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THE METHOD OF NORMAL SPLINE COLLOCATION*

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A projection method for solving linear one-dimensional integral and differential equations is described. The method is based on stating the problem of the normal solution (in Hilbert Sobolev space) of a collocation system and on transformation of the points and integral functionals to canonical form, i.e., a scalar product. The solution is obtained as a generalized spline.

Of the two basic (mesh and projection) classes of methods for the approximate solution of functional problems, the second has the potentially better properties. By suitably choosing the basis, we can develop different special features of the equations in the analytic working, and satisfactorily approximate the functional problem by a finite-dimensional problem of small dimensionality.

In the present paper we devise a projection method for solving linear integrodifferential equations, and initial-value and boundary value problems of the type of the collocation method /1/, though with the natural basis generated by the kernel of the integral operator and/or by the coefficients of the equation, and also, by the chosen space of solutions. As the latter we take a Hilbert space of Sobolev type. The initial equation is replaced by a finite system of equations at points of a division of the interval of the independent variable. Unlike traditional projection methods, the basis system is not postulated. The indeterminacy of the problem of solving the finite system of functional equations is removed by stating the problem of the normal solution of this system.

Our approach is based on the fact that, in space $W_2^l[a, b]$, the integrals of the wanted solution, with weights generated by the kernel of the initial integral equation, and the values of the function and of its derivatives up to order $l-1$ at the given points, are linear continuous functions. This fact follows /2/ from the Sobolev imbedding theorem for $W_2^l[a, b]$ into $C^{l-1}[a, b]$.

By Riesz's theorem, any linear continuous functional in Hilbert space can be written as a scalar product. This canonical form of the point and integral functionals, generated by the kernels and/or the coefficients of the initial equations, lies at the basis of the method of expressing our method in algorithmic form. The canonical images of the functionals are elements of W_2^l and form the basis system. Thus this basis, as in the Kupradze-Aleksidze method /3/ (in the case of integral equations), is determined by the given functions of the initial equation, except that the latter are transformed in accordance with the choice of the space of solutions.

This method gives a generalized spline /4-7/ as the approximate solution of the initial problem. In the well-known realizations of this approach, however, the minimized functional is a seminorm. By using the norm, we can simplify the study of the convergence and the computational scheme.

The method can be realized for the interpolation of functions, and for initial-value and boundary value problems for differential equations. There is then found to be a computational equivalence of the two norms generated by exponential and polynomial splines. We give a mesh condensation scheme which is suitable for solving "stiff" problems. The canonical form of the functionals was earlier used by the author when solving integral equations of the 1st kind /8, 9/.

1. *Equations in spaces* $W_2^l(a, b)$

Consider the problem of solving an equation of the type

$$A(t)\frac{dx(t)}{dt}+B(t)x(t)-\int_a^b K(t,s)x(s)\,ds=f(t) \tag{1.1}$$

with the conditions

$$Cx(a)+Dx(b)=g, \tag{1.2}$$

where $x, f, g\in \mathbb{R}_n$, and A, B, C, D, K are n-th order square matrices. In the case $A(t)\equiv 0$ we are concerned with an integral equations of the 2nd kind, and Conditions (1.2) are unnecessary. The matrix $B(t)$ is assumed to be non-degenerate for $a\leqslant t\leqslant b$.

The solvability of problems for Eq.(1.1) is usually considered in special cases for the

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Zh. vychisl. Mat. mat. Fiz., 29, 2, 212-224, 1989 functional spaces $C(a, b)$ or $L_2(a, b)$. We have in mind here the fact that the components of the vector function $x(t)$ belong to these spaces. If the problem is solvable in one of these spaces it is usually easy to show what extra conditions are needed for the solution to be differentiable, i.e., for its components to belong to the space $C^r(a, b)$ with some $r \geqslant 1$. For this, we have to show that the terms of Eq.(1.1) are differentiable a sufficient number of times.

Our variational method for solving approximately problems for Eq.(1.1) is based on the fact that the required solution can be seen as an element of Hilbert space of Sobolev type $W_2^l(a, b)$ with $l \geqslant 2$; if $A \equiv 0$, then $l \geqslant 1$. We shall consider two topologically equivalent norms

$$\|x\| = \left\{ \sum_{i=1}^{n} \int_a^b [x_i^2(s) + (x_i^{(l)})^2] ds \right\} . \tag{1.3a}$$

$$\|x\| = \left\{ \sum_{i=1}^{n} \left(\sum_{k=0}^{l-1} [x_i^{(k)}(a)]^2 + \int_a^b [x_i^{(l)}(s)]^2 \right) ds \right\}^{1/2} . \tag{1.3b}$$

In the case (1.3a), infinite intervals are admissible.

