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On a Least-Squares Collocation Method for Linear Differential-Algebraic Equations

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Summary. The aim of this note is to extend some results on least-squares collocation methods and to prove the convergence of a least-squares collocation method applied to linear differential-algebraic equations. Some numerical examples are presented.

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1. Introduction

In recent time, great interest has been spent on differential algebraic equations (DAE's). Such equations arise in many fields of applications. Whereas in former times equations describing electrical networks were mainly stimulating the investigation, nowadays the development of numerical methods is focused on equations describing constraint mechanical motions or kinetic reactions subject to balance invariants and on singular optimal control problems (cf. [2, 3, 9, 17]. In the present paper we deal with linear DAE's only, that is, with equations

$$A(t)\,x'(t)+B(t)\,x(t)=f(t), \qquad t\in(0,1) \tag{1.1}$$

$$D_0\,x(0)+D_1\,x(1)=\gamma \tag{1.2}$$

where $x(t)\in\mathbb{R}^n$, D_0, $D_1\in\mathbf{B}(\mathbb{R}^n,\mathbb{R}^r)$. Let $A(t)$, $B(t)\in\mathbf{B}(\mathbb{R}^n)$ be sufficiently smooth. Here, for Banach spaces X and Y, $\mathbf{B}(X, Y)$ denotes the Banach space of all bounded linear operators defined on X and mapping into Y. The analytic solution of (1.1), if A and B are constant, is reasonably well understood (cf. [2, 3, 8]). It turns out that so-called higher index DAE's include differentiation problems. This behaviour is preserved in the case of time-dependent coefficients (cf. [18, 19]). Therefore, they are essentially ill-posed [13, 16]. In this context it is not very surprising that usual finite-difference methods become unstable when applied to higher index equations [13]. To solve them numerically we should use some regularization procedure. Least-squares collocation methods are known to be regularizing algorithms [5–7, 15]. For (1.1)–(1.2) this method can be formulated as follows: For $\nu\in\mathbb{N}$, let $\pi_\nu\subset[0,1]$ be a finite set of collocation points. A function $x_\nu: [0,1] \to \mathbb{R}^n$ is called least-squares collocation solution of (1.1)–(1.2) on π_ν if x_ν satisfies (1.2) and

$$A(t)\,x_\nu'(t) + B(t)\,x_\nu(t) = f(t), \qquad t \in \pi_\nu \tag{1.3}$$

and x_ν minimizes some scalar product norm among all solutions of (1.3)–(1.2). The feasibility and convergence of this method will be proved. In Sect. 2 we develop the functional-analytic background. An essential tool of our proof is the theory of reproducing kernel Hilbert spaces (RKHS) [1]. Section 3 contains the application of these results to DAE's. Our construction follows some ideas of [20, 21]. Finally, some numerical are presented in Sect. 4.

2. Theoretical Background

Let X be a Hilbert space with the scalar product $(\cdot,\cdot)$. For $l \in X^*$, denote $\langle l, x\rangle = l(x)$ for $x \in X$. For a linear operator T, we denote the kernel and the range by $\mathbf{N}(T)$ and $\mathbf{R}(T)$, respectively. On X let the linear mappings T_j be defined:

$$T_j \in \mathbf{L}(X, (I_j, \mathbb{R})), \qquad j = 1, \ldots, m$$

with $\phi \neq I_j \subset \mathbb{R}$, $(I_j, \mathbb{R}) = \{f \mid f: I_j \to \mathbb{R}\}$. Denote Dirac's δ-functional by δ_t:

$$\langle \delta_t, x\rangle = x(t), \qquad t \in \mathbb{R}.$$

Throughout this section we assume the following supposition to be fulfilled.

Supposition. For $j = 1, \ldots, m$ and $t \in I_j$, $\delta_t \circ T_j$ is a continuous linear functional on X, i.e.,

$$\delta_t \circ T_j \in X^*, \qquad j = 1, \ldots, m, \ t \in I_j. \tag{2.1}$$

According to the Riesz representation theorem there are $\eta_t^j \in X$ such that

$$\langle \delta_t \circ T_j, x\rangle = T_j x(t) = (\eta_t^j, x), \qquad x \in X \tag{2.2}$$

for all $t \in I_j$, $j = 1, \ldots, m$. Denote $T = (T_1, \ldots, T_m)$. The equation to be solved is denoted by

$$Tx = f \tag{2.3}$$

for some vector function $f = (f_1, \ldots, f_m)$ and $f_j \in (I_j, \mathbb{R})$.

