Basic Fractal Analysis in Movement Science

To access slides and the tutorial code for this workshop:

1. Go to www.github.com and search "user:Nonlinear-Analysis-Core"
2. Download the "ASB2023_BasicFractalWorkshop" repository
3. Download the "NONANLibrary" repository

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Basic Fractal Analysis in Movement Science

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Background

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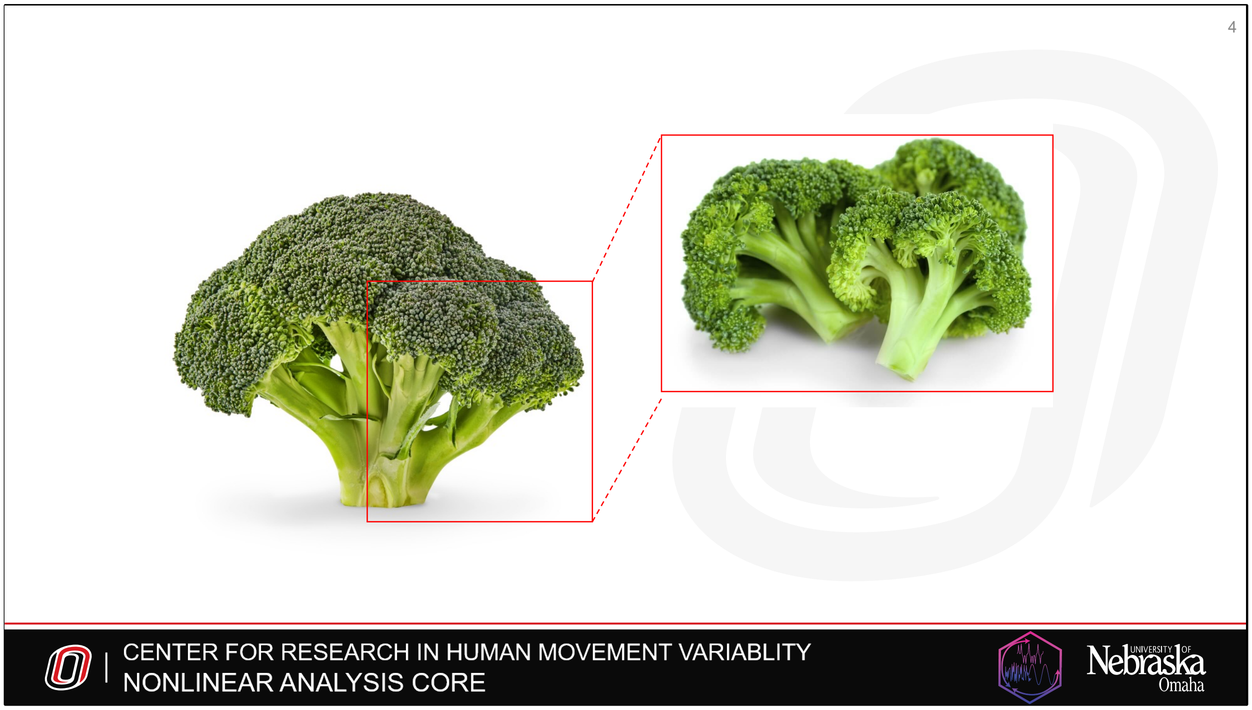
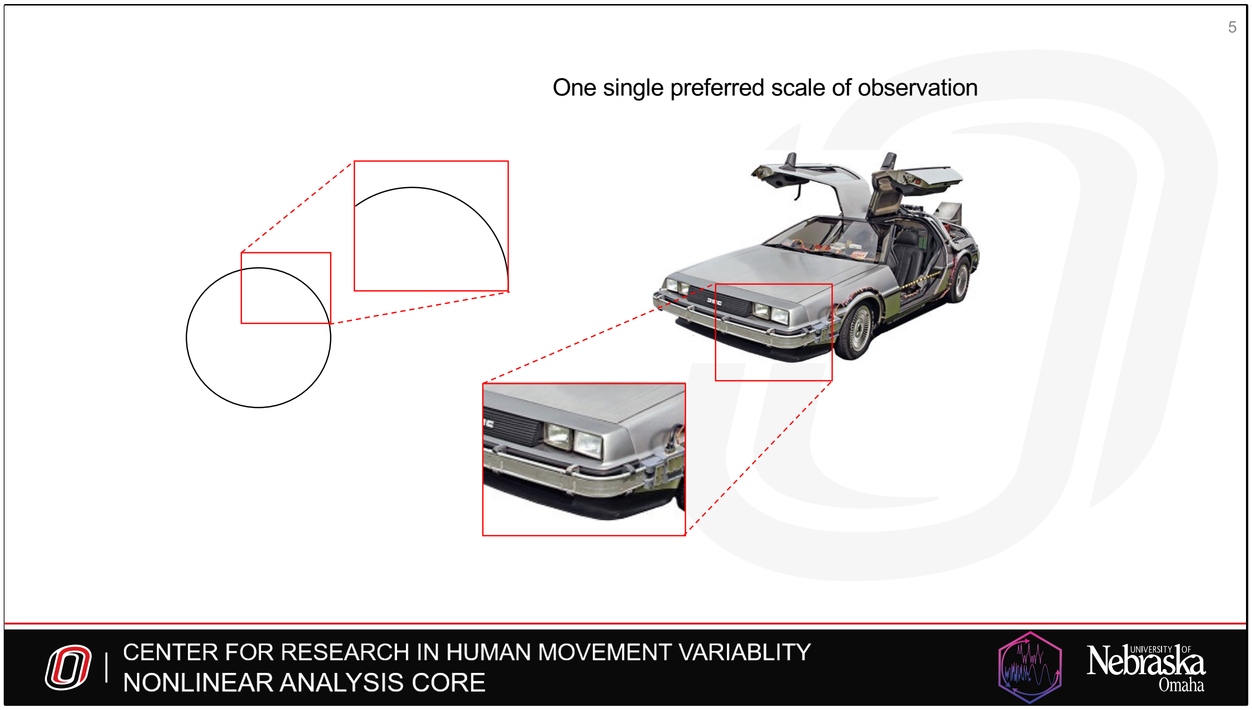
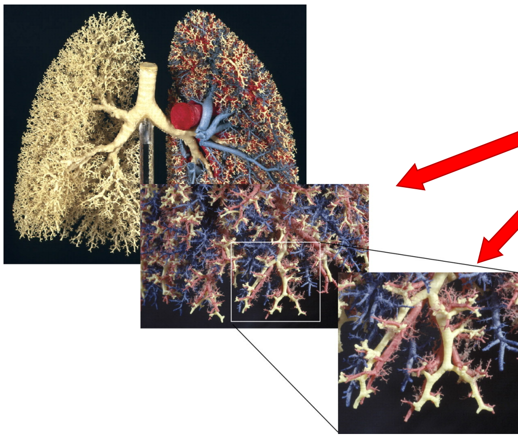


Broccoli is an example of a fractal object. When we zoom in on the broccoli, it still looks like broccoli, but there are new features that we discover.



When we zoom in on a circle, we do not discover any new features. The same thing with a car. There are no new features discovered when looking at it closer.



New details appears

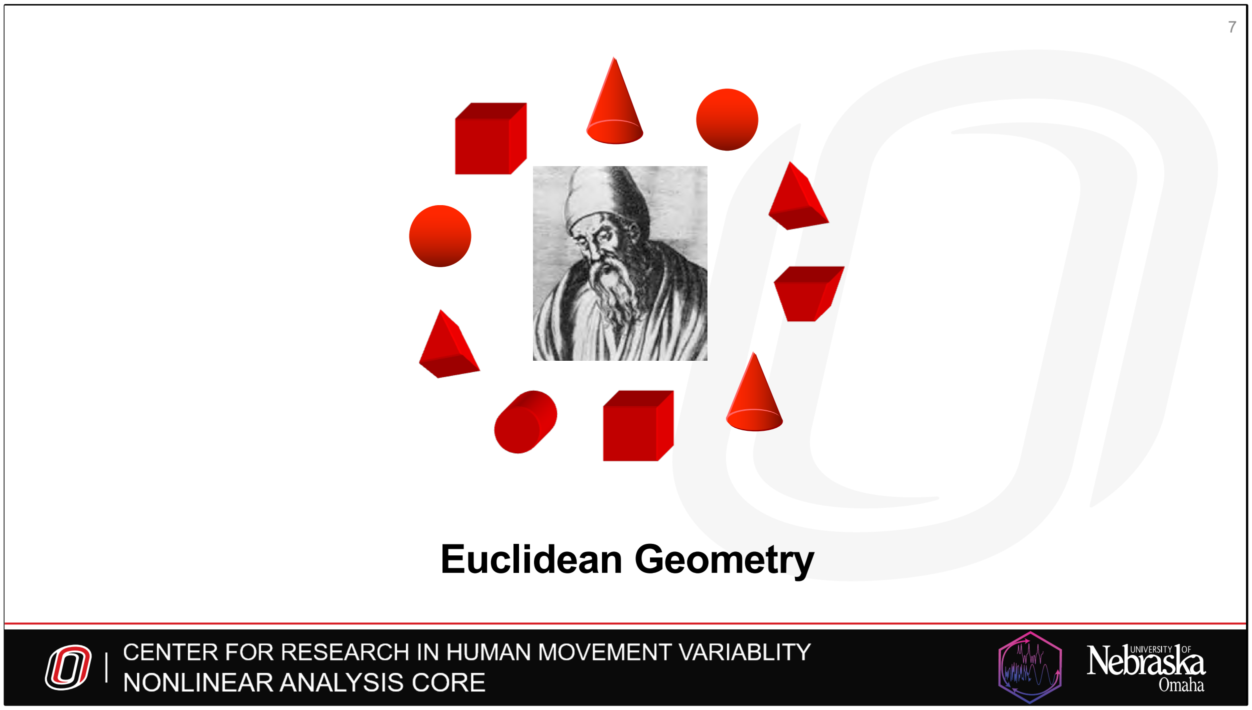
- Nested structure across scales of observation

Non-fractal

As it is magnified, there are no new features revealed
Size of the smallest feature is called its characteristic scale

Fractal

As it is magnified, finer features are revealed
Has features over a broad range of sizes

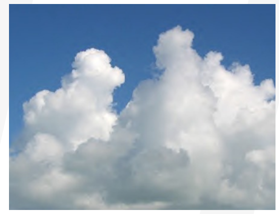


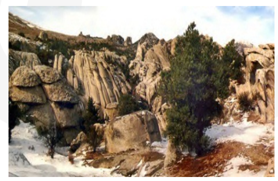
This geometry is not sufficient to describe objects with pieces that are repeated over and over again at different scales – shapes that are repeated within themselves.

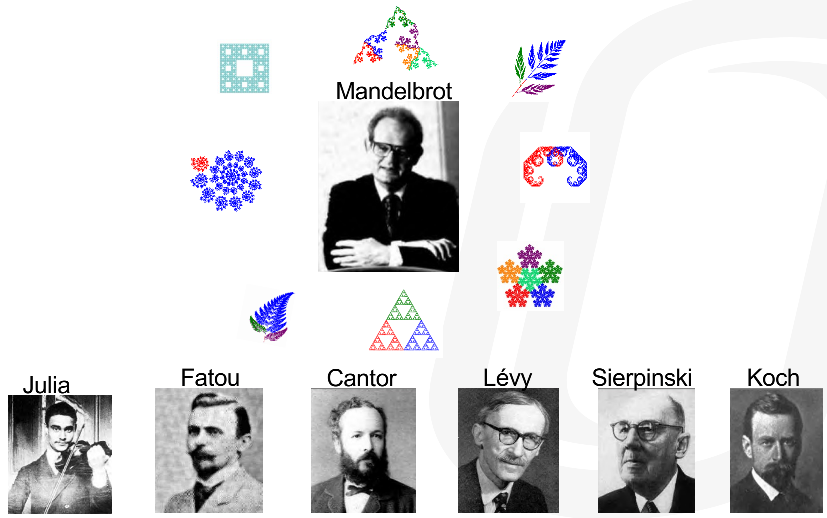
This geometry is not sufficient to describe objects with pieces that are repeated over and over again at different scales – shapes that are repeated within themselves.

“Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.”

Mandelbrot, introduction to *The Fractal Geometry of Nature*






Mandelbrot
Julia
Fatou
Cantor
Lévy
Sierpinski
Koch

Fractal Recap

- A fractal is an object that, when magnified more and more reveals finer and finer features. Also, the smaller features are kind of like the larger features.
- A fractal is a mathematical set that typically displays self-similar patterns, which means it is "the same from near as from far".

Self-similarity – As you enlarged a piece of a fractal you see the same kinds of shapes at smaller scales.

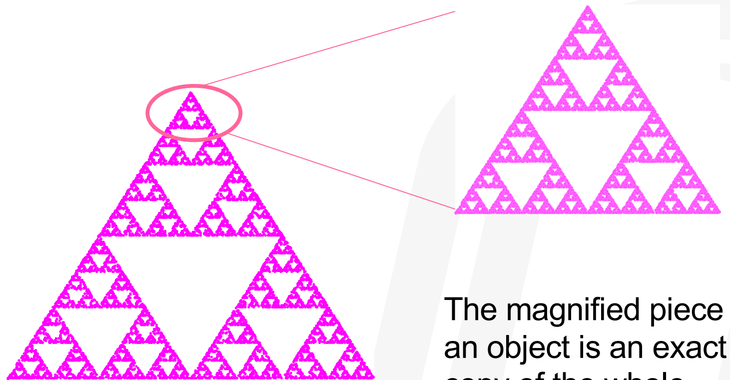
Scaling – Lets say you are going to measure some property of a fractal such as length or area or volume. When you measure these properties at smaller and smaller resolutions you include the finer and finer features of the fractal so the value of the property you are measuring changes depending on the scale of measurement.

Fractal Dimension – Characterizes how the fractal object fills space. It does not have to be an integer, although it can be. For example, a fractal can have a dimension between 1 and two. We will go more into these properties in the next few slides.

Self-Similarity

- Geometrical Self-Similarity
  - Smaller pieces of an object are **exact** smaller copies of the whole object
- Statistical Self-Similarity
  - Smaller pieces of real biological specimens are usually not exact copies of the whole object
  - Statistical properties of the smaller pieces can be geometrically similar to the statistical properties of the big pieces

Self-Similarity

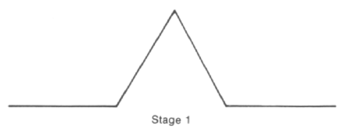


The magnified piece of an object is an exact copy of the whole object.

Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

Little pieces of an object are EXACTLY smaller copies of the whole object. This is usually true only for mathematically defined objects.