We assume that the integral operator in (1.1) is bounded in W_2^l and that the problem has a solution in this space. If the solution is not unique, then there is a unique normal solution; we call it x^0.

We introduce into the interval (a, b) the sequence of mesh divisions

$$a \leqslant t_1^m < \ldots < t_m^m \leqslant b, \qquad m \to \infty, \tag{1.4}$$

and we pass from Eq.(1.1) in the continuum (a, b) to the finite system of equations

$$A(t_k^m) \frac{dx(t_k^m)}{dt} + B(t_k^m) x(t_k^m) - \int_a^b K(t_k^m, s) x(s) ds = f(t_k^m), \tag{1.5}$$
$$1 \leqslant k \leqslant m,$$

for some Eq.(1.4); if $A(t) \not\equiv 0$, we consider the system jointly with Condition (1.2).

In the left-hand side of (1.5), the required solution $x(s)$ is represented by its values, and also be the values of the derivatives (if $A \neq 0$) at the points t_i^m and by integrals, linear in x. All these objects are linear continuous functionals in W_2^l, so that the set described by system (1.2), (1.5) is linear and closed. It obviously contains the solution x^0 and has an element which is the minimum norm

$$x_m = \operatorname{argmin}\{\|x\|^2 : (1.2), (1.5)\}. \tag{1.6}$$

In the case $A(t) \equiv 0$, Condition (1.2) has to be omitted in (1.6).

The well-known collocation method /1/ is based on the passage from the continual Eq.(1.1) to the finite system (1.5). In the original version of this method, the solution is written as an expansion in an a priori basis. There is a method of spline collocation in which the solution is sought as a cubic spline /10/. As distinct from the original version with classical algebraic or trigonometric bases, this modification ensures convergence on any condensing meshes.

Problem (1.6) is a special case of the abstract scheme is generalized splines /4-6/. If the norm is chosen as the minimized functional, the problem of the existence and uniqueness of the solution becomes trivial. From the point of view of the variational calculus, (1.6) is the classical isoperimetric problem and can be solved by Lagrange's method. However, this approach leads to complicated systems of integrodifferential equations in the wanted functions and the Lagrange multipliers. The specific feature of Problem (1.6), namely, the choice of norm-squared as the functional, means that a simple method of solution can be devised, based on the canonical transformation of linear continuous functionals, and described below.

2. *Convergence of normal solutions.*

Consider the convergence of solutions (1.6) to the solution of the initial problem. We denote by τ_m the maximum step of the division (1.4).

Lemma 1. Let $\psi(t)$ be a scalar function of space $W_2^l(a, b)$, which vanishes at the division points $a \leqslant t_1 < \ldots < t_m \leqslant b$, and, at a point $\theta \in (a, b)$, satisfies $|\psi(\theta)| = \varepsilon > 0$; then,

$$\|\psi\|^2 > \frac{2^{l-1}}{\tau^{2l-1}} \varepsilon^2, \tag{2.1}$$

where $\tau = \max\{t_i - t_{i-1} : 2 \leqslant i \leqslant m\}$.

Proof. Let $t_k < \theta < t_{k+1}$; then,

$$\|\psi\|^2 \geqslant \int_{t_k}^{t_{k+1}} |\psi^{(l)}(s)|^2 ds.$$

Consider the case $l=1$. Obviously,

$$\int_{t_k}^{t_{k+1}} |\psi'|^2 ds \geqslant \min_{\varphi \in W_2'} \left\{ \int_{t_k}^{\theta} |\varphi'|^2 ds : \varphi(t_k)=0,\ \varphi(\theta)=\varepsilon \right\}.$$

Euler's equation of the extremal problem on the right is $\varphi''=0$, and its solution is

$$\varphi(s) = \frac{s-t_k}{\theta-t_k}\varepsilon.$$

It follows from this that

$$\int_{t_k}^{t_{k+1}} |\psi^{(l)}(s)|^2 ds \geqslant \frac{2^{l-1}}{\tau^{2l-1}}\varepsilon^2, \tag{2.2}$$

i.e., we have inequality (2.1) for $l=1$.

Let (2.2) hold for $l=r\geqslant 1$. To prove the lemma, we only need to show that (2.2) likewise holds for $l=r+1$.

In the interval (t_k, t_{k+1}) there is an extremal point $\bar\theta$, at which $\psi(\bar\theta)\geqslant\psi(\theta)=\varepsilon$ and $\psi^{(r)}(\bar\theta)=0$, so that

$$\psi^{(r)}(t) = -\int_t^{\bar\theta} \psi^{(r+1)}(s)\,ds.$$

On squaring both sides and applying the Cauchy inequality, we have

$$|\psi^{(r)}(t)|^2 \leqslant (\bar\theta-t)\int_t^{\bar\theta} |\psi^{(r+1)}(s)|^2 ds \leqslant (\bar\theta-t)\int_{t_k}^{t_{k+1}} |\psi^{(r+1)}|^2 ds.$$

Integrating the extreme left- and right-hand expressions over $[t_k, t_{k+1}]$, we get

$$\int_{t_k}^{t_{k+1}} |\psi^{(r)}|^2 ds \leqslant \frac{\tau^2}{2}\int_{t_k}^{t_{k+1}} |\psi^{(r+1)}|^2 ds.$$

From this we obtain (2.2) for $l=r+1$; this proves the lemma.

