CWRU EMAE 485 – Lecture Notes

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Lecture 6: Control of Fully Actuated Mechanical Systems

Old School Robot Control

The general dynamics for a robot manipulator (and many mechanical systems) are:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{47}$$

Computed Torque

We define the control law as:

$$\tau = M(\ddot{q}_d - K_D\dot{e} - K_p e) + C\dot{q} + G \quad \text{where } e = q - q_d \tag{48}$$

Substituting this into the dynamics:

$$M(\ddot{e} + K_D\dot{e} + K_p e) = 0, \quad M > 0 \tag{49}$$

Defining the state $x = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}$, we get:

$$\dot{x} = \begin{bmatrix} 0 & I \\ -K_p & -K_D \end{bmatrix} x = Ax \tag{50}$$

Result: If A is Hurwitz, the system is **Globally Asymptotically Stable (G.A.S.)**.

Drawbacks: Requires a large τ. What happens if you have a bad model?

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PD+

A different control law:

$$\tau = M\ddot{q}_d + C\dot{q}_d + G - K_D\dot{e} - K_p e \tag{51}$$

This results in the error dynamics:

$$M\ddot{e} + C\dot{e} + K_D\dot{e} + K_p e = 0 \tag{52}$$

This is **nonlinear and time dependent** in e because $M(q) = M(e + q_d(t))$.

Lyapunov Candidate:

$$V = \frac{1}{2}\dot{e}^T M\dot{e} + \frac{1}{2}e^T K_p e \geq 0 \quad \checkmark \tag{53}$$

Derivative $\dot{V}$:

$$\dot{V} = \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + \dot{e}^T M\ddot{e} + \dot{e}^T K_p e \tag{54}$$

Substituting $M\ddot{e}$:

$$
\begin{aligned}
\dot{V} &= \frac{1}{2}\dot{e}^T\dot{M}\dot{e} - \dot{e}^T C\dot{e} - \dot{e}^T K_D\dot{e} - \dot{e}^T K_p e + \dot{e}^T K_p e \\
&= \frac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T K_D\dot{e} = -\dot{e}^T K_D\dot{e} \leq 0
\end{aligned}
$$

Note: Using the skew-symmetric property of $(\dot{M} - 2C)$, the first term is zero.

Result: Thus $\|e\| < \gamma_1$ and $\|\dot{e}\| < \gamma_2$. But the system is **time variant**.

Barbalat's Lemma

If $f(t)$ has a finite limit as $t \to \infty$ and $\dot{f}$ is uniformly continuous *(Kind of like Lipschitz; it is sufficient to show that $\ddot{f}$ is bounded).*, then $\dot{f}(t) \to 0$ as $t \to \infty$.

For Stability:

If $V(t, x)$ is lower bounded, $\dot{V}(t, x) \leq 0$, and $\ddot{V}$ is bounded, then $\dot{V} \to 0$ as $t \to \infty$. This allows us to essentially draw the same conclusions as Lasalle's Invariance principle.

Back to PD+:

$$
\begin{aligned}
\ddot{V} &= \frac{d}{dt}\left(-\dot{e}^T K_D\dot{e}\right) = -2\dot{e}^T K_D\ddot{e} \\
&= -2\dot{e}^T K_D M^{-1}(-C\dot{e} - K_D\dot{e} - K_p e) \\
\ddot{V} &= 2\dot{e}^T K_D M^{-1}(C\dot{e} + K_D\dot{e} + K_p e) \\
&\leq \gamma_{\dot{e}}\|K_D\|\gamma_{M^{-1}}(C\gamma_{\dot{e}} + \|K_p\|\gamma_e + \|K_D\|\gamma_{\dot{e}})
\end{aligned}
$$

As we see, we have already shown (through $\dot{V} \leq 0$ that the errors are bounded. From mechanics, we know that the physical parameters of our system are bounded (e.g. $C \leq (k_1 + k_2\gamma_e)\gamma_{\dot{e}}$). Thus, $\ddot{V}$ is bounded $\square$. Our system is G.A.S. (due to the radial unboundedness of V.