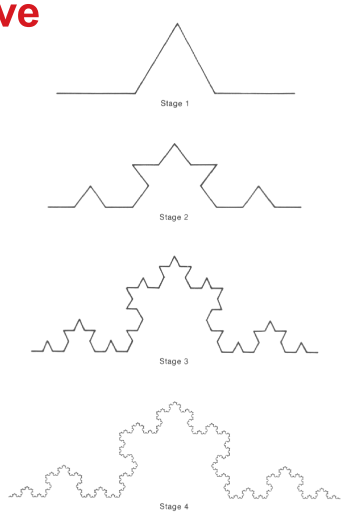
Koch Curve



- Iteration Process
  1. Divide the line segment into three segments of equal length
  2. Draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
  3. Remove the line segment that is the base of the triangle from step 2.

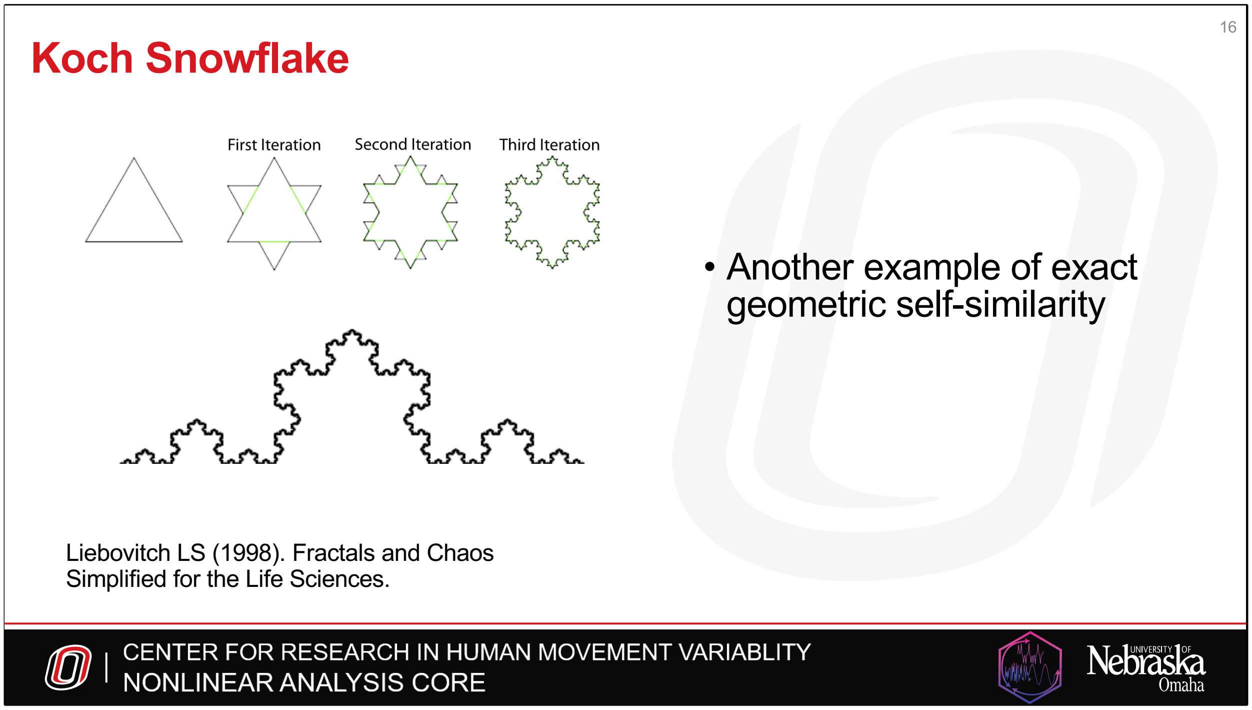
Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

Koch Curve



Adapted from Benoit Mandelbrot, *Fractals.*

- Keep repeating the iteration process!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Koch Snowflake
First Iteration
Second Iteration
Third Iteration
• Another example of exact geometric self-similarity
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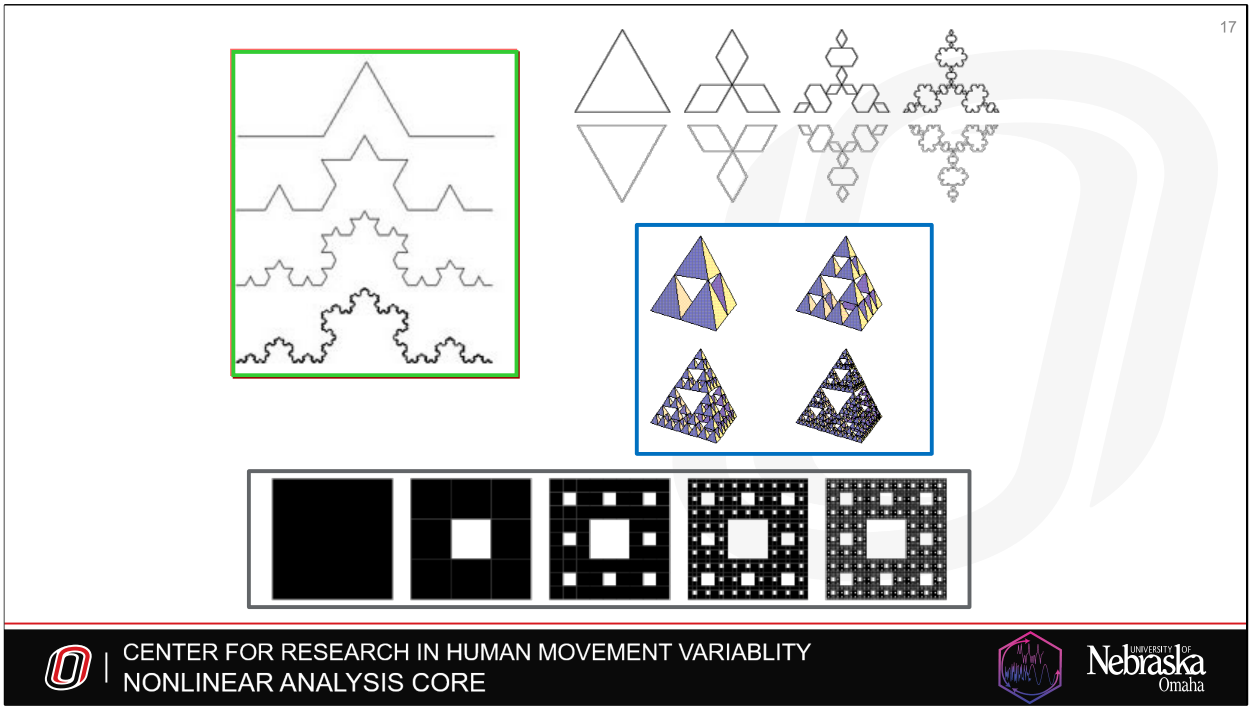
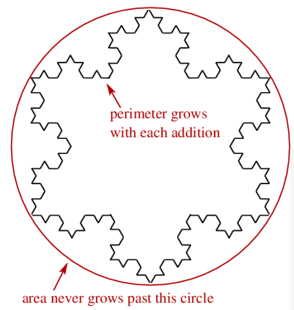


The Koch curve – green
The Koch antisnowflake
The Sierpinski carpet - red
Sierpinski tetrahedron - blue

Some examples of mathematically defined fractals.
Koch snowflake in the green box. The Koch anti-snowflake in the upper right. The Sierpinski tetrahedron in the yellow box. The Sierpinski Carpet in the blue box. Notice the koch snowflake and how each iteration introduces finer and finer scale pieces. Including each of these pieces causes the length of the line to grow. The final fractal has an infinite number of infinitely small pieces and measuring the length of the line including all these pieces leads to an infinitely long line length. This is a similar situation to measuring the coast of Britain, as we will see in a few slides.



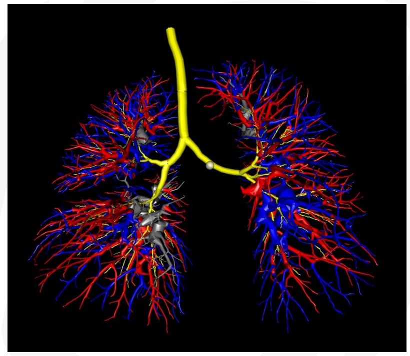
Notice the Koch snowflake and how each iteration introduces finer and finer scale pieces. Including each of these pieces causes the length of the line to grow. The final fractal has an infinite number of infinitely small pieces and measuring the length of the line including all these pieces leads to an **infinitely long line length, embedded within a finite space**…

Notice the Koch snowflake and how each iteration introduces finer and finer scale pieces. Including each of these pieces causes the length of the line to grow. The final fractal has an infinite number of infinitely small pieces and measuring the length of the line including all these pieces leads to an infinitely long line length. This is a similar situation to measuring the coast of Britain, as we will see in a few slides.

The statistical properties of the pieces are proportional to the statistical properties of the whole.

Example

- The average rate at which new vessels branch off from their parent vessels in a physiological structure can be the same for large and small vessels.



Statistical self-similarity in space of the lung's arterial tree.

Not geometrically exact but the statistical properties are the same as the whole. Biological specimens not usually exact smaller copies of the whole object. The little pieces are kind of like their whole. The statistical properties of the little pieces can be geometrically similar to the statistical properties of the big pieces.

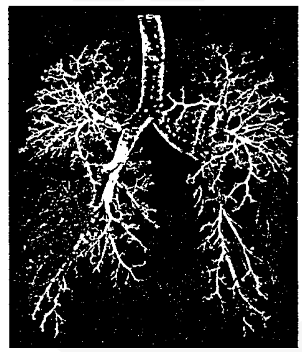
Unlike exact self-similarity, statistical self-similarity means that the smaller pieces have similar statistical properties to the whole. The arterial tree of the lungs is one example. The statistical properties of the pieces **are proportional** to the statistical properties of the whole.

blood vessels
in the retina

Family, Masters, and Platt 1989
Physica D38:98-103
Mainster 1990 Eye 4:235-241

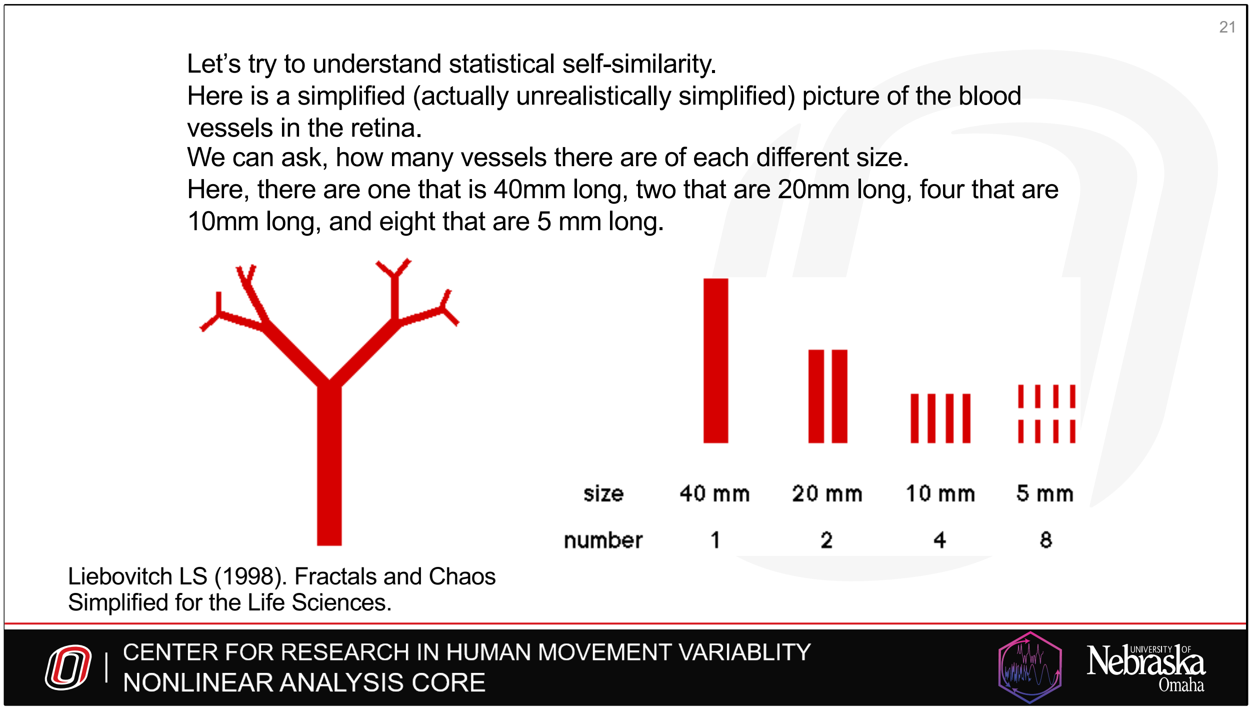
air ways
in the lungs

West and Goldberger 1987
Am. Sci. 75:354-365



Liebovitch LS (1998). Fractals and Chaos
Simplified for the Life Sciences.

The blood vessels in the retina are self-similar.
The branching of the larger vessels is like the branching of the smaller vessels.
The airways in the lung are self-similar.
The branching of the larger airways is like the branching of the smaller airways.
In real biologically objects like these, each little piece is not an exact copy of the whole object.
It is kind-of-like the whole object.
This is called Statistical Self-Similarity.

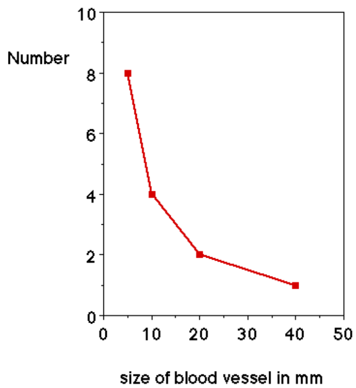


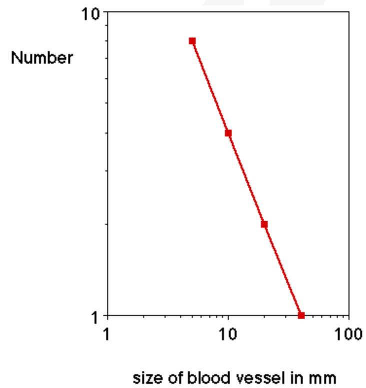
Let’s try to understand statistical self-similarity.
Here is a simplified (actually unrealistically simplified) picture of the blood vessels in the retina.
We can ask, how many vessels there are of each different size.
Here, there are one that is 40mm long, two that are 20mm long, four that are 10mm long, and eight that are 5 mm long.

HOW OFTEN there is THIS SIZE: We can plot HOW MANY vessels there are of each SIZE.

This is called the Probability Density Function, PDF.

This is a straight line on a plot of log(Number) versus Log(size).

