SPLINES MINIMIZING ROTATION-INVARIANT SEMI-NORMS IN SOBOLEV SPACES

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ABSTRACT

We define a family of semi-norms $\|u\|_{m,s} = \left(\int_{\mathbb{R}^n} |\tau|^{2s} |\mathcal{F} D^m u(\tau)|^2 \, d\tau \right)^{1/2}$. Minimizing such semi-norms, subject to some interpolating conditions, leads to functions of very simple forms, providing interpolation methods that : 1°) preserve polynomials of degree $\leq m-1$; 2°) commute with similarities as well as translations and rotations of $\mathbb{R}^n$; and 3°) converge in Sobolev spaces $H^{m+s}(\Omega)$.

Typical examples of such splines are : "thin plate" functions ($\sum_{a \in A} \lambda_a |t-a|^2 \mathrm{Log}|t-a| + \alpha.t + \beta$ with $\sum \lambda_a = 0$, $\sum \lambda_a a = 0$) , "multi-conic" functions ($\sum \lambda_a |t-a| + C$ with $\sum \lambda_a = 0$) , pseudo-cubic splines ($\sum \lambda_a |t-a|^3 + \alpha .t + \beta$ with $\sum \lambda_a = 0$, $\sum \lambda_a a = 0$), as well as usual polynomial splines in one dimension. In general, data functionals are only supposed to be distributions with compact supports, belonging to $H^{-m-s}(\mathbb{R}^n)$; there may be infinitely many of them. Splines are then expressed as convolutions $\mu * |t|^{2m+2s-n}$ (or $\mu * |t|^{2m+2s-n} \mathrm{Log}\,|t|$) + polynomials.

§ 0 - INTRODUCTION

Splines in more than one dimension are usually constructed from one-dimensional ones, via tensor products. We follow here another point of view (more analogous to the physical interpretation of elementary cubic splines as equilibrium positions of a beam) developed by M. ATTEIA [1][2][3] : his splines (minimizing $\int_\Omega |D^2 v|^2$, a functional similar to the bending energy of a thin plate) are uneasy to compute, because their characterization involves a kernel given by series. But things are much simpler if we replace Ω by the whole plane $\mathbb{R}^2$, as is shown in [5] (where present extensions are announced). This leads to what we call "thin plate" functions in $\mathbb{R}^2$ (engineers say "surface splines" [6]). Of course, it is possible to minimize $\int |D^m v|^2$ instead of $\int |D^2 v|^2$, and deal with $\mathbb{R}^n$ instead of $\mathbb{R}^2$ (with some restriction : $m > \frac{n}{2}$, if point values are used).

We notice that these functionals $\int_{\mathbb{R}^n} |D^m v|^2$ are invariant through translations and rotations. Moreover, if a similarity $t \mapsto \lambda t$ is applied to v, they are multiplied by some power of $|\lambda|$. Thus, corresponding interpolation methods will <u>commute with similarities</u> : interpolating on a contracted set of points λA gives the same result as interpolating on A (with same values) and then applying contraction $t \mapsto \lambda t$.

Now, since Fourier transform is isometric on $L^2(\mathbb{R}^n)$, we may write $\int_{\mathbb{R}^n} |D^m v(t)|^2 dt = \int_{\mathbb{R}^n} |\mathcal{F} D^m v(\tau)|^2 d\tau$. A natural idea, to get other interpolation methods, would be to introduce a weighting function w and minimize $\int_{\mathbb{R}^n} w(\tau) |\mathcal{F} D^m v(\tau)|^2 d\tau$. In view of the above invariance properties, it is natural to put $w(\tau) = |\tau|^\theta$ and try to minimize $\int_{\mathbb{R}^n} |\tau|^\theta \; |\mathcal{F} D^m v(\tau)|^2 d\tau$, a functional which is invariant through translations and rotations, and is multiplied by a constant if the variable t is changed into λt. This is actually possible (at least if $-2m-n < \theta < n$ and $2m + \theta$ is sufficiently large, depending on which kind of data are used), in a precise sense, as we shall see.

We first introduce some "Sobolev-type" spaces such as $\tilde{H}^s$, $D^{-m}\tilde{H}^s$, H^{m+s}_{loc} ,

H^{-m-s}_{comp} , and compare them. Central space is $D^{-m}\,\tilde{H}^s \,/\, P_{m-1}$ with Hilbert structure. Its dual contains $H^{-m-s}_{comp} \cap P^o_{m-1}$ as a dense subset (§1).

In §2 we prove existence and uniqueness for interpolation problems, and abstract characterization using reproducing kernels of semi-Hilbert spaces.

§3 is the crucial one. We compute the reproducing kernel of $D^{-m}\,\tilde{H}^s \,/\, P_{m-1}$ in $H^{m+s}_{loc} \,/\, P_{m-1}$, i.e. the natural isometric mapping from $H^{-m-s}_{comp} \cap P^o_{m-1}$, embedded in $(D^{-m}\,\tilde{H}^s \,/\, P_{m-1})$, into $D^{-m}\,\tilde{H}^s \,/\, P_{m-1}$.

§4 summarizes our main result (theorems 4 and 4 bis). In §5 we apply it to various examples.

§6 is concerned with convergence in Sobolev space $H^{m+s}(\Omega)$ (Ω bounded) for an interpolated function $f \in H^{m+s}(\Omega)$.

