied provided

$$\left. \begin{array}{l} \sum F_t = ma_t \\ \sum F_n = ma_n \\ \sum F_b = 0 \end{array} \right\} \longrightarrow (4.50)$$

Recall that  $a_t (= \frac{dv}{dt})$  represents the time rate of change in the magnitude of velocity. So if  $\sum F_t$  acts in the direction of motion, the particle's speed will increase whereas if it acts in the opposite direction, the particle will slow down. Likewise,  $a_n (= \frac{v^2}{r})$  represents the time rate of change in the velocity's direction. It is caused by  $\sum F_n$ , which always acts in the positive  $n$  direction, i.e., towards the path's centre of curvature. For this reason it is often referred to as the centripetal force.

### **Centripetal force:**

The force which acts along the radius of the circle at every point and is always directed towards the centre of the circle along which the body moves is known as centripetal force. Its magnitude is  $m\ddot{S}^2r$ .

### **Centrifugal force:**

The force which acts opposite to the centripetal force is known as centrifugal force. The centrifugal force always acts away from the centre of the path, or in other words the centrifugal force always tends to throw the body from the centre of circular path. Its magnitude is  $m\ddot{S}^2r$ .

### **Super elevation or Cant or Banking:**

We see that whenever a roadway ( or railway) is laid on a curved path, then both the edges are not at the same level. In such a case the outer edge is always higher than the inner one. This is done to keep the vehicle in equilibrium while in motion.

The body which is moving along a curved path is subjected to the following forces:

- a) Its own weight ( $mg$ )
- b) Centrifugal force ( $m\ddot{S}^2r$  or  $mv^2/r$ )

The resultant of these forces is inclined with the vertical as shown in fig.4. It may be noted that if the curved path be made level then the resultant force will be inclined at some angle with the vertical and thus the reactions on both the supports of a vehicle will not be equal which will affect the equilibrium of the vehicle.

To counterbalance this effect and to maintain equilibrium of the vehicle the surface of the path is made perpendicular to the resultant by raising the outer edge of

the roadway (or outer rail of the railway). The amount by which the outer edge of rail is raised is known as cant or super elevation.

### Need for banking of roads and super elevation of rails:

When a vehicle moves around a flat curve, a radial outward force  $mv^2/r$  is experienced increasing the pressure on the outer wheel and decreasing the pressure on the inner wheel. It creates discomfort to passengers. Apart from this, speed is to be reduced considerably on curved path to avoid skidding and over turning. If the road is banked or the track is given super elevation, the following may be achieved:

- a) Skidding and overturning avoided
- b) Higher speed permitted
- c) Lateral pressure eliminated giving comfort to passenger
- d) Excess wear and tear of wheels avoided

From the figure;

$$\tan \theta = \frac{\text{Centrifugal force}}{\text{Weight}}$$

$$= \frac{m \ddot{S}^2 r}{mg} = \frac{\ddot{S}^2 r}{g} = \frac{v^2}{rg} \rightarrow (4.51)$$

It may be noted from the above expression that the super elevation is independent of the mass of the body.

$$\text{For rails, } \tan \theta = \frac{S}{G} = \frac{v^2}{rg} \rightarrow (4.52)$$

Where,  $S$  – Height through which outer wheel is lifted

$G$  – Gauge length

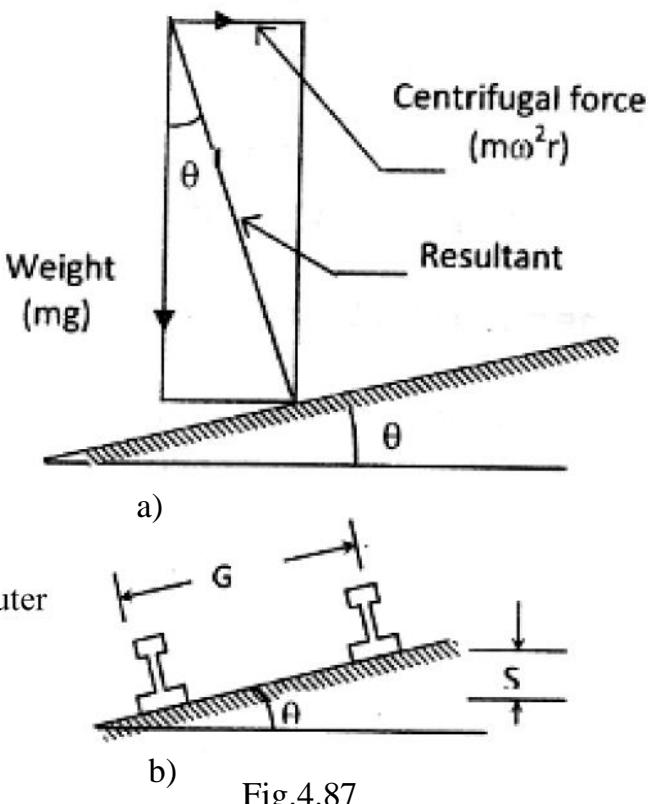


Fig.4.87

## Maximum velocity to avoid skidding away of a vehicle on a level circular path:

To avoid skidding:

$$Fr \geq \text{Centrifugal Force} \text{ or } \text{Centrifugal Force} \leq Fr$$

We know that,

$Fr$  – Friction force

= Coefficient of friction x normal reaction

$$= \mu \times R$$

Therefore,

$$\text{Centrifugal Force} \leq \mu \times R$$

$$\frac{mv^2}{r} \leq \mu \times mg$$

$$v^2 \leq \mu \times rg$$

$$v \leq \sqrt{\mu \times rg}$$

$$v_{\max} = \sqrt{\mu \times rg} \quad \text{---} \rightarrow (4.53)$$

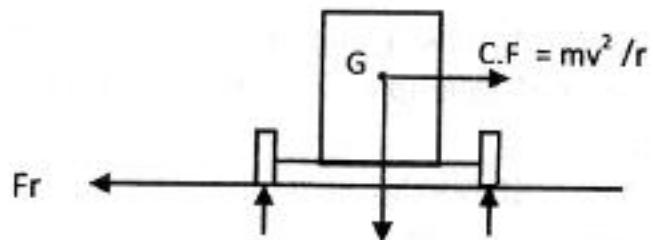


Fig.4.88

## Maximum velocity to avoid overturning of a vehicle on a level circular path:

Let  $h$  – height of centre of gravity of the vehicle from the road,

$R_A, R_B$  – reactions in the wheels A and B

$2a$  – distance between the two wheels

Taking moments about B, then

$$R_A \times 2a - mg \times a + \frac{mv^2 h}{r} = 0$$

$$R_A \times 2a = mg \times a - \frac{mv^2 h}{r}$$

$$R_A = \frac{\left(mga - \frac{mv^2h}{r}\right)}{2a}$$

$$R_A = \frac{mg}{2} \left[ 1 - \frac{v^2h}{rag} \right]$$

Taking moments about A, then

$$R_B \times 2a - mga - \frac{mv^2h}{r} = 0$$

$$R_B = \frac{\left(mga + \frac{mv^2h}{r}\right)}{2a}$$

$$R_B = \frac{mg}{2} \left[ 1 + \frac{v^2h}{rag} \right] \quad \text{-----} \rightarrow (4.54)$$

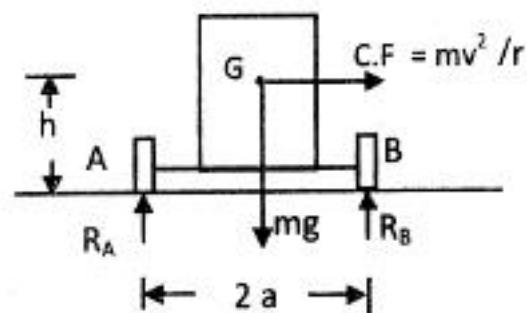


Fig.4.89

A little consideration will show that the reaction at B (i.e.,  $R_B$ ) can never be negative. But the reaction  $R_A$  may be negative if the value of  $\frac{v^2h}{rag}$  becomes more than unity. This is only possible by increasing the value of 'v'. When this condition reaches, the vehicle will overturn at the wheel B. Therefore in order to avoid overturning.

$$\frac{v^2h}{rag} < 1 \text{ or}$$

$$v < \sqrt{\frac{rag}{h}} \text{ or}$$

$$v_{\max} = \sqrt{\frac{rag}{h}} \quad \text{-----} \rightarrow (4.55)$$

### Problems:

**SP4.123:** Determine the banking angle  $\theta$  for the race track so that the wheels of the racing cars shown in fig.4.a will not have to depend upon friction to prevent nay car

from sliding up or down the track. Assume the cars have negligible size, a mass  $m$ , and travel around the curve of radius  $\rho$  with a constant speed  $v$ .

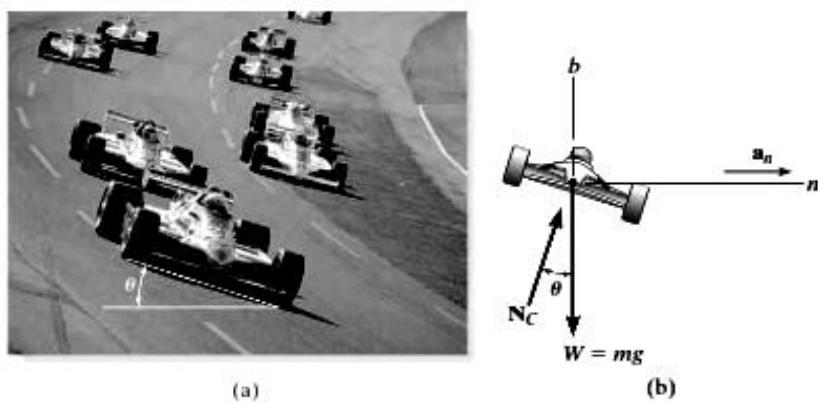


Fig.4.90

### Solution:

Before looking at the following solution, give some thought as to why it should be solved using  $t$ ,  $n$ ,  $b$  coordinates.

**Free-Body Diagram.** As shown in Fig. 13-12b, and as stated in the problem, no frictional force acts on the car. Here  $N_c$  represents the resultant of the ground on all four wheels. Since  $a_n$  can be calculated, the unknowns are  $N_c$  and  $\theta$ .

**Equations of Motion.** Using the  $n$ ,  $b$  axes shown,

$$\rightarrow \sum F_n = ma_n; \quad N_c \sin \theta = m \frac{v^2}{\rho} \quad (1)$$

$$+ \uparrow \sum F_b = 0; \quad N_c \cos \theta - mg = 0 \quad (2)$$

Eliminating  $N_c$  and  $m$  from these equations by dividing Eq. 1 by Eq. 2, we obtain

$$\tan \theta = \frac{v^2}{g\rho}$$

$$\theta = \tan^{-1} \left( \frac{v^2}{g\rho} \right) \quad \text{Ans}$$

**SP4.124:** Determine the maximum speed  $v$  which the sliding block may have as it passes point A without losing contact with the surface.

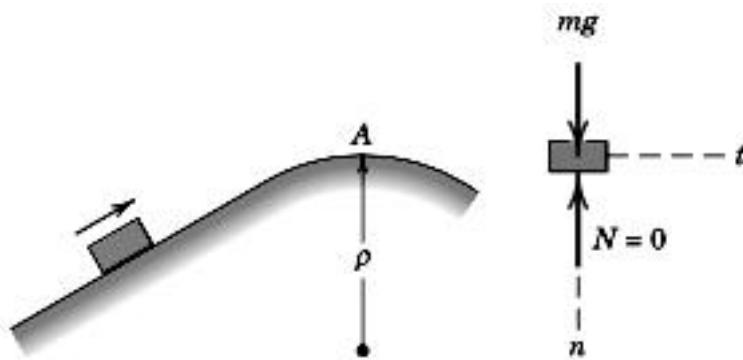


Fig.4.91

**Solution:**

The condition for loss of contact is that the normal force  $N$  which the surface exerts on the block goes to zero. Summing forces in the normal direction gives

$$[\sum F_n = ma_n] \quad mg = m \frac{v^2}{\rho} \quad v = \sqrt{g\rho} \quad \text{Ans.}$$

If the speed at  $A$  were less than  $\sqrt{g\rho}$ , then an upward normal force exerted by the surface on the block would exist. In order for the block to have a speed at  $A$  which is greater than  $\sqrt{g\rho}$ , some type of constraint, such as a second curved surface above the block, would have to be introduced to provide additional downward force.

**SP4.125:** Small objects are released from rest at  $A$  and slide down the smooth circular surface of radius  $R$  to a conveyor  $B$ . Determine the expression for the normal contact force  $N$  between the guide and each object in terms of  $\theta$  and specify the correct angular velocity  $\omega$  of the conveyor pulley of radius  $r$  to prevent any sliding on the belt as the objects transfer to the conveyor.

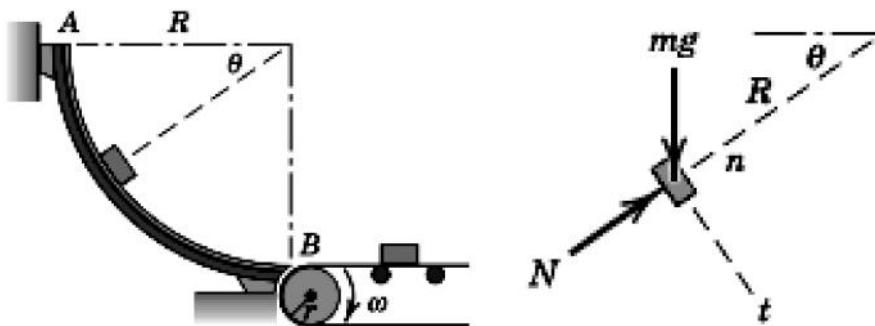


Fig.4.92

**Solution:**

The free-body diagram of the object is shown together with the coordinate directions  $n$  and  $t$ . The normal force  $N$  depends on the  $n$ -component of the acceleration which, in turn, depends on the velocity. The velocity will be cumulative according to the tangential acceleration  $a_t$ . Hence, we will find  $a_t$  first for any general position.

$$[\Sigma F_t = ma_t] \quad mg \cos \theta = ma_t \quad a_t = g \cos \theta$$

Now we can find the velocity by integrating

$$[v \, dv = a_t \, ds] \quad \int_0^v v \, dv = \int_0^\theta g \cos \theta \, d(R\theta) \quad v^2 = 2gR \sin \theta$$

We obtain the normal force by summing forces in the positive  $n$ -direction, which is the direction of the  $n$ -component of acceleration.

$$[\Sigma F_n = ma_n] \quad N - mg \sin \theta = m \frac{v^2}{R} \quad N = 3mg \sin \theta \quad \text{Ans.}$$

The conveyor pulley must turn at the rate  $\omega = rv$  for  $\theta = \pi/2$ , so that

$$\omega = \sqrt{2gR/r} \quad \text{Ans.}$$

**SP4.126:** A 1500kg car enters a section of curved road in the horizontal plane and slows down at a uniform rate from a speed of 100kmph at A to a speed of 50kmph as it passes C. The radius of curvature  $\rho$  of the road at A is 400m and at C is 80m. Determine the total horizontal force exerted by the road on the tires at positions A, B and C. Point B is the inflection point where the curvature changes direction.

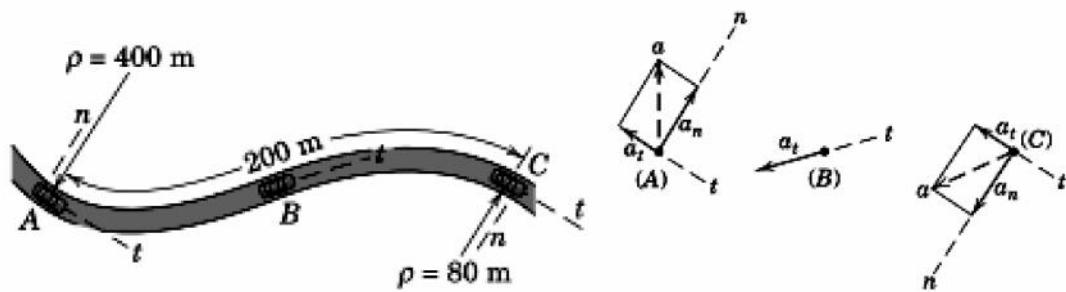


Fig.4.93

**Solution:**

The car will be treated as a particle so that the effect of all forces exerted by the road on the tires will be treated as a single force. Since the motion is described along the direction of the road, normal and tangential coordinates will be used to specify the acceleration of the car. We will then determine the forces from the accelerations.

The constant tangential acceleration is in the negative  $t$ -direction, and its magnitude is given by

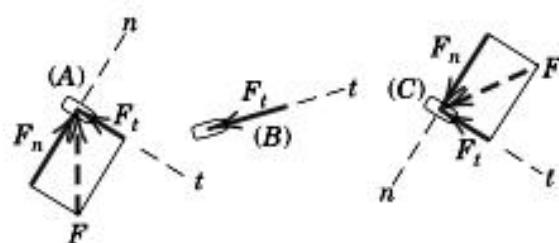
$$[v_C^2 = v_A^2 + 2a_t \Delta s] \quad a_t = \left| \frac{(50/3.6)^2 - (100/3.6)^2}{2(200)} \right| = 1.447 \text{ m/s}^2$$

The normal components of acceleration at  $A$ ,  $B$ , and  $C$  are

$$[a_n = v^2/\rho] \quad \text{At } A, \quad a_n = \frac{(100/3.6)^2}{400} = 1.929 \text{ m/s}^2$$

$$\text{At } B, \quad a_n = 0$$

$$\text{At } C, \quad a_n = \frac{(50/3.6)^2}{80} = 2.41 \text{ m/s}^2$$



Application of Newton's second law in both the *n*- and *t*-directions to the free-body diagrams of the car gives

$$[\Sigma F_t = ma_t] \quad F_t = 1500(1.447) = 2170 \text{ N}$$

$$[\Sigma F_n = ma_n] \quad \text{At A,} \quad F_n = 1500(1.929) = 2890 \text{ N}$$

$$\text{At B,} \quad F_n = 0$$

$$\text{At C,} \quad F_n = 1500(2.41) = 3620 \text{ N}$$

Thus, the total horizontal force acting on the tires becomes

$$\text{At A,} \quad F = \sqrt{F_n^2 + F_t^2} = \sqrt{(2890)^2 + (2170)^2} = 3620 \text{ N} \quad \text{Ans.}$$

$$\text{At B,} \quad F = F_t = 2170 \text{ N} \quad \text{Ans.}$$

$$\text{At C,} \quad F = \sqrt{F_n^2 + F_t^2} = \sqrt{(3620)^2 + (2170)^2} = 4220 \text{ N} \quad \text{Ans.}$$

**SP4.127:** Compute the magnitude *v* of the velocity required for the spacecraft *S* to maintain a circular orbit of altitude 200 mile above the surface of the earth.

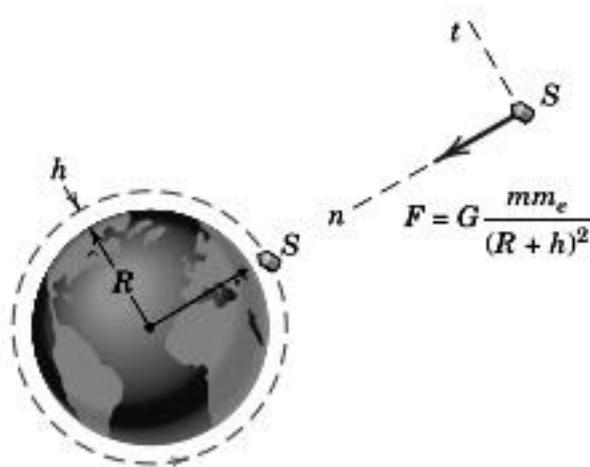


Fig.4.94

**Solution:**

**Solution.** The only external force acting on the spacecraft is the force of gravitational attraction to the earth (i.e., its weight), as shown in the free-body diagram. Summing forces in the normal direction yields

$$[\Sigma F_n = ma_n] \quad G \frac{mm_e}{(R + h)^2} = m \frac{v^2}{(R + h)}, \quad v = \sqrt{\frac{Gm_e}{(R + h)}} = R \sqrt{\frac{g}{(R + h)}}$$

where the substitution  $gR^2 = Gm_e$  has been made. Substitution of numbers gives

$$v = (3959)(5280) \sqrt{\frac{32.234}{(3959 + 200)(5280)}} = 25,326 \text{ ft/sec} \quad \text{Ans.}$$

**SP4.128:** A railway engine of mass 60 tonnes is moving in arc of radius 200m with a speed of 36kmph. Find the force exerted on the rails towards the centre of the circle.

**Solution:**

$$\text{Mass} = 60,000 \text{ kg}$$

$$\text{Radius} = 200 \text{ m}$$

$$\text{Speed } V = 36 \text{ kmph} = 10 \text{ m/s}$$

$$\omega = V/r = 10/200 = 0.05 \text{ rad/s}$$

Centrifugal force acting on the engine away from centre of rotation is equal to the reaction on the rail acting towards centre of rotation.

$$\begin{aligned} &= m \omega^2 r \\ &= 60,000 \times 0.05^2 \times 200 = 30,000 \text{ N} = 30 \text{ kN} \end{aligned}$$

**SP4.129:** In a circus show a motor cyclist is moving inside a spherical cage of radius 3m. The motor cycle and the man together have mass of 750kg. Find the least velocity with which the motor cyclist must pass the highest point on the cage, without losing contact inside the cage (underlined data is addition).

**Solution:**

At highest point on the cage, the centrifugal force acting away from centre of rotation should be greater than the self weight of the motorbike along with rider.

$$C.F = \text{Self weight}$$

$$m \omega^2 r = m \times g$$

$$750 \times \omega^2 \times 3 = 750 \times 9.81$$

$$\omega^2 = 3.27$$

$$\omega = 1.808$$

$$v = \omega \times r = 1.808 \times 3 = 5.429 \text{ m/s} = 19.52 \text{ kmph}$$

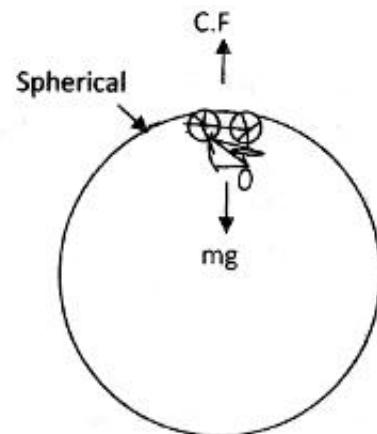


Fig.4.95

**SP4.130:** A circular automobile test track has a radius of 200m. The track is so designed that when a car travels at a speed of 90kmph, the force between the automobile and the track is normal to the surface of the track. Find the angle of bank.

**Solution:**

$$V = 90 \text{ kmph} = 25 \text{ m/s}$$

$$R = 200 \text{ m}; \theta = \text{angle of bank}$$

From the fig.  $\tan \theta = C.F/\text{Self weight}$

$$\begin{aligned} \tan \theta &= (mv^2/r) / mg = v^2/rg \\ &= 25^2 / 200 \times 9.81 = 0.3185 \end{aligned}$$

$$\theta = 17.67^\circ$$

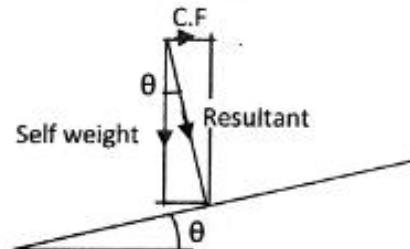


Fig.4.96

**SP4.131:** The distance between the rails of the track is 1.67m. How much the outer rail be elevated for a curve of 500m radius in order that the resultant force may be normal at a speed of 45kmph.

**Solution:**

$$V = 45 \text{ kmph} = 12.5 \text{ m/s}$$

$$R = 500 \text{ m}$$

$$\tan \theta = (mv^2/r) / mg = v^2/rg$$

$$= 12.5^2 / 500 \times 9.81$$

$$= 0.03185$$

$$\theta = 1.8245^\circ$$

$$\sin \theta = h / \text{gauge length}$$

$$\sin \theta = h/1.67$$

$$h = \sin \theta \times 1.67 = 0.05317 \text{ m}$$

$$= 53.17 \text{ mm}$$

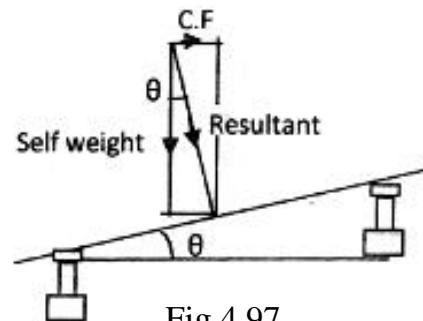


Fig.4.97

**SP4.132:** A cyclist riding at 5m/s has to turn a corner. What is the least radius of the curve he has to describe if the coefficient of friction between the tyre and the road be 0.25?

**Solution:**

$$v = 5 \text{ m/s}; \mu = 0.25$$

$$v = \sqrt{\mu gr}; 5 = \sqrt{0.25 \times 9.81 \times r}; r = 10.2 \text{ m}$$

**SP4.133:** A vehicle of mass 1000kg is to turn on a circular curve of 40m radius. The height of its centre of gravity above the road level is 0.75m and the distance between the centre lines of the wheel is 1.2m. Find the speed at which the vehicle should be run in order to avoid overturning.

**Solution:**

$$M = 1000 \text{ kg}$$

$$r = 40 \text{ m}$$

$$\text{We know that } v_{\max} = \sqrt{\frac{gra}{h}} = \sqrt{\frac{9.81 \times 40 \times 0.6}{0.75}} = 17.71 \text{ m/s} = 63.78 \text{ kmph}$$

**SP4.134:** A vehicle weighing 10kN is to turn a circular corner of radius 100m on a level road with a speed of 10m/s. The height of its centre of gravity above the road is

1m and the distance between its wheel is 1.5m. Find the reactions at the wheel and also find the maximum velocity to avoid overturning.

**Solution:**

$$mg = 10 \text{ kN} = 10,000 \text{ N}; r = 100 \text{ m}; v = 10 \text{ m/s}; h = 1 \text{ m}; 2a = 1.5 \text{ m}$$

We know that

$$R_A = \frac{mg}{2} \left[ 1 - \frac{v^2 h}{gra} \right] = \frac{10000}{2} \left[ 1 - \frac{10^2 \times 1}{9.81 \times 100 \times 0.75} \right] = 4320.4 \text{ N}$$

$$R_B = \frac{mg}{2} \left[ 1 + \frac{v^2 h}{gra} \right] = \frac{10000}{2} \left[ 1 + \frac{10^2 \times 1}{9.81 \times 100 \times 0.75} \right] = 5679.57 \text{ N}$$

$$v_{\max} = \sqrt{\frac{gra}{h}} = \sqrt{\frac{9.81 \times 100 \times 0.75}{1}} = 27.12 \text{ m/s} = 97.63 \text{ kmph}$$

**SP4.135:** A bob weighing 1N suspended by a cord from the ceiling of a railway carriage was found to make an angle  $8^\circ$  with the vertical when the railway carriage was negotiating a curve of 100m radius. Find the speed of the carriage and tension in the cord.

**Solution:**

Let T- Tension in the cord

$\theta$  – Inclination of the cord with vertical

C.F – Centrifugal force  $= mv^2/r$

mg – Self weight of the bob

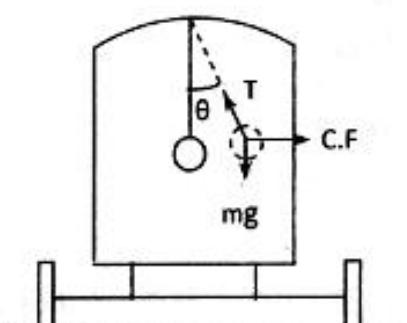
Resolving the forces horizontally

$$\sum F_x = (mv^2/r) - T \sin \theta = 0 \dots (1)$$

Resolving the forces vertically

$$\sum F_y = T \cos \theta - mg = 0$$

$$T = mg / \cos \theta \dots (2)$$



$$\tan \theta = \frac{mv^2}{rmg} = \frac{v^2}{rg}$$

$$v = \sqrt{gr \tan \theta}$$

Fig.4.98

Substituting (2) in (1)  
 $(mv^2/r) - mg \cdot \sin\theta / \cos\theta = 0$   
 $(mv^2/r) = mg \cdot \tan\theta$

$$v = \sqrt{9.81 \times 100 \times \tan 8} = 11.74 \text{ m/s}$$

$$= 42.26 \text{ kmph}$$

$$T = mg / \cos 8 = 1.01 \text{ N}$$

**SP4.136:** An automobile is negotiating a curve of radius 100m. The distance between the centre to centre of the wheel is 1.2m and height of the centre of gravity of the automobile is 0.75m above the ground. Find the maximum speed to avoid overturning.

**Solution:**

$$r = 100 \text{ m}$$

$$a = 0.6 \text{ m}$$

$$h = 0.75 \text{ m}$$

We know that  $v_{\max} = \sqrt{\frac{gra}{h}} = \sqrt{\frac{9.81 \times 100 \times 0.6}{0.75}} = 28.01 \text{ m/s} = 100.85 \text{ kmph}$

**SP4.137:** A motor cycle rider travelling at 18kmph has to turn a corner. Find the least radius of the curve he should follow for the same travelling if the coefficient of friction between the tyres and the road is 0.2.

**Solution:**

$$v = 18 \text{ kmph} = 5 \text{ m/s}$$

$$\mu = 0.2$$

$$v_{\max} = \sqrt{\mu gr} ; 5 = \sqrt{0.2 \times 9.81 \times r} ; r = 12.74 \text{ m}$$

**SP4.138:** A curve of 200m radius on a high way is banked for a speed of 72kmph so that at the speed the force exerted on a car by the road is normal to the surface of the road. A) Determine what the bank angle should be and B) if the  $\mu = 0.6$ , what is the maximum speed at which the car can travel without skidding?

**Solution:**

$$r = 200 \text{ m}$$

$$v = 72 \text{ kmph} = 20 \text{ m/s}$$

We know that when the resultant of centrifugal force and the self weight is acting normal to the banked road,

$$\tan \theta = C.F/mg = \frac{\left[\frac{mv^2}{r}\right]}{mg} = \frac{v^2}{rg} = \frac{20^2}{200 \times 9.81}$$

$$\theta = 11.52^\circ$$

$$v_{\max} = \sqrt{gr \tan(\phi + \theta)}$$

$$\text{But } \mu = \tan \Phi$$

$$0.6 = \tan \Phi$$

$$\Phi = 30.963^\circ$$

$$\text{Therefore } v_{\max} = \sqrt{9.81 \times 200 \tan(30.96 + 11.52)}$$

$$V_{\max} = 42.386 \text{ m/s} = 152.58 \text{ kmph}$$

**SP4.139:** A small block of weight rests on a horizontal turn table at a distance of  $r=1\text{m}$  from the centre of the turn table. Find the maximum uniform speed the turn table can have without slipping of the block. Assume  $\mu=0.5$ .

**Solution:**

The force acting on the blocks are its centrifugal force acting away from centre of rotation and the frictional force acting opposite direction of the motion.

Hence

$$(mv^2/r) = \mu mg$$

$$v = \sqrt{\mu gr} ; v = \sqrt{0.5 \times 9.81 \times 1} = 2.214 \text{ m/s}$$

$$\omega = v/r = 2.214$$

$$\frac{2\pi N}{60} = 2.214 \text{ m/s}$$

$$N = 21.149 \text{ rpm}$$

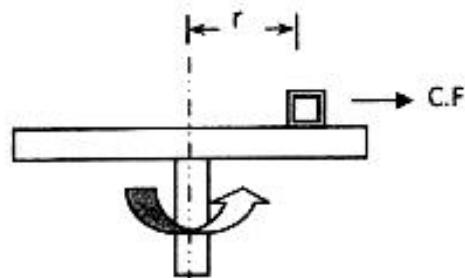


Fig.4.99

**Polar coordinates:**

When all the forces acting on a particle are resolved into polar components i.e along the unit vector direction  $e_r$  and  $e_\theta$  as shown in fig.4.100, the equation of motion can be expressed as

$$\sum F = ma$$

$$\sum F_r e_r + \sum F_\theta e_\theta = ma_r e_r + ma_\theta e_\theta$$

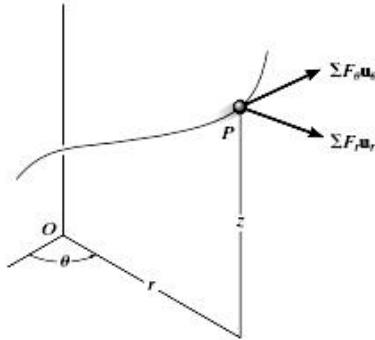


Fig.4.100

This equation is satisfied provided

$$\left. \begin{array}{l} \sum F_r = ma_r \\ \sum F_\theta = ma_\theta \end{array} \right\} \longrightarrow (4.56)$$

Where,  $a_r = r - r_\theta^2$  and  $a_\theta = r_\theta + 2r_\theta$

### Problems:

**SP4.140:** Tube A rotates about the vertical O axis with a constant angular rate  $\dot{\theta} = \omega$  and contains a small cylindrical plug B of mass m whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius b. Determine the tension T in the cord and the horizontal component  $F_\theta$  of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is  $\omega_0$  first in the direction for case (a) and second in the direction for case(b). Neglect friction.

### Solution:

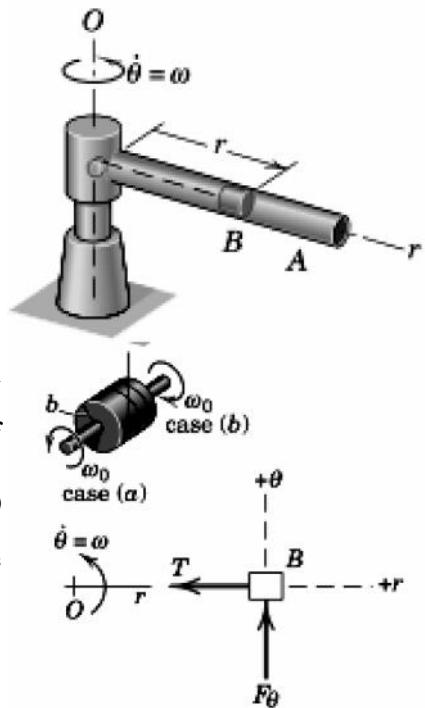


Fig.4.101

With  $r$  a variable, we use the polar-coordinate form of the equations of motion, Eqs. 3/8. The free-body diagram of  $B$  is shown in the horizontal plane and discloses only  $T$  and  $F_\theta$ . The equations of motion are

$$[\Sigma F_r = ma_r] \quad -T = m(\ddot{r} - r\dot{\theta}^2)$$

$$[\Sigma F_\theta = ma_\theta] \quad F_\theta = m(r\ddot{\theta} + 2r\dot{\theta})$$

**Case (a).** With  $\dot{r} = +b\omega_0$ ,  $\ddot{r} = 0$ , and  $\ddot{\theta} = 0$ , the forces become

$$T = mr\omega^2 \quad F_\theta = 2mb\omega_0\omega \quad \text{Ans.}$$

**Case (b).** With  $\dot{r} = -b\omega_0$ ,  $\ddot{r} = 0$ , and  $\ddot{\theta} = 0$ , the forces become

$$T = mr\omega^2 \quad F_\theta = -2mb\omega_0\omega \quad \text{Ans.}$$

**SP4.141:** The smooth 0.5kg double collar as in fig.4.102 can freely slide on arm AB and the circular guide rod. If the arm rotates with a constant angular velocity of  $\dot{\theta} = 3\text{ rad/s}$ , determine the force the arm exerts on the collar at the instant  $\theta=45^\circ$ . Motion is the horizontal plane.

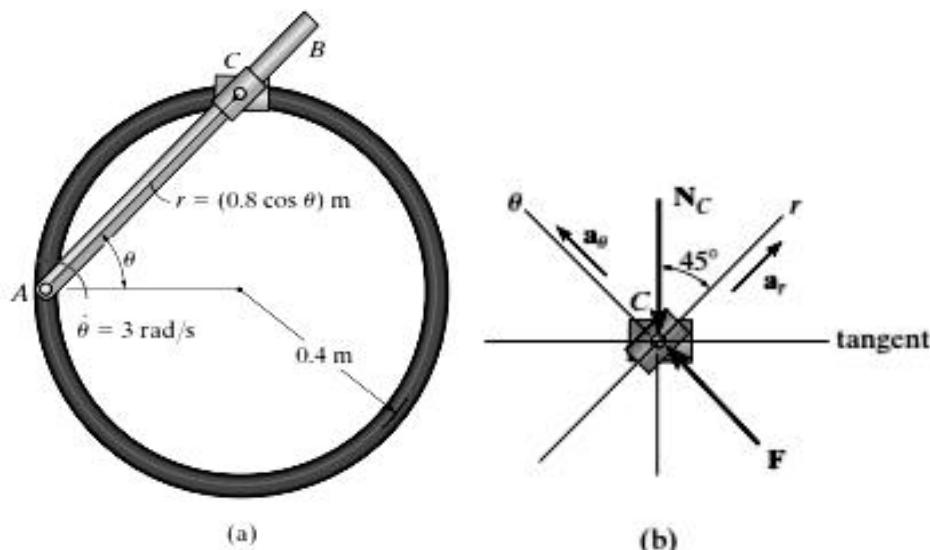


Fig.4.102

**Solution:**

**Free-Body Diagram.** The normal reaction  $N_c$  of the circular guide rod and the force  $F$  of arm  $AB$  act on the collar in the plane of motion, Fig. 13-19b. Note that  $F$  acts perpendicular to the axis of arm  $AB$ , that is, in the direction of the  $\theta$  axis, while  $N_c$  acts perpendicular to the tangent of the circular path at  $\theta = 45^\circ$ . The four unknowns are  $N_c$ ,  $F$ ,  $a_r$ ,  $a_\theta$ .

### Equations of Motion.

$$+\not\!\!\!\sum F_r = ma_r: \quad -N_c \cos 45^\circ = (0.5 \text{ kg}) a_r \quad (1)$$

$$+\not\!\!\!\sum F_\theta = ma_\theta: \quad F - N_c \sin 45^\circ = (0.5 \text{ kg}) a_\theta \quad (2)$$

**Kinematics.** Using the chain rule (see Appendix C), the first and second time derivatives of  $r$  when  $\theta = 45^\circ$ ,  $\dot{\theta} = 3 \text{ rad/s}$ ,  $\ddot{\theta} = 0$ , are

$$r = 0.8 \cos \theta = 0.8 \cos 45^\circ = 0.5657 \text{ m}$$

$$\dot{r} = -0.8 \sin \theta \dot{\theta} = -0.8 \sin 45^\circ (3) = -1.6971 \text{ m/s}$$

$$\begin{aligned} \ddot{r} &= -0.8 [\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2] \\ &= -0.8 [\sin 45^\circ (0) + \cos 45^\circ (3^2)] = -5.091 \text{ m/s}^2 \end{aligned}$$

We have

$$a_r = \ddot{r} - r\dot{\theta}^2 = -5.091 \text{ m/s}^2 - (0.5657 \text{ m})(3 \text{ rad/s})^2 = -10.18 \text{ m/s}^2$$

$$\begin{aligned} a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.5657 \text{ m})(0) + 2(-1.6971 \text{ m/s})(3 \text{ rad/s}) \\ &= -10.18 \text{ m/s}^2 \end{aligned}$$

Substituting these results into Eqs. (1) and (2) and solving, we get

$$N_c = 7.20 \text{ N}$$

$$F = 0$$

Ans.

**SP4.142:** The smooth 2kg cylinder C as shown in fig.4.103 has a pin P through its centre which passes through the slot in arm OA. If the arm is forced to rotate in the vertical plane at a constant rate  $\dot{\theta} = 0.5 \text{ rad/s}$ , determine the force that the arm exerts on the peg at the instant  $\theta = 60^\circ$ .

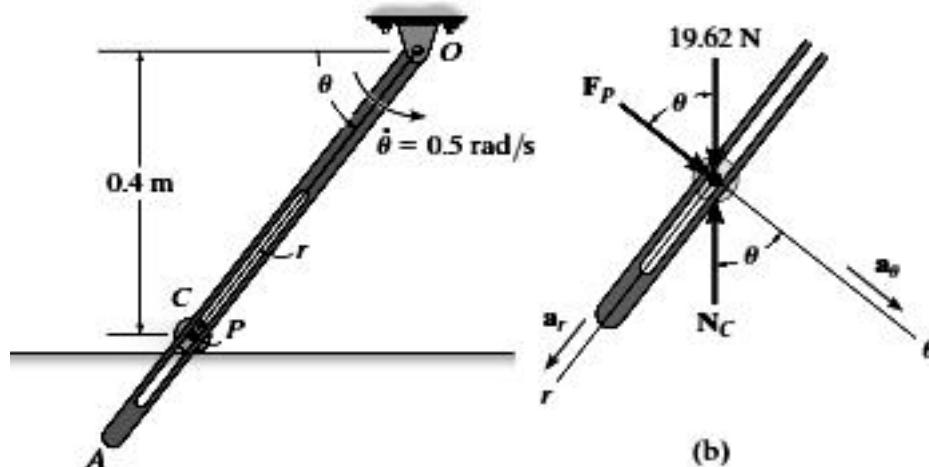


Fig.4.103

**Solution:**

**Why is it a good idea to use polar coordinates to solve this problem?**

**Free-Body Diagram.** The free-body diagram for the cylinder is shown in Fig. 13-20b. The force on the peg,  $\mathbf{F}_P$ , acts perpendicular to the slot in the arm. As usual,  $\mathbf{a}_r$  and  $\mathbf{a}_\theta$  are assumed to act in the directions of positive  $r$  and  $\theta$ , respectively. Identify the four unknowns.

**Equations of Motion.** Using the data in Fig. 13-20b, we have

$$+\not\sum F_r = ma_r; \quad 19.62 \sin \theta - N_C \sin \theta = 2a_r \quad (1)$$

$$+\not\sum F_\theta = ma_\theta; \quad 19.62 \cos \theta + F_P - N_C \cos \theta = 2a_\theta \quad (2)$$

**Kinematics.** From Fig. 13-20a,  $r$  can be related to  $\theta$  by the equation

$$r = \frac{0.4}{\sin \theta} = 0.4 \csc \theta$$

Since  $d(\csc \theta) = -(\csc \theta \cot \theta) d\theta$  and  $d(\cot \theta) = -(\csc^2 \theta) d\theta$ , then  $r$  and the necessary time derivatives become

$$\dot{\theta} = 0.5 \quad r = 0.4 \csc \theta$$

$$\ddot{\theta} = 0 \quad \dot{r} = -0.4(\csc \theta \cot \theta)\dot{\theta}$$

$$\begin{aligned}
 &= -0.2 \csc \theta \cot \theta \\
 \ddot{r} &= -0.2(-\csc \theta \cot \theta)(\dot{\theta}) \cot \theta - 0.2 \csc \theta (-\csc^2 \theta)\dot{\theta} \\
 &= 0.1 \csc \theta (\cot^2 \theta + \csc^2 \theta)
 \end{aligned}$$

Evaluating these formulas at  $\theta = 60^\circ$ , we get

$$\dot{\theta} = 0.5 \quad r = 0.462$$

$$\ddot{\theta} = 0 \quad \ddot{r} = -0.133$$

$$\ddot{r} = 0.192$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0.192 - 0.462(0.5)^2 = 0.0770$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2(-0.133)(0.5) = -0.133$$

Substituting these results into Eqs. 1 and 2 with  $\theta = 60^\circ$  and solving yields

$$N_c = 19.4 \text{ N} \quad F_r = -0.356 \text{ N} \quad \text{Ans.}$$

The negative sign indicates that  $F_r$  acts opposite to the direction shown in Fig. 13-20b.

**SP4.143:** A can C having a mass of 0.5kg moves along a grooved horizontal slot as shown in fig.4.104. The slot is in the form of a spiral which is defined by the equation  $r = (0.1\theta)m$  where  $\theta$  is in radians. If the arm OA rotates with a constant rate  $\dot{\theta} = 4 \text{ rad/s}$  in the horizontal plane, determine the force it exerts on the can at the instant  $\theta = \pi \text{ rad}$ . Neglect friction and the size of the can.

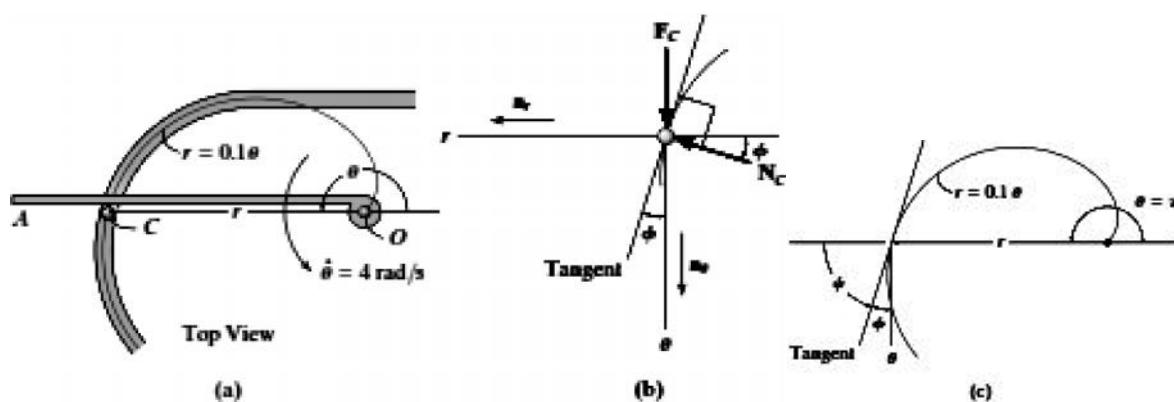


Fig.4.104

**Solution:**

**Free-Body Diagram.** The driving force  $F_c$  acts perpendicular to the arm  $OA$ , whereas the normal force of the wall of the slot on the can,  $N_c$ , acts perpendicular to the tangent to the curve at  $\theta = \pi$  rad, Fig. 13-21b. As usual,  $a_r$  and  $a_\theta$  are assumed to act in the positive directions of  $r$  and  $\theta$ , respectively. Since the path is specified, the angle  $\phi$  which the extended radial line  $r$  makes with the tangent, Fig. 13-21c, can be determined from Eq. 13-10. We have  $r = 0.1\theta$ , so that  $dr/d\theta = 0.1$ , and therefore

$$\tan \phi = \frac{r}{dr/d\theta} = \frac{0.1\theta}{0.1} = \theta$$

When  $\theta = \pi$ ,  $\phi = \tan^{-1}\pi = 72.3^\circ$ , so that  $\phi = 90^\circ - \phi = 17.7^\circ$ , as shown in Fig. 13-21c. Identify the four unknowns in Fig. 13-21b.

**Equations of Motion.** Using  $\phi = 17.7^\circ$  and the data shown in Fig. 13-21b, we have

$$\therefore \Sigma F_r = ma_r; \quad N_c \cos 17.7^\circ = 0.5a_r, \quad (1)$$

$$+\downarrow \Sigma F_\theta = ma_\theta; \quad F_c - N_c \sin 17.7^\circ = 0.5a_\theta \quad (2)$$

**Kinematics.** The time derivatives of  $r$  and  $\theta$  are

$$\dot{\theta} = 4 \text{ rad/s} \quad r = 0.1\theta$$

$$\ddot{\theta} = 0 \quad \dot{r} = 0.1\dot{\theta} = 0.1(4) = 0.4 \text{ m/s}$$

$$\ddot{r} = 0.1\ddot{\theta} = 0$$

At the instant  $\theta = \pi$  rad,

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0 - 0.1(\pi)(4)^2 = -5.03 \text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2(0.4)(4) = 3.20 \text{ m/s}^2$$

Substituting these results into Eqs. 1 and 2 and solving yields

$$N_c = -2.64 \text{ N}$$

$$F_c = 0.800 \text{ N} \quad \text{Ans.}$$

What does the negative sign for  $N_c$  indicate?

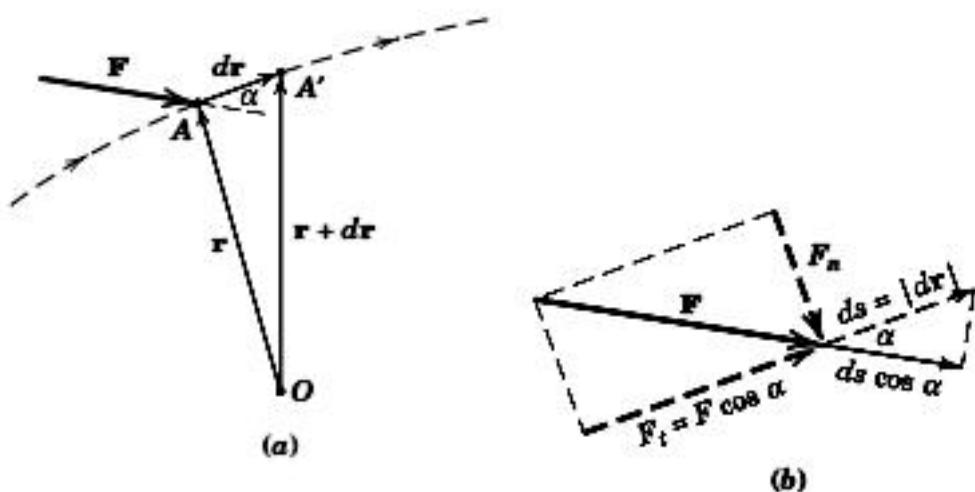
**WORK AND KINETIC ENERGY:**

Fig.4.105

Now develop the quantitative meaning of the term “work”. Fig.4.105 shows a force  $F$  acting on a particle at  $A$  which moves along the path shown. The position vector  $r$  measured from some convenient origin  $O$  locates the particle as it passes point  $A$ , and  $dr$  is the differential displacement associated with an infinitesimal movement from  $A$  to  $A'$ . The work done by the force  $F$  during the displacement  $dr$  is defined as

$$dU = F \cdot dr \quad \text{---} \rightarrow (4.57)$$

The magnitude of this dot product is  $dU = F ds \cos\alpha$ , where  $\alpha$  is the angle between  $F$  and  $dr$  and where  $ds$  is the magnitude of  $dr$ . This expression may be interpreted as the displacement multiplied by the force component  $F_t = F \cos\alpha$  in the direction of the displacement, as represented by the dashed lines in fig.4.105b. Alternatively, the work  $dU$  may be interpreted as the force multiplied by the displacement component  $ds \cos\alpha$  in the direction of the force, as represented by the full lines in fig.4.105b.

With this definition of work, it should be noted that the component  $F_n = F \sin \alpha$  normal to the displacement does no work. Thus, the work  $dU$  may be written as

$$dU = F_t ds$$

Work is positive if the working component  $F$  is in the direction of the displacement and negative if it is in the opposite direction. Forces which do work are termed active forces. Constraint forces which do no work are termed reactive forces.

Now formally compute the work associated with three frequently occurring forces: constant forces, spring forces, and weights.

### (1) Work Associated with a Constant External Force.

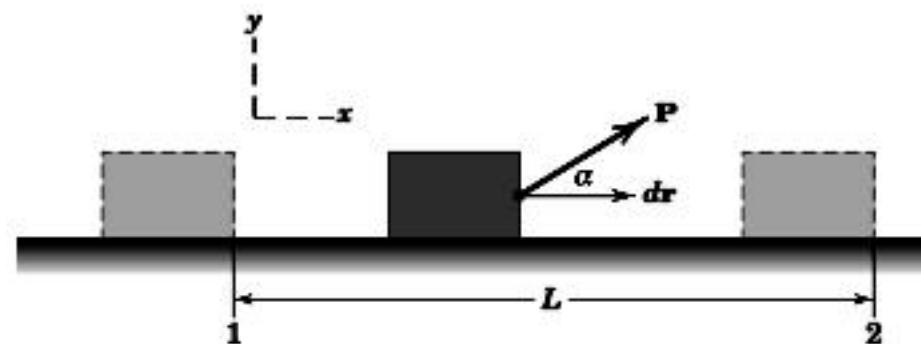


Fig.4.106

Consider the constant force  $P$  applied to the body as it moves from position 1 to position 2, fig.4.106. With the force  $P$  and the differential displacement  $dr$  written as vectors, the work done on the body by the force is

$$U_{1-2} = \int_1^2 F \cdot dr = \int [(P \cos \tau) i + (P \sin \tau) j] (dx) i$$

$$U_{1-2} = \int_{x_1}^{x_2} P \cos \tau dx = P \cos \tau (x_2 - x_1) = PL \cos \tau \quad \text{-----} \rightarrow (4.58)$$

As previously discussed, this work expression may be interpreted as the force component  $P \cos \alpha$  times the distance  $L$  travelled. Should  $\tau$  be between  $90^\circ$  and  $270^\circ$ , the work would be negative. The force component  $P \sin \tau$  normal to the displacement does no work.

### (2) Work Associated with a Spring Force.

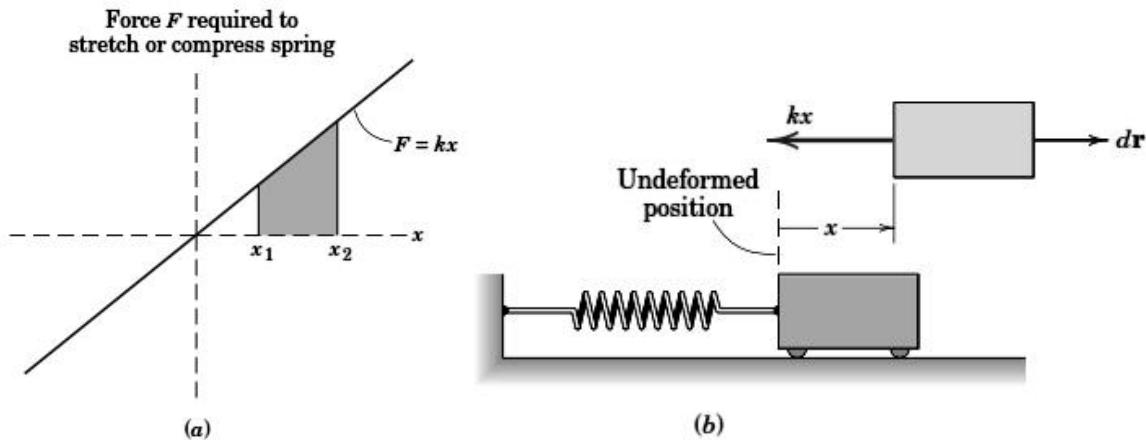


Fig.4.107

Now consider here the common linear spring of stiffness  $k$  where the force required to stretch or compress the spring is proportional to the deformation  $x$ , as shown in fig.4.107a. It is to determine the work done on the body by the spring force as the body undergoes an arbitrary displacement from an initial position  $x_1$  to a final position  $x_2$ . The force exerted by the spring on the body is  $F = -kx_i$ , as shown in fig.4.107b. From the definition of work, we have

$$U_{1-2} = \int F \cdot dr = \int_1^2 (-kx_i) \cdot (dx) i = - \int_{x_1}^{x_2} kx dx = \frac{1}{2} k(x_1^2 - x_2^2) \quad \text{-----} \rightarrow (4.59)$$

If the initial position is the position of zero spring deformation so that  $x_1 = 0$ , then the work is negative for any final position  $x_2 \neq 0$ . This is verified by recognizing that if the body begins at the un-deformed spring position and then moves to the right, the spring force is to the left; if the body begins at  $x_1 = 0$  and moves to the left, the spring force is to the right. On the other hand, if we move from an arbitrary initial position  $x_1 \neq 0$  to the un-deformed final position  $x_2 = 0$ , we see that the work is positive. In any movement toward the un-deformed spring position, the spring force and the displacement are in the same direction.

In the general case, of course, neither  $x_1$  nor  $x_2$  is zero. The magnitude of the work is equal to the shaded trapezoidal area of fig.4.107a. In calculating the work done on a body by a spring force, care must be taken to ensure that the units of  $k$  and  $x$  are consistent. If  $x$  is in meters (or feet),  $k$  must be in N/m (or lb/ft). In addition, be sure

to recognize that the variable  $x$  represents a deformation from the un-stretched spring length and not the total length of the spring.

The expression  $F = kx$  is actually a static relationship which is true only when elements of the spring have no acceleration. The dynamic behaviour of a spring when its mass is accounted for is a fairly complex problem which will not be treated here. We shall assume that the mass of the spring is small compared with the masses of other accelerating parts of the system, in which case the linear static relationship will not involve appreciable error.

### (3) Work Associated with Weight.

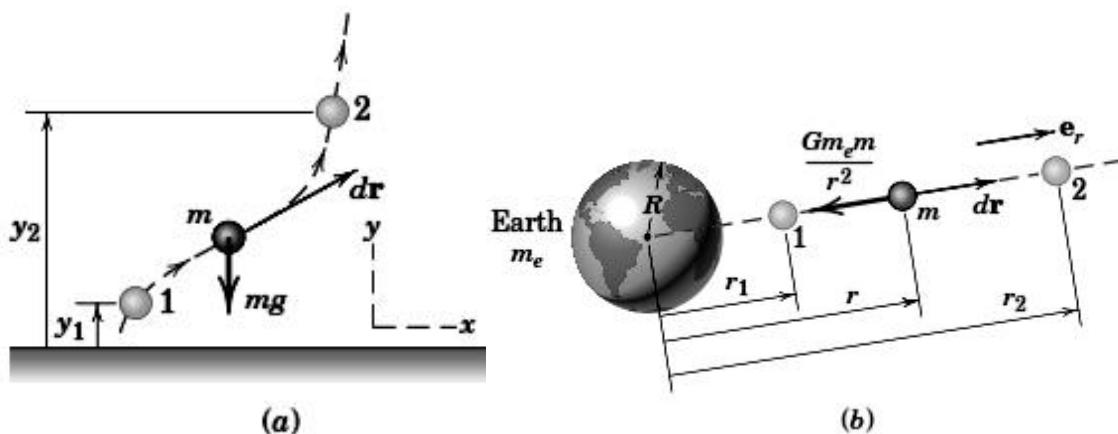


Fig.4.108

Case (a)  $g = \text{constant}$ . If the altitude variation is sufficiently small so that the acceleration of gravity  $g$  may be considered constant, the work done by the weight  $mg$  of the body shown in fig.4.108a as the body is displaced from an arbitrary altitude  $y_1$  to a final altitude  $y_2$  is

$$U_{1-2} = \int_1^2 F \cdot dr = \int_1^2 (-mgj) \cdot (dx_i + dy_j) = -mg \int_{y_1}^{y_2} dy = -mg(y_2 - y_1) \rightarrow (4.60)$$

Now see that horizontal movement does not contribute to this work. We also note that if the body rises (perhaps due to other forces not shown), then  $(y_2 - y_1) > 0$  and this work is negative. If the body falls,  $(y_2 - y_1) < 0$  and the work is positive.

Case (b)  $g$  constant. If large changes in altitude occur, then the weight (gravitational force) is no longer constant. We must therefore use the gravitational law and express the weight as a variable force of magnitude  $F = G \frac{m_e m}{r^2}$ , as indicated in fig.4.108b. Using the radial coordinate shown in the figure allows the work to be expressed as

$$U_{1-2} = \int_1^2 F \cdot dr = \int_1^2 -G \frac{m_e m}{r^2} e_r \cdot (dr) e_r = -G m_e m \int_{r_1}^{r_2} \frac{dr}{r^2} = G m_e m \left( \frac{1}{r_2} - \frac{1}{r_1} \right) = mgR^2 \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \rightarrow (4.61)$$

where the equivalence  $Gm_e = gR^2$  was established in previous, with  $g$  representing the acceleration of gravity at the earth's surface and  $R$  representing the radius of the earth. The student should verify that if a body rises to a higher altitude ( $r_2 > r_1$ ), this work is negative, as it was in case (a). If the body falls to a lower altitude ( $r_2 < r_1$ ), the work is positive. Be sure to realize that  $r$  represents a radial distance from the centre of the earth and not an altitude  $h = r - R$  above the surface of the earth. As in case (a), had we considered a transverse displacement in addition to the radial displacement shown in fig.4.108b, we would have concluded that the transverse displacement, because it is perpendicular to the weight, does not contribute to the work.

## Work and Curvilinear Motion

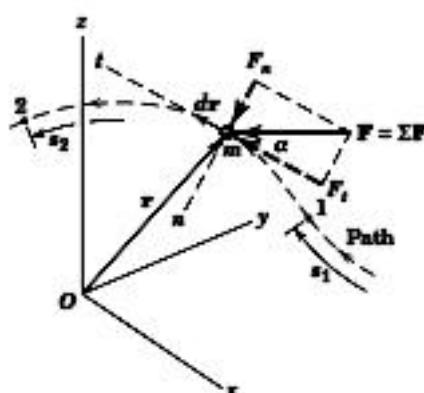


Fig.4.109

Now consider the work done on a particle of mass  $m$ , fig.4.109, moving along a curved path under the action of the force  $F$ , which stands for the resultant  $\Sigma F$  of all forces acting on the particle. The position of  $m$  is specified by the position vector  $r$ , and its displacement along its path during the time  $dt$  is represented by the change  $dr$  in its position vector. The work done by  $F$  during a finite movement of the particle from point 1 to point 2 is

$$u_{1-2} = \int_1^2 F \cdot dr = \int_{s1}^{s2} F_t ds \quad \longrightarrow \rightarrow (4.62)$$

where the limits specify the initial and final end points of the motion.

When we substitute Newton's second law  $F = ma$ , the expression for the work of all forces becomes

$$U_{1-2} = \int_1^2 F \cdot dr = \int_1^2 ma \cdot dr \quad \longrightarrow \rightarrow (4.63)$$

But  $a \cdot dr = a_t ds$ , where  $a_t$  is the tangential component of the acceleration of  $m$ . In terms of the velocity  $v$  of the particle, Eq.2/3 gives  $a_t ds = v dv$ . Thus, the expression for the work of  $F$  becomes

$$U_{1-2} = \int_1^2 F \cdot dr = \int_{v1}^{v2} mv dv = \frac{1}{2} m(v_2^2 - v_1^2) \quad \longrightarrow \rightarrow (4.64)$$

where the integration is carried out between points 1 and 2 along the curve, at which points the velocities have the magnitudes  $v_1$  and  $v_2$ , respectively.

## Principle of Work and Kinetic Energy

The kinetic energy  $T$  of the particle is defined as

$$T = \frac{1}{2} mv^2 \quad \longrightarrow \rightarrow (4.65)$$

and is the total work which must be done on the particle to bring it from a state of rest to a velocity  $v$ . Kinetic energy  $T$  is a scalar quantity with the units of N-m or joules (J)

in SI units and ft-lb in U.S. customary units. Kinetic energy is always positive, regardless of the direction of the velocity.

Equation  $U_{1-2} = \frac{1}{2}m(v_2^2 - v_1^2)$  may be restated as

$$U_{1-2} = T_2 - T_1 = \Delta T \quad \text{-----} \rightarrow (4.66)$$

which is the work-energy equation for a particle. The equation states that the total work done by all forces acting on a particle as it moves from point 1 to point 2 equals the corresponding change in kinetic energy of the particle. Although  $T$  is always positive, the change  $\Delta T$  may be positive, negative, or zero. When written in this concise form, Eq.  $U_{1-2} = T_2 - T_1 = \Delta T$  tells us that the work always results in a change of kinetic energy.

Alternatively, the work-energy relation may be expressed as the initial kinetic energy  $T_1$  plus the work done  $U_{1-2}$  equals the final kinetic energy  $T_2$ , or

$$T_1 + U_{1-2} = T_2 \quad \text{-----} \rightarrow (4.67)$$

When written in this form, the terms correspond to the natural sequence of events. Clearly, the two forms  $U_{1-2} = T_2 - T_1 = \Delta T$  and  $T_1 + U_{1-2} = T_2$  are equivalent.

## POWER:

The capacity of a machine is measured by the time rate at which it can do work or deliver energy. The total work or energy output is not a measure of this capacity since a motor, no matter how small, can deliver a large amount of energy if given sufficient time. On the other hand, a large and powerful machine is required to deliver a large amount of energy in a short period of time. Thus, the capacity of a machine is rated by its power, which is defined as the time rate of doing work.

Accordingly, the power  $P$  developed by a force  $F$  which does an amount of work  $U$  is  $P = \frac{dU}{dt} = F \cdot \frac{dr}{dt}$ . Because  $\frac{dr}{dt}$  is the velocity  $v$  of the point of application of the force, we have

$$P = F.v \quad \text{-----} \rightarrow (4.68)$$

Power is clearly a scalar quantity, and in SI it has the units of N-m/s = J/s. The special unit for power is the watt (W), which equals one joule per second (J/s). In U.S. customary units, the unit for mechanical power is the horsepower (hp). These units and their numerical equivalences are

$$1 \text{ W} = 1 \text{ J/s}$$

$$1 \text{ hp} = 550 \text{ ft-lb/sec} = 33,000 \text{ ft-lb/min}$$

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

## EFFICIENCY

The ratio of the work done by a machine to the work done on the machine during the same time interval is called the mechanical efficiency  $\gamma_m$  of the machine. This definition assumes that the machine operates uniformly so that there is no accumulation or depletion of energy within it. Efficiency is always less than unity since every device operates with some loss of energy and since energy cannot be created within the machine. In mechanical devices which involve moving parts, there will always be some loss of energy due to the negative work of kinetic friction forces. This work is converted to heat energy which, in turn, is dissipated to the surroundings. The mechanical efficiency at any instant of time may be expressed in terms of mechanical power  $P$  by

$$\gamma_m = \frac{P_{output}}{P_{input}} \quad \text{-----} \rightarrow (4.69)$$

In addition to energy loss by mechanical friction, there may also be electrical and thermal energy loss, in which case, the electrical efficiency  $\gamma_e$  and thermal efficiency  $\gamma_t$  are also involved. The overall efficiency  $\gamma$  in such instances is

$$\gamma = \gamma_m \gamma_e \gamma_t \quad \text{-----} \rightarrow (4.70)$$

**SP4.144:** Calculate the velocity  $v$  of the 150kg crate when it reaches the bottom of the chute at B if it is given an initial velocity of 4m/s down the chute at A. The coefficient of kinetic friction is 0.3.