Lemma 2.1. *For all j, $\mathbf{N}(T_j) \subset X$ is closed. Moreover, $\mathbf{N}(T) = \mathbf{N}(T_1) \cap \ldots \cap \mathbf{N}(T_m)$.*

Proof. Let $1 \leq j \leq m$. Let $\{x_l\} \subset \mathbf{N}(T_j)$ with $x_l \to x \in X$. Hence, $0 = T_j x_l(t) = \langle \delta_t \circ T_j, x_l\rangle \to \langle \delta_t \circ T_j, x\rangle = T_j x(t)$ for $t \in I_j$. Thus, $x \in \mathbf{N}(T_j)$. The identity $\mathbf{N}(T) = \mathbf{N}(T_1) \cap \ldots \cap \mathbf{N}(T_m)$ is obvious. □

Corollary. *The Moore-Penrose generalized inverses T_j^+ and T^+ of T_j and T, respectively, are well-defined on $\mathbf{R}(T_j)$ and $\mathbf{R}(T)$, respectively.*

For a definition of the Moore-Penrose generalized inverse see, e.g., [14]. The next lemma characterizes $\mathbf{R}(T_j)$.

Lemma 2.2. *Let* $Y_j := \mathbf{R}(T_j) \subset (I_j, \mathbb{R})$, $j=1, \ldots, m$. *Then:*

(i) *Every space* Y_j *is a RKHS with the reproducing kernel*

$$Q^j(t, t') = (\eta_t^j, \eta_{t'}^j), \qquad t, t' \in I_j. \tag{2.4}$$

The associated scalar product is given by

$$[f, g]_j = (T_j^+ f, T_j^+ g), \qquad f \in \mathbf{R}(T_j). \tag{2.5}$$

(ii) *Let* $Y = Y_1 \times \ldots \times Y_m$. *Then* $T \in \mathbf{B}(X, Y)$, *and* $\mathbf{R}(T) \subset Y$ *is closed.*

Proof. (i) Since T_j^+ is linear, (2.5) defines a positive semi-definite bilinear form $[\cdot, \cdot]_j$. If $f \in \mathbf{R}(T_j)$ and $[f, f]_j = 0$, $(T_j^+ f, T_j^+ f) = 0$. Hence, $T_j^+ f = 0$. Because of $T_j^+ f \in \mathbf{N}(T_j)^\perp$ we have $T_j T_j^+ f = f = 0$. Thus, $[\cdot, \cdot]_j$ is a scalar product.

Now, let $\{f_l\} \subset \mathbf{R}(T_j)$ be a Cauchy sequence with respect to $[\cdot, \cdot]_j$. Choose $x_l \in \mathbf{N}(T_j)^\perp$ such that $Tx_l = f_l$. Because of $[f_l - f_k, f_l - f_k] = (x_l - x_k, x_l - x_k)$, $\{x_l\}$ is a Cauchy sequence in $\mathbf{N}(T_j)^\perp$. Since $\mathbf{N}(T_j)$ is closed, there exists an $x \in \mathbf{N}(T_j)^\perp$ with $x_l \to x$. Now, $[f_l - T_j x, f_l - T_j x]_j = (x_l - x, x_l - x) = \|x_l - x\|^2$. Hence, $f_l \to T_j x \in \mathbf{R}(T_j)$.

Next we show that Q^j is the reproducing kernel of Y_j. Denote the orthogonal projection of X onto $\mathbf{N}(T_j)^\perp$ by P_j. Due to (2.2) and (2.4),

$$Q^j(t, t') = (\eta_t^j, \eta_{t'}^j) = T_j \eta_{t'}^j(t).$$

Let $f \in \mathbf{R}(T_j)$ and set $x = T_j^+ f \in \mathbf{N}(T_j)^\perp$. Compute

$$\begin{aligned}[f, Q^j(\cdot, t')]_j &= (T_j^+ f, T_j^+ Q^j(\cdot, t')) = (x, T_j^+ T_j \eta_{t'}^j) = (x, P_j \eta_{t'}^j) = (P_j x, \eta_{t'}^j) = (x, \eta_{t'}^j)\\ &= T_j x(t') = f(t').\end{aligned}$$

(ii) For all $x \in X$,

$$[Tx, Tx] = \sum_{j=1}^m [T_j x, T_j x]_j = \sum_{j=1}^m (T_j^+ T_j x, T_j^+ T_j x) = \sum_{j=1}^m (P_j x, P_j x) \leq m(x, x).$$

Hence, $T \in \mathbf{B}(X, Y)$. We compute T^*. Let $x \in X$ and $f \in Y$. $[Tx, f] = \sum_{j=1}^m [T_j x, f_j]_j = \sum_{j=1}^m (x, T_j^* f_j) = \left(x, \sum_{j=1}^m T_j^* f_j\right)$, thus, $T^* f = \sum_{j=1}^m T_j^* f_j$. This gives $\mathbf{R}(T^*) = \mathbf{R}(T_1^*) + \ldots + \mathbf{R}(T_m^*)$. Since $\mathbf{R}(T_j) = Y_j$ is closed $\mathbf{R}(T_j^*)$ is so. Hence, $\mathbf{R}(T^*)$ is closed and, consequently, $\mathbf{R}(T)$, too. $\square$

Remark. (i) From

$$[T_j x, f_j]_j = (T_j^+ T_j x, T_j^+ f_j) = (P_j x, T_j^+ f_j) = (x, P_j T_j^+ f_j) = (x, P_j T_j^+ f_j) = (x, T_j^+ f_j),$$

all $x \in X$, $f_j \in Y_j$, it follows that $T_j^+ = T_j^*$.