For brevity, we use classical or natural notations without explicit statement. Moreover, some familiarity with distributions is assumed (convolution, derivation, Fourier transform, multiplication with C^∞ functions, and their relations ; correspondence between locally summable functions and distributions, etc. ; standard reference is [9]). $D^m v(t)$ is the n^m-tuple of partial derivatives $(D_{i_1} \ldots D_{i_m} v(t) \;;\; i_1,\ldots,i_m = 1,\ldots,n)$, with Euclidian norm $|D^m v(t)| = (\Sigma|D_{i_1} \ldots D_{i_m} v(t)|^2)^{1/2}$. Pseudo-functions $Pf.|\tau|^\theta$ are to be found in [9], p. 44 and their Fourier transforms p. 257. Fourier transform is $\mathcal{F}$ or $\hat{}$.

The letter c is used to denote various constants, to avoid useless complication.

§ 1 - FUNCTIONAL SPACES

1.1. Sobolev spaces H^s, H^s_{loc}, H^{-s}_{comp} .

For any real s, $H^s(\mathbb{R}^n)$ is the set of tempered distributions u on $\mathbb{R}^n$ whose Fourier transform $\hat{u}$ is a (locally summable) function such that $\int_{\mathbb{R}^n} (1 + |\tau|^2)^s \,|\, \hat{u}(\tau)|^2 d\tau < \infty$. In other words $H^s = \mathcal{F}(1 + |\tau|^2)^{-s/2}\, L^2$. It is a Hilbert space. Its dual is naturally identified with $H^{-s}(\mathbb{R}^n)$. When s is a positive integer, $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \; ; \; D^\alpha u \in L^2(\mathbb{R}^n) \; , \; \forall \; |\alpha| \leq s\}$.

If K is a closed subset of $\mathbb{R}^n$, $H^s_K(\mathbb{R}^n)$ is the set of distributions $\in H^s(\mathbb{R}^n)$ whose support is contained in K. H^s_K is a closed linear subspace of H^s.

If Ω is an open subset of $\mathbb{R}^n$, $H^s(\Omega)$ is the set of restrictions, to Ω , of distributions $\in H^s(\mathbb{R}^n)$. It is isomorphic to the quotient space $H^s(\mathbb{R}^n) \; / \; H^s_{\Omega^c}(\mathbb{R}^n)$ hence a Hilbert space too. Its dual is naturally identified with $H^{-s}_{\overline{\Omega}}(\mathbb{R}^n)$.

If Ω is bounded and sufficiently regular (e.g. Ω satisfies some "uniform cone" condition) then (for non integer $s > 0$) $H^s(\Omega)$ is the set of distributions u on Ω whose derivatives of order [s] (integralpart of s) are in $L^2(\Omega)$, with

$$\iint_{\Omega\times\Omega} \frac{|D^{[s]}u(t) - D^{[s]}u(t')|^2}{|t-t'|^{n+2(s - [s])}} \, dt \, dt' < \infty$$

For integer $s \geq 0$ it suffices that $D^\alpha u \in L^2(\Omega)$, $\forall |\alpha| = s$.

Now $H^s_{loc}(\mathbb{R}^n)$ is the set of distributionson $\mathbb{R}^n$ whose restriction to any bounded open set Ω is in $H^s(\Omega)$. Of course $C^k \subset H^s_{loc}$ for integer $k \geq s$. On the other hand, Sobolev embedding theorems assert that $H^s_{loc}(\mathbb{R}^n) \subset C^k$ for $s > k + \frac{n}{2}$.

H^s_{loc} may be equipped with a Fréchet space structure : putting B_N = the open ball $\{ \; |t| < N\}$, one defines a countable family of semi-norms $u \mapsto$ norm of $u_{|B_N}$ in $H^s(B_N)$. It is reflexive, and its dual is naturally identified with $H^{-s}_{comp}(\mathbb{R}^n)$, the union of $H^{-s}_{\overline{B}_N}(\mathbb{R}^n)$, equipped with the topology of (strict) inductive limit of this countable family of Hilbert spaces.

All previous norms could have been replaced by equivalent ones, they do not play a role by themselves in our problem. We now come to definig spaces $\tilde{H}^s$ and $D^{-m} \tilde{H}^s$, whose semi-norms $\|.\|_{m,s}$ are fundamental.