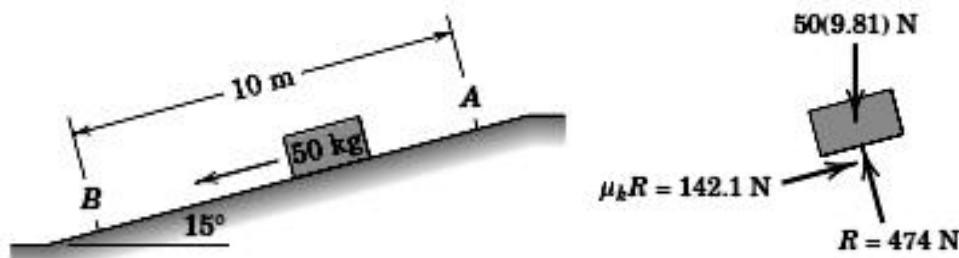


Fig.4.110

**Solution:**

The free-body diagram of the crate is drawn and includes the normal force  $R$  and the kinetic friction force  $F$  calculated in the usual manner. The work done by the weight is positive, whereas that done by the friction force is negative. The total work done on the crate during the motion is

$$[U = F_s] \quad U_{1-2} = 50(9.81)(10 \sin 15^\circ) - 142.1(10) = -151.9 \text{ J}$$

The work-energy equation gives

$$[T_1 + U_{1-2} = T_2] \quad \frac{1}{2}mv_1^2 + U_{1-2} = \frac{1}{2}mv_2^2$$

$$\frac{1}{2}(50)(4)^2 - 151.9 = \frac{1}{2}(50)v_2^2$$

$$v_2 = 3.15 \text{ m/s} \quad \text{Ans.}$$

Since the net work done is negative, we obtain a decrease in the kinetic energy.

**SP4.145:** The flatbed truck which carries an 80kg crate starts from rest and attains a speed of 72kmph in a distance of 75m on a level road with constant acceleration. Calculate the work done by the friction force acting on the crate during this interval if the static and kinetic coefficients of friction between the crate and the truck bed are a) 0.3 and 0.28 respectively and b) 0.25 and 0.20 respectively.



Fig.4.111

**Solution:**

If the crate does not slip on the bed, its acceleration will be that of the truck, which is

$$[v^2 = 2as] \quad a = \frac{v^2}{2s} = \frac{(72/3.6)^2}{2(75)} = 2.67 \text{ m/s}^2$$

**Case (a).** This acceleration requires a friction force on the block of

$$[F = ma] \quad F = 80(2.67) = 213 \text{ N}$$

which is less than the maximum possible value of  $\mu_s N = 0.30(80)(9.81) = 235 \text{ N}$ . Therefore, the crate does not slip and the work done by the actual static friction force of 213 N is

$$[U = Fs] \quad U_{1-2} = 213(75) = 16\,000 \text{ J} \quad \text{or} \quad 16 \text{ kJ} \quad \text{Ans.}$$

**Case (b).** For  $\mu_s = 0.25$ , the maximum possible friction force is  $0.25(80)(9.81) = 196.2 \text{ N}$ , which is slightly less than the value of 213 N required for no slipping. Therefore, we conclude that the crate slips, and the friction force is governed by the kinetic coefficient and is  $F = 0.20(80)(9.81) = 157.0 \text{ N}$ . The acceleration becomes

$$[F = ma] \quad a = F/m = 157.0/80 = 1.962 \text{ m/s}^2$$

The distances traveled by the crate and the truck are in proportion to their accelerations. Thus, the crate has a displacement of  $(1.962/2.67)75 = 55.2 \text{ m}$ , and the work done by kinetic friction is

$$[U = Fs] \quad U_{1-2} = 157.0(55.2) = 8660 \text{ J} \quad \text{or} \quad 8.66 \text{ kJ} \quad \text{Ans.}$$

**SP4.146:** The 50kg block at A is mounted on rollers so that it moves along the fixed horizontal rail with negligible friction under the action of the constant 300N force in the cable. The block is released from rest at A, with the spring to which it is attached

extended an initial amount  $x_1 = 0.233\text{m}$ . The spring has a stiffness  $k = 80\text{N/m}$ . Calculate the velocity  $v$  of the block as it reaches position B.

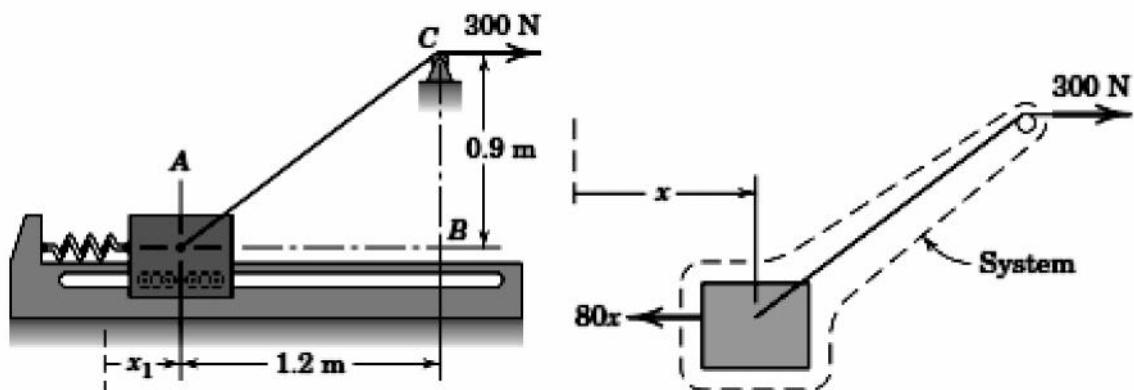


Fig.4.112

**Solution:**

It will be assumed initially that the stiffness of the spring is small enough to allow the block to reach position B. The active-force diagram for the system composed of both block and cable is shown for a general position. The spring force  $80x$  and the 300-N tension are the only forces external to this system which do work on the system. The force exerted on the block by the rail, the weight of the block, and the reaction of the small pulley on the cable do no work on the system and are not included on the active-force diagram.

As the block moves from  $x_1 = 0.233\text{ m}$  to  $x_2 = 0.233 + 1.2 = 1.433\text{ m}$ , the work done by the spring force acting on the block is

$$[U_{1,2} = \frac{1}{2}k(x_1^2 - x_2^2)] \quad U_{1,2} = \frac{1}{2}80[0.233^2 - (0.233 + 1.2)^2] \\ = -80.0\text{ J}$$

The work done on the system by the constant 300-N force in the cable is the force times the net horizontal movement of the cable over pulley C, which is  $\sqrt{(1.2)^2 + (0.9)^2} - 0.9 = 0.6\text{ m}$ . Thus, the work done is  $300(0.6) = 180\text{ J}$ . We now apply the work-energy equation to the system and get

$$[T_1 + U_{1,2} = T_2] \quad 0 - 80.0 + 180 = \frac{1}{2}(50)v^2 \quad v = 2.00\text{ m/s} \quad \text{Ans.}$$

We take special note of the advantage to our choice of system. If the block alone had constituted the system, the horizontal component of the 300-N cable tension on the block would have to be integrated over the 1.2-m displacement. This step would require considerably more effort than was needed in the solution as presented. If there had been appreciable friction between the block and

its guiding rail, we would have found it necessary to isolate the block alone in order to compute the variable normal force and, hence, the variable friction force. Integration of the friction force over the displacement would then be required to evaluate the negative work which it would do.

**SP4.147:** The power winch A hoists the 360kg log up the  $30^\circ$  incline at a constant speed of 1.2m/s. If the power output of the winch is 4kW, compute the coefficient of kinetic friction  $\mu_k$  between the log and the incline. If the power is suddenly increased to 6 kW, what is the corresponding instantaneous acceleration  $a$  of the log?

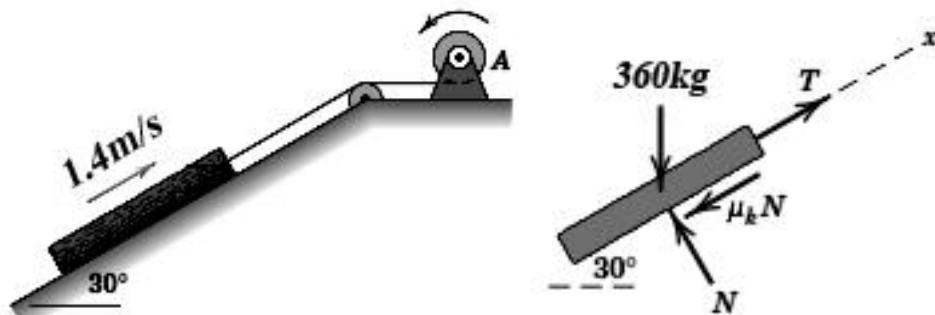


Fig.4.113

**Solution:**

From the free body diagram of the log we get  $N = 360(9.81) \cos 30^\circ = 3060\text{N}$  and the kinetic friction force becomes  $3060 \mu_k$ . For constant speed the forces are in equilibrium so that

$$\sum F_x = 0, \Rightarrow T - 3060\mu_k - 360(9.81) \sin 30^\circ = 0, \Rightarrow T = 3060\mu_k + 1766$$

The power output of the winch gives the tension in the cable

$$P = Tv, \Rightarrow T = \frac{P}{v} = \frac{4000}{1.2} = 3330\text{N}$$

Substituting T gives,  $3330 = 3060\mu_k + 1766, \Rightarrow \mu_k = 0.513$

When the power is increased, the tension momentarily becomes

$$P = Tv, T = \frac{P}{v} = \frac{6000}{1.2} = 5000\text{N}$$

and the corresponding acceleration is given by

$$\Sigma F_x = ma_x, \Rightarrow 5000 - 3060(0.513) - 360(9.81) \sin 30^\circ = 360a$$

$$a = 4.63 \text{ m/s}^2$$

**SP4.148:** A satellite of mass  $m$  is put into an elliptical orbit around the earth. At point A its distance from the earth is  $h_1 = 500$  km and it has a velocity  $v_1 = 30000 \text{ kmph}$ . Determine the velocity  $v_2$  of the satellite as it reaches point B, a distance  $h_2 = 1200 \text{ km}$  from the earth.

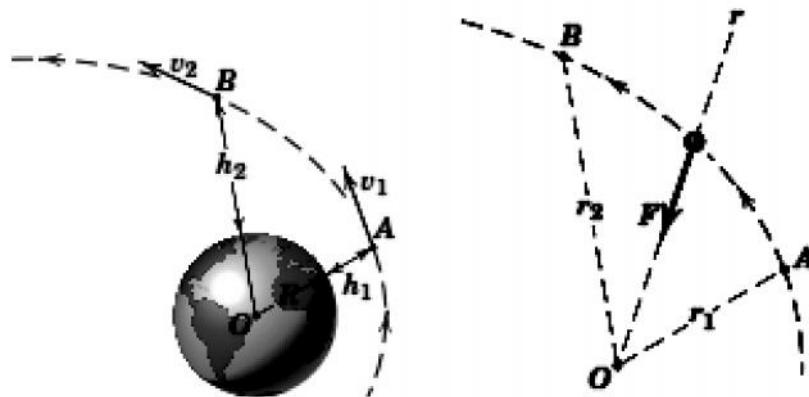


Fig.4.114

**Solution:**

The satellite is moving outside of the earth's atmosphere so that the only force acting on it is the gravitational attraction of the earth. For the large change in altitude of this problem, we cannot assume that the acceleration due to gravity is constant. Rather, we must use the work expression, derived in this article, which accounts for variation in the gravitational acceleration with altitude. Put another way, the work expression accounts for the variation of the

weight  $F = \frac{Gmm_e}{r^2}$  with altitude. This work expression is

$$U_{1,2} = mgR^2 \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

The work-energy equation  $T_1 + U_{1,2} = T_2$  gives

$$\frac{1}{2}mv_1^2 + mgR^2 \left( \frac{1}{r_2} - \frac{1}{r_1} \right) = \frac{1}{2}mv_2^2 \quad v_2^2 = v_1^2 + 2gR^2 \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$

Substituting the numerical values gives

$$v_2^2 = \left(\frac{30000}{3.6}\right)^2 + 2(9.81)[(6371)(10^3)]^2 \left(\frac{10^{-3}}{6371 + 1200} - \frac{10^{-3}}{6371 + 500}\right)$$

$$= 69.44(10^6) - 10.72(10^6) = 58.73(10^6) \text{ (m/s)}^2$$

$$v_2 = 7663 \text{ m/s} \quad \text{or} \quad v_2 = 7663(3.6) = 27590 \text{ km/h} \quad \text{Ans.}$$

**SP4.149:** A train of mass 100 tonnes is pulled by an engine on a level track at a constant speed of 45 kmph. The resistance due to friction is 1% of the train weight. Find the power of the engine.

**Solution:**

**Data given :** train mass =  $100 \times 1000 \text{ kg}$  ; Speed of the train = 45 kmph = 12.5 m/s  
**We know that** the forces acting on the train are tractive force and frictional force.

$$\text{Therefore } \Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R)s = \frac{1}{2}m(V_2^2 - V_1^2) = 0 \text{ as velocity remains constant}$$

$$\text{Hence } F_T = F_R \text{ but } F_R = 100 \times 1000 \times 9.81 \times (1/100) = 9810 \text{ N}$$

$$\text{Power} = F_T \times v = 9810 \times 12.5 = 122.625 \text{ kW}$$

**SP4.150:** The mass of a car is 4.5 tonnes. The tractive resistance on the level track is 50 N per tonne. Find the power required to propel the car at a uniform speed of 45 kmph; a) up an incline 1 in 300, b) on the level track and c) down an incline of 1 in 300. Take efficiency of the motors and drive as 70%.

**Solution:**

**Data Given:** mass of the car = 4.5 tons = 4500 kg; Tractive resistance  
 $F_R = 50 \times 4.5 = 225 \text{ N}$

Case i) Up the plane:

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R - mgsin\theta)s = 0 \text{ (since velocity is uniform change in kinetic energy = 0)}$$

$$F_T = F_R + mgsin\theta = 225 + 4500 \times 9.81 \times (1/300) = 372.15 \text{ N}$$

$$\text{Power} = F_T \times v = 372.15 \times 12.5 = 4.6518 \text{ kW}$$

$$\text{Actual power required by the car considering the efficiency of the drive} \\ = 4.6518 / 0.7 = 6.645 \text{ kW}$$

Case ii) on level track:

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R)s = 0$$

$$F_T = F_R = 225 \text{ N}$$

$$\text{Power} = F_T \times v = 225 \times 12.5 = 2.8125 \text{ kW}$$

Actual power required by the car considering the efficiency of the drive  
 $= 2.8125 / 0.7 = 4.017 \text{ kW}$

Case iii) down the plane:

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R + mgsin\theta)s = 0$$

$$F_T = F_R - mgsin\theta = 225 - 4500 \times 9.81 \times$$

$$(1/300) = 77.85 \text{ N}$$

$$\text{Power} = F_T \times v = 77.85 \times 12.5 = 0.973125 \text{ kW}$$

Actual power required by the car considering the efficiency of the drive  
 $= 0.973125 / 0.7 = 1.39 \text{ kW}$

**SP4.151:** A train of mass 300 tonnes moves down a slope of 1 in 200 at 45 kmph with the engine developing 75 kW. Find the power required to pull the train up the slope with the same velocity.

**Solution:**

Down the plane:

$$\text{Power} = F_T \times v = F_T \times 12.5 = 75000 \text{ W}$$

$$F_T = 75 / 12.5 = 6000 \text{ N}$$

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R + mgsin\theta)s = 0$$

$$6000 - F_R + 300000 \times 9.81 \times (1/200) = 0$$

$$F_R = 20715 \text{ N}$$

Up the plane:

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R - mgsin\theta)s = 0$$

$$F_T - 20715 - 300000 \times 9.81 \times (1/200) = 0$$

$$F_T = 35430 \text{ N}$$

$$\text{Power} = F_T \times v = 35430 \times 12.5 = 442.875 \text{ kW}$$

**SP4.152:** An engine of mass 50 tonnes pulls a train of mass 300 tonnes up an incline of 1 in 100. The train starts from rest and moves with a constant acceleration against a total resistance of 50N per tonne. If the train attains a speed of 36 kmph in a distance of 1km, find the tension in the coupling between the engine and the train. Also find the power exerted by the engine.

**Solution:**

***Data Given:***

$$V_1 = 0; V_2 = 36 \text{ kmph} = 10 \text{ m/s}; m = 50+300 \text{ tonnes} = 350,000 \text{ kg}; \text{Slope} = 1/100;$$

$$F_R = 50 \times 350 = 17500 \text{ N}; s = 1 \text{ km} = 1000 \text{ m}$$

To find the power of the engine:**Work done = Change in Kinetic energy**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R - mg \sin\theta)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$[F_T - 17500 - 350000 \times 9.81 \times (1/100)] \times 1000 = (350000/2) (10^2 - 0^2)$$

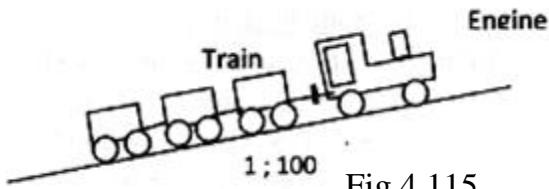


Fig.4.115

$$F_T = 69335 \text{ N}$$

$$\text{Power} = F_T \times v = 69335 \times 10 = 693350 \text{ W} = 693.35 \text{ kW}$$

To find the tension in the coupling

It is the tension which is responsible for driving the train only

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(T - F_R - mg \sin\theta)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$\text{Here } F_R = 300 \times 50 = 15000 \text{ N}; m = 300000 \text{ kg}$$

$$[T - 15000 - 300000 \times 9.81 \times (1/100)] \times 1000 = (300000/2) (10^2 - 0^2)$$

$$T = 59430 \text{ N}$$

$$\text{Tension in the coupling} = 59.43 \text{ kN}$$

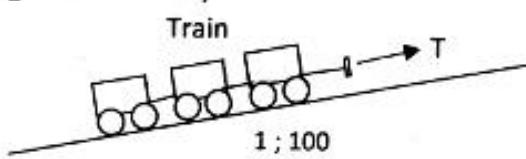


Fig.4.116

**SP4.153:** A locomotive draws a train of mass 400 tonnes, including its own mass on a level ground with a uniform acceleration until it acquires a velocity of 54 kmph in a distance of 2250m. If the frictional resistance is 40 N/tonne of mass and the air resistance varies with square of the velocity, find the power of the engine. Take air resistance as 500N at 18kmph. Hint: Air resistance at 54kmph is  $(54/18)^2 \times 500 = 4500\text{N}$ .

**Solution:****Data given:**

$$m = 400000 \text{ kg}; V_1 = 0 \text{ m/s}; V_2 = 54 \text{ kmph} = 15 \text{ m/s}$$

$$s = 2250 \text{ m}; F_R = 40 \times 400 = 16000 \text{ N}; \text{Air resistance } F_{AR} \text{ at } 18 \text{ kmph} = 500 \text{ N}; \text{Air resistance } F_{AR} \text{ at } 54 \text{ kmph} = (54/18)^2 \times 500 = 4500 \text{ N}$$

**W.D = Change in kinetic Energy**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R - F_{AR})s = \frac{1}{2}m(V_2^2 - V_1^2)$$

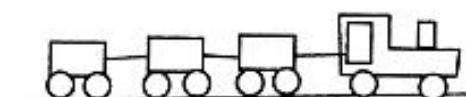


Fig.4.117

$$(F_T - 16000 - 4500)2250 = \frac{1}{2} \times 400000(15^2 - 0^2)$$

$$F_T = 40500 \text{ N}$$

$$\text{Power} = F_T \times v = 40500 \times 15 = 607500 \text{ W} = 607.5 \text{ kW}$$

**SP4.154:** A man wishes to move the solid block of wooden box of 1m cube to a distance of 5m with the least amount of work. If the block weighs 1 kN and the coefficient of friction is 0.3, find whether he should tip it or slide it.

**Solution:**

Work to be done in sliding:

Let  $P$  be the applied force to move through sliding

The frictional force acting on the body

$$= \mu R_N = 0.3 \times 1000 \text{ N} = 300 \text{ N}$$

The applied force  $P$  = force needed to overcome the friction = 300 N

Work to be done to move a distance of 5 m through sliding

$$P \times 5 = 300 \times 5 = 1500 \text{ N.m}$$

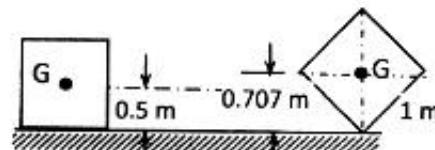


Fig.4.118

Work to be done in tipping:

For tipping once, the centre of gravity of the box is to be raised to distance of  $0.707 - 0.5 = 0.207 \text{ m}$ .

Work is to be done for one tipping = self weight x height raised  
 $= 1000 \times 0.207 = 207 \text{ Nm}$

To move a distance of 5 m, five tipplings are required. Hence work to be done for 5 tipplings  
 $= 5 \times 207 = 1035 \text{ Nm}$

Compared to sliding tipping requires less energy. Hence the man should move the box by tipping.

**SP4.155:** A man of mass 65kg jumps in a swimming pool from a tower of height 19.62m. He was found to go down in water by 2m and then started rising. Find the average resistance of water.

**Solution:**

Considering the downward motion of the man till it reaches the top surface of the water in the swimming pool

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(m.g)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(65 \times 9.81) 19.62 = \frac{1}{2} \times 65(V_2^2 - 0^2)$$

$$V_2 = 19.62 \text{ m/s}$$

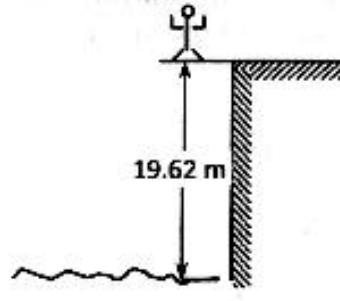


Fig.4.119

Considering the motion of the man inside the water

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(m.g - F_{RW})s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(65 \times 9.81 - F_{RW}) 2 = \frac{1}{2} \times 65 (0^2 - 19.62^2)$$

$$F_{RW} = 6892.99 \text{ N}$$

**SP4.156:** A bullet of mass 81 gram and moving with a velocity of 300m/s is fired into a block of wood and it penetrates to a depth of 100mm. If the bullet moving with the same velocity, were fired into a similar piece of wood 50mm thick, with what velocity would it emerge? Find also the force of resistance, assuming it to be uniform.

**Solution:**

Case(i)

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(-F_R)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(-F_R)0.1 = \frac{1}{2} \times 0.081(0^2 - 300^2)$$

$$F_R = 36450 \text{ N} = 36.45 \text{ kN}$$

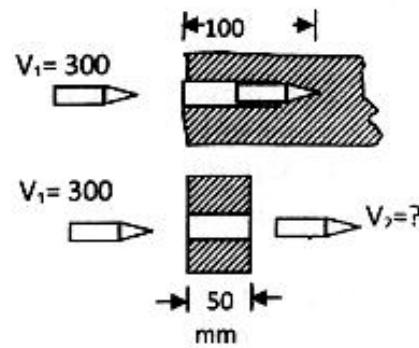


Fig.4.120

Case (ii)

$$(-F_R)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$-36450 \times 0.05 = \frac{1}{2} \times 0.081(V_2^2 - 300^2)$$

$$V_2 = 212.132 \text{ m/s}$$

**SP4.157:** In a police investigation of tyre marks, it was concluded that a car while in motion along a straight level road has skidded for a total of 60m after the brakes are applied. If the coefficient of friction is 0.5, what was the probable speed of the car just before the brakes were applied?

**Solution:**

**The only force acting on the tyre during skidding = frictional force =  $\mu R_N = \mu mg$**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2) \quad - \mu mg.s = \frac{1}{2}m(0^2 - v_1^2)$$

$$(-F_R)s = \frac{1}{2}m(V_2^2 - V_1^2) \quad - 0.5 \times 9.81 \times 60 = -\frac{1}{2} \times v_1^2 \quad (\text{Cancelling } m \text{ on both sides})$$

$$v_1 = 24.25 \text{ m/s} = 87.33 \text{ kmph}$$

**SP4.158:** A block of 2500N rests on a horizontal plane. The coefficient of friction between the block and the plane is 0.3. The block is pulled by a force of 1000N acting at an angle  $30^\circ$  to the horizontal. Find the velocity of the block after it moves over a distance of 30m starting from rest. If the force is then removed, how much further will it move?

**Solution:**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(1000 \cos 30 - F_R)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

**Resolving the forces vertically**

$$R_N + 1000 \sin 30 = 2500$$

$$R_N = 2500 - 500 = 2000 \text{ N}$$

$$F_R = \mu R_N = 0.3 \times 2000 = 600 \text{ N}$$

$$(1000 \cos 30 - 600) 30 = \frac{1}{2} \times (2500/9.81)(v_2^2 - 0^2)$$

$$V_2 = 7.914 \text{ m/s}$$

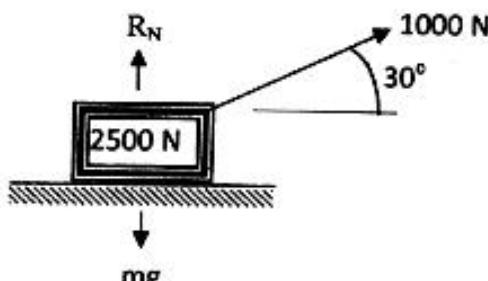


Fig.4.121

**Considering the motion of the block after removing the force**

**The only force acting on the block is the frictional force**

**Frictional force after removing the force  $F_R = \mu R_N = \mu mg = 0.3 \times 2500 = 750 \text{ N}$**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(-750)s = \frac{1}{2} \times (2500/9.81)(0^2 - 7.914^2)$$

$$s = 10.64 \text{ m}$$

**SP4.159:** A car of mass 1500 kg is uniformly accelerated. Its speed increased from 50kmph to 75kmph after travelling a distance of 200m. The resistance to the motion of the car is 0.2% of the weight of the car. Determine a) the maximum power required and b) the power required to maintain a constant speed of 75kmph.

**Solution:**

$$50 \text{ kmph} = 13.88 \text{ m/s}; 75 \text{ kmph} = 20.833 \text{ m/s}$$

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$(F_T - F_R)s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$F_R = (0.2/100) 1500 \times 9.81 = 29.43 \text{ N}$$

$$(F_T - 29.43) 200 = \frac{1}{2} \times 1500 (20.833^2 - 13.88^2)$$

$$F_T = 934.52 \text{ N}$$

$$\text{Max. power} = F_T \times \text{max. velocity}$$

$$= 941.88 \times 20.833 = 19666 \text{ N}$$

$$= 19.46 \text{ kW}$$

To find the power required to maintain the speed of 75 kmph

$(F_T - 29.43) = 0$  (velocity remains same)

$F_T = 29.43 \text{ N}$ ; Power =  $F_T \times v = 29.43 \times 20.833 = 613 \text{ W} = 0.613 \text{ kW}$

**SP4.160:** A gangman pushes a 250kg trolley for a distance of 50m along straight railway track with a force  $F = 100 - 2x$ , where  $x$  is the distance travelled in metres and  $F$  in N. The resistance to motion is 25N. Determine a) the velocity of the trolley when it has travelled 50m starting from rest and b) the total distance travelled by the trolley before coming to rest.

**Solution:**

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

Here the applied force 'F' is not constant and it is a function of 'x'

Hence the equation can be written as

$$\int_0^{50} ((F_{app} - F_R)dx) = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$\int F_{app.} dx - F_R \int dx = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$\int (100 - 2x)dx - F_R \cdot x = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$[100x - (2x^2/2)]_0^{50} - 25 \times 50 = \frac{1}{2} \times 250 (V_2^2 - 0)$$

$$[5000 - 2500] - 1250 = 125 V_2^2$$

$$V_2 = 3.162 \text{ m/s}$$

After travelling a distance of 50 m, the force acting on the trolley

$$F = 100 - 2x = 100 - 2 \times 50 = 0 \text{ N}$$

Hence the force acting on the trolley after travelling a distance of 50 m is only the friction force.

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

$$-25x = \frac{1}{2} \times 250 \times [0^4 - 3.162^4]$$

$$x = 50 \text{ m}$$

Therefore total distance travelled =  $50 + 50 = 100 \text{ m}$

**SP4.161:** Two bodies A and B of masses  $m_1$  and  $m_2$  ( $m_2 > m_1$ ) are connected by an inextensible string passing over a frictionless pulley. The system is initially at rest with the body A resting on the ground and the body B at a height 'h' from the ground. When B is released, determine the speed at which it will hit the ground.

**Solution:**

$$(v = \sqrt{[2gh(m_2 - m_1)/(m_1 + m_2)]})$$

$$\Sigma F.s = \frac{1}{2}m(V_2^2 - V_1^2)$$

Considering the downward motion of B

$$(m_2.g - T)h = \frac{1}{2}m_2(V_2^2 - V_1^2)$$

$$(m_2.g - T)h = \frac{1}{2}m_2(v_2^2 - 0)$$

$$(m_2.g - T)h = \frac{1}{2}m_2v_2^2 \dots (1)$$

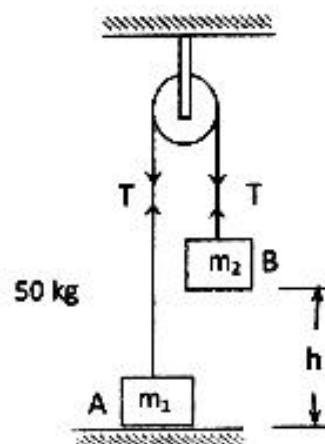


Fig.4.122

Considering the upward motion of A

$$(T - m_1.g)h = \frac{1}{2}m_1(v_2^2 - 0)$$

$$T = m_1.g + \frac{1}{2}(1/h)m_1v_2^2 \dots (2)$$

Substituting the value of T in equation (1)

$$\text{We get } (m_2.g - m_1.g - \frac{1}{2}(1/h)m_1v_2^2)h = \frac{1}{2}m_2v_2^2 \dots (1)$$

$$\begin{aligned} m_2.g.h - m_1.g.h - \frac{1}{2}m_1v_2^2 &= \frac{1}{2}m_2v_2^2 \\ 2g.h(m_2 - m_1) &= v_2^2(m_2 + m_1) \\ v_2^2 &= 2g.h(m_2 - m_1) / (m_2 + m_1) \\ v &= \sqrt{[2gh(m_2 - m_1)/(m_1 + m_2)]} \end{aligned}$$

**SP4.162:** Determine the total work done on a 5kg body that is pulled 6m up a rough plane inclined  $30^\circ$  with the horizontal as shown in fig.4. Assume the coefficient of friction as 0.2. What is the speed after travelling a distance of 6m from the start?

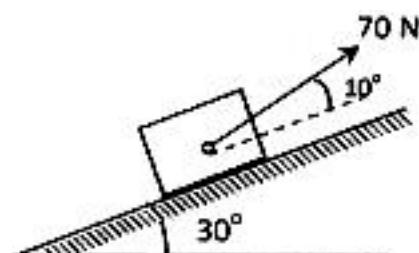


Fig.4.123

**Solution:**

**Resolving the forces acting perpendicular to the plane**

$$R_N + 70 \sin 10 - mg \cos 30 = 0$$

$$R_N + 12.15 - 5 g \cos 30 = 0$$

$$R_N + 12.15 - 42.47 = 0$$

$$R_N = 30.31 \text{ N}$$

$$\text{Frictional force} = \mu R_N = 0.2 \times 30.31 = 6.0629 \text{ N}$$

**Using the relation W.D = Change in K.E**

$$(70 \cos 10 - mg \sin 30 - F_R) s = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$(68.93 - 5 \times 9.81 \sin 30 - 6.0629) 6$$

$$= \frac{1}{2} \times 5 \times v_2^2 \quad (\text{since } v_1 = 0)$$

$$v_2 = 9.59 \text{ m/s}$$

$$W.D = \frac{1}{2} \times 5 \times v_2^2 = \frac{1}{2} \times 5 \times 9.59^2 = 230 \text{ Nm}$$

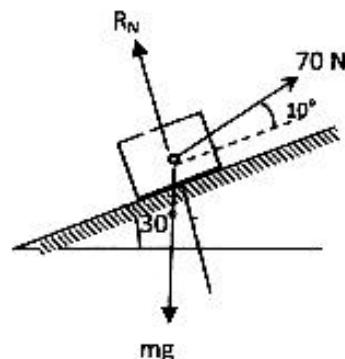


Fig.4.123a

**SP4.163:** Two block A and B of masses 100 kg and 150 kg are connected by a string as shown in fig.4. If the system is released from rest, find the velocity of the block A after it has moved a distance of 1m. Assume the coefficient of friction between the block A and the horizontal plane as 0.2.

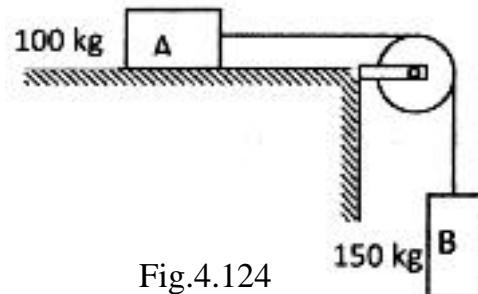


Fig.4.124

**Solution:**

#### Mass A

$$(T - \mu mg) s = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$(T - 0.2 \times 100 \times 9.81) 1 = \frac{1}{2} 100(v_2^2 - 0^2)$$

$$T - 196.2 = 50 v_2^2 \quad \dots(1)$$

#### Mass B

$$(mg - T) s = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$(150 \times 9.81 - T) 1 = \frac{1}{2} 150(v_2^2 - 0^2)$$

$$1471.5 - T = 75 v_2^2 \quad \dots(2)$$

Adding (1) and (2)

$$\text{We get } 1275.3 = 125 v_2^2$$

$$v_2^2 = 10.2024$$

$$v_2 = 3.19 \text{ m/s}$$

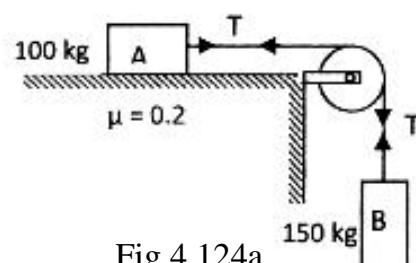


Fig.4.124a

**SP4.164:** A block of weight 'w' slides down an inclined plane from rest, from an elevation  $h$ , as shown in fig.4.125 Find the velocity of the block when it reaches the point B.

**Solution:**

From the fig we get

$$\sin\theta = h/AB ; AB = h/\sin\theta$$

Using the relation  $W.D = \text{Change in K.E}$

Net force  $\times$  distance moved = Change in K.E

$$(mg \sin\theta - \mu m g \cos\theta) AB = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$(\sin\theta - \mu \cos\theta) mg (h/\sin\theta) = \frac{1}{2} m v_2^2 ; \text{ here } (v_1 = 0)$$

Cancelling 'm' throughout and rearranging the equation

$$(1 - \mu \cot\theta) 2gh = v_2^2$$

$$v = \sqrt{2gh(1 - \mu \cot\theta)}$$

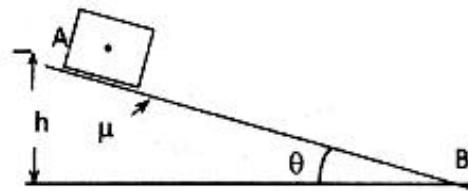


Fig.4.125

**SP4.165:** A 20kg package is projected up a  $20^\circ$  incline with an initial velocity of 12m/s.

Knowing that the coefficient of kinetic friction between the package and the incline is 0.15. Determine a) the maximum distance 'x' that the package will move up the plane, b) the velocity of the package as it returns to the original position and c) the total amount of energy dissipated due to friction.

**Solution:**

Using the relation  $W.D = \text{Change in K.E}$

Net force  $\times$  distance moved = Change in K.E

Considering the upward motion of the package

$$(-mg \sin\theta - \mu m g \cos\theta) x = \frac{1}{2} m (v_2^2 - v_1^2)$$

Cancelling 'm'

$$(-g \sin\theta - \mu g \cos\theta) x = \frac{1}{2} (v_2^2 - v_1^2)$$

$$(-9.81 \sin 20^\circ - 0.15 \times 9.81 \cos 20^\circ) x = \frac{1}{2} (0^2 - 12^2)$$

$$X = 15.196 \text{ m}$$

Considering the downward motion the package till it reaches the original position

$$(mg \sin\theta - \mu m g \cos\theta) x = \frac{1}{2} m (v_2^2 - v_1^2)$$

Cancelling 'm'

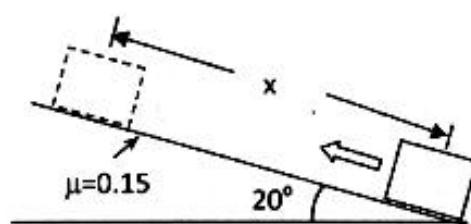


Fig.4.126

$$(g \sin \theta - \mu g \cos \theta) x = \frac{1}{2} (v_2^2 - 0^2)$$

$$(9.81 \sin 20 - 0.15 \times 9.81 \cos 20) 15.196 = \frac{1}{2} v^2$$

$$v^2 = 59.947 ; v = 7.742 \text{ m/s}$$

Energy dissipated due to friction = Frictional force x distance covered during up and down motion

$$= \mu m g \cos \theta \times 2 \times 15.196$$

$$= 0.15 \times 20 \times 9.81 \times \cos 20 \times 2 \times 15.196$$

$$= 840.49 \text{ Nm}$$

## POTENTIAL ENERGY:

In the previous section on work and kinetic energy, it is isolated a particle or a combination of joined particles and determined the work done by gravity forces, spring forces, and other externally applied forces acting on the particle or system. This to evaluate U in the work-energy equation. In the present section it will introduce the concept of potential energy to treat the work done by gravity forces and by spring forces. This concept will simplify the analysis of many problems.

### a) Gravitational Potential Energy

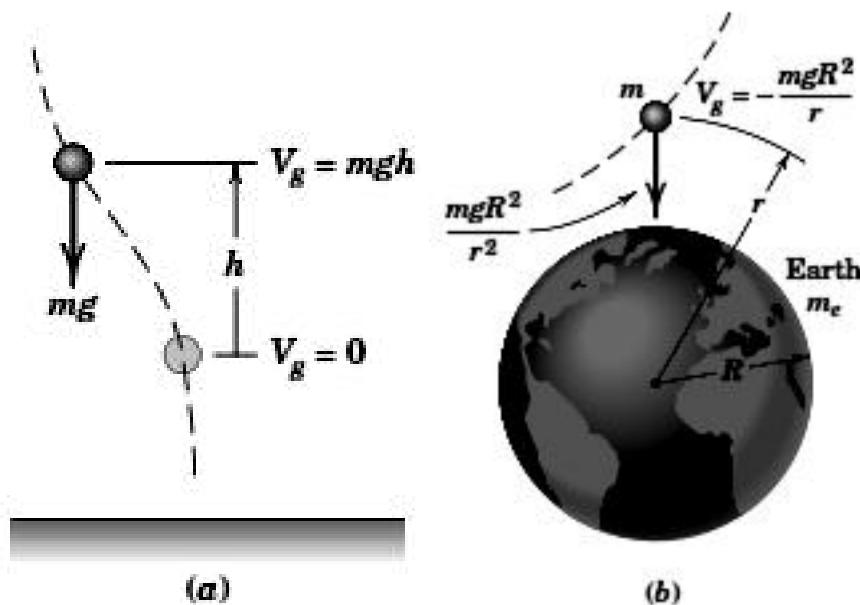


Fig.4.127

Now consider first the motion of a particle of mass  $m$  in close proximity to the surface of the earth, where the gravitational attraction (weight)  $mg$  is essentially

constant, fig.4.127a. The gravitational potential energy  $V_g$  of the particle is defined as the work  $mgh$  done against the gravitational field to elevate the particle a distance  $h$  above some arbitrary reference plane (called a datum), where  $V_g$  is taken to be zero. Thus, we write the potential energy as

$$V_g = mgh \quad \text{-----} \rightarrow (4.71)$$

This work is called potential energy because it may be converted into energy if the particle is allowed to do work on a supporting body while it returns to its lower original datum plane. In going from one level at  $h = h_1$  to a higher level at  $h = h_2$ , the change in potential energy becomes

$$\Delta V_g = mg(h_2 - h_1) = mg\Delta h$$

The corresponding work done by the gravitational force on the particle is  $-mg\Delta h$ . Thus, the work done by the gravitational force is the negative of the change in potential energy.

When large changes in altitude in the field of the earth are encountered, fig.4.3/8b, the gravitational force  $G \frac{m_e m}{r^2} = \frac{mgR^2}{r^2}$  is no longer constant. The work done against this force to change the radial position of the particle from  $r_1$  to  $r_2$  is the change  $(V_g)_2 - (V_g)_1$  in gravitational potential energy, which is

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2 \left( \frac{1}{r_1} - \frac{1}{r_2} \right) = (V_g)_2 - (V_g)_1$$

It is customary to take  $(V_g)_2 = 0$  when  $r_2 = \infty$ , so that with this datum we have

$$V_g = -\frac{mgR^2}{r} \quad \text{-----} \rightarrow (4.72)$$

In going from  $r_1$  to  $r_2$ , the corresponding change in potential energy is

$$\Delta V_g = mgR^2 \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

which, again, is the negative of the work done by the gravitational force. It notes that the potential energy of a given particle depends only on its position,  $h$  or  $r$ , and not on the particular path it followed in reaching that position.

### b) Elastic Potential Energy:

The second example of potential energy occurs in the deformation of an elastic body, such as a spring. The work which is done on the spring to deform it is stored in the spring and is called its elastic potential energy  $V_e$ . This energy is recoverable in the form of work done by the spring on the body attached to its movable end during the release of the deformation of the spring. For the one-dimensional linear spring of stiffness  $k$ , which already discussed in work – kinetic energy and illustrated in Fig.4.107, the force supported by the spring at any deformation  $x$ , tensile or compressive, from its undeformed position is  $F = kx$ . Thus, we define the elastic potential energy of the spring as the work done on it to deform it an amount  $x$ , and we have

$$V_e = \int_0^x kx dx = \frac{1}{2} kx^2 \quad \text{-----} \rightarrow (4.73)$$

If the deformation, either tensile or compressive, of a spring increases from  $x_1$  to  $x_2$  during the motion, then the change in potential energy of the spring is its final value minus its initial value or

$$\Delta V_e = \frac{1}{2} k(x_2^2 - x_1^2)$$

which is positive. Conversely, if the deformation of a spring decreases during the motion interval, then the change in potential energy of the spring becomes negative. The magnitude of these changes is represented by the shaded trapezoidal area in the  $F$ - $x$  diagram of Fig.4.107a.

Because the force exerted on the spring by the moving body is equal and opposite to the force  $F$  exerted by the spring on the body, it follows that the work done on the spring is the negative of the work done on the body. Therefore, we may replace the

work  $U$  done by the spring on the body by  $-V_e$ , the negative of the potential energy change for the spring, provided the spring is now included within the system.

**c) Work-Energy Equation:**

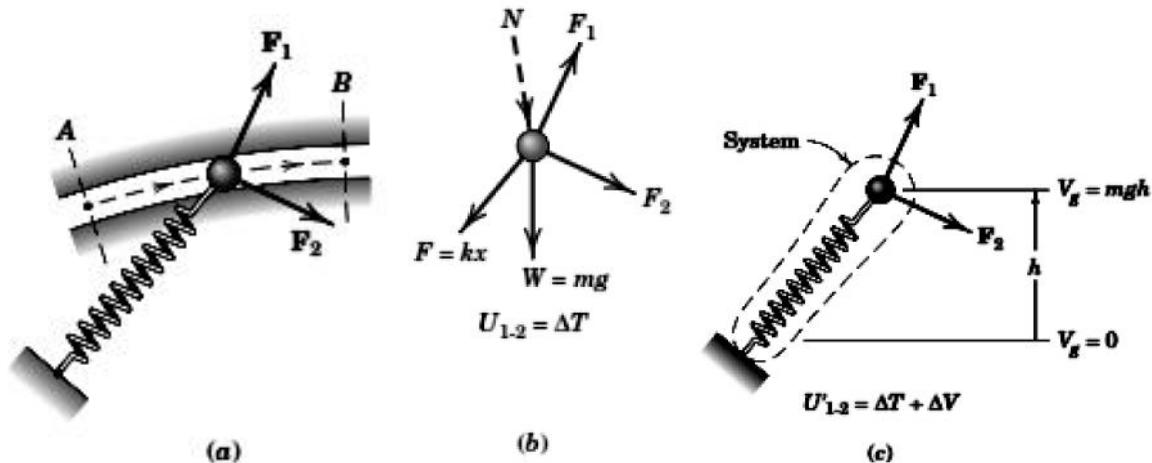


Fig.4.128

With the elastic member included in the system, we now modify the work-energy equation to account for the potential-energy terms. If  $U'_{1-2}$  stands for the work of all external forces other than gravitational forces and spring forces, we may write Eq.  $U_{1-2} = T_2 - T_1 = \Delta T$  as  $U'_{1-2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T$  or

$$U'_{1-2} = \Delta T + \Delta V \quad \text{-----} \rightarrow (4.74)$$

where  $V$  is the change in total potential energy, gravitational plus elastic.

This alternative form of the work-energy equation is often far more convenient to use than Eq.  $U_{1-2} = T_2 - T_1 = \Delta T$ , since the work of both gravity and spring forces is accounted for by focusing attention on the end-point positions of the particle and on the end-point lengths of the elastic spring. The path followed between these end-point positions is of no consequence in the evaluation of  $\Delta V_g$  and  $\Delta V_e$ .

$$T_1 + V_1 + U'_{1-2} = T_2 + V_2 \quad \text{-----} \rightarrow (4.75)$$

To help clarify the difference between the use of Eqs.  $U_{1-2} = T_2 - T_1 = \Delta T$  and  $U'_{1-2} = \Delta T + \Delta V$ , fig.4.128 shows schematically a particle of mass  $m$  constrained to

move along a fixed path under the action of forces  $F_1$  and  $F_2$ , the gravitational force  $W=mg$ , the spring force  $F$ , and the normal reaction  $N$ . In fig.4.128b, the particle is isolated with its free-body diagram. The work done by each of the forces  $F_1$ ,  $F_2$ ,  $W$ , and the spring force  $F = kx$  is evaluated, say, from A to B, and equated to the change  $T$  in kinetic energy using Eq.  $U_{1-2} = T_2 - T_1 = \Delta T$ . The constraint reaction  $N$ , if normal to the path, will do no work. The alternative approach is shown in fig.4.128c, where the spring is included as a part of the isolated system. The work done during the interval by  $F_1$  and  $F_2$  is the  $U'_{1-2}$ -term of Eq.  $U'_{1-2} = \Delta T + \Delta V$  with the changes in elastic and gravitational potential energies included on the energy side of the equation.

We note with the first approach that the work done by  $F = kx$  could require a somewhat awkward integration to account for the changes in magnitude and direction of  $F$  as the particle moves from A to B. With the second approach, however, only the initial and final lengths of the spring are required to evaluate  $V_e$ . This greatly simplifies the calculation.

For problems where the only forces are gravitational, elastic, and nonworking constraint forces, the  $U'$ -term of Eq.  $T_1 + V_1 + U'_{1-2} = T_2 + V_2$  is zero, and the energy equation becomes

$$T_1 + V_1 = T_2 + V_2 \text{ or } E_1 = E_2 \quad \longrightarrow \quad (4.76)$$

Where,  $E = T + V$  is the total mechanical energy of the particle and its attached spring. When  $E$  is constant we see that transfers of energy between kinetic and potential may take place as long as the total mechanical energy  $T+V$  does not change. Equation above expresses the law of conservation of dynamical energy.

### Problems:

**SP4.166:** The 3kg slider is released from rest at position 1 and slides with negligible friction in a vertical plane along the circular rod. The attached spring has a stiffness of 350N/m and has an un-stretched length of 0.6m. Determine the velocity of the slider as it passes position 2.

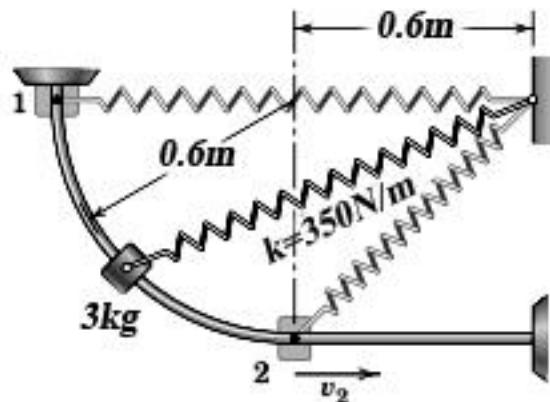


Fig.4.129

**Solution:**

The work done by the weight and the spring force on the slider will be treated using potential energy methods. The reaction of the rod on the slider is normal to the motion and does work no work. Hence  $U'_{1-2} = 0$ . We define the datum to be at the level of position 1, so that the gravitational potential energies are

$$V_1 = 0 \text{ and } V_2 = -mgh = -3(9.81)(0.6) = -17.66J$$

The initial and final elastic (Spring) potential energies are

$$V_1 = \frac{1}{2}kx_1^2 = \frac{1}{2}(350)(0.6)^2 = 63J$$

$$V_2 = \frac{1}{2}kx_2^2 = \frac{1}{2}(350)(0.6\sqrt{2} - 0.6)^2 = 10.81J$$

Substitution into the alternative work – energy equation yields

$$T_1 + V_1 + U'_{1-2} = T_2 + V_2, \Rightarrow 0 + 63 + 0 = \frac{1}{2}(3)(v_2^2) - 17.66 + 10.81$$

$$v_2 = 6.82m/s$$

**SP4.167:** The 10kg slider moves with negligible friction up the inclined guide. The attached spring has a stiffness of 60N/m and is stretched 0.6m in position A, where the slider is released from rest. The 250N force is constant and the pulley offers negligible

resistance to the motion of the cord. Calculated the velocity  $v_c$  of the slider as it passes point C.

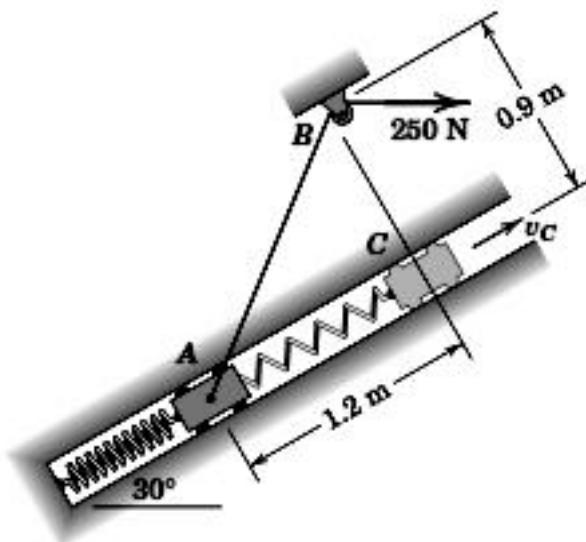


Fig.4.130

**Solution:**

The slider and inextensible cord together with the attached spring will be analyzed as a system, which permits the use of Eq. 3/21a. The only non-potential force doing work on this system is the 250-N tension applied to the cord. While the slider moves from A to C, the point of application of the 250-N force moves a distance of  $\overline{AB} - \overline{BC}$  or  $1.5 - 0.9 = 0.6$  m.

$$U'_{A-C} = 250(0.6) = 150 \text{ J}$$

We define a datum at position A so that the initial and final gravitational potential energies are

$$V_A = 0 \quad V_C = mgh = 10(9.81)(1.2 \sin 30^\circ) = 58.9 \text{ J}$$

The initial and final elastic potential energies are

$$V_A = \frac{1}{2}kx_A^2 = \frac{1}{2}(60)(0.6)^2 = 10.8 \text{ J}$$

$$V_C = \frac{1}{2}kx_B^2 = \frac{1}{2}60(0.6 + 1.2)^2 = 97.2 \text{ J}$$

Substitution into the alternative work-energy equation 3/21a gives

$$[T_A + V_A + U'_{A-C} = T_C + V_C] \quad 0 + 0 + 10.8 + 150 = \frac{1}{2}(10)v_C^2 + 58.9 + 97.2$$

$$v_C = 0.974 \text{ m/s} \quad \text{Ans.}$$

**SP4.168:** The system shown is released from rest with the lightweight slender bar OA in the vertical position as shown in fig.4.131. The torsional spring at O is undeflected in the initial position and exerts a restoring moment of magnitude  $k_T\theta$  on the bar, where

$\theta$  is the counter clockwise angular deflection of the bar. The string S is attached to point C of the bar and slips without friction through a vertical hole in the support surface. For the values  $m_A = 2\text{kg}$ ,  $m_B = 4\text{kg}$ ,  $L = 0.5\text{m}$  and  $k_T = 13\text{N}\cdot\text{m}/\text{rad}$ ; a) determine the speed  $v_A$  of particle A when  $\theta$  reaches  $90^\circ$ . and b) Plot  $v_A$  as a function of  $\theta$  over the range  $0^\circ \leq \theta \leq 90^\circ$ . Identify the maximum value of  $v_A$  and the value of  $\theta$  at which this maximum occurs.

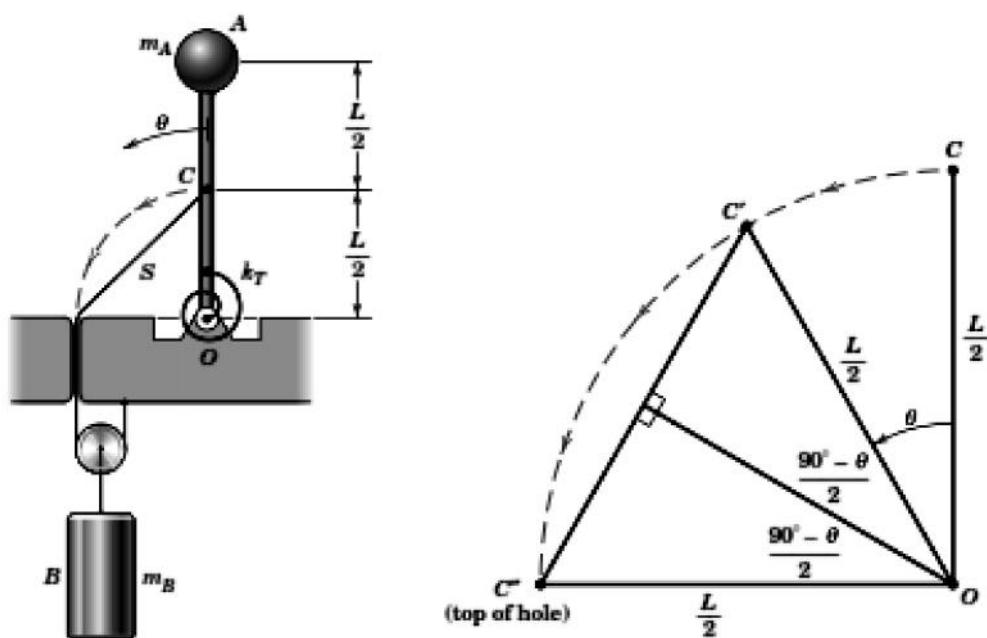


Fig.4.131

**Solution:**

**(a).** We begin by establishing a general relationship for the potential energy associated with the deflection of a torsional spring. Recalling that the change in potential energy is the work done on the spring to deform it, we write

$$V_r = \int_0^\theta k_T \theta \, d\theta = \frac{1}{2} k_T \theta^2$$

We also need to establish the relationship between  $v_A$  and  $v_B$  when  $\theta = 90^\circ$ . Noting that the speed of point C is always  $v_A/2$ , and further noting that the speed of cylinder B is one-half the speed of point C at  $\theta = 90^\circ$ , we conclude that at  $\theta = 90^\circ$ ,

$$v_B = \frac{1}{4} v_A$$

Establishing datums at the initial altitudes of bodies A and B, and with state 1 at  $\theta = 0$  and state 2 at  $\theta = 90^\circ$ , we write

$$[T_1 + V_1 + U'_{1,2} = T_2 + V_2]$$

$$0 + 0 + 0 = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 - m_A g L - m_B g \left(\frac{L\sqrt{2}}{4}\right) + \frac{1}{2}k_T \left(\frac{\pi}{2}\right)^2$$

With numbers:

$$0 = \frac{1}{2}(2)v_A^2 + \frac{1}{2}(4)\left(\frac{v_A}{4}\right)^2 - 2(9.81)(0.5) - 4(9.81)\left(\frac{0.5\sqrt{2}}{4}\right) + \frac{1}{2}(13)\left(\frac{\pi}{2}\right)^2$$

Solving,  $v_A = 0.794 \text{ m/s}$  Ans

(b). We leave our definition of the initial state 1 as is, but now redefine state 2 to be associated with an arbitrary value of  $\theta$ . From the accompanying diagram constructed for an arbitrary value of  $\theta$ , we see that the speed of cylinder B can be written as

$$\begin{aligned} v_B &= \frac{1}{2} \left| \frac{d}{dt} (\overline{C} \overline{C}^T) \right| = \frac{1}{2} \left| \frac{d}{dt} \left[ 2 \frac{L}{2} \sin\left(\frac{90^\circ - \theta}{2}\right) \right] \right| \\ &= \frac{1}{2} \left| L \left( -\frac{\theta}{2} \right) \cos\left(\frac{90^\circ - \theta}{2}\right) \right| = \frac{L\theta}{4} \cos\left(\frac{90^\circ - \theta}{2}\right) \end{aligned}$$

Finally, because  $v_A = L\dot{\theta}$ ,  $v_B = \frac{v_A}{4} \cos\left(\frac{90^\circ - \theta}{2}\right)$

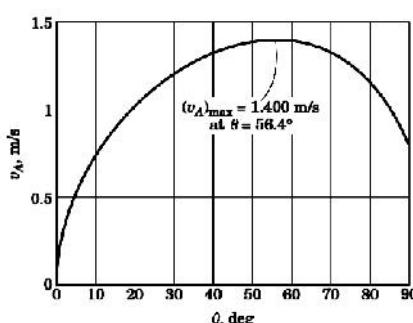
$$[T_1 + V_1 + U'_{1,2} = T_2 + V_2]$$

$$\begin{aligned} 0 + 0 + 0 &= \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B \left[ \frac{v_A}{4} \cos\left(\frac{90^\circ - \theta}{2}\right) \right]^2 - m_A g L (1 - \cos \theta) \\ &\quad - m_B g \left(\frac{1}{2}\right) \left[ \frac{L\sqrt{2}}{2} - 2 \frac{L}{2} \sin\left(\frac{90^\circ - \theta}{2}\right) \right] + \frac{1}{2}k_T \theta^2 \end{aligned}$$

Upon substitution of the given quantities, we vary  $\theta$  to produce the plot of  $v_A$  versus  $\theta$ . The maximum value of  $v_A$  is seen to be

$$(v_A)_{\max} = 1.400 \text{ m/s at } \theta = 56.4^\circ$$

Ans



**SP4.169:** A truck of mass 15 tonnes travelling at 1.6 m/s impacts with a buffer spring which compresses 1.25mm per kN. Find the maximum compression of the spring.

**Solution:**

Force needed by the spring to compress 1.25 mm = 1000 N

Force needed to compress 1 m =  $1000 \times 1000/1.25 = 800,000$  N

Stiffness of the spring  $k = 800$  kN/m

W.D on the spring =  $-\frac{1}{2} kx^2$

Using the relation W.D = Change in K.E

$$-\frac{1}{2} kx^2 = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$-\frac{1}{2} (800000) x^2 = \frac{1}{2} \times 15000(0^2 - 1.6^2)$$

$$-400 x^2 = -19200$$

$$x^2 = 48 \text{ m}; x = 0.219 \text{ m} = 219 \text{ mm}$$

**SP4.170:** A wagon of mass 50 tonnes starting from rest travels 30m down a 1% grade and strikes a post with bumper spring as shown in fig.4. If the rolling resistance of the track is 50N/tonne, find the velocity with which the wagon strikes the post. Also find the amount by which the spring will be compressed if the bumper spring compresses 1mm per 15kN.

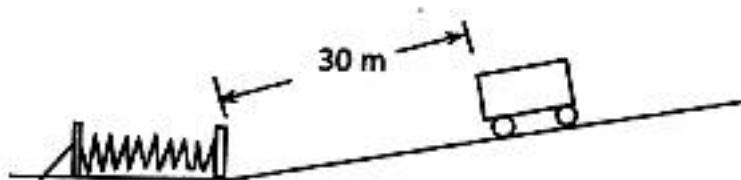


Fig.4.132

**Solution:**

Force needed to compress 1mm = 15000 N

Force needed to compress 1m

$$= 15000 \times 1000/1 = 15000000 \text{ N/m}$$

$$\text{i.e., } k = 15,000,000 \text{ N/m}$$

$$F_R = 50 \times 50 = 2500 \text{ N}$$

Consider the motion of the wagon till it reaches the spring

Using the relation W.D

= Change in K.E

Net force  $\times$  distance moved

= Change in K.E

$$(m g \sin\theta - F_R) x = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$(50,000 \times 9.81 \times 0.01 - 2500)30$$

$$= \frac{1}{2} \times 50,000 (v_2^2 - 0^2)$$

$$v^2 = 2.886; v = 1.698 \text{ m/s}$$

Considering the motion of the wagon during compressing the spring

Initial velocity of the wagon = 1.698 m/s

Final velocity of the wagon = 0 m/s

$$-\frac{1}{2} kx^2 = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$-\frac{1}{2} \times 15,000,000 x^2 = -\frac{1}{2} \times 50,000 \times 1.698^2$$

$$x = 0.09803 \text{ m} = 98 \text{ mm}$$

**SP4.171:** A block of mass 5 kg resting on a  $30^\circ$  inclined plane is released. The block after travelling a distance of 0.5m along the inclined plane hits a spring of stiffness 15N/cm. Find the maximum compression of the spring. Assume the coefficient of friction if 0.2.

**Solution:**

$$\text{Stiffness of the spring} = 15 \text{ N/cm} = 1500 \text{ N/m}$$

Let the total distance moved by the mass =  $(0.5 + x)$

Applying work energy equation

$$(mg \sin \theta - \mu mg \cos \theta)(0.5 + x) - \frac{1}{2} k x^2 = \frac{1}{2} m (v_2^2 - v_1^2) = 0$$

$$5 \times 9.81 \times 0.5 \sin 30 - 0.2 \times 5 \times 9.81 \times \cos 30 (0.5 + x) - \frac{1}{2} (1500x^2) = 0$$

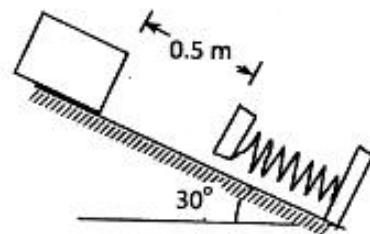


Fig.4.133

$$16.029x + 8.014 - 750x^2 = 0$$

$$750x^2 - 16.029x - 8.014 = 0$$

Solving this quadratic equation

$$x = 0.1146 \text{ m} = 114.6 \text{ mm}$$

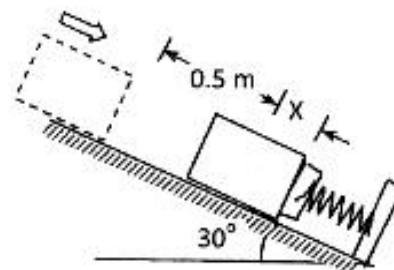
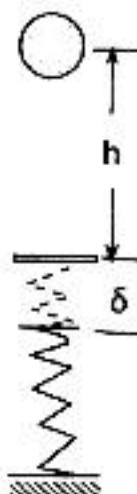


Fig.4.134

**SP4.172:** A ball of mass  $m$  is dropped on to a spring of stiffness  $k$  from a height  $h$ . Find the maximum deflection  $\delta$  of the spring. Assume  $m = 5\text{kg}$ ,  $k = 500 \text{ N/m}$ ,  $h = 10\text{cm}$ .



Let total distance moved by the ball =  $(h+\delta)$

Applying work energy equation

$$(mg)(h+\delta) - \frac{1}{2} k \delta^2 = 0$$

$$5 \times 9.81 (0.1 + \delta) - \frac{1}{2} \times 500 \times \delta^2 = 4.905 + 49.05 \delta - 250 \delta^2 = 0$$

$$250 \delta^2 - 49.05 \delta - 4.905 = 0$$

Solving the above quadratic equation

$$\delta = 0.269 \text{ m} = 269 \text{ mm}$$

Fig.4.135

**SP4.173:** If the spring of the previous problem is placed horizontal and the same ball now strikes the spring with a velocity equal to that attained by a vertical fall of height  $h = 10\text{cm}$ , find the maximum compression of the spring.

**Solution:**

**Work done against the spring = Change in K.E**

$$-\frac{1}{2} k x^2 = \frac{1}{2} m (v_2^2 - v_1^2)$$

$$-\frac{1}{2} (500) x^2 = \frac{1}{2} (5) (0^2 - v_1^2)$$

$$\text{Here } v_1 = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.1} \\ = 1.4 \text{ m/s}$$

$$-\frac{1}{2} (500) x^2 = \frac{1}{2} (5) (0^2 - 1.4^2)$$

$$250 x^2 = 4.9; x = 140 \text{ mm}$$

**SP4.174:** A 5 kg mass drops 2m upon a spring whose modulus is 10N/mm. a) what will be maximum deflection of the spring and b) find the speed of the block when the spring is deformed 100mm.

**Solution:**

$$k = 10 \text{ N/mm} = 10000 \text{ N/m}$$

$$\text{Let total distance moved by the ball} \\ = (2 + x)$$

Applying work energy equation

W.D = Change in K.E

$$(mg)(2 + x) - \frac{1}{2} k x^2 = 0$$

(RHS of the above equation is zero due to initial and final velocities are zero)

$$(5 \times 9.81)(2 + x) - \frac{1}{2} \times 10000 x^2 = 0$$

$$98.1 + 49.05 x - 5000 x^2 = 0$$

$$5000 x^2 - 49.05 x - 98.1 = 0$$

Solving the above quadratic equation

$$a = 5000; b = -49.05 \text{ & } c = -98.1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad x = \frac{-49.05 \pm \sqrt{49.05^2 + 4 \times 5000 \times 98.1}}{2 \times 5000}$$

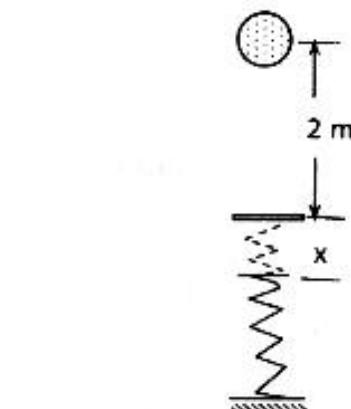


Fig.4.136

$$x = 0.145 \text{ m} = 145 \text{ mm}$$

Maximum deflection of the spring is

When the spring deflection is 100 mm

Applying work energy equation

W.D = Change in K.E

$$(mg)(2 + 0.1) - \frac{1}{2} k (0.1)^2 = \frac{1}{2} m(v^2 - 0^2)$$

$$5 \times 9.81 \times 2.1 - \frac{1}{2} \times 10000 \times 0.01 \\ = \frac{1}{2} \times 5 v^2$$

$$103.005 - 50 = 2.5 v^2$$

$$53.005 = 2.5 v^2$$

$$v = 4.604 \text{ m/s}$$

**SP4.175:** A 2kg block is released from rest on an inclined plane making an angle of  $30^\circ$  to the horizontal. The block starts from A and slides down a distance of 0.5m and strikes a spring with a stiffness of 4 kN/m. The coefficient of friction between the inclined plane and the block is 0.3. Determine a) the amount of spring will be compressed, b) the distance of block will be rebound up the plane from the compressed position and c) maximum velocity of the block.