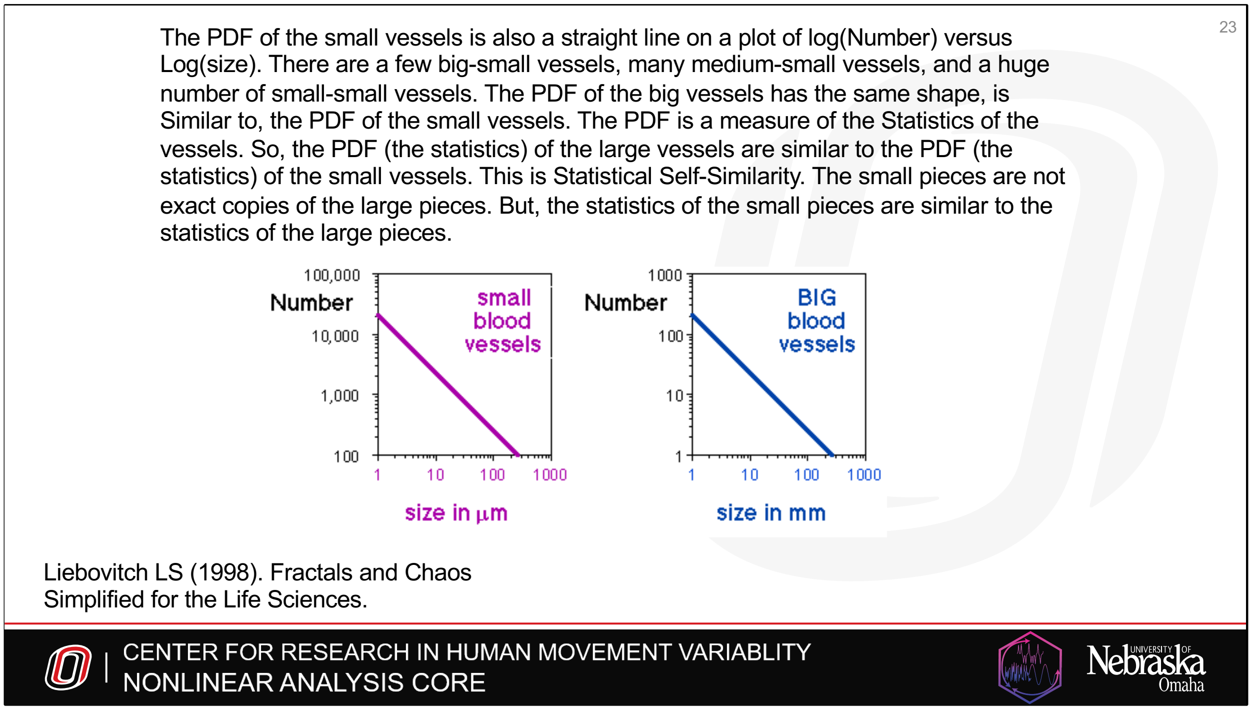




Straight line on log-log plot = Power Law

Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

We can plot HOW MANY vessels there are of each SIZE.
This is called the Probability Density Function, PDF.
This is a straight line on a plot of log(Number) versus Log(size).



The PDF of the large vessels is a straight line on a plot of log(Number) versus Log(size).
There are a few big-big vessels, many medium-big vessels, and a huge number of small-big vessels.
The PDF of the small vessels is also a straight line on a plot of log(Number) versus Log(size).
There are a few big-small vessels, many medium-small vessels, and a huge number of small-small vessels.
The PDF of the big vessels has the same shape, is Similar to, the PDF of the small vessels.
The PDF is a measure of the Statistics of the vessels.
So, the PDF (the statistics) of the large vessels are similar to the PDF (the statistics) of the small vessels.
This is Statistical Self-Similarity.
The small pieces are not exact copies of the large pieces.
But, the statistics of the small pieces are similar to the statistics of the large pieces.

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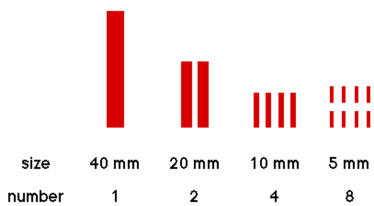
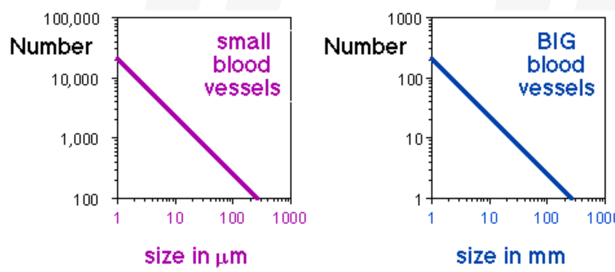
- There are a few big-small vessels, many medium-small vessels, and a huge number of small-small vessels
- The PDF of the big vessels has the same shape as the PDF of the small vessels.
- The PDF is a measure of the Statistics of the vessels
  - The PDF (the statistics) of the large vessels are similar to the PDF (the statistics) of the small vessels.
  - This is Statistical Self-Similarity!!





Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

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Self-Similarity

- Exact is also called geometric or deterministic.
  - These are more common in Mathematics.
- Statistical is also called natural or approximate or stochastic.
  - These are the fractals found in Nature.
  - They may have additional elements of randomness included. They exist only over a finite range of sizes.

Some more example of statistical self-similarity in biological systems are in the intestine and the placenta, the dendrite branching in neurons, the airways in the lungs, the ducts in the liver. Fractals can exist in time as well as in space. For example, the voltage across the cell membrane varies in a way that is fractal.

Self-similarity – As you enlarged a piece of a fractal you see the same kinds of shapes at smaller scales.
Scaling – Lets say you are going to measure some property of a fractal such as length or area or volume. When you measure these properties at smaller and smaller resolutions you include the finer and finer features of the fractal so the value of the property you are measuring changes depending on the scale of measurement.
Fractal Dimension – Characterizes how the fractal object fills space. It does not have to be an integer, although it can be. For example, a fractal can have a dimension between 1 and two. We will go more into these properties in the next few slides.

Self-Similarity Implies Scaling Relationship

- Smaller pieces of a fractal are seen at finer resolutions
- Value measured of a property will depend on the resolution used to make the measurement
  - Length, surface, or volume
- Self-similarity specifies how the small pieces are related to the large pieces, thus it determines the scaling relationship

Scaling of a non-fractal object

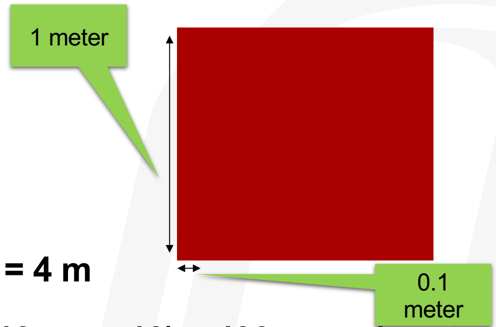
Q = q(s)

Q: Perimeter

q: 4 x length

s: 1 m → Q = 4 x 1m = 4 m

s: 10 cm → Q = 4 x (10 cm x 10) = 400 cm = 4 m



Estimates of Q → a single value

Scaling – How long is the coast of Britain?

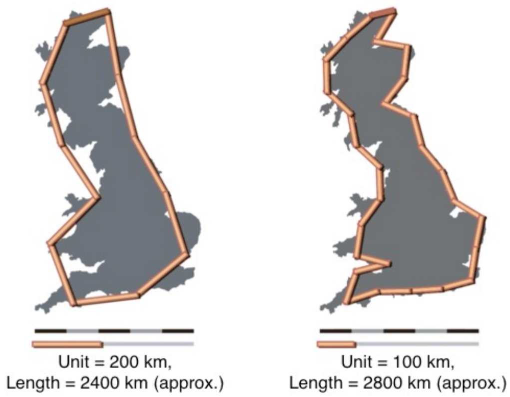
What is a fractal?

Q=q(s)

If we measure the length of the west coast of Britain with a large ruler, we get a certain value for the length of the coastline.

In 1961, Richardson measured the length of the coastline of Britain by laying small straight line segments of the same length, end to end, along the coastline. The length of these line segments set the spatial resolution of the measurement. The total length of the coastline was the combined length of all these line segments.

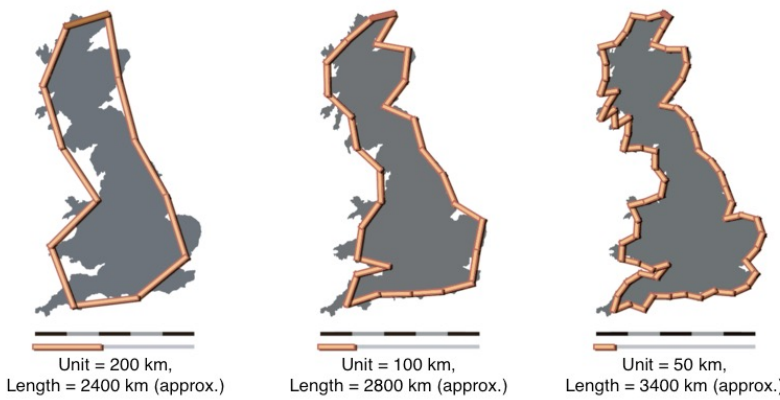
Scaling – How long is the coast of Britain?



When he measured the coastline at a finer resolution, the smaller line segments included smaller bays and peninsulas that were not included in the measurement at the coarser resolution. The addition of these smaller bays and peninsulas increased the total length of the coastline. s does make a difference b/c you have different lengths to consider. s value that is too large causes a large amount of error. Smaller lengths of s, the correctness of Q increases but becomes difficult to calculate. You can never have an exact value for Q.

Q = perimeter; s = length of each line; q = # of times you use s, designates the relationship.
The mathematical form of self-similarity determines the mathematical form of the scaling relationship.

Scaling – How long is the coast of Britain?



Scaling – How long is the coast of Britain?

$\uparrow Q = q(s\downarrow)$

Estimates of Q

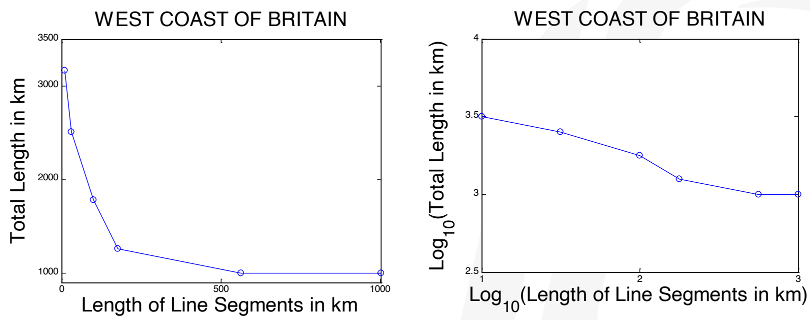
a single value

$Q = q(s)$

A power law scaling relationship with s

$Q = ps^{\varepsilon}$

Scaling



$$Q = s^{(-1/4)}$$

Implications of Scaling Relationships

- No unique or “correct” value for a measurement
  - The value measured for a property depends on the resolution used to make the measurement
- Measurements made at different resolutions will be different
  - Differences between the values measured by different people could be due to the fact that each person measured the property at a different resolution
- Importance of the scaling relationship
  - The measurement of the value of a property at only one resolution is not useful to characterize fractal objects or processes
  - Need to determine how the values measured for a property depend on the resolution used to make the measurement, namely, the scaling relationship

Self-similarity – As you enlarged a piece of a fractal you see the same kinds of shapes at smaller scales.
Scaling – Lets say you are going to measure some property of a fractal such as length or area or volume. When you measure these properties at smaller and smaller resolutions you include the finer and finer features of the fractal so the value of the property you are measuring changes depending on the scale of measurement.
Fractal Dimension – Characterizes how the fractal object fills space. It does not have to be an integer, although it can be. For example, a fractal can have a dimension between 1 and two. We will go more into these properties in the next few slides.

Fractal Dimension

- Provides a quantitative measure of the self-similarity and scaling. The dimension tells us how many new pieces we see when we look at finer resolution.
- Characterizes how the fractal object fills space. It does not have to be an integer, although it can be. For example, a fractal can have a dimension between 1 and two.

The dimension gives a quantitative measure of the fractal properties of self-similarity and scaling
The dimension tells us how many additional smaller pieces of an object are revealed when it is magnified by a certain amount
Fractal Dimension: describes how an object fills up space
Topological Dimension: describes how points within an object are connected to each other
Embedding Dimension: describes the space that contains the object

Fractal Dimension – What is it?

Fractal Dimension – What is it?

$$\text{Fractal Dimension} = \frac{\ln N}{\ln R}$$

N is the number of same figures necessary to get a new figure, R times larger

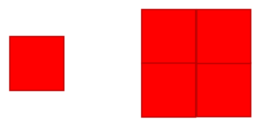
$$D = \frac{\ln 2}{\ln 2} = \frac{1}{1} = 1$$

To double the length of a line, you add a segment identical to the original line

Fractal Dimension – What is it?

Fractal Dimension = ln N/ ln R

N is the number of same figures necessary to get a new figure, R times larger



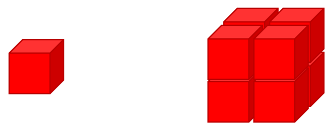
$$D = \frac{\ln 4}{\ln 2} = \frac{2}{1} = 2$$

To double the side of a square, you need 4 squares identical to the original square

Fractal Dimension – What is it?