1.2. $\tilde{H}^s(\mathbb{R}^n)$, $s < \frac{n}{2}$

We put $\tilde{H}^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \; ; \; \hat{u} \in L^1_{loc} \; , \; \int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau < \infty\}$. We equip it with norm $\|u\|_{0,s} = (\int_{\mathbb{R}^n} |\tau|^{2s} |\hat{u}(\tau)|^2 d\tau)^{1/2}$. $\tilde{H}^s(\mathbb{R}^n)$ is a Hilbert space if (and only if) $s < \frac{n}{2}$: for $f \in L^2(\mathbb{R}^n)$, the function $|\tau|^{-s} f(\tau)$ is locally summable, since $\int_K |\tau|^{-s} |f(\tau)| d\tau \leq (\int_K |\tau|^{-2s})^{1/2} (\int_K |f(\tau)|^2 d\tau)^{1/2}$ and $|\tau|^\lambda$ is locally summable if (and only if) $\lambda > -n$; therefore $|\tau|^{-s} f$ defines a distribution, easily seen to be tempered, so that $f \mapsto \mathcal{F}(|\tau|^{-s} f)$ is an isometry from $L^2(\mathbb{R}^n)$ onto $\tilde{H}^s(\mathbb{R}^n)$. Moreover if $f_j \to 0$ in L^2, one can prove that $|\tau|^{-s} f_j \to 0$ in $\mathcal{S}'$ (it suffices to show that $\int |\tau|^{-s} f_j(\tau) \, \varphi(\tau) d\tau \to 0$ for any C^∞ function φ rapidly decreasing at ∞). This implies that inclusion $\tilde{H}^s \hookrightarrow \mathcal{S}'$ (a fortiori $\tilde{H}^s \hookrightarrow \mathcal{D}'$) is continuous, or that $\tilde{H}^s$ is a Hilbert subspace of $\mathcal{D}'$.

We might even prove that, for $-\frac{n}{2} < s < \frac{n}{2}$, $\tilde{H}^s$ is a normal subspace of $\mathcal{S}'$ (i.e. contains $\mathcal{S}$ as a dense subspace) with dual $\tilde{H}^{-s}$.

(Some of these spaces are known and used in other fields. For example, $\tilde{H}^{-1}(\mathbb{R}^3)$ and $\tilde{H}^1(\mathbb{R}^3)$ are respectively spaces of charges and potentials of finite energy, in the theory of Newtonian potentials in $\mathbb{R}^3$. See [10], §11).

From definitions, it is obvious that $H^s(\mathbb{R}^n) \subset \tilde{H}^s(\mathbb{R}^n)$ if $s > 0$, the converse if $s < 0$ (and $\tilde{H}^0 = H^0 = L^2$). In general $H^s(\mathbb{R}^n) \neq \tilde{H}^s(\mathbb{R}^n)$ ($\tilde{H}^s$ is not contained in L^2, for $s > 0$, while $H^s \subset L^2$). But (at least for $-\frac{n}{2} < s < \frac{n}{2}$) equality holds on bounded subsets, as we now see.

1.3. $\tilde{H}^s(\Omega) = H^s(\Omega)$, if $-\frac{n}{2} < s < \frac{n}{2}$

Of course, $\tilde{H}^s(\Omega)$ is the set of restrictions, to Ω (bounded), of elements of $\tilde{H}^s(\mathbb{R}^n)$, with corresponding Hilbert structure. In fact, we can prove that any $u \in \tilde{H}^s(\mathbb{R}^n)$ $(s > 0)$ is sum of $u_1 \in H^s(\mathbb{R}^n)$ and a C^∞ function u_2. We first note that $|\tau|^{2s} \leq (1 + |\tau|^2)^s \leq 2 \, |\tau|^{2s}$ except for $\tau \in$ some ball B. Now for $u \in \tilde{H}^s$, $\hat{u}$ is a function. We write it as $v_1 + v_2$ where v_2 coïncides with $\hat{u}$ a.e. on B and is zero outside. v_1 satisfies $\int_{\mathbb{R}^n}(1+|\tau|^2)^s\,|v_1(\tau)|^2 d\tau =$ $\int_{B^c}(1+|\tau|^2)^s|v_1(\tau)|^2 d\tau \leq 2\int_{B^c}|\tau|^{2s}|v_1(\tau)|^2 d\tau < \infty$, so that its inverse Fourier transform $u_1 \in H^s(\mathbb{R}^n)$. On the other hand, v_2 has compact support, hence its (inverse) Fourier transform u_2 is a C^∞ function (in fact, an "entire function of exponential type", moreover $\to 0$ at ∞ since $v_2 \in L^1$). Case $s<0$ is exactly similar : we prove $H^s \subset \tilde{H}^s + C^\infty$.

We have already seen that $C^\infty \subset H^s_{loc}$, so that $\tilde{H}^s(\Omega) = H^s(\Omega)$ if $s>0$. For $s<0$ we must prove that $C^\infty \subset \tilde{H}^s_{loc}$; but for $s > -\frac{n}{2}$ we have $\mathcal{D} \subset \tilde{H}^s$, which gives the result. So in general $\tilde{H}^s(\Omega) = H^s(\Omega)$ for Ω bounded, whenever $-\frac{n}{2} < s < \frac{n}{2}$.

Inclusion $H^s \subsetneq \tilde{H}^s$ for $s>0$ ($\tilde{H}^s \subsetneq H^s$ for $s<0$) is continuous, so that the two Hilbert structures induced on $\tilde{H}^s(\Omega) = H^s(\Omega)$ are comparable, hence equivalent, from one of Banach's theorems.