(ii) Using the introduced norms, the problem $Tx=f$, $f\in Y$, is (conditionally) correctly posed since T^+ is continuous. Certainly, the Y-norm is very sharp and not adequate in practical cases.

For later convenience we introduce the notation $Q^j_{t'}(t)=Q^j(t,t')$. In order to approximate (2.3) we introduce a least-squares collocation method. Let $\pi_{j,\nu}\subset I_j$ be a finite set of collocation points for $\nu\in\mathbb{N}$, $1\leqq j\leqq m$. Without loss of generality we assume that $\pi_{j,\nu-1}\subset\pi_{j,\nu}$. For $f=(f_1,\dots,f_m)\in Y$, let $x_\nu\in X$ be the least-squares solution of

$$T_j x(t)=f_j(t),\qquad t\in\pi_{j,\nu},\ 1\leqq j\leqq m. \tag{2.6}$$

By Lemma 2.2, (2.6) is equivalent to

$$[T_j x, Q^j_t]_j=[f_j, Q^j_t]_j,\qquad t\in\pi_{j,\nu},\ 1\leqq j\leqq m. \tag{2.7}$$

Lemma 2.3. *Let Q^j be continuous and $\bigcup_{\nu=1}^{\infty}\pi_{j,\nu}$ dense in I_j. Then:*

(i) *$T_j x\in C(I_j)$ for all j.*

(ii) *Let $W_{j,\nu}=\operatorname{lin}\{Q^j_t\,|\,t\in\pi_{j,\nu}\}\subset Y_j$. Then $W_{j,\nu-1}\subset W_{j,\nu}$ and $\bigcup_{\nu=1}^{\infty}W_{j,\nu}$ is dense in Y_j.*

(iii) *Let $W_\nu=\operatorname{lin}\{e_jQ^j_t\,|\,t\in\pi_{j,\nu},\ 1\leqq j\leqq m\}\subset Y$. e_j be the j-th unit vector in $\mathbb{R}^m$. Then $W_{\nu-1}\subset W_\nu$ and $\bigcup_{\nu=1}^{\infty}W_\nu$ is dense in Y.*

Proof. Let $x\in X$, $1\leqq j\leqq m$. Then, for all $s,t\in I_j$,

$$|T_jx(s)-T_jx(t)|=|(x,\eta^j_s)-(x,\eta^j_t)|\leqq\|x\|\,\|\eta^j_s-\eta^j_t\|,$$

and $$\|\eta^j_s-\eta^j_t\|^2=(\eta^j_s,\eta^j_s)-2(\eta^j_s,\eta^j_t)+(\eta^j_t,\eta^j_t)=Q^j(s,s)-2Q^j(s,t)+Q^j(t,t).$$

Since Q^j is continuous, (i) follows.

In order to prove (ii) we show that $\left(\bigcup_{\nu=1}^{\infty}W_{j,\nu}\right)^{\perp}=\{0\}$. Let $f_j\in Y_j$ such that $f_j\perp W_{j,\nu}$ for all $\nu\in\mathbb{N}$. Hence, $0=[f_j,Q^j_t]_j=f_j(t)$, $t\in\pi_{j,\nu}$. Since $\bigcup_{\nu=1}^{\infty}\pi_{j,\nu}$ is dense in I_j and $f_j\in\mathbf{R}(T_j)$, we obtain $f_j=0$ by (i). (iii) follows immediately from (ii). □

The next theorem states the convergence of the method (2.6). If $W_\nu\subset\mathbf{R}(T)$, the convergence follows from well-known theorems (cf. [6, 7]). In our setting $W_\nu\subset\mathbf{R}(T)$ is not true in general.

Theorem 2.1. *Let Q^j be continuous and $\bigcup_{\nu=1}^{\infty}\pi_{j,\nu}$ dense in I_j. Let x_ν be defined by* (2.6).

(i) *If $f\in\mathbf{R}(T)$, $x_\nu\to T^+f$ for $\nu\to\infty$.*

(ii) *If $f\notin\mathbf{R}(T)$, $\{x_\nu\}$ does not contain a weakly convergent subsequence, i.e., $\|x_\nu\|\to\infty$ for $\nu\to\infty$.*

Proof. This proof is a slight modification of Groetsch [15, Chapter 4.3]. Let $\sigma_\nu : \{(j, t) \mid t \in \pi_{j,\nu}, 1 \leqq j \leqq m\} \to \{1, \ldots, \mu(\nu)\}$ be a one-to-one mapping and $q_l = e_j Q_t^j$ with $l = \sigma_\nu(j, t)$.