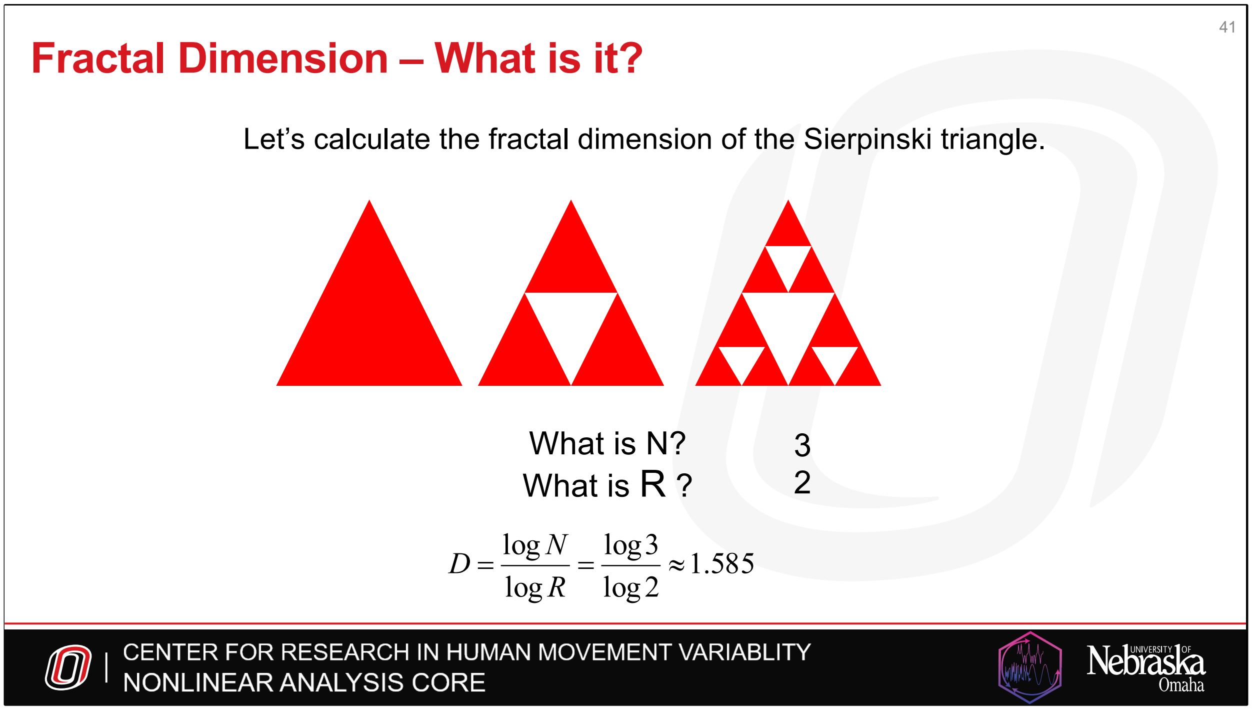
Fractal Dimension = ln N/ ln R

N is the number of same figures necessary to get a new figure, R times larger



$$D = \frac{\ln 8}{\ln 2} = \frac{3}{1} = 3$$

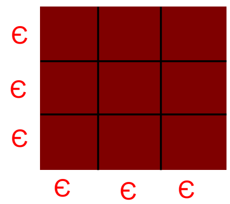
To double the side of a cube, you juxtapose 8 cubes identical to the original



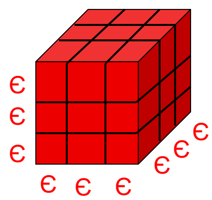
Length of scale is 2 – length of how many of the new triangles make up the length of the orignial triangle
V – 3 – number of new triangles

Similarity Dimension

Є = magnification factor = 1/3
N = # of segments = 3
$N = (1/\epsilon)^D$ $3 = 3^1$ d = 1



Є = 1/3
N = 9
$N = (1/\epsilon)^D$
$9 = 3^2$
d = 2



Є = 1/3
N = 27
$N = (1/\epsilon)^D$
$27 = 3^2$
d = 3

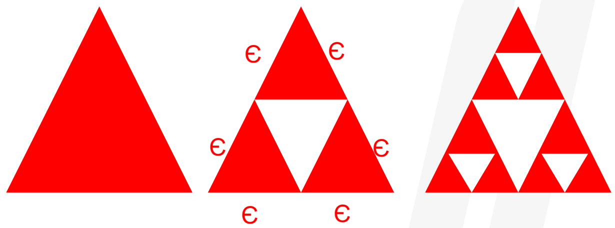
$1 = N * \epsilon^D$
$N = (1/\epsilon)^D$
$\ln N = D \ln (1/\epsilon)$
$D = \ln N / \ln (1/\epsilon)$

Fractal dimension given by the formula:

$D = \ln N / \ln R$

- Where R = 1/Є is the so-called scaling factor
- N is the number of components
- D is the fractal dimension

Let's calculate the similarity dimension of the Sierpinski triangle.



What is N? 3

What is Є ? 1/2

$$D = \frac{\log N}{\log R} = \frac{\log 3}{\log 2} \approx 1.585$$

Length of scale is 2 – length of how many of the new triangles make up the length of the orignial triangle
V – 3 – number of new triangles

Similarity Dimension

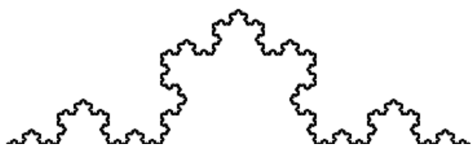
The similarity dimension can characterize fractals with exact and obvious self-similarity

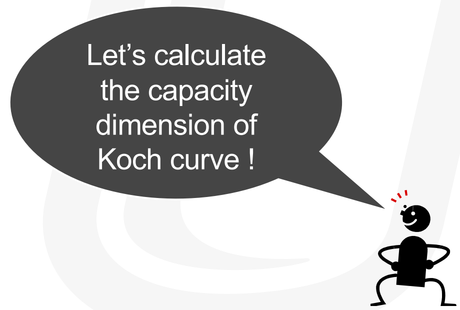
↓

How about irregularly shaped objects?

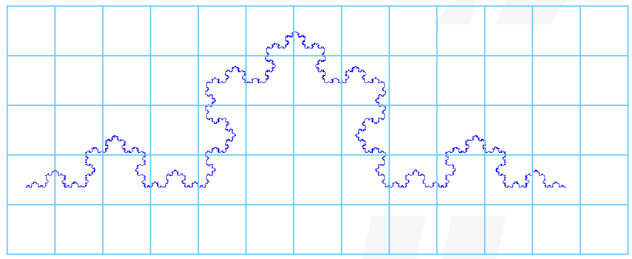
Need a general method to calculate the dimension.
The Capacity Dimension

Similarity Dimension Example





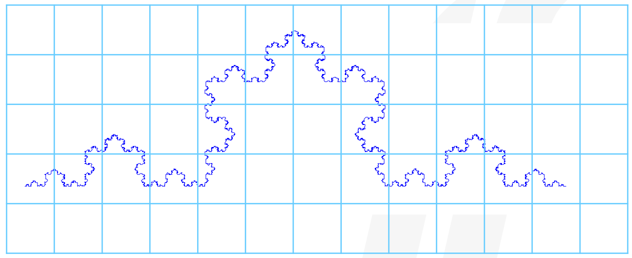
Similarity Dimension Example



L_i = the length of a side of a tile

N_i = the minimum number of tiles to cover the object

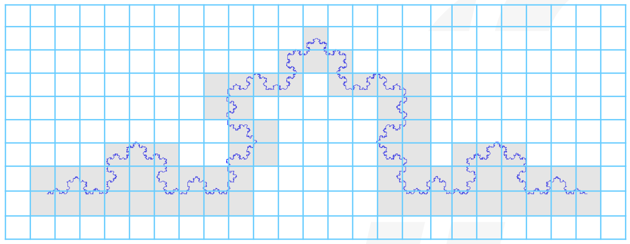
Similarity Dimension Example



L_1 = the length of a side of a tile = 1

N_1 = the minimum number of tiles to cover the object = 22

Similarity Dimension Example

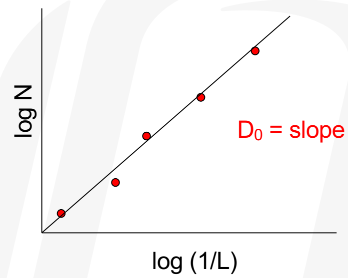


L_2 = the length of a side of a tile = 1/2

N_2 = the minimum number of tiles to cover the object =60

Similarity Dimension Example

- Repeat with tiles of different sizes.
- Plot N vs. L on a log-log scale.
- The object being covered. $\propto$
  - Line → N $\propto$ 1/L
  - Plane → N 1/L^2



$$D_0 = \lim_{L \to 0} \frac{\log N}{\log(1/L)}$$

$$D_0 = \lim_{L \to 0} \frac{-\log N}{\log L}$$

Similarity Dimension

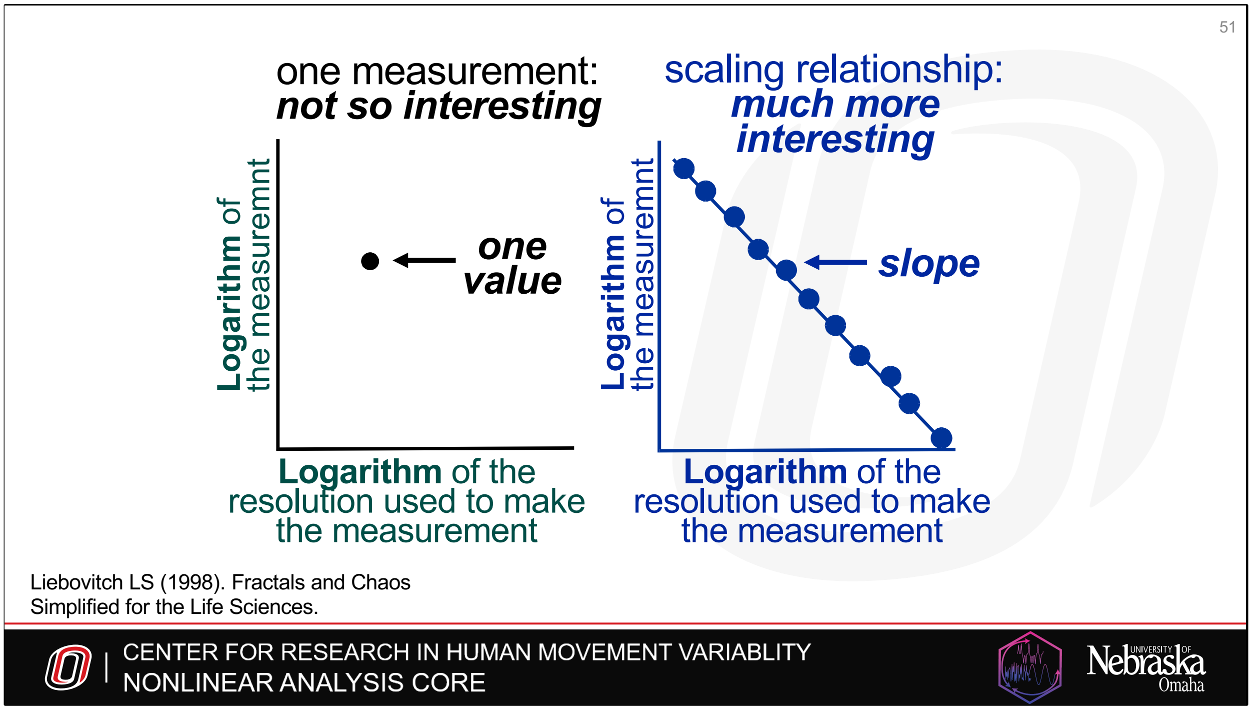
New viewpoint:

Analyze how a property depends on the effective time scale at which it is measured.

This Scaling Relationship:

We are using this to learn about the structure and motions in the parameter under study.

Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.



Take Home Lesson:
We are used to thinking that there is ONE measurement that best describes a property of an object.
For a fractal object that extends over many scales, a property depends on the scale at which it is measured.
There is no ONE measurement that best describes the object.
The object is best described by HOW THE PROPERTY MEASURED DEPENDS ON THE RESOLUTION AT WHICH IT IS MEASURED.
This relationship is characterized by a parameter called the Fractal Dimension.

Fractal Dimension Recap

- A fractal is an object in space or a process in time that has a fractal dimension greater than its topological dimension
- An example object has a fractal dimension of 1.2619. This dimension describes the space filling properties of the perimeter
- Since the dimension is larger than 1, the perimeter is so wiggly that it covers more than just a 1-D line, but since it is less than 2 the perimeter isn't so wiggly that it covers a 2-D area
- Topological dimension of the perimeter is 1. This describes how the points on the perimeter are connected together
  - No matter how wiggly the perimeter is, it is still a line with a topological dimension equal to 1

Fractal Dimension Recap

- Topological dimension tells us what kind of thing an object is, such as an edge, a surface, or a volume
- When the fractal dimension is larger than the topological dimension, it means that the edge, surface, or volume has more finer pieces than we would have expected of an object with its topological dimension

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Properties of a Fractal
Statistical Properties

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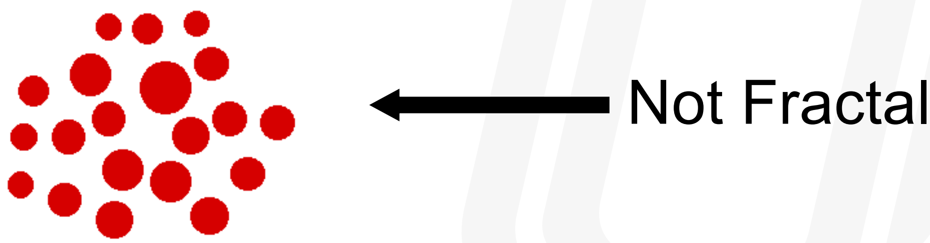
Statistical properties of fractals belong to a more general class of distributions called stable distributions

The average of a fractal may not exist

- Non-fractal
  - As more data is included, the averages of the data reach a limiting value which is therefore considered to be the "real" average
- Fractal
  - As more data is included, the averages of the data will continue to increase or continue to decrease
  - There is no limiting value that can be considered to be the "real" average

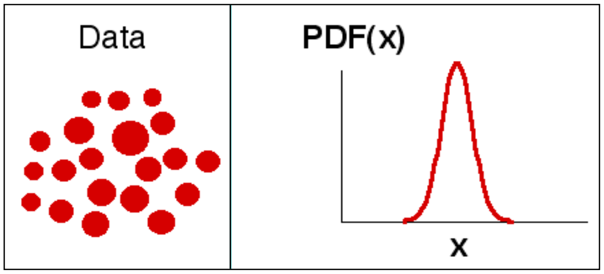
Statistical Properties – What is fractal?



Liebovitch LS (1998). Fractals and Chaos
Simplified for the Life Sciences.

Here is a set of numbers, maybe they are the values measured from an experiment.
I have drawn a circle to represent each number.
The diameter of the circle is proportional to the size of the number.
Here is a non-fractal set of numbers.
Most of them are about the size of an average number.
A few are a bit smaller than the average.
A few are bit larger than the average.