1.4. Semi-Hilbert spaces $D^{-m}\tilde{H}^s$

We put $D^{-m}\tilde{H}^s = \{u \in \mathcal{D}'(\mathbb{R}^n)\ ;\ D^\alpha u \in \tilde{H}^s(\mathbb{R}^n)\ ,\ \forall|\alpha| = m\}$, equipped with natural semi-norm $\|u\|_{m,s} = (\int_{\mathbb{R}^n}|\tau|^{2s}|\mathcal{F} D^m u(\tau)|^2 d\tau)^{1/2}$. We also consider the quotient space $D^{-m}\tilde{H}^s / P_{m-1}$ with corresponding norm. These are spaces of Beppo Levi type (see [4], where these spaces would be denoted $BL_m(\tilde{H}^s)$ and $BL^\cdot_m(\tilde{H}^s)$ respectively) : since $\tilde{H}^s$ is a Hilbert subspace of $\mathcal{D}'$, $D^{-m}\tilde{H}^s / P_{m-1}$ is a Hilbert subspace of $\mathcal{D}' / P_{m-1}$ (i.e. $D^{-m}\tilde{H}^s$ is a semi-Hilbert subspace of $\mathcal{D}'$, see §2). In fact, since $\tilde{H}^s \subset \mathcal{S}'$, and a distribution whose derivatives are tempered is tempered, we have $D^{-m}\tilde{H}^s \subset \mathcal{S}'$, hence $D^{-m}\tilde{H}^s / P_{m-1} \subset \mathcal{S}' / P_{m-1}$ and the closed graph theorem shows that this inclusion is continuous. So $D^{-m}\tilde{H}^s$ is even a semi-Hilbert subspace of $\mathcal{S}'$.

We now come to comparing $D^{-m}\tilde{H}^s$ with H^{m+s}. For Ω bounded we have :

1.5. $\underline{(D^{-m}\tilde{H}^s)(\Omega) = H^{m+s}(\Omega)}$, $\underline{\text{if } -m-\frac{n}{2} < s < \frac{n}{2}}$

We first prove that any $u \in H^{m+s}(\Omega)$ extends to an element of $D^{-m}\tilde{H}^s(\mathbb{R}^n)$. We know that u extends to some $Pu \in H^{m+s}(\mathbb{R}^n)$ with compact support. Then $\hat{Pu}$ is C^∞ and we can easily see that $\int|\tau|^{2s}|\tau^\alpha\hat{Pu}(\tau)|^2 d\tau < \infty$, $|\alpha| = m$, since $|\tau^\alpha|^2 \leq |\tau|^{2m}$ and $|\tau|^{2m+2s}$ is locally summable (since $2m + 2s > -n$) , and $|\tau|^{2m+2s} \leq 2(1+|\tau|^2)^{m+s}$ outside some ball. Thus $H^{m+s}(\Omega) \subset (D^{-m}\tilde{H}^s)(\Omega)$.

Conversely, we may write $(D^{-m}\tilde{H}^s)(\Omega) \subset D^{-m}(\tilde{H}^s(\Omega))$, with obvious meaning; on the other hand, $\tilde{H}^s(\Omega)$ is always contained in $H^s(\Omega)$ for $s < \frac{n}{2}$; and it is known that $D^{-m}(H^s(\Omega)) = H^{m+s}(\Omega)$.

This proves that $D^{-m}\tilde{H}^s(\mathbb{R}^n) \subset H^{m+s}_{loc}(\mathbb{R}^n)$, hence $D^{-m}\tilde{H}^s(\mathbb{R}^n) / P_{m-1} \subset H^{m+s}_{loc}(\mathbb{R}^n) / P_{m-1}$, and the closed graph theorem again tells us this inclusion is continuous. Thus $\underline{D^{-m}\tilde{H}^s(\mathbb{R}^n)}$ <u>is a (dense) semi-Hilbert subspace of</u> $\underline{H^{m+s}_{loc}(\mathbb{R}^n)}$.

Another point is that, on $(D^{-m}\tilde{H}^s)(\Omega) / P_{m-1} = H^{m+s}(\Omega) / P_{m-1}$, the two Hilbert structures are equivalent (both of them are Hilbert subspaces of $\mathcal{D}'(\Omega) / P_{m-1}$, so apply the closed graph theorem). From this one can easily deduce that $H^{-m-s}_{\overline{\Omega}}(\mathbb{R}^n) \cap P^o_{m-1}$, the dual of $H^{m+s}(\Omega) / P_{m-1}$, is closed in $(D^{-m}\tilde{H}^s / P_{m-1})'$.

§ 2 - <u>EXISTENCE AND UNIQUENESS</u>

We begin with stating an abstract frame : let E be a locally convex topological vector space, E' its dual, N a finite-dimensional (for simplicity) subspace of E, N^o its orthogonal in E'. N^o is naturally identified with the dual space of E / N .

A linear subspace X of E is called <u>a semi-Hilbert subspace of E with nullspace N</u> if X is equipped with a semi-norm $\|.\|$ (with nullspace N) deriving from a "semi-inner product" (nonnegative bilinear form) $((.,.))$ such that X / N equipped with the natural norm $\|x+N\| = \|x\|$ is a Hilbert space, and inclusion $X/N \hookrightarrow E/N$ is continuous. Equivalently, X / N is a Hilbert subspace of E / N in the sense of

L. SCHWARTZ [10]. We know that X / N has a (unique) reproducing kernel in E / N, which is a linear mapping L from $(E / N)'$ into X / N, satisfying $((y,x)) = \langle e', x + N \rangle$, $\forall x \in X$, $\forall y \in Le'$. To be slightly simpler, we will say that a linear mapping $\theta : N^\circ \to E$ is reproducing kernel of X in E, if $\theta e' \in Le'$ $\forall e' \in N^\circ \underset{\sim}{\sim} (E / N)'$. Equivalently, a mapping θ from N° into X is a reproducing kernel of X in E, if and only if $((\theta e', x)) = \langle e', x \rangle$, $\forall x \in X$.