**Solution:**

Consider the downward motion of the block

Applying work energy equation

W.D = Change in K.E

$$(mg \sin \alpha - \mu mg \cos \alpha)(0.5 + x) - \frac{1}{2} k x^2 = 0$$

$$(2 \times 9.81 \times \sin 30 - 0.3 \times 2 \times 9.81 \times \cos 30)$$

$$(0.5 + x) - \frac{1}{2} \times 4000x^2 = 0$$

$$(9.81 - 5.0974)(0.5 + x) - 2000x^2 = 0$$

$$2.356 + 4.7125x - 2000x^2 = 0$$

$$2000x^2 - 4.7125 - 2.356 = 0$$

Solving the quadratic equation

$$a = 2000; b = -4.7125; c = -2.356$$

$$x = \frac{-4.7125 \pm \sqrt{4.7125^2 + 4 \times 2000 \times 2.356}}{2 \times 2000} \\ x = 0.0355 \text{ m} = 35.5 \text{ mm}$$

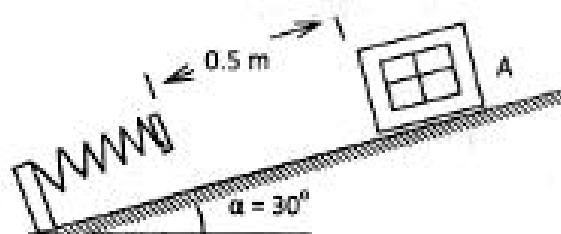


Fig.4.137

When the block rebounds up

Let the distance moved by the block = x

Applying work energy equation

W.D = Change in K.E

$$\frac{1}{2} k x^2 + (-mg \sin \alpha - \mu mg \cos \alpha)(x) = 0$$

$$\frac{1}{2} \times 4000 (0.0355)^2 + (-2 \times 9.81 \times \sin 30 - 0.3 \times 2 \times 9.81 \times \cos 30)(x) = 0$$

$$2.52 - 14.9x = 0$$

$$x = 0.169 \text{ m} ; 169 \text{ mm}$$

The velocity will be maximum when acceleration is zero. Acceleration is zero when net force acting on the body is zero. The net force in this case will be zero when the spring force balance the downward force.

Let 'x' be the distance moved by the spring when the net force on the body in the direction of motion is zero.

$$(mg \sin \alpha - \mu mg \cos \alpha) - \frac{1}{2} k x = 0$$

$$(mg \sin \alpha - \mu mg \cos \alpha) = \frac{1}{2} k x$$

$$(9.81 - 5.0974) = 4000 x$$

$$4.7126 = 4000x$$

$$x = 0.00117815 \text{ m}$$

Maximum velocity can be obtained by substituting this 'x' in the following equation

$$(mg \sin \alpha - \mu mg \cos \alpha)(0.5 + x) - \frac{1}{2} k x^2 = \frac{1}{2} m(v^2 - 0^2)$$

$$(9.81 - 5.0974)(0.5 + x) - 2000 x^2 = \frac{1}{2} x 2xv^2 = v^2$$

$$(9.81 - 5.0974)(0.5 + 0.001178) - 2000 (0.001178)^2 = v^2$$

$$2.359 = v^2; v = 1.535 \text{ m/s}$$

$$v_{\max} = 1.535 \text{ m/s}$$

**SP4.176:** In the above problem determine the initial velocity of block should be given at A so that the block rebounds and comes to rest at A.

**Solution:**

Consider the downward motion of the block. Applying the work energy relation

W.D = Change in K.E

$$(mg \sin \alpha - \mu mg \cos \alpha)(0.5 + x) - \frac{1}{2} k x^2 = \frac{1}{2} m(0^2 - v^2)$$

$$(2 \times 9.81 \times \sin 30 - 0.3 \times 2 \times 9.81 \times \cos 30)(0.5 + x) - \frac{1}{2} \times 4000 x^2 = \frac{1}{2} m(0^2 - v^2)$$

$$(9.81 - 5.0974)(0.5 + x) - 2000 x^2 = - \frac{1}{2} \times 2 (-v^2)$$

$$2.356 + 4.7125 x - 2000 x^2 = -v^2$$

$$2000 x^2 - 4.7125 x - 2.356 = v^2 \quad \dots\dots(1)$$

Consider the upward motion of the block (i.e., rebound motion)

Applying the work energy relation

W.D = Change in K.E

$$(- mg \sin \alpha - \mu mg \cos \alpha)(0.5 + x) + \frac{1}{2} k x^2 = \frac{1}{2} m(0^2 - 0^2)$$

$$(- 2 \times 9.81 \times \sin 30 - 0.3 \times 2 \times 9.81 \times \cos 30)$$

$$(0.5 + x) + \frac{1}{2} \times 4000 x^2 = 0$$

$$(- 9.81 - 5.0974)(0.5 + x) + 2000 x^2 = 0$$

$$- 14.9(0.5 + x) + 2000 x^2 = 0$$

$$- 7.45 - 14.9 x + 2000 x^2 = 0$$

$$2000 x^2 - 14.9 x - 7.45 = 0$$

Solving the quadratic equation

$$a = 2000; b = - 14.9; c = - 7.45$$

$$x = 0.06487 \text{ m}$$

Substituting the value of 'x' in eqn (1)

$$2000 (0.06487)^2 - 4.7125 (0.06487) - 2.356$$

$$= v^2 \quad \dots\dots(1)$$

$$v = 2.39 \text{ m/s}$$

**SP4.177:** A 30 kg block is dropped from a height of 2m onto the 10kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The spring constant is 20kN/m.

**Solution:**

Consider the motion of 'A' till it contact with 'B'

Applying work energy equation

W.D = Change in K.E

$$mgh = \frac{1}{2} m(v^2 - 0^2)$$

$$v^2 = 2gh = 2 \times 9.81 \times 2 = 39.24$$

$$v = 6.264 \text{ m/s}$$

Let 'V' be the common velocity of A & B after impact

Considering the principle of conservation of momentum

Momentum before impact = Momentum after impact

$$m_1u = (m_1 + m_2)v$$

$$30 \times 6.264 = (30 + 10) v ; V = 4.698 \text{ m/s}$$

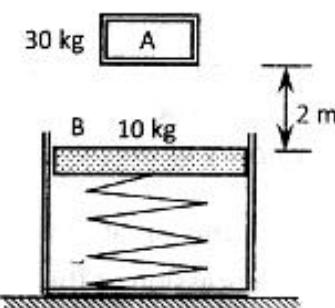


Fig.4.138

Let 'x' be the maximum deflection

Applying work energy equation

W.D = Change in K.E

$$mg(x) - \frac{1}{2} k x^2 = \frac{1}{2} m (0^2 - v^2)$$

$$40 \times 9.81 x - \frac{1}{2} 20000 x^2 = - \frac{1}{2} \times 40 \times 4.698^2$$

$$392.4 x - 10000 x^2 = - 441.424$$

$$10000 x^2 - 392 x - 441.42 = 0$$

Solving the above quadratic equation

$$x = 0.2306 \text{ m} = 230.6 \text{ mm}$$

**SP4.178:** A collar of mass 5kg can slide along a vertical bar as shown in fig.4. The spring attached to the collar is in undeformed state of length 200mm and stiffness 500 N/m. If the collar is suddenly released, find the velocity of the collar if it moves 150mm down as shown in fig.4.

**Solution:**

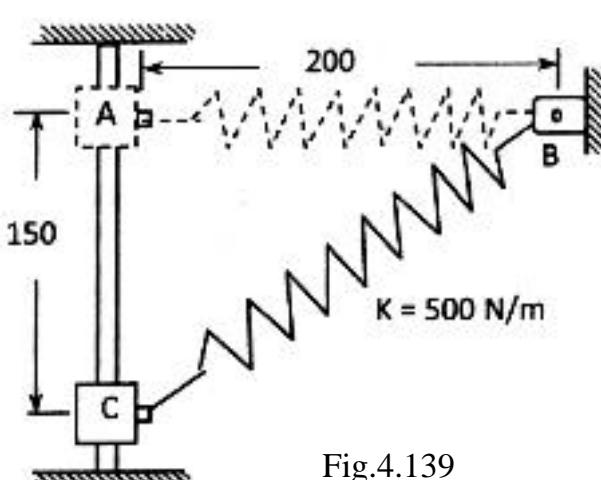


Fig.4.139

At Position 1( at A)

Potential Energy for mass =  $mgh$   
 $= 5 \times 9.81 \times 0.15 = 7.3575$

For spring = 0 (No elongation)

Kinetic energy = 0

Total energy at position 1

$$= 7.3575 \text{ N.m}$$

At Position 2 ( at C)

Potential Energy for the mass =  $mgh = 0$

Potential Energy for the spring =  $\frac{1}{2} k x^2$   
 $(x$  is the extension of the spring)

Length of the spring in position

$$2 = \sqrt{200^2 + 150^2} = 250 \text{ mm}$$

Hence the extension =  $250 - 200 = 50 \text{ mm}$

Potential Energy for the spring

$$= \frac{1}{2} 500(0.05)^2$$

$$= 0.625 \text{ N.m}$$

$$\text{Kinetic energy} = \frac{1}{2} mv^2 = \frac{1}{2} \times 5 v^2 = 2.5 v^2$$

Total energy at position 2

$$= 0.625 + 2.5v^2$$

As per energy conservation principle

Energy at 1 = Energy at 2

$$7.3575 = 0.625 + 2.5v^2$$

$$V = 1.641 \text{ m/s}$$

**SP4.179:** A spring is used to stop a 60kg package which is sliding on a horizontal surface. The spring has a constant  $k = 20\text{kN/m}$  and is held by cables so that it is initially compressed 120mm. Knowing that the package has a velocity of 2.5m/s in the position as shown in fig.4. and that the maximum additional deflection of the spring is 40mm, determine a) the coefficient of kinetic friction between the package and surface, and b) the velocity of the package as it passes again through the position as shown in fig.4.

**Solution:**

Let  $x_1$  be the initial compression and  $x_2$  be the compression of the spring after impact  
 $x_1 = 120 \text{ mm} ; x_2 = 120 + 40 = 160 \text{ mm}$

Applying work energy principle

W.D = Change in K.E

$$(-\mu m g) (0.6 + x_2) - \frac{1}{2} k(x_2^2 - x_1^2) = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$(-\mu \times 60 \times 9.81)(0.6 + 0.4) - \frac{1}{2} \times 20000(0.16^2 - 0.12^2) = \frac{1}{2} \times 60 \times (0^2 - 2.5^2)$$

$$-376.7 \mu = -75.5 ; \mu = 0.2$$

Considering the motion of the package during return

$$(-\mu m g) (0.6 + x_2) + \frac{1}{2} k(x_2^2 - x_1^2) = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$(-0.2 \times 60 \times 9.81)(0.6 + 0.4) + \frac{1}{2} \times 20000(0.16^2 - 0.12^2) = \frac{1}{2} \times 60 \times (v^2 - 0^2)$$

$$-75.34 + 112 = 30 v^2$$

$$v = 1.104 \text{ m/s}$$

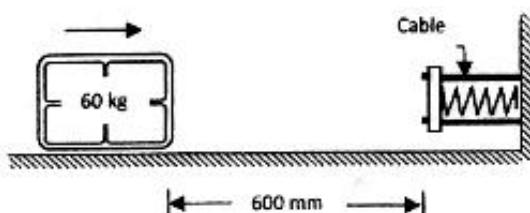


Fig.4.140

**SP4.180:** A bullet of mass 30 gram is fired into a body of mass 10kg which is suspended by a string 0.8m long. Due to this impact the body swings through an angle 30°. Find the velocity of the bullet.

**Solution:**

From the geometry we can find

$$h = 0.8 - 0.8 \cos 30^\circ$$

$$h = 0.107179 \text{ m}$$

Let  $v_1$  - velocity of the bullet before impact

$v_2$  - velocity of the bullet and body after impact

Consider the impact of the bullet with the hanging body

Applying law of conservation of momentum

Momentum before impact = momentum after impact

Let  $m_1$  - mass of the bullet and  $m_2$  - mass of the hanging mass

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v$$

$$0.03 u_1 + 10 \times 0 = (10 + 0.03) v$$

$$0.03 u_1 = 10.03 v \dots (1)$$

Total energy possessed by the body

at A = P.E + K.E

$$= 0 + \frac{1}{2} m v^2 = \frac{1}{2} (10.03) v^2 = 5.015 v^2 \text{ N.m}$$

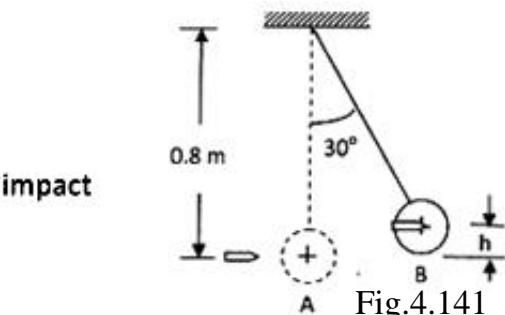


Fig.4.141

Total energy possessed by the body at

$$B = P.E + K.E$$

$$= mgh + 0 = 10.03 \times 9.81 \times 0.107179$$

$$= 10.5459 \text{ N.m}$$

Energy at A = Energy at B

$$5.015 v^2 = 10.5459 \text{ N.m}$$

$$v = 1.45 \text{ m/s}$$

Substituting v in equation (1)

$$0.03 u_1 = 10.03 \times 1.45$$

$$u_1 = 484.82 \text{ m/s}$$

Hence the velocity of the bullet

$$= 484.82 \text{ m/s}$$

**SP4.181:** A hammer of mass 500 gram hits a nail of mass 25 gram with a velocity of 5m/s and drives it into a fixed wooden block by 25mm. Find the resistance offered by the wooden block.

**Solution:**

Hammer mass,  $M = 0.5 \text{ kg}$

Nail mass,  $m = 0.025 \text{ kg}$

Velocity of hammer = 5 m/s

Applying law of conservation of momentum

Applying work energy relation

W.D = Change in K.E

$$(-R + mg) x = \frac{1}{2} m(v_2^2 - v_1^2)$$

Momentum before impact =  
momentum after impact

$$M u + m \times 0 = (M + m) v$$

$$0.5 \times 5 = (0.5 + 0.025)v$$

$$v = 4.7619 \text{ m/s}$$

$$(-R + 0.525 \times 9.81) 0.025 = \frac{1}{2} \times 0.525(0^2 - 4.7619^2)$$

$$-0.025 R + 0.12875625 = -5.95236$$

$$R = 243.24 \text{ N}$$

**SP4.182:** A hammer of mass 1500kg drops from a height of 0.6m on a pile of mass 750kg. Find a) the common velocity after impact assuming plastic impact, and b) the average resistance of the ground if the pile comes to rest after penetrating 50mm into the ground.

**Solution:**

$$\text{Hammer mass, } M = 1500 \text{ kg}$$

$$\text{Pile mass, } m = 750 \text{ kg}$$

Velocity of the hammer before impact =  $\sqrt{2gh} = \sqrt{2 \times 9.81 \times 0.6}$

$$= 3.43 \text{ m/s}$$

Applying law of conservation of momentum

Momentum before impact = momentum after impact

$$M u + m \times 0 = (M + m) v$$

$$1500 \times 3.43 = (1500 + 750) v$$

$$v = 2.2866 \text{ m/s}$$

Consider the penetration motion of the pile and hammer  
Applying work energy relation

W.D = Change in K.E

$$(-R + mg) x = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$(-R + 2250 \times 9.81) 0.05$$

$$= \frac{1}{2} \times 2250 (0^2 - 2.2866^2)$$

$$-0.05 R + 1103.625 = -5882.1$$

$$R = 139.71 \text{ kN}$$

**SP4.183:** The block A shown in fig.4.142 is moving down at 1.5m/s. The cylinder B is considered a homogeneous solid and moves in frictionless bearings. The spring is originally compressed 150mm and has a modulus of 875N/m. What will be the speed 'v' of A after dropping 1.25m?

**Solution:**

Mass moment of Inertia of the solid cylinder

$$I_B = \frac{1}{2} m r^2 = \frac{1}{2} \times 87.5 \times 0.15^2 = 0.984375 \text{ kg.m}^2$$

$$s_A = 1.25 \text{ m} ; \theta_B = s_A / r_B = 1.25 / 0.15 = 8.3333 \text{ rad}$$

$$v_A = 1.5 \text{ m/s} ; \omega = v_A / r_B = 1.5 / 0.15 = 10 \text{ rad/s}$$

Consider the downward motion of mass A

$$(mg - T_A) s_A = \frac{1}{2} m_A(v_2^2 - 1.5^2)$$

$$(735.75 - T_A) 1.25 = \frac{1}{2} \times 75(v_2^2 - 2.25)$$

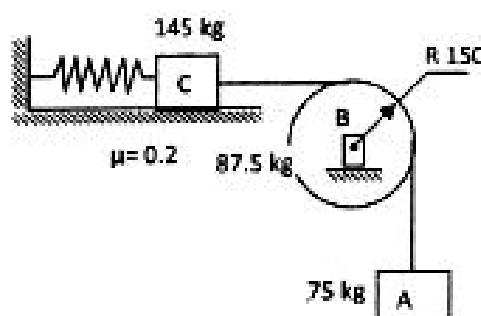


Fig.4.142a

$$919.6875 - 1.25 T_A = 37.5 v_2^2 - 84.375$$

$$1004.0625 - 1.25 T_A = 37.5 v_2^2$$

$$803.25 - T_A = 30 v_2^2$$

$$T_A = 803.25 - 30 v_2^2 \dots\dots(1)$$

Consider the motion of the mass 'C' from L to R

$$(T_B - \mu mg) S_c + \frac{1}{2} kx_1^2 - \frac{1}{2} kx_2^2 = \frac{1}{2} m_C(v_2^2 - 1.5^2)$$

$$(T_B - 0.2 \times 145 \times 9.81) 1.25 + \frac{1}{2} \times 875 (0.15)^2 - \frac{1}{2}$$

$$875 (1.1)^2 = \frac{1}{2} \times 145 (v_2^2 - 1.5^2)$$

$$1.25 T_B - 355.6125 + 9.84375 - 529.375$$

$$= 72.5 v_2^2 - 163.125$$

$$1.25 T_B - 355.6125 + 9.84375 - 529.375 + 163.125 = 72.5 v_2^2$$

$$1.25 T_B - 712 = 72.5 v_2^2$$

$$T_B = 569.6 + 58 v_2^2 \dots\dots(2)$$

Consider the motion of the cylinder

W.D = Change in K.E

$$(T_A - T_B) r_B (\theta_B) = \frac{1}{2} I (\omega_2^2 - \omega_1^2)$$

$$I = \frac{1}{2} m r^2 = \frac{1}{2} \times 87.5 \times 0.15^2 = 0.984375 \text{ kg.m}^2; \omega_1 = v/r = 1.5/0.15 = 10 \text{ rad/s};$$

$$\omega_2 = v/r = v_2/0.15$$

$$(T_A - T_B) 0.15 (1.5/0.15) = \frac{1}{2} \times 0.984375 \times [(v_2/0.15)^2 - 10^2]$$

$$(T_A - T_B) = 0.39375 [(v_2^2/0.0225) - 100]$$

$$(T_A - T_B) = 17.5 v_2^2 - 39.375$$

$$(T_A - T_B) + 39.375 = 17.5 v_2^2 \dots\dots(3)$$

Substituting (1) and (2) in equation (3)

We get  $v_2 = 1.608 \text{ m/s}$

**SP4.184:** Block A and B have masses of 45kg and 60kg respectively. The drum has a moment of inertia  $I = 16 \text{ kg.m}^2$ . Through what distance will A fall before it reaches a speed of 2m/s?

**Solution:**

Consider the downward motion of the mass 'A'

Applying work energy relation

$$(m_A \cdot g - T_A) s_A = \frac{1}{2} m_A (v_2^2 - v_1^2)$$

$$(45 \times 9.81 - T_A) s_A = \frac{1}{2} \times 45 (2^2 - 0^2)$$

$$(441.45 - T_A) s_A = 90$$

$$(441.45 - T_A) = 90 / s_A \dots\dots(1)$$

Consider the upward motion of the mass 'B'

$$(T_B - m_B \cdot g) s_B = \frac{1}{2} m_B (v_2^2 - v_1^2)$$

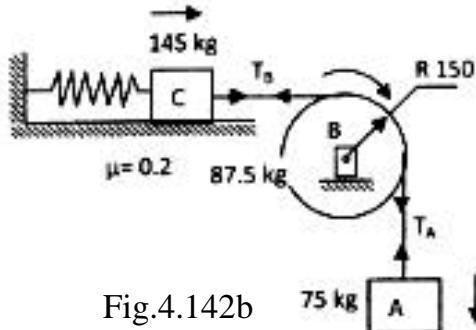


Fig.4.142b

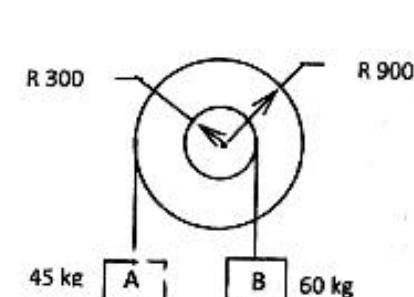


Fig.4.143a

$$(T_B - 60 \times 9.81) s_B = \frac{1}{2} \times 60 (v_2^2 - 0^2)$$

$$s_A/r_A = s_B/r_B = \theta$$

$$s_A/0.9 = s_B/0.3$$

$$s_B = s_A/3$$

$$v_A/r_A = v_B/r_B$$

$$2/0.9 = v_B/0.3$$

$$v_B = 2/3 \text{ m/s}$$

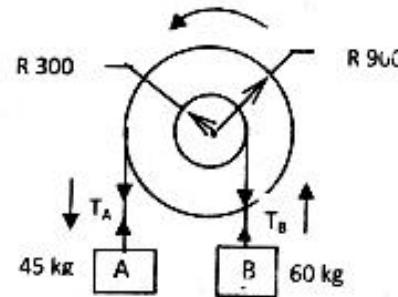


Fig.4.143b

$$(T_B - 60 \times 9.81) s_B = \frac{1}{2} \times 60 (0.666^2 - 0^2)$$

$$(T_B - 588.6) (s_A/3) = 13.333$$

$$(T_B - 588.6) = 40/s_A \dots (2)$$

Considering the motion of the drum in the anticlockwise direction

$$(T_A \cdot r_A - T_B \cdot r_B) \theta = \frac{1}{2} I (\omega^2 - 0^2)$$

$$(T_A \cdot 0.9 - T_B \cdot 0.3) (s_A/r_A) = \frac{1}{2} I \omega^2$$

$$(0.9 T_A - 0.3 T_B) (s_A/0.9) = \frac{1}{2} \times 16 (2/0.9)^2$$

$$(0.9 T_A - 0.3 T_B) = 35.5555/s_A \dots (3)$$

Solving the three equations we can get  $s_A = 0.582 \text{ m.}$

## IMPULSE AND MOMENTUM:

In the previous two sections, it focused attention on the equations of work and energy, which are obtained by integrating the equation of motion  $F = ma$  with respect to the displacement of the particle. It found that the velocity changes could be expressed directly in terms of the work done or in terms of the overall changes in energy. In the next section, it will integrate the equation of motion with respect to time rather than displacement. This approach leads to the equations of impulse and momentum. These equations greatly facilitate the solution of many problems in which the applied forces act during extremely short periods of time (as in impact problems) or over specified intervals of time.

### 1) LINEAR IMPULSE AND LINEAR MOMENTUM

Consider again the general curvilinear motion in space of a particle of mass  $m$ , fig.4.144, where the particle is located by its position vector  $r$  measured from a fixed origin  $O$ . The velocity of the particle is  $v = \dot{r}$  and is tangent to its path (shown as a dashed

line). The resultant  $\Sigma F$  of all forces on  $m$  is in the direction of its acceleration  $\dot{v}$ . It may now write the basic equation of motion for the particle, Eq.  $\Sigma F = ma$ , as

$$\Sigma F = m \dot{v} = \frac{d}{dt}(mv) \text{ or } \Sigma F = \dot{G} \quad \text{---} \rightarrow (4.77)$$

where the product of the mass and velocity is defined as the linear momentum  $G = mv$  of the particle.

Equation 4.77 states that the resultant of all forces acting on a particle equals its time rate of change of linear momentum. In SI the units of linear momentum  $mv$  are seen to be  $\text{kg}\cdot\text{m/s}$ , which also equals  $\text{N}\cdot\text{s}$ . In U.S. customary units, the units of linear momentum  $mv$  are  $[\text{lb}/(\text{ft/sec}^2)][\text{ft/sec}] = \text{lb}\cdot\text{sec}$ .

Because Eq. 4.77 is a vector equation, we recognize that, in addition to the equality of the magnitudes of  $\Sigma F$  and  $\dot{G}$ , the direction of the resultant force coincides with the direction of the rate of change in linear momentum, which is the direction of the rate of change in velocity. Equation 4.77 is one of the most useful and important relationships in dynamics, and it is valid as long as the mass  $m$  of the particle is not changing with time.

We now write the three scalar components of Eq. 4.77 as

$$\Sigma F_x = \dot{G}_x, \Sigma F_y = \dot{G}_y \text{ and } \Sigma F_z = \dot{G}_z \quad \text{---} \rightarrow (4.78)$$

These equations may be applied independently of one another.

### a) The Linear Impulse-Momentum Principle:

All that we have done so far in this section is to rewrite Newton's second law in an alternative form in terms of momentum. But we are now able to describe the effect of the resultant force  $\Sigma F$  on the linear momentum of the particle over a finite period of

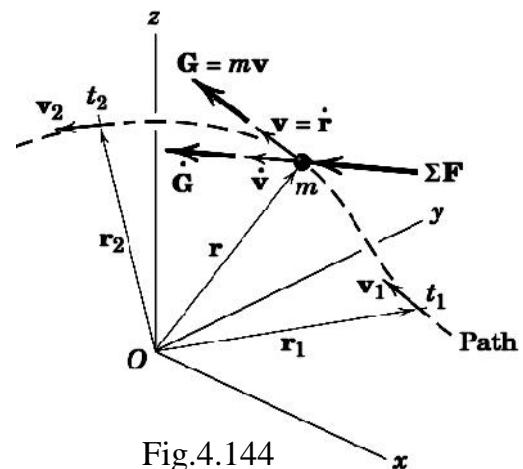


Fig.4.144

time simply by integrating Eq.4.77 with respect to the time  $t$ . Multiplying the equation by  $\sum Fdt = dG$ , which we integrate from time  $t_1$  to the time  $t_2$  to obtain

$$\int_{t_1}^{t_2} \sum Fdt = G_2 - G_1 = \Delta G \quad \longrightarrow \rightarrow (4.79)$$

Here the linear momentum at time  $t_2$  is  $G_2 = mv_2$  and the linear momentum at time  $t_1$  is  $G_1 = mv_1$ . The product of force and time is defined as the linear impulse of the force, and Eq.4.79 states that the total linear impulse on  $m$  equals the corresponding change in linear momentum of  $m$ . Alternatively, we may write Eq.4.79 as

$$G_1 + \int_{t_1}^{t_2} \sum Fdt = G_2 \quad \longrightarrow \rightarrow (4.80)$$

which says that the initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum. The impulse integral is a vector which, in general, may involve changes in both magnitude and direction during the time interval. Under these conditions, it will be necessary to express  $\Sigma F$  and  $G$  in component form and then combine the integrated components. The components of Eq.4.80 are the scalar equations

$$\left. \begin{aligned} m(v_1)_x + \int_{t_1}^{t_2} \sum F_x dt &= m(v_2)_x \\ m(v_1)_y + \int_{t_1}^{t_2} \sum F_y dt &= m(v_2)_y \\ m(v_1)_z + \int_{t_1}^{t_2} \sum F_z dt &= m(v_2)_z \end{aligned} \right\} \quad \longrightarrow \rightarrow (4.81)$$

These three scalar impulse-momentum equations are completely independent.

Whereas Eq.4.79 clearly stresses that the external linear impulse causes a change in the linear momentum, the order of the terms in Eqs.4.80 and 4.81 corresponds to the natural sequence of events. While the form of Eq.4.79 may be best for the experienced dynamicist, the form of Eqs.4.80 and 4.81 is very effective for the beginner.

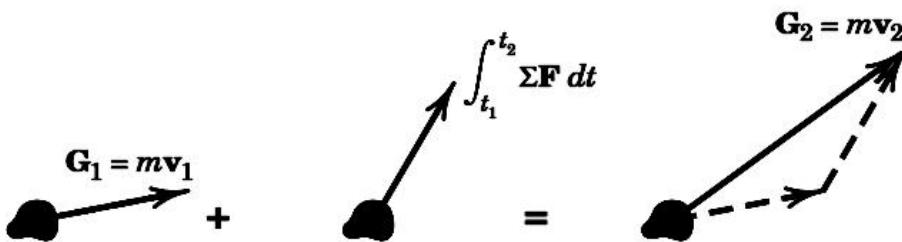


Fig.4.145

We now introduce the concept of the impulse-momentum diagram. Once the body to be analysed has been clearly identified and isolated, we construct three drawings of the body as shown in Fig.4.145. In the first drawing, we show the initial momentum  $mv_1$ , or components thereof. In the second or middle drawing, we show all the external linear impulses (or components thereof). In the final drawing, we show the final linear momentum  $mv_2$  (or its components). The writing of the impulse-momentum equations 4.81 then follows directly from these drawings, with a clear one-to-one correspondence between diagrams and equation terms.

We note that the centre diagram is very much like a free-body diagram, except that the impulses of the forces appear rather than the forces themselves. As with the free-body diagram, it is necessary to include the effects of all forces acting on the body, except those forces whose magnitudes are negligible.

In some cases, certain forces are very large and of short duration. Such forces are called impulsive forces. An example is a force of sharp impact. We frequently assume that impulsive forces are constant over their time of duration, so that they can be brought outside the linear-impulse integral. In addition, we frequently assume that non-impulsive forces can be neglected in comparison with impulsive forces. An example of a non-impulsive force is the weight of a baseball during its collision with a bat—the weight of the ball (about 5 oz) is small compared with the force (which could be several hundred pounds in magnitude) exerted on the ball by the bat.

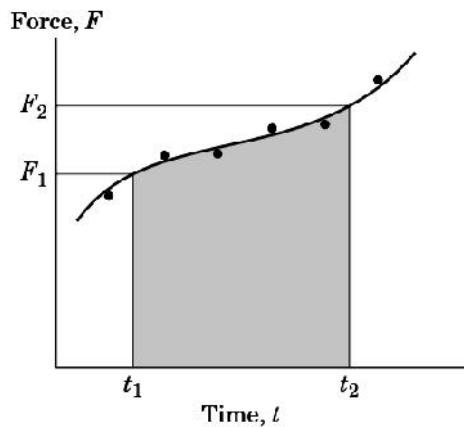


Fig.4.146

There are cases where a force acting on a particle varies with the time in a manner determined by experimental measurements or by other approximate means. In this case a graphical or numerical integration must be performed. If, for example, a force  $F$  acting on a particle in a given direction varies with the time  $t$  as indicated in Fig.4.146, then the impulse,  $\int_{t_1}^{t_2} F dt$ , of this force from  $t_1$  to  $t_2$  is the shaded area under the curve.

### b) Conservation of Linear Momentum

If the resultant force on a particle is zero during an interval of time, we see that Eq. 4.77 requires that its linear momentum  $G$  remain constant. In this case, the linear momentum of the particle is said to be conserved. Linear momentum may be conserved in one coordinate direction, such as  $x$ , but not necessarily in the  $y$ - or  $z$ -direction. A careful examination of the impulse-momentum diagram of the particle will disclose whether the total linear impulse on the particle in a particular direction is zero. If it is, the corresponding linear momentum is unchanged (conserved) in that direction.

Consider now the motion of two particles  $a$  and  $b$  which interact during an interval of time. If the interactive forces  $F$  and  $-F$  between them are the only unbalanced forces acting on the particles during the interval, it follows that the linear impulse on particle  $a$  is the negative of the linear impulse on particle  $b$ . Therefore, from Eq.4.79, the change in linear momentum  $G_a$  of particle  $a$  is the negative of the change  $G_b$  in linear momentum of particle  $b$ . So we have  $G_a = -G_b$  or  $(G_a + G_b) = 0$ . Thus, the

total linear momentum  $G = G_a + G_b$  of the two particles remains constant during the interval, and we write

$$\Delta G = 0 \text{ or } G_1 = G_2 \quad \text{-----} \rightarrow (4.82)$$

Equation 4.82 expresses the principle of conservation of linear momentum.

### Problems:

**SP4.185:** A tennis player strikes the tennis ball with her racket when the ball is at the uppermost point of its trajectory as shown in fig.4.147. The horizontal velocity of the ball just before impact with the racket is  $v_1 = 15\text{m/s}$ , and just after impact its velocity is  $v_2 = 21\text{m/s}$  directed at the  $15^\circ$  angle as shown. If the 60 gram ball is in contact with the racket for 0.02 seconds, determine the magnitude of the average force  $R$  exerted by the racket on the ball. Also determine the angle  $s$  made by  $R$  with the horizontal.

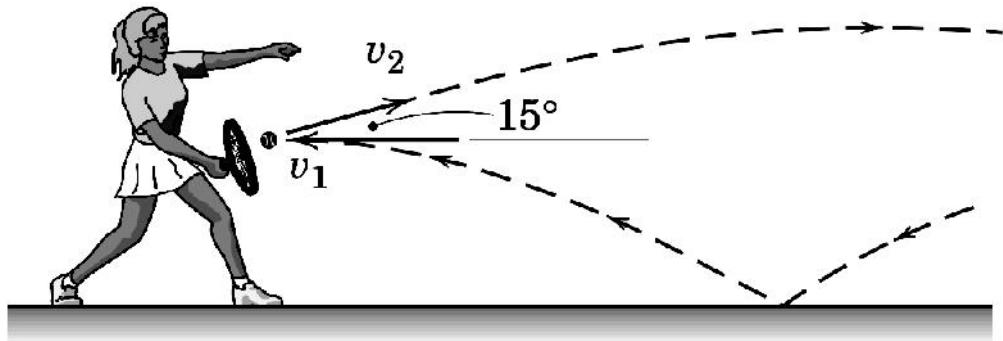


Fig.4.147

### Solution:

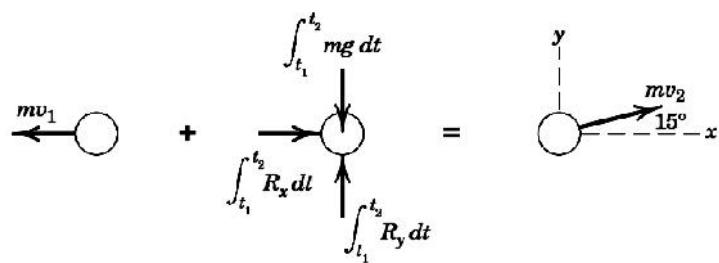


Fig.4.147a

$$m(v_x)_1 + \int_{t1}^{t2} \sum F_x dt = m(v_x)_2, \Rightarrow -0.06(15) + R_x(0.02) = 0.06(21 \cos 15^\circ)$$

$$R_x = 105.9 N$$

$$m(v_y)_1 + \int_{t1}^{t2} \sum F_y dt = m(v_y)_2,$$

$$\Rightarrow -0.06(0) + R_y(0.02) - 0.06(9.81)(0.02) = 0.06(21 \sin 15^\circ)$$

$$R_y = 16.89 N$$

We note that the impact force  $R_y = 16.89 N$  is considerably larger than the  $0.06(9.81) = 0.589 N$  weight of the ball. Thus the weight  $mg$  a non-impulsive force could have been neglected as small in comparison with  $R_y$ . Had we neglect the weight the computed value of  $R_y$  would have been  $16.31 N$ .

We now determine the magnitude and direction of  $R$  as

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{105.9^2 + 16.89^2} = 107.2 N$$

$$\theta = \tan^{-1} \left( \frac{R_y}{R_x} \right) = \tan^{-1} \left( \frac{16.89}{105.9} \right) = 9.07^\circ$$

**SP4.186:** A 0.2kg particle moves in the vertical y-z plane (z up, y horizontal) under the action of its weight and a force  $F$  which varies with time. The linear momentum of the particle in newton-seconds is given by the expression  $G = \frac{3}{2}(t^2 + 3)j - \frac{2}{3}(t^3 - 4)k$ , where  $t$  is the time in seconds. Determine  $F$  and its magnitude for the instant when  $t=2$  seconds.

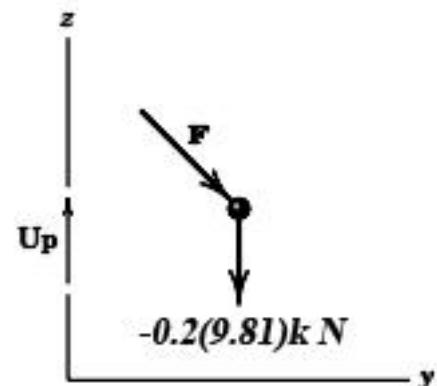


Fig.4.148

### Solution:

The weight expressed as a vector is  $-0.2(9.81)k N$ . Thus the force momentum equation becomes.

$$\sum F = \dot{G}, \Rightarrow F - 0.2(9.81)k = \frac{d}{dt} \left[ \frac{3}{2} (t^2 + 3)j - \frac{2}{3} (t^3 - 4)k \right] = 3tj - 2t^2k$$

$$\text{For } t = 2\text{ sec, } F = 0.2(9.81)k + 3(2)j - 2(2^2)k = 6j - 6.04k \text{ N}$$

$$\text{Thus, } F = \sqrt{6^2 + 6.04^2} = 8.51 \text{ N}$$

**SP4.187:** A particle with a mass of 0.5kg has a velocity of 10m/s in the x – direction at time t=0. Forces F<sub>1</sub> and F<sub>2</sub> act on the particle and their magnitudes change with time according to the graphical schedule shown in fig.4.149. Determine the velocity v<sub>2</sub> of the particle at the end of the 3s interval. The motion occurs in the horizontal x-y plane.

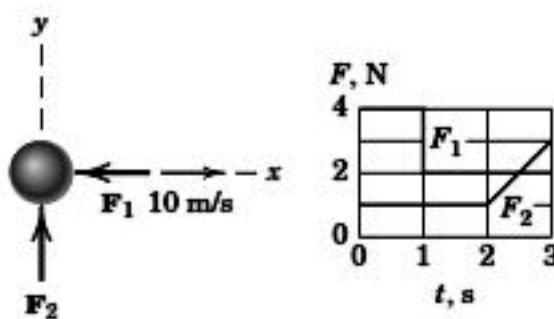
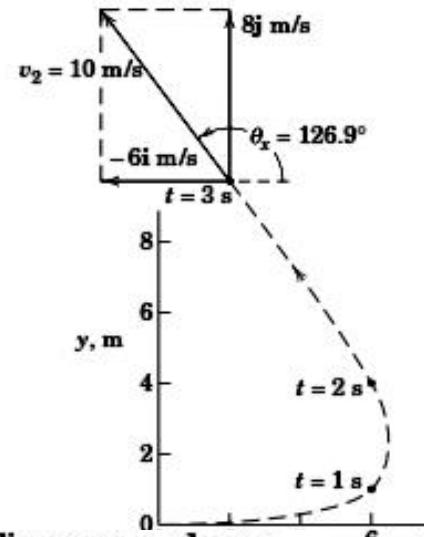
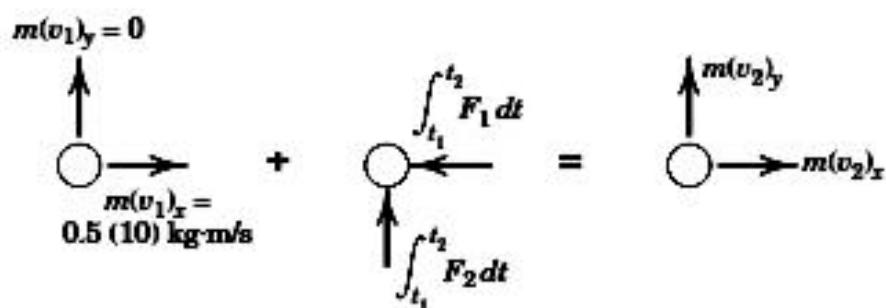


Fig.4.149



**Solution:**

First, we construct the impulse-momentum diagrams as shown.



Then the impulse-momentum equations follow as

$$[m(v_1)_x + \int_{t_1}^{t_2} \sum F_x dt = m(v_2)_x] \quad 0.5(10) - [4(1) + 2(3 - 1)] = 0.5(v_2)_x \\ (v_2)_x = -6 \text{ m/s}$$

$$[m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y dt = m(v_2)_y] \quad 0.5(0) + [1(2) + 2(3 - 2)] = 0.5(v_2)_y, \\ (v_2)_y = 8 \text{ m/s}$$

Thus,

$$\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s} \quad \text{and} \quad v_2 = \sqrt{6^2 + 8^2} = 10 \text{ m/s}$$

$$\theta_x = \tan^{-1} \frac{8}{-6} = 126.9^\circ \quad \text{Ans.}$$

Although not called for, the path of the particle for the first 3 seconds is plotted in the figure. The velocity at  $t = 3 \text{ s}$  is shown together with its components.

**SP4.188:** The loaded 150kg skip is rolling down the incline at 4m/s when a force  $P$  is applied to the cable as shown at time  $t=0$ . The force  $P$  is increased uniformly with the time until it reaches 600N at  $t=4\text{s}$ , after which time it remains constant at this value. Calculate a) the time  $t'$  at which the skip reverses its direction and b) the velocity  $v$  of the skip at  $t=8\text{s}$ . Treat the skip as a particle.

**Solution:**

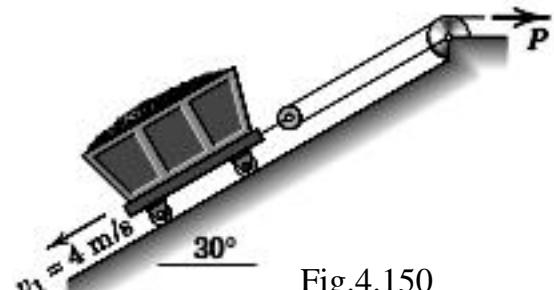
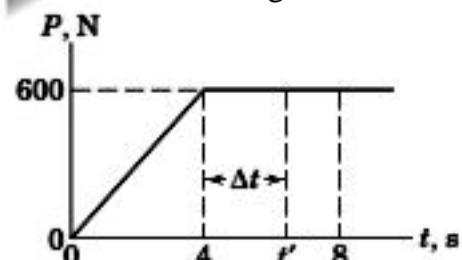
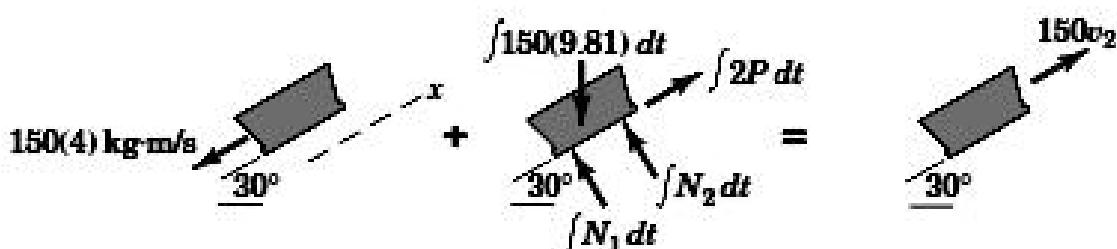


Fig.4.150



The stated variation of  $P$  with the time is plotted, and the impulse-momentum diagrams of the skip are drawn.



**Part (a).** The skip reverses direction when its velocity becomes zero. We will assume that this condition occurs at  $t = 4 + \Delta t$  s. The impulse-momentum equation applied consistently in the positive  $x$ -direction gives

$$[m(v_1)_x + \int \Sigma F_x dt = m(v_2)_x]$$

$$150(-4) + \frac{1}{2}(4)(2)(600) + 2(600)\Delta t - 150(9.81) \sin 30^\circ(4 + \Delta t) = 150(0)$$

$$\Delta t = 2.46 \text{ s} \quad t' = 4 + 2.46 = 6.46 \text{ s} \quad \text{Ans.}$$

**Part (b).** Applying the momentum equation to the entire 8-s interval gives

$$[m(v_1)_x + \int \Sigma F_x dt = m(v_2)_x]$$

$$150(-4) + \frac{1}{2}(4)(2)(600) + 4(2)(600) - 150(9.81) \sin 30^\circ(8) = 150(v_2)_x$$

$$(v_2)_x = 4.76 \text{ m/s} \quad \text{Ans.}$$

The same result is obtained by analyzing the interval from  $t'$  to 8 s.

**SP4.189:** The 50kg bullet travelling at 600m/s strikes the 4kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12m/s in the direction shown prior to impact, determine the velocity  $v_2$  of the block and embedded bullet immediately after impact.

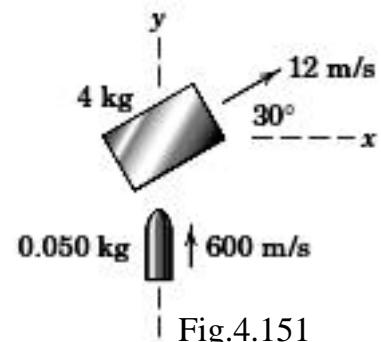


Fig.4.151

**Solution:**

Since the force of impact is internal to the system composed of the block and bullet and since there are no other external forces acting on the system in the plane of motion, it follows that the linear momentum of the system is conserved. Thus,

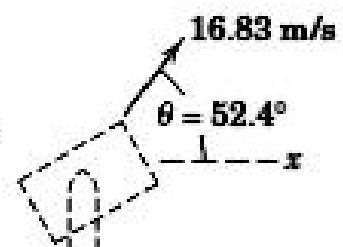
$$[G_1 = G_2] 0.050(600j) + 4(12)(\cos 30^\circ i + \sin 30^\circ j) = (4 + 0.050)v_2$$

$$v_2 = 10.26i + 13.33j \text{ m/s}$$

The final velocity and its direction are given by

$$[v = \sqrt{v_x^2 + v_y^2}] \quad v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83 \text{ m/s}$$

$$[\tan \theta = v_y/v_x] \quad \tan \theta = \frac{13.33}{10.26} = 1.299 \quad \theta = 52.4^\circ$$



**SP4.190:** A 2000 kg wagon is pushed along the rails with a constant force of 400N. If the resistance due to friction is 1000N, determine the time taken for the wagon to reach a speed of 6m/s.

**Solution:**

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$(400 - 100) \cdot t = 2000 (6 - 0)$$

$$300 t = 12000; t = 40 \text{ s}$$

**SP4.191:** A 10 kg block slides from rest down a plane inclined  $25^\circ$  with the horizontal. Assuming a coefficient of kinetic friction between the block and plane of  $1/4$ , what will be the speed of the block at the end of 3 seconds.

**Solution:**

Considering downward motion of the block

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$(mgsin\alpha - \mu mg \cos\alpha) t = m (v_2 - v_1)$$

$$(10 \times 9.81 \times \sin 25^\circ - 0.25 \times 10 \times 9.81 \cos 25^\circ) t = 10 (v_2 - 0)$$

$$(41.458 - 22.225) 3 = 10 v_2$$

$$v_2 = 5.769 \text{ m/s}$$

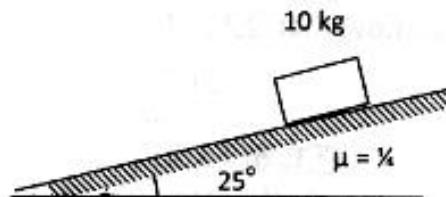


Fig.4.152

**SP4.192:** A 50 kg block rests on a horizontal floor. It is acted upon by a horizontal force that varies from zero according to the law  $F = 61.3125t$  where 'F' is the force in Newton's and 't' is the time in seconds. If the force acts for 10seconds, what is the speed of the block? Take the coefficient of friction as 0.25.

**Solution:**

Here the applied force is varying from '0'. The block will move only when the applied force is greater than the frictional force.

$$\text{The frictional force} = \mu mg = 0.25 \times 50 \times 9.81 \\ = 122.625 \text{ N}$$

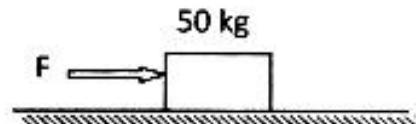


Fig.4.153

The block will not move till the applied force reaches the force of 122.625 N. Since the applied force is vary with time as per equation ' $F = 61.3125 t$ ', the time to reach this force  $= 122.625/61.3125 = 2 \text{ sec}$ . Hence the block will move when  $t = 2$  to 10 s

Here applied force is variable. i.e not constant.

Applying impulse momentum relation

Impulse = Change in momentum

$$\int F dt = \int m dv$$

$$\int_2^{10} (61.3125t - 122.625) dt = m(v_2 - v_1)$$

$$\left[ \frac{61.3125t^2}{2} \right]_2^{10} - [122.625t]_2^{10} = 50 v_2$$

$$61.3125(100/2) - 61.3125(4/2) - 122.625 \times (10 - 2) = 50 v_2$$

$$61.3125 \times 48 - 122.625 \times 8 = 50 v_2$$

$$v_2 = 39.2 \text{ m/s}$$

**SP4.193:** A 1.5 kg block starts from rest on a smooth horizontal plane under the action of horizontal force F which varies according to the equation  $F = 3t - 5t^2$ . Determine the maximum speed. Since the surface is smooth, friction force is zero.

**Solution:**

Applying impulse momentum relation

Impulse = Change in momentum

$$\int F dt = \int m dv$$

$$\int_0^t (3t - 5t^2) dt = m(v_2 - v_1)$$

$$3t^2/2 - 5t^3/3 = mv_2 \quad (\text{Since } v_1 = 0)$$

$$1.5t^2 - 1.6666t^3 = 1.5v_2$$

$$t^2 - 1.111t^3 = v_2 \quad (\text{dividing by 1.5}) \dots (1)$$

To get max speed, differentiating

wr to t and equating to zero

$$2t - 1.111 \times 3t^2 = 0$$

$$2 - 3.333t = 0; t = 2/3.333 = 0.6$$

Substituting  $t = 0.6$ , in eqn (1)

$$(0.6)^2 - 1.1111(0.6)^3 = v_2$$

$$v_2 = v_{\max} = 0.12 \text{ m/s}$$

**SP4.194:** The rolling resistance for the car shown in fig.4. is 150N/tonne. The initial velocity of the car is 5m/s up the incline. Determine the time in which the car will come to rest.

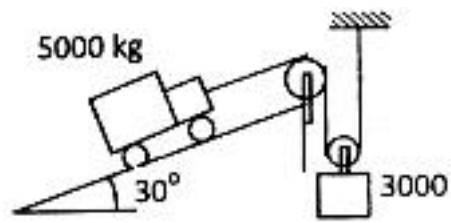


Fig.4.154

**Solution:**

Let 'T' be the tension acting on the rope at all points.

$$\text{Rolling resistance} = 150 \times 5 = 750 \text{ N}$$

Considering the upward motion of the car

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$(T - mgsin30^\circ - 750) t = 5000(0 - 5)$$

$$(T - 5000 \times 9.81 \times 0.5 - 750)t = -25000$$

$$(T - 25725)t = -25000 \dots (1)$$

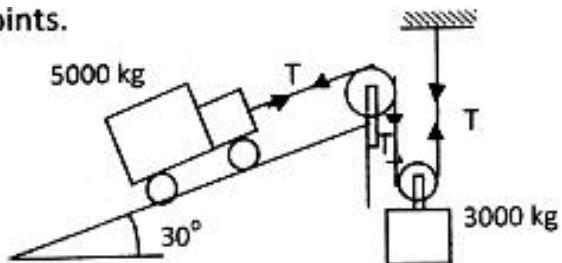


Fig.4.154a

Considering the downward motion of the 3000 kg block

The initial velocity of the 3000 kg block

$$= \frac{1}{2} (\text{velocity of the car}) = 5/2 = 2.5 \text{ m/s}$$

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$(mg - 2T)t = m (v_2 - v_1)$$

$$(3000 \times 9.81 - 2T)t = 3000(0 - 2.5)$$

$$(29430 - 2T)t = -7500 \dots (2)$$

Solving (1) and (2)

We get  $T = 16093.7 \text{ N}$

$$t = 2.72 \text{ s}$$

**SP4.195:** The 40kg block as shown in fig.4. is moving up initially with a speed of 2.5m/s. What constant value of P will give it an upward speed of 5m/s in 12 seconds? Assume that the pulleys are frictionless and the coefficient of friction between the blocks and the plane is 0.1.

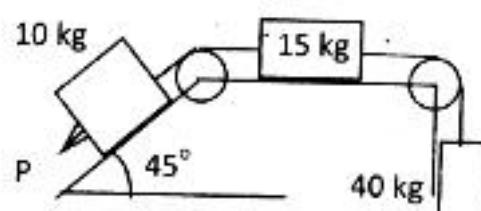


Fig.4.155

**Solution:**

All the three blocks are serially connected. Hence net tension will be equal to zero.  
Applying impulse momentum relation for all the three blocks

Impulse = Change in momentum

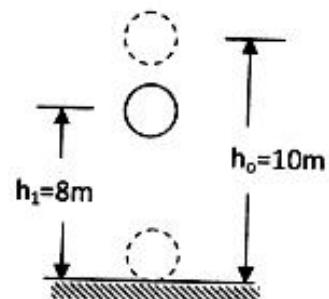
$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$(P + m_1 g \sin 45 - \mu m_1 g \cos 45 - \mu m_2 g - m_3 g) t = (m_1 + m_2 + m_3) (v_2 - v_1)$$

$$(P + 10 \times 9.81 \times \sin 45 - 0.1 \times 10 \times 9.81 \times \cos 45 - 0.1 \times 15 \times 9.81 - 40 \times 9.81) t = (10 + 15 + 40) (5 - 2.5)$$

$$(P + 69.367 - 6.936 - 14.715 - 392.4) 12 = 65 \times 2.5$$

$$P = 358 \text{ N}$$



**SP4.196:** A glass of marble, whose weight is 0.2N, falls a height of 10m and rebounds to a height of 8m. Find the impulse and the average force between the marble and the floor, if the time during which they are in contact is 0.1 seconds.

**Solution:**

Fig.4.156

The velocity with which the marble strikes the floor

$$= \sqrt{2gh_0} = \sqrt{2 \times 9.81 \times 10} \\ = 14 \text{ m/s}$$

Velocity of rebound =  $\sqrt{2gh_1} = \sqrt{2 \times 9.81 \times 8} = 12.52 \text{ m/s}$

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$\Sigma F \cdot t = (0.2/9.81) [12.52 - (-14)]$$

$$\text{Impulse} = 0.541 \text{ N.s}$$

$$\text{Average force } F = 0.54/0.1 = 5.41 \text{ N}$$

**SP4.197:** A ball of mass 100 gram is moving towards a bat with a velocity of 25m/s as shown in fig.4. When hit by a bat the ball attains a velocity of 40m/s. If the bat and the ball are in contact for a period of 0.015seconds, determine the average impulse force exerted by the bat on the ball during the impact.

**Solution:**

Applying impulse momentum relation in horizontal direction

Impulse = Change in momentum

$$\Sigma F_x \cdot t = m (v_2 - v_1)$$

$$\Sigma F_x \cdot 0.015 = 0.1 [40 \cos 40 - (-25)]$$

$$F_x = 370.945 \text{ N}$$

Applying impulse momentum relation in vertical direction

Impulse = Change in momentum

$$\Sigma F_y \cdot t = m (v_2 - v_1)$$

$$\Sigma F_y \cdot 0.015 = 0.1 (40 \sin 40 - 0)$$

$$F_y = 171.41 \text{ N}$$

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{370.94^2 + 171.31^2} = 408.6 \text{ N}$$

Average impulsive force exerted = 408.6 N

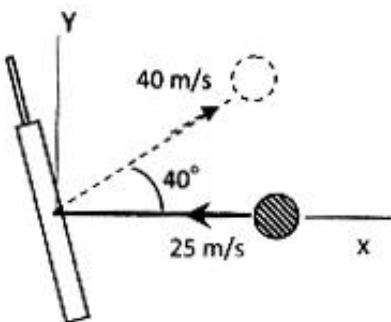


Fig.4.157

**SP4.198:** A gun of mass 3 tonne fires horizontally a shell of mass 50kg with a velocity of 300m/s. What is the velocity with which the gun will recoil? Also determine the uniform force required to stop the gun in 0.25seconds.

**Solution:**

Consider the gun as mass '1' and shell as mass '2'

According to the conservation of linear momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$3000 \times 0 + 50 \times 0 = 3000 v_1 + 50 \times 300$$

$$3000 v_1 = -50 \times 300$$

$$v_1 = -5 \text{ m/s}$$

Applying impulse momentum relation

Impulse = Change in momentum

$$\Sigma F \cdot t = m (v_2 - v_1)$$

$$\Sigma F \cdot 0.25 = 3000[0 - (-5)]$$

$$F_{ave} = 60,000 \text{ N} = 60 \text{ kN}$$

**SP4.199:** A bullet weighing 0.3N and moving at 600m/s penetrates the 40N body shown in fig.4. and emerges with a velocity of 180m/s. How far and how long will the block move, if the coefficient of friction between the body and the horizontal floor is 0.3.

**Solution:**

Let  $m_1$  – mass of the bullet =  $0.3/9.81$  kg

Let  $m_2$  – mass of the body =  $40/9.81$  kg

Since there is no external force is acting on the body, law of conservation of momentum can be applied.

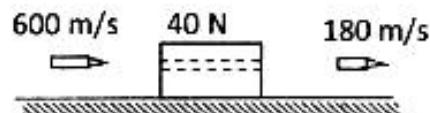


Fig.4.158

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$0.3/9.81 \times 600 + 40/9.81 \times 0 = 0.3/9.81 \times 180 + 40/9.81 \times v_2$$

$$v_2 = 3.15 \text{ m/s}$$

Considering the motion of the block

The block gains a velocity of 3.15 m/s due to impact and comes to rest due to frictional force.

To find the time:

Applying impulse momentum relation

Impulse = Change in momentum

$$(-\mu mg) t = m (v_2 - v_1)$$

$$-0.3 \times 40 \times t = (40/9.81) (0 - 3.15)$$

$$t = 1.0703 \text{ s}$$

To find the distance moved:

Applying work energy relation

Net force x distance = Change in K.E

$$(-\mu mg) \times s = \frac{1}{2} m(v_2^2 - v_1^2)$$

$$-0.3 \times 40 \times s = \frac{1}{2} \times (40/9.81)(0^2 - 3.15^2)$$

$$s = 1.686 \text{ m}$$

**SP4.200:** A car weighing 50kN and moving at 54kmph along a main road collides with a lorry of weight 100kN which emerges at 18kmph from a cross road at right angles to main road. If the two vehicle lock after collision, what will be the magnitude and direction of the resulting velocity.

**Solution:**

$$54 \text{ kmph} = 15 \text{ m/s}; 18 \text{ kmph} = 5 \text{ m/s}$$

Since no external force is acting on it we can apply law of conservation of linear momentum

In the X- direction

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$50,000/9.81 \times 15 + 100,000/9.81 \times 0$$

$$= (50000+100000)/9.81 \times v_x$$

$$v_x = 5 \text{ m/s}$$

In the Y- direction

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$50,000/9.81 \times 0 + 100,000/9.81 \times 5$$

$$= (50000+100000)/9.81 \times v_y$$

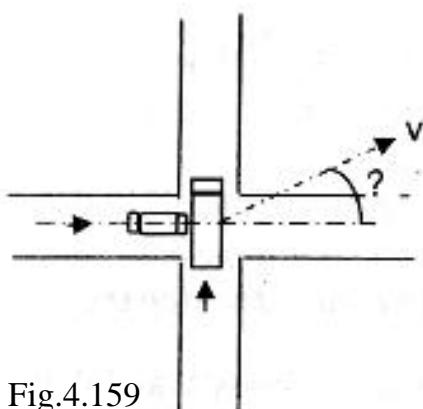


Fig.4.159

$$v_y = 3.33 \text{ m/s}$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5^2 + 3.33^2} = 6.009 \text{ m/s}$$

**= 21.633 kmph**

$$\theta = \tan^{-1} \frac{v_y}{v_x} = \frac{3.33}{5} = 33.69^\circ$$

**SP4.201:** A man of mass 50kg stands at the one end of a 5m long floating boat of mass 250kg. If the man walks towards the other end of the boat at a steady rate of 1m/s w.r.to the boat, determine a) the velocity of the boat as observed from the ground and b) the distance by which the boat get shifted.

**Solution:**

When the man walks on the boat the boat will move in the opposite direction. If we assume the direction of the boat is +ve, then the velocity of man is -ve.

Therefore absolute velocity of man

$$= V_{m/b} + V_b$$

$$= (-1 + V_b)$$

$$m_m = 50 \text{ kg}; m_b = 250 \text{ kg};$$

No external force is acting on the boat.

Hence we can apply law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$50 \times 0 + 250 \times 0 = 50 (-1 + V_b) + 250 \times V_b$$

$$0 = -50 + 50 V_b + 250 V_b = 300 V_b - 50$$

$$V_b = 50/300 = 0.16666 \text{ m/s i.e., the boat is moving to the right}$$

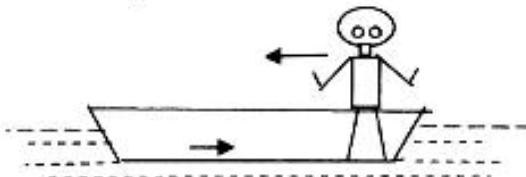


Fig.4.160

Time taken by the man to move to the other end of the boat = Distance travelled inside the boat /velocity of the man w.r.t boat = 5/ 1 = 5 s

The distance travelled by the boat = velocity of the boat x time = 0.1666 x 5 = 0.835 m.

**SP4.202:** Two men  $m_1$  of mass 50kg and  $m_2$  of mass 75kg dive off the end of a boat of mass  $M = 250\text{kg}$  so that their relative velocity with respect to the boat is 4m/s. If the boat is initially at rest, find its velocity if a) two men dive simultaneously, b) the man of mass 75kg dives first followed by the man of mass 50kg and c) the man of mass 50kg dives first followed by the man of mass 75kg.

**Solution:**

*a) two men dive simultaneously*

Considering both the men as a single body

When both men dives from the boat, the boat will move in the opposite direction. If we assume the direction of the boat is +ve, then the velocity of men are -ve.

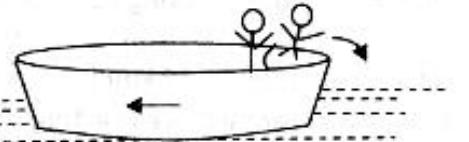


Fig.4.161

Therefore absolute velocity of men =  $V_{m/b} + V_b = (-4 + V_b)$

Applying law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$(50 + 75) \times 0 + 250 \times 0 = (50 + 75) \times (-4 + V_b) + 250 V_b$$

$$0 = 125 (-4 + V_b) + 250 V_b$$

$$0 = -500 + 125 V_b + 250 V_b$$

$$375 V_b = 500; V_b = 500/375 = 1.3333 \text{ m/s}$$

*b) the man of mass 75 kg dives first*

When man of mass 75kg dives, the other man is on the boat. Hence the mass of the boat and the man is  $(250 + 50 = 300)$  kg.

Applying law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$75 \times 0 + 300 \times 0 = 75 (-4 + V_b) + 300 V_b$$

$$0 = -300 + 75 V_b + 300 V_b$$

$$375 V_b = 300$$

$$V_b = 300/375 = 0.8 \text{ m/s}$$

When the second man jumps

Applying law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$50 \times 0.8 + 250 \times 0.8 = 50 (-4 + V_b) + 250 V_b$$

$$300 \times 0.8 = -200 + 50 V_b + 250 V_b$$

$$300 V_b = 440$$

$$V_b = 440/300 = 1.4666 \text{ m/s}$$

*c) the man of mass 50 kg dives first*

When man of mass 50 kg dives, the other man is on the boat. Hence the mass of the boat is increased to  $(250 + 75 = 325)$  kg.

Applying law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$50 \times 0 + 325 \times 0 = 50 (-4 + V_b) + 325 V_b$$

$$0 = -200 + 50 V_b + 325 V_b$$

$$375 V_b = 200$$

$$V_b = 200/375 = 0.5333 \text{ m/s}$$

When the second man jumps

Applying law of conservation of momentum

$$m_m u_m + m_b u_b = m_m v_m + m_b v_b$$

$$75 \times 0.533 + 250 \times 0.533 = 75 (-4 + V_b) + 250 V_b$$

$$325 \times 0.533 = -300 + 75 V_b + 250 V_b$$

$$325 V_b = 473.3225$$

$$V_b = 473.3225/325 = 1.4563 \text{ m/s}$$

**2) ANGULAR IMPULSE AND ANGULAR MOMENTUM:**

In addition to the equations of linear impulse and linear momentum, there exists a parallel set of equations for angular impulse and angular momentum. First, we define the term angular momentum. Fig.4.162a shows a particle P of mass m moving along a

curve in space. The particle is located by its position vector  $\mathbf{r}$  with respect to a convenient origin O of fixed coordinates x-y-z. The velocity of the particle is  $\mathbf{v} = \dot{\mathbf{r}}$ , and its linear momentum is  $\mathbf{G} = m\mathbf{v}$ . The moment of the linear momentum vector  $m\mathbf{v}$  about the origin O is defined as the angular momentum  $\mathbf{H}_O$  of P about O and is given by the cross-product relation for the moment of a vector

$$H_O = \mathbf{r} \times m\mathbf{v} \quad \longrightarrow \quad (4.83)$$

The angular momentum then is a vector perpendicular to the plane A defined by  $\mathbf{r}$  and  $\mathbf{v}$ . The sense of  $\mathbf{H}_O$  is clearly defined by the right-hand rule for cross products.

The scalar components of angular momentum may be obtained from the expansion

$$H_O = \mathbf{r} \times m\mathbf{v} = m(v_z y - v_y z)\mathbf{i} + m(v_x z - v_z x)\mathbf{j} + m(v_y x - v_x y)\mathbf{k}$$

$$H_O = m \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} \quad \longrightarrow \quad (4.84)$$

So that

$$H_x = m|v_z y - v_y z|, \quad H_y = m|v_x z - v_z x| \text{ and} \quad H_z = m|v_y x - v_x y|$$

Each of these expressions for angular momentum may be checked easily from fig.4.163, which shows the three linear-momentum components, by taking the moments of these components about the respective axes.