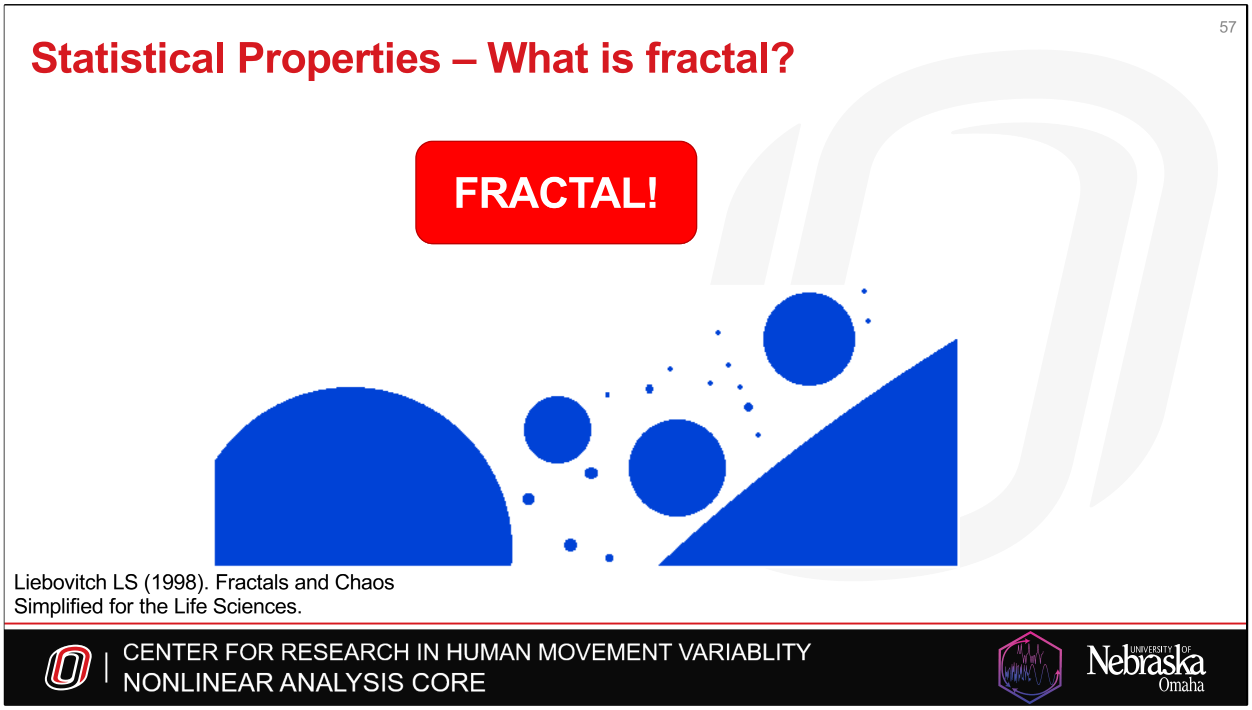
Statistical Properties – What is fractal?



Not Fractal

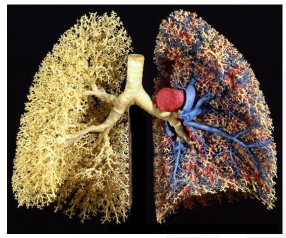
Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

Here is the PDF of these non-fractal numbers.
The PDF is how many numbers there are of each size.
The PDF here is called a "Bell Curve", a "Gaussian Distribution", or a "Normal Distribution".
What an interestingly unloaded word to use, a "NORMAL" distribution.
We are about to see that much of the world is definitely not "normal".



Here is a set of numbers from a fractal distribution.
The diameter of each circle is proportional to the size of the number.
These numbers could be from the room around you.
Look around your room.
There are few big things: people and chairs.
Many medium sized things: pens and coins.
And a huge number of tiny things: dust and bacteria.
It is NOT at all like that "Normal" distribution.
There are a few big things, many medium sized things, and a huge number of tiny things.
Sets of data from many things in the real world are just like this.
We call this a FRACTAL distribution of numbers because it has the same statistical properties as the sizes of the pieces in fractal objects.

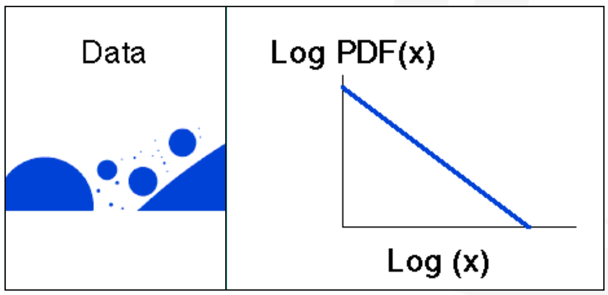
Fractals are ubiquitous in nature!



Many natural structures exhibit ***self-similarity***, where their structure repeat itself over many scales

Statistical Properties – What is fractal?

FRACTAL!

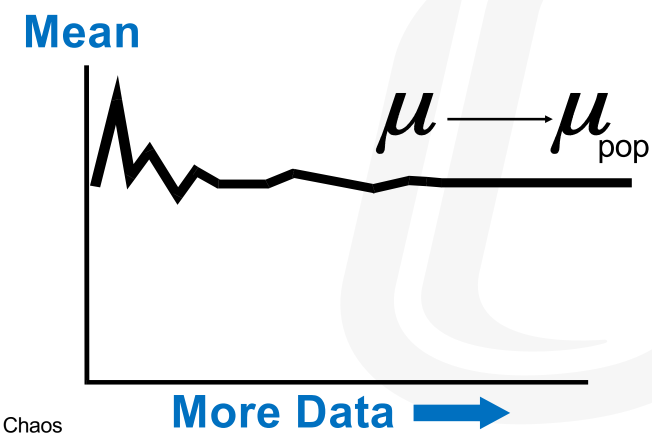


Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

Here is the PDF of these fractal numbers.
The PDF is how many numbers there are of each size.
There are a few big numbers, many medium sized numbers, and a huge amount of tiny numbers.
The PDF is a straight line on a plot of log(How Many Numbers, the PDF) versus log(value of the numbers).

Statistical Properties – What is fractal?

Non - Fractal

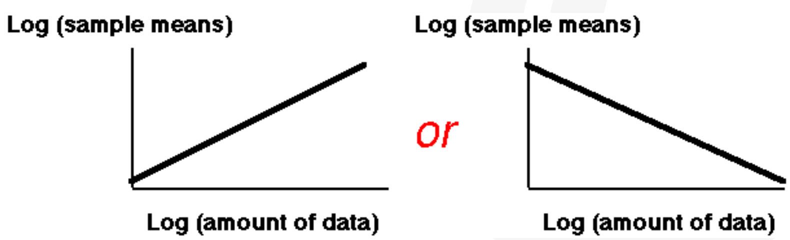


Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

The statistics of a fractal set of numbers is VERY DIFFERENT from the statistics of "normal" numbers that they taught you about in Statistics 101.
The statistics you learned in Statistics 101 is only about non-fractal numbers.
Take the average of a sample of non-fractal numbers.
This is called the Sample Mean.
As you include ever more data, the sample means get ever closer to one value.
We call that value the Population Mean.
We think that the population mean is the "real" value of the mean.

Statistical Properties – What is fractal?

The Average Depends on the Amount of Data Analyzed



Liebovitch LS (1998). Fractals and Chaos Simplified for the Life Sciences.

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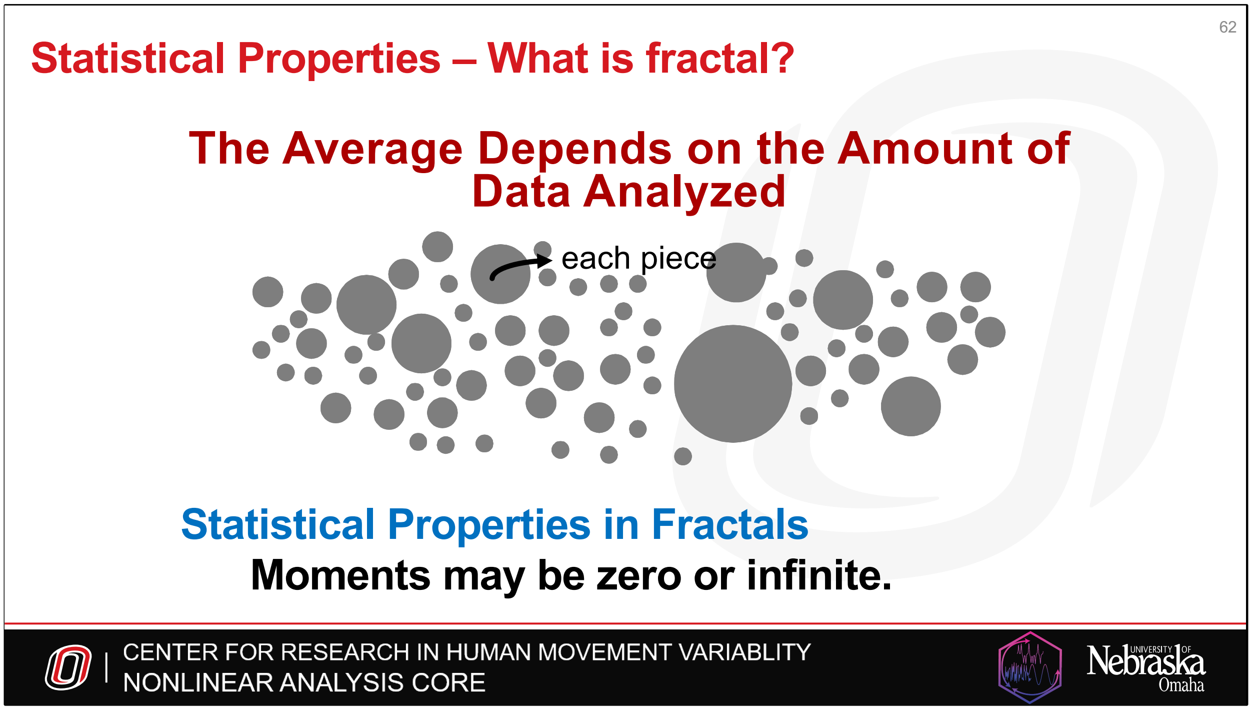
But the statistics of fractal numbers is very different.
Take the average of a sample of fractal numbers.
This is called the Sample Mean.
As you include ever more data, the sample means do NOT get ever closer to one value.
Either the sample means keep increasing OR the sample means keep decreasing as you include more data.
There is NO Population Mean.
There is NO one value that best describes the data.
The data extends over a range of many different values.



Here is why that happens.
Again, here is a set of fractal numbers.
The diameter of the circles are proportional to the size of the numbers.
As you include more data, either:

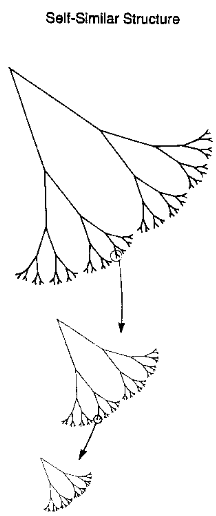
-you include many smaller values and so the sample means get smaller and smaller, or

-you include a few bigger values and so the sample means get bigger and bigger.

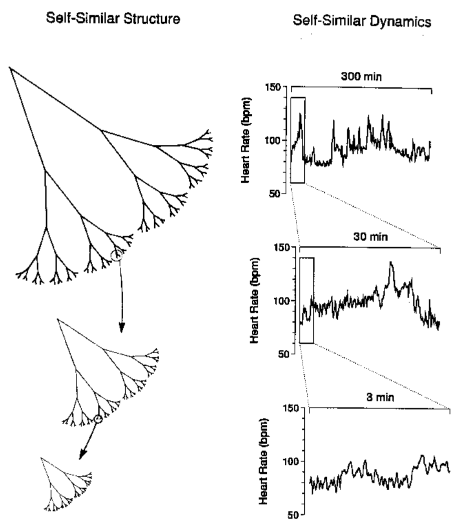
You can kind-of-see that which happens depends on the ratio of the amount of small numbers to the amount of big numbers.
That ratio is characterized by a parameter called the Fractal Dimension.

Self-Similarity Effect on Moments

- When determining the average of the values of the sizes of the pieces, the most significant contribution to the average will come from either:
  - The frequent number of small values, which the average decreases as more data are included
  - The infrequent large values, which the average increases as more data are included
- Moment is the average of a property raised to a power
- Mean is the first moment
  - Self-similarity means that the mean will increase or decrease as more data are analyzed, depending on the relative contribution of the frequent small values vs the infrequent large values
- Variance is the second moment
  - Self-similarity means that the variance will increase as more data are analyzed because there are an ever larger number of self-similar fluctuations



Some examples of systems that exhibit this long range interdependency are heart-rate, breathing patterns, ion-channel kinetics,



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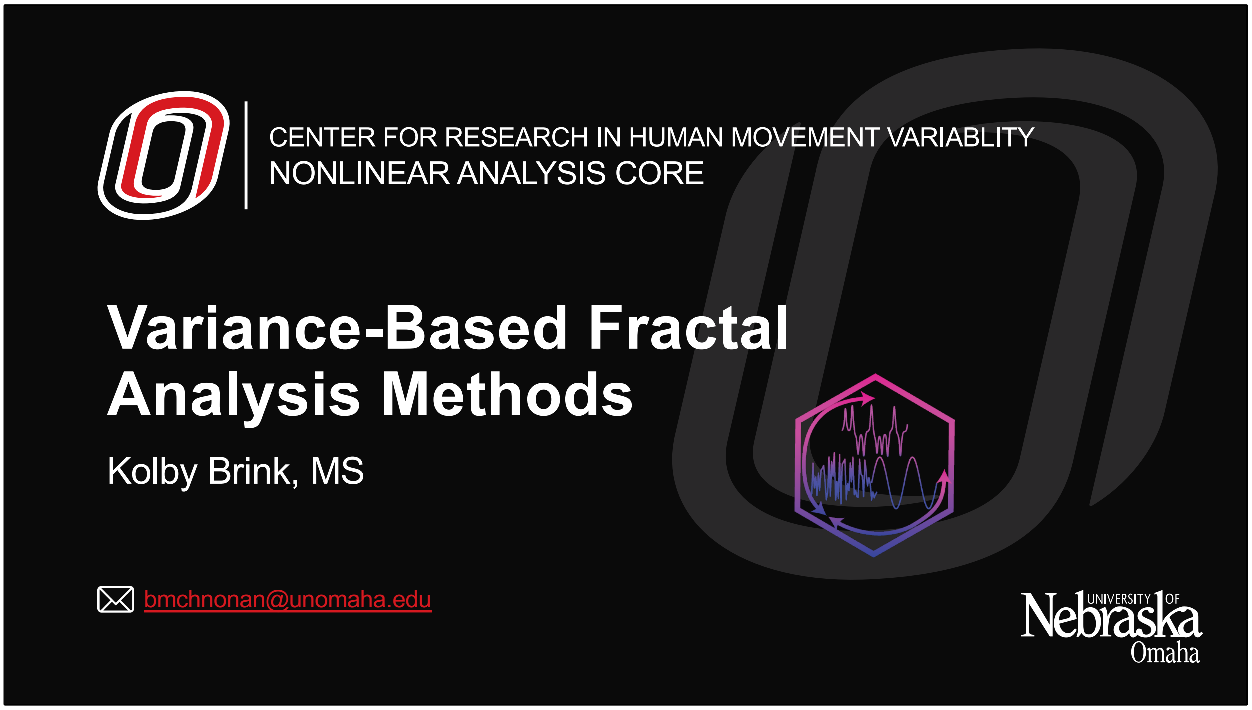


Talk about Power Spectral Density

Overview

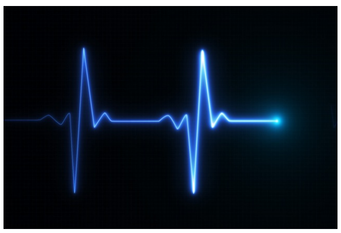
- Why Do We Want to Measure Fractals?
- The Basics of Signal Processing
- Detrended Fluctuation Analysis Steps:
  1. Mean Subtraction and Integration
  2. Box Division
  3. Local Trend Estimation
  4. Average Fluctuation Calculation
  5. Scaling Analysis
- A Novel Method for Fractal Analysis

Talk about Power Spectral Density

Quick Refresher: Why do we want to measure fractals?