Semi-Hilbert spaces and reproducing kernels provide a simple and convenient language for splines. However, following theorems 2.1. and 2.2. could be easily deduced from [8], chapter 4.

<u>THEOREM 2.1.</u>

Let M be a linear subspace of E' such that, if $x \in N$ and $\langle e', x \rangle = 0$ $\forall e' \in M$, then $x = 0$. Let $f \in X$. There exists a unique element f^M in X satisfying $\langle e', f^M \rangle = \langle e', f \rangle$ $\forall e' \in M$, with $\|f^M\|$ minimum.

<u>Proof</u> : The set $f + M^\circ \cap X + N$ is a nonempty closed affine subspace of X / N, it has an element of minimum norm which is exactly $f^M + N$, which in turn contains one element $f^M \in f + M^\circ \cap X$, i.e. f^M satisfies $\langle e', f^M \rangle = \langle e', f \rangle$ $\forall e' \in M$.

<u>THEOREM 2.2.</u>

Let us suppose that X is dense is E, so that $N^\circ \subset (X / N)'$; and assume that $M \cap N^\circ$ is closed in $(X / N)'$. Let θ be a reproducing kernel of X in E. Then f^M coïncides with the only $g \in \theta(M \cap N^\circ) + N$ satisfying $\langle e', g \rangle = \langle e', f \rangle$ $\forall e' \in M$.

<u>Proof</u> : $f^M + N$ is the orthogonal projection of 0 ($\in X / N$) onto $f + M^\circ \cap X + N$ and therefore is the only element of $f + M^\circ \cap X + N$ belonging to the orthogonal of $M^\circ \cap X + N$ (in X / N for Hilbert structure), which is the image of $(M^\circ \cap X + N)^\circ$ (orthogonal in $(X / N)'$) by the canonical isometry from $(X / N)'$ onto X / N. But

if $M \cap N^\circ$ is closed in $(X / N)'$, $(M^\circ \cap X + N)^\circ$ is exactly $M \cap N^\circ$, whose image is $\theta(M \cap N^\circ) + N$.

REMARK : Frequently M is finite-dimensional (finitely many data to interpolate) so that $M \cap N^\circ$ is automatically closed in $(X / N)'$.

Applying this to $X = D^{-m} \tilde{H}^s$ with $N = P_{m-1}$ and $E = H^{m+s}_{loc}$, $E' = H^{-m-s}_{comp}$, we get :

THEOREM 2.3.

Let M be a closed linear subspace of $H^{-m-s}_{\bar{\Omega}}(\mathbb{R}^n)$ (Ω bounded), such that if $p \in P_{m-1}$ and $< \mu , p > = 0 \ \forall \mu \in M$ then $p = 0$. Let $f \in H^{m+s}(\Omega)$. There exists exactly one element $f^M \in D^{-m} \tilde{H}^s$ satisfying $< \mu, f^M > = < \mu, f >$, $\forall \mu \in M$, with minimum semi-norm $\|f^M\|_{m,s}$. Let θ be a reproducing kernel of $D^{-m} \tilde{H}^s$ in H^{m+s}_{loc}. Then the only g of the form $\theta\mu + p$ with $\mu \in M \cap P^\circ_{m-1}$ and $p \in P_{m-1}$, satisfying $< \mu, g > = < \mu, f >$, $\forall \mu \in M$, is f^M.

Proof : We just have to check that $M \cap P^\circ_{m-1}$ is closed in $(D^{-m} \tilde{H}^s / P_{m-1})'$. But M is closed in $H^{-m-s}_{\bar{\Omega}}$, hence $M \cap N^\circ$ is closed in $H^{-m-s}_{\bar{\Omega}} \cap N^\circ$, which is exactly the orthogonal, in $(D^{-m} \tilde{H}^s / P_{m-1})'$, of the subset of (equivalence classes mod P_{m-1} of) elements in $D^{-m} \tilde{H}^s$ which are zero on Ω. Thus $M \cap N^\circ$ is closed in a closed subset of $(D^{-m} \tilde{H}^s / P_{m-1})'$, hence closed itself. All this is useless if M is finite-dimensional.

§ 3 - REPRODUCING KERNEL OF $D^{-m}\tilde{H}^s$.

THEOREM 3

$\theta : \mu \mapsto (2\pi)^{-2m}\,\mu * \mathcal{F}\,\mathrm{Pf}.|\tau|^{-2m-2s}$ maps $H^{-m-s}_{comp} \cap P^o_{m-1}$ into $D^{-m}\tilde{H}^s$ and is a reproducing kernel of $D^{-m}\tilde{H}^s$ as a semi-Hilbert subspace of $H^{m+s}_{loc}(\mathbb{R}^n)$.