(i) Denote the mapping $f \to ([f, q_1], \ldots, [f, q_{\mu(\nu)}])$ by r_ν. Setting $T_\nu = r_\nu T$ we have $x_\nu = T_\nu^+ r_\nu f$. Set $V_\nu = \operatorname{lin}\{T^* q_1, \ldots, T^* q_{\mu(\nu)}\}$. Let $f \in \mathbf{R}(T)$ be fixed. Because of $T^+ f \in \mathbf{N}(T)^\perp = \mathbf{R}(T^*)$ there is a $z \in \mathbf{N}(T^*)^\perp = \mathbf{R}(T)$ such that $T^* z = T^+ f$. Since $W_\nu = \operatorname{lin}\{q_1, \ldots, q_{\mu(\nu)}\}$, there are a $\kappa \in \mathbb{N}$ and a $u \in W_\kappa \subset W_\nu$ $(\nu \geqq \kappa)$ such that $\|u - z\| < \varepsilon / \|T\|$ for given $\varepsilon > 0$. Therefore, $\|T^* u - T^+ f\| = \|T^* u - T^* z\| < \varepsilon$. Since $T^* u \in V_\nu$ $(\nu \geqq \kappa)$,

$$P_\nu T^+ f \to T^+ f \quad (\nu \to \infty) \tag{2.8}$$

where P_ν denotes the orthogonal projection onto V_ν. Furthermore,

$$T_\nu T^+ f = r_\nu T T^+ f = r_\nu P f = r_\nu f \tag{2.9}$$

with the orthogonal projection P onto $\mathbf{R}(T)$. But

$$\begin{aligned} x \in \mathbf{N}(T_\nu) \quad & \text{iff } T_\nu x = 0 \\ & \text{iff } [Tx, q_l] = 0, \quad l = 1, \ldots, \mu(\nu) \\ & \text{iff } (x, T^* q_l) = 0, \quad l = 1, \ldots, \mu(\nu) \\ & \text{iff } x \in V_\nu^\perp. \end{aligned}$$

Hence, by (2.8)–(2.9),

$$x_\nu = P_\nu T^+ f \to T^+ f. \tag{2.10}$$

(ii) By (2.7),

$$[Tx_\nu - f, q_l] = 0, \qquad l = 1, \ldots, \mu(\nu). \tag{2.11}$$

Let $\{x_{\nu_k}\}$ be a weakly convergent subsequence of $\{x_\nu\}$, $x_{\nu_k} \rightharpoonup z$ for $k \to \infty$. Since T is continuous, $Tx_{\nu_k} \rightharpoonup Tz$. By (2.11), $[Tz - f, q_l] = 0$ for $l = 1, \ldots, \mu(\nu)$. Hence, $Tz - f = 0$ because of Lemma 2.3. Thus, $f \in \mathbf{R}(T)$. This contradiction proves the assertion. □

By (2.10), $x_\nu = P_\nu T^+ f$ where P_ν is the orthogonal projection of X onto $V_\nu = T^*(W_\nu)$. If $e_j Q_t^j \in W_\nu$ and $x \in X$,

$$(T^* e_j Q_t^j, x) = [e_j Q_t^j, Tx] = [Q_t^j, T_j x]_j = T_j x(t) = (\eta_t^j, x).$$

Hence, $T^* e_j Q_t^j = \eta_t^j$ and $V_\nu = \operatorname{lin}\{\eta_t^j \mid t \in \pi_{j,\nu}, 1 \leqq j \leqq m\}$. From now on we suppose that the set $\{\eta_t^j \mid t \in \pi_{j,\nu}, 1 \leqq j \leqq m\}$ forms a basis of V_ν. This assumption must be checked in actual problems since η_t^j depends essentially on T. For integral equations, η_t^j is connected with the kernel of the integral operator. Integral kernels which have this property for every $\pi_{j,\nu}$ are called "quasi-Čebyšev" in [12]. Using some index mapping σ_ν we define $p_l = \eta_t^j$, $l = \sigma_\nu(j, t)$, as above. Now,

$$x_\nu = \sum_{k=1}^{\mu(\nu)} c_k p_k, \qquad c_k \in \mathbb{R}. \tag{2.12}$$

Multiplying (2.12) by p_l, $l=1, \ldots, \mu(v)$, we obtain $\sum_{k=1}^{\mu(v)} c_k(p_k, p_l)=(x_v, p_l) =(x_v, \eta_t^j)=[f, e_j Q_t^j]$, i.e.,

$$\sum_{k=1}^{\mu(v)} c_k(p_k, p_l)=f_j(t) \tag{2.13}$$

where $l=\sigma_v(j, t)$. (2.13) is a linear system of equations for determing the coefficients c_k. The matrix $A_v=(p_k, p_l)_{k,l=1,\ldots,\mu(v)}$ is a Grammian matrix of linearly independent elements, therefore, A_v is positive definite. We investigate the influence of perturbations of the right-hand side on the least-squares solution. Denote by $d_l=f_j(t)$ the exact data in (2.13) and $\tilde{d}_l \in \mathbb{R}$ erroneous data with

$$\|(\tilde{d}_1, \ldots, \tilde{d}_{\mu(v)})-(d_1, \ldots, d_{\mu(v)})\| \leqq \delta_v. \tag{2.14}$$

The following theorem is due to Engl [6].