Theorem 1. If $-\infty<a<b<\infty$, $\tau_m\to 0$, then

$$\lim_{m\to\infty} \|x_m - x^0\| = 0. \tag{2.3}$$

Proof. The solution x^0 satisfies any system (1.5), so that $\|x_m\|\leqslant\|x^0\|$ and the sequence x_m has a weak limit $\bar x$, while $\|\bar x\|\leqslant\|x^0\|$.

Consider the discrepancies of Eq.(1.1) on the functions $x_m, \bar x$, i.e., the vector functions

$$\varphi^m(t) = A(t)\frac{dx_m(t)}{dt} + B(t)x_m(t) - \int_a^b K(t,s)x_m(s)ds - f(t),$$

$$\varphi(t) = A(t)\frac{d\bar x(t)}{dt} + B(t)\bar x(t) - \int_a^b K(t,s)\bar x(s)ds - f(t).$$

Since the operator of Eq.(1.1) is bounded from $W_2^l(a,b)$ into $W_2^{l-1}(a,b)$ when $A\neq 0$, and is bounded in W_2^l when $A\equiv 0$, these discrepancies are elements at least of space $W_2^1(a,b)$, and the sequence φ^m is bounded and weakly convergent to φ. This implies the pointwise convergence $\varphi^m(t)\to\varphi(t)$, $a\leqslant t\leqslant b$.

Assume that the component of the discrepancy φ_i reaches the value $2\varepsilon>0$ at some point $\theta\in(a,b)$ for $i\in\{1,2,\ldots,n\}$. Then, for sufficiently large m, we have $\varphi_i^m(\theta)\geqslant\varepsilon$, and by the lemma, and the condition $\tau_m\to 0$, the sequence of discrepancies φ_i^m will be unbounded. This contradiction leads to the identity $\varphi_i(t)=0$, $a\leqslant t\leqslant b$, whence it follows that $\bar x$ is a solution of (1.1).

If follows from $\|\bar x\|\leqslant\|x^0\|$ and the reverse inequality, the $\|\bar x\|=\|x^0\|$, and since the normal solution x^0 is unique, then $\bar{x}=x^0$.

We know that the norm is weakly lower semicontinuous, i.e.,

$$\|x^0\|=\liminf_{m\to\infty}\|x_m\|\leqslant\limsup_{m\to\infty}\|x_m\|\leqslant\|x^0\|,$$

so that $\|x_m\|\to\|x^0\|$. For a Hilbert space, the weak convergence and the convergence of the norms imply the strong convergence, i.e., Eq.(2.3).

Theorem 2. If the interval (a, b) of Eq.(1.1) is unbounded, the divisions (1.4) are localized in a finite interval $[\alpha, \beta] \subset (a, b)$, and $\tau_m\to 0$, then the sequence x_m is convergent in the metric of $W_2^l(a, b)$ to the normal solution $\bar{x}$ of the equation

$$A(t)\frac{dx(t)}{dt}+B(t)x(t)-\int_a^b K(t,s)x(s)ds=f(t), \qquad \alpha\leqslant t\leqslant\beta. \tag{2.4}$$

The proof is basically a repetition of the proof of Theorem 1, with the following differences. It follows from the weak convergence $x_m\to\bar{x}$ in $W_2^l(a, b)$ with an unbounded interval, that we have pointwise convergence in any bounded part of (a, b), and vanishing of the discrepancy $\bar{x}$ will only occur in $[\alpha, \beta]$. Consequently, the weak limit of $\bar{x}$ only satisfies Eq.(2.4). Meantime, $\|x_m\|\leqslant\bar{x}$, whence it follows, as in Theorem 1, that we have the convergence $x_m\to\bar{x}$ in the norm of $W_2^l(a, b)$. Since any solution of (2.4) satisfies system (1.5), i.e., has a norm of not less than $\|x_m\|$, then $\bar{x}$, being the limit of the sequence x_m, is the solution of (2.4) with the minimum norm.

In short, passage to problem (1.6) is an approximate method of solving Eq.(1.1). Recall that convergence in $W_2^l(a, b)$ implies uniform convergence of the derivatives $x^{(k)}(s)$, $0\leqslant k\leqslant l-1$, in any finite part of (a, b).

