CWRU EMAE 485 – Lecture Notes

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Lecture 3: Lyapunov Stability

Existence and Uniqueness

Consider the system:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{18}$$

If not for existence and uniqueness, small differences in data will result in very different solutions. We use the **Lipschitz condition**:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \tag{19}$$

Theorem 3.1: Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition for all $x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ and $t \in [t_0, t_f]$. Then there exists some $\delta > 0$ such that the system has a unique solution over $[t_0, t_0 + \delta]$.

Local vs. Global: For a 1D case, the condition holds if:

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L \tag{20}$$

What does the left side of this function look like to you (hint: think back to your basic calculus). Thus, as it turns out, if the derivative is bounded, the function is Lipschitz.

For vector-valued functions, this implies the norm of the Jacobian is bounded: $\|J\| \leq L$. This is a stronger condition than continuity (it implies continuity).

xStability and Lyapunov Theory

Comparison Lemma (3.4)

If $\dot{v}(t) \leq f(t, v(t))$ and $\dot{u}(t) = f(t, u(t))$, then $v(t) \leq u(t)$. This allows us to use known functions to bound unknown ones.

Stability Definitions

An equilibrium point is **stable** (i.s.L. - in the sense of Lyapunov) if all solutions starting nearby stay nearby.

Lyapunov Stability: This is a **sufficient**, but not necessary, condition (a system may be stable even if we can't show stability i.s.L.). Essentially, we use it to show the boundedness of a solution, even without equilibrium points.

We will begin with analysis of Autonomous Systems: $\dot{x} = f(x)$, where f is a locally Lipschitz map $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\overline{x} = 0 \in D$ is an equilibrium point (we choose $\overline{x} = 0$ for convenience).

Definition 4.1:

- **Stable**: If for each $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ s.t. $\|x(0)\| < \delta \implies \|x(t)\| < \epsilon, \forall t > 0$.

- **Asymptotically Stable (A.S.)**: If it is stable and δ can be chosen s.t. $\|x(0)\| < \delta \implies \lim_{t\to\infty} x(t) = 0$.

Think about these definitions in the context of our pendulum example. Recall that the phase portrait took the form of a center or a stable focus, depending on the presence of damping. Which form of stability applies to which of these cases? What would δ be for this system?

Let's try to come up with an analytical way to say this.

Recall our simple pendulum dynamics:

$$ml^2\ddot{\theta} + mgl\sin\theta = \tau_p. \tag{21}$$

In state-space, let $x_1 = \theta, x_2 = \dot{\theta}$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - bx_2. \tag{22}$$

The energies are: KE: $T = \frac{1}{2}ml^2\dot{\theta}^2$; PE: $U = -mgl\cos\theta$.

Thus, energy with $E(0) = 0$ can be written as:

$$E(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2, \tag{23}$$

where $a = g/\ell$.

Let's see how energy changes with time.

$$\dot{E} = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = ax_2\sin x_1 + x_2(-a\sin x_1 - bx_2) = -bx_2^2 \tag{24}$$

- $b = 0$: E is conserved and thus we can infer that this is a closed orbit and $x = 0$ is **stable**.
- $b > 0$: With friction, E must eventually go to 0, which implies $x \to 0$ as $t \to \infty$.

Other functions can play the same role.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function defined in D, which contains the origin.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial f}{\partial t} = \begin{bmatrix}\frac{\partial V}{\partial x_1} \dots \frac{\partial V}{\partial x_n}\end{bmatrix}\begin{bmatrix}f_1(x)\\ \vdots \\ f_n(x)\end{bmatrix} = \frac{\partial V}{\partial x}f(x) \tag{25}$$

Note that if $\dot{V} < 0$, then V decreases along $x(t)$ (the solutions of the system).

Theorem 4.1

(1) Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$ in $D \subset \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that:

(2) $V(0) = 0$ and $V(x) > 0$ in $D \setminus \{0\}$ (**Positive Definite**)

(3) $\dot{V}(x) \leq 0$ in D (**Negative Semi-Definite**)

Then $x = 0$ is **stable**. Further, if $\dot{V}(x) < 0$ in $D \setminus \{0\}$, then $x = 0$ is **asymptotically stable**.

Proof of Theorem 4.1

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ s.t. $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq D$ (the "Ball" with radius r).

Let $\alpha = \min_{\|x\|=r} V(x)$ (this is the smallest value of V on the edge of the ball).

Then $\alpha > 0$ (by **(2)**). Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$.

Thus Ω_β is in the interior of B_r.

If it were not, it must have a point p on the boundary of B_r. This implies $V(p) \geq \alpha > \beta$, but for $x \in \Omega_\beta, V(x) \leq \beta$, a contradiction $\square$.

Any trajectory starting in Ω_β stays in $\Omega_\beta, \forall t \geq 0$ (from **(3)**):

$$\dot{V} \leq 0 \implies V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0 \tag{26}$$

(Because V does not increase).

Ω_β is a compact set (closed and bounded) and therefore we know $\dot{x} = f(x)$ has a unique solution (Lipschitz) $\forall t \geq 0$ when $x(0) \in \Omega_\beta$.

As V is continuous and $V(0) = 0, \exists \delta > 0$ s.t. $\|x\| \leq \delta \implies V < \beta$.

Then, $B_\delta \subset \Omega_\beta \subset B_r$ and:

$$x(0) \in B_\delta \implies x(0) \in \Omega_\beta \implies x(t) \in \Omega_\beta \implies x(t) \in B_r, \forall t \geq 0 \tag{27}$$

Therefore $\|x(0)\| < \delta \implies \|x(t)\| < r \leq \epsilon, \forall t \geq 0$.

This concludes the proof $\square$.

If $\dot{V} < 0$, then it is **Asymptotically Stable (A.S.)** (see Khalil for the proof).

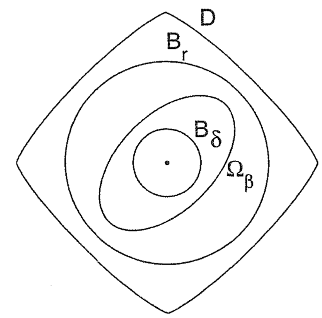

Figure 4: Figure 4.1 from Khalil's Nonlinear Systems. It is a geometric representation of the proof of Thm. 4.1.