Proof : Since $D^{-m}\tilde{H}^s / P_{m-1}$ is a Hilbert subspace of H^{m+s}_{loc} / P_{m-1}, it has a reproducing kernel, say L, which maps $H^{-m-s}_{comp} \cap P^o_{m-1}$ (dual space of $D^{-m}\tilde{H}^s / P_{m-1}$) into $D^{-m}\tilde{H}^s / P_{m-1}$, and is characterized by : $L\mu$ is the set of distributions u $\in D^{-m}\tilde{H}^s$ satisfying :

$$(*) \qquad (2\pi)^{2m} \int_{\mathbb{R}^n} |\tau|^{2s}(\tau^m\hat{u})(\tau).\overline{(\tau^m\hat{w})(\tau)}d\tau = \langle \mu,w \rangle , \forall\, w \in D^{-m}\tilde{H}^s$$

We first deduce that $|2\pi\tau|^{2m}\,\hat{u} = \hat{\mu}|\tau|^{-2s}$ (lemma 1). Now it is easily seen that $v = (2\pi)^{-2m}\,\mu * \mathcal{F}\,\mathrm{Pf}.|\tau|^{-2m-2s}$ satisfies $|2\pi\tau|^{2m}\,\hat{v} = \hat{\mu}|\tau|^{-2s}$, since $\hat{v} = (2\pi)^{-2m}\,\hat{\mu}\,\mathrm{Pf}.|\tau|^{-2m-2s}$ and $|\tau|^{2m}\mathrm{Pf}.|\tau|^{-2m-2s} = |\tau|^{-2s}$. On the other hand, lemma 2 proves that $\tau^\alpha\,\hat{v} \in L^1_{loc}$, $\forall\, |\alpha| = m$. So that for any $u \in L\mu$ we have $|\alpha|^{2m}\,(\hat{u}-\hat{v}) = 0$ and $\tau^\alpha(\hat{u}-\hat{v}) \in L^1_{loc}$, $\forall\, |\alpha| = m$. Lemma 3 then shows that $u-v \in P_{m-1}$, i.e. $v \in L\mu$ since $L\mu$ is an equivalence class modulo P_{m-1}. In other words $L\mu = (2\pi)^{-2m}\,\mu * \mathcal{F}\,\mathrm{Pf}.|\tau|^{-2m-2s} + P_{m-1}$, i.e. θ is a reproducing kernel of $D^{-m}\tilde{H}^s$.

Lemma 1 : $|2\pi\tau|^{2m}\,\hat{u} = \hat{\mu}|\tau|^{-2s}$ if $u \in L\mu$.

Proof : Condition $s > -m - \frac{n}{2}$ implies $\mathcal{S} \subset D^{-m}\tilde{H}^s$, so that we may apply (*) whith $w = \hat{\varphi}$, $\varphi \in \mathcal{D}$, and get $(2\pi)^{2m}\int|\tau|^{2s}(\tau^m\,\hat{u})(\tau).\tau^m\,\varphi(\tau)d\tau = \langle \mu, \hat{\varphi} \rangle = \langle \hat{\mu},\varphi \rangle$, $\forall\, \varphi \in \mathcal{D}$. This implies $(2\pi)^{2m}|\tau|^{2s}\tau^m.(\tau^m\,\hat{u})(\tau) = \hat{\mu}(\tau)$ a.e., hence $(2\pi)^{2m}\tau^m.(\tau^m\,\hat{u})(\tau) = |\tau|^{-2s}\,\hat{\mu}(\tau)$ a.e. and $|2\pi\tau|^{2m}\,\hat{u} = \hat{\mu}|\tau|^{-2s}$ as distributions.

Lemma 2 : If μ is a distribution with compact support, orthogonal to P_{m-1}, i.e. $\mu \in \mathcal{E}' \cap P^{o}_{m-1}$, then $\tau^{\alpha}\,\hat{\mu}\,\mathrm{Pf.}|\tau|^{-2m-2s} \in L^{1}_{loc}$, $\forall |\alpha| = m$.

Proof : It suffices to show that the function (in usual sense) $\tau^{\alpha}\,\hat{\mu}(\tau)\,|\tau|^{-2m-2s}$ is locally summable. But, since μ is orthogonal to P_{m-1}, the C^{∞} function $\hat{\mu}$ has derivatives of order $\leq$ m-1 vanishing at 0, so that $|\hat{\mu}(\tau)| \leq c|\tau|^{m}$ on a neighbourhood of 0. Then $\tau^{\alpha}\;\hat{\mu}(\tau)|\tau|^{-2m-2s} \leq c|\tau|^{-2s}$ on that neighbourhood of 0, and is C^{∞} elsewhere, so is locally summable since $s < \frac{n}{2}$.

Lemma 3 : Any tempered distribution T such that $|\tau|^{2m}\,\hat{T} = 0$ and $\tau^{\alpha}\,\hat{T} \in L^{1}_{loc}$ is in P_{m-1} .

Proof : $\hat{T}$ is supported by $\{0\}$, since $|\tau|^{2m}$ vanishes only for $\tau = 0$. Then $\tau^{\alpha}\,\hat{T}$ is also supported by $\{0\}$ and should be in L^{1}_{loc}, which is possible only if $\tau^{\alpha}\,\hat{T} = 0$, and then $D^{\alpha}T = 0$, $\forall|\tau| = m$, i.e. $T \in P_{m-1}$.

§ 4 - A GENERAL CHARACTERIZATION RESULT

To be more explicit, we now use formulas giving Fourier transforms of pseudo-functions $\mathrm{Pf.}|\tau|^{\lambda}$. In general $\mathrm{Pf.}|\tau|^{\lambda} = c\,\mathrm{Pf.}|t|^{-n-\lambda}$ except if λ or $-n-\lambda$ is an even positive integer 2k : $\mathcal{F}|\tau|^{2k} = c\,\Delta^{k}\delta$, $\mathcal{F}\,\mathrm{Pf.}|\tau|^{-n-2k} = c|t|^{2k}\log|t| + c|t|^{2k}$.