3. *Transformation of integral and point functionals.*

Let us dwell on the transformation of integral and point functionals to the canonical form (scalar product); we consider the point values of the derivatives of $x(s)$ up to order $l-1$. These functionals are also continuous, in view of the well-known imbeddings

$$W_2^l\to C^k, \qquad 0\leqslant k\leqslant l-1.$$

In short, we consider the linear functionals

$$\int_a^b k(s)x(s)ds, \qquad \left.\frac{d^rx(s)}{ds^r}\right| , \qquad 0\leqslant r\leqslant l-1,$$

where k is a summable function. We start with the integral functional. By Riesz's theorem, to the function k there corresponds the unique function $h\in W_2^l$, such that, for any $x\in W_2^l$,

$$\left.\begin{aligned}&\int_a^b [h(s)x(s)+h^{(l)}(s)x^{(l)}(s)]ds \qquad (3.1a)\\ &\sum_{r=0}^{l-1} h^{(r)}(a)x^{(r)}(a)+\int_a^b h^{(l)}x^{(l)}ds \qquad (3.1b)\end{aligned}\right\}=\int_a^b k(s)x(s)ds,$$

in the norms (1.3a) and (1.3b) respectively.

Using integration by parts, we obtain in the case of (3.1a) for the wanted function $h(s)$ the boundary value problem

$$(-1)^lh^{(2l)}(t)+h(t)=k(t), \qquad h^{(r)}(a)=h^{(r)}(b)=0, \qquad l\leqslant r\leqslant 2l-1, \tag{3.2a}$$

and in the case of (3.1b) the problem

$$\begin{aligned}&(-1)^lh^{(2l)}(t)=k(t), \qquad h^{(r)}(a)-(-1)^{l-1-r}h^{(2l-1-r)}(a)=\\ &k^{(2l-1-r)}(b)=0, \qquad 0\leqslant r\leqslant l-1.\end{aligned} \tag{3.2b}$$

These problems are meaningful provided that the function $h^{(2l)}$ exists and is summable, which is not a necessary property of a function of W_2^l. The proof of this property, and hence, of the passage from the variational equations (identities in x) to Problems (3.2), is given by:

Theorem 3. The solution of each of the boundary value Problems (3.2) exists for any summable function $k(t)$, is unique, and can be written in terms of the appropriate Green's function $G(s, t)$:

$$h(s)=\int_a^b G(s,t)k(t)dt. \tag{3.3}$$

The proof amounts to showing that zero is not an eigenvalue of problems (3.2), i.e., the corresponding homogeneous problems have no non-trivial solutions.

In the case of (3.1a), the general solution of the equation

$$(-1)^l h^{(2l)}(s)+h(s)=0$$

has the form

$$h(s)=\sum_{r=1}^{2l} c_r \exp(\lambda_r s),$$

where λ_r, the roots of degree $2l$ of the number $(-1)^{l+1}$, are distinct non-zero complex numbers. Substituting this expression into the boundary conditions of (3.2a), where we can take $a=0$, $b=1$, we obtain for the coefficients c_r the system of equations

$$\sum_{r=1}^{2l} \lambda_r^{k+l} c_r=0, \qquad \sum_{r=1}^{2l} \lambda_r^{k+l} \exp(\lambda_r) c_r=0, \qquad 0\leqslant k\leqslant l-1.$$

We consider a linear combination of any column of the matrix of this system with coefficients α_k, β_k:

$$\sum_{k=0}^{l-1} [\alpha_k+\beta_k \exp(\lambda_r)]\lambda_r^{k+l}=0.$$

Since the system $\{\lambda^l,\dots,\lambda^{2l-1}\}$ is linearly independent, we must have $\alpha_k=-\beta_k \exp(\lambda_r)$. Since this equation is satisfied for all $k=0, 1,\dots, l-1$ with different λ_r, we must have $\alpha_k=\beta_k=0$. Thus the system of equations that defines the coefficients of the general solution is not degenerate and can only have the trivial solution. The theorem is proved for the case (3.1a).

In the case (3.1b), the general solution of the homogeneous equation is

$$h(s)=\sum_{r=0}^{2l-1} c_r s^r,$$

and the coefficients which satisfy the boundary conditons are given by

$$k!c_k-(-1)^{l-1-k}(2l-k-1)!c_{2l-k-1}=0,$$

$$\sum_{r=l+k}^{2l-1} \frac{r!}{(r-l-k)!} c_r=0, \qquad 0\leqslant k\leqslant l-1.$$

It is easily seen that the matrix of this system is upper triangular and non-degenerate, so that the system has only the trivial solution. The theorem is proved.

Corollary. If the function $k\in C^r(a, b)$, $r\geqslant -1$ (C^{-1} is the class of summable functions), then its image (3.3) belongs to $C^{2l+r}(a, b)$.