Theorem 2.2. *Let the suppositions of Theorem* 2.1 *be true and* $f \in \mathbf{R}(T)$. *Denote by* λ_v *the smallest eigenvalue of* A_v. *If* $\limsup \delta_v^2/\lambda_v=0$, *then* $\tilde{x}_v \to T^+ f$ *where* $\tilde{x}_v$ *be the solution of* (2.13) *with* d_l *replaced by* $\tilde{d}_l$.

Remark. A generalization of an analogous result on weak convergence is possible by introducing further suppositions: If $\limsup \delta_v^2/\lambda_v < \infty$ and there exist $f_v \in \mathbf{R}(T)$ such that $f_{v,j}(t)=d_l$ for every v, l, then $x_v \rightharpoonup T^+ f$. The additional assumption seems to be very restrictive.

Using the notations of Theorem 2.2 we obtain the error estimate

$$\|\tilde{x}_v - T^+ f\| = \|x_v - T^+ f\| + \delta_v \lambda_v^{-1/2}. \tag{2.15}$$

The first term of the right-hand side is the discretization error. Regarding the fact that $x_v \in \mathbf{N}(T)^\perp$ and Remark (ii) of Lemma 2.2 we obtain

$$\begin{aligned} \|x_v - T^+ f\| &= \|T^+(Tx_v - f)\| \leqq \|T^+\| \, \|Tx_v - f\| \\ &\leqq \|T^+\| \sum_{j=1}^{m} \|(\tilde{P}_v f)_j - f_j\|_j \\ &\leqq T^+\| \sum_{j=1}^{m} \|\tilde{P}_{j,v} f_j - f_j\|_j \end{aligned}$$

where $\tilde{P}_v$ and $\tilde{P}_{j,v}$ are the orthogonal projections onto W_v and $W_{j,v}$, respectively. Estimates of the convergence order of the terms of the right-hand side in terms of the mesh size of $\pi_{j,v}$ and the smoothness of the reproducing kernels Q^j are well-known (cf. [20, 21]).

The second term of (2.15) represents the ill-posedness of the problem. Generally, $\lambda_v \to 0$ for $v \to \infty$. But until now, no estimates of λ_v in terms of X, Q^j, $\pi_{j,v}$, and T have been known.

3. Least-Squares Collocation for DAE's

Now we return to the DAE (1.1)–(1.2). In this section we show how the method (1.3) fits into the abstract setting introduced in the previous section. Let $m=n+1$. With $I_j=[0, 1]$, $1\leqq j\leqq n$, and $I_m=\{1, \ldots, r\}$ we define

$$T_j x(t)=\sum_{i=1}^{n}(a_{ji}(t)\,x_i'(t)+b_{ji}(t)\,x_i(t)), \qquad 1\leqq j\leqq n \tag{3.1}$$

$$T_m x(t)=D_0\,x(0)+D_1\,x(1). \tag{3.2}$$

The underlying Hilbert space X be some Sobolev space $[H^k(0, 1)]^n$. For (2.1) to hold, $k\geqq 2$ should be chosen. In order to reduce the computational expense we select the minimal value for k and set

$$X=[H^2(0, 1)]^n \tag{3.3}$$

endowed with the scalar product

$$(x, y)=\sum_{i=1}^{n}\left\{\int_0^1 x_i''(t)\,y_i''(t)\,dt+x_i(0)\,y_i(0)+x_i(1)\,y_i(1)\right\}. \tag{3.4}$$

The construction of the functions η_t^j is based on the following lemma. It is an extension of a procedure given in [21].

Lemma 3.1. *Let $X=X_1\times\ldots\times X_n$, where X_i is an RKHS on sets $I^i\subset\mathbb{R}$ with the reproducing kernels R^i, i.e., $(x, R_s^i)_i=x(s)$ $(x\in X_i, s\in I^i)$. Let T_j be given such that* (2.1) *holds. Then* (2.2) *holds for $\eta_t^j=(\eta_t^{1j}, \ldots, \eta_t^{nj})$ where $\eta_t^{ij}(s)=T_j\,e_i\,R_s^i(t)\in X_i$.*

Proof. Let δ_{it} be defined by $x_i\in X_i\mapsto T_j\,e_i\,x_i(t)$. Because of (2.1), $\delta_{it}\in X_i^*$. Hence, there are $\eta_t^{ij}\in X_i$ such that

$$T_j\,e_i\,x_i(t)=(x_i, \eta_t^{ij})_i \tag{3.5}$$

for all $x_i\in X_i$. Denote $\eta^j=(\eta_t^{1j}\ldots, \eta_t^{nj})\in X$. Hence, for all $x\in X$,

$$T_j\,x(t)=T_j\left(\sum_{i=1}^{n}e_i\,x_i(t)\right)=\sum_{i=1}^{n}(x_i, \eta_t^{ij})_i=(x, \eta_t^j),$$

i.e. (2.2). Moreover, by (3.5),

$$\eta_t^{ij}(s)=(\eta_t^{ij}, R_s^i)_i=(R_s^i, \eta_t^{ij})_i=T_j\,e_i\,R_s^i(t). \qquad \square$$