To help visualize angular momentum, we show in fig.4.162b a two dimensional representation in plane A of the vectors shown in part a of the figure. The motion is viewed in plane A defined by  $\mathbf{r}$  and  $\mathbf{v}$ . The magnitude of the moment of  $m\mathbf{v}$  about O is

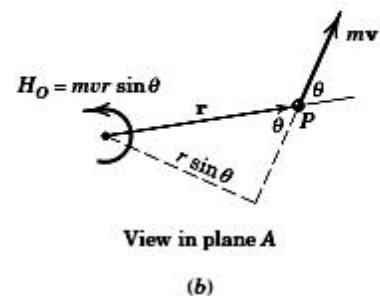
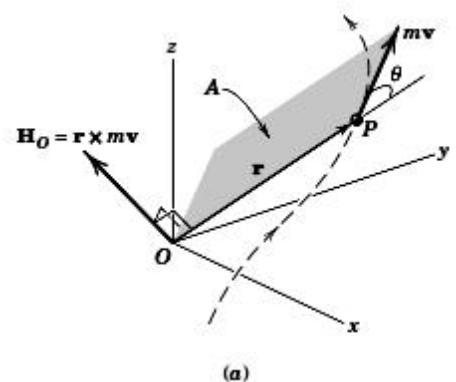


Fig.4.162

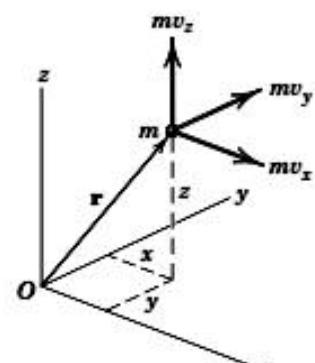


Fig.4.163

simply the linear momentum  $mv$  times the moment arm  $r \sin \theta$ , or  $mv r \sin \theta$ , which is the magnitude of the cross product  $\mathbf{H}_O = \mathbf{r} \times \mathbf{mv}$ .

Angular momentum is the moment of linear momentum and must not be confused with linear momentum. In SI units, angular momentum has the units  $\text{kg} \cdot (\text{m/s}) \cdot \text{m} = \text{kg} \cdot \text{m}^2/\text{s} = \text{N} \cdot \text{m} \cdot \text{s}$ . In the U.S. customary system, angular momentum has the units  $[\text{lb} \cdot (\text{ft} / \text{sec}^2)][\text{ft} / \text{sec}][\text{ft}] = \text{lb} \cdot \text{ft} \cdot \text{sec}$ .

### a) Rate of Change of Angular Momentum:

We are now ready to relate the moment of the forces acting on the particle P to its angular momentum. If  $\Sigma F$  represents the resultant of all forces acting on the particle P of fig. 4.3/14, the moment  $M_O$  about the origin O is the vector cross product

$$\sum M_O = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \dot{\mathbf{m}}\mathbf{v}$$

where Newton's second law  $\sum \mathbf{F} = \dot{\mathbf{m}}\mathbf{v}$  has been substituted. We now differentiate Eq. 4.83 with time, using the rule for the differentiation of a cross product and obtain

$$\dot{M}_O = \dot{\mathbf{r}} \times \dot{\mathbf{m}}\mathbf{v} + \mathbf{r} \times \dot{\dot{\mathbf{m}}}\mathbf{v} = \mathbf{v} \times \dot{\mathbf{m}}\mathbf{v} + \mathbf{r} \times \dot{\dot{\mathbf{m}}}\mathbf{v}$$

The term  $\mathbf{v} \times \dot{\mathbf{m}}\mathbf{v}$  is zero since the cross product of parallel vectors is identically zero. Substitution into the expression for  $\sum M_O$  gives

$$\sum M_O = \dot{M}_O \quad \text{-----} \rightarrow (4.85)$$

Equation 4.85 states that the moment about the fixed point O of all forces acting on m equals the time rate of change of angular momentum of m about O. This relation, particularly when extended to a system of particles, rigid or non-rigid, provides one of the most powerful tools of analysis in dynamics.

Equation 4.85 is a vector equation with scalar components

$$\sum M_{O_x} = \dot{M}_{O_x}, \sum M_{O_y} = \dot{M}_{O_y} \text{ and } \sum M_{O_z} = \dot{M}_{O_z} \quad \text{-----} \rightarrow (4.86) 3/32$$

**b) The Angular Impulse-Momentum Principle:**

Equation 4.85 gives the instantaneous relation between the moment and the time rate of change of angular momentum. To obtain the effect of the moment  $\Sigma M_O$  on the angular momentum of the particle over a finite period of time, we integrate Eq. 4.85 from time  $t_1$  to time  $t_2$ . Multiplying the equation by  $dt$  gives  $\Sigma M_O dt = dH_O$ , which we integrate to obtain

$$\int_{t_1}^{t_2} \Sigma M_O dt = (H_O)_2 - (H_O)_1 = \Delta H_O \quad \rightarrow (4.87)$$

Where  $(H_O)_2 = r_2 \times mv_2$  and  $(H_O)_1 = r_1 \times mv_1$ . The product of moment and time is defined as angular impulse, and Eq. 4.87 states that the total angular impulse on  $m$  about the fixed point  $O$  equals the corresponding change in angular momentum of  $m$  about  $O$ .

Alternatively, we may write Eq. 4.87 as

$$(H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O dt = (H_O)_2 \quad \rightarrow (4.88)$$

which states that the initial angular momentum of the particle plus the angular impulse applied to it equals its final angular momentum. The units of angular impulse are clearly those of angular momentum, which are N-m-s or kg-m<sup>2</sup>/s in SI units and lb-ft-sec in U.S. customary units.

As in the case of linear impulse and linear momentum, the equation of angular impulse and angular momentum is a vector equation where changes in direction as well as magnitude may occur during the interval of integration. Under these conditions, it is necessary to express  $\Sigma M_O$  and  $H_O$  in component form and then combine the integrated components. The x-component of Eq. 4.88 is

$$(H_{O_x})_1 + \int_{t_1}^{t_2} \Sigma M_{O_x} dt = (H_{O_x})_2 \text{ or}$$

$$m(v_z y - v_y z)_1 + \int_{t_1}^{t_2} \Sigma M_{O_x} dt = m(v_z y - v_y z)_2 \quad \rightarrow (4.89)$$

where the subscripts 1 and 2 refer to the values of the respective quantities at times  $t_1$  and  $t_2$ . Similar expressions exist for the y- and z-components of the angular impulse-momentum equation.

### c) Plane-Motion Applications

The foregoing angular-impulse and angular-momentum relations have been developed in their general three-dimensional forms. Most of the applications of interest to us, however, can be analysed as plane motion problems where moments are taken about a single axis normal to the plane of motion. In this case, the angular momentum may change magnitude and sense, but the direction of the vector remains unaltered.

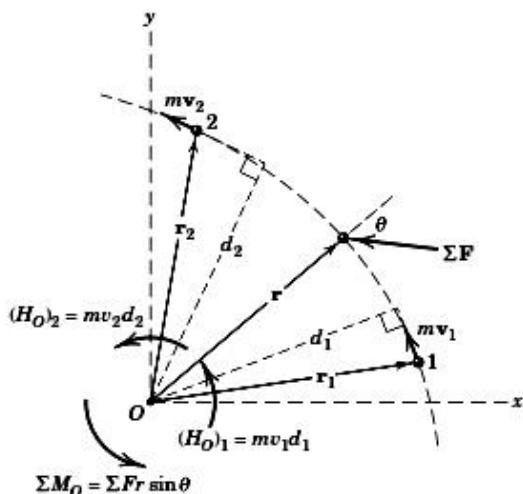


Fig.4.164

Thus, for a particle of mass  $m$  moving along a curved path in the x-y plane, fig.4.164, the angular momenta about O at points 1 and 2 have the magnitudes  $(H_o)_1 = |r_1 \times mv_1| = mv_1d_1$  and  $(H_o)_2 = |r_2 \times mv_2| = mv_2d_2$ , respectively. In the illustration both  $(H_o)_1$  and  $(H_o)_2$  are represented in the counter clockwise sense in accord with the direction of the moment of the linear momentum. The scalar form of Eq. 4.88 applied to the motion between points 1 and 2 during the time interval  $t_1$  to  $t_2$  becomes

$$(H_o)_1 + \int_{t_1}^{t_2} \sum M_o dt = (H_o)_2 \text{ or } mv_1d_1 + \int_{t_1}^{t_2} \sum Fr \sin \theta dt = mv_2d_2$$

This example should help clarify the relation between the scalar and vector forms of the angular impulse-momentum relations.

Whereas Eq. 4.87 clearly stresses that the external angular impulse causes a change in the angular momentum, the order of the terms in Eqs. 4.88 and 4.89 corresponds to the natural sequence of events. Equation 4.88 is analogous to Eq. 4.80, just as Eq. 4.85 is analogous to Eq. 4.77.

As was the case for linear-momentum problems, we encounter impulsive (large magnitude, short duration) and non-impulsive forces in angular-momentum problems.

#### d) Conservation of Angular Momentum

If the resultant moment about a fixed point O of all forces acting on a particle is zero during an interval of time, Eq. 4.85 requires that its angular momentum  $H_o$  about that point remain constant. In this case, the angular momentum of the particle is said to be conserved. Angular momentum may be conserved about one axis but not about another axis. A careful examination of the free-body diagram of the particle will disclose whether the moment of the resultant force on the particle about a fixed point is zero, in which case, the angular momentum about that point is unchanged (conserved).

Consider now the motion of two particles a and b which interact during an interval of time. If the interactive forces  $F$  and  $-F$  between them are the only unbalanced forces acting on the particles during the interval, it follows that the moments of the equal and opposite forces about any fixed point O not on their line of action are equal and opposite. If we apply Eq. 4.87 to particle a and then to particle b and add the two equations, we obtain  $H_a + H_b = 0$  (where all angular momenta are referred to point O). Thus, the total angular momentum for the system of the two particles remains constant during the interval, and we write

$$\Delta H_o = 0 \text{ or } (H_o)_1 = (H_o)_2 \quad \text{-----} \rightarrow (4.90)$$

which expresses the principle of conservation of angular momentum.

**Problems:**

**SP4.203:** A small sphere has the position and velocity indicated in the fig.4.165 and is acted upon by the force  $F$ . Determine the angular momentum  $H_o$  about point O and the time derivative  $\dot{H}_o$ .

**Solution:**

We begin with definition of angular momentum and write

$$H_o = r \times mv = (3i + 6j + 4k) \times 2(5j) = (-40i + 30k) \text{ N-m/s}$$

$$\text{From, } \dot{H}_o = M_o = r \times F = (3i + 6j + 4k) \times 10k = (60i - 30j) \text{ N-m}$$

As with moments of force, the position vector must run from the reference point (O in this case) to the line of action of the linear momentum  $mv$ . Here  $r$  runs directly to the particle.

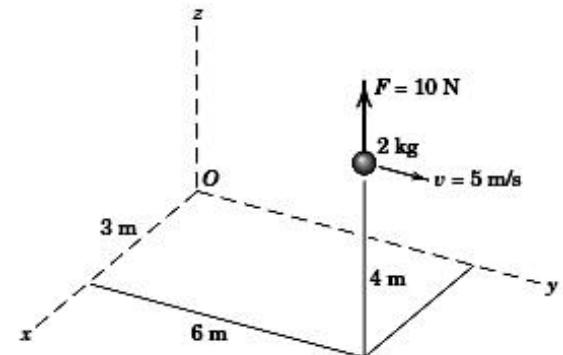


Fig.4.165

**SP4.204:** A comet is in the highly eccentric orbit shown in the figure. Its speed at the most distant point A, which is at the outer edge of the solar system is  $v_A = 740 \text{ m/s}$ . Determine its speed at the point B of closest approach to the sun.

**Solution:**

Because the only significant force acting on the comet, the gravitational force exerted on it by the sun, is central (point to the sun centre O), angular momentum about O is conserved.

$$(H_o)_A = (H_o)_B \quad , \Rightarrow mr_A v_A = mr_B v_B$$

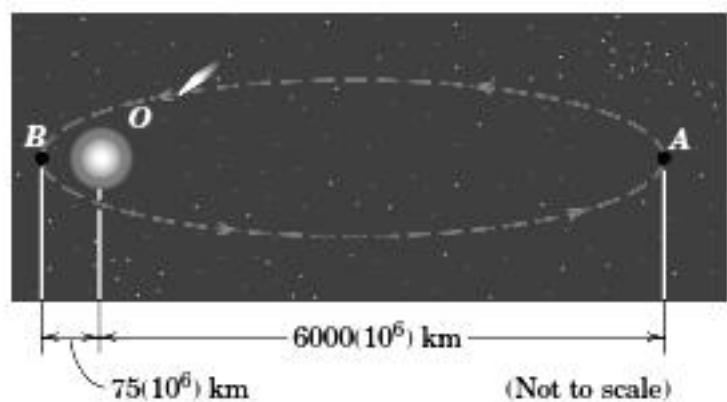


Fig.4.166

$$v_B = \frac{r_A v_A}{r_B} = \frac{6000(10^6)740}{75(10^6)} = 59200 \text{ m/s}$$

**SP4.205:** The assembly of the light rod and two end masses is at rest when it is struck by the falling wad of putty travelling with speed  $v_1$  as shown. The putty adheres to and travels with the right-hand end mass. Determine the angular velocity  $\dot{\theta}_2$  of the assembly just after impact. The pivot at O is frictionless and all three masses may be assumed to be particles.

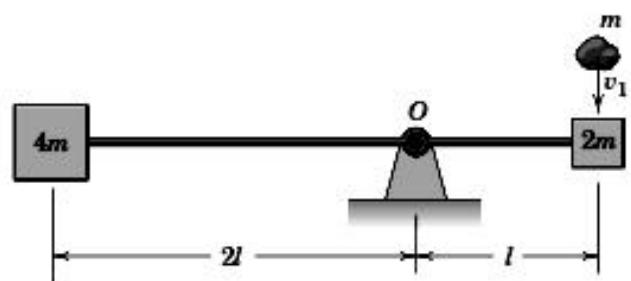


Fig.4.167

### Solution:

If we ignore the angular impulses associated with the weights during the collision process, then system angular momentum about O is conserved during the impact.

$$\begin{aligned} (H_O)_1 &= (H_O)_2 \\ mv_1 l &= (m + 2m)(l\dot{\theta}_2)l + 4m(2l\dot{\theta}_2)2l \\ \dot{\theta}_2 &= \frac{v_1}{19l} \text{ CW} \end{aligned}$$

Ans.

Note that each angular-momentum term is written in the form  $mvd$ , and the final transverse velocities are expressed as radial distances times the common final angular velocity  $\dot{\theta}_2$ .

**SP4.206:** A small mass particle is given an initial velocity  $v_0$  tangent to the horizontal rim of a smooth hemispherical bowl at a radius  $r_0$  from the vertical centre line as shown at point A. As the particle slides past point B, a distance  $h$  below A and a distance  $r$  from the vertical centreline, its velocity  $v$  makes an angle  $\theta$  with the horizontal tangent to the bowl through B. Determine  $\theta$ .

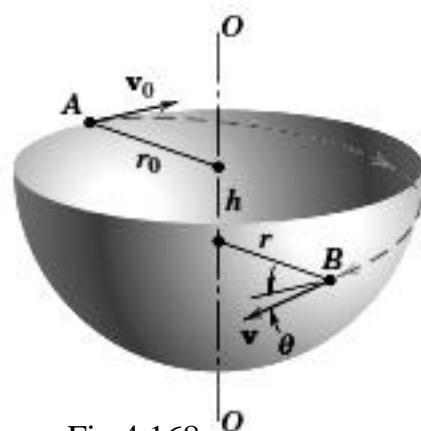


Fig.4.168

### Solution:

The forces on the particle are its weight and the normal reaction exerted by the smooth surface of the bowl. Neither force exerts a moment about the axis  $O-O$ , so that angular momentum is conserved about that axis. Thus,

$$[(H_O)_1 = (H_O)_2] \quad mv_0 r_0 = mvr \cos \theta$$

Also, energy is conserved so that  $E_1 = E_2$ . Thus

$$[T_1 + V_1 = T_2 + V_2] \quad \frac{1}{2}mv_0^2 + mgh = \frac{1}{2}mv^2 + 0$$

$$v = \sqrt{v_0^2 + 2gh}$$

Eliminating  $v$  and substituting  $r^2 = r_0^2 - h^2$  give

$$v_0 r_0 = \sqrt{v_0^2 + 2gh} \sqrt{r_0^2 - h^2} \cos \theta$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{1 + \frac{2gh}{v_0^2}} \sqrt{1 - \frac{h^2}{r_0^2}}}$$

Ana.

**SP4.207:** A flywheel having mass 500kg has hub 100mm in diameter supported on bearings. The flywheel has a radius of gyration of 500mm and it is turning at 240rpm. The coefficient of friction is 0.08. Determine the elapsed time until the flywheel comes to rest because of bearing friction.

**Solution:**

$$\text{Radius of gyration } K = 500 \text{ mm} = 0.5 \text{ m}$$

$$I = m K^2 = 500 \times 0.5^2 = 125 \text{ kg.m}^2$$

$$N = 240 \text{ rpm}; \omega = 2\pi N/60 = 2 \times 3.14 \times 240/60 \\ = 25.132 \text{ rad/s}$$

$$\text{Frictional force at the bearing} = \mu R_n = \mu mg \\ = 0.08 \times 500 \times 9.81 = 392.4 \text{ N}$$

$$\text{Frictional torque} = \text{frictional force} \times \text{radius of the hub} = 392.4 \times 0.05 = 19.62 \text{ Nm}$$

Applying angular impulse momentum relation

(Net torque)  $\times$  time = Change in angular momentum

The only torque acting on it is frictional torque which is acting in the opposite direction of the rotation

$$\Sigma T_q \times t = I (\omega_2 - \omega_1) \\ -19.62 \times t = 125(0 - 25.132); t = 160.12 \text{ s}$$

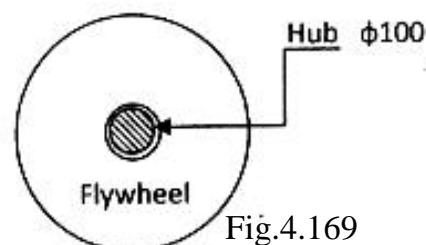


Fig.4.169

**SP4.208:** A solid cylinder as shown in fig.4. is supported on frictionless bearings. Find the angular velocity of the cylinder and the tension  $T$  in the rope 8 seconds after the system is released from rest.

**Solution:**

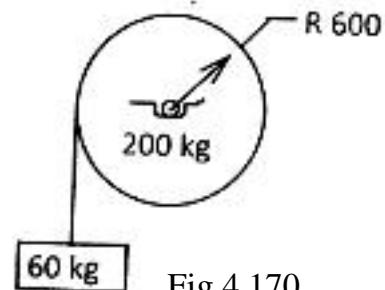


Fig.4.170

When the system is released from rest, the mass 60 kg will go down and the solid cylinder will rotate in the anticlockwise direction.

$$I = \frac{1}{2} mr^2 = \frac{1}{2} \times 200 \times 0.6^2 = 36 \text{ kg.m}^2$$

Considering the downward motion of the mass 60 kg

Applying impulse momentum relation

Impulse = Change in momentum

$$(mg - T)t = m(v_2 - v_1)$$

$$(60 \times 9.81 - T)8 = 60(v_2 - 0)$$

$$4708.8 - 8T = 60v_2 = 60\omega_2 \times 0.6 = 36\omega_2 \text{ (Since } v = \omega r\text{)}$$

$$4708.8 - 8T = 60v_2 = 36\omega_2 \dots\dots(1)$$

Considering the rotation of the solid cylinder in anticlockwise direction

Applying angular impulse momentum relation

(Net torque)  $\times$  time = Change in angular momentum

$$(T \times r)t = I(\omega_2 - \omega_1)$$

$$0.6T \times 8 = 36\omega_2; \omega_1 = 0$$

$$4.8T = 36\omega_2 \dots\dots(2)$$

Solving the equation (1) and (2)

$$T = 367.875 \text{ N}$$

$$\omega_2 = 49.05 \text{ rad/s}$$

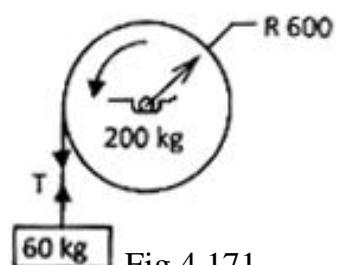


Fig.4.171

**SP4.209:** In fig.4. a mass less rope carries two masses of 5kg and 7 kg when hanging on a pulley of mass 5kg radius 600mm and radius of gyration of 450mm. How long will it take to change the speed of the masses from 3m/s to 6m/s. a) when the friction in the bearing is not considered and b) when considering the friction in the bearing of radius 100mm and  $\mu$  is 0.05.

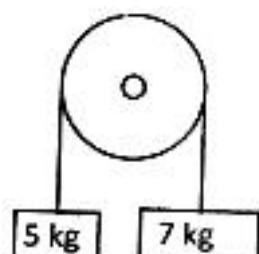


Fig.4.172

**Solution:**

i) when the friction in the bearing is not considered

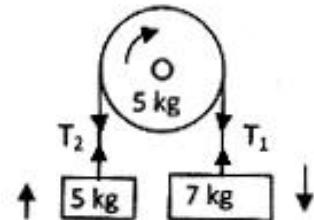
Moment of Inertia of Pulley  $I = mk^2 = 5 \times 0.45^2 = 1.0125 \text{ kgm}^2$

$$v_1 = 3 \text{ m/s} = \omega_1 r = \omega_1 \times 0.6$$

$$\omega_1 = 3/0.6 = 5 \text{ rad/s}$$

$$v_2 = 6 \text{ m/s} = \omega_2 r = \omega_2 \times 0.6$$

$$\omega_2 = 6/0.6 = 10 \text{ rad/s}$$



Considering the downward motion of heavier mass 7 kg

Applying impulse momentum relation

Impulse = Change in momentum

$$(mg - T_1)t = m(v_2 - v_1)$$

$$(7 \times 9.81 - T_1)t = 7(6 - 3)$$

$$68.67 - T_1 = 21/t \dots\dots(1)$$

Considering the upward motion of mass 5 kg

$$(T_2 - mg)t = 5(6 - 3)$$

$$T_2 - 5 \times 9.81 = 15/t$$

$$T_2 - 49.05 = 15/t \dots\dots(2)$$

Fig.4.172a

Considering the clockwise rotation of the pulley 5 kg

Applying angular impulse momentum relation

(Net torque) x time = Change in angular momentum

$$(T_1 - T_2)r \times t = I(\omega_2 - \omega_1)$$

$$(T_1 - T_2)0.6 \times t = 1.0125(10 - 5)$$

$$(T_1 - T_2)0.6 \times t = 5.0625$$

$$(T_1 - T_2) = 8.4375/t \dots\dots(3)$$

Solving the equations (1), (2) and (3)

We get  $t = 2.27 \text{ s}$

ii) when the friction in the bearing is considered

By considering friction in the bearing, the equation (3) has to be changed. Where as the equation (1) and (2) remains same

Considering the clockwise rotation of the pulley 5 kg

Applying angular impulse momentum relation

(Net torque) x time = Change in angular momentum

$$[T_1 r - T_2 r - F_r r_1]t = I(\omega_2 - \omega_1)$$

In the above equation  $F_r r_1 = \text{Frictional force at the bearing} \times \text{bearing radius}$

Frictional force at the bearing =  $\mu mg$

Total mass acting on the bearing = mass of the pulley + mass hanging on the pulley  
 $= 5+5+7 = 17 \text{ kg}$

Frictional force at the bearing =  $\mu mg = 0.05 \times 17 \times 9.81 = 8.3385 \text{ N}$

Frictional torque = Frictional force x bearing radius =  $8.3385 \times 0.1 = 0.83385 \text{ N.m}$

$$[T_1 \times 0.6 - T_2 \times 0.6 - 0.83385] t = 5.0625$$

$$[T_1 - T_2] = (5.0625/t + 0.83385)/0.6$$

$$[T_1 - T_2] = 8.4375/t + 1.38975 \dots (3)$$

Solving the equations (1), (2) and (3)

We get  $t = 2.434 \text{ s}$

**SP4.210:** Determine the mass of B necessary to cause the 50kg mass A to change its speed from 4m/s to 8m/s in 6 seconds. Assume the drum rotates in frictionless bearings.

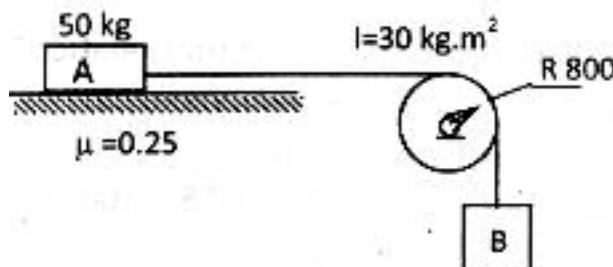


Fig.4.173

**Solution:**

Considering the downward motion of the block B

Applying impulse momentum relation

Impulse = Change in momentum

$$(mg - T_B) t = m (v_2 - v_1)$$

$$(9.81m - T_B) 6 = m (8 - 4)$$

$$(9.81m - T_B) 6 = 4 m$$

$$(9.81m - T_B) = 0.667 m$$

$$T_B = 9.1437 m \dots (1)$$

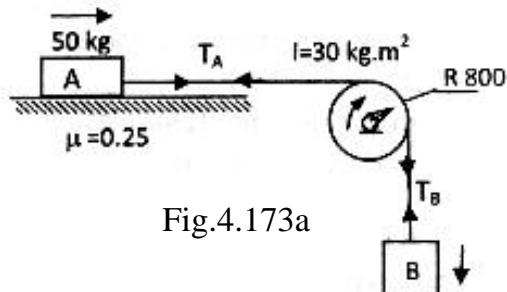


Fig.4.173a

Considering the motion of the block 'A' to RHS

Applying impulse momentum relation

Impulse = Change in momentum

$$(T_A - \mu mg) t = m (v_2 - v_1)$$

$$(T_A - 0.25 \times 50 \times 9.81) 6 = 50 (8-4)$$

$$T_A - 122.625 = 33.333$$

$$T_A = 155.958 \text{ N}$$

Considering the clockwise rotation of the pulley

Applying angular impulse momentum relation

Net torque) x time = Change in angular momentum

$$[T_B - T_A] r t = I (\omega_2 - \omega_1)$$

$$\omega_2 = v_2/r = 8/0.8 = 10 \text{ rad/s}; \omega_1 = v_1/r = 4/0.8 = 5 \text{ rad/s};$$

$$[T_B - T_A] 0.8 \times 6 = 30 (10 - 5)$$

$$[T_B - T_A] = 31.25 \dots (3)$$

Solving the above 3 equations we get  $m = 20.475 \text{ kg}$

**SP4.211:** The drum shown in fig.4.174 has a mass of 300kg and a radius of gyration with respect to its mass centre G of 500mm. What will be the time for the angular speed to change uniformly from 100 to 300 rad/s?

**Solution:**

When the drum is in static condition, the torque due to mass 75 kg

$$= 75 \times 9.81 \times 0.5 = 367.875 \text{ N}$$

The torque due to mass 30 kg

$$= 30 \times 9.81 \times 0.9 = 264.87 \text{ N}$$

Since torque due to mass 75 kg is greater than 30 kg mass, the 75 kg mass tends to move down when it is released from rest.

Considering the downward motion of the 75 kg mass

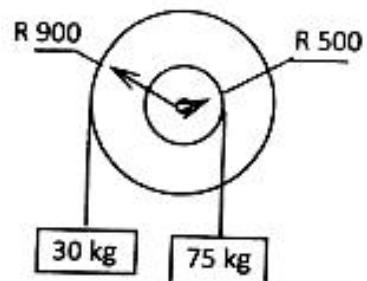


Fig.4.174

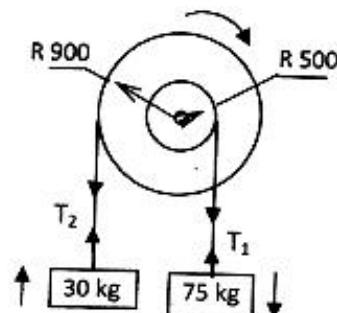


Fig.4.174a

Applying impulse momentum relation

Impulse = Change in momentum

$$(mg - T_1)t = m(v_2 - v_1)$$

$$(75 \times 9.81 - T_1)t = 75(300 \times 0.5 - 100 \times 0.5)$$

$$735.75 - T_1 = 7500/t \dots(1)$$

Considering the upward motion of the 30 kg mass

Applying impulse momentum relation

Impulse = Change in momentum

$$(T_2 - mg)t = m(v_2 - v_1)$$

$$(T_2 - 30 \times 9.81)t$$

$$= 30(300 \times 0.9 - 100 \times 0.9)$$

$$T_2 - 294.3 = 5400/t \dots(2)$$

$$I = mk^2 = 300 \times 0.5^2 = 75 \text{ kg.m}^2$$

Considering the clockwise motion of the drum

Applying angular impulse momentum relation

(Net torque) x time = Change in angular momentum

$$[T_1r_1 - T_2r_2]t = I(\omega_2 - \omega_1)$$

$$[T_1r_1 - T_2r_2]t = I(\omega_2 - \omega_1)$$

$$0.5T_1 - 0.9T_2 = 75 \times 200/t = 1500/t \dots(3)$$

Solving the above equations we get  $t = 229.6 \text{ s}$

**SP4.212:** A jet of water 50mm diameter exerts a force of 1200N on a flat vane perpendicular to the stream. What is the nozzle speed of the jet?

**Solution:**

**Mass flow rate of water striking the vane**

$$\dot{m} = \rho A V = 1000 \times \pi \times 0.025^2 \times 30 = 1.963495 \text{ kg/s}$$

**Applying impulse momentum relation**

*Impulse = Change in momentum*

$$F \Delta t = \dot{m} \Delta t (v_2 - v_1)$$

$F = 1.963495 \text{ v} (0 - v)$  (after hitting the vertical plate the velocity of stream is zero and the force exerted will be acting in the opposite direction of the jet of water.

$$F = -1.963495 \text{ v}^2$$

$$-1200 = -1.963495 \text{ v}^2$$

$$V = 24.72 \text{ m/s}$$

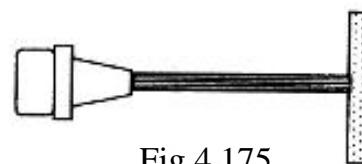


Fig.4.175

**SP4.213:** A jet of water issues from a 25mm diameter nozzle at 30m/s. Determine the total force against a circularly curved vane where the direction of the jet changes 45°. Assume no friction.

**Solution:**

**Mass flow rate of water**  $\dot{m} = \rho A V$

$$= 1000 \times \pi \times 0.0125^2 \times 30$$

$$= 14.7262 \text{ kg/s}$$

**Applying impulse momentum relation along X-direction**

*Impulse = Change in momentum*



Fig.4.176

$$-F_x \Delta t = \dot{m} \Delta t (v_2 - v_1)$$

$$-F_x = \dot{m} (v_2 - v_1)$$

$$-F_x = 14.7262 (30 \cos 45^\circ - 30)$$

$$-F_x = -129.39 \text{ N}$$

**Applying impulse momentum relation along Y-direction**

$$F_y \Delta t = \dot{m} \Delta t (v_2 - v_1)$$

$$F_y = \dot{m} (v_2 - v_1)$$

$$F_y = 14.7262 (30 \sin 45^\circ - 0)$$

$$F_y = 312.38 \text{ N}$$

$$F = \sqrt{312.38^2 + 129.39^2} = 338 \text{ N}$$

**SP4.214:** A jet of water of cross sectional area 1000mm<sup>2</sup> is directed into a vane with a constant velocity 6m/s and is deflected 120° from the initial direction. Determine the force on the vane is a) the vane is fixed and b) the vane is moving with velocity of 1m/s to the right.

**Solution:**

a) the vane is fixed

$$\begin{aligned}\text{Mass flow rate of water } \dot{m} &= \rho A V \\ &= 1000 \times 1000 \times 10^{-6} \times 6 \\ &= 6 \text{ kg/s}\end{aligned}$$

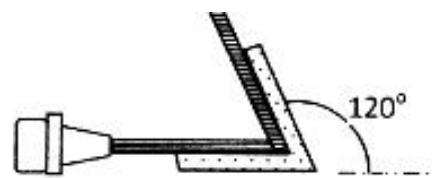


Fig.4.177

Applying impulse momentum relation along X-direction

$$\begin{aligned}F_x \Delta t &= \dot{m} \Delta t (v_2 - v_1) \\ F_x &= \dot{m} (v_2 - v_1) = 6 (6 \cos 120 - 6) = -54 \text{ N} \\ \text{Similarly in Y-direction} \\ F_y &= \dot{m} (v_2 - v_1) = 6 (6 \sin 120 - 0) = 31.176 \text{ N} \\ F &= \sqrt{-54^2 + 31.176^2} = 62.35 \text{ N}\end{aligned}$$

b) the vane is moving with velocity of 1 m/s to the right

The relative velocity of water with respect to the vane =  $6 - 1 = 5 \text{ m/s}$

$$\text{Mass flow rate of water hitting the vane } \dot{m} = \rho A V = 1000 \times 1000 \times 10^{-6} \times 5 = 5 \text{ kg/s}$$

Applying impulse momentum relation along X-direction

$$\begin{aligned}F_x \Delta t &= \dot{m} \Delta t (v_2 - v_1) \\ F_x &= \dot{m} (v_2 - v_1) = 5 (5 \cos 120 - 5) = -37.5 \text{ N} \\ \text{Similarly in Y-direction} \\ F_y &= \dot{m} (v_2 - v_1) = 5 (5 \sin 120 - 0) = 21.65 \text{ N} \\ F &= \sqrt{-37.5^2 + 21.65^2} = 43.3 \text{ N}\end{aligned}$$

**SP4.215:** A stream of water  $2700 \text{ mm}^2$  in cross section and moving horizontally with a speed of  $30 \text{ m/s}$  splits into two equal parts against the fixed vane. Assume no friction between the water and the blade, determine the force of water on the blade.

**Solution:**

Mass flow rate of water which hits the vane  
 $\dot{m} = \rho A V = 1000 \times 2700 \times 10^{-6} \times 30 = 81 \text{ kg/s}$

$$F_x = \dot{m} (v_2 - v_1) = 2(81/2) (30 \cos 135 - 30) = -4148.2 \text{ N}$$

Force is acting in the opposite direction of the water flow.

$$F_y = \dot{m} (v_2 - v_1)$$

Acting upward direction

$$= (81/2) (30 \sin 135 - 0) = 859.13 \text{ N}$$

Acting downward direction

$$= (81/2) (30 \sin 225 - 0) = -859.13 \text{ N}$$

+ve sign represents upward direction and -ve direction represents downward direction. Since both the magnitudes are equal, net force = 0

Hence total force = 4148 N

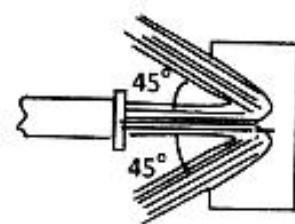


Fig.4.178

## IMPACT:

The principles of impulse and momentum have important use in describing the behaviour of colliding bodies. Impact refers to the collision between two bodies and is characterized by the generation of relatively large contact forces which act over a very short interval of time. It is important to realize that an impact is a very complex event involving material deformation and recovery and the generation of heat and sound. Small changes in the impact conditions may cause large changes in the impact process and thus in the conditions immediately following the impact. Therefore, we must be careful not to rely heavily on the results of impact calculations.

### 1) Direct Central Impact:

As an introduction to impact, we consider the collinear motion of two spheres of masses  $m_1$  and  $m_2$ , fig.4.179a, traveling with velocities  $v_1$  and  $v_2$ . If  $v_1$  is greater than  $v_2$ , collision occurs with the contact forces directed along the line of centres. This condition is called **direct central impact**.

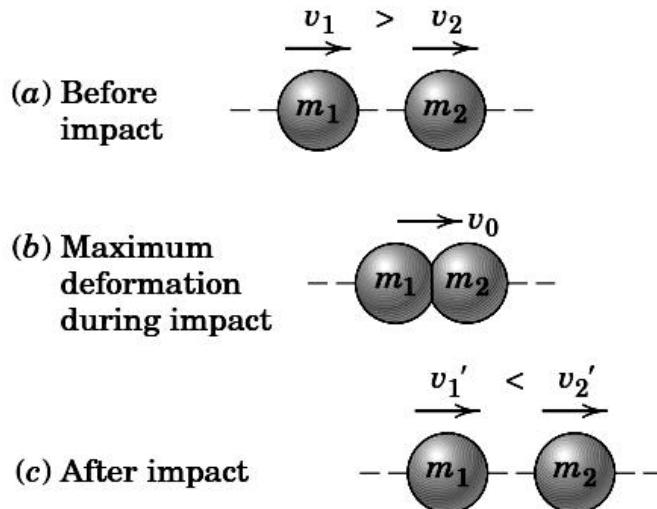


Fig.4.179

Following initial contact, a short period of increasing deformation takes place until the contact area between the spheres ceases to increase. At this instant, both spheres, fig.4.179b, are moving with the same velocity  $v_0$ . During the remainder of contact, a period of restoration occurs during which the contact area decreases to zero. In the final condition shown in part c of the figure, the spheres now have new velocities  $v_1'$  and  $v_2'$ , where  $v_1'$  must be less than  $v_2'$ . All velocities are arbitrarily assumed positive to the right, so that with this scalar notation a velocity to the left would carry a negative sign. If the impact is not overly severe and if the spheres are highly elastic, they will regain their original shape following the restoration. With a more severe impact and with less elastic bodies, a permanent deformation may result.

Because the contact forces are equal and opposite during impact, the linear momentum of the system remains unchanged. Thus, we apply the law of conservation of linear momentum and write

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \quad \longrightarrow (4.91)$$

We assume that any forces acting on the spheres during impact, other than the large internal forces of contact, are relatively small and produce negligible impulses compared with the impulse associated with each internal impact force. In addition, we

assume that no appreciable change in the positions of the mass centres occurs during the short duration of the impact.

### a) Coefficient of Restitution:

For given masses and initial conditions, the momentum equation contains two unknowns,  $v_1'$  and  $v_2'$ . Clearly, we need an additional relationship to find the final velocities. This relationship must reflect the capacity of the contacting bodies to recover from the impact and can be expressed by the ratio  $e$  of the magnitude of the restoration impulse to the magnitude of the deformation impulse. This ratio is called the coefficient of restitution.

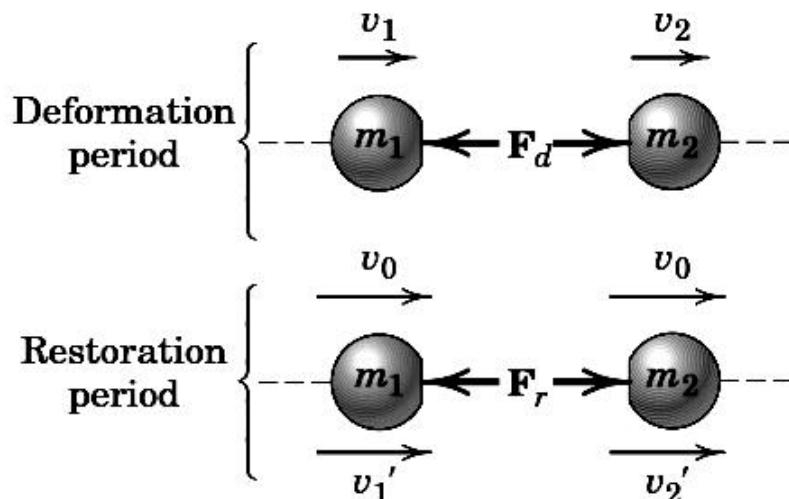


Fig.4.180

Let  $F_r$  and  $F_d$  represent the magnitudes of the contact forces during the restoration and deformation periods, respectively, as shown in fig.4.180. For particle 1 the definition of  $e$  together with the impulse-momentum equation give us

$$e = \frac{\int_{t_0}^t F_r dt}{\int_0^{t_0} F_d dt} = \frac{m_1[-v_1' - (-v_0)]}{m_1[-v_0 - (-v_1)]} = \frac{v_0 - v_1'}{v_1 - v_0}$$

Similarly, for particle 2 we have

$$e = \frac{\int_0^t F_r dt}{\int_0^t F_d dt} = \frac{m_2 [v_2' - (v_0)]}{m_2 [v_0 - (v_2)]} = \frac{v_2' - v_0}{v_0 - v_2}$$

We are careful in these equations to express the change of momentum (and therefore  $v$ ) in the same direction as the impulse (and thus the force). The time for the deformation is taken as  $t_0$  and the total time of contact is  $t$ . Eliminating  $v_0$  between the two expressions for  $e$  gives us

$$e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|} \quad \rightarrow (4.92)$$

If the two initial velocities  $v_1$  and  $v_2$  and the coefficient of restitution  $e$  are known, then Eqs. 4.91 and 4.92 give us two equations in the two unknown final velocities  $v_1'$  and  $v_2'$ .

### b) Energy Loss during Impact

Impact phenomena are almost always accompanied by energy loss, which may be calculated by subtracting the kinetic energy of the system just after impact from that just before impact. Energy is lost through the generation of heat during the localized inelastic deformation of the material, through the generation and dissipation of elastic stress waves within the bodies, and through the generation of sound energy.

According to this classical theory of impact, the value  $e = 1$  means that the capacity of the two particles to recover equals their tendency to deform. This condition is one of elastic impact with no energy loss. The value  $e = 0$ , on the other hand, describes inelastic or plastic impact where the particles cling together after collision and the loss of energy is a maximum. All impact conditions lie somewhere between these two extremes.

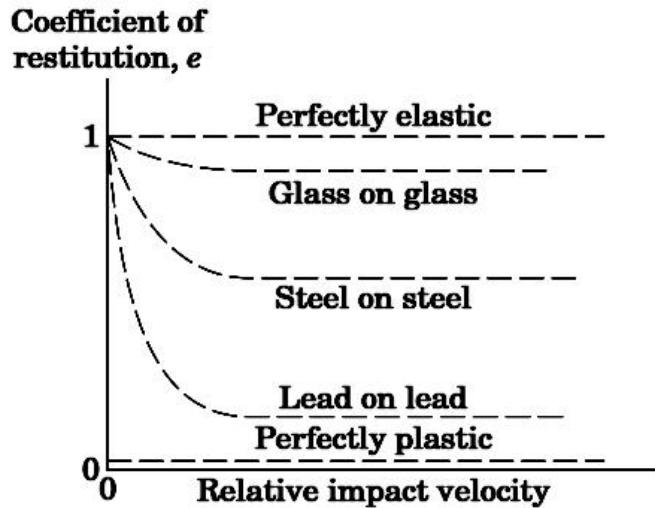


Fig.4.181

Also, it should be noted that a coefficient of restitution must be associated with a pair of contacting bodies. The coefficient of restitution is frequently considered a constant for given geometries and a given combination of contacting materials. Actually, it depends on the impact velocity and approaches unity as the impact velocity approaches zero as shown schematically in fig.4.181. A handbook value for  $e$  is generally unreliable.

## 2) Oblique Central Impact

We now extend the relationships developed for direct central impact to the case where the initial and final velocities are not parallel, fig.4.182. Here spherical particles of mass  $m_1$  and  $m_2$  have initial velocities  $v_1$  and  $v_2$  in the same plane and approach each other on a collision course, as shown in part a of the figure. The directions of the velocity vectors are measured from the direction tangent to the contacting surfaces, fig.4.182b. Thus, the initial velocity components along the  $t$ - and  $n$ -axes are  $(v_1)_n = -v_1 \sin \theta_1$ ,  $(v_1)_t = v_1 \cos \theta_1$ ,  $(v_2)_n = v_2 \sin \theta_2$  and  $(v_2)_t = v_2 \cos \theta_2$ . Note that  $(v_1)_n$  is a negative quantity for the particular coordinate system and initial velocities shown.

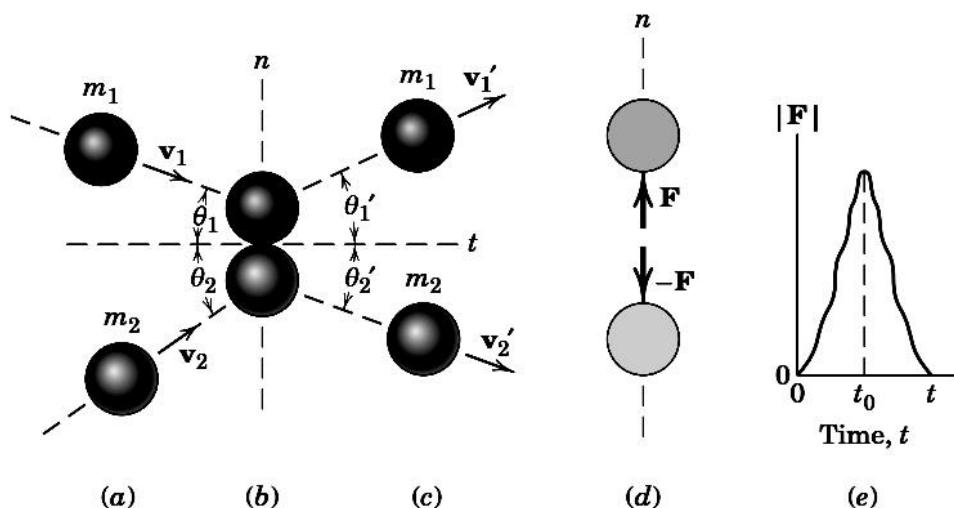


Fig.4.182

The final rebound conditions are shown in part c of the figure. The impact forces are  $F$  and  $-F$ , as seen in part d of the figure. They vary from zero to their peak value during the deformation portion of the impact and back again to zero during the restoration period, as indicated in part e of the figure where  $t$  is the duration of the impact interval.

For given initial conditions of  $m_1$ ,  $m_2$ ,  $(v_1)_n$ ,  $(v_1)_t$ ,  $(v_2)_n$  and  $(v_2)_t$  there will be four unknowns, namely,  $(v_1')_n$ ,  $(v_1')_t$ ,  $(v_2')_n$  and  $(v_2')_t$ . The four needed equations are obtained as follows:

(1) Momentum of the system is conserved in the  $n$ -direction. This gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n$$

(2) and (3) The momentum for each particle is conserved in the  $t$ -direction since there is no impulse on either particle in the  $t$ -direction. Thus,

$$m_1(v_1)_t = m_1(v_1')_t$$

$$m_2(v_2)_t = m_2(v_2')_t$$

(4) The coefficient of restitution, as in the case of direct central impact, is the positive ratio of the recovery impulse to the deformation impulse. Equation 3 /36 applies, then,

to the velocity components in the n-direction. For the notation adopted with fig.4.182, we have

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n}$$

Once the four final velocity components are found, the angles  $\alpha_1'$  and  $\alpha_2'$  of fig.4.182 may be easily determined.

### Problems:

**SP4.216:** The ram of a pile driver has a mass of 800kg and is released from rest 2m above the top of the 2400kg pile. If the ram rebounds to a height of 0.1m after impact with the pile, calculate a) the velocity  $v_p'$  of the pile immediately after impact, b) the coefficient of restitution e and c) the percentage loss of energy due to the impact.

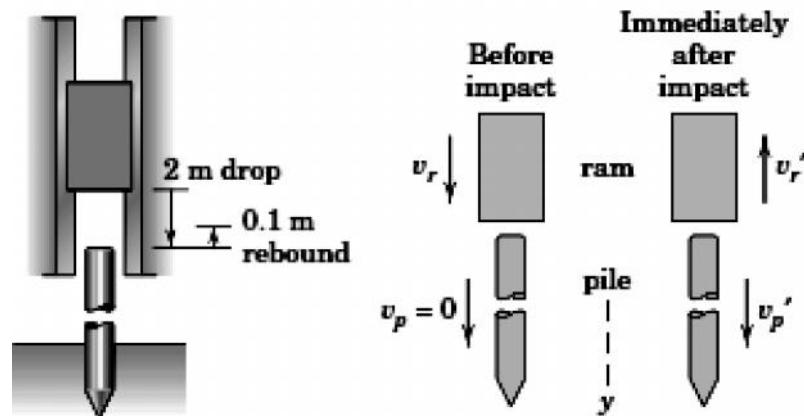


Fig.4.183

### Solution:

Conservation of energy during free fall gives the initial and final velocities of the ram from  $v = \sqrt{2gh}$ . Thus,

$$v_r = \sqrt{2(9.81)(2)} = 6.26 \text{ m/s} \quad v_r' = \sqrt{2(9.81)(0.1)} = 1.40 \text{ m/s}$$

(a) Conservation of momentum ( $G_1 = G_2$ ) for the system of the ram and pile gives

$$800(6.26) + 0 = 800(-1.40) + 2400v_p' \quad v_p' = 2.55 \text{ m/s} \quad \text{Ans.}$$

(b) The coefficient of restitution yields

$$e = \frac{|\text{rel. vel. separation}|}{|\text{rel. vel. approach}|} \quad e = \frac{2.55 + 1.401}{6.26 + 0} = 0.631 \quad \text{Ans.}$$

(c) The kinetic energy of the system just before impact is the same as the potential energy of the ram above the pile and is

$$T = V_g = mgh = 800(9.81)(2) = 15700 \text{ J}$$

The kinetic energy  $T'$  just after impact is

$$T' = \frac{1}{2}(800)(1.401)^2 + \frac{1}{2}(2400)(2.55)^2 = 8620 \text{ J}$$

The percentage loss of energy is, therefore,

$$\frac{15700 - 8620}{15700} (100) = 45.1\% \quad \text{Ans.}$$

**SP4.217:** A ball is projected onto the heavy plate with a velocity of 16m/s at the  $30^\circ$  angle shown. If the effective coefficient of restitution is 0.5, compute the rebound velocity  $v'$  and its angle  $\theta'$ .

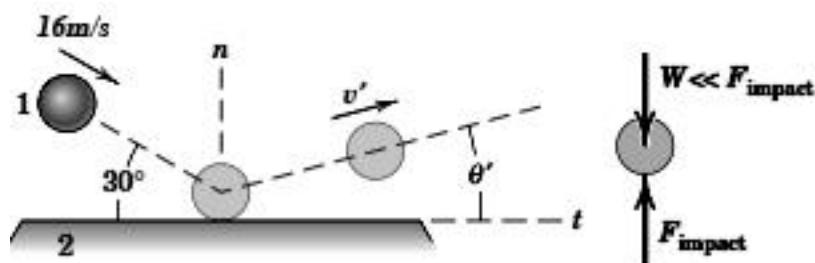


Fig.4.184

**Solution:**

Let the ball be denoted body 1 and the plate body 2. The mass of heavy plate may be considered infinite and its corresponding velocity zero after impact. The coefficient of restitution is applied to the velocity components normal to the plate in the direction of the impact force and gives

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n}, \Rightarrow 0.5 = \frac{0 - (v_1')_n}{-16 \sin 30^\circ - 0}, \Rightarrow (v_1')_n = 4 \text{ m/s}$$

Momentum of the ball in the  $t$ -direction is unchanged since, with assumed smooth surface there is no force acting on the ball in that direction. Thus,

$$m(v_1)_t = m(v_1')_t, \Rightarrow (v_1')_t = (v_1)_t = 16 \cos 30^\circ = 13.86 \text{ m/s}$$

The rebound velocity  $v'$  and its angle  $\theta'$  are then

$$v' = \sqrt{(v_1')_n^2 + (v_1')_t^2} = \sqrt{4^2 + 13.86^2} = 14.42 \text{ m/s}$$

$$\theta' = \tan^{-1} \frac{(v_1')_n}{(v_1')_t} = \tan^{-1} \frac{4}{13.86} = 16.10^\circ$$

**SP4.218:** Spherical particle 1 has a velocity  $v_1 = 6 \text{ m/s}$  in the direction shown and collides with spherical particle 2 of equal mass and diameter and initially at rest. If the coefficient of restitution for these conditions is  $e = 0.6$ , determine the resulting motion of each particle following impact. Also calculate the percentage loss of energy due to the impact.

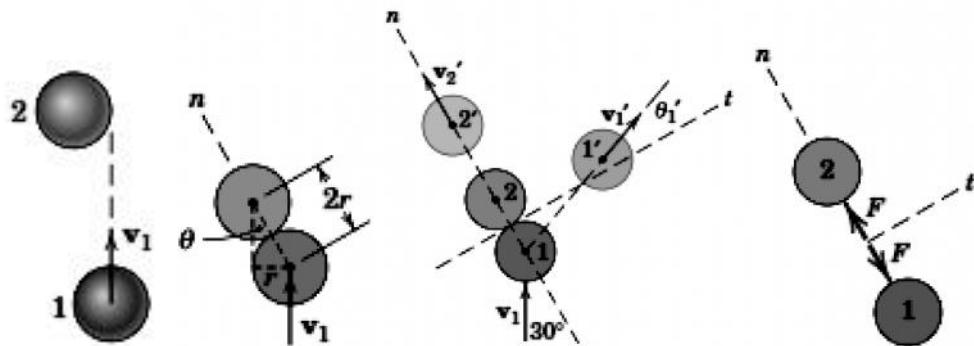


Fig.4.185

**Solution:**

The geometry at impact indicates that the normal  $n$  to the contacting surfaces makes an angle  $\theta = 30^\circ$  with the direction of  $v_1$ , as indicated in the figure. Thus, the initial velocity components are  $(v_1)_n = v_1 \cos 30^\circ = 6 \cos 30^\circ = 5.20 \text{ m/s}$ ,  $(v_1)_t = v_1 \sin 30^\circ = 6 \sin 30^\circ = 3 \text{ m/s}$ , and  $(v_2)_n = (v_2)_t = 0$ .

Momentum conservation for the two-particle system in the  $n$ -direction gives

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n$$

or, with  $m_1 = m_2$ ,

$$5.20 + 0 = (v_1')_n + (v_2')_n \quad (a)$$

The coefficient-of-restitution relationship is

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \quad 0.6 = \frac{(v_2')_n - (v_1')_n}{5.20 - 0} \quad (b)$$

Simultaneous solution of Eqs. *a* and *b* yields

$$(v_1')_n = 1.039 \text{ m/s} \quad (v_2')_n = 4.16 \text{ m/s}$$

Conservation of momentum for each particle holds in the *t*-direction because, with assumed smooth surfaces, there is no force in the *t*-direction. Thus for particles 1 and 2, we have

$$m_1(v_1)_t = m_1(v_1')_t \quad (v_1')_t = (v_1)_t = 3 \text{ m/s}$$

$$m_2(v_2)_t = m_2(v_2')_t \quad (v_2')_t = (v_2)_t = 0$$

The final speeds of the particles are

$$v_1' = \sqrt{(v_1')_n^2 + (v_1')_t^2} = \sqrt{(1.039)^2 + 3^2} = 3.17 \text{ m/s} \quad \text{Ans.}$$

$$v_2' = \sqrt{(v_2')_n^2 + (v_2')_t^2} = \sqrt{(4.16)^2 + 0^2} = 4.16 \text{ m/s} \quad \text{Ans.}$$

The angle  $\theta'$  which  $\mathbf{v}_1'$  makes with the *t*-direction is

$$\theta' = \tan^{-1} \left( \frac{(v_1')_n}{(v_1')_t} \right) = \tan^{-1} \left( \frac{1.039}{3} \right) = 19.11^\circ \quad \text{Ans.}$$

The kinetic energies just before and just after impact, with  $m = m_1 = m_2$ , are

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m(6)^2 + 0 = 18m$$

$$T' = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 = \frac{1}{2}m(3.17)^2 + \frac{1}{2}m(4.16)^2 = 13.68m$$

The percentage energy loss is then

$$\frac{|\Delta E|}{E} (100) = \frac{T - T'}{T} (100) = \frac{18m - 13.68m}{18m} (100) = 24.0\% \quad \text{Ans.}$$

**SP4.219:** A ball of mass 1 kg moving with a velocity of 2m/s impinges directly on a ball of mass 2kg at rest. The first ball, after impinging comes to rest. Find the velocity of the second ball after the impact and the coefficient of restitution.

**Solution:**

Data Given are:

$$m_1 = 1 \text{ kg}; u_1 = 2 \text{ m/s}; m_2 = 2 \text{ kg}; u_2 = 0 \text{ m/s}; v_1 = 0 \text{ m/s};$$

Applying Law of conservation of Momentum:

Momentum before impact = momentum after impact

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$(1 \times 2) + (2 \times 0) = (1 \times 0) + (2 \times v_2)$$

$$2 = 2 v_2; v_2 = 1 \text{ m/s}$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(1 - 0) = e (2 - 0)$$

$$2e = 1; e = 0.5$$

**SP4.220:** A ball overtakes another ball of twice its mass and moving with  $1/7^{\text{th}}$  of its velocity. If the coefficient of restitution between the two balls is 0.75, show that the first ball will come to rest after impact.

**Solution:**

$$m_1 = m \text{ kg}; u_1 = u \text{ m/s}; m_2 = 2m \text{ kg}; u_2 = u/7 \text{ m/s}; e = 0.75$$

Applying Law of conservation of Momentum:

Momentum before impact = momentum after impact

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$(m \times u) + (2m \times u/7) = (m \times v_1) + (2m \times v_2)$$

$$9/7(mu) = m v_1 + 2m v_2$$

Cancelling 'm' throughout, we get

$$(9/7)u = v_1 + 2 v_2 \dots \dots (1)$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$= 0.75 (u - u/7) = 9/14(u)$$

$$v_2 = v_1 + 9/14(u) \dots \dots (2)$$

Substituting eqn (2 in eqn (1) we get

$$(9/7)u = v_1 + 2 (v_1 + 9/14(u))$$

$$(9/7)u = 3v_1 + 9/7(u)$$

$$3v_1 = 0; v_1 = 0$$

**SP4.221:** A sphere of mass 1kg moving at 3m/s overtakes another sphere of mass 5kg moving in the same line at 0.6m/s. Find the loss of kinematic energy during impact and show that the direction of motion of the first sphere is reversed. Take coefficient of restitution as 0.75.

**Solution:**

$$m_1 = 1 \text{ kg}; u_1 = 3 \text{ m/s}; m_2 = 5 \text{ kg};$$

$$u_2 = 0.6 \text{ m/s}; e = 0.75$$

Applying Law of conservation of

Momentum:

Momentum before impact = momentum after impact

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$(1 \times 3) + (5 \times 0.6) = (1 \times v_1) + (5 \times v_2)$$

$$v_1 + 5 v_2 = 6 \dots (1)$$

$v_1 = -0.5 \text{ m/s}$  (-ve sign indicates the direction is reversed)

$$v_2 = 1.3 \text{ m/s}$$

Loss of K.E

Kinetic energy before impact

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$= \frac{1}{2} \times 1 \times 3^2 + \frac{1}{2} \times 5 \times (0.6)^2$$

$$= 4.5 + 0.9 = 5.4 \text{ Nm}$$

$$v_1 + v_2 = v_2 - v_1$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(v_2 - v_1) = 0.75 (3 - 0.6)$$

$$(v_2 - v_1) = 1.8 \dots (2)$$

Solving (1) and (2)

We get

Kinetic energy after impact

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} \times 1 \times (-0.5)^2 + \frac{1}{2} \times 5 \times (1.3)^2$$

$$= 4.35 \text{ Nm}$$

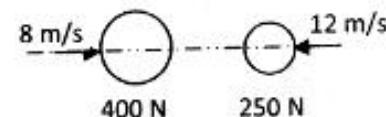
$$\text{Loss of K.E} = 5.4 - 4.35 = 1.05 \text{ Nm}$$

**SP4.222:** Two bodies one of which is 400N with a velocity of 8m/s and the other of 250N with a velocity of 12m/s move towards each other along a straight line and impinge centrally. Find the velocity of each body after impact if the coefficient of restitution is 0.8.

**Solution:**

$$0.8. (-5.846 \text{ m/s}, 10.153 \text{ m/s})$$

$$m_1 = 400/\text{g kg}; u_1 = 8 \text{ m/s}; m_2 = 250/\text{g kg}; u_2 = -12 \text{ m/s};$$



$$e = 0.8$$

Fig.4.186

Applying Law of conservation of Momentum:

Momentum before impact = momentum after impact

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$[(400/\text{g}) \times 8] + [(250/\text{g}) \times (-12)] = [(400/\text{g}) \times v_1] + [(250/\text{g}) \times v_2]$$

Cancelling 'g' throughout

$$3200 - 3000 = 400v_1 + 250v_2$$

$$200 = 400v_1 + 250v_2$$

$$8v_1 + 5v_2 = 4 \dots (1)$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(v_2 - v_1) = 0.8 (8 + 12)$$

$$(v_2 - v_1) = 16 \dots (2)$$

Solving (1) and (2)

We get

$$v_1 = -5.846 \text{ m/s}; v_2 = 10.153 \text{ m/s}$$

(Direction of motion of both the balls reversed after impact)

**SP4.223:** Three spherical balls of mass 2 kg, 6 kg and 12 kg are moving in the same direction with velocity 12 m/s, 4 m/s and 2 m/s respectively. If the ball of mass 2 kg impinges with the ball of mass 6 kg which in turn impinges with the ball of mass 12 kg. Prove that the balls of mass 2 kg and 6 kg will be brought to rest by the impacts. Assume the balls to be perfectly elastic.

**Solution:**

$$m_1 = 2 \text{ kg}; u_1 = 12 \text{ m/s}; m_2 = 6 \text{ kg}; u_2 = 4 \text{ m/s}; \\ m_3 = 12 \text{ kg}; u_3 = 2 \text{ m/s}; e = 1$$

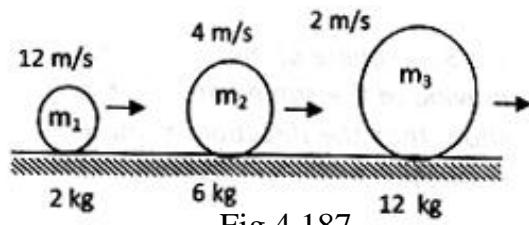


Fig.4.187

Considering the impact between  $m_1$  &  $m_2$ 

Applying Law of conservation of Momentum:

Momentum before impact = momentum after impact

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$2 \times 12 + 6 \times 4 = 2 \times v_1 + 6 \times v_2$$

$$v_1 + 3v_2 = 24 \dots (1)$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(v_2 - v_1) = 1 (12 - 4)$$

$$v_2 - v_1 = 8 \dots (2)$$

Solving (1) and (2) we get

$$v_1 = 0; v_2 = 8 \text{ m/s}$$

Considering the impact between  $m_2$  &  $m_3$ 

Here  $u_2 = 8 \text{ m/s}$

Applying Law of conservation of Momentum:

Momentum before impact = momentum after impact

$$m_2 u_2 + m_3 u_3 = m_2 v_2 + m_3 v_3$$

$$6 \times 8 + 12 \times 2 = 6 \times v_2 + 12 \times v_3$$

$$v_2 + 2v_3 = 12 \dots (3)$$

Applying Newton's law of collision

$$(v_3 - v_2) = e (u_2 - u_3)$$

$$(v_3 - v_2) = 1 (8 - 2)$$

$$v_3 - v_2 = 6 \dots (4)$$

Solving (3) and (4) we get

$$v_2 = 0; v_3 = 6 \text{ m/s}$$

**SP4.224:** Two identical balls each of mass 'm' collide with velocities as shown in fig.4.

Find the final velocities of the balls after the impact. Assume  $e = 0.9$ .

**Solution:**

Along the line of impact :

$$m_1 = m_2 = m; u_1 = 1.6 \text{ m/s};$$

$$u_2 = -3.5 \cos 45 \text{ m/s};$$

momentum before impact = momentum after impact

$$m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 = m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2$$

Cancelling 'm' throughout

$$u_1 \cos \alpha_1 + u_2 \cos \alpha_2 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$1.6 \cos 0 - 3.5 \cos 45 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$1.6 - 2.4748 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$v_1 \cos \theta_1 + v_2 \cos \theta_2 = -0.8748 \dots (1)$$

Applying Newton's law of collision

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = e (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

$$= 0.9(1.6 + 3.5 \cos 45)$$

$$= 0.9(1.6 + 2.4748)$$

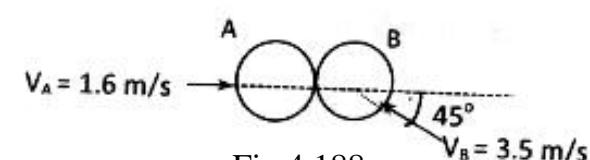


Fig.4.188

Along normal to the line of impact:

$$u_1 \sin \alpha_1 = v_1 \sin \theta_1;$$

$$0 = v_1 \sin \theta_1 \dots (3)$$

$$u_2 \sin \alpha_2 = v_2 \sin \theta_2;$$

$$3.5 \sin 45 = v_2 \sin \theta_2$$

$$2.4748 = v_2 \sin \theta_2 \dots (4)$$

$$v_1 = \sqrt{v_1 \sin \theta_1^2 + v_1 \cos \theta_1^2}$$

$$= 0.9(1.6 + 2.474)$$

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = 3.6673 \dots (2)$$

Adding (1) and (2)

We get

$$2 v_2 \cos \theta_2 = 2.798$$

$$v_2 \cos \theta_2 = 1.399$$

$$v_1 \cos \theta_1 = -2.268$$

$$v_1 = \sqrt{v_1^2 + v_2^2}$$

$$= \sqrt{0^2 + (-2.268)^2} = -2.268 \text{ m/s}$$

$$\tan \theta_1 = 0 / -2.268 = 0$$

$$\theta_1 = 0$$

$$v_2 = \sqrt{v_2 \sin \theta_2^2 + v_2 \cos \theta_2^2}$$

$$= \sqrt{2.4748^2 + 1.399^2} = 2.842 \text{ m/s}$$

$$\tan \theta_2 = 2.4748 / 1.399 = 1.7689$$

$$\theta_2 = 60.5^\circ$$

**SP4.225:** Two identical frictionless balls strike each other as shown in fig.4. Assuming  $e=0.9$  determine the magnitude and direction of the velocity of each ball after impact.

**Solution:**

Consider ball A = 1; B = 2

There fore  $u_1 = 10 \text{ m/s}$ ;  $u_2 = 15 \text{ m/s}$

Let us assume that the direction of the velocity is +ve when it is acting in the direction from left to right and is -ve when it acts in the direction from right to left.

