CWRU EMAE 485 – Lecture Notes

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Lecture 1:

Feedback Control Loop

- **Plant/Environment**: The system being controlled.
- **Control Input/Action** (u/a): Sent to the Environment (Robot/Plant).
- **Output/Observations** (y/o): Measured from the Plant (e.g., sensors).
- **Controller/Policy** (π): The Controller determining u based on y.

Nonlinear System

The general form of a continuous-time nonlinear system is given by:

$$\dot{x} = f(t, x, u) \implies \begin{cases} \dot{x}_1 = f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\ \vdots \\ \dot{x}_n = f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \end{cases} \tag{1}$$

$$y = h(t, x, u) \tag{2}$$

Where:

- x: **state**
- u: **input**
- t: **time**
- y: **observation** (usually from sensors)

Special Cases

- **Linear**:
$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{3}$$
- **Unforced**: $u = \gamma(t, x) \implies \dot{x} = f(t, x)$
- **Autonomous**: $\dot{x} = f(x)$

Mechanical Systems

For mechanical systems (robots, planes, etc.), we follow Newton's second law ($F = ma$), which is second order:

$$\ddot{q} = f(q, \dot{q}, \tau) \quad (\tau = u) \tag{4}$$

To convert this to state-space form, let $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \bar{f}(x, \tau) \tag{5}$$

Control Affine Systems

Mechanical systems are almost always **control affine**:

$$\ddot{q} = f(q, \dot{q}) + g(q, \dot{q})\tau \tag{6}$$

Where $g \in \mathbb{R}^{n \times p}$.

Actuation Authority

1. **Fully Actuated**: If g is full row rank.

   $$\text{rank}(g) = n \implies p \geq n \tag{7}$$

   i.e., the system is **right invertible**.

2. **Underactuated**: If the condition above is not met. For humans, it may seem $p >> n$, but this is often not true due to constraints.

Feedback Linearization and Model Mismatch

Fully actuated makes things easy (but only without *model mismatch*). i.e., take $\ddot{q}_d$ and $u = g^{-1}(\ddot{q}_d - f(q, \dot{q}))$ (dropping the dependency of g on $q, \dot{q}$ for clarity.

Now:

$$\ddot{q} = f(q, \dot{q}) + gg^{-1}(\ddot{q}_d - f(q, \dot{q})) \tag{8}$$

Resulting in:

$$\ddot{q} = \ddot{q}_d \tag{9}$$

See demo code for an example of this.

Drawbacks: Not robust, input saturations, and constraints.

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Manipulator Equations

A common form for rigid body dynamics is given by:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = B\tau \tag{10}$$

Where:

- $M(q)$: **Inertia** matrix
- $C(q, \dot{q})$: **Coriolis** and centrifugal terms

- $G(q)$: **Gravity** vector
- B: Input mapping matrix (relates to g from before)

Additional terms can include $F_k(q)$, $D(q)\dot{q}$, F_{ext}, $J_c^T\lambda$.

In state-space form, where $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$:

$$\dot{x} = \begin{bmatrix} \dot{q} \\ M^{-1}(-C\dot{q} - G + B\tau) \end{bmatrix} \tag{11}$$