In our setting we have $X_i=H^2(0, 1)$ with the scalar product

$$(x, y)_i=\int_0^1 x''(t)\,y''(t)\,dt+x(0)\,y(0)+x(1)\,y(1).$$

The reproducing kernel is given by

$$R_s(t)=\tfrac{1}{6}(s-1)\,t^3+(s+\tfrac{5}{6}(s-1)-\tfrac{1}{6}(1-s)^3)\,t+1-s+\begin{cases}0, & t\leqq s\\ \tfrac{1}{6}(t-s)^3, & t>s\end{cases} \tag{3.6}$$

Lemma 3.1 gives

$$\begin{aligned}\eta_t^j(s)&=a_j(t)\,R_s'(t)+b_j(t)\,R_s'(t), && j=1,\dots,n,\ t\in I\\ \eta_\iota^m(s)&=d_\iota^0\,R_s(0)+d_\iota^1\,R_s(1), && \iota\in I_m\end{aligned}$$

with $a_j(t)=(a_{j1}(t),\dots,a_{jn}(t))$, $b_j(t)=(b_{j1}(t),\dots,b_{jn}(t))$, $d_\iota^k=(d_{\iota 1}^k,\dots,d_{\iota n}^k)$ and $D_k=(d_{\iota\kappa}^k)$ for $k=0,1$. Now, by (2.4) for $1\leqq j\leqq n$,

$$\begin{aligned}Q^j(t,t')=\sum_{i=1}^n\{&a_{ji}(t)\,a_{ji}(t')\,[\tfrac{1}{2}t^2+\tfrac{1}{2}t'^2-t'+\tfrac{7}{3}]\\ &+a_{ji}(t)\,b_{ji}(t')\,[\tfrac{1}{6}t'^3-\tfrac{1}{2}t'^2+\tfrac{1}{2}t^2(t'-1)+\tfrac{7}{3}t'-1]\\ &+b_{ji}(t)\,a_{ji}(t')\,[\tfrac{1}{6}t^3+\tfrac{1}{2}t'^2\,t-tt'+\tfrac{7}{3}t-1]\\ &+b_{ji}(t)\,b_{ji}(t')\,[\tfrac{1}{6}t^3(t'-1)+t(\tfrac{1}{6}t'^3-\tfrac{1}{2}t'^2+\tfrac{1}{3}t')+1-t'-t+2tt']\},\ t\leqq t'.\end{aligned}$$

$$Q^m(\iota,\kappa)=\sum_{i=1}^n(d_{\iota i}^0\,d_{\kappa i}^0+d_{\iota i}^1\,d_{\kappa i}^1),\qquad \iota,\kappa=1,\dots,r.$$

Obviously, the reproducing kernels Q^j are continuous. For a properly chosen set of collocation points, Theorem 2.1 ensures the convergence of the method. Let $\pi_{j,\nu}\subset I=[0,1]$, $j=1,\dots,n$, be sets of collocation points. Let $\bigcup_{\nu=1}^{\infty}\pi_{j,\nu}$ be dense in I. Without loss of generality we may assume that $\pi_{m,\nu}=I_m$, $\nu\in\mathbb{N}$. For Theorem 2.2 to hold we have to check the assumption on the linear independence of the set $\{\eta_t^j\,|\,t\in\pi_{j,\nu},\ 1\leqq j\leqq m\}$. It turns out that very weak suppositions on the matrices A and B are sufficient for this property to hold.

Lemma 3.2. *Let* $0,1\notin\pi_{j,\nu}$ *for* $j=1,\dots,n$. *Then* $\{\eta_t^j\,|\,t\in\pi_{j,\nu},\ 1\leqq j\leqq m\}$ *is linearly independent if the matrix pencil* $(A(t),B(t))$ *is regular for all* $t\in(0,1)$ *and* $\operatorname{rank}(D_0,D_1)=r$.

Proof. Without loss of generality we assume that $\pi_{j,\nu}\equiv\pi_\nu=\{t_1,\dots,t_\kappa\}$ for $j=1,\dots,n$. It is known that the set $\{R_s(0),R_s(t_1),\dots,R_s(t_\kappa),R_s(1),R_s'(t_1),\dots,R_s'(t_\kappa)\}$ is linearly independent (functions with respect to s!). Consider

$$\begin{aligned}0&=\sum_{k=1}^{\kappa}\sum_{j=1}^{n}\alpha_{jk}\,\eta_{t_k}^j(s)+\sum_{\iota=1}^{r}\alpha_{m\iota}\,\eta_\iota^m(s)\\ &=\sum_{k=1}^{\kappa}\left\{R_s'(t_k)\left[\sum_{j=1}^{n}\alpha_{jk}\,a_j(t_k)\right]+R_s'(t_k)\left[\sum_{j=1}^{n}\alpha_{jk}\,b_j(t_k)\right]\right\}\\ &\quad+R_s(0)\sum_{\iota=1}^{r}\alpha_{m\iota}\,d_\iota^0+R_s(1)\sum_{\iota=1}^{r}\alpha_{m\iota}\,d_\iota^1.\end{aligned}$$