Applying Law of conservation of Momentum along the line of impact

$$m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 = m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2$$

Since both the balls are identical mass remains same. Cancelling the mass 'm' throughout

$$u_1 \cos \alpha_1 + u_2 \cos \alpha_2 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$10 \cos 30 - 15 \cos 60 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$8.66 - 7.5 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$1.16 = v_1 \cos \theta_1 + v_2 \cos \theta_2 \dots (1)$$

Applying Newton's law of collision

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = e (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = e (8.66 - (-7.5))$$

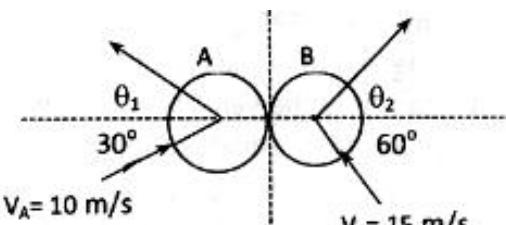


Fig.4.189

$$= 0.9 \times 16.16 = 14.544$$

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = 14.544 \dots (2)$$

Adding (1) and (2) we get

$$2 v_2 \cos \theta_2 = 15.704$$

$$v_2 \cos \theta_2 = 7.852$$

By substituting this in equation (2) we get

$$v_1 \cos \theta_1 = -6.692$$

Along the normal direction to the line of impact

$$u_1 \sin \alpha_1 = v_1 \sin \theta_1$$

$$10 \sin 30 = v_1 \sin \theta_1 = 5$$

$$U_2 \sin \alpha_2 = v_2 \sin \theta_2$$

$$15 \sin 60 = v_2 \sin \theta_2 = 12.99$$

$$\text{Therefore } V_1 = \sqrt{v_1 \sin \theta_1^2 + v_1 \cos \theta_1^2}$$

$$v_1 = \sqrt{5^2 + (-6.692)^2} = 8.35 \text{ m/s}$$

$$\tan \theta_1 = 5 / (-6.692) = -0.7471$$

$$\theta_1 = -36.76^\circ$$

$$V_2 = \sqrt{v_2 \sin \theta_2^2 + v_2 \cos \theta_2^2}$$

$$= \sqrt{12.99^2 + 7.857^2} = 15.178 \text{ m/s}$$

$$\tan \theta_2 = 12.99 / 7.852 = 1.6543$$

$$\theta_2 = 58.84^\circ$$

**SP4.226:** A smooth spherical ball A of mass 120 gram is moving from left to right with a velocity of 2m/s in a horizontal plane. Another identical ball B travelling in a perpendicular direction with a velocity 6 m/s collides with A in such a way that the line of impact is in the direction of motion of ball B. Assuming  $e=0.8$ , determine the velocity of the balls A and B after the impact.

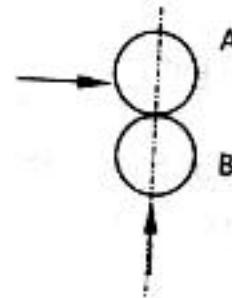


Fig.4.190

**Solution:**

Consider ball A = 1; B = 2

$$m_1 = m_2 = m = 120 \text{ g} ; u_1 = 2 \text{ m/s} ; u_2 = 6 \text{ m/s} ;$$

Here the line of impact is vertical. Hence  $\alpha_1 = 90^\circ$ ;  $\alpha_2 = 0^\circ$

Let us assume the velocity is +ve when direction is upward

Applying Law of conservation of Momentum along the line of impact

$$m_1 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 = m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2$$

Since both the balls are identical, mass 'm' remains same

Cancelling the mass 'm' throughout

$$u_1 \cos \alpha_1 + u_2 \cos \alpha_2 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$2 \cos 90 + 6 \cos 0 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$0 + 6 = v_1 \cos \theta_1 + v_2 \cos \theta_2$$

$$v_1 \cos \theta_1 + v_2 \cos \theta_2 = 6 \dots (1)$$

Along the normal direction to the line of impact

$$u_1 \sin \alpha_1 = v_1 \sin \theta_1$$

$$2 \sin 90 = v_1 \sin \theta_1 = 2$$

$$U_2 \sin \alpha_2 = v_2 \sin \theta_2$$

$$6 \sin 0 = v_2 \sin \theta_2 = 0$$

Applying Newton's law of collision

$$v_2 \cos \theta_2 - v_1 \cos \theta_1$$

$$= e (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = 0.8 (0 - 6) = - 4.8$$

$$v_2 \cos \theta_2 - v_1 \cos \theta_1 = - 4.8 \dots (2)$$

Solving (1) and (2) We get

$$v_2 \cos \theta_2 = 0.6$$

$$v_1 \cos \theta_1 = 5.4$$

$$\text{Therefore } V_1 = \sqrt{v_1 \sin \theta_1^2 + v_1 \cos \theta_1^2}$$

$$v_1 = \sqrt{2^2 + 5.4^2} = 5.75 \text{ m/s}$$

$$\tan \theta_1 = 2/5.4 = 0.37$$

$$\theta_1 = 20.32^\circ$$

$$V_2 = \sqrt{v_2 \sin \theta_2^2 + v_2 \cos \theta_2^2}$$

$$= \sqrt{0^2 + 0.6^2} = 0$$

$$\tan \theta_2 = 0/0.6 = 0$$

$$\theta_2 = 0$$

**SP4.227:** From a point on a smooth floor of a room, a toy ball is shot to hit a wall by rolling on the floor. The ball then returns back to the point of projection. If the time taken by the ball in returning is twice the time taken in reaching the wall, find the coefficient of restitution between the ball and wall.

**Solution:**

Let 's' be the distance between the point of projection and the wall

't' be time taken by the ball in reaching the wall.

Therefore time taken by the ball in returning to the point of projection is  $2t$ .

Let 'e' – coefficient of restitution

Velocity with which the ball will hit the wall

$$U = \text{distance / time} = s/t$$

And the velocity with which the ball will return after hitting

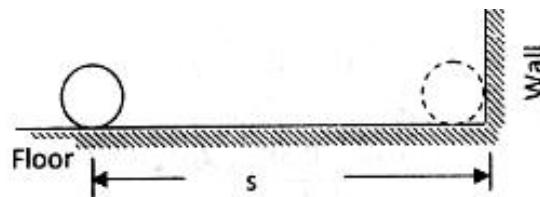


Fig.4.191

$$V = \text{distance / time} = s / 2t$$

Applying Newton's law of collision

$$V = e U$$

$$(s/2t) = e (s/t)$$

$$e = 0.5$$

**SP4.228:** A ball is dropped from a height  $h_0 = 1 \text{ m}$  on a smooth floor.

Knowing that the height of the first bounce is  $h_1 = 0.81 \text{ m}$ , determine

- the coefficient of restitution and b) the expected height of the second bounce.

**Solution:**

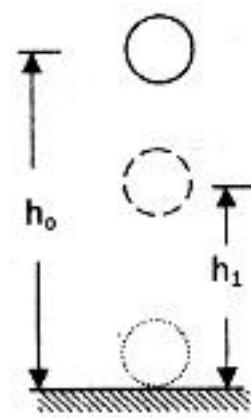


Fig.4.192

The velocity with which the ball impinges on the floor,

$$u = \sqrt{2gh_o} = \sqrt{2g \times 1} = \sqrt{2g} \text{ m/s}$$

The velocity with which the ball rebounds

$$V = \sqrt{2gh_1} = \sqrt{2g \times 0.81} = \sqrt{1.62g}$$

$$V = e \cdot u$$

$$\sqrt{1.62g} = e \sqrt{2g}$$

$$e = \sqrt{1.62g} / \sqrt{2g} = 0.9$$

For the second bounce

$$u = \sqrt{1.62g}$$

$$V = \sqrt{2gh_2}$$

$$V = e \cdot u$$

$$\sqrt{2gh_2} = e \sqrt{1.62g}$$

Squaring both sides

$$2gh_2 = e^2 \times 1.62 \times g$$

$$h_2 = e^2 \times 0.81$$

$$= 0.9^2 \times 0.81 = 0.656 \text{ m}$$

**SP4.229:** From what height must a heavy elastic ball be dropped on a floor so that after rebounding thrice, it will reach a height of 16m. Take  $e = (0.5)^{1/3}$ .

**Solution:**

Let  $h_o$  be the initial height before it is dropped

$h_1$  be the height of the ball reached after 1<sup>st</sup> bounce

$h_2, h_3$  be height reached by the ball respectively after 2<sup>nd</sup> and 3<sup>rd</sup> bounce.

Considering the first bounce

$$\text{Velocity before impact } u = \sqrt{2gh_o}$$

$$\text{Velocity after impact } v = \sqrt{2gh_1}$$

Applying Newton's law of collision

$$v = e \cdot u$$

$$\sqrt{2gh_1} = (0.5)^{1/3} \sqrt{2gh_o} \quad \dots \dots (1)$$

Similarly during second bounce

$$\sqrt{2gh_2} = (0.5)^{1/3} \sqrt{2gh_1}$$

$$\sqrt{2gh_2} = (0.5)^{1/3} (0.5)^{1/3} \sqrt{2gh_o} = (0.5)^{2/3} \sqrt{2gh_o}$$

Similarly during third bounce

$$\sqrt{2gh_3} = (0.5)^{1/3} \sqrt{2gh_2}$$

$$= (0.5)^{1/3} (0.5)^{2/3} \sqrt{2gh_o} = (0.5)^{3/3} \sqrt{2gh_o}$$

$$= 0.5 \sqrt{2gh_o}$$

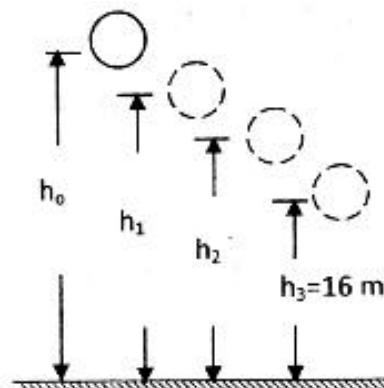


Fig.4.193

$$\sqrt{2gh_3} = 0.5 \sqrt{2gh_o}$$

$$\sqrt{2g \times 16} = 0.5 \sqrt{2gh_o}$$

$$4\sqrt{2g} = 0.5 \sqrt{2gh_o}$$

$$\sqrt{2gh_o} = 8\sqrt{2g}$$

$$\sqrt{h_o} = 8$$

$$h_o = 64 \text{ m}$$

**SP4.230:** A ball moving with a velocity of 4m/s, impinges on a fixed plane at an angle of  $30^\circ$ . If the coefficient of restitution is 0.5, find a) direction of the ball after impact and b) velocity of the ball after impact.

**Solution:**

The angle between the initial velocity and the line of impact ' $\alpha$ ' =  $90 - 30 = 60^\circ$

Applying Newton's law of collision along the line of impact  $v \cos \theta = e u \cos \alpha$

$$v \cos \theta = 0.5 \times 4 \cos 60$$

$$v \cos \theta = 1$$

In the perpendicular direction of the line of impact

$$u \sin \alpha = v \sin \theta$$

$$4 \sin 60 = v \sin \theta = 3.4641$$

$$v = \sqrt{(v \sin \theta)^2 + (v \cos \theta)^2}$$

$$v = \sqrt{3.4641^2 + (1)^2} = 3.605 \text{ m/s}$$

$$\tan \theta = v \sin \theta / v \cos \theta = 3.464 / 1 = 3.464$$

$$\theta = 73.89^\circ$$

$$\text{Direction of the ball with the plane after impact} = 90 - 73.89 = 16.1^\circ$$

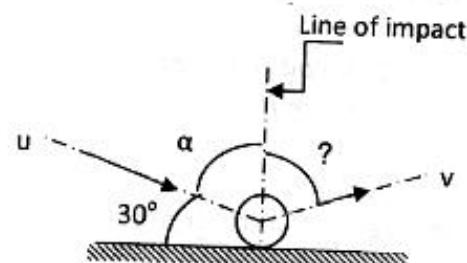


Fig.4.194

**SP4.231:** A spherical ball A of mass 'm' when released from rest slides down the surface of a smooth bowl and strikes another spherical ball of mass  $m/4$  resting at the bottom of the bowl. Determine the height 'h' from which the ball A should be released so that after the impact the ball B just leaves the bowl. The coefficient of restitution may be assumed to be 0.8.

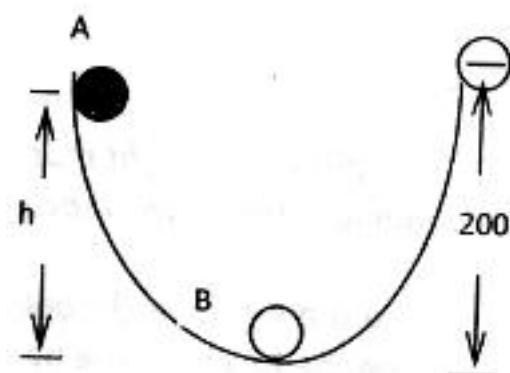


Fig.4.195

**Solution:**

The ball 'A' slides down a height 'h' along a frictionless surface from position 'A' to position 'B'. The velocity  $u_{A1}$  of the ball A when it strikes the ball B can be determined using the principle of conservation of energy.

$$mgh = \frac{1}{2} m u_{A1}^2$$

$$u_{A1} = \sqrt{2gh}$$

Hence the velocity of the ball 'A' before

$$\sqrt{2gh} - 0.495 = v_{A1} \dots (1)$$

Applying Newton's law of collision

$$\text{impact, } u_{A1} = \sqrt{2gh}$$

The ball 'B' just leaves the bowl at a height of 200 mm. Hence the velocity of the ball 'B' after impact  $v_{B2} = \sqrt{2g \times 0.2} = 1.98 \text{ m/s}$

Applying Law of conservation of Momentum:

$$m_{A1} u_{A1} + m_{B2} u_{B2} = m_{A1} v_{A1} + m_{B2} v_{B2}$$

$$m \times \sqrt{2gh} + 0 = m v_{A1} + (m/4) 1.98$$

Cancelling m throughout

$$(v_{B2} - v_{A1}) = e (u_{A1} - u_{B2})$$

$$(1.98 - v_{A1}) = 0.8 (\sqrt{2gh} - 0)$$

$$(1.98 - v_{A1}) = 0.8 \sqrt{2gh}$$

Substituting eqn(1)

$$1.98 - \sqrt{2gh} + 0.495 = 0.8 \sqrt{2gh}$$

$$2.475 = 1.8 \sqrt{2gh}$$

$$\sqrt{2gh} = 1.375$$

$$h = 0.0963 \text{ m}$$

**SP4.232:** Two equal balls are in contact on a table; a third equal ball strikes them simultaneously and remains at rest after the impact. Show that the coefficient of restitution is 2/3.

**Solution:**

Let us assume that the ball 'c' hits the balls A and B simultaneously. Applying the law of conservation of momentum in the direction of 'C' (considering the ball 'C' as mass '1' and balls A & B as mass '2') we can write;

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$\text{Here } m_1 = m; m_2 = (m+m); u_1 = u_1; u_2 = 0; v_1 = 0; v_2 = v_2 \cos 30$$

$$m u_1 + (m+m)0 = m_1 0 + (m+m) v_2 \cos 30$$

Cancelling 'm' throughout

$$u_1 = 2 v_2 \cos 30$$

$$u_1 = 1.732 v_2 \dots (1)$$

Considering the impact between balls C and A

Applying Newton's law of collision along the line of impact of C & A

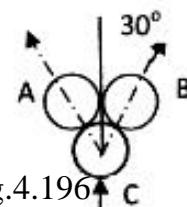


Fig.4.196

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(v_2 - 0) = e (u_1 \cos 30 - 0)$$

$$v_2 = e u_1 \cos 30$$

$$v_2 = e 0.866 u_1$$

$$\text{Substituting } u_1 = 1.732 v_2$$

$$v_2 = e 0.866 \times 1.732 v_2$$

$$v_2 = e 1.5 v_2$$

$$e = 1/1.5 = 0.66$$

**SP4.233:** A 1 kg ball traverses a frictionless tube shown in fig.4 falling through a height of 2m. It then strikes a 2 kg ball hung from a rope 1.2 m long. Determine the height to which the hanging ball will rise if the coefficient of restitution is 0.7.

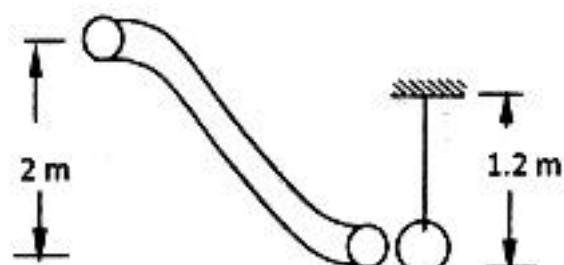


Fig.4.197

**Solution:**

The velocity of 1 kg ball after falling a height of 2m =  $\sqrt{2gh} = \sqrt{2g \times 2} = 6.264 \text{ m/s}$

Let  $u_1 = 6.264 \text{ m/s}$ ;  $u_2 = 0$

Applying the law of conservation of momentum

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$1 \times 6.264 + 2 \times 0 = 1 \times v_1 + 2 \times v_2$$

$$6.264 = v_1 + 2 v_2 \dots (1)$$

Applying Newton's law of collision

$$(v_2 - v_1) = e (u_1 - u_2)$$

$$(v_2 - v_1) = 0.7 (6.264 - 0)$$

$$(v_2 - v_1) = 0.7 (6.264 - 0) = 4.382$$

$$(v_2 - v_1) = 4.382 \dots (2)$$

Solving the equations (1) and (2)

We get

$$v_2 = 3.548 \text{ m/s}; v_1 = -0.833 \text{ m/s}$$

Let 'h' be the height to which hanging ball will rise after impact. Applying work energy equation

Change in K.E = Work done

Final K.E - initial K.E =  $-mg h$

$$0 - \frac{1}{2} mv^2 = -mgh \text{ (-ve sign for moving up)}$$

$$0 - 0.5 \times 2 \times 3.548^2 = -2 \times 9.81 \times h$$

$$h = 0.6416 \text{ m}$$

**SP4.234:** A ball is dropped from a height of ' $h_0$ ' = 1.2 m on a smooth floor as shown in fig.4. Knowing that the first bounce  $h_1$  = 1m and  $D_1$  = 0.4m, determine a) the coefficient of restitution and b) the height and the range of the second bounce.

**Solution:**

Line of impact being vertical, horizontal component of velocity is not affected by impact.

The ball is dropped from a height of  $h_0 = 1.2 \text{ m}$

Hence vertical component of velocity before first impact  $u_{1y} = \sqrt{2gh_0}$  downward

Let the vertical component of velocity after first impact be  $v_{1y}$

Since after first bounce, the ball has raised to  $h_1 = 1 \text{ m}$

$$v_{1y} = \sqrt{2gh_1} \text{ upward}$$

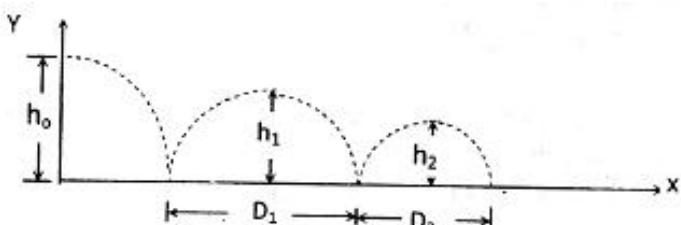


Fig.4.198

As per Newton's law of collision

$$v_{1y} = e u_{1y}$$

$$e = v_{1y} / u_{1y} = \frac{\sqrt{2gh_1}}{\sqrt{2gh_0}} = \sqrt{h_1/h_0} \sqrt{1/1.2} = 0.9128$$

$$\text{Time of flight } t_1 \text{ in first bounce} = 2 \sqrt{2gh_1}/g = 2 \sqrt{2 \times 9.81 \times 1}/9.81 = 0.903 \text{ s}$$

$$\text{Range, } D_1 = u_x \times t_1$$

$$0.4 = u_x \times 0.903$$

$$u_x = 0.44296 \text{ m/s}$$

Vertical component of velocity after second bounce

$$V_{2y} = e\sqrt{2gh_1}$$
$$= 0.9128 \sqrt{2 \times 9.81 \times 1} = 4 \text{ m/s}$$

$$h_2 = V_{2y}^2/2g = 4^2/(2 \times 9.81) = 0.8155 \text{ m}$$

Time of flight  $t_2$  in second bounce

$$t_2 = 2 \cdot v_{2y} / g = 2 \times 4 / 9.81 = 0.8155 \text{ s}$$

$$\text{Range, } D_2 = u_x \times t_2 = 0.44296 \times 0.8155 = 0.361$$

**LECTURE NOTES**

**UNIT – 5**

**2021**

# **CEES303-Engineering Mechanics**



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**Dec-2021**

## Unit - 5

Introduction to Kinetics of Rigid Bodies covering, Basic terms, general principles in dynamics; Types of motion, Instantaneous centre of rotation in plane motion and simple problems; D'Alembert's principle and its applications in plane motion and connected bodies; Work energy principle and its application in plane motion of connected bodies; Kinetics of rigid body rotation

Mechanical Vibrations covering, Basic terminology, free and forced vibrations, resonance and its effects; Degree of freedom; Derivation for frequency and amplitude of free vibrations without damping and single degree of freedom system, simple problems, types of pendulum, use of simple, compound and torsion pendulums;

### Reference for the preparation of course material:

1. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.
3. J.L. Meriam and L. G. Kraige, Engineering Mechanics –Statics, volume I, Seventh edition, John Wiley & Sons, Inc.
4. S.S.Bhavikatti and K.G. Rajashekharappa, Engineering Mechanics (Third Edition), New Age International (P) Limited, Publishers.

### TEXT BOOKS

1. Irving H. Shames (2006), Engineering Mechanics, 4th Edition, Prentice Hall
2. F. P. Beer and E. R. Johnston (2011), Vector Mechanics for Engineers, Vol I - Statics, Vol II, – Dynamics, 9th Ed, Tata McGraw Hill

### REFERENCES

1. R. C. Hibbler (2006), Engineering Mechanics: Principles of Statics and Dynamics, Pearson Press.
2. Khurmi R.S. (2010), Engineering Mechanics, S. Chand & Co.

**Unit - 5****Dynamics of Rigid Bodies****BASIC TERMS:****1) Motion:**

A body is said to be in motion if it is changing its position with respect to a reference point. A person on a scooter is in motion when referred to the road but is at rest when referred to the scooter itself. In most problems, the reference point is implied. For all engineering problems, any fixed point on the earth is an implied reference point. Other examples of implied reference points are centre of the earth for the study of satellite motion, centre of the sun for the study of motion of the solar system and mass centre of the solar system for the study of interplanetary motion. In this only engineering problems have been taken up and hence any fixed point on the earth is an implied reference point while considering motion of the body.

**2) Distance and displacement:**

Let a body move along the path as shown in fig.5.1 by hatched line and the time taken by the body to move from A to B be  $t$ , then the distance moved by the body in time  $t$  is the distance measured along the hatched line. Distance is a scalar quantity since it has only magnitude.

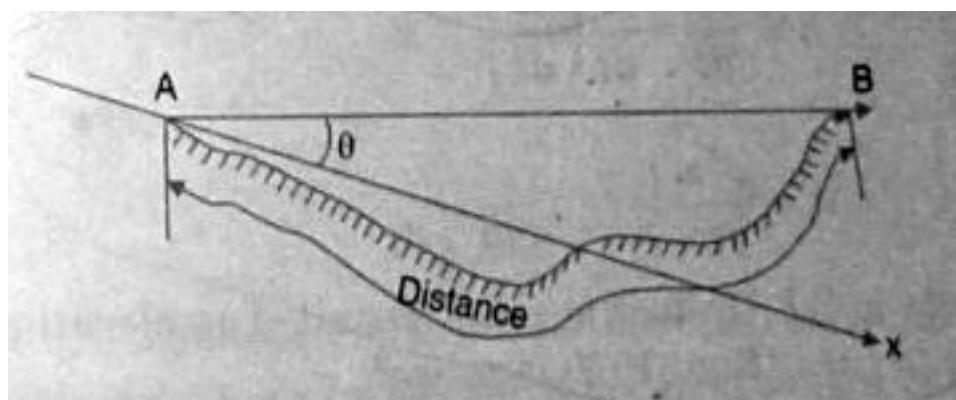


Fig.5.1 Distance

Displacement in this time interval is the linear distance AB which makes angle  $\theta$  with the axis. Displacement of the body in a time interval may be defined as the linear distance between the two positions of the body in the beginning and at the end of the time interval. Since, displacement has the magnitude and direction, it is a vector quantity.

### 3) Speed and Velocity:

The rate of change of distance with respect to time is defined as speed whereas the rate of change of displacement with respect to time is called velocity. Speed has the magnitude only, whereas velocity has both magnitude and direction. Hence speed is a scalar quantity while velocity is a vector quantity.

If 's' is the displacement in interval 't', the average velocity 'v' is given by  $\frac{s}{t}$ .

Velocity of a particle at a given instant is called instantaneous velocity and is given by the limiting value of the ratio  $\frac{s}{t}$  at time 't' when both s and t are very small. Let  $\delta s$  be the small displacement in a small time  $\delta t$ . the instantaneous velocity is given by the expression

$$v = \lim_{\delta t \rightarrow 0} \frac{\delta s}{\delta t} = \frac{ds}{dt} \quad \text{-----} \rightarrow (5.1)$$

In SI units, metre is the unit for displacement and second is the unit of time. Hence from equation 5.1 it may be observed that unit of velocity is m/s. Many a times kilometre per hour (kmph) is also used as unit of velocity. The relation between the two units is given by:

$$1 \text{ kmph} = \frac{1 \times 1000 \text{ m}}{60 \times 60 \text{ s}} = \frac{5}{18} \text{ m/s}$$

### 4) Acceleration and retardation:

Rate of change of velocity with respect to time is called acceleration. Hence mathematically acceleration a is given by

$$a = \frac{dv}{dt} \quad \longrightarrow \rightarrow (5.2)$$

The acceleration may be positive or negative. The positive acceleration is simply referred as acceleration and the negative acceleration is called as retardation or deceleration.

From equations 5.1 and 5.2, we get

$$a = \frac{dv}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \right) = \frac{d^2 s}{dt^2} \quad \longrightarrow \rightarrow (5.3)$$

The unit of velocity in SI units is m/s. The unit of acceleration which is rate of change of velocity with respect to time (Eqn.5.2) is m/s<sup>2</sup>.

## GENERAL PRINCIPLES IN DYNAMICS:

Study of dynamics needs not only the devices to measure the force and length but also time. Lack of availability of accurate time measuring devices delayed the development of dynamics as compared to that of statics. Galileo (1564 – 1642) made useful observations which led to development of dynamics. Sir Isaac Newton (1642-1727) generalized Galileo's observations and came out with experimental proof for what are now known as Newton's first law and second law.

Newton's first law has been stated in unit -1 as everybody continues in its state of rest or of uniform motion, unless it is acted by some external agency. In other words, if a body is subjected to a balanced system of force, it will continue to be in the state of rest or of uniform motion. Thus,

A body acted by a balanced system of forces has no acceleration.

In unit- 1 and 4, Newton's second law has been stated as the rate of change of momentum is directly proportional to the impressed force, and takes in the direction, in which the force acts. This statement and the definition of unit of a force lead to the famous equation.

Force = mass x acceleration

$$F = ma \quad \text{-----} \rightarrow (5.4)$$

Hence many a times engineers state Newton's second law as when an unbalanced system forces acts on a particle, the particle moves with an acceleration proportional to the resultant force and it takes place in the direction of the resultant force.

It may be observed that the Newton's first law is only a particular case of the second law. That is when the resultant force is zero, the acceleration also will be zero.

Newton's third law states that for every action there is an equal and opposite reaction. This has been frequently used in statics for drawing free body diagrams and will be used in dynamics also while treating kinetics problems.

Newton's law of gravitation is another important principle in dynamics. This law states that any two particles of masses  $m_1$  and  $m_2$  separated by distance  $r$ , attract each other with a force directly proportional to their masses and inversely proportional to square of their distance. Thus the force  $F$  in fig.5.2 is given by

$$F \propto \frac{m_1 m_2}{r^2}$$

$$F = G \frac{m_1 m_2}{r^2} \quad \text{-----} \rightarrow (5.5)$$

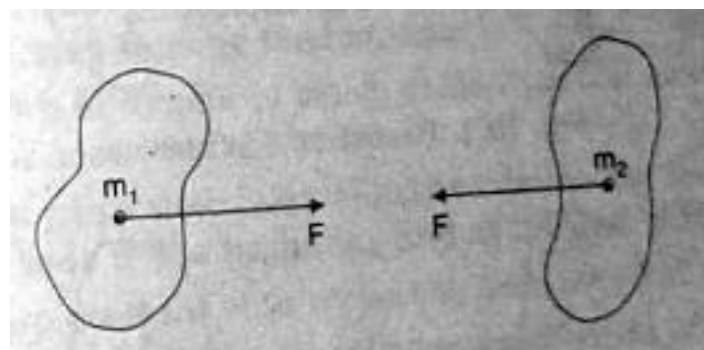


Fig.5.2

Where,  $G$  is constant of proportionality and is called universal constant of gravitation.

The force of attraction on a body by the earth is called weight of the body. Let  $m$  be the mass of the body and  $W$  its weight. Then substituting in equation 5.5.

$$F = W$$

$$m_1 = m$$

$$m_2 = M$$

$$r = R$$

Where, M is mass of the earth and R is its radius, we get

$$W = \frac{mMG}{R^2} = mg \quad \longrightarrow \rightarrow (5.6)$$

Where,  $g = \frac{MG}{R^2}$   $\longrightarrow \rightarrow (5.7)$

The radius of earth along polar axis is 63, 56400 m and along equatorial axis is 63, 28375 m. Hence weight of a body varies from place to place. It is slightly more when the body is near the pole compared to when it is at equator. However, the variation is small. Hence the earth radius R is considered to be the same in all directions. The variation in distances of the bodies considered in engineering problems is very small compared to radius of the earth. Hence it is sufficiently accurate in most engineering computations to assume value of g, and hence that of weight of a body, as constant. Comparing equation 5.6 with equation 5.4 we can call g as acceleration due to gravity and its value is found to be  $9.81 \text{ m/s}^2$ . The weight W is the force always directed towards centre of the earth, i.e. in vertically downward direction.

### **TYPES OF MOTION:**

A body may move in any direction in space. In this unit, motion in a single plane only is considered. This type of motion is called plane motion. Plane motion may be classified as

- a) Translation
- b) Rotation
- c) General plane motion

### **a) Translation:**

A motion is said to be translation, if a straight line drawn on the moving body remains parallel to its original position at any time. During translation if the path traced by a point is straight line it is called rectilinear translation as shown in fig 5.3 and if the path is a curve one it is called curvilinear translation.

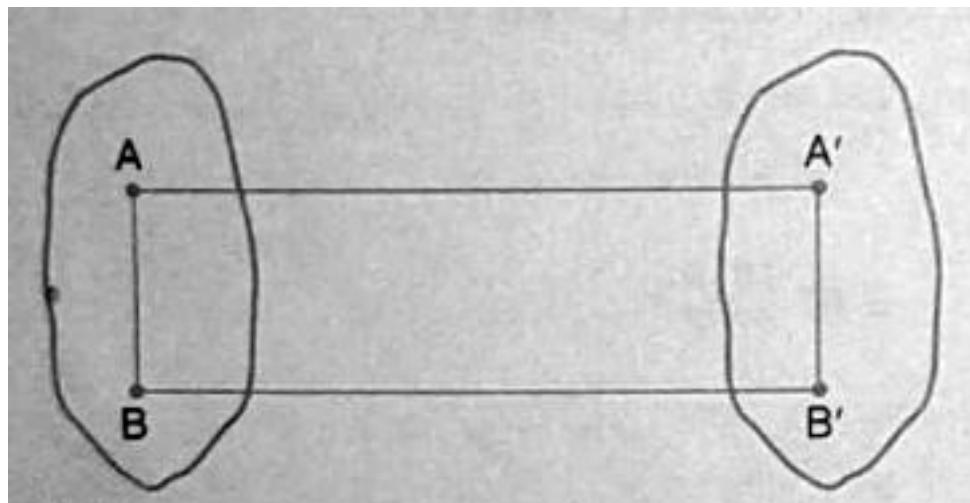


Fig.5.3 Rectilinear translation

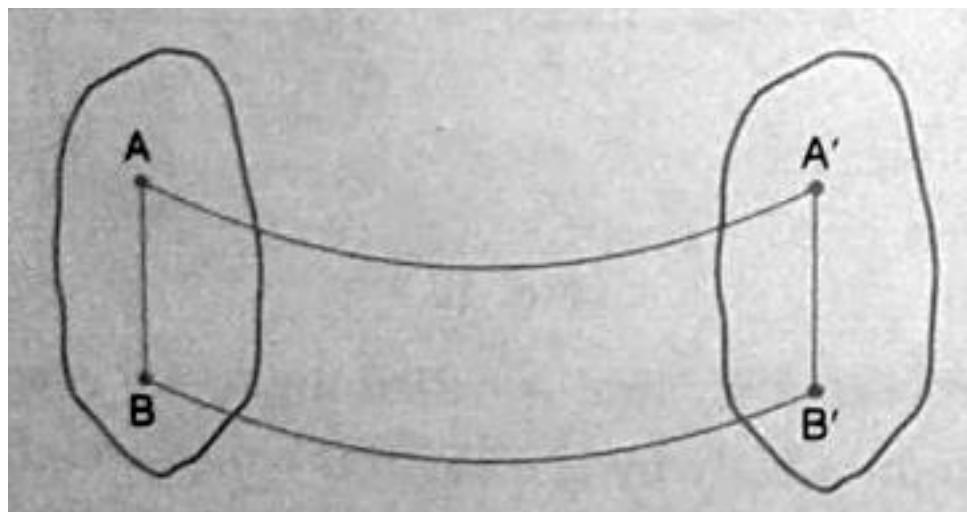


Fig.5.4 Curvilinear translation

In the study of the motion of a particles, rectilinear translation and curvilinear translation are usually referred as linear motion and curvilinear motion.

### b) Rotation:

A motion is said to be rotation if all particles of a rigid body move in a concentric circle as shown in fig.5.5

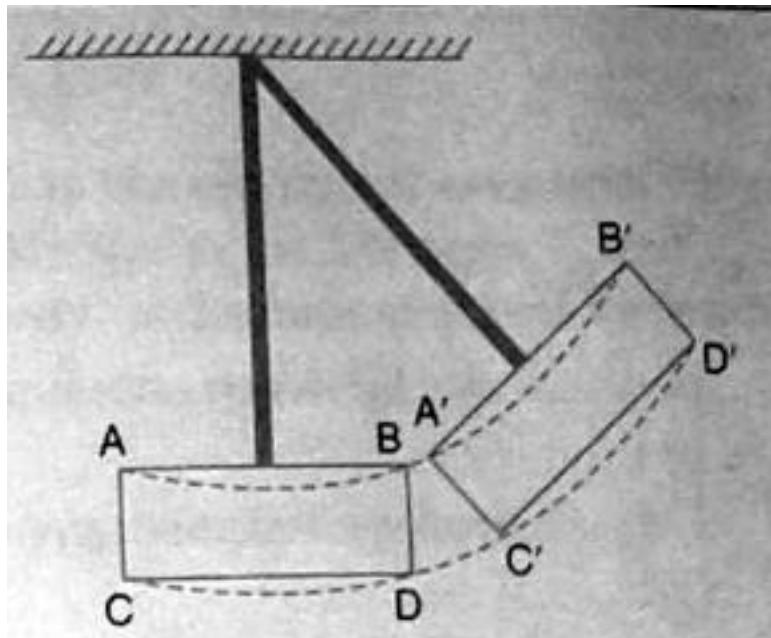


Fig.5.5 Rotation of a rigid body

### c) General plane motion:

The general plane motion is a combination of both translation and rotation. Common examples of such motion are points on wheels of moving vehicles, ladder sliding down from its position against wall etc.

### INSTANTANEOUS CENTRE OF ROTATION IN PLANE MOTION:

In the previous sections, the velocity of a point on a rigid body in plane motion by adding the relative velocity due to rotation about a convenient reference point to the velocity of the reference point. Now solve the problem by choosing a unique reference point which momentarily has zero velocity. As far as velocities are concerned, the body may be considered to be in pure rotation about an axis, normal to the plane of motion, passing through this point. This axis is called the instantaneous axis of zero velocity, and the intersection of this axis with the plane of motion is known as the instantaneous centre of zero velocity or the instantaneous centre of rotation. This approach provides us with a valuable means for visualizing and analysing velocities in plane motion.

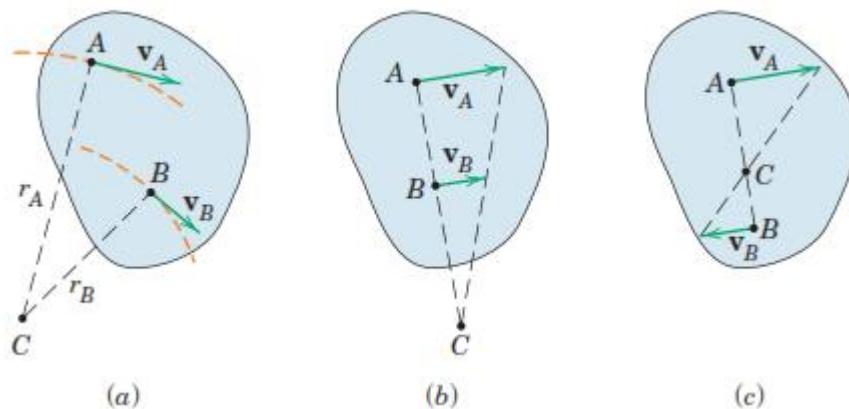


Fig. 5.6 Location of instantaneous centre

### a) Locating the Instantaneous Centre

The existence of the instantaneous centre is easily shown. For the body in Fig. 5.6, assume that the directions of the absolute velocities of any two points A and B on the body are known and are not parallel. If there is a point about which A has absolute circular motion at the instant considered, this point must lie on the normal to  $v$  through A. Similar reasoning applies to B, and the intersection of the two perpendiculars fulfils the requirement for an absolute centre of rotation at the instant considered. Point C is the instantaneous centre of zero velocity and may lie on or off the body. If it lies off the body, it may be visualized as lying on an imaginary extension of the body. The instantaneous centre need not be a fixed point in the body or a fixed point in the plane.

If also know the magnitude of the velocity of one of the points, say,  $v_A$ , we may easily obtain the angular velocity  $\omega$  of the body and the linear velocity of every point in the body. Thus, the angular velocity of the body, Fig. 5.6a, is

$$\ddot{S} = \frac{v_A}{r_A} \quad \text{-----} \rightarrow (5.8)$$

which, of course, is also the angular velocity of every line in the body. Therefore, the velocity of B is  $v_B = r_B \ddot{S} = (r_B / r_A) v_A$ . Once the instantaneous centre is located, the direction of the instantaneous velocity of every point in the body is readily found since it must be perpendicular to the radial line joining the point in question with C. If the

velocities of two points in a body having plane motion are parallel, Fig. 5.6b or 5.6c, and the line joining the points is perpendicular to the direction of the velocities, the instantaneous centre is located by direct proportion as shown. We can readily see from Fig. 5.6b that as the parallel velocities become equal in magnitude, the instantaneous centre moves farther away from the body and approaches infinity in the limit as the body stops rotating and translates only.

### b) Motion of the Instantaneous Centre

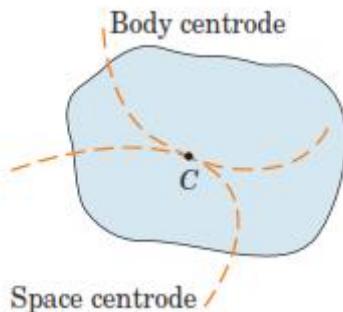


Fig. 5.7 Motion of the instantaneous centre

As the body changes its position, the instantaneous centre C also changes its position both in space and on the body. The locus of the instantaneous centres in space is known as the space centrode, and the locus of the positions of the instantaneous centres on the body is known as the body centrode. At the instant considered, the two curves are tangent at the position of point C. It can be shown that the body-centrode curve rolls on the space-centrode curve during the motion of the body, as indicated schematically in Fig. 5.7. Although the instantaneous center of zero velocity is momentarily at rest, its acceleration generally is not zero. Thus, this point may not be used as an instantaneous centre of zero acceleration in a manner analogous to its use for finding velocity. An instantaneous centre of zero acceleration does exist for bodies in general plane motion, but its location and use represent a specialized topic in mechanism kinematics and will not be treated here.

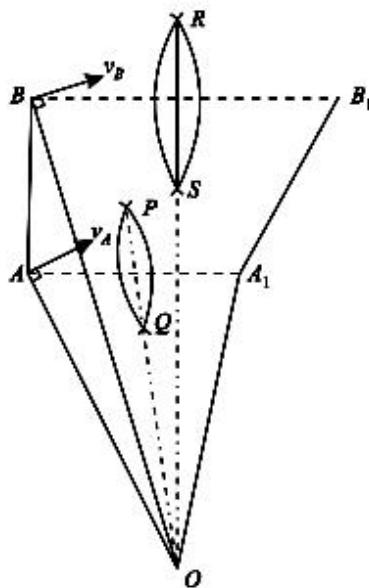
**Procedure:**

Fig.5.6

- First of all draw the initial and final positions of the rigid link as  $AB$  and  $A_1B_1$  respectively.
- Join  $AA_1$  and  $BB_1$ .
- Draw  $PQ$  right bisector of  $AA_1$  and  $RS$  right bisector of  $BB_1$ .
- Extend the two right bisector  $PQ$  and  $RS$  to meet at  $O$ . Which is required centre of rotation or instantaneous centre as shown in fig. 5.6.

Let  $S$  = Angular velocity of the rod  $AB$  about  $A$ .

$\therefore$  Linear velocity of point  $A$ .

$$v_A = \dot{S} \times OA \text{ or } \dot{S} = \frac{v_A}{OA} \quad \xrightarrow{\text{---}} \text{(i)}$$

$$\text{Similarly, } \dot{S} = \frac{v_B}{OB} \quad \xrightarrow{\text{---}} \text{(ii)}$$

$$\text{Equating equations (i) and (ii), } \frac{v_B}{v_A} = \frac{OB}{OA}$$

The direction of  $v_A$  and  $v_B$  will be at right angles to  $OA$  and  $OB$  respectively as shown in fig.5.6

**Problems:**

**SP5.1:** A link AB is moving in a vertical plane. At a certain instant, when the link is inclined at  $60^\circ$  to the horizontal, the point A is moving horizontally at 2m/s, while B is moving in a vertical direction, Find the velocity of B.

**Solution.** Given : Inclination of the link with horizontal =  $60^\circ$  and velocity of point A in horizontal direction ( $v_A$ ) = 2 m/s

Let  $v_B$  = Velocity of B in the vertical direction.

First of all, let us locate the position of instantaneous centre O, graphically, as shown in Fig.5.7 and as discussed below :

1. First of all draw the position of the link AB, such that it is inclined at an angle of  $60^\circ$  with the horizontal.
  2. Now draw the lines indicating the directions of motions of points A (in horizontal direction) and B (in vertical direction).
  3. Now draw perpendiculars at A and B on the directions of motion  $v_A$  and  $v_B$ .
  4. Let these perpendiculars meet at O, which is the instantaneous centre of the link AB.
- From the geometry of velocity of B, the triangle AOB, we find that

$$\frac{OB}{OA} = \cot 60^\circ = 0.577$$

We know that

$$\frac{v_B}{v_A} = \frac{OB}{OA} = 0.577$$

$$\therefore v_B = v_A \times 0.577 = 2 \times 0.577 = 1.15 \text{ m/s Ans.}$$

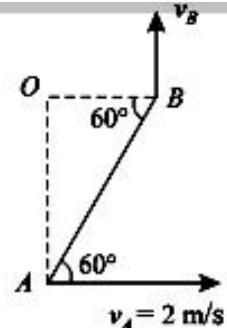


Fig.5.7

**SP5.2:** Three links are hinged together to form a triangle ABC as shown in fig.5.8 At a certain instant, the point A is moving towards the mid-point of BC with a velocity of 5m/s and B is moving at a perpendicular direction to AC. Find the velocity of C.

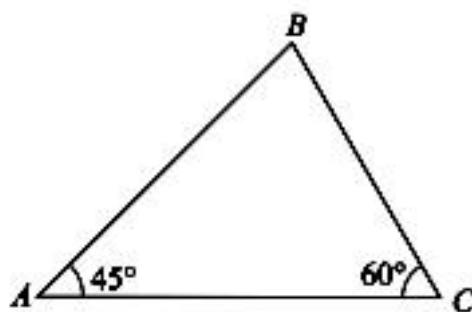


Fig.5.8

**Solution:** Given velocity of A( $v_A$ ) = 5m/s

First of all let us locate the position of instantaneous centre of the points A and B graphically as shown in fig.5.8 and as discussed below.

1. Draw the triangle **ABC** with the given data.
2. Now draw the lines indicating the directions of motions of points A (towards mid-point of BC) and B (at right angles to AC).
3. Now draw perpendiculars at A and B on the directions of motion of  $v_A$  and  $v_B$ .
4. Let these perpendiculars meet at **O**, which is the instantaneous centre of the link **AB** and **BC**.
5. Now join **OC** and draw a line at right angle to **OC** indicating the direction of motion of the point **C**.

Measuring the diagram to some scale, we find that  $OA = 2.6$  cm and  $OC = 5.4$  cm

We know that  $\frac{v_C}{v_A} = \frac{OC}{OA} = \frac{5.4}{2.6} = 2.08$

∴ Velocity of **C**,

$$v_C = v_A \times 2.08 = 5 \times 2.08 = 10.4 \text{ m/s Ans.}$$

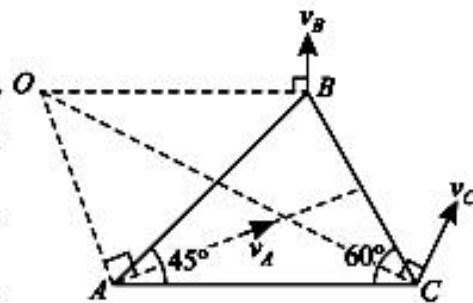


Fig.5.8

**SP5.3:** Determine the angular velocity of a rolling wheel motion without slipping using instantaneous centre.

Solution:

Consider a wheel rolling along a straight path, in such a way that there is no slipping of the wheel on its path. A little consideration will show, that the centre of the wheel **O** moves with some linear velocity. And each particle on the periphery of the wheel rotates with some angular velocity.

Thus the motion of any particle on the periphery of the wheel is a combination of linear and angular velocity as shown in Fig. 22.17

Let  $v$  = Linear velocity of the centre of the wheel

$\omega$  = Angular velocity of the wheel, and

$r$  = Radius of the wheel

Now consider any particle **P** on the periphery of the wheel making an angle  $\theta$  with the vertical through the centre of the wheel as shown in Fig. 22.17. We know that particle (**P**) is subjected to the following two motions simultaneously.

1. Linear velocity ( $v$ ) acting in the horizontal direction.

2. Tangential velocity (equal to linear velocity, such that  $v = \omega r$ ) acting at right angles to  $OP$ .

From the geometry of the figure, we find that the angle between these two velocities is equal to  $\theta$ . The resultant of these velocities ( $R$ ) will act along the bisector of the angle between the two forces, such that

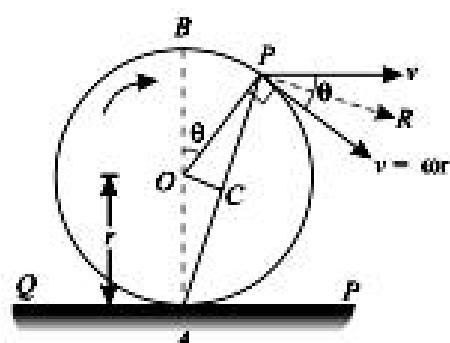


Fig.5.9

Motion of a rolling wheel without slipping.

$$R = 2v \cos\left(\frac{\theta}{2}\right)$$

We know that angular velocity of  $P$  about  $A$

$$= \frac{v_P}{AP}$$

We know that from the triangle  $AOP$ ,

$$AP = 2AC = 2r \cos\left(\frac{\theta}{2}\right)$$

Fig.5.9

$\therefore$  Angular velocity of  $P$  about  $A$  Fig.5.9

$$= \frac{2v \cos\left(\frac{\theta}{2}\right)}{AP} = \frac{2v \cos\left(\frac{\theta}{2}\right)}{2r \cos\left(\frac{\theta}{2}\right)} = \frac{v}{r}$$

It is thus obvious, that any point on the wheel rotates about the lowest point  $A$  (which is in touch with the ground) with the same angular velocity ( $\omega = \frac{v}{r}$ ).

**Notes.** 1. The particle  $A$  is subjected to the following two velocities :

- (i) linear velocity ( $v$ ) towards  $P$ , and
- (ii) tangential velocity ( $v = \omega r$ ) towards  $Q$

Since these two velocities are equal, opposite and collinear, therefore their resultant is zero. Or in other words, the particle  $A$  will be momentarily at rest.

2. Similarly, it can be proved the particle  $B$  (i.e. highest point of the wheel) will be subjected to a resultant velocity of  $2v$  and acts perpendicular to  $OB$ .
3. The lowest point ( $A$ ) is the instantaneous centre of the motion of the wheel.

**SP5.4:** The wheel as shown in fig.5.10 rolls to the right without slipping with its centre  $O$  having a velocity and use it to find the velocity of point  $A$  for the position indicated.

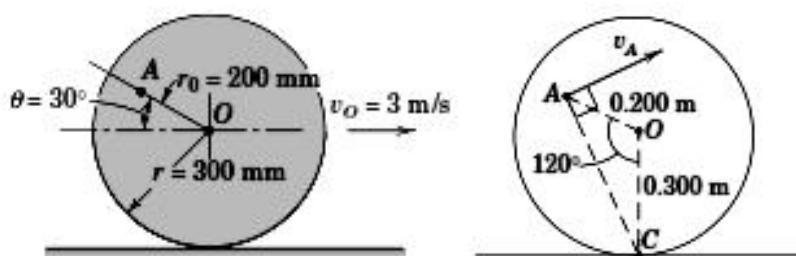


Fig.5.10

**Solution.** The point on the rim of the wheel in contact with the ground has no velocity if the wheel is not slipping; it is, therefore, the instantaneous center  $C$  of zero velocity. The angular velocity of the wheel becomes

$$[\omega = v/r] \quad \omega = v_O / \overline{OC} = 3/0.300 = 10 \text{ rad/s}$$

The distance from *A* to *C* is

$$\overline{AC} = \sqrt{(0.300)^2 + (0.200)^2 - 2(0.300)(0.200) \cos 120^\circ} = 0.436 \text{ m}$$

The velocity of *A* becomes

$$[v = r\omega] \quad v_A = \overline{AC}\omega = 0.436(10) = 4.36 \text{ m/s} \quad \text{Ans.}$$

The direction of  $\mathbf{v}_A$  is perpendicular to  $AC$  as shown.

**SP5.5:** Arm OB of the linkage has a clockwise angular velocity of 10 rad/sec in the position shown where  $\theta = 45^\circ$ . Determine the velocity of *A*, the velocity of *D* and the angular velocity of link AB for the position as shown in fig.5.11

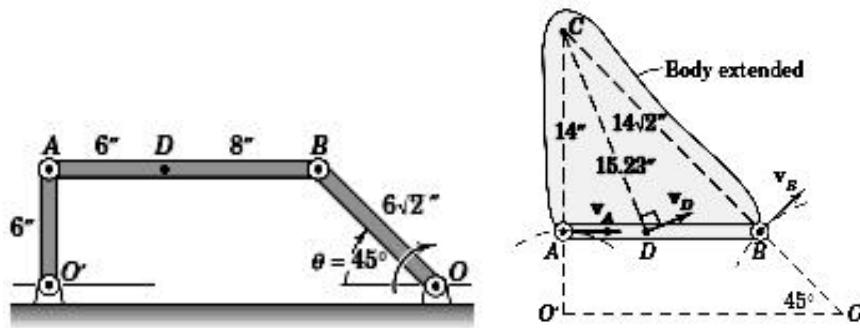


Fig.5.11

**Solution.** The directions of the velocities of *A* and *B* are tangent to their circular paths about the fixed centers  $O'$  and  $O$  as shown. The intersection of the two perpendiculars to the velocities from *A* and *B* locates the instantaneous center *C* for the link *AB*. The distances  $\overline{AC}$ ,  $\overline{BC}$ , and  $\overline{DC}$  shown on the diagram are computed or scaled from the drawing. The angular velocity of *BC*, considered a line on the body extended, is equal to the angular velocity of *AC*, *DC*, and *AB* and is

$$[\omega = v/r] \quad \omega_{BC} = \frac{v_B}{\overline{BC}} = \frac{\overline{OB}\omega_{OB}}{\overline{BC}} = \frac{6\sqrt{2}(10)}{14\sqrt{2}} = 4.29 \text{ rad/sec CCW} \quad \text{Ans.}$$

Thus, the velocities of *A* and *D* are

$$[v = r\omega] \quad v_A = \frac{14}{12} (4.29) = 5.00 \text{ ft/sec} \quad \text{Ans.}$$

**SP5.6:** Determine the velocity of piston of a reciprocating pump using instantaneous centre method as shown in fig.5.12

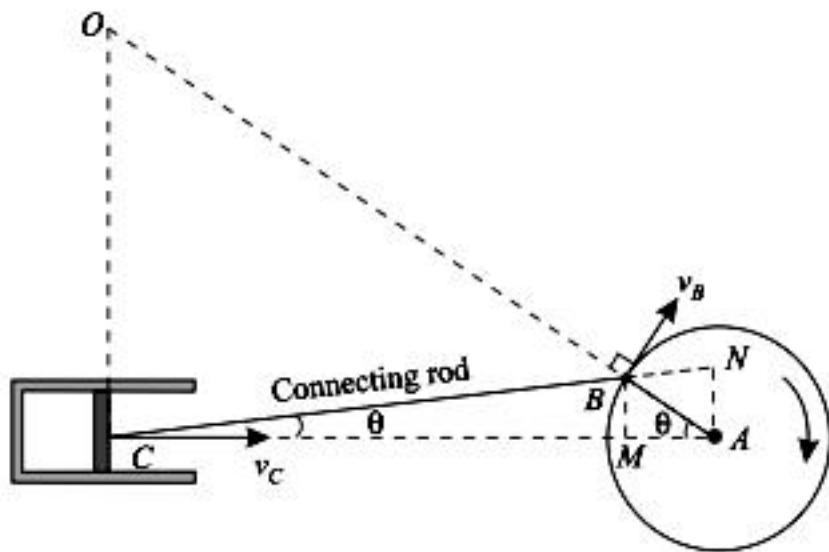


Fig.5.12 velocity of a piston of reciprocating pump

**Solution:**

Consider the mechanism of a reciprocating pump, in which  $AB$  be the crank,  $BC$  the connecting rod, and  $C$  the piston. Let  $O$  be the instantaneous centre of the connecting rod  $BC$ . Produce the line  $CB$  to meet the vertical line through  $A$  at  $N$  as shown in Fig.5.12. Now through  $B$ , draw  $BM$  perpendicular to  $CA$ .

From the geometry of the triangle  $OCB$  and  $ABN$ , we find that these two triangles are similar. Therefore

$$\frac{OC}{OB} = \frac{AN}{AB}$$

Let

$\omega_1$  = Angular velocity of the crank  $AB$  in rad/s.

$\omega_2$  = Angular velocity of the connecting rod  $BC$  about  $O$ , in rad/s.

$l$  = Length of the connecting rod  $BC$ ,

$r$  = Radius of the crank  $AB$ ,

$\theta$  = Angle traversed by the crank in degrees,

$\phi$  = Angle, which the connecting rod makes with  $CA$  (known as obliquity of the connecting rod).

We know that velocity of piston or  $C$ ,

$$v_C = \omega_2 \times OC \quad \dots(i)$$

and velocity of  $B$ ,

$$v_B = \omega_2 \times OB \quad \dots(ii)$$

We also know that velocity of  $B$ ,

$$v_B = \omega_1 \times AB \quad \dots(iii)$$

Dividing equation (i) by (ii),

$$\frac{v_C}{v_B} = \frac{\omega_2 \times OC}{\omega_2 \times OB} = \frac{OC}{OB} = \frac{AN}{AB}$$

$$\begin{aligned} \text{or } v_C &= v_B \times \frac{AN}{AB} = \omega_1 \times AB \times \frac{AN}{AB} \\ &= \omega_1 \times r \times \frac{AN}{r} = \omega_1 \times AN \end{aligned} \quad \dots(iv)$$

Now from the geometry of the figure, we find that

$$CA = CM + MA = CB \cos \phi + AB \cos \theta = l \cos \phi + r \cos \theta$$

and

$$\begin{aligned} AN &= CA \tan \phi = (l \cos \phi + r \cos \theta) \tan \phi \\ &= l \cos \phi \tan \phi + r \cos \theta \tan \phi \\ &= l \sin \phi + r \cos \theta \tan \phi \end{aligned}$$

Substituting this value of  $AN$  in equation (iv) velocity of  $C$ ,

$$v_C = \omega_1 (l \sin \phi + r \cos \theta \tan \phi)$$

**SP5.7:** In a crank and connecting rod mechanism, the radius of crank and length of the connecting rod are 300mm and 1200mm respectively. The crank is rotating at 180 rpm. Find the velocity of the piston, when the crank is at angle of  $45^\circ$ , with the horizontal.

**Solution.** Given : Radius of the crank ( $r$ ) = 300 mm = 0.3 m ; Length of connecting rod ( $l$ ) = 1200 mm = 1.2 m ; Angular rotation of crank ( $N$ ) = 180 r.p.m and angle traversed by the crank ( $\theta$ ) =  $45^\circ$

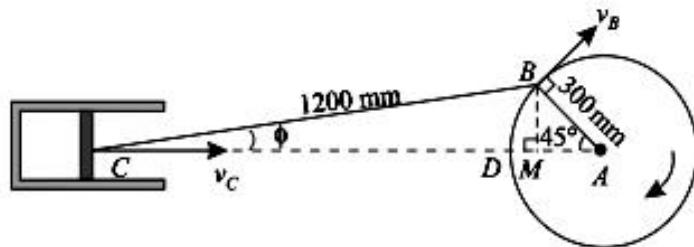


Fig.5.13

We know that Angular velocity of the crank,

$$\omega_1 = \frac{2\pi \times 180}{60} = 6\pi \text{ rad/s}$$

From the geometry of the figure, we find that

$$\begin{aligned} \sin \phi &= \frac{BM}{BC} = \frac{AB \sin 45^\circ}{BC} = \frac{0.3 \times 0.707}{1.2} \\ &= 0.1768 \quad \text{or} \quad \phi = 10.18^\circ \end{aligned}$$

We also know that velocity of the piston,

$$\begin{aligned} v_C &= \omega_1 (l \sin \phi + r \cos \theta \tan \phi) \\ &= 6\pi (1.2 \sin 10.18^\circ + 0.3 \cos 45^\circ \tan 10.18^\circ) \text{ m/s} \\ &= 6\pi [(1.2 \times 0.1768) + (0.3 \times 0.707 \times 0.1796)] \text{ m/s} \\ &= 4.72 \text{ m/s Ans.} \end{aligned}$$

## D'ALEMBERT'S PRINCIPLE AND ITS APPLICATIONS IN PLANE MOTION AND CONNECTED BODIES:

French mathematician Jean le Rond d' Alembert proved in 1743 that the newton's second law of motion is applicable not only to the motion of a particle but also to the motion of a body and looked at equation  $\sum F = m a$  from different angle. The equation  $\sum F = m a$  may be written as

$$\sum F - m a = 0 \quad \text{-----} \rightarrow (5.9)$$

The term “ $- m a$ ” may be looked as a force of magnitude  $m \times a$ , applied in the opposite direction of motion and is termed as the inertia force or reverse effective force. D'Alembert looks at equation 5.9 as an equation of equilibrium and **states that the system of forces acting on a body in motion is in dynamic equilibrium with the inertia force of the body. This is known as D'Alembert's principle.**

Let the body shown in fig.5.14 be subjected to a system of forces causing the body to move with an acceleration  $a$  in the direction of the resultant. Then apply a force equal to “ $m a$ ” in the reversed direction of acceleration as shown in fig.5.14. Now according to D'Alembert's principle, the equation of equilibrium  $\sum F_x = 0$  and  $\sum F_y = 0$  may be used for the system of forces shown in fig.5.15.

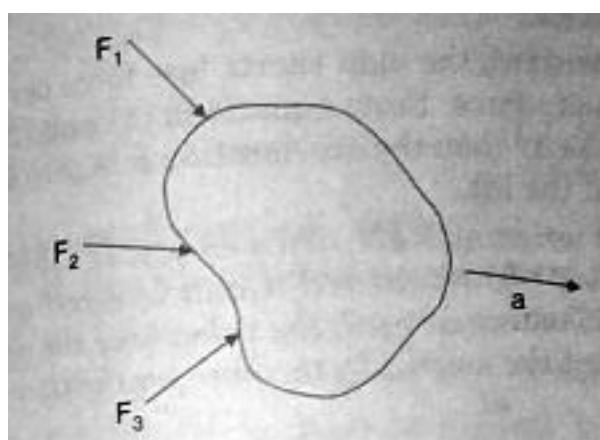


Fig.5.14 Motion of a body

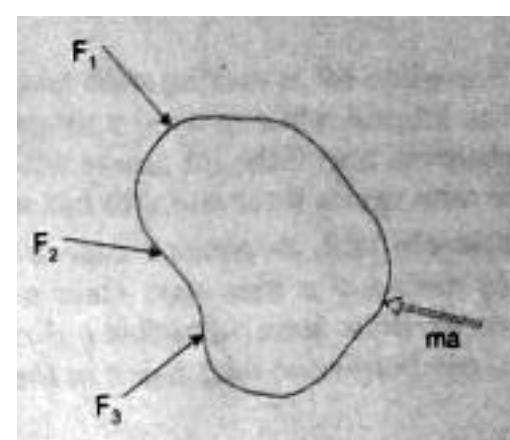


Fig.5.15 The system of forces in equilibrium

The inertia force “ $-ma$ ” has a physical meaning. According to Newton's first law of motion, a body continues to be in the state of rest or uniform motion in a straight line unless acted by an external force. That means every body has a tendency to continue in its state of rest or of uniform motion. This tendency is called inertia. Hence inertia force is the resistance offered by a body to the change in its state of rest or of uniform motion.

Many scientists are critical of D'Alembert's principle, usually equilibrium equations are applied to a system of forces acting on a body. Inertia force is not acting on the moving body. Actually this is the force exerted by the moving body to resist the change in its state and hence D'Alembert's is criticized for messing up the concept of equations of equilibrium.

However, many engineers prefer to use D'Alembert's principle since just by applying a reverse effective force, the moving body can be treated as a body in equilibrium and can be analysed using equations of static equilibrium.

### Problems:

**SP5.8:** A man weighing  $W$  Newton entered a lift which moves with an acceleration of  $a \text{ m/s}^2$ . Find the force exerted by the man on the floor of lift when a) lift is moving downward and b) lift is moving upward.

### Solution:

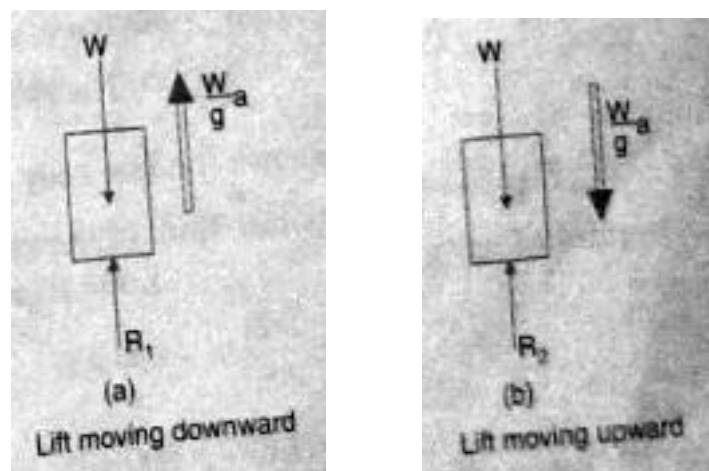


Fig.5.16

a) When the lift is moving downward,

The inertia force  $ma = \frac{W}{g}a$  , should be applied in upward direction as shown in

figure.

$$\sum F_V = 0 \text{ for the lift gives as } \Rightarrow R_1 - W + \frac{W}{g}a = 0$$

$$R_1 = W \left( 1 - \frac{a}{g} \right)$$

b) When lift is moving upwards,

Inertia force  $\frac{W}{g}a$  should be applied in the downwards direction as shown in

above figure. Now applying the equation of equilibrium, we get,

$$R_2 - W - \frac{W}{g}a = 0$$

$$R_2 = W \left( 1 + \frac{a}{g} \right)$$

Thus when lift is moving with acceleration downwards, the man exerts less force on the floor of the lift and while moving upwards he exerts more force. From the above equations it may be observed that if the lift moves with uniform velocity then the acceleration  $a$  is zero and hence the man exerts force equal to his weight on the lift.

**SP5.9:** An elevator cage of a mine shaft weighing 8 kN, when empty is lifted or lowered by means of a wire rope. Once a man weighing 600N entered it and lowered with uniform acceleration such that when a distance 187.5m was covered, the velocity of the cage was 25m/s. Determine the tension in the rope and the force exerted by the man on the floor of the cage.

**Solution.** In this problem

$$\begin{array}{ll} \text{initial velocity} & u = 0 \\ \text{final velocity} & v = 25 \text{ m/sec} \\ \text{and} & \text{distance covered, } s = 187.5 \text{ m} \\ \text{Using the equation of motion,} & \\ v^2 - u^2 = 2as & \end{array}$$

we get,

$$\begin{aligned} 25^2 - 0 &= 2a \cdot 187.5 \\ a &= 1.667 \text{ m/sec}^2 \end{aligned}$$

or

Figure 14.4 (a) shows free body diagram of elevator cage and the man with inertia force  $\frac{W}{g} a = \frac{8000 + 600}{9.81} a$  applied in upward direction (since the motion is downward).

Summing up the forces in vertical direction

$$\begin{aligned} T + \frac{8600}{9.81} 1.667 - 8600 &= 0 \\ T &= 7138.90 \text{ N} \end{aligned} \quad \text{Ans.}$$

Figure 14.4 (b) shows the free body diagram of the man along with inertia force  $\frac{600}{9.81} a$  applied in upward direction. Sum of the vertical forces should be zero. That is,  $\sum V = 0$  gives

$$\begin{aligned} R + \frac{600}{9.81} a - 600 &= 0 \\ \text{i.e. } R + \frac{600}{9.81} \times 1.667 - 600 &= 0 \\ R &= 498.06 \text{ N} \end{aligned} \quad \text{Ans.}$$

or

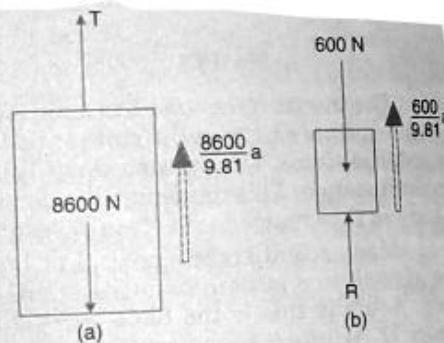


Fig.5.17

**SP5.10:** A motorist travelling at a speed of 70 kmph suddenly applies brakes and halts after skidding 50m. Determine, 1) the time required to stop the car and 2) the coefficient of friction between the tyres and the road.