This is an obvious consequence of the differential equations of (3.2), which are satisfied by the images of the h integral functionals (3.1), by Theorem 3.

The problem of canonical transformation of the integral functionals can thus be solved by finding the Green's function $G(s, t)$ of the corresponding boundary value Problem (3.2). For $a<t<b$, this function is given in the case (3.1a) by the boundary value problem

$$\begin{gathered}(-1)^l \frac{\partial^{2l} G(s,t)}{\partial s^{2l}}+G(s,t)=0, \qquad s\neq t,\\ \frac{\partial^r G(a,t)}{\partial s^r}=\frac{\partial^r G(b,t)}{\partial s^r}=0, \qquad l\leqslant r\leqslant 2l-1,\end{gathered} \tag{3.4a}$$

and in the case (3.1b), by the problem

$$\frac{\partial^{2l} G(s,t)}{\partial s^{2l}}=0, \tag{3.4b}$$

$$\frac{\partial^r G(a,t)}{\partial s^r} - (-1)^{l-1-r} \frac{\partial^{2l-1-r} G(a,t)}{\partial s^{2l-1-r}} = \frac{\partial^{2l-1-r} G(b,t)}{\partial s^{2l-1-r}} = 0,$$

$$0 \leqslant r \leqslant l-1,$$

where in both cases we have the jump condition

$$\frac{\partial^{2l-1} G(t-0,t)}{\partial s^{2l-1}} - \frac{\partial^{2l-1} G(t+0,t)}{\partial s^{2l-1}} = (-1)^l \tag{3.5}$$

and the derivatives of lower orders are continuous in the first argument.

By Theorem 3, each of Problems (3.4), (3.5) is uniquely solvable. Since the problems are selfadjoint, the Green's functions are symmetric.

Determination of Green's function also solves the problem of transforming the point functionals, since we have

$$x(t) = \int_a^b \left[G(s,t)x(s) + \frac{\partial^l G(s,t)}{\partial s^l} x^{(l)}(s) \right] ds \tag{3.6a}$$

or

$$x(t) = \sum_{k=0}^{l-1} x^{(k)}(a) \frac{\partial^k G(a,t)}{\partial s^k} + \int_a^b \frac{\partial^l G(s,t)}{\partial s^l} x^{(l)}(s)\, ds \tag{3.6b}$$

for any $x \in W_2^2(a,b)$, $a \leqslant t \leqslant b$. The right-hand sides are scalar products of the functions $G(s,t)$, regarded as elements of $W_2^l(a,b)$ for any $t \in [a,b]$, with x. The identity is easily proved by integrating the right-hand side by parts, using Eqs. (3.4) and (3.5). The images of the functionals, i.e., the values at points of the derivatives of order up to $l-1$, are obtained by differentiating the identity (3.6) with respect to t.

We shall in future use the notation $\langle x, y \rangle$ for the scalar product of elements $x, y \in W_2^l$, corresponding to the norms (1.3). The point functionals then have the form

$$\frac{d^r x(t)}{dt^r} = \left\langle \frac{\partial^r G(\cdot,t)}{\partial t^r}, x \right\rangle, \qquad 0 \leqslant r \leqslant 1. \tag{3.7}$$

Transformation of the linear functionals to the canonical form reduces the conditions of problem (1.6) to the form of scalar products. This simplifies their analysis and makes them easier to write in algorithmic form. The following lemma is useful here.

Lemma 2. If $G(s,t)$ is the Green's function which defines the canonical transformation of linear functionals in space W_2^l, $a \leqslant t_i, t_j \leqslant b$, then

$$\langle G(\cdot, t_i), G(\cdot, t_j) \rangle = G(t_i, t_j), \tag{3.8a}$$

$$\langle G(\cdot, t_i), G_t'(\cdot, t_j) \rangle = G_t'(t_i, t_j), \tag{3.8b}$$

$$\langle G_t'(\cdot, t_i), G_t'(\cdot, t_j) \rangle = G_{st}''(t_i, t_j). \tag{3.8c}$$

Proof. Since $G(s,t)$ and the scalar product are symmetric, Eqs. (3.8a, b) follow at once from (3.7) with $r=0$, $t=t_i$, $x(s)=G(s,t_j)$ and $x'(s)=G_t'(s,t_j)$.

Further, using (3.8b), we obtain

$$\langle G_t'(\cdot,t_i), G_t'(\cdot,t_j) \rangle = \frac{d}{dt} \langle G(\cdot,t), G_t'(\cdot,t_j) \rangle |_{t=t_i} =$$

$$\frac{d}{dt} G_t'(t,t_j)|_{t=t_i} = G_{ts}''(t_i,t_j),$$

i.e., Eq. (3.8c).