Hence, for all $k=1, \ldots, \kappa$

$$\sum_{j=1}^{n} \alpha_{jk}\, a_j(t_k) = \sum_{j=1}^{n} \alpha_{jk}\, b_j(t_k) = 0.$$

If one of the α_{jk}'s is nonzero, the vectors $\{\lambda a_j(t_k) + b_j(t_k) \mid 1 \leqq j \leqq n\}$ are linearly dependent for all $\lambda \in \mathbb{R}$ in contradiction to the regularity of the matrix pencil $(A(t_k), B(t_k))$. Analogously, from

$$\sum_{\iota=1}^{r} \alpha_{m\iota}\, d_\iota^0 = \sum_{\iota=1}^{r} \alpha_{m\iota}\, d_\iota^1 = 0$$

we obtain $\alpha_{m1} = \ldots = \alpha_{mr} = 0$ because of rank $(D_0, D_1) = r$. $\square$

4. Examples

Finally, I add some examples for illustrating the behaviour of the method developed above. In all examples, the problem is uniquely solvable. For every example, the number of equispaced gridpoints ν, the minimal resp. maximal eigenvalue $\lambda_{\min}$ ($\lambda_{\max}$) of the matrix A_ν, and the error of the approximation in the norms of $C[0, 1](e_c)$ and $L^2(0, 1)(e_0)$ are recorded. The examples were run in double precision (about 16 decimal digits) on an EC 1055 M computer.

Example 4.1.

$$A(t) = \begin{pmatrix} 0 & 0 \\ 1 & \eta t \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & \eta t \\ 0 & 1+\eta \end{pmatrix}. \tag{4.1}$$

This example is extensively used in the literatur (e.g., [4, 10, 11]). It has several interesting features. The implicit Euler method (i.e., the simplest BDF) is unstable if $\eta < -1/2$ and convergent and weak unstable if $\eta > -1/2$. For $\eta = -1$, the pencil $(A(t), B(t))$ is singular, hence, the implicit Euler method is not applicable. The tables show the results given by our method for $\eta = -3, -1, 0$ with the right-hand side $f(t) = (e^t, t^2)$. For $\eta = -1$, Lemma 3.2 does not apply.

$\eta = -3$

ν	$\lambda_{\min}$	$\lambda_{\max}$	e_c	e_0
3	1.8(−2)	5.3(+1)	1.8(+0)	9.6(−1)
5	9.7(−4)	7.6(+1)	1.2(+0)	5.8(−1)
9	7.0(−5)	1.2(+2)	7.9(−1)	3.5(−1)
17	6.6(−6)	2.3(+2)	4.0(−1)	1.5(−1)
33	7.3(−7)	4.2(+2)	2.0(−1)	6.0(−2)
65	5.2(−8)	8.3(+2)	1.0(−1)	2.5(−2)

$\eta = -1$

ν	λ_{min}	λ_{max}	e_c	e_0
3	1.2(−1)	9.3(+0)	1.1(−1)	9.0(−2)
5	4.9(−4)	1.4(+1)	3.9(−2)	2.7(−2)
9	4.9(−5)	2.6(+1)	2.4(−2)	1.2(−2)
17	5.6(−6)	4.8(+1)	1.2(−2)	5.1(−3)
33	6.8(−7)	9.2(+1)	6.4(−3)	1.8(−3)
65	8.2(−8)	1.8(+2)	3.6(−3)	7.1(−4)

$\eta = 0$

ν	λ_{min}	λ_{max}	e_c	e_0
3	1.0(−2)	8.4(+0)	3.1(−1)	1.6(−1)
5	3.7(−4)	1.4(+1)	1.0(−1)	4.3(−2)
9	4.2(−5)	2.4(+1)	4.0(−2)	1.3(−2)
17	5.1(−6)	4.5(+1)	1.8(−2)	5.2(−3)
33	6.3(−7)	8.7(+1)	8.5(−3)	2.2(−3)
65	7.6(−8)	1.7(+2)	3.9(−3)	8.8(−4)

Remark. $a.b(c)$ should be read $a.b \times 10^c$.