**Solution.** Initial velocity  $u = 70 \text{ kmph}$

$$= \frac{70 \times 1000}{60 \times 60} = 19.44 \text{ m/sec}$$

Final velocity  $v = 0$

Displacement  $s = 50 \text{ m}$

Using the equation of linear motion

$$v^2 = u^2 + 2as, \text{ we get}$$

$$0 = 19.44^2 + 2a \times 50$$

$$a = -3.78 \text{ m/sec}^2$$

i.e. the retardation is  $3.78 \text{ m/sec}^2$ .

Using the relation  $v = u + at$ , we get

$$0 = 19.44 - 3.78 t$$

$$t = 5.14 \text{ sec}$$

Inertia force must be applied in the opposite direction of acceleration, which means, it should be applied in the direction of motion while retarding. Figure 14.5 shows the free body of the motor along with inertia force.

$\Sigma$  Forces normal to road = 0

$$N = W$$

From the law of friction,  $F = \mu N = \mu W$

$\Sigma$  Forces in the direction of motion = 0, gives

$$F = \frac{W}{9.81} \times 3.78$$

$$\mu W = \frac{W \times 3.78}{9.81}$$

$$\mu = 0.385$$

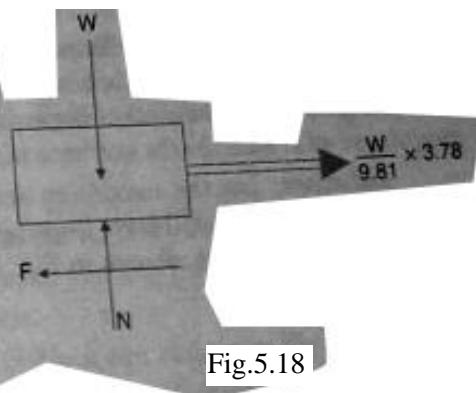


Fig.5.18

**SP5.11:** A block weighing 1 kN rests on a horizontal plane as shown in fig.5.19 Find the magnitude of the force P required to give the block an acceleration of  $3 \text{ m/s}^2$  to the right. The coefficient of friction between the block and the plane is 0.25.

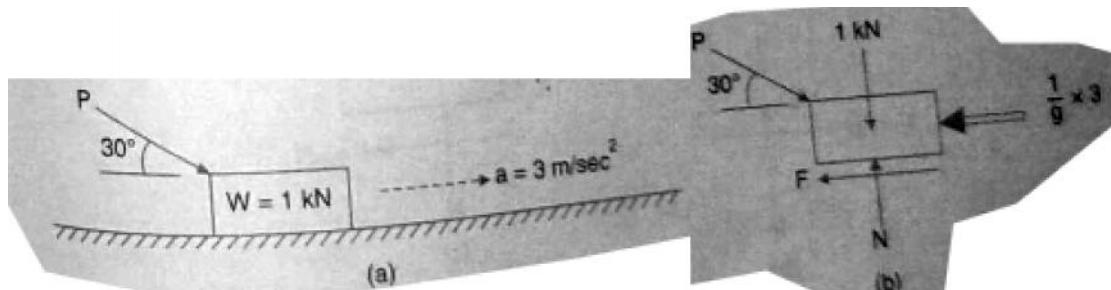


Fig.5.19

**Solution.** Free body diagram of the block along with inertia force  $ma = \frac{1}{g} \times 3 = \frac{3}{9.81} \text{ kN}$  is shown in Fig. 14.6(b). In the figure,  $N$  is the normal reaction and  $F$  is the frictional force.

$$\Sigma V = 0$$

$$N - 1 - P \sin 30^\circ = 0$$

or

$$N = 1 + \frac{P}{2}$$

From the law of friction.

$$F = \mu N = 0.25 \left( 1 + \frac{P}{2} \right) \quad \dots(2)$$

$$\Sigma H = 0$$

$$P \cos 30^\circ - F - \frac{3}{9.81} = 0$$

$$P \cos 30^\circ - 0.25 \left( 1 + \frac{P}{2} \right) - \frac{3}{9.81} = 0$$

$$P = 0.561 \text{ kN}$$

**SP5.12:** A 750 N crate rests on a 599 N cart. The coefficient of friction between the crate and the cart is 0.3 and between cart and the road is 0.2. If the cart is to be pulled by a force  $P$  such that the crate does not slip, determine a) the maximum allowable magnitude of  $P$  and b) the corresponding acceleration of the cart.

**Solution.** Let the maximum acceleration be  $a$ , at which 750 N crate is about to slip. Hence frictional force will have limiting value ( $= \mu N$ ). Consider the free body diagram of crate along with inertia force, shown in Fig.5.20 (b).

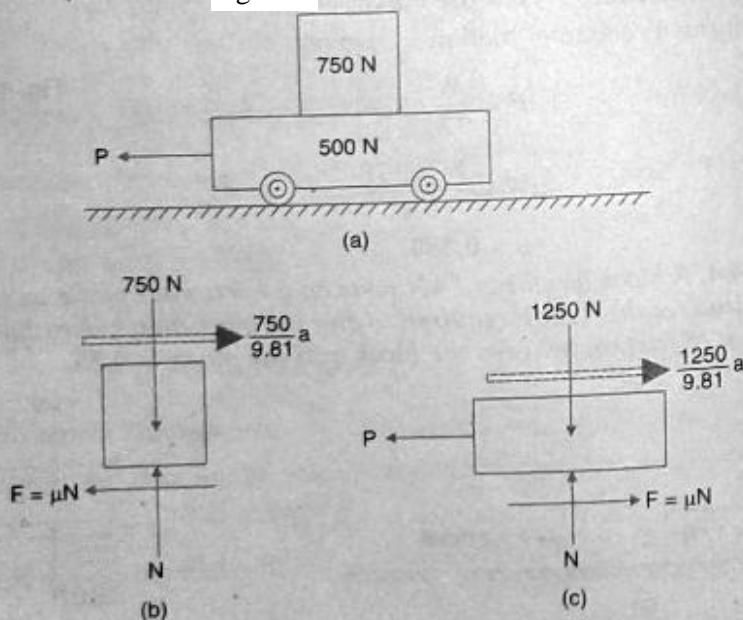


Fig.5.20

$$\Sigma V = 0$$

$$N = W = 750 \text{ Newton}$$

$$\begin{aligned} \text{Frictional force, } F &= \mu N = 0.3 \times 750 \\ &= 225 \text{ N} \end{aligned}$$

$$\Sigma H = 0$$

$$225 = \frac{750}{9.81} a$$

$$a = 2.943 \text{ m/sec}^2$$

Ans.

Consider now dynamic equilibrium of crate and the cart shown as a single body in Fig.5.20 (c).

$$\Sigma V = 0$$

$$N = 1250 \text{ Newton}$$

$$\text{Frictional force} = \mu N = 0.2 \times 1250 = 250 \text{ Newton}$$

$$\Sigma H = 0$$

$$P - 250 - \frac{1250}{9.81} \times 2.943 = 0$$

$$P = 625 \text{ N}$$

Ans.

**SP5.13:** A body weighing 1200 N rests on a rough plane inclined at  $12^\circ$  to the horizontal. It is pulled up the plane by means of a light flexible rope running parallel to the plane and passing over a light frictionless pulley at the top of the plane as shown in fig.5.21. The portion of the rope beyond the pulley hangs vertically down and carries a weight of 800 N at its end. If the coefficient of friction for the plane and the body is 0.2 find a) tension in the rope, b) acceleration with which the body moves up the plane and c) the distance moved by the body in 3 seconds after starting from rest.

**Solution.** Let  $a$  be the acceleration of the system.

Free body diagrams of 1200 N block and 800 N block are shown in Figs. 14.8(b) and 14.8(c) along with inertia forces. According to D' Alembert, we can treat these bodies as in static equilibrium.

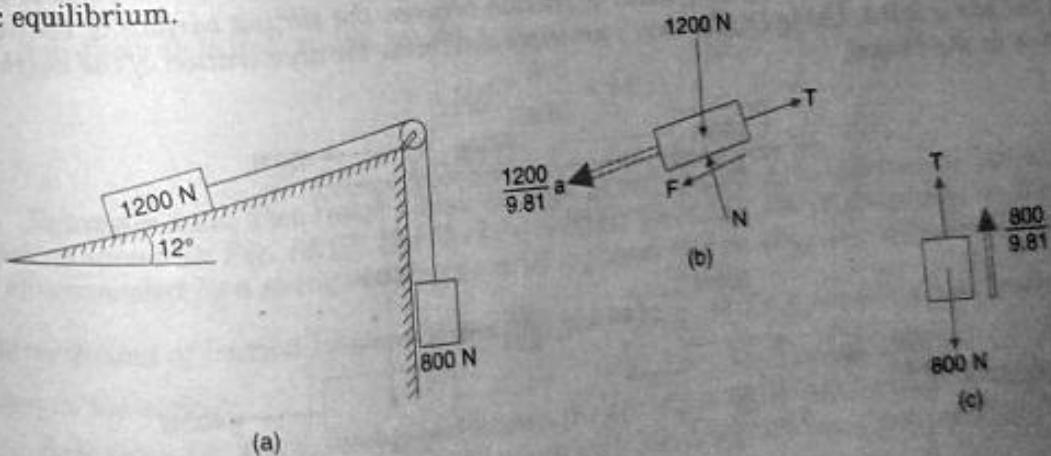


Fig.5.21

Consider 1200 N block.

$$\begin{aligned} \Sigma \text{ Forces normal to the plane} &= 0, \text{ gives} \\ N - 1200 \cos 12^\circ &= 0 \end{aligned}$$

$$N = 1173.77 \text{ Newton}$$

From the law of friction

$$F = \mu N = 0.2 \times 1173.77 = 234.76 \text{ N}$$

$\Sigma$  Forces parallel to the inclined plane = 0, gives

$$\frac{1200}{9.81} a + 1200 \sin 12^\circ + F - T = 0$$

$$\text{i.e. } 122.32 a - T = -484.25 \quad \dots(1)$$

Consider the free body diagram of 800 N block shown in Fig. 14.8(c).

$$T + \frac{800}{9.81} a = 800 \quad \dots(2)$$

Adding equations (1) and (2), we get,

$$\left( 122.32 + \frac{800}{9.81} \right) a = 800 - 484.25$$

$$a = 1.549 \text{ m/sec}^2$$

Ans.

Substituting it in (2), we get,

$$T = 800 - \frac{800}{9.81} \times 1.549$$

i.e.

$$T = 673.68 \text{ N}$$

Ans.

Initial velocity = 0

$$a = 1.549 \text{ m/sec}^2$$

$$t = 3 \text{ sec}$$

Using the equation  $s = ut + \frac{1}{2} at^2$ , we get **distance moved in 3 seconds as**

$$= 0 \times 3 + \frac{1}{2} \times 1.549 \times 3^2$$

$$= 6.971 \text{ m}$$

Ans.

**SP5.14:** Two weights 800 N and 200 N are connected by a thread and they move along a rough horizontal plane under the action of a force 400 N applied to the 800 N weight as shown in fig. 5.22. The coefficient of friction between the sliding surface of the weights and the plane is 0.3. Using D'Alembert's principle determine the acceleration of the weight and tension in the thread.

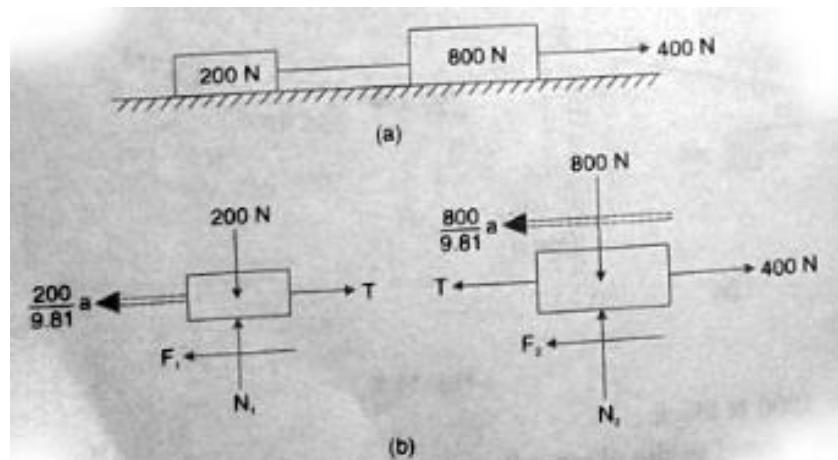


Fig.5.22

**Solution.** Free body diagrams of 200 N and 800 N blocks along with inertia forces are shown in Fig.5.22 (b), in which  $a$  is acceleration of the system.

Consider the dynamic equilibrium of 200 N weight

$$\Sigma V = 0$$

$$N_1 = 200 \text{ N.}$$

From law of friction,  $F_1 = \mu N_1 = 0.3 \times 200 = 60 \text{ N}$  (1)

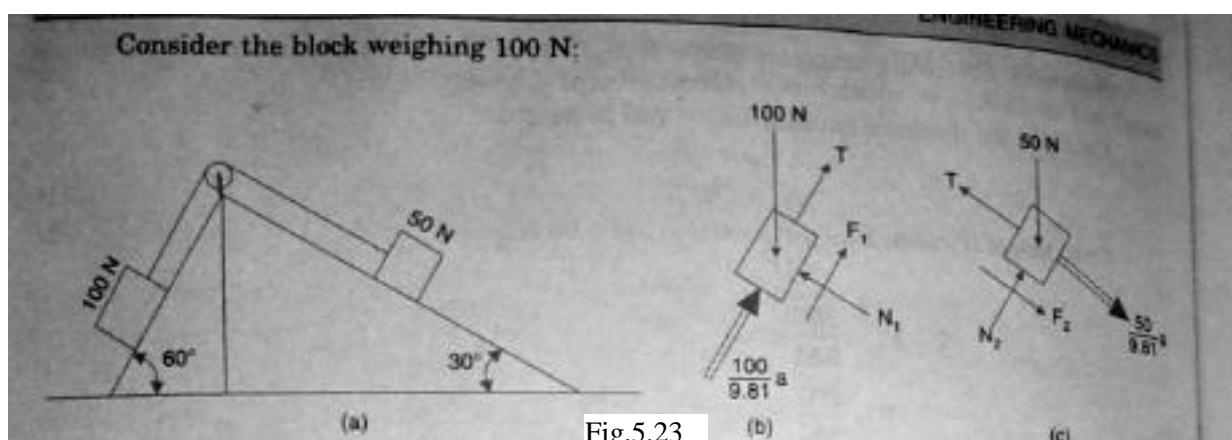
$$\Sigma H = 0$$

$$T - F_1 - \frac{200}{9.81} a = 0$$
 (2)

i.e.  $T - \frac{200}{9.81} a = 60$  .....(3)  
 since  $F_1 = 60 \text{ N}$ .  
 Consider 800 N body.  
 $\Sigma V = 0$   
 $N_2 = 800 \text{ N}$  .....(4)  
 From law of friction,  $F_2 = \mu N_2 = 0.3 \times 800$   
 $= 240 \text{ N}$  .....(5)  
 $\Sigma H = 0$   
 $- T - \frac{800}{9.81} a - F_2 + 400 = 0$   
 or  $T + \frac{800}{9.81} a = 160$  .....(6)  
 Since  $F_2 = 240 \text{ N}$ .  
 Subtracting equations (3) from (6), we get,  
 $\left( \frac{200}{9.81} + \frac{800}{9.81} \right) a = 160 - 60$   
 $a = 0.981 \text{ m/sec}^2$  .....Ans.  
 Substituting it in Eqn. (6), we get,  
 $T = 160 - \frac{800}{9.81} \times 0.981$   
 i.e.  $T = 80 \text{ N}$  .....Ans.

**SP5.15;** Two rough planes inclined at  $30^\circ$  and  $60^\circ$  to horizontal are placed back to back as shown in fig.5.23. The blocks of weights 50N and 100N are placed on the faces and are connected by a string running parallel to planes and passing over a frictionless pulley. If the coefficient of friction between planes and blocks is  $1/3$ , find the resulting acceleration and tension in the string.

Solution: Let the assembly move down the  $60^\circ$  plane by an acceleration 'a'  $\text{m/s}^2$ . Free body diagrams of 100N and 50N blocks along with inertia forces are shown in figure.



$\Sigma$  Forces normal to the plane = 0, gives

$$N_1 = 100 \cos 60^\circ = 50 \text{ N} \quad \dots(1)$$

From the law of friction,

$$F_1 = \mu N = \frac{1}{3} \times 50 = 16.67 \text{ N} \quad \dots(2)$$

$\Sigma$  Forces parallel to the plane = 0, gives

$$T + \frac{100}{9.81} a - 100 \sin 60^\circ + F_1 = 0$$

$$T + \frac{100}{9.81} a = 69.93 \quad \dots(3)$$

since  $F_1 = 16.67$

Now consider 50 N block:

$\Sigma$  Forces normal to plane = 0, gives

$$N_2 = 50 \cos 30^\circ = 43.30 \text{ N} \quad \dots(4)$$

From the law of friction,  $F_2 = \mu N_2$

$$= \frac{1}{3} \times 43.3 = 14.43 \text{ N} \quad \dots(5)$$

$\Sigma$  Forces parallel to  $30^\circ$  plane = 0

$$\frac{50}{9.81} a + F_2 + 50 \sin 30^\circ - T = 0 \quad \dots(6)$$

$$\frac{50}{9.81} a - T = -39.43$$

since  $F_2 = 14.43$

Adding equations (3) and (6), we get,

$$\left( \frac{100}{9.81} + \frac{50}{9.81} \right) a = 69.93 - 39.43 \quad \text{Ans}$$

$$a = 1.9947 \text{ m/sec}^2$$

or

From eqn. (3),

$$T = 69.93 - 100/9.81 \times 1.9947 = 49.6 \text{ N} \quad \text{Ans}$$

**SP5.16:** Two blocks A and B released from rest on a  $30^\circ$  incline when they are 18m apart. The coefficient of friction under the upper block A is 0.2 and that under the lower block B is 0.4. In what time block A reaches the block B? After they touch and move as a single unit, what will be the contact force between them? Weights of the block A and B are 100N and 80N respectively.

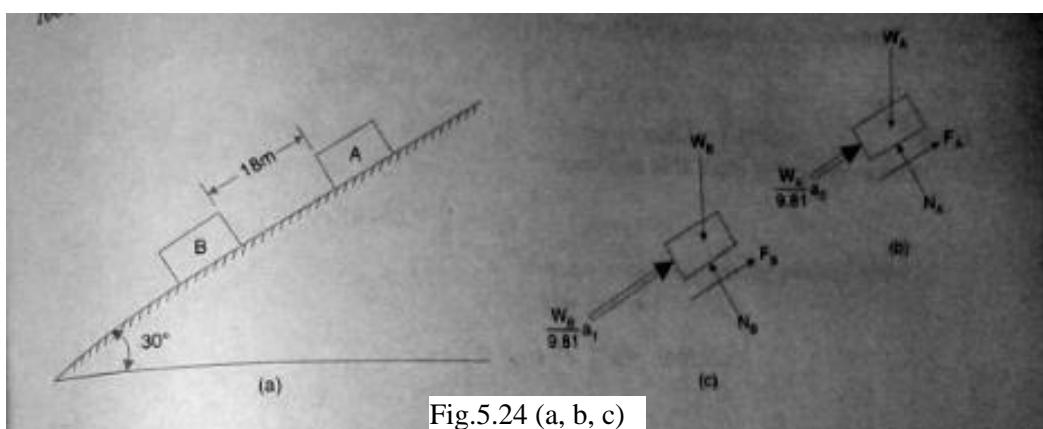


Fig.5.24 (a, b, c)

**Solution.** Let block A move with an acceleration  $a_1$  and block B with an acceleration  $a_2$ .  
 The free body diagrams of the blocks A and B along with inertia forces are shown in Fig. 5.24 (b) and 5.24 (c) respectively.

Consider block A.

$\Sigma$  Forces normal to the plane = 0, gives

$$N_A = W_A \cos \theta = W_A \cos 30^\circ \quad \dots(1)$$

$$\begin{aligned} \text{From the law of friction} \quad F_A &= \mu N_A \\ &= 0.2 W_A \cos 30^\circ \end{aligned} \quad \dots(2)$$

$\Sigma$  Forces parallel to the plane = 0, gives

$$\frac{W_A}{9.81} a_1 + F_A - W_A \sin 30^\circ = 0$$

$$\therefore \frac{W_A}{9.81} a_1 + 0.2 W_A \cos 30^\circ - W_A \sin 30^\circ = 0$$

$$\frac{a_1}{9.81} + 0.2 \cos 30^\circ - \sin 30^\circ = 0 \quad \dots(3)$$

$$a_1 = 3.2058 \text{ m/sec}^2$$

Consider block B.

$\Sigma$  Forces normal to the plane = 0, gives

$$N_B = W_B \cos 30^\circ \quad \dots(4)$$

$$\begin{aligned} \text{From the law of friction} \quad F_B &= \mu N_B \\ F_B &= 0.4 W_B \cos 30^\circ \end{aligned} \quad \dots(5)$$

$\Sigma$  Forces parallel to the plane = 0, gives

$$\frac{W_B}{9.81} a_2 + F_B - W_B \sin 30^\circ = 0$$

$$\frac{W_B}{9.81} a_2 + 0.4 W_B \cos 30^\circ - W_B \sin 30^\circ = 0$$

$$\therefore a_2 = 1.5067 \text{ m/sec}^2$$

Let  $t$  be the time elapsed until the blocks touch each other.

Displacement of block A in this period

$$s_1 = u_1 t + \frac{1}{2} a_1 t^2 = \frac{1}{2} \times 3.2058 t^2$$

since

Initial velocity  $u_1 = 0$

Displacement of block B in this time

$$s_2 = u_2 t + \frac{1}{2} a_2 t^2 = \frac{1}{2} \times 1.5067 t^2$$

When the two blocks touch each other

$$s_1 = s_2 + 18$$

$$\frac{1}{2} \times 3.2058 t^2 = \frac{1}{2} \times 1.5067 t^2 + 18$$

$$\therefore t = 4.60 \text{ sec}$$

Ans.

After the blocks touch each other, let the common acceleration be  $a$ . Summing up the forces including inertia forces along the inclined plane.

$$\frac{100}{9.81} a + 0.2 \times 100 \cos 30^\circ - 100 \sin 30^\circ + \frac{80}{9.81} a + 0.4 \times 80 \cos 30^\circ - 80 \sin 30^\circ = 0$$

$$a = 2.45 \text{ m/sec}^2$$

Considering the free body diagram of any one of the blocks, contact force  $P$  can be obtained. Free body diagram of block A along with inertia force is shown in Fig. 14.11(d).

Now  $\sum$  forces parallel to plane = 0, gives

$$P - 100 \sin 30^\circ + F + \frac{100}{9.81} a = 0$$

$$P - 100 \sin 30^\circ + 0.2 \times 100 \cos 30^\circ + \frac{100}{9.81} \times 2.45 = 0$$

$$P = 7.7 \text{ N}$$

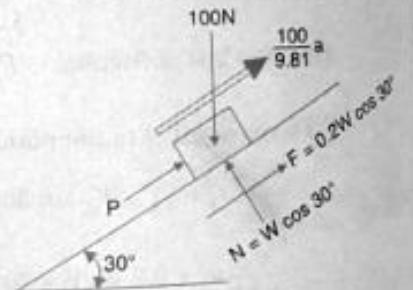
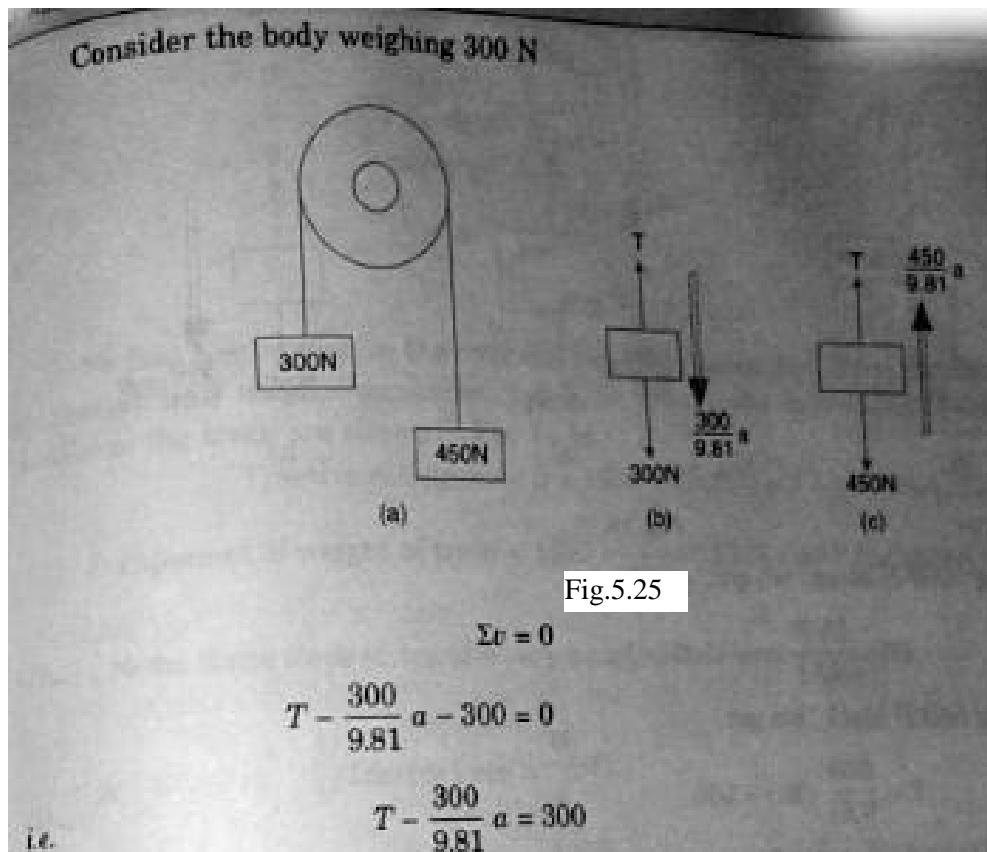


Fig.5.24 (d)

Ans.

**SP5.17:** Two bodies weighing 300N and 450N are hung to the ends of a rope passing over an ideal pulley as shown in fig.5.25 with what acceleration the heavier body comes down? What is the tension in the string?

**Solution:** Let  $a$  be the acceleration with which the system moves and  $T$  be the tension in the string. Free body diagrams of 300N block and 450N blocks along with inertia forces are shown in figure.



Considering the body weighing 450 N, we get

$$T + \frac{450}{9.81} a = 450$$

Subtracting eqn. (1) from eqn. (2) we get,

$$\frac{450}{9.81} a + \frac{300}{9.81} a = 450 - 300$$

$$a = 1.962 \text{ m/sec}^2$$

Substituting this value in eqn. (1), we get,

$$T = 300 + \frac{300}{9.81} \times 1.962$$

$$T = 360 \text{ N}$$

**SP5.18:** Determine the tension in the string and accelerations of blocks A and B weighing 1500N and 500N connected by an inextensible string as shown in fig.5.26. Assume pulleys as frictionless and weightless.

**Solution:** In this pulley system, it may be observed that if 1500N moves downward by distance x, 500 N block moves up by 2x. Hence if acceleration of 1500N block is a that of 500N block is 2a. The free body diagrams of 1500N and 500N blocks are shown in figure along with inertia forces. According to D'Alembert's principle the system of forces shown in figure may be treated to be in equilibrium.

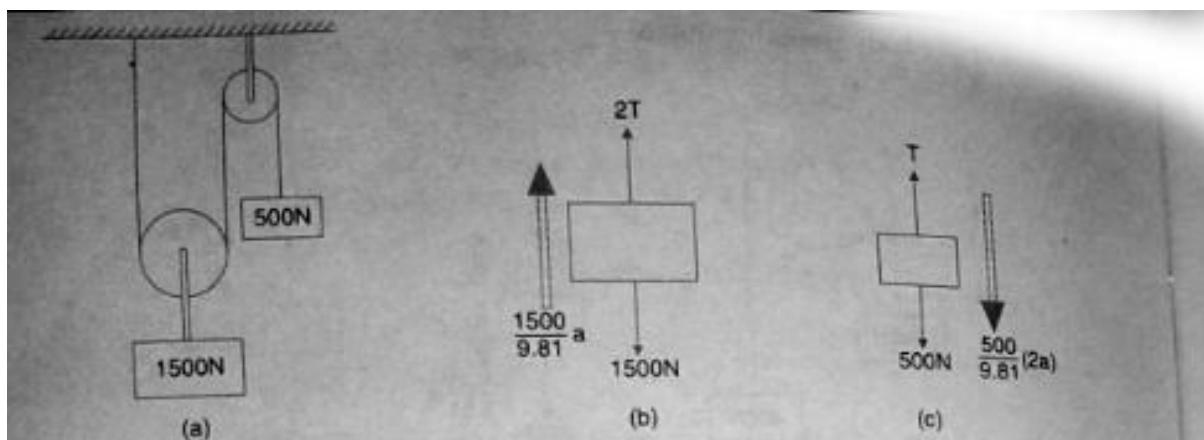


Fig.5.26

Considering 1500 N block, we get,

$$2T + \frac{1500}{9.81} a = 1500 \quad \dots(1)$$

Considering 500 N block, we get,

$$T - \frac{500}{9.81} (2a) = 500 \quad \dots(2)$$

From eqns. (1) and (2), we get,

$$\left( \frac{1500}{9.81} + \frac{2000}{9.81} \right) a = 500$$

$$a = 1.401 \text{ m/sec}^2 \quad \text{Ans.}$$

∴ Substituting it in (1), we get

$$2T = 1500 - \frac{1500}{9.81} \times 1.401 \quad \text{Ans.}$$

$$T = 642.89 \text{ N}$$

**SP5.19:** An engine of weight 500 kN pulls a carriage weighing 1500 kN up an incline of 1 in 100. The train starts from rest and moves with a constant acceleration against a resistance of 5 N/kN. It attains a maximum speed of 36 kmph in km distance. Determine the tension in the coupling between the carriage and the engine and the tractive force developed by the engine.

**Solution:** Initial velocity,  $u=0$ , Final velocity,  $v=36 \text{ kmph} = 36/3.6 \text{ m/s} = 10 \text{ m/s}$  and Displacement,  $S = 1 \text{ km} = 1000 \text{ m}$ .

From the kinematic equation  $v^2 = u^2 + 2aS$ , we get

$$10^2 = 0 + 2 \times a \times 1000, \Rightarrow a = 0.05 \frac{m}{s^2}$$

To find the tension in the coupling between engine and train, dynamic equilibrium of the coaches only may be considered. Let  $T$  be the tension in coupling. Various forces acting parallel to the track are shown in  

Tractive resistance =  $5 \times 1500 = 7500 \text{ N} = 7.5 \text{ kN}$  down the plane.

Component of weight of train =  $1500 \times \frac{1}{100} = 15 \text{ kN}$  down the plane.

(Note: Since slope of track is very small,  $\sin \theta = \tan \theta = \frac{1}{100}$ )

$$\text{Inertia force} = \frac{W}{g} a$$

$$= \frac{1500}{9.81} \times 0.05 = 7.645 \text{ kN} \text{ (down the plane)}$$

Dynamic equilibrium equation is

$$T - 7.5 - 15 - 7.645 = 0 \quad \text{Ans.}$$

$$T = 30.145 \text{ kN}$$

Q5 To find tractive force developed by the engine, consider dynamic equilibrium of entire train.

Fig.5.27 shows various forces in the direction of track acting on the entire train along with inertia force. Let  $P$  be tractive force developed.

$$\text{Total tractive resistance} = 5 \times 2000 = 10,000 \text{ N}$$

$$= 10 \text{ kN, down the plane.}$$

$$\text{Inertia force} = \frac{W}{g} \times a$$

$$= \frac{2000}{9.81} \times 0.05 = 10.194 \text{ kN, down the plane.}$$

Component of weight down the plane

$$= 2000 \times \frac{1}{100} = 20 \text{ kN}$$

Dynamic equilibrium equation along the track is

$$P - 10 - 10.194 - 20 = 0$$

$$P = 40.194 \text{ kN}$$

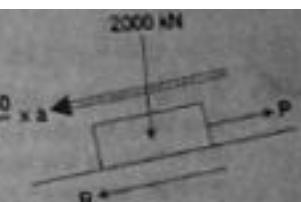


Fig.5.27

## WORK ENERGY PRINCIPLE AND ITS APPLICATIONS IN PLANE MOTION OF CONNECTED BODIES:

In previous section, kinetic problems were solved using D'Alembert's dynamic equilibrium condition. In this section, another approach called Work – Energy approach is used to solve kinetic problems. This method is advantageous over D'Alembert's method when the problem involves velocities rather than acceleration. The terms work, energy and power are explained first, then the work energy equation is derived. Using this equation a number of kinetic problems are solved.

### a) Work:

The work done by a force on a moving body is defined as the product of the force and the distance moved in the direction of the force. In fig.5. 28, various forces acting on a particle are shown. If the particle moves a distance  $s$  in  $x$  direction, from A to B, then the work done by various forces are as given below:

Force	Work done
$P_1$	$P_1 s$
$P_2$	$+ P_2 (-s) = - P_2 s$
$P_3$	$P_3 \times 0 = 0$
$P_4$	$P_4 s \cos \theta$

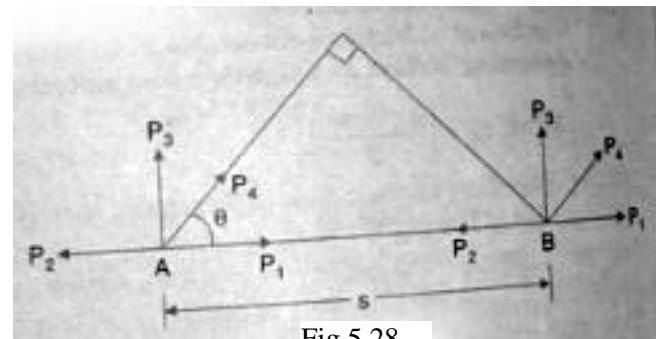


Fig.5.28

The general expression for work done by a force  $P$  is  $P s \cos\theta$ , where  $\theta$  is the angle between the force and the direction of motion. This expression may be rearranged as  $P \cos\theta \times s$  giving the new definition of work done. Thus work done by a force may be defined as the product of component of force in the direction of motion and the distance moved.

From the definition of work, it is obvious that unit of work is obtained by multiplying unit of force by unit of length. Hence if newton is unit of force and metre is unit of displacement, unit of work will be N-m. One N-m of work is denoted by the term Joule (J). Hence one Joule may be defined as the amount of work done by the one Newton force when the particle moves 1 metre in the direction of that force.

The other commonly used units are: kilo joules kJ (i.e., kN-m) or milli Joule mJ (N-mm) etc.

### b) Work done by a varying force:

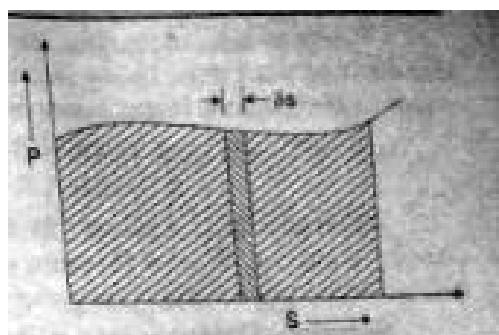


Fig.5.29 Varying force

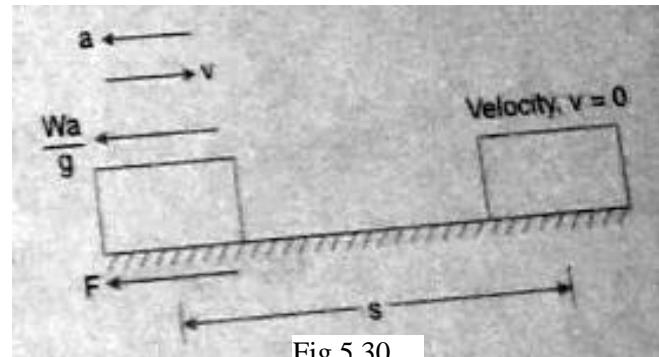
Let the varying force acting at any instance on the particle be  $P$ . Now if the particle moves a small distance  $\delta s$ , then the work done by the force is  $P \times \delta s$ . Work done by the force in moving the body by a distance  $s$  is  $\sum P \cdot s$ . Thus if a force versus displacement curve is drawn as shown in fig.5.29, the area under the curve gives the work done by the force. If the variation of  $P$  is in a regular fashion, then

$$\sum Ps = \int Pds \quad \text{-----} \rightarrow (5.10)$$

### c) Energy:

Energy is defined as the capacity to do work. There are many forms of energy like heat energy, mechanical energy, electrical energy and chemical energy. In engineering mechanics we are interested in mechanical energy. This energy may be classified into potential energy and kinetic energy.

Potential energy is the capacity to do work due to the position of the body. A body of weight 'W' held at a height  $h$  possesses an energy  $Wh$ .



Kinetic energy is the capacity to do work due to motion of the body. Consider a car moving with a velocity  $v$  m/s. If the engine is stopped, it still moves forward, doing work against frictional resistance and stops at a certain distance  $s$ . From the kinematics of the motion, we have

$$0 - u^2 = 2aS, \Rightarrow a = \frac{u^2}{2S}$$

From D'Alembert's principle,

$$F + \frac{W}{g}a = 0$$

$$F - \frac{W}{g} \times \frac{u^2}{2S} = 0, \quad \Rightarrow F = \frac{Wu^2}{2gS}$$

$$Work\ done = F \times S = \frac{Wu^2}{2gS} \quad \rightarrow (5.11)$$

This work is done by the energy stored initially in the body.

$$\text{Therefore, Kinetic energy} = \frac{1}{2} \times \frac{W}{g} v^2, \quad \rightarrow (5.12)$$

Where  $v$  is the velocity of the body

Unit of energy is same as that of work, since it is nothing but capacity to do work.

It is measured in Joules J (N-m) or kilo Joules kJ (i.e., kN-m).

#### d) Power:

Power is defined as time rate of doing work. Unit of power is Watt (w) and is defined as one Joule of work done in one second. In practice kilowatt is the commonly used unit which is equal to 1000 watts. Horse power is the unit used in MKS and FPS systems.

1 metric H.P = 735.75 watts and 1 British H.P = 745.8 watts

#### e) Work energy equation for translation:

Consider the body as shown in fig.5.31 subject to a system of forces  $F_1, F_2, \dots$  and moving with an acceleration  $a$  in  $x$  – direction.

Let its initial velocity at A be  $u$  and final velocity when it moves distance  $AB = s$  be  $v$ . Then the resultant of system of the forces must be in  $x$  – direction. Let

$$R = \sum F_x \quad \rightarrow (5.13)$$

From Newton's second law of motion,

$$R = \frac{W}{g} a$$

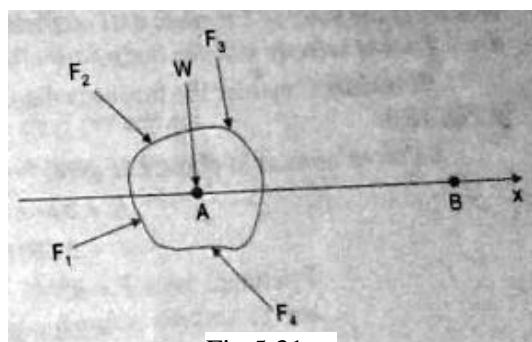


Fig.5.31

Multiplying both sides by elementary distance  $ds$ , we get

$$Rds = \frac{W}{g} ads$$

$$Rds = \frac{W}{g} v \frac{dv}{ds} ds \quad [\because a = v \frac{dv}{ds}]$$

$$Rds = \frac{W}{g} v dv$$

Integrating both sides for the motion from A to B we get,

$$\int_0^s Rds = \int_u^v \frac{W}{g} v dv$$

$$RS = \frac{W}{g} \left( \frac{v^2}{2} \right)_u^v$$

$$RS = \frac{W}{2g} (v^2 - u^2)$$

Now R.S is the work done by the forces acting on the body.  $\frac{W}{2g} v^2$  is final kinetic energy and  $\frac{Wu^2}{2g}$  is initial kinetic energy. Hence we can say, work done in a motion is

equal to change in kinetic energy. That is,

$$\text{Work done} = \text{Final kinetic energy} - \text{Initial kinetic energy} \quad \rightarrow (5.14)$$

The equation 5.14 Is called Work Energy Equation.

This work energy equation may be stated as the work done by a system of forces acting on a body during a displacement is equal to the change in kinetic energy of the body during the same displacement.

Using this work energy equation a number of kinetic problems can be solved. This will be found more useful than D'Alembert's principle when we are not interested in finding acceleration in the problem, but mainly interested in velocity and distance.

### f) Work done by a spring:

Consider a body attached to a spring as shown in fig.5.32. It is obvious that if the body moves out from its un-deformed position, tensile force develops and if it moves towards the supports compressive force develops. In other words the force of a spring is always directed towards its normal position. Experimental results have shown that the magnitude of the force developed in the spring is directly proportional to its displacement from the un-deformed position. Thus, if  $F$  is the force in the spring due to deformation  $x$  from its un-deformed position.

$$F \propto x$$

$$F = kx \quad \text{-----} \rightarrow (5.15)$$

Where the constant of proportionality  $k$  is called spring constant and is defined as a force required for unit deformation of the spring. Hence the unit of spring constant is N/m or kN/m.

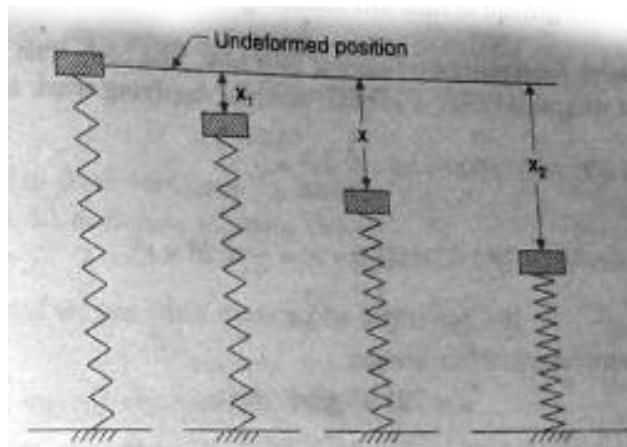


Fig.5.32

At any instant if the displacement is  $dx$ , the work done by spring force  $dU$  is given by

$$dU = -Fdx = -kx dx$$

Note: The negative sign is used since the force of spring is in the opposite direction of displacement.

Therefore, work done in displacement of the body from  $x_1$  to  $x_2$  is given by

$$U = \int_{S1}^{S2} -kx dx = -k \left[ \frac{x^2}{2} \right]_{S1}^{S2} = -\frac{1}{2}k(x_2^2 - x_1^2)$$

If the work done is to be found in moving from un-deformed position to displacement x, Then

$$U = \frac{1}{2}k(x^2 - 0^2) = -\frac{1}{2}kx^2 \quad \text{-----} \rightarrow (5.16)$$

Note: The negative sign is used with the expression for work done by the spring since whenever spring is deformed the force of spring is in the opposite direction of deformation. However, if a deformed spring is allowed to move towards its normal position work done will be positive, since the movement and the force of spring are in the same direction.

### Problems:

**SP5.20:** A pump lifts 40 m<sup>3</sup> of water to a height of 50m and delivers it with a velocity of 5 m/s. What is the amount of energy spent during this process? If the job is done in half an hour, what is the input power of the pump which has an overall efficiency of 70%?

**Solution.** Output energy of the pump is spent in lifting 40 m<sup>3</sup> of water to a height of 50 m and delivering it with the given kinetic energy of delivery.

Work done in lifting 40 m<sup>3</sup> of water to a height of 50 m is,

$$\begin{aligned} &= Wh \text{ where } W \text{ is weight of } 40 \text{ m}^3 \text{ of water} \\ &= 40 \times 9810 \times 50 \\ &= 1,96,20000 \text{ N-m} \end{aligned}$$

[Note: 1 m<sup>3</sup> of water weighs = 9810 Newtons]

$$\text{Kinetic energy at delivery} = \frac{1}{2} \times \frac{W}{g} v^2$$

$$= \frac{1}{2} \times \frac{40 \times 9810}{9.81} \times 5^2 \\ = 5,00000 \text{ N-m}$$

$$\text{Total energy spent} = 1,96,20000 + 5,00000$$

$$\text{Energy spent} = 2,01,20000 \text{ N-m Ans.}$$

This energy is spent by the pump in half an hour i.e., in  $30 \times 60 = 1800$  sec.

∴ Output power of pump = output energy spent per second

$$= \frac{2,01,20000}{1800} \\ = 11177.8 \text{ watts} \\ = 11.1778 \text{ kw.}$$

$$\text{Input power} = \frac{\text{output power}}{\text{efficiency}} \\ = \frac{11.1778}{0.7} \\ = 15.9683 \text{ kw Ans.}$$

**SP5.21:** A bullet of mass 30 gram is fired into body of mass 10kg which is suspended by a spring 0.8m long. Due to this impact, the body swings through an angle  $30^\circ$ . Find the velocity of the bullet.

**Solution.** Given : Mass of bullet ( $m$ ) = 30 g = 0.03 kg and mass of body ( $M$ ) = 10 kg.

Let

$u$  = Initial velocity of the bullet, and

$v$  = Velocity of the body after impact.

From the geometry of the figure, we find that when the body swings through  $30^\circ$  i.e. from  $A$  to  $B$ , it has gone up by a distance,

$$h = 0.8 - 0.8 \cos 30^\circ \\ = 0.8 - (0.8 \times 0.866) \text{ m} \\ = 0.1072 \text{ m.}$$

We know that Kinetic energy of the body and bullet after impact at  $A$ .

$$= \frac{(m+M)v^2}{2} = \frac{(0.03+10) \times v^2}{2} \text{ N-m} \\ = 5.015 v^2 \text{ N-m} \quad \dots(i)$$

and potential energy of the body at  $B$

$$= (m+M)gh = (10+0.03) 9.8 \times 0.1072 \text{ N-m} \\ = 10.54 \text{ N-m} \quad \dots(ii)$$

Since entire kinetic energy of the body and bullet is used in raising the body (from  $A$  to  $B$ ), therefore equating equations (i) and (ii),

$$5.015 v^2 = 10.54 \quad \text{or} \quad v = 1.45 \text{ m/s}$$

We also know that momentum of the body and bullet just after impact

$$= (10+0.03) 1.45 = 14.54 \text{ kg-m/s} \quad \dots(iii)$$

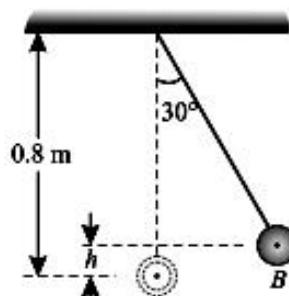


Fig.5.33

and momentum of the bullet just before impact

$$= 0.03 u \text{ kg-m/s} \quad \dots \text{(iv)}$$

Now equating equations (iii) and (iv),

$$14.54 = 0.03 u$$

$$\therefore u = \frac{14.54}{0.03} = 484.7 \text{ m/s} \quad \text{Ans.}$$

## KINETICS OF RIGID BODY ROTATION:

Consider the wheel as shown in fig.5.34 rotating about its axis in clockwise direction with an acceleration  $\alpha$ . Let  $\delta m$  be mass of an element at a distance  $r$  from the axis of rotation. If  $\delta p$  be the resulting force on this element.

$$u_p = u_m \times a, \text{ where } a \text{ is tangential acceleration}$$

$$\text{But } a = r\alpha, \text{ where } \alpha \text{ is angular acceleration}$$

$$\therefore u_p = u_m r \alpha$$

Rotational moment  $\delta M_t$  due to this force  $\delta p$  is given by

$$uM_t = u_p \times r$$

$$uM_t = u_m r^2 \alpha$$

$$M_t = \sum uM_t = \sum u_m r^2 \alpha$$

$$M_t = r \sum u_m r^2$$

$$M_t = r I \alpha, \text{ where } I \text{ is mass moment of inertia of the rotating body}$$

$$\text{Thus, } M_t = I \alpha \quad \text{-----} \rightarrow (5.17)$$

Note the similarity between the expression  $M_t = I \alpha$  and  $F = m a$  used in linear motion.

Force causes linear motion while rotational moment cause angular motion. The force is equal to the product of mass and the linear acceleration whereas rotational moment is the product of mass moment of inertia and the angular acceleration.

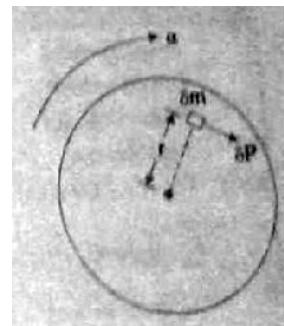


Fig.5.34

The product of the mass moment of inertia and the angular velocity of a rotating body is called Angular momentum. Thus

$$\text{Angular momentum} = I \dot{S} \quad \longrightarrow \rightarrow (5.18)$$

### Kinetic Energy of Rotating Bodies:

Consider the rotating body as shown in fig.5.35 with angular velocity  $\omega$ . Let  $\delta m$  be mass of an element which is at a distance  $r$  from the axis of rotation. Hence, if  $v$  is the linear velocity

$$v = r \dot{S}$$

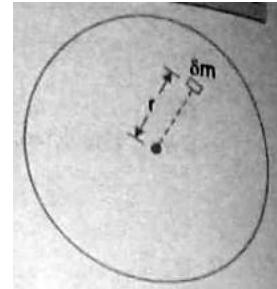


Fig.5.35

Now, Kinetic energy of the elemental mass

$$= \frac{1}{2} \text{um} v^2$$

Therefore, K.E. of the rotating body

$$= \sum \frac{1}{2} \text{um} v^2$$

$$= \sum \frac{1}{2} \text{um} r^2 \dot{S}^2$$

$$= \frac{1}{2} \dot{S}^2 \sum \text{um} r^2$$

But from the definition of mass moment of inertia,

$$I = \sum \text{um} r^2$$

$$\text{Therefore, } K.E. = \frac{1}{2} I \dot{S}^2$$

	Particulars	Linear motion	Angular motion
1	Displacement	$S$	$\theta$
2	Initial velocity	$u$	$\omega_0$
3	Final velocity	$V$	$\omega$
4	Acceleration	$a$	$\alpha$
5	Formula for final velocity	$v = u + at$	$\check{S} = \check{S}_0 + r t$
6	Formula for displacement	$S = ut + \frac{1}{2}at^2$	$\check{S} = \check{S}_0 t + \frac{1}{2}r t^2$
7	Formula in terms of displacement, velocity and acceleration	$v^2 = u^2 + 2aS$	$\check{S}^2 = \check{S}_0^2 + 2r \check{S}$
8	Force causing motion	$F = ma$	$M_t = I\check{r}$
9	Momentum	$mv$	$I\check{S}$
10	Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\check{S}^2$

## Problems,

**SP5.22:** Derive an expression for the fluctuation energy of flywheel.

It is a circular heavy wheel, generally, fitted to a rotating engine shaft to control variation in its speed during each cycle. Strictly speaking, it serves as a reservoir to store and restore energy by virtue of its inertia. Or in other words, it shares its energy during the period, when the supply of energy is more than the requirement; and releases it during the period when the supply of energy is less than the requirement.

It will be interesting to know that in all types of steam engines and I.C. engines, the power generated and supplied to the shaft is variable. As a matter of fact, the flywheel speed increases when it absorbs energy and decreases when it releases energy. Thus a flywheel is designed to keep the engine speed within the prescribed limits during each cycle.

The difference between the maximum and minimum speeds of an engine during a cycle is called the fluctuation of speeds. And the ratio of fluctuation of speed, to the mean engine speed, is called *coefficient of fluctuation of speed*.

Let

$M$  = Mass of the flywheel,

$I$  = Mass moment of inertia of the flywheel,

$N_1$  = Maximum speed of flywheel during a cycle,

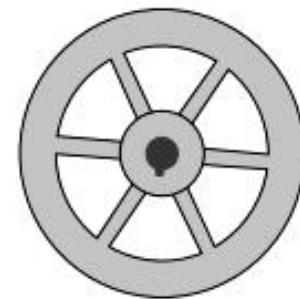


Fig.5.36 Flywheel.

$N_2$  = Minimum speed of flywheel during a cycle,

$\omega_1$  = Maximum angular velocity of flywheel during a cycle, and

$\omega_2$  = Minimum angular velocity of flywheel during a cycle.

We know that average speed of the flywheel,

$$N = \frac{1}{2} (N_1 + N_2) \omega = \frac{1}{2} (\omega_1 + \omega_2) = 0.5 (\omega_1 + \omega_2)$$

and average kinetic energy of the flywheel,

$$E = \frac{I \omega^2}{2}$$

∴ Fluctuation of energy,

$$\begin{aligned} E &= \text{Maximum K.E.} - \text{Minimum K.E.} = \frac{I \omega_1^2}{2} - \frac{I \omega_2^2}{2} = \frac{I}{2} (\omega_1^2 - \omega_2^2) \\ &= \frac{I}{2} (\omega_1 + \omega_2) (\omega_1 - \omega_2) = I \omega (\omega_1 - \omega_2) \quad \dots [\because \omega = 0.5 (\omega_1 + \omega_2)] \\ &= \frac{I \times 2\pi N}{60} \left( \frac{2\pi N_1}{60} - \frac{2\pi N_2}{60} \right) = \frac{4\pi^2}{3600} \times I N (N_1 - N_2) \\ &= \frac{\pi^2}{900} \times I N (N_1 - N_2). \end{aligned}$$

**SP5.23:** A flywheel of an engine has a mass of 6.5 tonnes and radius of gyration 1.8 metres. If the maximum and minimum speeds of the flywheel are 120 rpm and 118 rpm respectively, find the fluctuation of energy.

**Solution.** Given: Mass of flywheel ( $M$ ) = 6.5 t = 6500 kg; Radius of gyration ( $k$ ) = 1.8 m; Maximum speed of flywheel ( $N_1$ ) = 120 r.p.m. and minimum speed of flywheel ( $N_2$ ) = 118 r.p.m.

We know that average speed of the flywheel,

$$N = \frac{1}{2} (N_1 + N_2) = \frac{1}{2} (120 + 118) = 119 \text{ r.p.m.}$$

and mass moment of inertia,

$$I = Mk^2 = 6500 (1.8)^2 = 21060 \text{ kg-m}^2$$

∴ Fluctuation of energy,

$$\begin{aligned} E &= \frac{\pi^2}{900} \times IN (N_1 - N_2) = \frac{\pi^2}{900} \times 21060 \times 119 (120 - 118) \text{ N-m} \\ &= 54970 \text{ N-m} = 54.97 \text{ kN-m} = 54.97 \text{ kJ} \quad \text{Ans.} \end{aligned}$$

**SP5.24:** A flywheel of mass 8 tonnes starts from rest and gets up a speed of 180 rpm in 3 minutes. Find the average torque exerted on it, if the radius of gyration of the flywheel is 60cm.

**Solution.** Given: Mass of the flywheel ( $M$ ) = 8 t = 8000 kg; Initial angular speed ( $\omega_0$ ) = 0 (because it starts from rest); Final angular speed ( $\omega$ ) = 180 r.p.m. =  $\frac{180 \times 2\pi}{60} = 6\pi$  rad/s.; Time ( $t$ ) = 3 min = 180 s and radius of the gyration of the flywheel ( $k$ ) = 60 cm = 0.6 m.

Let  $\alpha$  = Constant angular acceleration of the flywheel

We know that the mass moment of inertia of the flywheel,

$$I = Mk^2 = 8000 \times (0.6)^2 = 2880 \text{ kg-m}^2$$

and final angular velocity of the flywheel ( $\omega$ ),

$$6\pi = \omega_0 + \alpha t = 0 + \alpha \times 180$$

$$\text{or} \quad \alpha = \frac{6\pi}{180} = 0.105 \text{ rad/s}^2$$

∴ Average torque exerted by the flywheel.

$$T = I\alpha = 2880 \times 0.105 = 302.4 \text{ N-m} \quad \text{Ans.}$$

**SP5.25:** A flywheel is made up of steel ring 40mm thick and 200mm wide plate with mean diameter of 2 metres. If initially the flywheel is rotating at 300 rpm, find the time taken by the wheel in coming to rest due to frictional couple of 100 N-m. Take mass density of the steel as 7900 kg/m<sup>3</sup>. Neglect the effect of the spokes.

**Solution.** Given: Thickness of flywheel = 40 mm = 0.04 m; Width of flywheel = 200 mm = 0.2 m; Mean diameter of flywheel = 2 m or mean radius ( $r$ ) = 1 m; Initial angular speed ( $\omega_0$ ) = 300 r.p.m. = 5 r.p.s. =  $10\pi$  rad/s; Frictional couple = 100 N-m and density of steel = 7900 kg/m<sup>3</sup>.

Let

$\alpha$  = Constant angular acceleration of flywheel and

$t$  = Time taken by the flywheel in coming to rest.

We know that volume of flywheel,

$$= \pi \times 2 \times 0.2 \times 0.04 = 0.05 \text{ m}^3$$

∴ Mass of the flywheel,

$$M = 0.05 \times 7900 = 395 \text{ kg}$$

and mass moment of inertia,  $I = Mr^2 = 395 \times (1)^2 = 395 \text{ kg-m}^2$

We know that frictional couple (or torque  $T$ )

$$100 = I\alpha = 395 \alpha$$

$$\therefore \alpha = \frac{100}{395} = 0.253 \text{ rad/s}^2$$

and final angular velocity of the flywheel,

$$0 = \omega_0 - \alpha t = 10\pi - 0.253 t \quad \dots(\text{Minus sign due to retardation})$$

$$\therefore t = \frac{10\pi}{0.253} = 124.2 \text{ s} \quad \text{Ans.}$$

**SP5.26:** A flywheel weighing 50 kN and having radius of gyration 1metre loses its speed from 400 rpm to 280 rpm in 2 minutes. Calculate i) the retarding torque acting on it, ii) change in its kinetic energy during the above period and iii) change in its angular momentum during the same period.

Solution:

$$\omega_0 = 400 \text{ rpm} = \frac{400 \times 2\pi}{60} = 41.888 \text{ rad/sec}$$

$$\omega = 280 \text{ rpm} = \frac{280 \times 2\pi}{60} = 29.322 \text{ rad/sec}$$

but,

$$\begin{aligned}
 t &= 2 \text{ min} = 120 \text{ sec} \\
 \omega &= \omega_0 + \alpha t \\
 \alpha &= \frac{\omega - \omega_0}{t} = \frac{29.3224 - 41.888}{120} \\
 &= -0.1047 \text{ rad/sec}^2
 \end{aligned}$$

i.e., retardation is  $0.1047 \text{ rad/sec}^2$ Weight of flywheel =  $50 \text{ kN} = 50,000 \text{ N}$ Radius of gyration  $k = 1 \text{ m}$ 

$$I = mk^2 = \frac{50,000}{9.81} \times 1^2 = 5096.84$$

**(i) Retarding Torque Acting on the Flywheel**

$$\begin{aligned}
 &= I\alpha = 5096.84 \times 0.1047 \\
 &= 533.74 \text{ N}\cdot\text{m}
 \end{aligned}$$

Ans.

**(ii) Change in Kinetic Energy**

= Initial K.E. - Final K.E.

$$\begin{aligned}
 &= \frac{1}{2} I\omega_0^2 - \frac{1}{2} I\omega^2 \\
 &= \frac{1}{2} \times 5096.84 (41.888^2 - 27.322^2) \\
 &= 22,80442.9 \text{ N}\cdot\text{m}
 \end{aligned}$$

Ans.

**(iii) Change in its Angular Momentum**

$$\begin{aligned}
 &= I(\omega_0 - \omega) \\
 &= 5096.84 (41.888 - 27.322) \\
 &= 64048.93 \text{ N}\cdot\text{sec.}
 \end{aligned}$$

Ans.

**SP5.27:** A pulley of weight 400N has a radius of 0.6m. A block of 600N is suspended by a tight rope wound round the pulley, the other end being attached to the pulley as shown in fig.5.37 Determine the resulting acceleration of the weight and the tension in the rope.

**Solution.** Let  $a$  be the resulting acceleration and  $T$  be the tension in the rope. Hence angular acceleration of pulley.

$$\alpha = a/r = a/0.6 = 1.667a \text{ m/sec}^2 \quad \dots(1)$$

An inertia force of  $(600/g) a$  may be considered and the dynamic equilibrium condition can be written for the block as:

$$T = \left( 600 - \frac{600}{9.81} a \right) \quad \dots(2)$$

From Kinetic equation for pulley, we get,

$$M_t = I\alpha \text{ i.e., } T(0.6) = I \times 1.667a \quad \dots(3)$$

$$I = \frac{Mr^2}{2} = \frac{400}{9.81} \times \frac{0.6^2}{2}$$

From (3),

$$T = \frac{400}{9.81} \times \frac{0.6^2}{2} \times \frac{1667 a}{0.6}$$

$$= \frac{200}{9.81} a \text{ N}\cdot\text{m}$$

Substituting it in (2), we get

$$\frac{200}{9.81} a = 600 - \frac{600}{9.81} a$$

$$a = \frac{600 \times 9.81}{800}$$

$$= 7.358 \text{ m/sec}^2 \quad \text{Ans.}$$

$$\therefore T = \frac{200}{9.81} \times 7.358$$

$$= 150 \text{ N} \quad \text{Ans.}$$

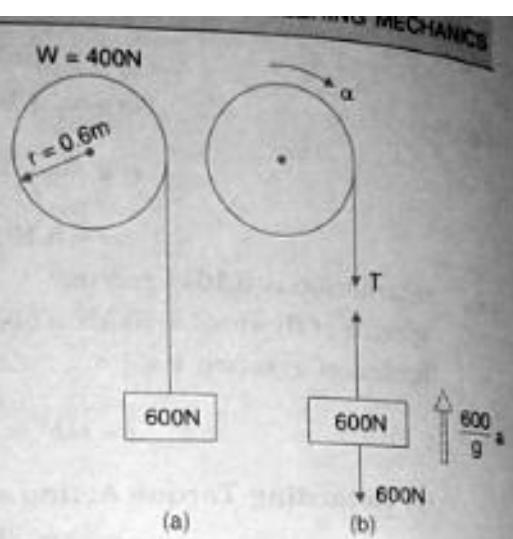


Fig.5.37

**SP5.28:** The composite pulley shown in fig.5. weighs 800 N and has a radius of gyration of 0.6 m. The 2000 N and 4000 N blocks are attached to the pulley by inextensible strings as shown in the figure. Neglecting weight of the strings, determine the tension in the strings and angular acceleration of the pulley.

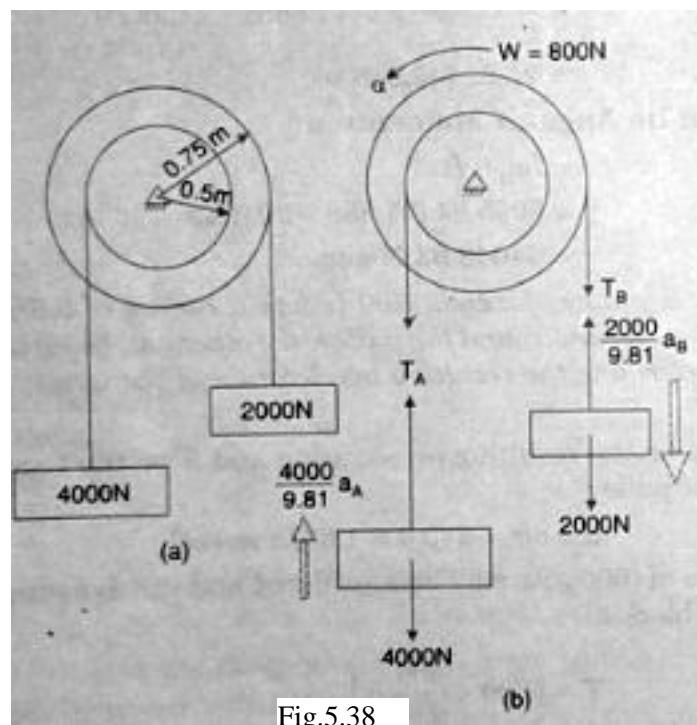


Fig.5.38

(b)

**Solution.** Since the moment of 4000 N block is more than that of 2000 N block about the axis of rotation, the pulley rotates in anticlockwise direction as shown in Fig. 19.7 (b). Let  $a_A$  be acceleration of 4000 N block and  $a_B$  that of 2000 N block. Then  
 $a_A = 0.5 \alpha$  and  $a_B = 0.75\alpha$   
where,  $\alpha$  - angular acceleration of pulley.

Writing dynamic equilibrium equation for the two blocks, we get

$$T_A = 4000 \left( 1 - \frac{a_A}{9.81} \right) = 4000 \left( 1 - \frac{0.5 \alpha}{9.81} \right)$$

$$T_B = 2000 \left( 1 + \frac{a_B}{9.81} \right) = 2000 \left( 1 + \frac{0.75 \alpha}{9.81} \right)$$

From Kinetic equation of pulley, we have

$$M_I = I\alpha$$

i.e.,  $T_A \times 0.5 - T_B \times 0.75 = \frac{800}{9.81} \cdot 0.6^2 \alpha$

$$4000 \left( 1 - \frac{0.5}{9.81} \alpha \right) 0.5 - 2000 \left( 1 + \frac{0.75}{9.81} \alpha \right) 0.75 = \frac{800}{9.81} (0.6)^2 \alpha$$

$$245.97 \alpha = 500$$

$$\alpha = 2.033 \text{ rad/sec}^2$$

Ans.

$$T_A = 4000 \left( 1 - \frac{0.5}{9.81} \times 2.033 \right) = 3585.58 \text{ N}$$

Ans.

$$T_B = 2000 \left( 1 + \frac{0.75}{9.81} \times 2.033 \right) = 2310.83 \text{ N}$$

Ans.

**SP5.29:** A cylinder weighing 500N is welded to a 1m long uniform bar of 200N as shown in fig.5.39 Determine the acceleration with which the assembly will rotate about point A, if released from rest in horizontal position. Determine the reaction of A at this instant.

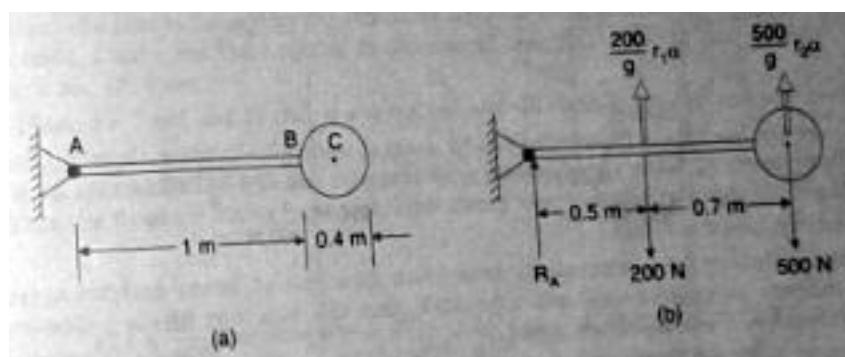


Fig.5.39

**Solution.** Let  $\alpha$  be the angular acceleration with which the assembly will rotate. Let  $I$  be the mass moment of inertia of the assembly about the axis of rotation A. Using the transfer formula  $I = Ig + Md^2$  we can assemble mass moment of inertia about the axis of rotation through A.