The solution of Problems (3.4), (3.5) on the Green's functions for the cases $l=1$, $l=2$ and the norms (1.3a), is given in /8/. This norm is generally used to obtain the stabilizing functionals in the theory and methods of solving ill-posed problems. The norm (1.3b) is equivalent to it, but leads to simpler Green's functions. These functions are easily obtained for $l=1$:

$$G(s,t) = 1+s, \qquad 0 \leqslant s \leqslant t \leqslant 1,$$

and for $l=2$:

$$G(s,t) = 1 + (s+s^2/2)t - s^3/6, \qquad 0 \leqslant s \leqslant t \leqslant 1.$$

The norm (1.3b) is only meaningful for finite intervals, whereas (1.3a) preserves its value for spaces $W_2^l[0, \infty)$ and $W_2^l(-\infty, \infty)$. The corresponding Green's functions are easily obained from the fundamental solutions of Eqs.(3.4a), which are given in /8/ for $l=1$ and 2, and from the boundary conditions, which signify the limit at infinity.

4. *Realization of the method.*

The method of normal spline collocation is realized for the interpolation of functions, and initial-value and boundary value problems for a second-order differential equation. For the interpolation of functions we pose three problems on the normal solution of the system

$$x(t_i)=f_i, \qquad 1\leqslant i\leqslant m, \tag{4.1}$$

where $a=t_1<\ldots\leqslant t_m=b$: in space $W_2^1[a, b]$, in space $W_2^2[a, b]$, and in space $W_2^2[a, b]$ with the auxiliary conditions

$$x'(t_1)=f_{m+1}, \quad x'(t_m)=f_{m+2}. \tag{4.2}$$

The two norms (1.3) are used in all these cases.

System (4.1), (4.2) can be written in canonical form as

$$\langle g_i, x\rangle=f_i, \qquad 1\leqslant i\leqslant m+2, \tag{4.3}$$

where

$$g_i(s)=G(s, t_i), \qquad 1\leqslant i\leqslant m, \tag{4.4a}$$

$$g_{m+1}(s)=\frac{\partial G(s,0)}{\partial t}, \qquad g_{m+2}(s)=\frac{\partial G(s,1)}{\partial t}. \tag{4.4b}$$

Functions (4.4) are obviously linearly independent. The normal solution of system (4.3) can be written as

$$x_m(t)=\sum_{i=1}^{m+2} u_i g_i(t), \tag{4.5}$$

and the coefficients of the expansion are here given by the system of equations

$$\sum_{j=1}^{m+2} g_{ij}u_j=f_j, \qquad 1\leqslant j\leqslant m+2, \tag{4.6}$$

where g_{ij} are the coefficients of the symmetric positive definite Gram matrix of the relevant basis of (4.4). By Lemma 2,

$$g_{ij}=G(t_i, t_j), \qquad 1\leqslant i, j\leqslant m,$$

$$g_{i, m+1}=G_t'(t_i, t_1), \qquad g_{i, m+2}=G_t'(t_i, t_m),$$

$$g_{m+1, m+1}=G_{st}''(t_1, t_1), \qquad g_{m+1, m+2}=G_{st}''(t_1, t_m),$$

$$g_{m+2, m+2}=G_{st}''(t_m, t_m).$$

It follows from (4.5) and the properties of the Green's function that the interpolant x_m is a spline of class C^{2l-2} of deficiency 1. In the case of the norm (1.3b), these are classical polynomial splines; for $l=1$ they are linear, and for $l=2$ are cubic.

In the case of the norm (1.3b), our present approach gives a new representation of polynomial splines in the form (4.5). For $l\geqslant 2$, this representation has minimal requirements on the amount of information, similar to the case of B-splines /10/: two blocks have to be stored: the division points, and the coefficients of the expansion in a basis given by the appropriate Green's function.

The properties of the approximation of polynomial splines for $l=1$ and 2, as obtained e.g., in /10/, can be extended to the splines (4.5) corresponding to the norm (1.3b). The splines generated by the norm (1.3a) are described by exponential and trigonometric functions, and we cannot make direct use here of the methods of obtaining approximation estimates for polynomial splines. To compare the approximation properties of the two types of splines, we carried out numerical experiments on the test functions of /10/. On uniform meshes with steps of 0.1 and 0.05, the errors computed on a mesh with a step of 0.01 differ by not more than 8%. An anomaly is perceptible when interpolating the function e^t. Here, in the first problem, with the norm (1.3a), we obtain an accuracy of order 10^{-9} on a mesh of step 0.2. This is explained by the fact that the interpolated function belongs to the subspace of basis hyperbolic functions.