Fractals are found (almost) everywhere!

- Environmental Systems
- Brain Activity
- Human movement
- Human Physiology

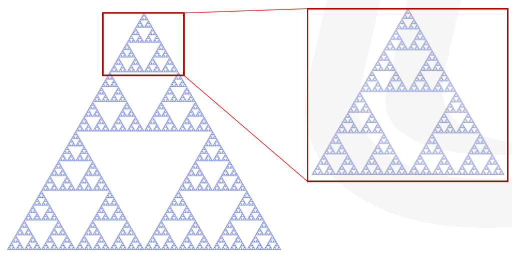


Talk about Power Spectral Density

Quick Refresher: Why do we want to measure fractals?

The goal:

- Quantify and understand the underlying patterns of complexity, self-similarity, and scaling properties in a given system or dataset



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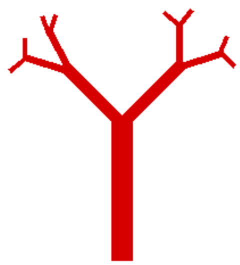
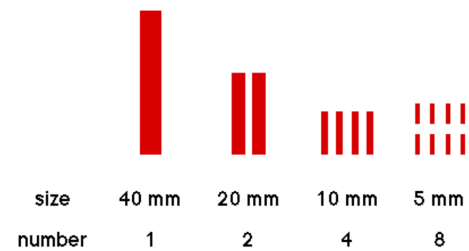
Here is a typical example of a fractal – the sierpinski triangle.

Quick Refresher: Why do we want to measure fractals?

One example: Blood vessels in the retina

- How many vessels are there at each size?





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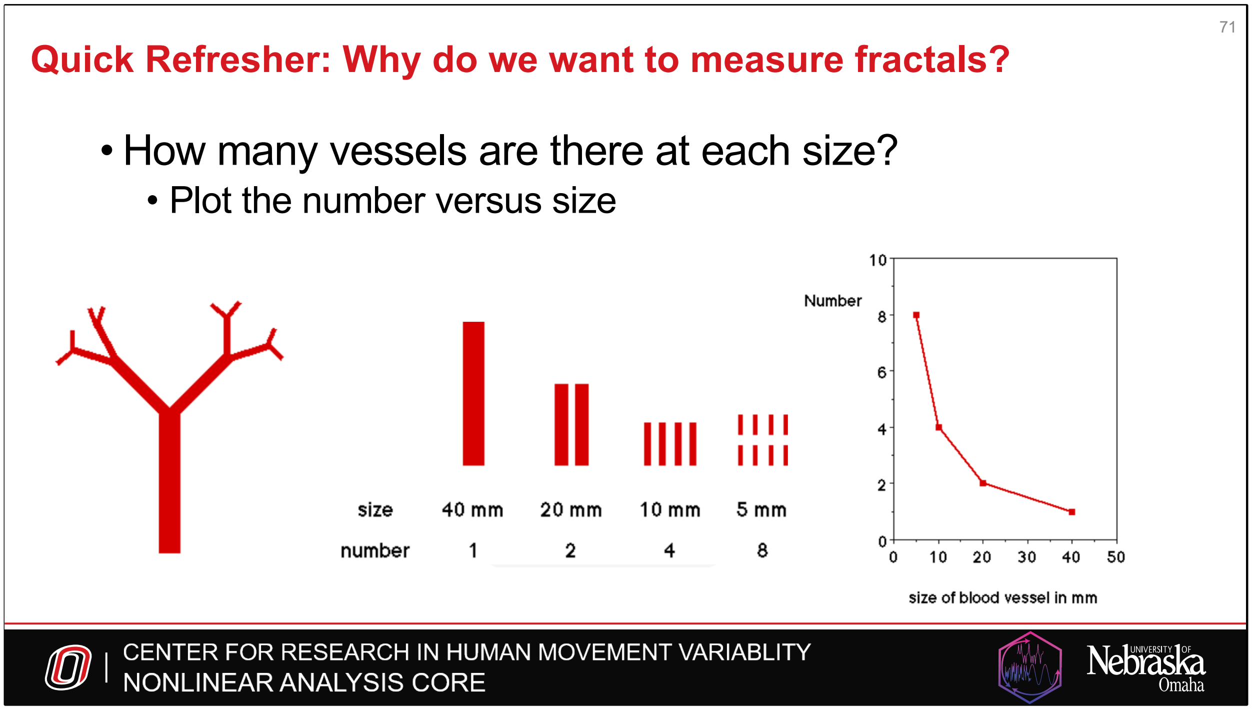
How do does the concept of fractals and self-similarity at smaller scales make its way to our physiology?

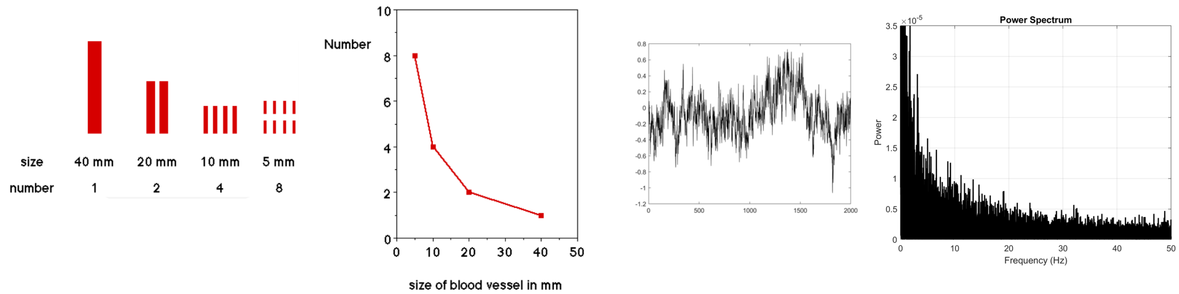


The basic idea here is that there are a few big vessels and a lot of the smaller blood vessels

When you plot the number of blood vessels and the size of blood vessels you see a power law type of distribution. That is- the number of blood vessels drops and seems to slowly decrease as the size of the blood vessels get larger.

Quick Refresher: Why do we want to measure fractals?

- In a similar manner, we can look at the occurrence of big and small fluctuations within movements



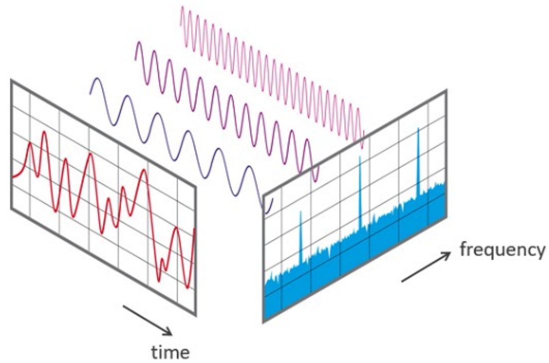
So in a similar way that we can look at the relationship between the number of blood vessels relative to the size of blood vessels –

We can also look at the time series of our movements or physiological data see how different sizes of fluctuations are interacting to compose the overall noisy structure of the time series.

Which in this case – appears to show a similar pattern such that there are a few signals with larger fluctuations and a higher number of signals that have smaller flucutations

The Basics of Signal Processing: Power Spectral Density

- Power Spectral Density (PSD) is a tool used to analyze the frequency content of a signal or time series data
- It quantifies the distribution of power or energy across different frequencies in the signal



One of the earlier methods to measure time series signals – which I am sure most of you are familiar with is power spectral density

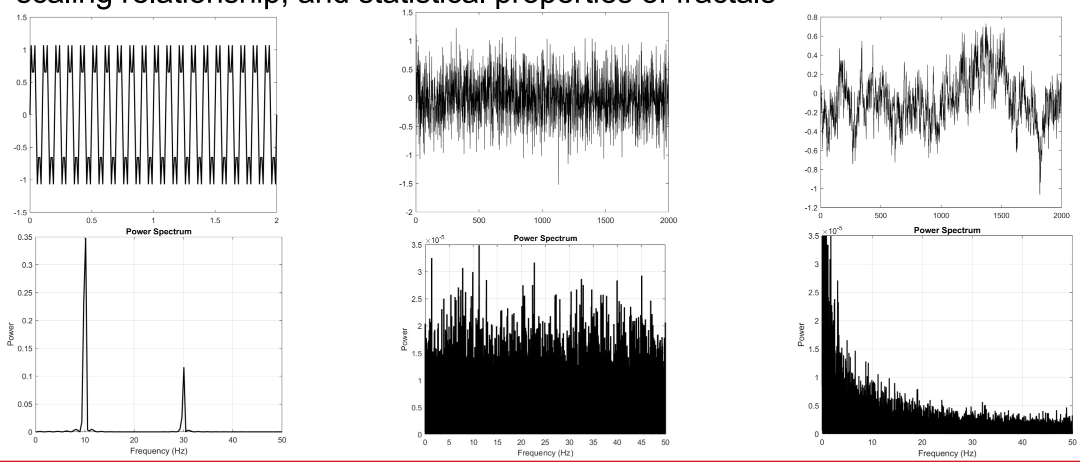
Power spectral density characterizes the noise in our signal by breaking the signal down into smaller and smaller scales

Read second bullet point

our ultimate goal is to transform a time series from the time domain into the frequency domain and thus gain insights into the distribution of its signal power across different frequency components

The Basics of Signal Processing: Power Spectral Density

- Power spectra of fractal objects in space or time reflect the self-similarity, scaling relationship, and statistical properties of fractals



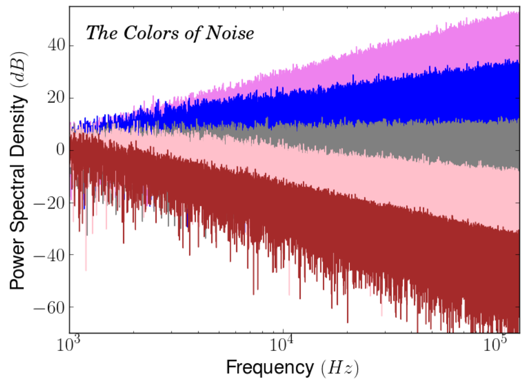
Here is an example of a signal with two different frequencies of different powers. It looks like a simple double peak sinusoid. The power spectrum correspondingly shows us this.

But when we look at data that better represents what we see in real life - we see that transforming the time series into the frequency components is not always so simple and clean.

However, even with our noisier time series we can see certain trends in the power spectral density plots that indicate fractal behavior

Power Spectra and the relationship to 'noise'

- Noises are called the power spectra of statistical time series.



These relationships between the frequency components and power give rise to why we describe the noise of our time series with certain colors

Each noise type has a characteristic frequency distribution. For example, white noise has a flat frequency distribution, pink noise has a 1/f distribution (more power in lower frequencies)

Problems with PSD

- PSD requires stationary signals, which may not be applicable to all real-world signals with non-stationary characteristics
- It assumes linearity, which may not hold true for complex signals with nonlinear behavior, such as chaotic or turbulent systems
- Power Spectral Density is not as accurate as other methods
  - Specifically, ***Detrended Fluctuation Analysis***

How can we measure fractals in our data?

Detrended Fluctuation Analysis (DFA)

- Introduced by Peng et al (1994)
- DFA analyzes fractal-like (i.e., self-similar) relationships between patterns of fluctuation across progressively longer time scales
- Permits the detection of long-range correlations embedded in a nonstationary time series
- Avoids spurious detection of apparent long-range correlations that are an artifact of nonstationarity

What is DFA??

How can we measure fractals in our data?

Detrended Fluctuation Analysis (DFA)

- DFA is arguably the most common form of fractal analysis
- It is used by researchers in many scientific domains

Why DFA??

Why use DFA?

1. DFA is relatively simple.
   - You can perform the analysis with virtually any programming language.
2. DFA is versatile.
   - It can be used with data that is nonstationary – a common occurrence in movement science
3. DFA's extensive use allows for more robust comparisons
4. It is relatively unbiases with sufficient time series length
5. It outperforms all other fractal analysis in terms of bias and consistency

Detrended Fluctuation Analysis: The Steps

The 5 Steps of DFA

1. Mean Subtraction and Integration
2. Windowing
3. Local Trend Fitting and Subtraction
4. Average Fluctuation Calculation
5. Fluctuation Function / Diffusion Plot

Detrended Fluctuation Analysis: The Steps

Step 1: Mean Subtraction and Integration

- The original time series $y(i)$ is subtracted from its mean, $\bar{y}$, for every time point i, such that:
- Integrate the time series, $y(k)$

$$y(k) = \sum_{i=1}^{k} [y(i) - \bar{y}] \qquad \text{for } k = 1, \ldots, N$$

  - where N is the length of the time series

Change phrasing on bullet 1 – subtracting mean

Detrended Fluctuation Analysis: The Steps

Step 1: Mean Subtraction and Integration

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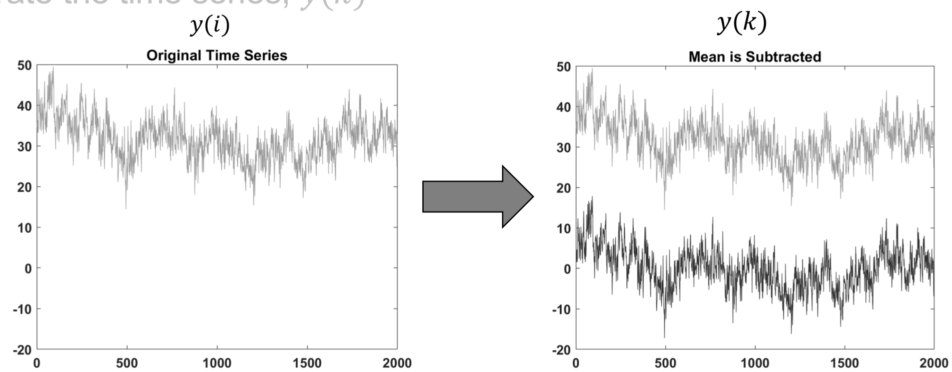

Add in time on x-axis and have ‘amplitude’