For simplicity, we assume $s > -m$. Then $\mathcal{F}\,\mathrm{Pf.}|\tau|^{-2m-2s}$ is $c|t|^{2m+2s-n}\,\mathrm{Log}|t| + c|t|^{2m+2s-n}$ if 2m+2s-n is an even positive integer, $c|t|^{2m+2s-n}$ if not . It is easyly seen that, in the first case (2m+2s-n = 2k), if $\mu \in P^{o}_{m-1}$ then $\mu * |t|^{2m+2s-n} \in P_{m-1}$. So that, putting $K_{\lambda}(t) = |t|^{\lambda}\mathrm{Log}|t|$ if λ is an even positive integer, $K_{\lambda}(t) = |t|^{\lambda}$ otherwise, the mapping $\mu \mapsto \mu * K_{2m+2s-n}$ is proportional to a reproducing kernel of $D^{-m}\tilde{H}^s$ when $m+s > 0$.

Now we are able to explicit theorem 2.3. (in the case $m+s > 0$, that is, H^{m+s}_{loc} is a space of (classes of locally summable) functions) :

THEOREM 4 :

Let M be a closed linear subspace of some $H^{-m-s}_{\overline{\Omega}}(\mathbb{R}^n)$ (Ω bounded), satisfying : if $p \in P_{m-1}$ and $< \mu, p > = 0$, $\forall \mu \in M$, then $p = 0$. Let $f \in H^{m+s}(\Omega)$. Then there exists a unique function $f^M \in D^{-m}\tilde{H}^s(\mathbb{R}^n)$ satisfying $< \mu, f^M > = < \mu, f > \ \forall \mu \in M$, with minimum semi-norm $\|f^M\|_{m,s}$. Moreover, if $g = \nu * K_{2m+2s-n} + p$ (with $\nu \in M \cap P^o_{m-1}$ and $p \in P_{m-1}$) satisfies $< \mu, g > = < \mu, f >$, $\forall \mu \in M$, then $g = f^M$.

Let us now restrict ourselves to the important case where $m + s > \frac{n}{2}$, so that H^{m+s}_{loc} is a space of continuous functions (Sobolev theorem), and data are finitely many point values.

THEOREM 4 bis :

Let A be a finite subset of $\mathbb{R}^n$, containing a P_{m-1} - unisolvent subset. Then there exists exactly one function of the form $\sigma(t) = \sum_{a \in A} \lambda_a K_{2m+2s-n}(t-a) + p(t)$ with $p \in P_{m-1}$ and $\sum \lambda_a q(a) = 0$, $\forall q \in P_{m-1}$, taking prescribed values on A. Moreover, if f is another function taking the same values on A, one has $\|f\|_{m,s} \geq \|\sigma\|_{m,s}$.

Actually, $\sum \lambda_a K_{2m+2s-n}(t-a)$ is $(\sum \lambda_a \delta_a * K_{2m+2s-n})(t)$. Existence of a function $f \in D^{-m}\tilde{H}^s$ taking prescribed values on A (finite) is obvious : f may even be chosen in $\mathcal{D}$.

§ 5 - EXAMPLES

5.1. Pseudo-polynomial splines

We put $s = \frac{n-1}{2}$ and consider a finite set $A \subset \mathbb{R}^n$ containing some P_{m-1} - unisolvent subset. Then there exists exactly one function of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^{2m-1} + p(t)$ where $p \in P_{m-1}$ and $\sum \lambda_a q(a) = 0$, $\forall q \in P_{m-1}$, taking prescribed values on A. For all f taking the same values on A one has $\|f\|_{m,\frac{n-1}{2}} \geq \|\sigma\|_{m,\frac{n-1}{2}}$.

For m = 1 we get multi-conic functions $\sum_{a\in A} \lambda_a |t-a| + C$ with $\sum \lambda_a = 0$, and the set A must only contain two distinct points. The functional minimized is $\int_{\mathbb{R}^n} |\tau|^{n-1} |\mathcal{F} Dv(\tau)|^2 d\tau$.

For m = 2 we get pseudo-cubic splines, if A is not contained in a hyperplane (a line if n = 2) : functions of the form $\sum \lambda_a |t-a|^3 + \alpha.t + \beta$ with $\sum \lambda_a = 0$ and $\sum \lambda_a a = 0$. Coefficients $(\lambda_a ; a \in A)$ and α_1, α_2, β may be computed from the linear system :

$$\begin{cases} \sum_{a\in A} |a-b|^3 \lambda_a + b_1\alpha_1 + b_2\alpha_2 + \beta = f(b) & (b \in A) \\ \sum_{a\in A} \lambda_a a_1 = 0 & \\ \sum_{a\in A} \lambda_a a_2 = 0 & \\ \sum_{a\in A} \lambda_a = 0 & \end{cases}$$

We notice that, for n = 1, we get simply polynomial splines : polynomials of degree $\leq 2m-1$ on intervals, C^2, and degenerating to polynomials of degree $\leq m-1$ at both ends (thanks to conditions $\sum \lambda_a a^k = 0$, $k=0,\ldots,m-1$).