In conclusion, let us demonstrate our method on the final problem for the equation

$$\ddot{x}(t)+q(t)\dot{x}(t)+r(t)x(t)=f(t) \tag{4.7}$$

in the interval [0, 1] with the conditions

$$c_{11}x(0)+c_{12}\dot{x}(0)=d_1, \quad c_{21}x(1)+c_{22}\dot{x}(1)=d_2.$$

The method of collocation has several possible versions here, based on the transformation of Eq.(4.7) to systems of differential, integrodifferential, or integral, equations. We shall dwell on two versions. In the first, we transform to a system of differential equations in the variables $x_1=x$ and $x_2=\dot{x}_1$:

$$\dot{x}_2(t)+q(t)x_2(t)+r(t)x_1(t)=f(t), \quad \dot{x}_1(t)-x_2(t)=0;$$

in the second, we integrate (4.7) from 0 to t, thus transforming to the integrodifferential equation

$$[\dot{x}(s)+q(s)x(s)]_0^t+\int_0^t[r(s)-\dot{q}(s)]x(s)\,ds=\int_0^t f(s)\,ds. \tag{4.8}$$

We write the system of collocations of the first version as

$$r_i x_1(t_i)+\dot{x}_2(t_i)+q_i x_2(t_i)=f_i, \quad 1\leqslant i\leqslant m, \tag{4.9a}$$

$$c_{11}x_1(t_1)+c_{12}x_2(t_1)=d_1, \quad \dot{x}_1(t_i)-x_2(t_i)=0, \tag{4.9b}$$

$$c_{21}x_1(t_m)+c_{22}x_2(t_m)=d_2. \tag{4.9c}$$

Here and henceforth, $q_i=q(t_i)$, $r_i=r(t_i)$, $f_i=f(t_i)$. $f_{m+1}=d_1$, $f_{m+2}=d_2$.

Table 1

k from	k to	$h_{1k}(s)$	$h_{2k}(s)$
1	m	$r_k G(s, t_k)$	$G_t'(s, t_k)+q_k G(s, t_k)$
$m+1$		$c_{11}G(s, t_1)$	$c_{12}G(s, t_1)$
$m+2$	$2m+1$	$G_t'(s, t_{k-m-1})$	$-G(s, t_{k-m-1})$
$2m+2$		$c_{21}G(s, t_m)$	$c_{22}G(s, t_m)$

We show the canonical form of the functionals of system (4.9) in Table 1.

The elements of the Gram matrix of this vector system are given by $g_{ij}=\langle h_{1i}, h_{1j}\rangle+\langle h_{2i}, h_{2j}\rangle$, $1\leqslant i\leqslant j\leqslant 2m+2$.

The normal solution of system (4.9) is

$$x_\nu(s)=\sum_{k=1}^{2m+2}u_k h_{\nu k}(s), \quad \nu=1, 2,$$

and the coefficients u_k are given by system (4.6) of order $2m+2$.

We take the collocation system of the second version in the form

$$\dot{x}(t_{i+1})-\dot{x}(t_i)+q_{i+1}x(t_{i+1})-q_i x(t_i)+\int_{t_i}^{t_{i+1}}k(s)x(s)\,ds=\int_{t_i}^{t_{i+1}}f(s)\,ds, \tag{4.10a}$$

$1\leqslant i\leqslant m-1,$

$$c_{11}x(t_1)+c_{12}\dot{x}(t_1)=d_1, \quad c_{21}x(t_m)+c_{22}\dot{x}(t_m)=d_2. \tag{4.10b}$$

We use here the notation $k(s)=r(s)-\dot{q}(s)$.

The canonical images of the functionals of system (4.10) are

$$h_i(s)=G_t'(s, t_{i+1})-G_t'(s, t_i)+q_{i+1}G(s, t_{i+1})- \tag{4.11a}$$

$$q_i G(s, t_i)+\hat{h}_i(s), \quad 1\leqslant i\leqslant m-1,$$

$$h_{m+1}(s)=c_{11}G(s, t_1)+c_{12}G_t'(s, t_1), \tag{4.11b}$$

$$h_{m+2}(s)=c_{21}G(s, t_m)+c_{22}G_t'(s, t_m). \tag{4.11c}$$

Here,

$$\hat{h}_i(s)=\int_{t_i}^{t_{i+1}}G(s, \tau)k(\tau)\,d\tau. \tag{4.12}$$

Notice that the basis system of the first version (Table 1) and its Gram matrix depend only on the discrete values of the coefficients of the initial Eq.(4.7). The second version, based on transformation to the integrodifferential Eq.(4.8), requires the evaluation of integrals (4.12) for the basis system (4.11) and of the corresponding scalar products $\langle \hat{h}_i, \hat{h}_j \rangle$.