Detrended Fluctuation Analysis

Step 1: Mean Subtraction and Integration

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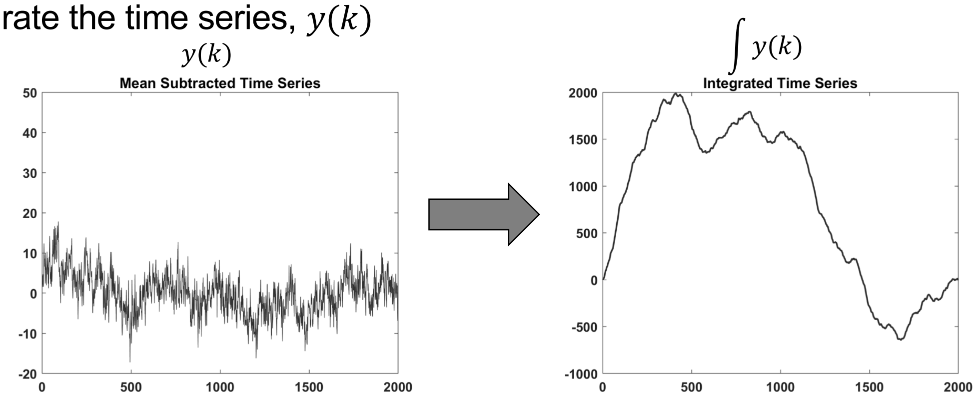
- Integrate the time series, $y(k)$

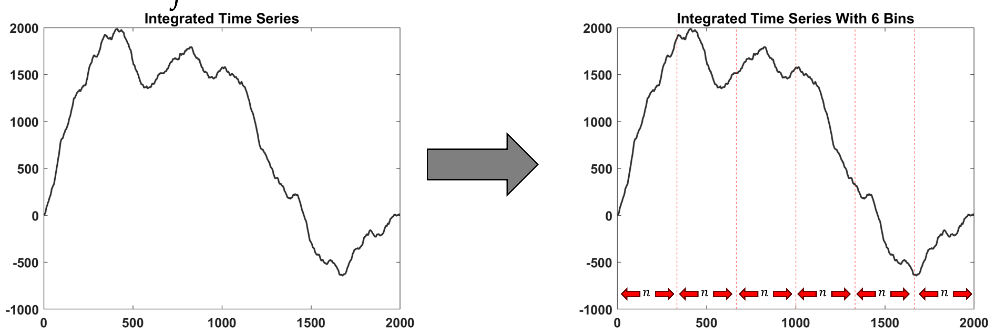


Detrended Fluctuation Analysis

Step 2: Windowing

- Divide the time series into nonoverlapping window sizes of length n

$$\int y(k)$$

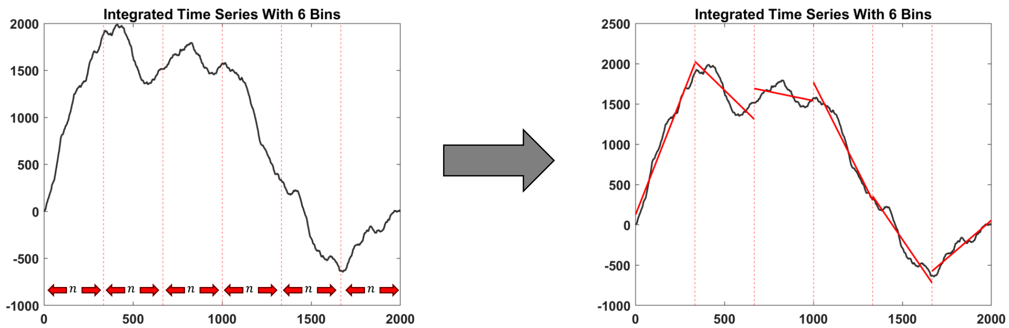


Make it all window

Detrended Fluctuation Analysis

Step 3: Local Trend Fitting and Subtraction

- *Detrend* each window
  - This simply entails fitting a regression line within each of the windows
- Subtract the fitted trend line from the data in each window



Detrended Fluctuation Analysis

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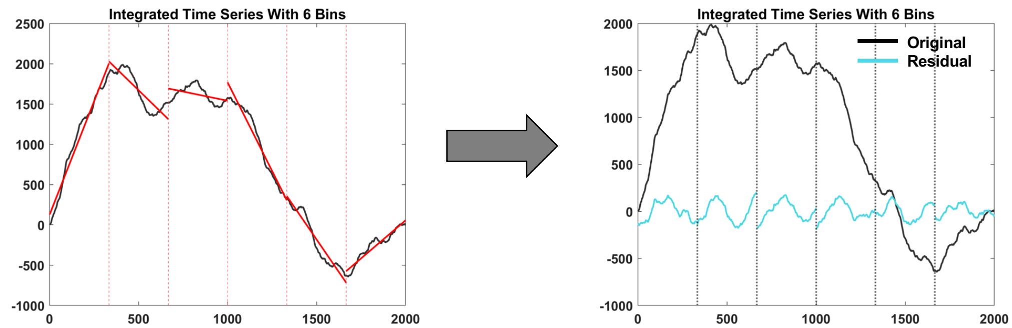


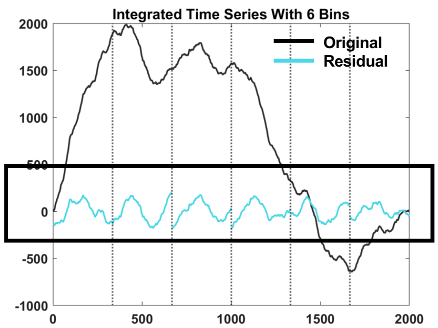
Test in R

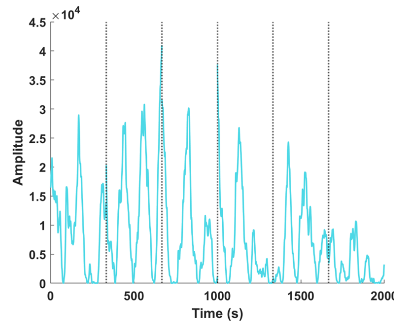
Make blue darker

Detrended Fluctuation Analysis

Step 4: Average Fluctuation Calculation

- Square the residual
- Find the root mean square, $F(s)$, of the local squared residuals
  - $F(s)$ represents the average distance of each point in a time series from a local trend line estimated at a given scale, s





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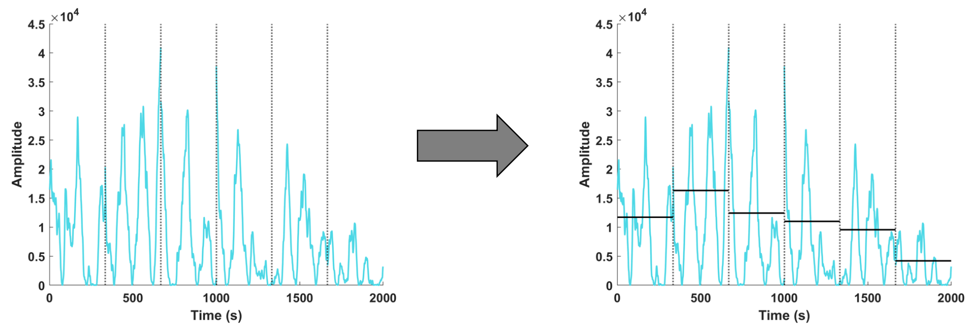

Detrended Fluctuation Analysis

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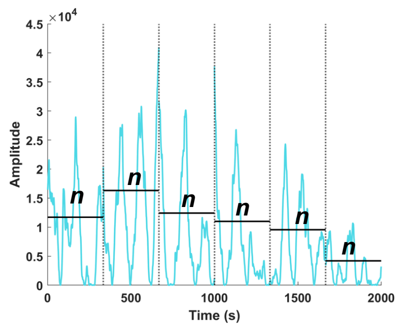
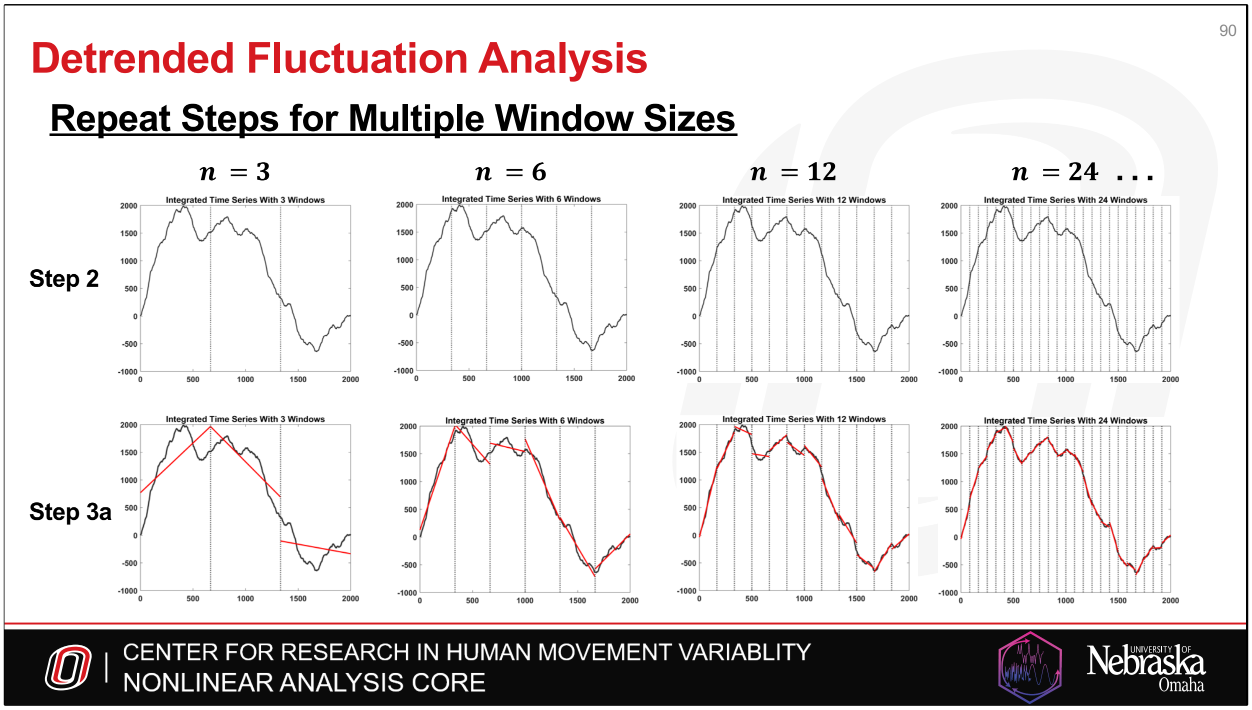


$$\sqrt{\frac{\qquad}{\text{\# of Windows}}}$$



Dot lines instead of dash? Maybe solid? Make them light grey? Make change in power of 2