5.2. Thin plate functions

Putting s = 0 and, as an example, n = 2 and m = 2, we get functions of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^2 \mathrm{Log}\, |t-a| + \alpha.t + \beta$ with $\sum \lambda_a = 0$ and $\sum \lambda_a a = 0$ (function $|t|^2 \mathrm{Log}\, |t|$ is extended to 0 at 0, so as to be continuous). In this case we have $\int_{\mathbb{R}^2} |D^2\sigma|^2 \leq \int_{\mathbb{R}^2} |D^2 f|^2$ for all f that coïncides with σ on A. The set A must not be contained in a line.

5.3. Hermite interpolation

Since $H_{loc}^{2+\frac{n-1}{2}}(\mathbb{R}^n) \subset C^1$, we may minimize semi-norm $\|\cdot\|_{2,\frac{n-1}{2}}$ subject to Hermite conditions : values and gradients prescribed on a finite set A. We get functions of the form $\sigma(t) = \sum_{a\in A} \lambda_a |t-a|^3 + \sum \lambda'_a \cdot (t-a)|t-a| + \alpha.t + \beta$, with $\sum \lambda_a = 0$, $\sum \lambda_a\, a + \frac{1}{3} \sum \lambda'_a = 0$. In one dimension this corresponds to ordinary piecewise cubic Hermite interpolation.

§ 6 - CONVERGENCE IN $H^{m+s}(\Omega)$

Let $f \in H^{m+s}(\Omega)$, and let (M_k) be a sequence of closed linear subspaces of $H_{\overline{\Omega}}^{-m-s}(\mathbb{R}^n)$. We suppose that for any $\mu \in H_{\overline{\Omega}}^{-m-s}(\mathbb{R}^n)$, the distance from μ to M_k converges to 0. Then :

1°) For k sufficiently large, M_k is such that : if $p \in P_{m-1}$ satisfies $<\mu,p> = 0$, $\forall \mu \in M_k$, then $p = 0$. So that there exists a unique $f_k \in D^{-m}\tilde{H}^s$ satisfying $< \mu,f_k > = < \mu ,f >$, $\forall \mu \in M_k$, with $\|f_k\|_{m,s}$ minimum.

2°) $f_k \to f$ in $H^{m+s}(\Omega)$.

This is a straightforward consequence of a general result of J.L. JOLY [7], putting $X = H^{m+s}(\Omega)$, $Y = H^{m+s}(\Omega) / P_{m-1}$ with a norm derived from $\|\cdot\|_{m,s}$. Another way to see it (partially) is the following : put f^{Ω} = the minimal extension of f, relatively to $\|\cdot\|_{m,s}$, i.e. the unique element in $D^{-m}\tilde{H}^s$ that coïncides with f on Ω with minimum semi-norm $\|\cdot\|_{m,s}$. It is uniquely written $\mu * K + p$ with $p \in P_{m-1}$ and $\mu \in H_{\overline{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P^o_{m-1}$ ($K = \mathcal{F}\mathrm{Pf}.|\tau|^{-2m-2s}$). Same thing for $f_k = \mu_k * A + p_k$ with $p_k \in P_{m-1}$ and $\mu_k \in M_k \cap P^o_{m-1}$. And μ_k is simply the orthogonal projection of μ onto $M_k \cap P^o_{m-1}$, in Hilbert space $H_{\overline{\Omega}}^{-m-s}(\mathbb{R}^n) \cap P^o_{m-1}$ equipped with norm induced by $(D^{-m}\tilde{H}^s / P_{m-1})'$ (equivalent to that induced by

$H^{-m-s}(\mathbb{R}^n))$. So that $\|\mu_k - \mu\| \leq c\, d(\mu, M_k) \to 0$, hence $\|\mu_k * K - \mu * K\|_{m,s} \to 0$ and this proves that $f_k + P_{m-1} \to f + P_{m-1}$ in $H^{m+s}(\Omega) / P_{m-1}$.

Let us now specialize to the case where $m + s > \frac{n}{2}$ and M_k is spanned by Dirac masses $(\delta_a \; ; \; a \in A_k)$ where (A_k) is a sequence of subsets of $\bar{\Omega}$. Then the condition $d(\mu, M_k) \to 0$, $\forall \mu \in H^{-m-s}_{\bar{\Omega}}$ is equivalent to saying that any point in Ω is limit of a sequence $(a_k \in A_k)$, or that Hausdorff distance from A_k to Ω tends to zero. This results from complete continuity of inclusion $H^{m+s}(\Omega) \subsetneq C(\bar{\Omega})$ (a bounded subset of $H^{m+s}(\Omega)$ is an equicontinuous set of functions on $\bar{\Omega}$). We then get :

<u>THEOREM 6</u> :

<u>If (A_k) is a sequence of subsets of $\bar{\Omega}$ (Ω bounded open subset of $\mathbb{R}^n$) such that $d(t, A_k) \to 0$, $\forall t \in \Omega$, and $f \in H^{m+s}(\Omega)$ with $m + s > \frac{n}{2}$, then the sequence (f_k) of functions coïnciding with f on A_k with minimum semi-norm $\|\cdot\|_{m,s}$ (uniquely determined for sufficiently large k) satisfies $f_k \to f$ in $H^{m+s}(\Omega)$.</u>

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