Let us quote the results of solving the model example of /1/:

$$\varepsilon\ddot{x}(t)-\dot{x}(t)=-e^t, \quad 0\leqslant t\leqslant 1, \qquad x(0)=0, \quad \dot{x}(1)=z,$$

where z corresponds to the condition $x(1)=0$. The exact solution is

$$x(t)=\frac{1}{1-\varepsilon}\left(e^t-\frac{e-1}{e^{(1-t)/\varepsilon}-e^{-t/\varepsilon}}-\frac{1-e^{(\varepsilon-1)/\varepsilon}}{1-e^{-1/\varepsilon}}\right)$$

and is stiff for small $\varepsilon>0$. In the greater part of the interval [0, 1] it differs little from the solution of the degenerate problem e^t-1 and at the end drops sharply to zero, i.e., has a "boundary layer" at $t=1$.

Table 2

ε	DFEQ2	IDEQX	P2	PZ
0.2	0.89E-3	0.30E-3	0.0038	0.027
0.02	0.072	0.03	0.37	0.029
0.002	3.54	2.0	71.0	0.035

In Table 2 we give the errors of the solution of the example on a uniform mesh with step 0.02 ($m=51$) by the DFEQ2 program (the normal solution of sytem (4.9)) and the IDEQX program (the normal solution of system (4.10)) with the norm (1.3b). The components x_1, x_2 of the first version and the variable x of the second, are elements of space $W_2^2[0,1]$. The error was computed on a double mesh. In the columns P2 and PZ we show the corresponding results of /11/ of solving the example by a difference method. In the case of P2 the derivative in the boundary condition was approximated to the second order (using the equation), and in the case of PZ we used a special approximation, exact on the boundary layer function.

The first version (DFEQ2) was also realized for the norm (1.3a). The corresponding errors are virtually the same as shown in Table 2. We can conclude from this that the use of the norm (1.3a) is undesirable, since the realization becomes more difficult, for problems in bounded intervals.

Let us dwell on the boundary layer problem. One way of solving it is to construct a non-uniform mesh which condenses at points of sharp variation of the solution. The methods of spline collocation enable us to construct such meshes in stages, first solving the problem on a sparse mesh, and adding at the subsequent stages nodes which depend on the discrepancy of the equations, evaluated at the initial and the additional intermediate nodes of the original mesh.

The following scheme is one strategy for mesh condensation. Let N be the number of points added to the initial mesh.

$$\varphi_m(t)=\dot{x}_m(t)+q(t)x_m(t)+r(t)x_m(t)-f(t),$$

$$\delta_{mi}=|\varphi_m(t_i)|+|\varphi_m(t_{i+1})|+\left|\varphi_m\left(\frac{t_i+t_{i+1}}{2}\right)\right|,$$

$$\Delta_m=\sum_{i=1}^{m-1}\delta_{mi}, \qquad p_i=\delta_{mi}/\Delta_m.$$

As the initial mesh $\{t_i\}$ we can take a uniform mesh, and at the next stage, in each interval (t_i, t_{i+1}), and $n_i=Np_i$ uniformly distributed additional nodes.

This scheme is idealized, and when it is computerized, we have to ensure that the block of n_i is not empty (is non-zero), by averaging the actual results of multiplication by Np_i up to an integer n_i, and using the equation $N=n_1+\ldots+n_{m-1}$.

Table 3

m	$\varepsilon=0.02$		$\varepsilon=0.002$	
	$\alpha=0.1$	$\alpha=1$	$\alpha=0.1$	$\alpha=1$
4	3.90	3.90	44.8	44.8
10	0.51	0.34	10.8	8.5
16	0.49E-1	0.12E-1	2.8	1.1
22	0.55E-2	0.30E-2	0.54	0.55E-1
28	0.23E-2	0.28E-2	0.75E-1	0.26E-2

After addition of the nodes, we obtain a "piecewise constant" mesh $\{\bar{t}_i\}$, which can be "smoothed" by minimizing the functional

$$\sum_{i=1}^{m-1} (t_{i+1}-t_i)^2+\alpha\sum_{i=2}^{m-1} (t_i-\bar{t}_i)^2$$

with respect to $t_2,\ldots,t_{m-1}$ for $t_1=0$, $t_m=1$. The parameters $\alpha>0$ prevents the transition to a uniform mesh.

The condensation strategy is defined by two parameters: the initial number m of nodes, and the number N of nodes added at each stage. The meshes obviously become better for smaller values of these parameters, whereas the number of stages then increases. We are therefore faced with task of optimizing the strategy so as to limit the number of operations required.

In Table 3 we show the errors of solving our example on condensing meshes by the IDEQX program. Here $m=4$, $N=6$, and the initial mesh is uniform. In the case $\varepsilon=0.002$, $\alpha=1.0$, the last mesh has a minimum (first) step of 0.171 and a minimum (last) step 0.55E-3. The error of the special approximation /11/ on a condensing mesh of 50 steps is 0.24E-1.

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