Change numbers at top to match properly

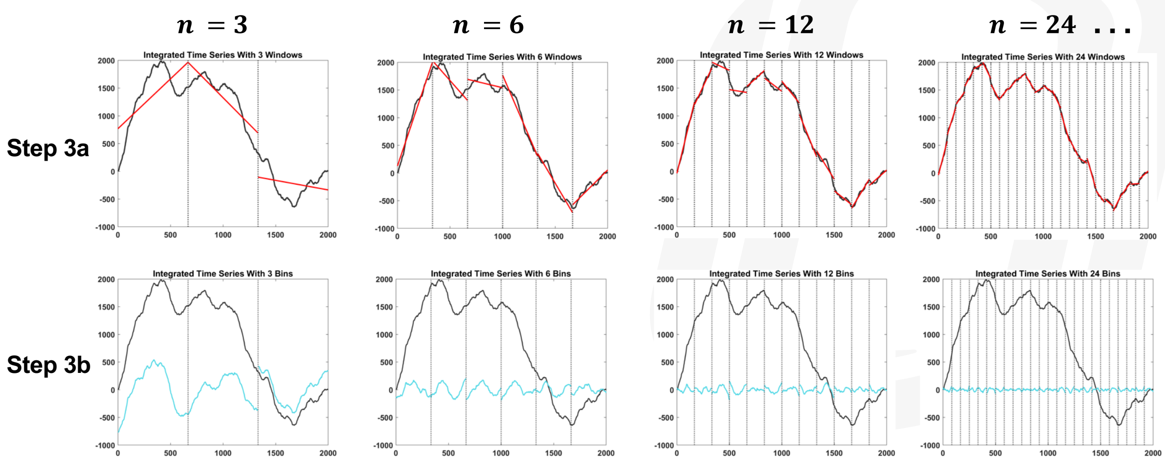
Constant of proportionality

Use subplot on these plots add axes

Use morph transition to shift graphs upward

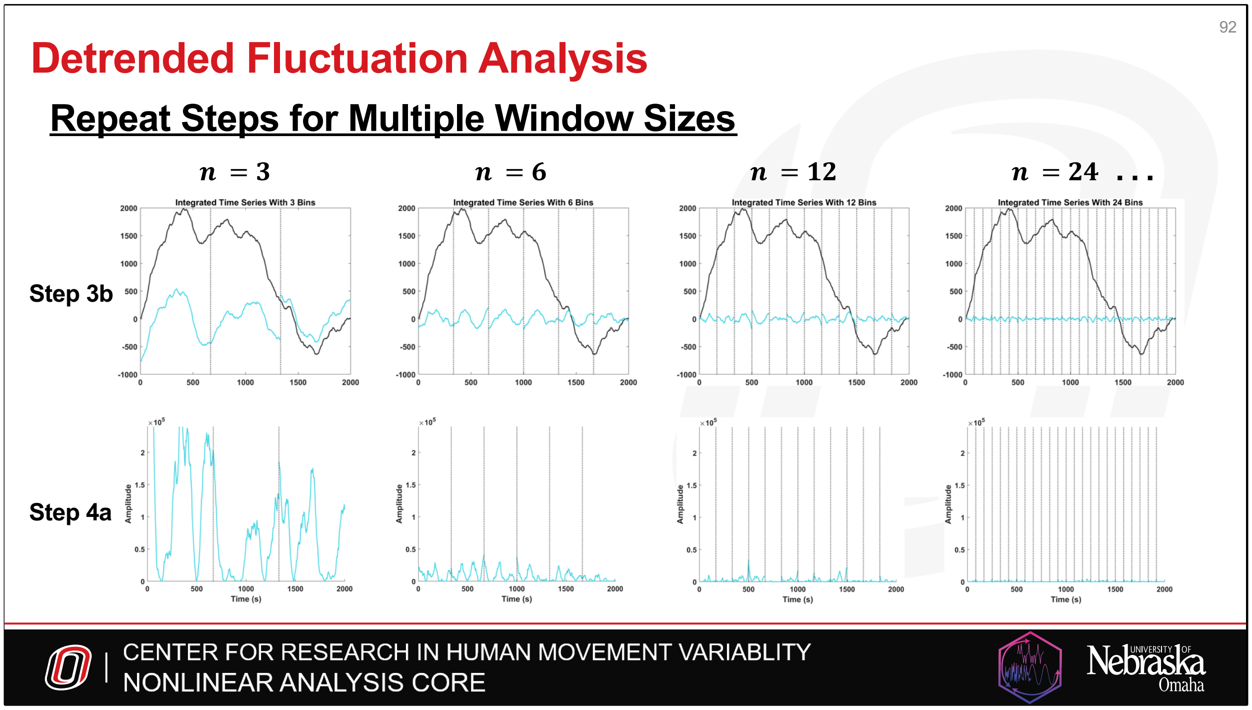
Detrended Fluctuation Analysis

Repeat Steps for Multiple Window Sizes



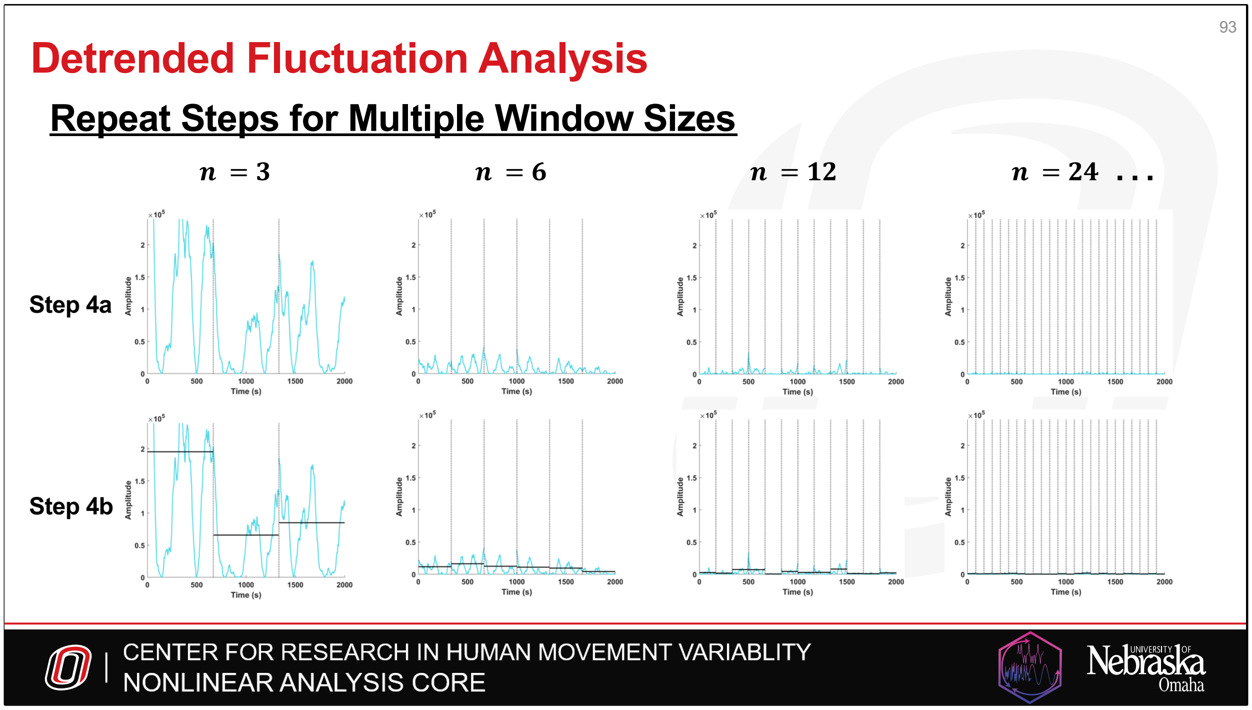
Add in slide that shows the averaging of the lines

Add in this point: “look at how this residual changes as a function of time scale – where we have very little variation at small time scales and large variation at large time scale – the maginitude of the statistic changes as a function of a constant of proportionatiy – changes as the scale of the window changes



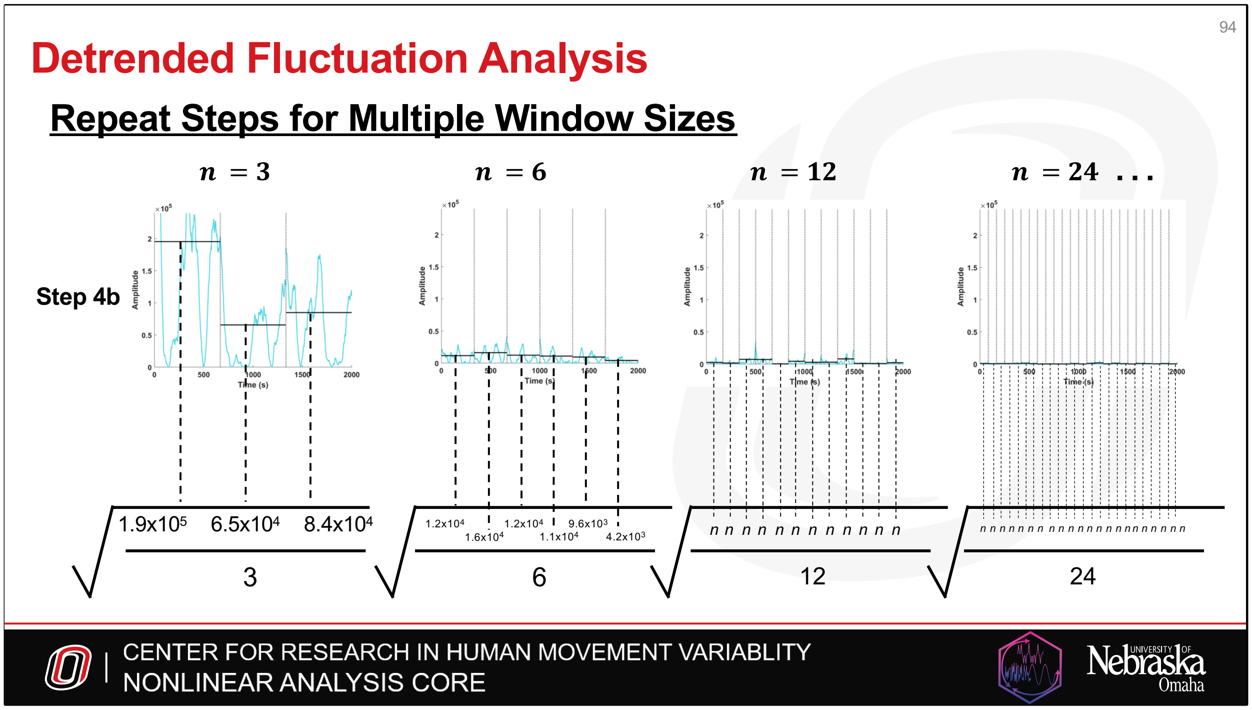
Add in slide that shows the averaging of the lines

Make sure its all on same axis



Add in slide that shows the averaging of the lines

Add in slide of averi ng all the lines to get our averaged value ➔ then you square root it ➔ then its plotted on a log log scale



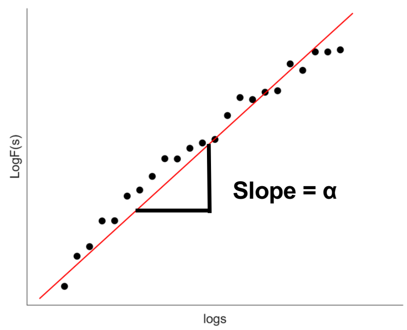
Add in slide that shows the averaging of the lines

Add in slide of averi ng all the lines to get our averaged value ➔ then you square root it ➔ then its plotted on a log log scale

Detrended Fluctuation Analysis

Step 5: Fluctuation Function / Diffusion Plot

- A regression is performed in order to estimate the scaling exponent, α



- α provides a measure of how fast the standard deviation changes as a function of the timescale

The smallest window is at bottom – largest window is at top right –

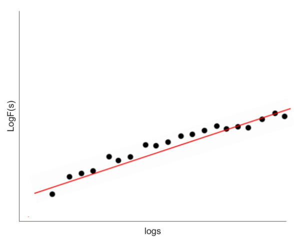
Maybe: add in slide on table?

Detrended Fluctuation Analysis

- Interpreting alpha

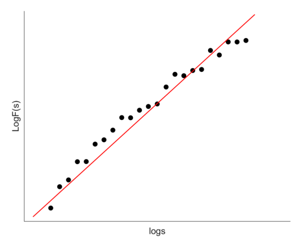
α < 0.5

Anticorrelated



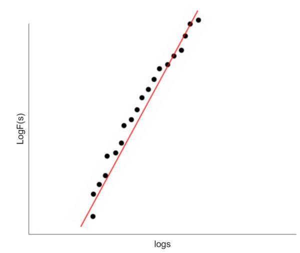
α > 0.5

Correlated



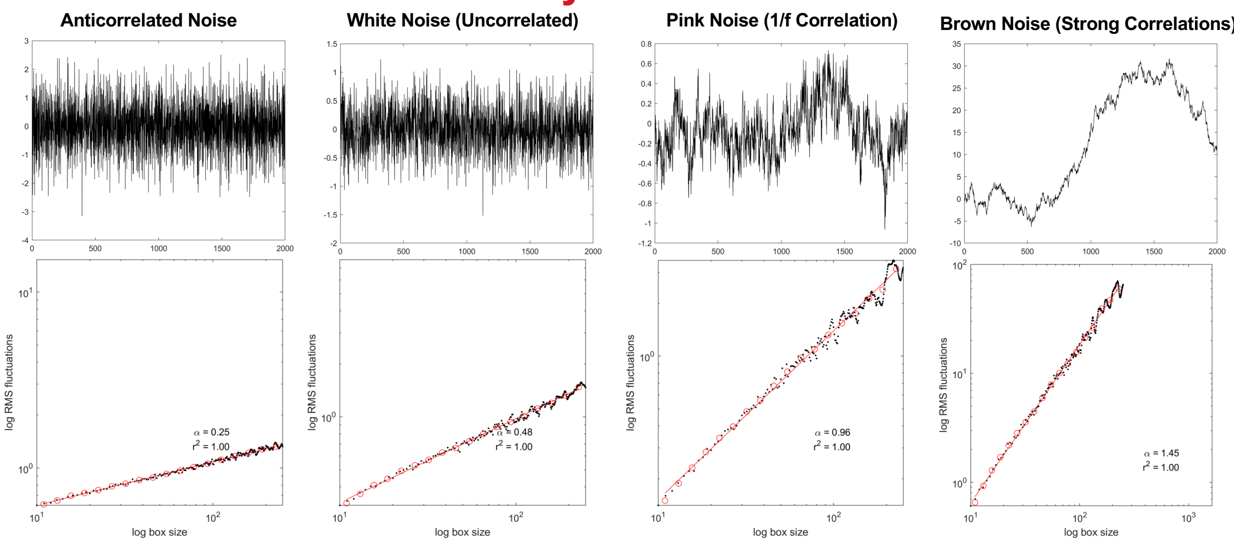
α > 1

Nonstationary, unbounded



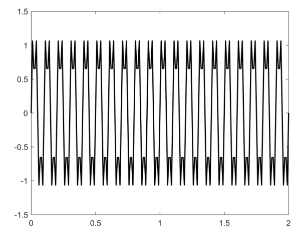
Add in white noise

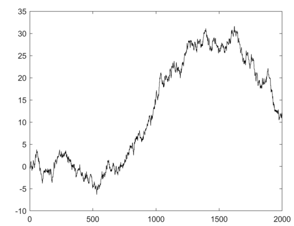
Detrended Fluctuation Analysis



Detrended Fluctuation Analysis – Best Practices

- Plot your data
  - Visually inspect for:
    - ↑ or ↓ trends
    - Structure of roughness





- Time series length
  - More is almost always better
  - ≥ 512 datapoints for stable α estimate
  - i.e., 512 steps, taps, beats, etc…
  - Options exist for shorter time series

Add in plots that showcase these i

Detrended Fluctuation Analysis – Best Practices

- Plot your data
  - Visually inspect for:
    - ↑ or ↓ trends
    - Structure of roughness
- Time series length
  - More is almost always better
  - ≥ 512 datapoints for stable α estimate
  - i.e., 512 steps, taps, beats, etc…
  - ***Options exist for shorter time series***

Make dfa dots black

The Hurst-Kolmogorov (HK) Method

- The HK (Hurst-Kolmogorov) process is another method used to estimate the **Hurst exponent (H)** in time series data

How does the Hurst exponent relate to α?

- When H sits on the interval (0-1) ➔ it is equivalent to α
- When α = H = (0-0.5) ➔ reflects an antipersistent time series
- When α = H = (0.5-1.0) ➔ reflects a persistent time series
- If α is > 1, H = α - 1

Make last thing bullet point

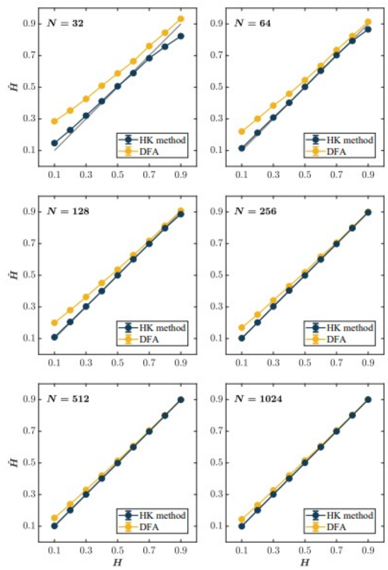
The Hurst-Kolmogorov (HK) Method

- The HK (Hurst-Kolmogorov) process is another method used to estimate the Hurst exponent (H) in time series data
- The HK method is particularly useful when dealing with short time series data
- See Likens et al. 2021 *Better than DFA?* for further details on calculation

HK Vs. DFA

Simulated Hurst Values

- When N = 64, the HK method closely approximates the a priori known values of H, while DFA consistently produces a positive bias in mean $\widehat{H}$ across the entire range of H.
- For N = 128 and higher, the mean $\widehat{H}$ values for the HK method are virtually indistinguishable from nominal values, while DFA remains positively biased.
- **Overall,** the same general trend is observed for all time series where the HK method outperforms DFA in accurately estimating H on known simulated data



Bring in fractal regression – use Aarons slides

Each panel plots the Mean estimated values of Hˆ for 1, 000 synthetic time series of length N = 32, 64, 128, 256, 512, 1024 with a priori known values of H. The grey line indicates the ideal case where the estimated value is the same as the actual value, i.e., Hˆ = H.

Thank you!

Questions?

CENTER FOR RESEARCH IN HUMAN MOVEMENT VARIABLITY
NONLINEAR ANALYSIS CORE

University of Nebraska Omaha