Example 4.2

$$A(t) = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ n_2 & -n_1 & 0 & 0 \\ 0 & 0 & n_1 & n_2 \end{pmatrix},$$

$$F(t) = (0, 0, -n_2 u_e, 0),$$

$$D_0 = (1, 0, 0, 0), \quad D_1 = (0, 0, 0, 0), \quad \gamma = 1.$$

This equation describes a simple electrical network:

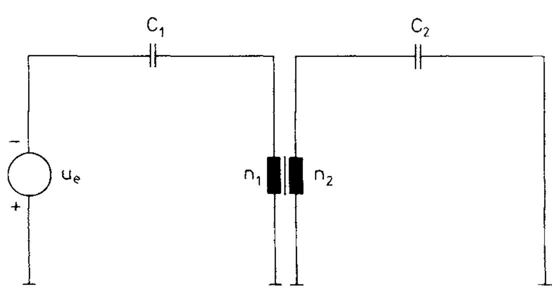

Fig. 1

The index of the system is 2. For simplicity we choose $C_1 = C_2 = n_1 = 1$ and $n_2 = 2$. In a first run we assumed that there is no voltage source, i.e., $u_e = 0$. A second run was made assuming $u_e = \sin 2\pi t$. The results are given in the following table.

v			$u_e = 0$		$u_e = \sin 2\pi t$	
	λ_{min}	λ_{max}	e_c	e_0	e_c	e_0
3	5.0(−2)	1.2(+1)	4.5(−2)	2.7(−2)	5.5(+0)	4.0(+0)
5	1.8(−3)	1.8(+1)	1.1(−2)	6.7(−3)	4.5(+0)	3.2(+0)
9	2.1(−4)	3.3(+1)	2.9(−3)	1.7(−3)	2.8(+0)	1.8(+0)
17	2.6(−5)	6.2(+1)	7.3(−4)	4.2(−4)	1.4(+0)	8.7(−1)
33	3.2(−6)	1.2(+2)	1.8(−4)	1.0(−4)	1.0(+0)	4.7(−1)

References

1. Aronszajn, N.: Theory of reproducing kernels. Trans. Amer. Math. Soc. **68**, 337–404 (1950)
2. Campbell, S.L.: Singular systems of differential equations. Marshfield: Pitman 1980 (Research Notes in Mathematics, 40)
3. Campbell, S.L.: Singular systems of differential equations II. Marshfield: Pitman 1982 (Research Notes in Mathematics, 61)
4. Campbell, S.L.: The numerical solution of higher index linear time varying singular systems of differential equations. SIAM J. Sci. Stat. Comput. **6** (2), 334–348 (1985)
5. Engl, H.W.: On least-squares collocation for solving linear integral equations of the first kind with noisy right-hand side. Boll. Geod. Sci. Affini **XLI** (3), 291–313 (1982)
6. Engl, H.W.: Regularization by least-squares collocation. In: Deuflhard, P., Hairer, E. (eds.), Numerical treatment of inverse problems in differential and integral equations, pp. 345–354 (Progress in Scientific Computing, 2). Boston: Birkhäuser 1983
7. Engl, H.W.: On the convergence of regularization methods for ill-posed linear operator equations. In: Hämmerlin, G., Hoffmann, K.H. (eds.), Improperly posed problems and their numerical treatment, pp. 81–95 (ISNM, 63). Basel: Birkhäuser 1983
8. Gantmacher, F.R.: Matrizenrechnung II. Berlin: Dt. Verlag Wiss. 1959
9. Gear, C.W.: Maintaining solution invariants in the numerical solution of ODE's. SIAM J. Sci. Stat. Comput. **7** (3), 734–743 (1986)
10. Gear, C.W., Hsu, H.H., Petzold, L.R.: Differential-algebraic equations revisited. Proc. of the Oberwolfach Conf. on Stiff Equations. Inst. für Geom. u. Praktische Math. der Rhein.-Westfälischen TH: Aachen 1981 (Bericht, 9)
11. Gear, C.W., Petzold, L.R.: ODE methods for the solution of differential-algebraic systems. SIAM J. Numer. Anal. **21** (4), 716–728 (1984)
12. Gorbunov, V.K.: Methods of reduction of unstable numerical problems (in Russian). Frunze: Ilim 1984
13. Griepentrog, E., März, R.: Differential-algebraic equations and their numerical treatment (Teubner-Texte zur Mathematik, 88). Leipzig: Teubner 1986
14. Groetsch, C.W.: Generalized inverses of linear operators: Representation and approximation, 1st Ed. New York: Dekker 1977
15. Groetsch, C.W.: The theory of Tikhonov regularization for Fredholm equations of the first kind (Research Notes in Mathematics, 105). Boston: Pitman 1984
16. Hanke, M.: Differential-algebraic equations in spaces of integrable functions. J. Differ. Equations (to appear)
17. Lötstedt, P., Petzold, L.R.: Numerical solution of nonlinear differential equations with algebraic constraints I: Convergence results for BDF. Math. Comput. **46** (174), 491–516 (1986)

18. März, R.: Index-2 differential-algebraic equations. Berlin, Humboldt-Universität, Sektion Math. 1987 (Preprint 135)
19. März, R.: Some new results concerning index-3 differential-algebraic equations. J. Math. Anal. Appl. (to appear)
20. Wahba, G.: Convergence rates for certain approximate solutions to Fredholm integral equations of the first kind. J. Approx. Theory **7**, 167–185 (1973)
21. Wahba, G.: A class of approximate solutions to linear operator equations. J. Approx. Theory **9**, 61–77 (1973)

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