Mass moment of inertia of the bar about A

$$= \frac{1}{2} \times \frac{200}{9.81} \times 1^2 + \frac{200}{9.81} \times (0.5)^2 = 6.7968$$

Now moment of inertia of the cylinder about A

$$= \frac{1}{2} \times \frac{500}{9.81} \times 0.2^2 + \frac{500}{9.81} \times 1.2^2 = 74.414$$

∴ Mass moment of inertia of the system about A

$$I = 6.7958 + 74.41 = 81.2097$$

Rotational moment about A (Ref. Fig. 19.8(b))

$$M_r = 200 \times 0.5 + 500 \times 1.2 = 700 \text{ N}\cdot\text{m}$$

equating it to  $I\alpha$ , we get,

$$81.2097 \alpha = 700 \quad \text{or} \quad \alpha = 8.6197 \text{ rad/sec}$$

Instantaneous acceleration of rod AB is vertical and its magnitude is given by

$$= r_1 \alpha = 0.5 \times 8.6197 \\ = 4.310 \text{ m/sec}$$

Similarly the instantaneous acceleration of the cylinder is also vertical and is equal to

$$r_2 \alpha = 1.2 \times 8.6197 = 10.344 \text{ m/sec}$$

Applying D'Alembert's dynamic equilibrium equation to the system of forces shown in Fig. 19.8 (b), we get

$$R_A = 200 + 500 - \frac{200}{9.81} \times 4.3100 - \frac{500}{9.81} \times 10.344$$

i.e.,

$$R_A = 84.934 \text{ N}$$

Ans.

Ans.

**SP5.30:** Road AB weighing 200N is welded to the rod CD weighing 100N as shown in fig.5.40 The assembly is hinged at A and is freely held. Determine the instantaneous vertical and horizontal reactions of A when a horizontal force of 300N acts at a distance of 0.75m from A.

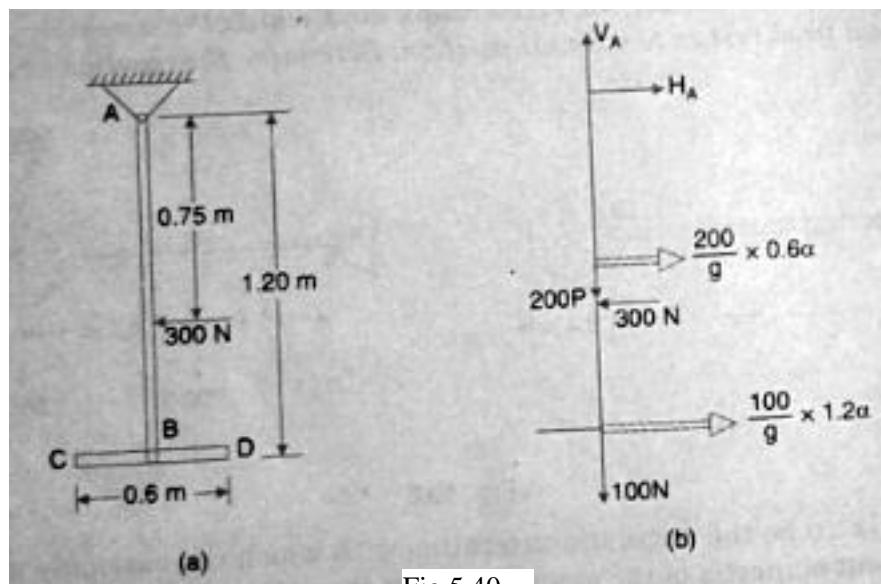


Fig.5.40

**Solution.** Mass moment of inertia of AB about axis of rotation

$$A = \frac{1}{12} \times \frac{200}{9.81} \times 12^2 + \frac{200}{9.81} \times 0.6^2 = 9.786$$

Mass moment of inertia of rod CD about A

$$= \frac{1}{12} \times \frac{100}{9.81} \times 0.6^2 + \frac{100}{9.81} \times 1.2^2 = 147.0$$

∴ Total mass moment of the system about A

$$9.786 + 147.0 = 156.78$$

Let  $\alpha$  be the instantaneous angular acceleration. Writing the kinetic equation for the rotation about A, we get

$$I\alpha = M$$

$$156.786 \alpha = 300 \times 0.75$$

$$\alpha = 1.4351 \text{ rad/sec}$$

At the instant 300 N force is applied, the linear accelerations of AB and CD are horizontal and are equal to  $0.6 \alpha$  and  $1.2 \alpha$  respectively. Hence the inertia forces are as shown in Fig. 19.9 (b) by dotted arrows. Let  $V_A$  and  $H_A$  be the vertical and horizontal reactions at A. Writing the dynamic equilibrium conditions, we get,

$$V_A = 200 + 100 = 300 \text{ N}$$

Ans.

and

$$H_A = 300 - \frac{200}{9.81} \times 0.6 \alpha - \frac{100}{9.81} \times 1.2 \alpha$$

Substituting the value of  $\alpha$ , we get,

$$H_A = 264.891 \text{ N}$$

Ans.

**SP5.31:** A train loaded with cars has a total mass of 6 tonnes is hauled up an incline of 1 in 30 by a rope coiled on a winding drum at the top of the incline. The drum is 1.0m in diameter and has a mass of 1 tonnes and has a radius of gyration of 0.45m. The drum

axle friction is negligible. If a torque of 3 kN-m is applied to the drum from a driving motor, determine the tension in the rope.

**Solution.** Given: Mass of train and cars ( $M$ ) = 6 t; Inclination ( $\sin \theta$ ) = 1/30; Diameter of drum ( $d$ ) = 1 m or radius ( $r$ ) = 0.5 m; Mass of drawn ( $m$ ) = 1 t, Radius of gyration ( $k$ ) = 0.45 m and torque applied ( $T$ ) = 3 kN-m.

Let

$P$  = Tension in the rope

$a$  = Linear acceleration of the train, and

$\alpha$  = Angular acceleration of the drum.

First of all, consider the motion of the cage, which is being pulled by the rope due to the torque on the drum.

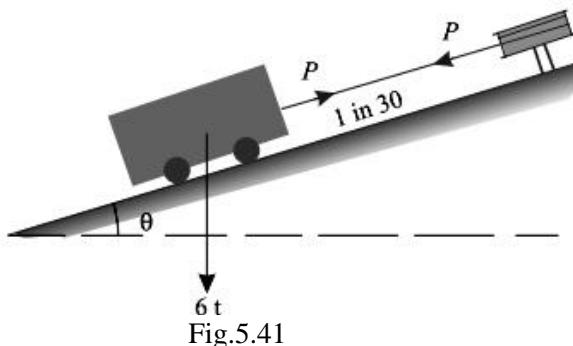


Fig.5.41

We know that component of the weight of the cage along the plane

$$= Mg \sin \theta = 6 \times 9.8 \times \frac{1}{30} = 1.96 \text{ kN}$$

As the cage loaded with cars is moving upwards, therefore resultant force along the plane

$$= P - 1.96 \quad \dots(i)$$

Since the cage is moving with an acceleration of ( $a$ ), therefore force acting on it

$$= Ma = 6 a \quad \dots(ii)$$

Equating equations (i) and (ii),

$$P - 1.96 = 6 a \quad \dots(iii)$$

Now consider motion of the drum, which is rotating about its axis due to torque. We know that moment of inertia of the drum,

$$I = mk^2 = 1 \times (0.45)^2 = 0.2 \text{ kg-m}^2$$

We know that linear acceleration of the train is equal to the angular acceleration of the drum.

$$\therefore a = r\alpha = 0.5 \alpha \quad \text{or} \quad \alpha = \frac{a}{0.5} = 2a$$

Now accelerating torque,

$$T_1 = I\alpha = 0.2 \times 2a = 0.4a \text{ kN-m}$$

and torque due to tension in the rope,

$$T_2 = P \times r = P \times 0.5 = 0.5P$$

$$\therefore \text{Total torque, } T = T_1 + T_2 = 0.4a + 0.5P$$

We know that this total torque is equal to the torque applied to the drum.

$$3 = 0.4a + 0.5P$$

$$\text{or } P = \frac{3 - 0.4a}{0.5} = 6 - 0.8a$$

Substituting this value of  $P$  in equation (iii),

$$6 - 0.8 a - 1.96 = 6 a \quad \text{or} \quad 6.8 a = 4.04$$

$$\therefore a = \frac{4.04}{6.8} = 0.59 \text{ m/s}^2$$

Now substituting this value of  $a$  in equation (iii),

$$P - 1.96 = 6 \times 0.59 = 3.54$$

or

$$P = 3.54 + 1.96 = 5.5 \text{ kN}$$

**SP5.32:** Two bodies  $A$  and  $B$ , of mass 150 kg and 75 kg respectively are supported by a string of negligible mass and pass over a composite pulley. The bodies rest on two smooth inclined planes as shown in Fig.5.42

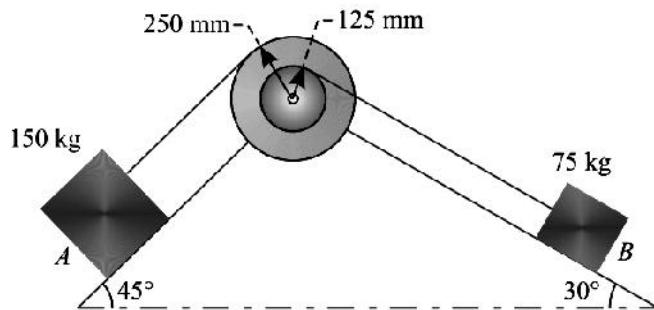


Fig.5.42

If the pulley has a mass of 75 kg and radius of gyration of 100 mm, find the accelerations of the masses  $A$  and  $B$  and pulls in the string. Neglect friction in the bearings.

**Solution.** Given: Mass of the body  $A$  ( $m_1$ ) = 150 kg; Mass of the body  $B$  ( $m_2$ ) = 75 kg; Mass of the pulley ( $M$ ) = 75 kg; Radius of gyration ( $k$ ) = 100 mm = 0.1 m; External radius of the pulley ( $r_1$ ) = 250 mm = 0.25 m and internal radius of the pulley ( $r_2$ ) = 125 mm = 0.125 m.

*Pulls in the strings*

Let

$P_1$  = Pull in the string carrying 150 kg mass, and

$P_2$  = Pull in the string carrying 75 kg mass.

From the geometry of the masses, we find that turning moment of mass 150 kg (i.e.,  $150 \sin 45^\circ \times 0.25 = 150 \times 0.707 \times 0.25 = 26.5 \text{ kg-m}$ ) is more than that of the mass 75 kg (i.e.,  $75 \sin 30^\circ \times 0.125 = 75 \times 0.5 \times 0.125 = 4.7 \text{ kg-m}$ ). It is thus obvious that the 150 kg mass will come downwards and the 75 kg mass will go upwards, when the system is released.

Let

$a_1$  = Acceleration of the 150 kg mass,

$a_2$  = Acceleration of the 75 kg mass, and

$\alpha$  = Angular acceleration of the pulley.

We know that mass moment of inertia of the pulley,

$$I = MK^2 = 75 (0.1)^2 = 0.75 \text{ kg-m}^2$$

First of all, consider the motion of 150 kg mass, which is coming down. We know that the force acting on it, along the plane, =  $m_1 g \sin \theta_1 = 150 \times 9.8 \sin 45^\circ = 150 \times 9.8 \times 0.707 = 1039 \text{ N}$ .

(downwards) and  $P_1$  (upwards). As the mass is moving downwards, therefore resultant force

$$= 1039 - P_1 \quad \dots(i)$$

Since the mass is moving downwards with an acceleration ( $a_1$ ), therefore force acting on the body

$$= 150 a_1 \quad \dots(ii)$$

Equating equations (i) and (ii),

$$1039 - P_1 = 150 a_1 \quad \dots(iii)$$

Now consider the motion of 75 kg mass, which is going upwards. We know that the force acting on it, along the plane,  $= m_2 g \sin \theta_2 = 75 \times 9.8 \sin 30^\circ = 75 \times 9.8 \times 0.5 = 367.5 \text{ N}$  (downwards) and  $P_2$  (upwards). As the mass is moving upwards, therefore resultant force

$$= P_2 - 367.5 \quad \dots(iv)$$

Since the mass is moving upwards with an acceleration ( $a_2$ ), therefore force acting on the body

$$= 75 a_2 \quad \dots(v)$$

Equating equations (iv) and (v),

$$P_2 - 367.5 = 75 a_2 \quad \dots(vi)$$

Now consider the motion of the pulley, which is rotating about its axis due to downward motion of the 150 kg mass tied to the string. We know that linear acceleration of the 150 kg mass is equal to the angular acceleration of the pulley.

$$\therefore a_1 = r_1 \alpha = 0.25 \alpha$$

$$\text{Similarly } a_2 = r_2 \alpha = 0.125 \alpha$$

$$\text{and torque, } T = P_1 r_1 - P_2 r_2 = P_1 \times 0.25 - P_2 \times 0.125 \quad \dots(vii)$$

We also know that torque on the pulley,

$$T = I \alpha = 0.75 \alpha \quad \dots(viii)$$

Equating equations (vii) and (viii),

$$0.25 P_1 - 0.125 P_2 = 0.75 \alpha$$

$$\text{or } P_1 - 0.5 P_2 = 3 \alpha \quad \dots(\text{Multiplying by 4})$$

$$\therefore P_1 = 0.5 P_2 + 3 \alpha \quad \dots(ix)$$

Substituting the value of  $P_1$  in equation (iii),

$$1039 - (0.5 P_2 + 3 \alpha) = 150 a_1 = 150 \times 0.25 \alpha$$

$$1039 - 0.5 P_2 - 3 \alpha = 37.5 \alpha$$

$$\therefore 1039 - 0.5 P_2 = 37.5 \alpha + 3\alpha = 40.5 \alpha$$

Multiplying both sides by 2

$$2078 - P_2 = 81 \alpha \quad \dots(x)$$

From equation (vi), we find that

$$P_2 - 367.5 = 75 a_2 = 75 \times 0.125 \alpha = 9.4 \alpha \quad \dots(xi)$$

Adding equations (x) and (xi),

$$1710.5 = 90.4 \alpha$$

$$\therefore \alpha = \frac{1710.5}{90.4} = 18.9 \text{ rad/s}^2$$

Now substituting the value of  $\alpha$  in equation (x),

$$2078 - P_2 = 81 \alpha = 81 \times 18.9 = 1531$$

$$\therefore P_2 = 2078 - 1531 = 547 \text{ N} \quad \text{Ans.}$$

Again substituting the value of  $\alpha$  and  $P_2$  in equation (ix),

$$P_1 = (0.5 \times 547) + (3 \times 18.9) = 330 \text{ N} \quad \text{Ans.}$$

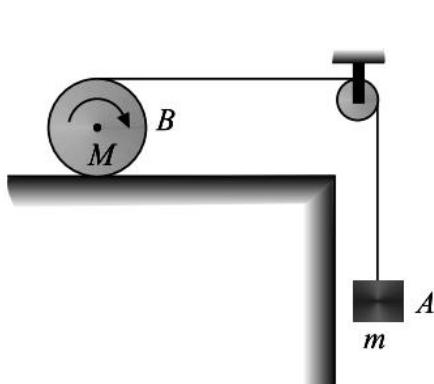
#### Acceleration of the masses A and B

We know that the acceleration of mass A (i.e., 150 kg),

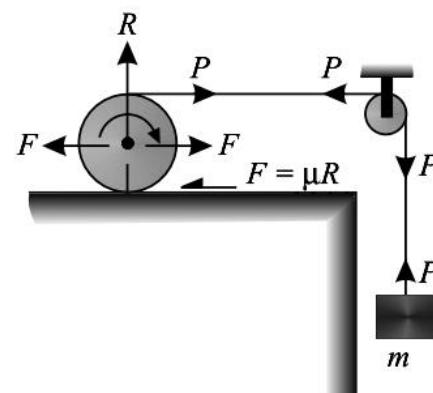
$$a_1 = r_1 \alpha = 0.25 \times 18.9 = 4.72 \text{ m/s}^2 \quad \text{Ans.}$$

$$\text{Similarly } a_2 = r_2 \alpha = 0.125 \times 18.9 = 2.36 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.33:** Derive the force of friction and acceleration relation for motion of a body rolling on a rough horizontal plane without slipping.



(a)



(b)

Fig.5.43 Body rolling on a rough horizontal plane without slipping.

Consider a body A tied to a string passing over a smooth pulley and pulling another circular body B, which rolls on a rough horizontal plane without slipping as shown in Fig. (a).

Let

$m$  = Mass of the hanging body A,

$P$  = Tension in the string,

$M$  = Mass of the rolling body B,

$I$  = Moment of inertia of the rolling body B,

$k$  = Radius of gyration of the rolling body B,

$r$  = Radius of the rolling body  $B$ ,

$\mu$  = Coefficient of friction between the plane and the body,

$a$  = Linear acceleration of the rolling body  $B$ , and

$\alpha$  = Angular acceleration of the rolling body  $B$ .

From the geometry of the motion, we find that the acceleration of the hanging body  $A$  will be  $(2a)$ . We know that the normal reaction on the horizontal plane for the body  $B$ ,

$$R = Mg$$

and force of friction,  $F = \mu R = \mu Mg$

Since the rolling body tends to roll towards right, therefore force of friction will act towards left as shown in Fig. 31.14 (b). Let us introduce two equal and opposite forces (each equal to the force of friction,  $F$ ) through the centre of the rolling body as shown in Fig. 31.14. (b).

A little consideration will show, that these two forces will not effect the motion of the system. Now the rolling body is subjected to the following forces :

1. A force equal to  $P - F$  (acting towards right)
2. A couple whose moment is equal to  $F \times r$  (responsible for rolling the body).

First of all, consider motion of the hanging body  $A$ , which is coming down. We know that the forces acting on it are  $mg$  (downwards) and  $P$  (upwards). As the body is moving downwards, therefore resultant force

$$= mg - P \quad \dots(i)$$

Since the body is moving downwards with an acceleration of  $(2a)$ , therefore force acting on this body

$$= m \times 2 a = 2 ma \quad \dots(ii)$$

Equating equations (i) and (ii),

$$mg - P = 2 ma$$

or  $P = mg - 2 ma \quad \dots(iii)$

Now consider the linear motion (neglecting rolling for the time being) of the body  $B$ , on the rough horizontal plane due to the force  $(P - F)$  acting on it. Since the body is moving with an acceleration  $(a)$  towards right, therefore force acting on it.

$$= Ma \quad (\text{towards right})$$

We know that the force acting towards right is responsible for this motion. Therefore

$$P - F = Ma \quad \dots(iv)$$

Now consider the circular motion (i.e. rolling) of the rolling body due to the couple (equal to  $P \times r$ ) which is responsible for rolling the body. We know that linear acceleration of the body is equal to its angular acceleration.

∴

$$a = r\alpha$$

and torque acting on the body,

$$T = I\alpha$$

Now equating the couple (responsible for rolling) and torque on the body,

$$F \times r = I\alpha$$

$$\begin{aligned} F \times r^2 &= I\alpha r && \dots(\text{Multiplying both sides by } r) \\ &= Ia && \dots(\because a = r\alpha) \end{aligned}$$

Now equating the couple (responsible for rolling) and torque on the body,

$$F \times r = I\alpha$$

$$F \times r^2 = I\alpha r$$

$$= Ia$$

...(Multiplying both sides by  $r$ )  
...( $\because a = r\alpha$ )

$$\therefore F = \frac{Ia}{r^2} = \frac{Mk^2a}{r^2}$$

...( $\because I = mk^2$ )

Now substituting the value of  $F$  in equation (iv),

$$P - \frac{Mk^2a}{r^2} = Ma$$

and now substituting the value of  $P$  from equation (iii) in the above equation,

$$mg - 2ma - \frac{Mk^2a}{r^2} = Ma$$

or  $mg = Ma + 2ma + \frac{Mk^2a}{r^2} = a \left( M + 2m + \frac{Mk^2}{r^2} \right)$

$$\therefore a = \frac{mg}{M + 2m + \frac{Mk^2}{r^2}}$$

The above expression shows that the acceleration of the rolling body is independent of the value of coefficient of friction.

### SP5.34:

*Find the acceleration (a) on a solid right circular roller A of mass 20 kg when it is being pulled by another body B of mass 10 kg along a horizontal plane as shown in*

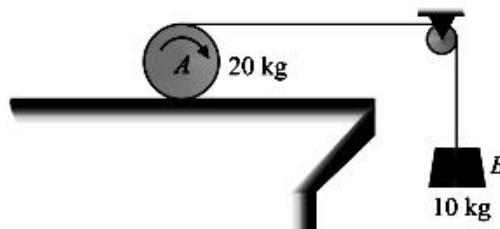


Fig.5.44

*The mass B is attached to the end of a string wound round the circumference of the roller. Assume that there is no slipping of the roller and the string is inextensible.*

**Solution.** Given: Mass of roller ( $M$ ) = 20 kg and mass of hanging body  $B$  ( $m$ ) = 10 kg

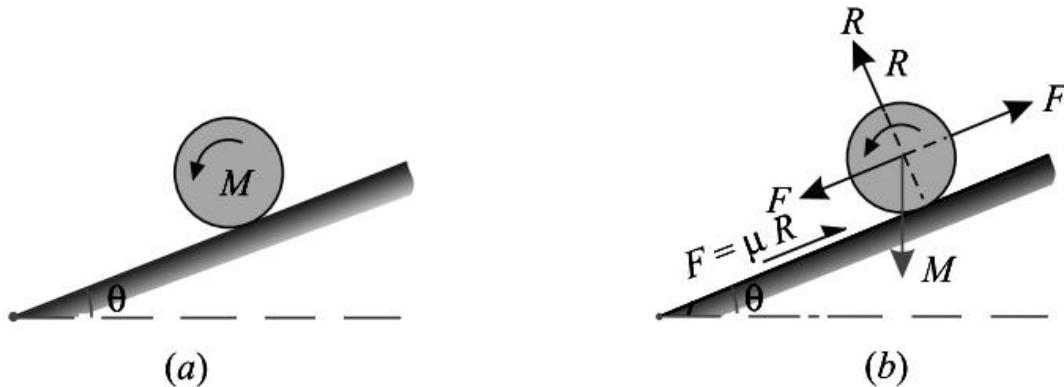
We know that for a solid right circular roller of radius  $r$ ,  $3k^2 = 0.5r^2$

and acceleration of the roller on the horizontal plane,

$$a = \frac{mg}{M + 2m + \frac{Mk^2}{r^2}} = \frac{10 \times 9.8}{20 + (2 \times 10) + \frac{20 \times 0.5r^2}{r^2}} \text{ m/s}^2$$

$$= 1.96 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.35:** Derive an expression of coefficient of friction and radius of gyration for the motion of a body rolling down a rough inclined plane without slipping.



**Fig. 5.45 Body rolling down on a rough inclined plane without slipping.**

Consider a circular body rolling down an inclined rough plane without slipping as shown in Fig. 5.45 (a).

Let

$M$  = Mass of the body,

$I$  = Moment of inertia of the body,

$k$  = Radius of gyration of the body,

$r$  = Radius of the body,

$\theta$  = Inclination of the plane with the horizontal,

$\mu$  = Coefficient of friction between the plane and the body,

$a$  = Linear acceleration of the body, and

$\alpha$  = Angular acceleration of the body.

We know that the normal reaction,

$$R = Mg \cos \theta$$

$$\text{force of friction, } F = \mu R = \mu Mg \cos \theta$$

Since the body tends to roll downwards, therefore the force of friction will act upwards as shown in Fig. 31.16 (b). Let us introduce two equal and opposite forces (each equal to the force of friction  $F$ ), through the centre of the body as shown in Fig. 31.16 (b).

A little consideration will show, that these two forces will not affect the motion of the system. Now the rolling body is subjected to the following two forces :

1. A force equal to  $Mg \sin \theta - F$  (acting downwards)
2. A couple whose moment is equal to  $F \times r$  (responsible for rolling down the body).

First of all, consider downward motion (neglecting rolling for the time being) of the body due to force  $(Mg \sin \theta - F)$  acting on it. Since the body is moving with an acceleration ( $a$ ), therefore force acting on it

$$= Ma$$

We know that the force acting on it is responsible for this motion. Therefore

$$Mg \sin \theta - F = Ma \quad \dots(i)$$

Now consider the circular motion (i.e., rolling) of the body due to the couple (equal to  $P \times r$ ) which is responsible for the rolling of the body. We know that linear acceleration of the body is equal to its angular acceleration.

$$\therefore a = r\alpha$$

$$\text{and torque on the body, } T = I\alpha$$

Now equating the couple (responsible for rolling) and torque acting on the body

$$F \times r = I\alpha$$

$$F \times r^2 = I\alpha r \quad \dots(\text{Multiplying both sides by } r) (\because a = r\alpha) \\ = Ia$$

$$\therefore F = \frac{Ia}{r^2} = \frac{Mk^2 a}{r^2} \quad \dots (\because I = mk^2) \quad \dots(ii)$$

and now substituting the value of  $F$  in equation (i),

$$Mg \sin \theta - \frac{Mk^2 a}{r^2} = ma$$

$$Mg \sin \theta = Ma + \frac{Mk^2 a}{r^2} = Ma \left( 1 + \frac{k^2}{r^2} \right)$$

$$\therefore a = \frac{g \sin \theta}{1 + \frac{k^2}{r^2}} = \frac{g \sin \theta}{\frac{r^2 + k^2}{r^2}}$$

The above expression shows that the acceleration of the rolling body is independent of its mass. Now substituting the value of  $(a)$  in equation (ii),

$$F = \frac{Mk^2}{r^2} \times \frac{g \sin \theta}{\frac{r^2 + k^2}{r^2}} = \frac{Mk^2 g \sin \theta}{k^2 + r^2}$$

Now for the body (or wheel) to roll down without slipping, the applied force must be less than (or equal to) the available force of friction, in order to fulfil the condition of rolling without slipping. Or in other words

$$F \leq \mu mg \cos \theta$$

$$\frac{Mk^2 g \sin \theta}{k^2 + r^2} \leq \mu Mg \cos \theta$$

$$\tan \theta \leq \mu \left( \frac{k^2 + r^2}{k^2} \right)$$

The above equation may also be written as :

$$\mu \geq \frac{\tan \theta}{\left( \frac{k^2 + r^2}{k^2} \right)}$$

**Note:** The above expression shows that the value of ( $\mu$ ) should be more or equal to the value obtained from the right hand side. Thus for the minimum value of ( $\mu$ ), the above expression may be written as :

$$\mu = \frac{\tan \theta}{\left( \frac{k^2 + r^2}{k^2} \right)}$$

**SP5.36:** A sphere rolls down a plane inclined at  $30^\circ$  to the horizontal. Find the minimum value of the coefficient of friction between the sphere and the plane, so that the sphere may roll without slipping.

**Solution.** Given: Inclination of plane  $\theta = 30^\circ$

We know that for a sphere of radius  $r$ ,

$$k^2 = 0.4 r^2$$

and minimum value of coefficient of friction,

$$\mu = \frac{\tan \theta}{\frac{k^2 + r^2}{k^2}} = \frac{\tan 30^\circ}{\frac{0.4 r^2 + r^2}{0.4 r^2}} = \frac{0.5774}{3.5} = 0.165 \quad \text{Ans.}$$

**SP5.37:** A solid uniformly thick wheel of radius 1 m and mass 40 kg is released with no initial velocity at the top of an inclined plane, which makes an angle of  $30^\circ$  with the horizontal. It rolls down without slipping. Determine i) the minimum value of coefficient of friction, ii) the velocity of the centre of the wheel after it has travelled a distance 4m down the inclined plane.

**Solution.** Given: Radius of the wheel ( $r$ ) = 1 m; \*Mass of the wheel ( $m$ ) = 40 kg and inclination of plane ( $\theta$ ) =  $30^\circ$

(i) *Minimum value of coefficient of friction*

We know that for a uniformly thick wheel,

$$k^2 = 0.5 r^2$$

and minimum value of coefficient of friction,

$$\mu = \frac{\tan \theta}{\frac{k^2 + r^2}{k^2}} = \frac{\tan 30^\circ}{\frac{0.5 r^2 + r^2}{0.5 r^2}} = \frac{0.5774}{3} = 0.192 \quad \text{Ans.}$$

(ii) *Velocity of the centre of the wheel after it has travelled a distance of 4 m*

Let  $v$  = Velocity of the centre of the wheel.

We know that acceleration of the wheel when it rolls down the plane,

$$a = \frac{g \sin \theta}{\frac{k^2 + r^2}{r^2}} = \frac{9.8 \sin 30^\circ}{\frac{0.5 r^2 + r^2}{r^2}} = \frac{9.8 \times 0.5}{1.5} = 3.27 \text{ m/s}^2$$

and

$$v^2 = u^2 + 2as = 0 + 2 \times 3.27 \times 4 = 26.16$$

$$v = 5.1 \text{ m/s} \quad \text{Ans.}$$

## MECHANICAL VIBRATIONS COVERING:

A mechanical vibration is the motion of a particle or a body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable because of the increased stresses and energy losses which accompany them. They should therefore be eliminated or reduced as much as possible by appropriate design. The analysis of vibrations has become increasingly important in recent years owing to the current trend toward higher-speed machines and lighter structures. There is every reason to expect that this trend will continue and that an even greater need for vibration analysis will develop in the future.

The analysis of vibrations is a very extensive subject to which entire texts have been devoted. This study will therefore be limited to the simpler types of vibrations, namely, the vibrations of a body or a system of bodies with one degree of freedom. A mechanical vibration generally results when a system is displaced from a position of stable equilibrium. The system tends to return to this position under the action of restoring forces (either elastic forces, as in the case of a mass attached to a spring, or gravitational forces, as in the case of a pendulum). But the system generally reaches its original position with a certain acquired velocity which carries it beyond that position. Since the process can be repeated indefinitely, the system keeps moving back and forth across its position of equilibrium. The time interval required for the system to complete a full cycle of motion is called the period of the vibration. The number of cycles per unit time defines the frequency, and the maximum displacement of the system from its position of equilibrium is called the amplitude of the vibration.

When the motion is maintained by the restoring forces only, the vibration is said to be a free vibration. When a periodic force is applied to the system, the resulting motion is described as a forced vibration. When the effects of friction can be neglected, the vibrations are said to be un-damped. However, all vibrations are actually damped to some degree. If a free vibration is only slightly damped, its amplitude slowly decreases until, after a certain time, the motion comes to a stop. But if damping is large enough to prevent any true vibration, the system then slowly regains its original position. A

damped forced vibration is maintained as long as the periodic force which produces the vibration is applied. The amplitude of the vibration, however, is affected by the magnitude of the damping forces.

### **Basic terminology:**

- 1. Periodic Motion**
- 2. Time Period**
- 3. Cycle**
- 4. Frequency**
- 5. Amplitude**
- 6. Natural Frequency**
- 7. The fundamental mode of vibration**
- 8. Degree of freedom**
- 9. Simple harmonic motion**
- 10. Damping**
- 11. Resonance**
- 12. Mechanical System**

#### **1. Periodic Motion:**

The motion which repeats itself after an equal interval of times is called Periodic motion.

#### **2. Time Period:**

This is defined as time to complete one cycle is called a Time period and it is normally expressed in seconds (s).

#### **3. Cycle:**

It is motion completed during one time period.

#### **4. Frequency:**

This is defined as the number of cycles describes in one second. This SI unit is Hertz.

#### **5. Amplitude:**

The maximum displacement of the vibrating body from its equilibrium position.

#### **6. Natural frequency:**

When no external force acts on the system after giving its an initial displacement, the body vibrates. The vibration is called as free vibration and the frequency is called Natural frequency. This is expressed in rad/sec or Hertz.

**7. The fundamental mode of vibration:**

The fundamental mode of vibration is the model having the lowest natural frequency

**8. Degree of Freedom:**

The minimum number of independent coordinates required to specify the motion of a system at any instant is known as the degree of freedom.

**9. Simple Harmonic Motion:**

The motion of the body to and fro about a fixed point is called simple harmonic motion.

**10. Damping:**

It is resistance to the motion of a vibrating body. The vibration associated with the resistance is known as Damped Vibration.

**11. Resonance:**

When the frequency of external excitation is equal to the natural frequency of the vibrating body, the amplitude of vibration becomes excessively large. This concept is called resonance.

**12. Mechanical System:**

The system consists of Mass, stiffness, and damping are known as a mechanical system.

**FREE AND FORCED VIBRATIONS:****1) Free vibration:**

If a system, after an initial disturbance is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

**Types of free vibrations:**

- A) Linear / Longitudinal vibrations: When the disc is displaced vertically downwards by an external force and released as shown in the figure 4, all the particles of the rod and disc move parallel to the axis of shaft. The rod is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the rod. The vibration occurs is known as Linear/Longitudinal vibrations.
- B) Transverse vibrations: When the rod is displaced in the transverse direction by an external force and released as shown in the figure 5, all the particles of rod and disc move approximately perpendicular to the axis of the rod. The shaft is straight and bends alternately inducing bending stresses in the rod. The vibration occurs is known as transverse vibrations.
- C) Torsional vibrations: When the rod is twisted about its axis by an external force and released as shown in the figure 6, all the particles of the rod and disc move in a circle about the axis of the rod. The rod is subjected to twist and torsional shear stress is induced. The vibration occurs is known as torsional vibrations.

**2) Forced vibration:**

If a system is subjected to an external force (often a repeating type of force), the resulting vibration is known as forced vibration.

**RESONANCE AND ITS EFFECTS:**

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as resonance occurs and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines and airplane wings have been associated with the occurrence of resonance.

**DEGREE OF FREEDOM:**

The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines **the degree of freedom of the system**.

**A single degree of freedom system** requires only one coordinate to describe its position at any instant of time.

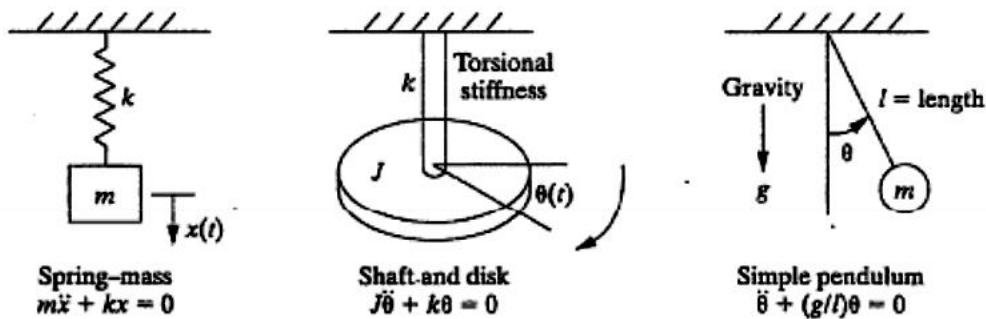


Fig.5.46 Single degree of freedom

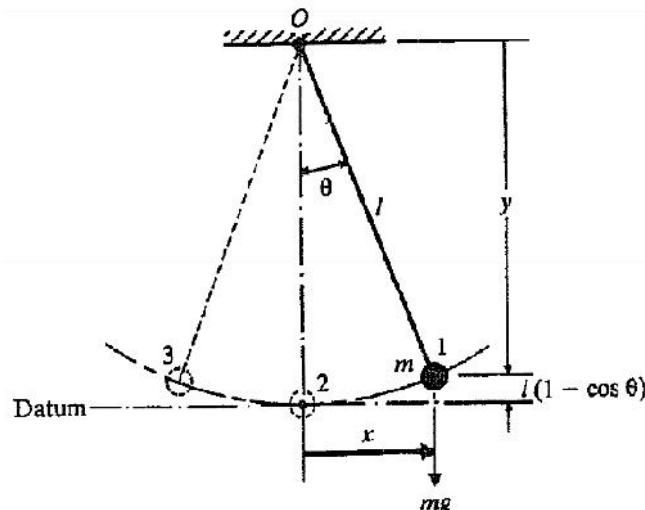


Fig.5.47 Simple pendulum

For the simple pendulum in the fig.5. , the motion can be stated either in terms of  $\theta$  or  $x$  and  $y$ . If the coordinates  $x$  and  $y$  are used to describe the motion, it must be recognized that these coordinates are not independent. They are related to each other through the relation.

$$x^2 + y^2 = l^2 \quad \text{-----} \rightarrow (5.19)$$

Where,  $l$  is the constant length of the pendulum. Thus any one coordinate can be describe the motion of the pendulum. In this example, we find that the choice of  $\theta$  as the independent coordinate will be more convenient than the choice of  $x$  and  $y$ .

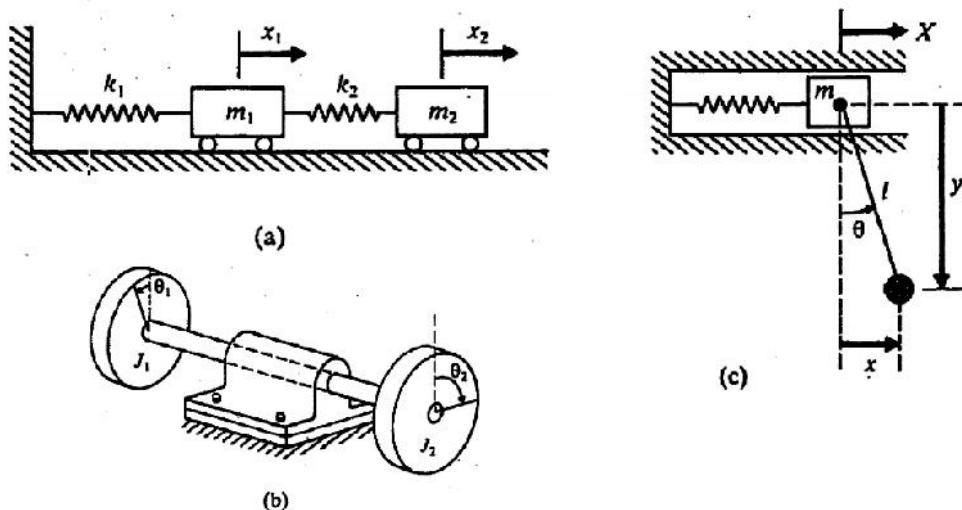


Fig.5.48 Two degree of freedom systems

Some examples of two degree of freedom systems are shown in fig.5. The fig.5. a shows a two mass – two spring system that is described by two linear coordinates  $x_1$  and  $x_2$ . Fig.5.b denotes the motion of system either by  $x$  and  $\theta$  or by  $x$ ,  $y$  and  $x$ . Fig.5.c shows that a two rotor system whose motion can be specified in terms of  $\theta_1$  and  $\theta_2$ .

## DERIVATION FOR FREQUENCY AND AMPLITUDE OF FREE VIBRATION WITHOUT DAMPING AND SINGLE DEGREE OF FREEDOM:

### 1) Free vibrations of particles (Simple harmonic motion):

Consider a body of mass  $m$  attached to a spring of constant  $k$  (Fig.5.49a). Since at the present time we are concerned only with the motion of its mass centre, we will refer to this body as a particle. When the particle is in static equilibrium, the forces acting on it are its weight  $\mathbf{W}$  and the force  $\mathbf{T}$  exerted by the spring, of magnitude  $T = kd_{st}$ , where  $d_{st}$  denotes the elongation of the spring. We have, therefore,

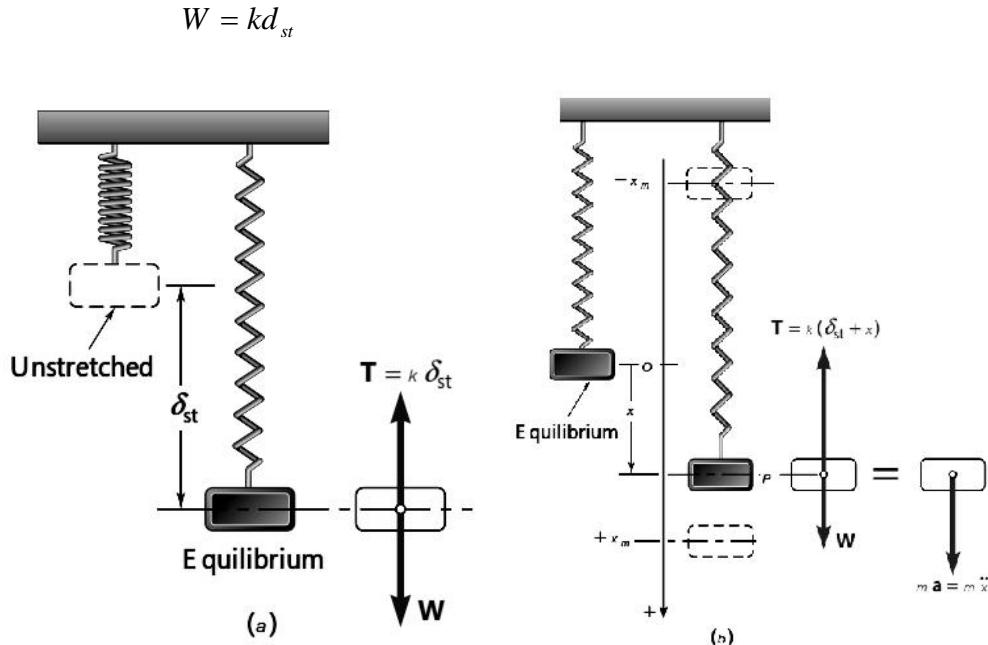


Fig.5.49

Suppose now that the particle is displaced through a distance  $x_m$  from its equilibrium position and released with no initial velocity. If  $x_m$  has been chosen smaller than  $\delta_{st}$ , the particle will move back and forth through its equilibrium position; a vibration of amplitude  $x_m$  has been generated. Note that the vibration can also be produced by imparting a certain initial velocity to the particle when it is in its equilibrium position  $x=0$  or, more generally, by starting the particle from any given position  $x=x_0$  with a given initial velocity  $v_0$ .

To analyze the vibration, let us consider the particle in a position P at some arbitrary time t (Fig. 5.49b). Denoting by x the displacement OP measured from the equilibrium position O (positive downward), we note that the forces acting on the particle are its weight **W** and the force **T** exerted by the spring which, in this position, has a magnitude  $T = k(d_{st} + x)$ . Recalling that  $W = kd_{st}$ , we find that the magnitude of the resultant **F** of the two forces (positive downward) is

$$F = W - k(d_{st} + x) = -kx \quad \text{-----} \rightarrow (5.20)$$

Thus the resultant of the forces exerted on the particle is proportional to the displacement OP measured from the equilibrium position. Recalling the sign

convention, we note that  $\mathbf{F}$  is always directed toward the equilibrium position O. Substituting for  $\mathbf{F}$  into the fundamental equation  $\mathbf{F} = m \mathbf{a}$  and recalling that  $\mathbf{a}$  is the second derivative  $\ddot{\mathbf{x}}$  of  $\mathbf{x}$  with respect to  $t$ , we write

$$m \ddot{x} + kx = 0 \quad \longrightarrow \quad (5.21)$$

Note that the same sign convention should be used for the acceleration  $\ddot{x}$  and for the displacement  $x$ , namely, positive downward.

The motion defined by Eq. 5.21 is called a simple harmonic motion. It is characterized by the fact that the acceleration is proportional to the displacement and of opposite direction. We can verify that each of the functions  $x_1 = \sin\left(\sqrt{\frac{k}{m}} t\right)$  and  $x_2 = \cos\left(\sqrt{\frac{k}{m}} t\right)$  satisfies Eq. 5.21. These functions, therefore, constitute two particular solutions of the differential equation 5.21. The general solution of Eq. 5.21 is obtained by multiplying each of the particular solutions by an arbitrary constant and adding. Thus, the general solution is expressed as

$$x = C_1 x_1 + C_2 x_2 = C_1 \sin\left(\sqrt{\frac{k}{m}} t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}} t\right) \quad \longrightarrow \quad (5.22)$$

We note that  $x$  is a periodic function of the time  $t$  and does, therefore, represent a vibration of the particle P. The coefficient of  $t$  in the expression we have obtained is referred to as the natural circular frequency of the vibration and is denoted by  $\nu_n$ . We have

$$\text{Natural circular frequency} = \nu_n = \sqrt{\frac{k}{m}} \quad \longrightarrow \quad (5.23)$$

Substituting for  $\sqrt{\frac{k}{m}}$  into Eq. 5.22, we write

$$x = C_1 \sin \nu_n t + C_2 \cos \nu_n t \quad \longrightarrow \quad (5.24)$$

This is the general solution of the differential equation

$$\ddot{x} + V_n^2 x = 0 \quad \text{-----} \rightarrow (5.25)$$

which can be obtained from Eq. 5.21 by dividing both terms by  $m$  and observing that

$\frac{k}{m} = V_n^2$ . Differentiating twice both members of Eq. 5.24 with respect to  $t$ , we obtain

the following expressions for the velocity and the acceleration at time  $t$ :

$$\dot{v} = \dot{x} = C_1 V_n \cos V_n t - C_2 V_n \sin V_n t \quad \text{-----} \rightarrow (5.26)$$

$$\ddot{a} = \ddot{x} = -C_1 V_n^2 \sin V_n t - C_2 V_n^2 \cos V_n t \quad \text{-----} \rightarrow (5.27)$$

The values of the constants  $C_1$  and  $C_2$  depend upon the initial conditions of the motion. For example, we have  $C_1=0$  if the particle is displaced from its equilibrium position and released at  $t = 0$  with no initial velocity, and we have  $C_2 = 0$  if the particle is started from  $O$  at  $t = 0$  with a certain initial velocity. In general, substituting  $t = 0$  and the initial values  $x_0$  and  $v_0$  of the displacement and the velocity into Eqs. 5.24 and 5.26, we find that  $C_1 = \frac{v_0}{V_n}$  and  $C_2 = x_0$ .

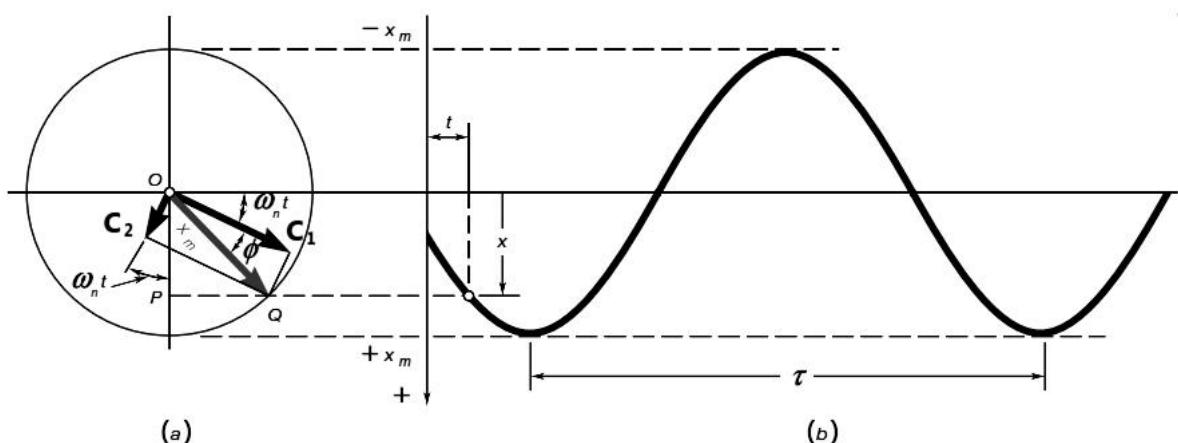


Fig.5.50 Simple harmonic motion

The expressions obtained for the displacement, velocity, and acceleration of a particle can be written in a more compact form if we observe that Eq.5.24 expresses that the displacement  $x = OP$  is the sum of the  $x$  components of two vectors  $\mathbf{C}_1$  and  $\mathbf{C}_2$ , respectively, of magnitude  $C_1$  and  $C_2$ , directed as shown in Fig. 5.49a. As  $t$  varies, both

vectors rotate clockwise; we also note that the magnitude of their resultant  $\vec{OQ}$  is equal to the maximum displacement  $x_m$ . The simple harmonic motion of P along the x axis can thus be obtained by projecting on this axis the motion of a point Q describing an auxiliary circle of radius  $x_m$  with a constant angular velocity  $V_n$  (which explains the name of natural circular frequency given to  $V_n$ ). Denoting by  $f$  the angle formed by the vectors  $\vec{OQ}$  and  $\mathbf{C}_1$ , we write

$$OP = OQ \sin(V_n t + f) \quad \text{-----} \rightarrow (5.28)$$

which leads to new expressions for the displacement, velocity, and acceleration of  $\pi$ :

$$x = x_m \sin(V_n t + f) \quad \text{-----} \rightarrow (5.29)$$

$$v = \dot{x} = x_m V_n \cos(V_n t + f) \quad \text{-----} \rightarrow (5.30)$$

$$a = \ddot{x} = -x_m V_n^2 \sin(V_n t + f) \quad \text{-----} \rightarrow (5.31)$$

The displacement time curve is represented by a sine curve (fig.5.50b); the maximum value  $x_m$  of the displacement is called the amplitude of the vibration, and the angle  $f$  which defines the initial position of Q on the circle is called the phase angle. We note from fig. 5.50 that a full cycle is described as the angle  $V_n t$  increases by  $2\pi$  rad. The corresponding value of  $t$ , denoted by  $t_n$ , is called the period of the free vibration and is measured in seconds. We have

$$\text{Period} = t_n = \frac{2\pi}{V_n} \quad \text{-----} \rightarrow (5.32)$$

The number of cycles described per unit of time is denoted by  $f_n$  and is known as the natural frequency of the vibration. We write

$$\text{Natural frequency} = f_n = \frac{1}{t_n} = \frac{V_n}{2\pi} \quad \text{-----} \rightarrow (5.33)$$

The unit of frequency is a frequency of 1 cycle per second, corresponding to a period of 1 s. In terms of base units the unit of frequency is thus  $1/s$  or  $s^{-1}$ . It is called a

hertz (Hz) in the SI system of units. It also follows from Eq. 5.33 that a frequency of 1 s<sup>-1</sup> or 1 Hz corresponds to a circular frequency of  $2\pi$  rad/s. In problems involving angular velocities expressed in revolutions per minute (rpm), we have

$$1\text{ rpm} = \frac{1}{60} \text{ s}^{-1} = \frac{1}{60} \text{ Hz}, \text{ or } 1 \text{ rpm} = (2\pi/60) \text{ rad/s.}$$

Recalling that  $V_n$  was defined in Eqn.5.23 in terms of the constant  $k$  of the spring and the mass  $m$  of the particle, we observe that the period and the frequency are independent of the initial conditions and of the amplitude of the vibration. Note that  $t_n$  and  $f_n$  depend on the mass rather than on the weight of the particle and thus are independent of the value of  $g$ .

The velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve, but with different phase angles. From Eqs. 5.30 and 5.31, we note that the maximum values of the magnitudes of the velocity and acceleration are

$$v_m = x_m V_n \text{ and } a_m = x_m V_n^2 \quad \text{-----} \rightarrow (5.34)$$

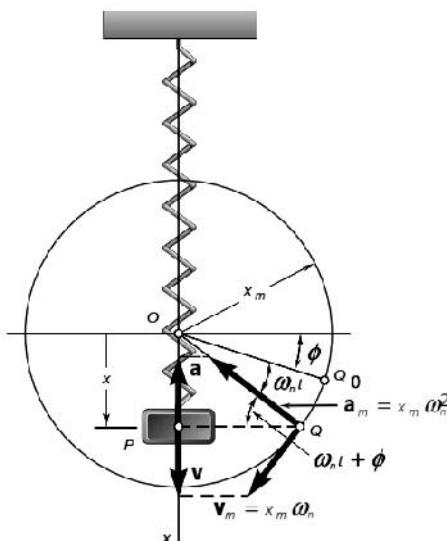


Fig.5. 51

Since the point Q describes the auxiliary circle, of radius  $x_m$ , at the constant angular velocity  $V_n$ , its velocity and acceleration are equal, respectively, to the expressions (5.34). Recalling Eqs. (5.30) and (5.31), we find, therefore, that the velocity

and acceleration of P can be obtained at any instant by projecting on the x axis vectors of magnitudes  $v_m = x_m V_n$  and  $a_m = x_m V_n^2$  representing, respectively, the velocity and acceleration of Q at the same instant (Fig.5.51).

The results obtained are not limited to the solution of the problem of a mass attached to a spring. They can be used to analyze the rectilinear motion of a particle whenever the resultant F of the forces acting on the particle is proportional to the displacement x and directed toward O. The fundamental equation of motion  $F = ma$  can then be written in the form of Eq. (5.25), which is characteristic of a simple harmonic motion. Observing that the coefficient of x must be equal to  $V_n^2$ , we can easily determine the natural circular frequency  $V_n$  of the motion. Substituting the value obtained for  $V_n$  into Eqs. (5.32) and (5.33), we then obtain the period  $t_n$  and the natural frequency  $f_n$  of the motion.

## 2. Free vibrations of rigid bodies

The analysis of the vibrations of a rigid body or of a system of rigid bodies possessing a single degree of freedom is similar to the analysis of the vibrations of a particle. An appropriate variable, such as a distance x or an angle u, is chosen to define the position of the body or system of bodies, and an equation relating this variable and its second derivative with respect to t is written. If the equation obtained is of the same form as (5.25), i.e., if we have

$$\ddot{x} + V_n^2 x = 0 \quad \text{or} \quad \ddot{u} + V_n^2 u = 0 \quad \longrightarrow \rightarrow (5.35)$$

the vibration considered is a simple harmonic motion. The period and natural frequency of the vibration can then be obtained by identifying  $V_n$  and substituting its value into Eqs. (5.32) and (5.33).

In general, a simple way to obtain one of Eqs. (5.35) is to express that the system of the external forces is equivalent to the system of the effective forces by drawing a free-body-diagram equation for an arbitrary value of the variable and writing the

appropriate equation of motion. We recall that our goal should be the determination of the coefficient of the variable  $x$  or  $u$ , not the determination of the variable itself or of the derivative  $\ddot{x}$  or  $\ddot{u}$ . Setting this coefficient equal  $V_n^2$ , we obtain the natural circular frequency  $v_n$ , from which  $t_n$  and  $f_n$  can be determined.

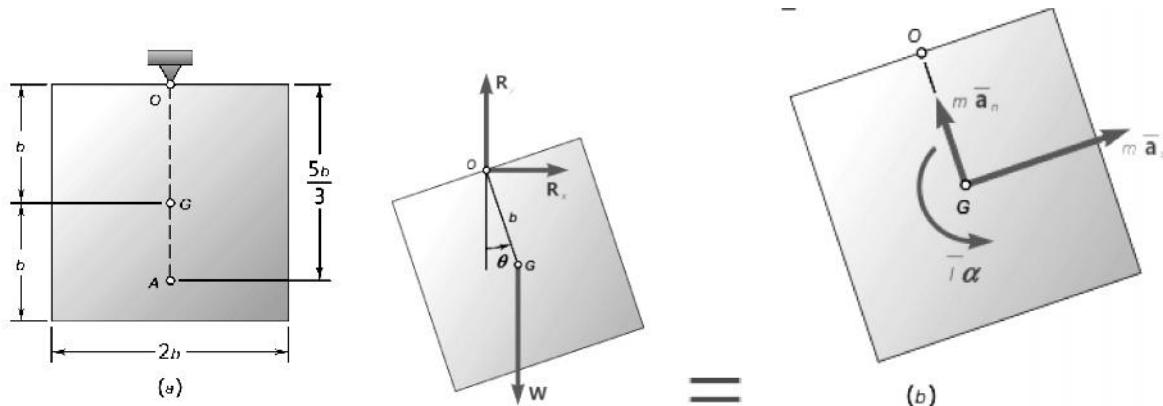


Fig.5.52

The method we have outlined can be used to analyze vibrations which are truly represented by a simple harmonic motion, or vibrations of small amplitude which can be approximated by a simple harmonic motion. As an example, let us determine the period of the small oscillations of a square plate of side  $2b$  which is suspended from the midpoint  $O$  of one of its sides (Fig. 5.52a). We consider the plate in an arbitrary position defined by the angle  $u$  that the line  $OG$  forms with the vertical and draw a free-body-diagram equation to express that the weight  $W$  of the plate and the components  $R_x$  and  $R_y$  of the reaction at  $O$  are equivalent to the vectors  $ma_t$  and  $ma_n$  and to the couple  $\bar{I}A$  (Fig. 5.52b). Since the angular velocity and angular acceleration of the plate are equal, respectively, to  $\dot{u}$  and  $\ddot{u}$  magnitudes of the two vectors are, respectively,  $mb\dot{u}$  and  $mb\dot{u}^2$  while the moment of the couple is  $\bar{I}\ddot{u}$ . In previous applications of this method, we tried whenever possible to assume the correct sense for the acceleration. Here, however, we must assume the same positive sense for  $u$  and  $\dot{u}$  in order to obtain an equation of the form (5.35). Consequently, the angular acceleration  $\ddot{u}$  will be assumed positive counter clockwise, even though this assumption is obviously unrealistic. Equating moments about  $O$ , we write

$$-W(b \sin u) = (mb \ddot{u})b + \bar{I} \ddot{u}$$

Noting that  $\bar{I} = \frac{1}{12}m[(2b)^2 + (2b)^2] = \frac{2}{3}mb^2$  and  $W = mg$ , we obtain

$$\ddot{u} + \frac{3}{5} \frac{g}{b} \sin u = 0 \quad \rightarrow (5.36)$$

Comparison with (5.35) shows that the equation obtained is that of a simple harmonic motion and that the natural circular frequency  $V_n$  of the oscillations is equal to  $(3g/5b)^{1/2}$ . Substituting into (5.32), we find that the period of the oscillations is

$$T_n = \frac{2\pi}{V_n} = 2\pi \sqrt{\frac{5b}{3g}} \quad \rightarrow (5.37)$$

The result obtained is valid only for oscillations of small amplitude. A more accurate description of the motion of the plate is obtained by comparing Eqs. (5.35). We note that the two equations are identical if we choose  $l$  equal to  $5b/3$ . This means that the plate will oscillate as a simple pendulum of length  $l = 5b/3$ . The point A of the plate located on line OG at a distance  $l = 5b/3$  from O is defined as the centre of oscillation corresponding to O (Fig. 5.52a).

## TYPES OF PENDULUMS:

It is common to differentiate between four types of pendulums:

### 1. Simple or Mathematical Pendulum:

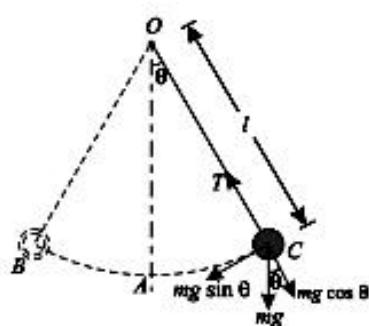


Fig.5.53 Simple pendulum

A simple pendulum, in its simplest form, consists of a heavy bob suspended at the end of a light inextensible, flexible string and the other end of the string is fixed at  $O$  as shown in fig5.53

Let  $l$  = Length of the string in metres, and

$m$  = Mass of the bob in kg (such that its weight is  $m.g$  newtons).

We know that the pendulum is in equilibrium, when the bob is at  $A$ . If the bob is brought to  $B$  or  $C$  and released, it will start vibrating between the positions  $B$  and  $C$  with  $A$  as the mean position.

It has been observed that if the angle  $\angle AOC$  is very small, the bob will move with simple harmonic motion.

Consider the equilibrium of the system at  $C$ . The weight  $mg$  of the bob can be resolved into two components *i.e.*,  $mg \cos \theta$  and  $mg \sin \theta$  at right angles to each other.

The component  $mg \cos \theta$  will act along the thread. This will balance the tension in the string as shown in fig. 5.53. The other component  $mg \sin \theta$ , being unbalanced, will give rise to an acceleration in the direction  $CA$ .

Let  $a$  = Acceleration caused by the component.

$\therefore$  Force responsible for this acceleration

$$= mg \sin \theta \quad \text{-----} \rightarrow (5.38)$$

We also know that the force

$$= \text{Mass} \times \text{Acceleration} = m.a \quad \text{-----} \rightarrow (5.39)$$

Equating equations (5.38) and (5.39),

$$ma = mg \sin \theta$$

$$\therefore a = g \sin \theta$$

Since the angle  $\theta$  is very small, therefore substituting  $\sin \theta \approx \theta$  in the above equation,

$$a = g \quad \text{-----} \rightarrow (5.40)$$

From the geometry of the figure, we know that

$$" = \frac{\text{Length of the arc}}{\text{Radius}} = \frac{AC}{l}$$

$$a = g" = g \times \frac{AC}{l} \quad \text{or}$$

$$\frac{AC}{a} = \frac{l}{g} \quad \text{[where AC = Displacement of the body]}$$

We know that in a simple harmonic motion, the time period,

$$t = 2f \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2f \sqrt{\frac{AC}{a}} = 2f \sqrt{\frac{l}{g}}$$

**Notes 1.** The motion of the bob from one extremity to the other (i.e. from B to C or C to B) is known as a beat or swing. It is thus obvious, that one beat =  $\frac{1}{2}$  oscillation.

Therefore time period for one beat  $f \sqrt{\frac{l}{g}}$

2. A pendulum, which executes one beat per second, is known as a second's pendulum.

### LAWS OF SIMPLE PENDULUM:

The following laws of a simple pendulum are important from the subject point of view:

**1. Law of isochronism.** It states, “The time period (t) of simple pendulum does not depend on its amplitude of vibrations, and remain the same provided the angular amplitude (θ) does not exceed  $4^\circ$ .”

**2. Law of mass.** It states, “The time period (t) of a simple pendulum does not depend upon the mass of the body suspended at the free end of the string.”

**3. Law of Length.** It states, “The time period (t) of a simple pendulum is proportional to  $\sqrt{l}$ , where l is the length of the string.”

**4. Law of gravity.** It states, “The time period (t) of a simple pendulum is inversely proportional to  $\sqrt{g}$  . where g is the acceleration due to gravity.”

**Notes.** The above laws of a simple pendulum are true from the equation of the time period i.e.

$$t = 2\pi \sqrt{\frac{l}{g}}$$

**2. Compound or Physical Pendulum :** A rigid body suspended from a fixed horizontal axis. The body may oscillate in a vertical plane due to the action of gravity.

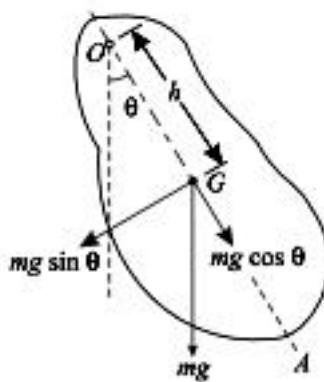


Fig.5.54 Compound pendulum

A compound pendulum, in its simplest form, consists of a rigid body suspended vertically at  $O$  and oscillating with a small amplitude under the action of the force of gravity. At some instant, let the position of the rigid body be at its extreme position as shown in fig.5.54.

Let  $m$  = mass of the pendulum

$h$  = Distance between point of suspension ( $O$ ) and the centre of gravity ( $G$ ) of the body.

A little consideration will show, that if the pendulum is given a small angular displacement , then the moment of the couple tending to restore the pendulum in the equilibrium position  $OA$

$$= mgh \sin \theta = mgh \quad \text{-----} \rightarrow (5.41)$$

(Since  $\theta$  is very small, therefore substituting  $\sin \theta = \theta$  )

Now let  $I_0$  = Mass moment of inertia of the body about  $O$

$\alpha$  = Angular acceleration of the body.

$\therefore$  Disturbing moment due to angular displacement

$$= Ir \quad \text{-----} \rightarrow (5.42)$$

Equating two moments *i.e.* equations (5.41) and (5.42),

$$mgh = I_0 r$$

$$r = \frac{mgh}{I_0}$$

We know that periodic time,

$$t = 2f \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2f \sqrt{\frac{r}{r}} = 2f \sqrt{\frac{mgh}{I_0}} = 2f \sqrt{\frac{I_0}{mgh}}$$

$$\text{-----} \rightarrow (5.43)$$

and frequency of motion,

$$f_n = \frac{1}{t} = \frac{1}{2f} \sqrt{\frac{mgh}{I_0}} \quad \text{-----} \rightarrow (5.44)$$

**Note.** The above formula for periodic time (or frequency of motion) may also be expressed in terms of radius of gyration ( $k$ ) as discussed below :

We know from the theorem of parallel axis that the mass moment of inertia of the pendulum about  $O$ ,

$$I_0 = I_g + mh^2 = mk^2 + mh^2 = m(k^2 + h^2) \quad I$$

Now substituting the value of  $I_0$  in equation (5.43),

$$t = 2f \sqrt{\frac{m(k^2 + h^2)}{mgh}} = 2f \sqrt{\frac{k^2 + h^2}{gh}}$$

$$\text{Similarly, } f_n = \frac{1}{t} = \frac{1}{2f} \sqrt{\frac{gh}{k^2 + h^2}}$$

**3. Conical Pendulum:** Similar to the simple pendulum except that the weight suspended by the chord moves in uniform speed around the circumference of a circle in the horizontal plane.

A conical pendulum, in its simplest form, consists of a heavy bob suspended at the end of a light inextensible flexible string; and the other end of the string is fixed at  $O$  as shown in fig.5.55. The pendulum bob rotates about the vertical axis with a uniform angular velocity. A little consideration will show, that the bob moves in a horizontal plane and describes a circle.

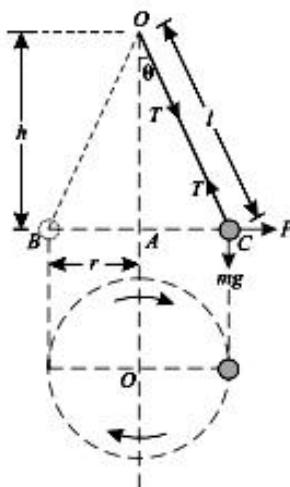


Fig.5.55 Conical pendulum

Let  $l$  = Length of the string,

=Angular velocity of the bob,

$r$  = Radius of the horizontal circle described by the bob,

$h$  = Vertical distance of the bob from  $O$ ,

=Inclination of the string with the vertical

$m$  = Mass of the bob in kg (such that its weight is  $m.g$ . newtons).

We know that the pendulum is in equilibrium, when the bob is at *A*. But when the bob is made to rotate, it will move outwards due to centrifugal force. Now consider equilibrium of the bob at *C*. We know that at this point, the forces acting on it are:

1. Weight (equal to  $mg$ ) of the bob acting downwards in Newton's.
2. Tension in the string (equal to  $T$ ) in Newton's.
3. Centrifugal force ( $P$  equal to  $m \omega^2 r$ ) in Newton's.

Resolving the forces horizontally and equating the same,

$$T \sin \theta = m \omega^2 r \quad \text{-----} \rightarrow (5.45)$$

or

$$T = \frac{m \omega^2 r}{\sin \theta} = \frac{m \omega^2 r}{\left(\frac{r}{l}\right)} = m \omega^2 l \quad \text{-----} \rightarrow (5.46)$$

Now taking moments about *O*, and equating the same,

$$(m \omega^2 r)h = mgr$$

$$\therefore \omega^2 = \frac{g}{h}$$

We know that the periodic time (*i.e.* time taken by the bob for one revolution),

$$t = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{h}{g}}$$

and now resolving the forces vertically and equating the same,

$$T \cos \theta = mg \quad \text{-----} \rightarrow (5.47)$$

or  $T = \frac{mg}{\cos \theta} = \frac{mg}{\sqrt{1 - \frac{h}{l}}} = \frac{mgl}{h} \quad \text{-----} \rightarrow (5.48)$

Now dividing equation (5.45) by (5.47),

$$\frac{T \sin \theta}{T \cos \theta} = \frac{m \omega^2 r}{mg}$$

$$\tan \alpha = \frac{S^2 r}{g} \quad \longrightarrow \rightarrow (5.49)$$

**4. Torsional Pendulum:** A disk fixed to a slender rod. The rod is fastened to a fixed frame. If the disk is twisted it will oscillate back and forth about the axis of the rod.

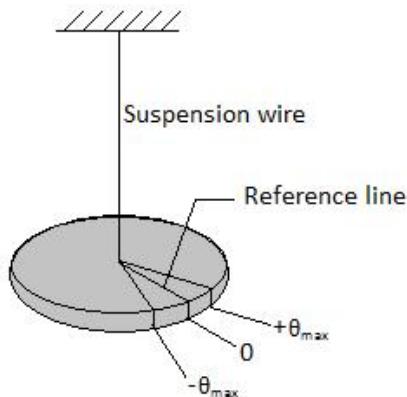


Fig.5.56 Torsional pendulum

### USE OF SIMPLE PENDULUM:

The applications of simple pendulums are

- a) Clock
- b) Car Shock Absorbers
- c) Musical Instruments
- d) Bungee Jumping
- e) Diving Board
- f) The Process of Hearing
- g) Earthquake-proof buildings
- h) Metronome

#### a) Clock:

Depending on the type of clock and the era that it was made, either a large pendulum or a vibrating quartz crystal will keep time. This motion of the pendulum or crystal must be periodic, in order to ensure that the time shown will be accurate. The

oscillators have a constant period because they are in Simple Harmonic Motion. This allows them to keep time accurately. This accurate time is more than just for convenience. It is a necessity when navigating using celestial marks like the Sun or stars because the position of the Sun and stars in the sky changes with the time of day.

It is also important that the period is independent of the amplitude of oscillation. This is because the conditions in a grandfather clock can vary, and affect the amplitude of oscillation. No matter how the amplitude varies, the time shown is always constant.

### **b) Car Shock Absorbers:**

Springs attached to the wheels of a car help ensure a smoother ride for the passengers. When the car hits a bump in the road, the wheels of the car rise up. If the car did not have shock absorbers, the whole car will be thrown up and down, causing an extremely uncomfortable and possibly dangerous ride.

When there are springs in the car, the wheel will rise, compressing the spring while the car body remains relatively stationary. The restoring force on the compressed spring will push the car wheel back into place.

However, when the spring is put into SHM by the bump in the road, it continues to bounce up and down even though the unevenness that triggered the motion is gone. This is why cars contain shock absorbers, which are usually pistons filled with a viscous liquid, to damp the oscillations quickly. This is an example of over damping.

### **c) Musical Instruments**

Sound itself is produced from oscillations of the air. In a string instrument such as a violin or a guitar, bowing or plucking the string provides the force needed to make the string oscillate and produce sound. In a wind instrument like a trumpet, the vibrations are caused by the player's lips while the sound is caused by exciting the air molecules by blowing across the opening in a flute. In a percussion instrument like the triangle, the vibrations occur when the instrument is struck.

The vibration produced in the string, column of air or body of the instrument causes standing waves to be formed, which produces sound.

**d) Bungee Jumping**

When a long, elastic cord is tied to the ankles of a person, who then jumps off a bridge or a platform, setting off a series of vertical oscillations.

The amplitude of the oscillations is calculated very carefully, because an error could mean the loss of a life. This sport must be undertaken only with high safety precautions to prevent injury.

**e) Diving Board**

This oscillating system is known as a cantilever, which is defined as a beam fixed at only one end.

As the person on the edge of the board jumps up and down on the end of the diving board, the board and the person undergo SHM. This motion is affected by the force constant of the board (a stiffer board will not oscillate as much) and the weight of the person (a heavier person will cause greater amplitude of oscillation).

**f) Hearing**

The basic function of our ears, hearing, cannot be possible without SHM. The sound waves travel through the air and when they hit the eardrum, they cause the eardrum to vibrate. This movement is passed to the brain, which then translates these signals into complex sounds that we are able to hear clearly.

**g) Metronome**

A metronome generates rhythmic ticks in order to help musicians to play a piece at a constant speed. The structure is a variation of the pendulum. In this case, the oscillating arm is anchored at the bottom. A counterweight is fixed within the metronome to counter-balance a second weight that is on the oscillating arm and the period can be varied by sliding the weight up and down the arm, thus changing the moment of the oscillation.

The restoring force in this case is gravity acting on the fixed counterweight. The nearer the weight is to the bottom of the arm, the faster it will swing and generate ticks.

## USE OF COMPOUND PENDULUM:

### 1. Measurement of Moment of Inertia of Human Body:

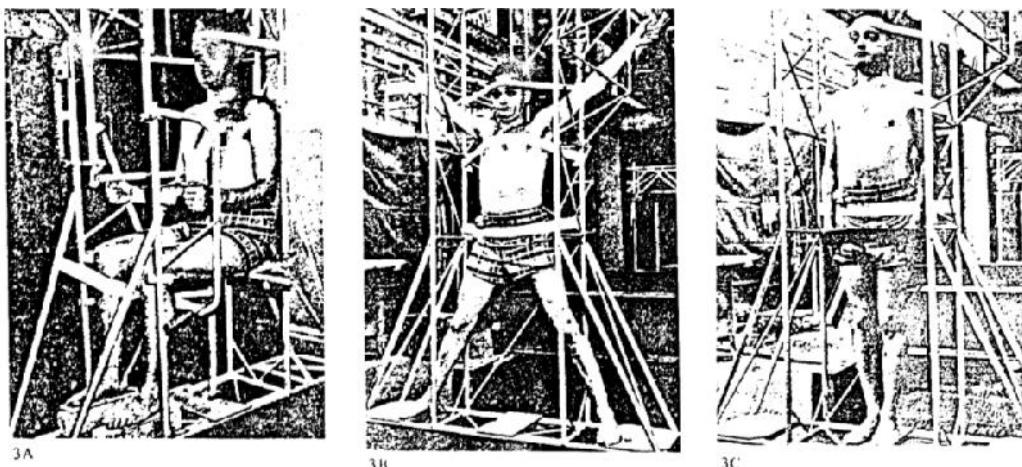


Fig.5.57 Measurement of moment of inertia

The above picture displays an Air force “Volunteer” attached to a compound pendulum for measurement of his **Moment of Inertia**. Why the measurement of Moment of Inertia of Human Body is important:

- a) **Human moment of inertia:** The moment of inertia of the human body about an axis through its centre of mass is important in the application of biomechanics to sports such as diving and gymnastics. We can measure the body's moment of inertia in a particular position while a person remains in that position on a horizontal turn table with the body's centre of mass on the turntable's rotational axis as shown in fig.5.58. The turntable with the person on it is then accelerated from rest by a torque produced by using a rope wound around a pulley on the shaft of the turntable. From the measured tension in the rope and angular acceleration, we can calculate the body's moment of inertia about an axis through its centre of mass.



Fig.5.58 Turntable with a person

- b) **Astronaut moment of inertia:** The occupant of an aircraft ejection seat contributes significantly to the moment of inertia and therefore be included in the design. Space-walking astronaut's maneuverer using small rocket engines. If the thrust is not through the centre of mass it will exert a torque as well as a force and produce an unwanted tumbling motion. To predict such motions the astronaut's moment of inertia must be known for a variety of postures.
- c) **Human motion of moment of inertia:** In the investigation of automobile safety anthropomorphic dummies are filmed to study the effects of restraints and improvements in interior design. For the dummies to model human motions not only must the masses of the body segments be correct but also the moment of inertia, otherwise the angular motions are not reproduced. Computer models are extensively used in the design of aircraft cockpits and other control systems where the human response must be optimized. The parameters required for such models of human motion include the moment of inertia.

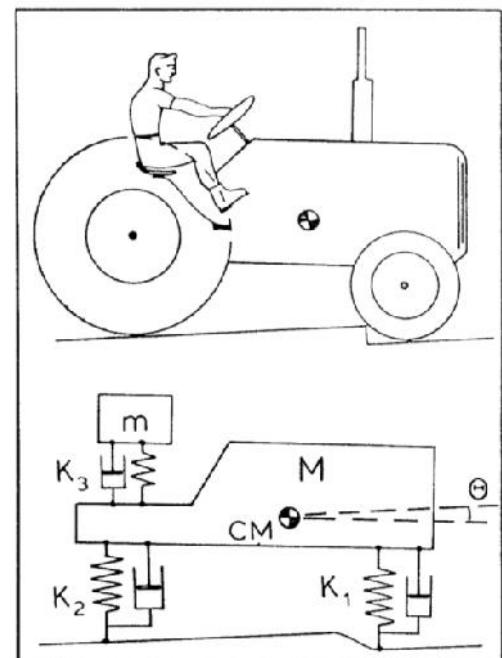


Fig.5.59 Farm tractor

## 2) Farm Tractors:

Form tractors determining conditions of operation and human comfortability. These are used for slow speed use over uneven ground and therefore, unlike cars and trucks, do not have spring suspension. The rough ride can lead back to injuries and health problem. A computer model is shown in fig.5.59 The figure shows a dynamic model of a tractor. The tractor of mass  $M$  and moment of inertia  $I$  is supported by the tries, represented by a damped spring  $K_1$  and  $K_2$ . The motion of the operator  $m$  supported by the seat, damped spring  $k_3$  is studied as the tractor goes on uneven ground.

## 3. Sail – Boat:

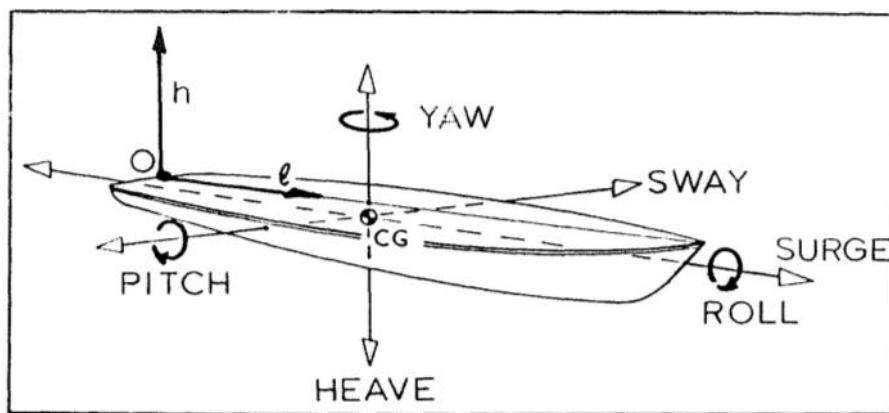


Fig.5.60 Sail boat

The total weight of the sailboat has a major effect on its speed. A heavy boat floats lower and causes a disturbance while moving. The compound pendulum concept helps in determining conditions for operation.

## USE OF TORSION PENDULUM:

Torsion pendulums are often used for time-keeping purposes. For instance, the balance wheel in a mechanical wristwatch is a torsion pendulum in which the restoring torque is provided by a coiled spring.

There are two important applications of torsional pendulum:

1. Bodies which are having geometrically defined shape are called regular bodies. It is possible to evaluate the moment of inertia of regular bodies

mathematically. But, to find the moment of inertia of irregular bodies, using torsional pendulum is the only viable method as on today.

2. The rigidity modulus of a material can be found by taking the material in the form of a wire and setting up a torsional pendulum using any regular body of suitable mass.

### Problems:

**SP5.38:** Derive an expression for velocity and acceleration of a particle moving with simple harmonic motion.

Solution:

Consider a particle moving along the circumference of a circle, of radius  $r$  with a uniform angular velocity of  $\omega$  radians/sec as shown in Fig. 23.2.

Let  $P$  be the position of the particle at some instant after  $t$  sec from  $X$ . Therefore, angle turned by the particle,

$$\theta = \omega t \text{ rad}$$

and displacement of the point  $N$  (i.e. projection of  $P$  on the vertical diameter  $Y-Y'$  of the circle),

$$\begin{aligned} y &= ON = r \sin \theta \\ &= r \sin \omega t \end{aligned} \quad \dots(i)$$

Differentiating this equation with respect to time  $t$ ,

$$\frac{dy}{dt} = r \omega \cos \omega t \quad \dots(ii)$$

$$\text{or velocity, } v = r \omega \sqrt{1 - \sin^2 \omega t} \quad \dots(\because \sin^2 \theta + \cos^2 \theta = 1)$$

From equation (i), we find that

$$\sin \omega t = \frac{y}{r}$$

Substituting this value of  $\sin \omega t$  in the above equation,

$$v = r \omega \sqrt{1 - \left(\frac{y^2}{r^2}\right)}$$

$$\text{or velocity, } v = \omega \sqrt{r^2 - y^2} \quad \dots(iii)$$

Now differentiating equation (ii) with respect to time  $t$ ,

$$\frac{d^2 y}{dt^2} = -r \omega^2 \sin \omega t$$

$$\text{or acceleration, } a = -\omega^2 y \quad \dots(\text{Substituting } y = r \sin \omega t)$$

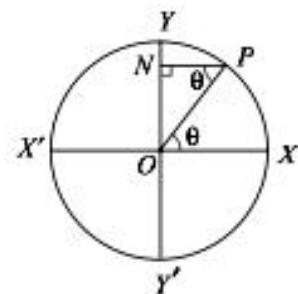


Fig. 23.2. Particle moving with S.H.M.

**Note.** The minus sign shows that the direction of acceleration is opposite to the direction in which  $y$  increases i.e., the acceleration is always directed towards the point  $O$ . But in actual practice, this relation is used as  $a = \omega^2 y$

**SP5.39:** *The piston of a steam engine moves with simple harmonic motion. The crank rotates at 120 r.p.m. and the stroke length is 2 metres. Find the velocity and acceleration of the piston, when it is at a distance of 0.75 metre from the centre.*

**Solution.** Given : Frequency of piston ( $N$ ) = 120 r.p.m ; stroke length  $l$  = 2 m or radius ( $r$ ) = 1 m and distance of piston from the centre ( $y$ ) = 0.75 m

**Velocity of piston**

We know that angular velocity of piston,

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 120}{60} = 4\pi \text{ rad/sec}$$

$$\therefore \text{Velocity of piston, } v = \omega \sqrt{r^2 - y^2} = 4\pi \sqrt{(1)^2 - (0.75)^2} = 8.3 \text{ m/s} \quad \text{Ans.}$$

**Acceleration of piston**

We know that acceleration of piston,

$$a = \omega^2 y = (4\pi)^2 \times 0.75 = 118.4 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.40:** *A body, moving with simple harmonic motion, has an amplitude of 1 meter and the period of complete oscillation is 2 seconds. What will be the velocity and acceleration of the body after 0.4 second from the extreme position?*

**Solution.** Given : Amplitude ( $r$ ) = 1 m; Periodic time ( $T$ ) = 2 s and time taken by the body from extreme position = 0.4 s

**Velocity of the body**

Now let  $O$  be the centre,  $Y$  an extremity of the motion and  $P$  the position of the body after 0.4 sec from  $Y$  as shown in Fig. 23.3. Therefore time required by the body to travel from  $Y$  to  $P$

$$= 0.4 \text{ s}$$

We know that time required by the body to travel from  $O$  to  $Y$

$$= \frac{1}{4} \times T = \frac{1}{4} \times 2 = 0.5 \text{ s}$$

$\therefore$  Time required by the body to travel from  $O$  to  $N$

$$t = 0.5 - 0.4 = 0.1 \text{ s} \quad \dots(i)$$

We know that angular velocity of the body

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \text{ rad/s} \quad \dots(ii)$$

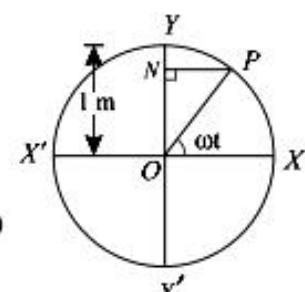


Fig. 23.3.

$\therefore$  Displacement of the body after 0.4 sec from the extreme position (or 0.1 second from the mean position),

$$\begin{aligned} y &= r \cos \omega t = 1 \cos (\pi \times 0.1) = 1 \cos 18^\circ \text{ m} \\ &\quad \dots(\pi \times 0.1 = 180 \times 0.1 = 18^\circ) \\ &= 0.95 \text{ m} \end{aligned}$$

$\therefore$  Velocity of the body,

$$v = \omega \sqrt{r^2 - y^2} = \pi \sqrt{l^2 - (0.95)^2} = 0.98 \text{ m/s} \quad \text{Ans.}$$

### Acceleration of the body

We know that acceleration of the body,

$$a = \omega^2 y = (\pi)^2 \times 0.95 = 9.38 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.41:** Find amplitude and time period of a particle moving with simple harmonic motion, which has a velocity of 9 m/s and 4 m/s at the distance of 2 m and 3 m respectively from the centre.

**Solution.** Given : When velocity ( $v_1$ ) = 9 m/s, distance from centre ( $y_1$ ) = 2 m and when velocity ( $v_2$ ) = 4 m/s, distance from centre ( $y_2$ ) = 3 m

### Amplitude of the particle

Let

$r$  = Amplitude of the particle, and

$\omega$  = Angular velocity of the particle.

We know that velocity of the particle,

$$v = \omega \sqrt{r^2 - y^2}$$

$$\therefore 9 = \omega \sqrt{r^2 - (2)^2} = \omega \sqrt{r^2 - 4} \quad \dots(i)$$

$$\text{and} \quad 4 = \omega \sqrt{r^2 - (3)^2} = \omega \sqrt{r^2 - 9} \quad \dots(ii)$$

Dividing equation (i) by (ii)

$$\frac{9}{4} = \frac{\sqrt{r^2 - 4}}{\sqrt{r^2 - 9}}$$

or

$$\frac{81}{16} = \frac{r^2 - 4}{r^2 - 9} \quad \dots(\text{Squaring both sides})$$

$$81r^2 - 729 = 16r^2 - 64 \quad \text{or} \quad 65r^2 = 665$$

$$\therefore r = \sqrt{\frac{665}{65}} = \sqrt{10.23} = 3.2 \text{ m}$$

### Time-period of the particle

Substituting this value of  $r$  in equation (i),

$$9 = \omega \sqrt{(3.2)^2 - 4} = \omega \sqrt{6.24} = \omega \times 2.5$$

$$\therefore \omega = \frac{9}{2.5} = 3.6 \text{ rad/s}$$

We know that time period,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3.6} = 1.75 \text{ s} \quad \text{Ans.}$$

**SP5.42:** A body performing simple harmonic motion has a velocity of 12 m/s when the displacement is 50 mm, and 3 m/s when the displacement is 100 mm, the displacement being measured from the mid-point. Calculate the frequency and amplitude of the motion. What is the acceleration when the displacement is 75 mm?

**Solution.** Given : Velocity ( $v_1$ ) = 12 m/s, when displacement ( $y_1$ ) = 50 mm = 0.05 m ; and velocity ( $v_2$ ) = 3 m/s, when displacement ( $y_2$ ) = 100 mm = 0.1 m

*Amplitude of the motion*

Let

$r$  = Amplitude of the motion, and

$\omega$  = Angular displacement of the body,

We know that velocity of the body,

$$v = \omega \sqrt{r^2 - y^2}$$

$$\therefore 12 = \omega \sqrt{r^2 - (0.05)^2} = \omega \sqrt{r^2 - 0.0025} \quad \dots(i)$$

$$\text{Similarly} \quad 3 = \omega \sqrt{r^2 - (0.1)^2} = \omega \sqrt{r^2 - 0.01} \quad \dots(ii)$$

Dividing equation (i) and (ii)

$$\frac{12}{3} = \frac{\omega \sqrt{r^2 - 0.0025}}{\omega \sqrt{r^2 - 0.01}} \quad \text{or} \quad 4 = \frac{\sqrt{r^2 - 0.0025}}{\sqrt{r^2 - 0.01}}$$

$$16 = \frac{r^2 - 0.0025}{r^2 - 0.01} \quad \dots(\text{Squaring both sides})$$

$$16r^2 - 0.16 = r^2 - 0.0025$$

$$15r^2 = -0.0025 + 0.16 = 0.1575$$

$$\therefore r^2 = \frac{0.1575}{15} = 0.0105$$

or  $r = 0.1025 \text{ m} \quad \text{Ans.}$

*Frequency of the motion*

Substituting the value of  $r$  in equation (i),

$$12 = \omega \sqrt{(0.1025)^2 - 0.0025} = \omega \times 0.09$$

$$\therefore \omega = \frac{12}{0.09} = 133.3 \text{ rad/sec}$$

We know that frequency of the motion,

$$N = \frac{1}{T} = \frac{133.3}{2\pi} = 21.2 \text{ Hz} \quad \text{Ans.}$$

*Acceleration when the displacement is 75 mm*

We know that acceleration of the body when  $y$  is 75 mm or 0.075 m,

$$a = \omega^2 y = (133.3)^2 \times 0.075 = 1332.6 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.43:** In a system, the amplitude of the motion is 5 m and the time is 4 seconds. Find the time required by the particle in passing between points which are at distances of 4 m and 2 m from the centre of force and are on the same side of it.

**Solution.** Given : Amplitude ( $r$ ) = 5 m ; Time taken ( $T$ ) = 4 sec and distances of the point ( $y_1$ ) = 4 m and ( $y_2$ ) = 2 m

We know that angular velocity of the particle,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4} = 90^\circ/\text{s}$$

and displacement of particle ( $y$ ),

$$4 = r \sin \omega t_1 = 5 \sin \omega t_1$$

$$\therefore \sin \omega t_1 = \frac{4}{5} = 0.8$$

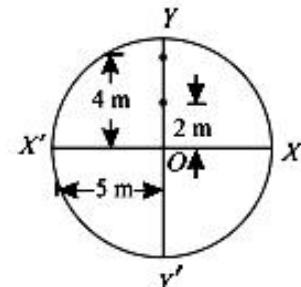


Fig. 23.4.

$$\omega t_1 = 53.1^\circ$$

$$\text{or } t_1 = \frac{53.1}{90} = 0.59 \text{ s}$$

$$\text{Similarly } 2 = r \sin \omega t_2$$

$$\text{or } \sin \omega t_2 = \frac{2}{5} = 0.4$$

$$\therefore \omega t_2 = 23.6^\circ$$

$$\text{or } t_2 = \frac{23.6}{90} = 0.26 \text{ s}$$

Time required in passing between the two points,

$$t = t_1 - t_2 = 0.59 - 0.26 = 0.33 \text{ s} \quad \text{Ans.}$$

**SP5.44:** A body is vibrating with simple harmonic motion of amplitude 100 mm, and frequency 2 vibrations/sec. Calculate the maximum velocity and acceleration.

**Solution.** Given : Amplitude ( $r$ ) = 100 mm = 0.1 m and frequency of body ( $N$ ) = 2 vib/sec.

**Maximum velocity**

We know that angular velocity of the body,

$$\omega = 2\pi N = 2\pi \times 2 = 4\pi \text{ rad/s}$$

and maximum velocity,  $v_{max} = r\omega = 0.1 \times 4\pi = 1.257 \text{ m/s} \quad \text{Ans.}$

**Maximum acceleration**

We also know that maximum acceleration,

$$a_{max} = \omega^2 r = (4\pi)^2 \times 0.1 = 15.79 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.45:** Derive an expression for maximum velocity and acceleration of a particle moving with simple harmonic motion.

**Solution:**

We have already discussed in that the velocity of a particle moving with simple harmonic motion,

$$v = \omega \sqrt{r^2 - y^2} \quad \dots(i)$$

A little consideration will show, that the velocity is maximum, when  $y = 0$  or when  $N$  passes through  $O$  i.e. its mean position. Therefore, maximum velocity

$$v_{max} = \omega r \quad \dots(ii)$$

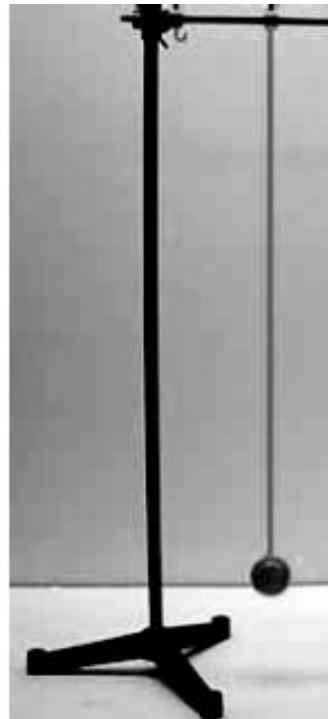
It may be noted from equation (i) that its velocity is zero when  $y = r$ , i.e. when  $N$  passes through  $Y'$  or  $Y$  as shown in At these points,  $N$  is momentarily at rest. We have also discussed that the acceleration of a particle moving with simple harmonic motion,

$$a = \omega^2 y \quad \dots(iii)$$

A little consideration will show, that the acceleration is maximum when the value of  $y$  is maximum or  $y = r$  i.e. when  $N$  passes through  $Y$  or  $Y'$ . Therefore maximum acceleration,

$$a_{max} = \omega^2 r \quad \dots(iv)$$

It may also be noted from equation (iii) that the acceleration is zero, when  $y = 0$  or when  $N$  passes through  $O$  i.e. its mean position. It is thus obvious, that the acceleration is proportional to the distance from  $O$ , i.e. mean position.



Simple pendulum is the most common example for SHM.

**SP5.47:** A particle, moving with simple harmonic motion, performs 10 complete oscillation per minute and its speed, is 60% of the maximum speed when it is at a distance of 8 cm from the centre of oscillation. Find amplitude, maximum acceleration of the particle. Also find speed of the particle, when it is 6 cm far from the centre of oscillation.

**Solution.** Given : No. of oscillation/min = 10 and when displacement ( $y$ ) = 8 cm velocity ( $v$ ) = 60%  $v_{max}$  = 0.6  $v_{max}$

**SP5.46:**

**Amplitude of the particle**

We know that no. of oscillations per sec

$$= \frac{10}{60} = \frac{1}{6}$$

∴ Time-period of the motion

$$(T) = \frac{6}{1} = 6 \text{ s}$$

and angular velocity,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ rad/s}$$

∴ Linear velocity,

$$v = \omega \sqrt{r^2 - y^2}$$

or

$$0.6 v_{max} = \omega \sqrt{r^2 - (8)^2} \quad \dots (\because y = 8 \text{ cm})$$

$$0.6 \omega r = \omega \sqrt{r^2 - 64}$$

$$0.6 r = \sqrt{r^2 - 64}$$

Squaring both sides,

$$0.36r^2 = r^2 - 64 \quad \text{or} \quad 0.64r^2 = 64$$

∴

$$r^2 = \frac{64}{0.64} = 100$$

or

$$r = \sqrt{100} = 10 \text{ cm} \quad \text{Ans.}$$

**Maximum acceleration of the particle**

We know that maximum acceleration of the particle,

We know that maximum acceleration of the particle,

$$a_{max} = \omega r = \left(\frac{\pi}{3}\right)^2 \times 10 = 10.97 \text{ cm}^2/\text{s} \quad \text{Ans.}$$

**Speed of the particle when it is 6 cm from the centre of oscillation**

We know that speed of the particle when it is 6 cm from the centre of oscillation,

$$v = \omega \sqrt{r^2 - y^2} = \frac{\pi}{3} \sqrt{(10)^2 - (6)^2} = 8.38 \text{ cm/s} \quad \text{Ans.}$$

**SP5.48:** A prismatic bar  $AB$  of weight  $W$  is resting on rough rollers rotating with equal angular velocity in opposite directions as shown below :

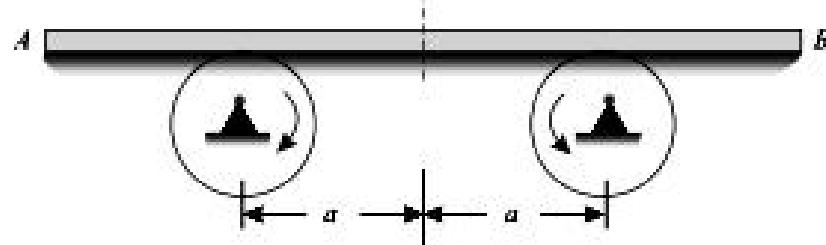


Fig. 23.5.

If the bar is so placed on the rollers that its c.g. is displaced from the middle plane and released, show that the bar executes a simple harmonic motion. Also find the time period, if the coefficient of friction is  $\mu$  and the distance between the two rollers is  $2a$ .

**Solution.** Let the weight  $W$  of the bar  $AB$  acting at its centre of gravity  $G$  be at a distance  $x$  from its mid point as  $M$  as shown in Fig. 23.6.

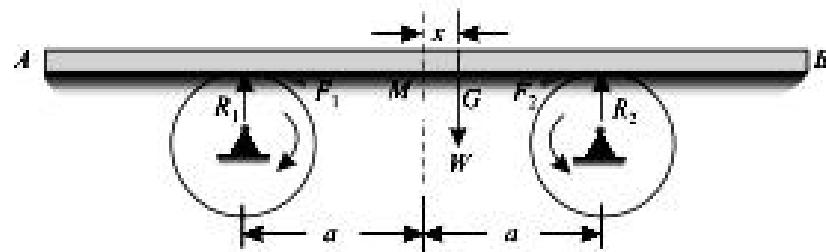


Fig. 23.6.

Let

$R_1$  = Vertical reaction of the left roller,

$F_1$  = Force of friction between the bar and left roller, and

$R_2, F_2$  = Corresponding values for the right roller.

Taking moments about the left support and equating the same,

$$R_2 \times 2a = W(a + x)$$

$$\therefore R_2 = \frac{W(a + x)}{2a}$$

Similarly

$$R_1 = \frac{W(a - x)}{2a}$$

We know that force of friction at the left roller,

$$F_1 = \mu R_1 = \frac{\mu W (a - x)}{2a}$$

Similarly

$$F_2 = \mu R_2 = \frac{\mu W (a + x)}{2a}$$

and resultant forces of friction.

$$F = F_2 - F_1 = \frac{\mu W (a + x)}{2a} - \frac{\mu W (a - x)}{2a} = \frac{\mu W x}{a}$$

We know that this resultant force of friction is equal to the product of mass of the bar and its acceleration.

$$\therefore \frac{\mu Wx}{a} = \frac{W}{g} \times a \quad \text{or} \quad a = \frac{\mu gx}{a}$$

Since the acceleration in the above equation is proportional to (i.e.  $x$  distance from the centre of the bar), thus the bar executes a simple harmonic motion. Ans.

#### Periodic time

We know that periodic time in case of a simple harmonic motion,

$$T = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{\mu gx}{a}} = 2\pi \sqrt{\frac{a}{\mu g}} \quad \text{Ans.}$$

**SP5.49:** Derive an expression for angular velocity of helical spring mass system.

#### Solution:

Let a body be attached to the lower end. Let A-A be the equilibrium position of the spring, after the body is attached. If the spring is stretched up to B-B and then released, the body will move up and down with a simple harmonic motion.

Let

$m$  = Mass of the body in kg (such that its weight ( $W$ ) is  $mg$  newtons)

$s$  = Stiffness of the spring in N/m

$x$  = Displacement of the load below the equilibrium position in metres.

$a$  = Acceleration of the body in  $m/s^2$

$g$  = Gravitational acceleration, and

$t$  = Periodic time.

We know that the deflection of spring,

$$\delta = \frac{W}{s} = \frac{mg}{s}$$

Then disturbing force  
and restoring force

$$= \text{Mass} \times \text{Acceleration} = m.a \quad \dots(i)$$

$$= s.x \quad \dots(ii)$$

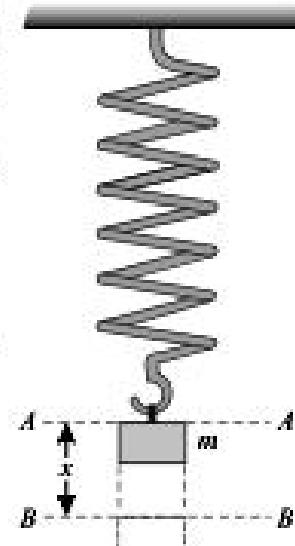


Fig. 26.1 Helical spring.

Equating equations (i) and (ii),

$$ma = sx \quad \text{or} \quad \frac{m}{s} = \frac{x}{a} \quad \dots(iii)$$

We know that in simple harmonic motion, time period,

$$t = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{x}{a}} = 2\pi \sqrt{\frac{m}{s}}$$

$$\frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\delta}{s}} \quad \dots(\because t = \frac{2\pi}{\omega} \text{ and } \frac{mg}{s} = \delta)$$

$$\therefore \omega = \sqrt{\frac{s}{\delta}}$$

where  $\omega$  is the angular velocity in rad/sec.

**Notes.** 1. Frequency of motion,

$$n = \frac{1}{t} = \frac{1}{2\pi} \sqrt{\frac{s}{m}} = \frac{1}{2\pi} \sqrt{\frac{s}{\delta}}$$

2. From equation (iii) we find that

$$x = \frac{ma}{s}$$

Thus we see that  $x$  is directly proportional to  $m/s$ .

**SP5.49:** A 4 kg mass hung at one end of a helical spring and is set vibrating vertically. The mass makes 2 vibrations per second. Determine the stiffness of the spring.

**Solution.** Given : Mass ( $m$ ) = 4 kg and frequency ( $n$ ) = 2 vib/s = 2 Hz

Let

$s$  = Stiffness of the spring.

We know that periodic time,

$$t = \frac{1}{n} = \frac{1}{2} = 0.5 \text{ s}$$

We also know that periodic time,

$$0.5 = 2\pi \sqrt{\frac{m}{s}} = 2\pi \sqrt{\frac{4}{s}}$$

Squaring both sides,

$$0.25 = (2\pi)^2 \times \frac{4}{s} = \frac{157.9}{s}$$

$$s = \frac{157.9}{0.25} = 631.6 \text{ N/m} \quad \text{Ans.}$$

**SP5.50:** *A spiral spring hung up at one end, and carrying a mass of 7 kg at the other is made to vibrate. Find the period of oscillation, if the spring is found to extend 10 mm for each 0.5 kg of mass.*

**Solution.** Given : Mass = 7 kg and when mass = 0.5 kg, then deflection ( $\delta$ ) = 10 mm = 0.01 m.

We know that when mass is equal to 7 kg, then deflection of the spring

$$\delta = \frac{0.01}{0.5} \times 7 = 0.14 \text{ m}$$

and period of oscillation,  $t = 2\pi \sqrt{\frac{\delta}{g}} = 2\pi \sqrt{\frac{0.14}{9.8}} = 0.75 \text{ s}$  Ans.

**SP5.51:** *A mass supported by a spring has a static deflection of 0.5 mm. Determine its natural frequency of oscillation.*

**Solution.** Given: Deflection ( $\delta$ ) = 0.5 mm = 0.0005 m

We know that natural frequency of oscillation,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{9.8}{0.0005}} = 22.3 \text{ vib/s} = 22.3 \text{ Hz}$$
 Ans.

**SP5.52:** *A body of mass 3 kg, suspended from a vertically mounted spring, deflects it by 12 mm. Determine the no. of oscillations of the body.*

*Also determine the maximum force in the spring, when it is displaced through a distance of 25 mm from its rest position and then released.*

**Solution.** Given : Mass of the body ( $m$ ) = 3 kg ; Deflection ( $\delta$ ) = 12 mm = 0.012 m and displacement ( $x$ ) = 25 mm = 0.025 m

*No. of oscillations of the body*

We know that the no. of oscillations of the body,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{9.8}{0.012}} = 4.55 \text{ vib/s} = 4.55 \text{ Hz}$$
 Ans.

*Maximum force in the spring*

We know that the angular velocity of the body,

$$\omega = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.8}{0.012}} = 28.6 \text{ rad/s}$$

and maximum acceleration,  $a_{\text{max}} = \omega^2 x = (28.6)^2 \times 0.025 = 20.4 \text{ m/s}^2$

$\therefore$  Maximum inertia force

$$= \text{Mass} \times \text{Acceleration} = 3 \times 20.4 = 61.2 \text{ N}$$

We also know that maximum force in the spring occurs when the mass is at its lowest position. In this position, the force in the spring is equal to the sum of weight of the body and the inertia force. Therefore maximum force in the spring

$$= (3 \times 9.8) + 61.2 = 90.6 \text{ N}$$
 Ans.

**SP5.53:** The weight of an empty railway wagon is 240 kN. On loading it with goods weighing 320 kN, its spring gets compressed by 80 mm.

(a) Calculate its natural period of vibrations when the railway wagon is (i) empty and (ii) loaded as above.

(b) It is set into natural vibrations with an amplitude of 100 mm when empty. Calculate the velocity of the railway wagon when its displacement is 40 mm from statical equilibrium position.

**Solution.** Given : Weight of empty wagon = 240 kN and when the load (W) is 320 kN, then deflection ( $\delta$ ) = 80 mm = 0.08 m

(i) *Period of vibrations when the wagon is empty*

We know that deflection of the spring, when wagon is empty,

$$\delta_1 = \frac{0.08}{320} \times 240 = 0.06 \text{ m}$$

and period of vibrations,  $t_1 = 2\pi \sqrt{\frac{\delta_1}{g}} = 2\pi \sqrt{\frac{0.06}{9.8}} = 0.49 \text{ s}$  Ans.

(ii) *Period of vibrations when the wagon is loaded*

We know that total load on the springs when the wagon is loaded

$$= 240 + 320 = 560 \text{ kN}$$

∴ Deflection of the spring when the wagon is loaded

$$\delta_2 = \frac{0.08}{320} \times 560 = 0.14 \text{ m}$$

and period of vibrations,  $t_2 = 2\pi \sqrt{\frac{\delta_2}{g}} = 2\pi \sqrt{\frac{0.14}{9.8}} = 0.75 \text{ s}$

(b) *Velocity of the railway wagon when it is empty*

When the wagon is empty, amplitude ( $r$ ) = 100 mm = 0.1 m and displacement ( $y$ ) = 40 mm = 0.04 m

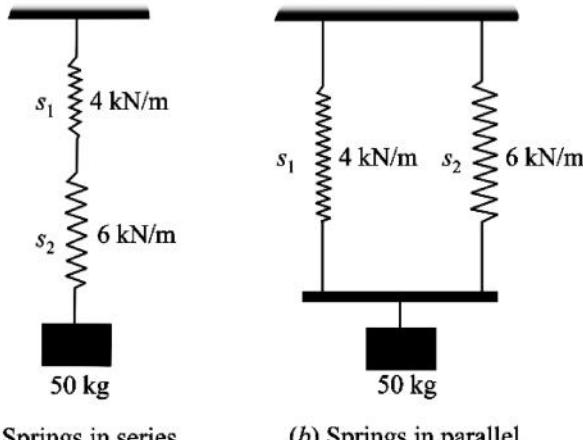
We know that angular velocity of the wagon,

$$\omega = \frac{2\pi}{t_1} = \frac{2\pi}{0.49} = 12.82 \text{ rad/s}$$

and velocity,  $v = \omega \sqrt{r^2 - y^2} = 12.82 \sqrt{(0.1)^2 - (0.04)^2} = 1.175 \text{ m/s}$  Ans.

**SP5.54:**

A block of mass 50 kg supported by two springs connected in series hangs from the ceiling. It can move between smooth vertical guides. The spring constants are 4 kN/m and 6 kN/m as shown in Fig. 26.3.

**Fig. 26.3.**

The block is pulled 40 mm down from its position of equilibrium and then released. Determine (a) period of vibrations, maximum velocity and acceleration of the block.

(b) quantities in (a) above, when the block is supported by the springs connected in parallel.

**Solution.** Given : Mass ( $m$ ) = 50 kg = 0.005 t, Stiffness of first spring ( $s_1$ ) = 4 kN/m ; Stiffness of second spring ( $s_2$ ) = 6 kN /m and displacement ( $r$ ) = 40 mm = 0.04 m.

(a) When the springs are connected in series

We know that spring constant of an equivalent spring,

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{1}{2.4}$$

or

$$s = 2.4 \text{ kN/m}$$

and deflection of the spring  $\delta = \frac{mg}{s} = \frac{0.05 \times 9.8}{2.4} = 0.204 \text{ m}$

$$\therefore \text{Period of vibrations, } t = 2\pi \sqrt{\frac{\delta}{g}} = 2\pi \sqrt{\frac{0.204}{9.8}} = 0.91 \text{ s} \quad \text{Ans.}$$

We know that the angular velocity of the block,

$$\omega = \frac{2\pi}{t} = \frac{2\pi}{0.91} = 6.9 \text{ rad/s}$$

$\therefore$  Maximum velocity,

$$v_{max} = \omega r = 6.9 \times 0.04 = 0.276 \text{ m/s} \quad \text{Ans.}$$

and maximum acceleration,  $a_{max} = \omega^2 r = (6.9)^2 \times 0.04 = 1.9 \text{ m/s}^2 \quad \text{Ans.}$

**(b) When the springs are connected in parallel**

We know that spring constant of an equivalent spring,

$$s = s_1 + s_2 = 4 + 6 = 10 \text{ kN/m}$$

and deflection of the spring due to block of weight 0.49 kN

$$\delta = \frac{0.05 \times 9.8}{10} = 0.049 \text{ m}$$

$$\therefore \text{Period of vibrations} \quad t = 2\pi \sqrt{\frac{\delta}{g}} = 2\pi \times \sqrt{\frac{0.049}{9.8}} = 0.44 \text{ s} \quad \text{Ans.}$$

We know that angular velocity of the block,

$$\omega = \frac{2\pi}{t} = \frac{2\pi}{0.44} = 14.28 \text{ rad/s}$$

$$\therefore \text{Maximum velocity, } v_{max} = \omega r = 14.28 \times 0.04 = 0.57 \text{ m/s} \quad \text{Ans.}$$

$$\text{and maximum acceleration, } = \omega^2 r = (14.28)^2 \times 0.04 = 8.16 \text{ m/s}^2 \quad \text{Ans.}$$

**SP5.55:** A weight  $P$  is attached to springs of stiffness  $C_1$  and  $C_2$  in two different cases as shown in Fig. 26.4.

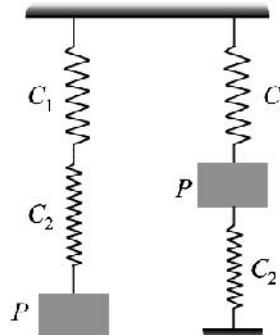


Fig. 26.4.

Determine the period of vibrations in both the cases.

**Solution.** Given : Weight  $= P$

Period of vibrations in the first case

We know that in this case both the springs will be subjected to the weight  $P$ . Therefore total displacement of the spring

$$= \frac{P}{C_1} + \frac{P}{C_2} = P \left( \frac{1}{C_1} + \frac{1}{C_2} \right) = \frac{P(C_1 + C_2)}{C_1 C_2}$$

$$\text{and period of vibration} = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{P(C_1 + C_2)}{g C_1 C_2}} \quad \text{Ans.}$$

*Period of vibrations in the second case*

We know that in this case, the upper spring will be subjected to tension, whereas the lower one will be subjected to compression.

Let

$P_1$  = Weight shared by the upper spring, and

$P_2$  = Weight shared by the lower spring.

∴ Elongation of the upper spring,

$$\delta_1 = \frac{P_1}{C_1} \quad \dots(i)$$

and shortening of the lower spring,

$$\delta_2 = \frac{P_2}{C_2} \quad \dots(ii)$$

Since elongation of the upper spring is equal to shortening of the lower spring, therefore equating the values of  $\delta_1$  and  $\delta_2$  from the above equations.

$$\frac{P_1}{C_1} = \frac{P_2}{C_2}$$

∴

$$P_1 C_2 = P_2 C_1$$

$$(P - P_2) C_2 = P_2 C_1 \quad \dots(\because P_1 + P_2 = P)$$

$$P C_2 - P_2 C_2 = P_2 C_1$$

$$P_2 (C_1 + C_2) = P C_2$$

or

$$\frac{P_2}{C_2} = \frac{P}{C_1 + C_2}$$

∴ Displacement

$$= \frac{P}{C_1 + C_2}$$

$$\dots \left( \because \frac{P_2}{C_2} = \delta_2 \right)$$

We know that the period of vibration,

$$= 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{P}{g(C_1 + C_2)}} \quad \text{Ans.}$$

**SP5.56:** *A simple pendulum of amplitude  $4^\circ$  performs 24 oscillations in one minute. Find (a) length of the pendulum (b) maximum acceleration of the bob, (c) maximum linear velocity of the bob; and (d) maximum angular velocity of the bob.*

**Solution.** Given : Angular amplitude ( $\theta$ ) =  $4^\circ$  =  $\frac{4\pi}{180}$  rad Frequency ( $n$ ) = 24 Hz and time ( $t$ ) = 1 min = 60 s.

(a) *Length of the pendulum*

Let

$l$  = Length of the pendulum

We know that time period for one oscillation,

$$t = \frac{60}{24} = 2.5 \text{ s}$$

$$\therefore 2.5 = 2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{l}{9.8}}$$

Squaring both sides of the equation

$$6.25 = (2\pi)^2 \frac{l}{9.8}$$

or  $l = \frac{6.25 \times 9.8}{(2\pi)^2} = 1.55 \text{ m Ans.}$

**(b) Maximum linear acceleration of the bob**

We know that maximum linear acceleration of the bob takes place, when it is at its extreme position (or in other words, the displacement is maximum). We also know that displacement of the bob (as per Fig. 26.6).

$$= \text{Arc } AC = OC \times \theta \text{ (in radians)} = 1.55 \times \frac{4\pi}{180} = 0.108 \text{ m}$$

and angular velocity,  $\omega = \frac{2\pi}{t} = \frac{2\pi}{2.5} = 2.51 \text{ rads/s}$

$\therefore$  Maximum linear acceleration of the bob

$$a_{\max} = \omega^2 \times AC = (2.51)^2 \times 0.108 = 0.68 \text{ m/s}^2 \text{ Ans.}$$

**(c) Maximum linear velocity of the bob**

We know that maximum linear velocity of the bob,

$$v_{\max} = \omega \times AC = 2.51 \times 0.108 = 0.27 \text{ m/s Ans.}$$

**(d) Maximum angular velocity of the bob**

We also know that maximum angular velocity of the bob,

$$= \frac{v_{\max}}{l} = \frac{0.27}{1.55} = 0.174 \text{ rad/s Ans.}$$

**SP5.57:**

*A simple pendulum consists of a 600 mm long cord and a bob of mass 2 kg.*

*Find the no. of oscillations made by the bob per second.*

*If the same pendulum is suspended inside a train, accelerating smoothly on a level track at the rate of 3 m/s<sup>2</sup>, find the angle which the cord will make with the vertical. Also find the tension in the cord.*

**Solution.** Given: Length of cord ( $l$ ) = 600 mm = 0.6 m ; Mass of bob ( $m$ ) = 2 kg and acceleration ( $a$ ) = 3 m/s<sup>2</sup>

**Number of oscillations made by the bob per second**

We know that time-period for one oscillation,

$$t = 2\pi \sqrt{\frac{l}{g}} = 2\pi \sqrt{\frac{0.6}{9.8}} = 1.55 \text{ s}$$

and no. of oscillations made by the bob per second,

$$n = \frac{l}{t} = \frac{1}{1.55} = 0.65 \text{ Hz Ans.}$$

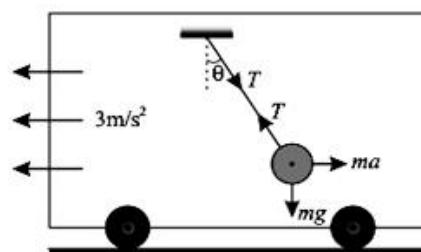


Fig. 26.6.

*Angle, which the cord will make with the vertical*

Let  $\theta$  = Angle, which the cord will make with the vertical.

We know that weight of the bob (acting downwards)

$$\begin{aligned} &= mg = 2 \times 9.8 \\ &= 19.6 \text{ N} \end{aligned}$$

and inertia force acting on the bob (opposite to the acceleration of the train)

$$\begin{aligned} &= ma = 2 \times 3 \\ &= 6 \text{ N} \end{aligned}$$

$$\therefore \tan \theta = \frac{6}{19.6} = 0.3061 \quad \text{or} \quad \theta = 17^\circ$$

*Tension in the cord*

We know that tension in the cord,

$$T = \sqrt{(19.6)^2 + (6)^2} = 20.5 \text{ N} \quad \text{Ans.}$$

**SP5.58:** Derive an expression for gain or loss in the no. of oscillation due to change in the length of string or acceleration due to gravity of a simple pendulum.

**Solution:**

Consider a simple pendulum, oscillating with a simple harmonic motion.

Let  $l$  = Length of the string,

$t$  = Time period for one oscillation and

$n$  = No. of beats or swings in time  $t$ .

We have seen in Art. 26.4 that the time period for one beat.

$$= \pi \sqrt{\frac{l}{g}}$$

$$\therefore \text{Time-period for } n \text{ beats} \quad t = n\pi \sqrt{\frac{l}{g}}$$

$$\text{or} \quad n = \frac{t}{\pi \sqrt{\frac{l}{g}}} = \frac{t}{\pi} \sqrt{\frac{g}{l}} \quad \dots(i)$$

Taking logs of both sides,

$$\log n = \log t - \log \pi + \frac{1}{2} (\log g - \log l)$$

Taking differential coefficients of variable (considering  $t$  and  $\pi$  as constants),

$$\frac{dn}{n} = \frac{1}{2} \left( \frac{dg}{g} - \frac{dl}{l} \right) = \frac{dg}{2g} - \frac{dl}{2l} \quad \dots(ii)$$

where  $dn$  is the change in time in  $n$  seconds.

Now, if the gravity changes (keeping length of pendulum constant), then

$$\frac{dn}{n} = \frac{dg}{2g} \quad \dots \left( \because \frac{dl}{l} = 0 \right)$$

and if the length of pendulum changes (keeping g constant) then

$$\frac{dn}{n} = -\frac{dl}{2l} \quad \dots \left( \because \frac{dg}{g} = 0 \right)$$

**Note.** The minus sign indicates that as the value of  $dn/n$  increases, the value of  $dl/l$  decreases, and vice versa. Or in other words, if  $dn$  increases,  $dl$  decreases (keeping  $n$  and  $l$  constant) i.e., if the no. of beats in a clock increases, it is due to decrease in length of the pendulum. Similarly, if the no. of beats decreases it means that it is due to increase in length of the pendulum.

Thus in order to set the pendulum correct, we have to reduce its length. Similarly, if the no. of beats increase (or the pendulum gains time), it is due to decrease in the length of the pendulum. Thus in order to set the pendulum correct, we have to increase its length.

**SP5.59:** *The gravity at the poles exceeds the gravity at the equator in the ratio of 301 : 300. If a pendulum regulated at the poles is taken to the equator, find how many seconds a day will it lose ?*

**Solution.** Given : Gravity at pole ( $g_p$ ) = 301 and gravity at the equator ( $g_e$ ) = 300

$\therefore$  Change in gravity, when the pendulum is taken from the poles to the equator,

$$dg = 300 - 301 = -1$$

Let

$dn$  = No. of seconds the pendulum will lose in one day.

We know that no. of seconds in one day or 24 hours

$$n = 24 \times 60 \times 60 = 86400$$

We also know that

$$\frac{dn}{n} = \frac{dg}{2g}$$

$$\frac{dn}{86400} = \frac{-1}{2 \times 301}$$

$$\therefore dn = -\frac{86400}{2 \times 301} = -143.5 \text{ s} \quad \text{Ans.}$$

Minus sign means that the pendulum will lose 143.5 seconds per day.

**SP5.60:** *Find the length of a pendulum, which will have one beat per second. If such a pendulum loses 5 seconds a day, by how much length must it be shortened to keep the correct time?*

**Solution.** Given : Time ( $t$ ) = 1 s and no. of second the pendulum loses in one day ( $dn$ ) = -5 (Minus sign due is loss of seconds.)

*Length of the pendulum which will have one beat per second*

Let

$l$  = Length of the pendulum.

We know that time for one beat ( $t$ ).

$$1 = \pi \sqrt{\frac{l}{g}} = \pi \sqrt{\frac{l}{9.8}}$$

Squaring both sides,  $1 = \pi^2 \times \frac{l}{9.8}$   
 $\therefore l = \frac{9.8}{\pi^2} = 0.993 \text{ m} = 993 \text{ mm} \quad \text{Ans.}$

*Length, by which the pendulum should be shortened*

Let  $dl$  = Length in mm by which the pendulum should be shortened, to keep the correct time.

We know that no. of seconds in one day or 24 hours,

$$n = 24 \times 60 \times 60 = 86400$$

We also know that  $\frac{dn}{n} = -\frac{dl}{2l}$   
 $\frac{-5}{86400} = -\frac{dl}{2 \times 993}$   
 $\therefore dl = \frac{5 \times 2 \times 993}{86400} = 0.115 \text{ mm} \quad \text{Ans.}$

It means that the length of the pendulum is 0.115 mm more than the correct length. Thus, for correct time, the length of the pendulum should be shortened by 0.115 mm Ans.

**SP5.61:** Derive an expression for gain or loss in the no. of oscillation due to change in the position of a simple pendulum.

**Solution:**

Consider a simple pendulum oscillating with a simple harmonic motion.

Let  $l$  = Length of the string,  
 $t$  = Time period,  
 $n$  = No. of beats in time  $t$ , and  
 $r$  = Radius of the earth

We know that the value of  $g$  (i.e., acceleration due to gravity) varies inversely as  $r^2$ . Mathematically :

$$g \propto \frac{1}{r^2}$$

$$\therefore g = k \times \frac{1}{r^2} = \frac{k}{r^2} \quad \dots(\text{where } k \text{ is constant})$$

Taking logs of both sides,

$$\log g = \log k - 2 \log r$$

Now taking differentials,

$$\frac{dg}{g} = -2 \frac{dr}{r}$$

or

$$\frac{dg}{2g} = -\frac{dr}{r}$$

Now if  $h$  is height of the point where pendulum is placed above the earth's surface, then  $dr = h$ .  
Therefore

$$\frac{dg}{2g} = -\frac{h}{r}$$

We have studied in Art. 26.6. that

$$\frac{dg}{2g} = \frac{dn}{n}$$

$$\therefore \frac{dn}{n} = -\frac{h}{r}$$

- Notes :**
1. Minus sign indicates that as  $\frac{dn}{n}$  increases, the value of  $\frac{h}{r}$  decreases and *vice versa*.
  2. Similarly, it can be proved that if the value of  $h$  is taken as negative (i.e., the pendulum is taken below the surface of the earth as in the case of mines etc.) the value of  $dn/n$  will increase.

**SP5.62:** Find the approximate height of a mountain, at the top of which a pendulum, which beats seconds at sea level, loses 20 seconds a day. Take radius of the earth as 6400 km and the acceleration due to gravity varies as the square of the distance from the centre of the earth.

**Solution.** Given : No. of seconds the pendulum loses in one day ( $dn$ ) = -20 (Minus sign due to loss of seconds) and radius of the earth ( $r$ ) = 6400 km.

Let  $h$  = Height of mountain in km.

We know that no. of seconds in one day or 24 hours

$$n = 24 \times 60 \times 60 = 86400$$

We also know that  $\frac{dn}{n} = -\frac{h}{r}$

$$\frac{-20}{86400} = -\frac{h}{6400}$$

$$\therefore h = \frac{20 \times 6400}{86400} = 1.48 \text{ km} \quad \text{Ans.}$$

**SP5.63:** A uniform straight rod of length 600 mm and mass 250 g is smoothly pivoted about a point, which is 40 mm from one end. Find the period of small oscillation about the pivot, if the rod can turn freely in the vertical plane.

**Solution.** Given : Length of rod ( $l$ ) = 600 mm = 0.6 m ; Mass of rod ( $m$ ) = 250 g = 0.25 kg. and distance between the point of suspension and the centre of gravity of the body ( $h$ ) = 300 - 40 = 260 mm = 0.26 m

We know that mass moment of inertia of the rod about the pivot  $G$ ,

$$I_0 = \frac{0.25 (0.3)^2}{3} + 0.25 (0.26)^2 = 0.0244 \text{ kg-m}^2$$

$$\text{and period of small oscillation, } t = 2\pi \sqrt{\frac{I_0}{mgh}} = 2\pi \sqrt{\frac{0.0244}{0.25 \times 9.8 \times 0.26}} \text{ s} \\ = 1.23 \text{ s} \quad \text{Ans.}$$

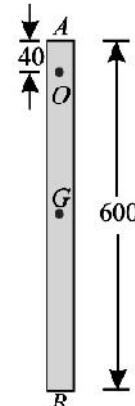


Fig. 26.8.

**SP5.64:** A block whose weight =  $W$  is suspended by means of a spring whose spring constant =  $k_2$  from the end of a rigid weightless beam, which is of length  $l$  and hinged to a wall by a frictionless connection at its left. It is held in a horizontal position by a spring whose spring constant =  $k_1$ , attached to it at a distance  $b$  from the hinge end to the ceiling as shown in Fig. 26.9.

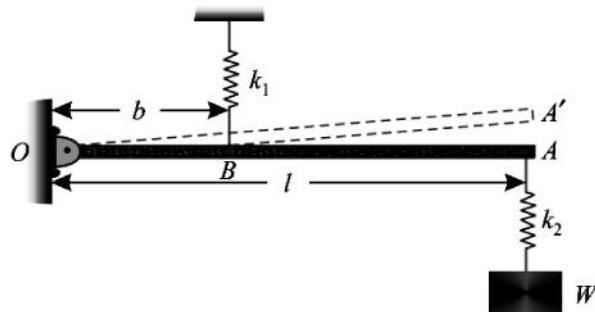


Fig. 26.9.

Prove that (f), the natural frequency of vibration of the system in Hz is given by :

$$n = \frac{1}{2\pi} \sqrt{\frac{gk_1k_2}{W[k_1 + (l/b)^2 k_2]}}$$

**Solution.** Fig. 26.9 shows  $OA$  the cantilever beam in the horizontal position with springs of constant  $k_1$  and  $k_2$  and weight  $W$  attached at  $A$ . A little consideration will show, that if the weight  $W$  is removed, the cantilever will spring upwards and  $A$  will occupy a new position  $A'$  as shown in the figure.

Let

$T$  = Tension in the spring  $k_1$ .

Taking moments about  $O$  and equating the same,

$$T \times b = W \times l$$

or

$$T = \frac{Wl}{b} \quad \dots(i)$$

$\therefore$  Deflection of spring  $k_1$  (at  $B$ ) due to tension  $T$

$$= \frac{T}{k_1} = \frac{Wl}{k_1 b} \quad \dots(ii)$$

From the geometry of the figure, we find that

$$\frac{\text{Deflection at } B}{b} = \frac{\text{Deflection at } A}{l}$$

$$\therefore \text{Deflection at } A = \frac{l \times \text{Deflection at } B}{b} = \frac{lWl}{k_1 b} = \frac{W}{k_1} \left( \frac{l}{b} \right)^2$$

and deflection of spring  $k_2$  (at A) due to weight W

$$= \frac{W}{k_2}$$

$\therefore$  Total deflection of cantilever at A,

$$\delta = \frac{W}{k_1} \left( \frac{l}{b} \right)^2 + \frac{W}{k_2} = W \left[ \frac{1}{k_1} \left( \frac{l}{b} \right)^2 + \frac{1}{k_2} \right] = W \left[ \frac{k_2 \left( \frac{l}{b} \right)^2 + k_1}{k_1 k_2} \right]$$

and natural frequency of vibration of the system,

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} = \frac{1}{2\pi} \sqrt{\frac{gk_1 k_2}{W \left[ k_1 + \left( \frac{l}{b} \right)^2 k_2 \right]}} \text{ Hz} \quad \text{Ans.}$$

**SP5.65:** A uniform thin rod as shown in Fig. 26.10 has a mass of 1 kg and carries a concentrated mass of 2.5 kg at B. The rod is hinged at A, and is maintained in the horizontal position by a spring of stiffness 1.8 kN/m at C.

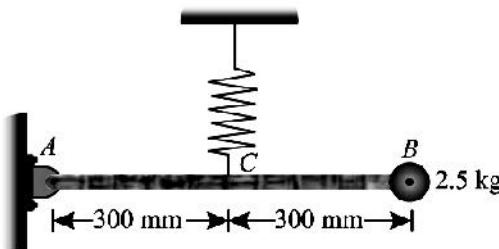


Fig. 26.10.

Find the frequency of oscillation. Neglect the effect of the mass of the spring.

**Solution.** Given : Mass of rod = 1 kg ; Mass at B = 2.5 kg ; Length of rod AB = 300 + 300 = 600 mm = 0.6 m ; Stiffness of spring (s) = 1.8 kN/m = 1800 N/m.

Let

$n$  = Frequency of oscillation.

$\theta$  = Small angular displacement of the rod, and

$\alpha$  = Angular acceleration of the rod AB.

We know that mass moment of inertia of the system about A,

$$I_A = \text{M.I. of 1 kg about } A + \text{M.I. of 2.5 kg about } A$$

$$= \frac{1(0.6)^2}{3} + 2.5(0.6)^2 = 1.02 \text{ kg-m}^2$$

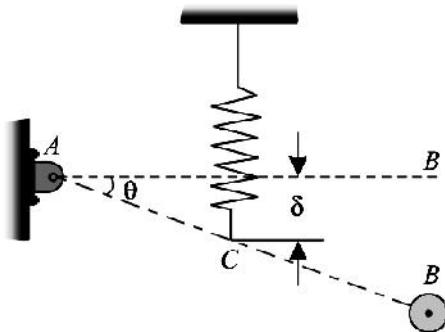


Fig. 26.11.

If the rod  $AB$  is given a small angular displacement as shown in Fig. 26.11 and then released, it will start oscillating about  $A$  with simple harmonic motion.

We know that extension of the spring,

$$\delta = 0.3 \sin \theta = 0.3 \times \theta \text{ m}$$

... (since  $\theta$  is very small, therefore substituting  $\sin \theta = \theta$ )

and restoring force

$$= s \delta = 1800 \times 0.3 \theta = 540 \theta \text{ N}$$

Therefore restoring moment about  $A$

$$= 540 \theta \times 0.3 = 162 \theta \text{ N-m} \quad \dots(i)$$

and disturbing moment about  $A$

$$= I_A \alpha = 1.02 \alpha \text{ N-m} \quad \dots(ii)$$

Equating equations (i) and (ii)

$$1.02 \alpha = 162 \theta$$

$$\therefore \frac{\alpha}{\theta} = \frac{162}{1.02} = 158.8$$

We know that frequency of oscillation,

$$n = \frac{1}{2\pi} \sqrt{\frac{\alpha}{\theta}} = \frac{1}{2\pi} \sqrt{158.8} = 2.01 \text{ Hz} \quad \text{Ans.}$$

**SP5.66:** A body of mass 0.5 kg oscillates about an axis at a distance 300 mm from the centre of gravity. If the mass moment of inertia about the centroidal axis, parallel to the axis of rotation, be 0.125 kg-m<sup>2</sup>, find the length of the equivalent simple pendulum.

**Solution.** Given : Mass of the body ( $m$ ) = 0.5 kg ; Distance of Centre of oscillation from the centre of gravity ( $h$ ) = 300 mm = 0.3 m and moment of inertia about centroidal axis ( $I_G$ ) = 0.125 kg-m<sup>2</sup>

Let

$k_G$  = Radius of gyration about the centroidal axis.

We know that mass moment of inertia about the centroidal axis ( $I_G$ )

$$0.125 = m k_G^2 = 0.5 k_G^2$$

$$\therefore k_G^2 = \frac{0.125}{0.5} = 0.25$$

We also know that the length of equivalent simple pendulum,

$$L = h + \frac{k_G^2}{h} = 0.3 + \frac{0.25}{0.3} = 1.133 \text{ m} \quad \text{Ans.}$$

**SP5.67:** *The pendulum AB of an Izod impact testing machine makes 40 oscillations per minute about a knife edge through the hole A. The distance of centre of gravity from the centre of oscillation 450 mm. Find the radius of gyration of the pendulum about an axis through its centre of gravity and parallel to the knife edge.*

*Also find the number of oscillations per minute, which the pendulum will make, if supported on a knife edge through another hole at B such that AB = 1.5 m. Take g as 9.81 m/s<sup>2</sup>.*

**Solution.** Given : No. of oscillations/ min = 40 ; Distance of centre of gravity from the centre of oscillation ( $l$ ) = 450 mm = 0.45 m ; Length AB = 1.5 m and centre of gravity ( $g$ ) = 9.81 m/s<sup>2</sup>.

*Radius of gyration*

Let

$k_G$  = Radius of gyration about an axis through centre of gravity and parallel to the knife edge, and

$L$  = Length of the equivalent simple pendulum.

We know that frequency of pendulum

$$n = \frac{40}{60} = \frac{2}{3}$$

$$\therefore \text{Time period } (T) = \frac{3}{2} = 1.5 \text{ s}$$

We also know that time period ( $T$ ).

$$1.5 = 2\pi \sqrt{\frac{L}{g}} = 2\pi \times \sqrt{\frac{L}{9.81}}$$

$$\text{Squaring both sides, } 2.25 = 4\pi^2 \times \frac{L}{9.81} = 4.024 L$$

$$\therefore L = \frac{2.25}{4.024} = 0.559 \text{ m}$$

and length of equivalent simple pendulum ( $L$ ),

$$0.559 = h + \frac{k_G^2}{h} = 0.45 + \frac{k_G^2}{0.45}$$

$$\therefore k_G^2 = (0.559 - 0.45) \times 0.45 = 0.049$$

$$\text{or } k_G = 0.22 \text{ m} = 220 \text{ mm} \quad \text{Ans.}$$

*No. of oscillations per minute*

Given. Distance between A and B = 1.5 m

$$\text{or } h = AB - l = 1.5 - 0.45 = 1.05 \text{ m}$$

We know that length of the equivalent simple pendulum,

$$L = h + \frac{k_G^2}{h} = 1.05 + \frac{(0.22)^2}{1.05} = 1.1 \text{ m}$$

and no. of oscillations per second of the pendulum when supported at B.

$$n_1 = \frac{1}{2\pi} \sqrt{\frac{g}{L}} = \frac{1}{2\pi} \sqrt{\frac{9.81}{1.1}} = 0.475$$

∴ No. of oscillations per minute  
=  $0.475 \times 60 = 28.5$  Ans.