CWRU EMAE 485 – Lecture Notes

Instructor: Zach Patterson, Spring 2026

Lecture 5: Lyapunov Stability & Invariance

Invariance Implications

1. Gives us new ROA estimate (Ω) different from Ω_c.
2. Equilibrium set instead of equilibrium pt.
3. V does not have to be positive definite.

Example: Adaptive Control

Consider the system: $\dot{y} = ay + u$, with control $u = -ky$, and adaptation law $\dot{k} = \alpha y^2, \alpha > 0$.

Step 1. Define State Variables

Let the state variables be defined as:

$$\begin{aligned} x_1 = y &\implies \dot{x}_1 = ax_1 - x_2 x_1 \\ x_2 = k &\implies \dot{x}_2 = \alpha x_1^2 \end{aligned}$$

Step 2. Find Equilibrium

$\dot{x} = 0 \implies x_1 = 0$ (Note: this is a line in the state space, not a single point).

Step 3. Choose a candidate Lyapunov function (given in this case)

$$V = \frac{1}{2}x_1^2 + \frac{1}{2\alpha}(x_2 - b)^2, \quad b > a \tag{41}$$

Step 4. Take the time derivative of V

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + \frac{1}{\alpha}(x_2 - b)\dot{x}_2 \\ &= ax_1^2 - x_2x_1^2 + \frac{1}{\alpha}(x_2 - b)\alpha x_1^2 \\ &= ax_1^2 - x_2x_1^2 + x_2x_1^2 - bx_1^2 \\ &= -(b-a)x_1^2 \le 0 \end{aligned}$$

Step 5. If $\dot{V}$ is negative semi-definite, apply Invariance Theorem

$\dot{V} = 0$ iff $x_1 = 0$. *(Interesting that we didn't need to know b).*

Step 6. Determine region of attraction

$$\lim_{\|x\| \to \infty} V = \infty \implies \text{radially unbounded, G.A.S.} \tag{42}$$

How to find Lyapunov Functions?

1. Energy of physical systems
2. Quadratic functions of the state
3. Combinations of the above
4. “Variable Gradient Method” (see Khalil pg 120-121)
5. Sum of Squares Programming (see Tedrake chapter 9.2).

LTI Systems

$$\dot{x} = Ax \tag{43}$$

For linear systems, we can always find a quadratic Lyapunov function of the form:

$$V(x) = x^T P x \tag{44}$$

Where P is a symmetric positive definite matrix ($P = P^T > 0$).

The derivative is:

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P (Ax) \\ &= x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA) x \end{aligned}$$

To show that the system is stable, we want $\dot{V} = -x^T Q x$ where Q is a positive definite matrix ($Q > 0$).

This leads to the **Lyapunov Equation**:

$$A^T P + PA = -Q \tag{45}$$

Theorem 4.6

A matrix A is **Hurwitz** (all eigenvalues have negative real parts) if and only if for any $Q > 0$, there exists a unique $P > 0$ that satisfies the Lyapunov equation.

So, all we have to do is:

1. Pick a simple Q (like $Q = I$).
2. Solve the Lyapunov Equation for the elements of P.
3. Check if P is positive definite (e.g., using Sylvester’s Criterion).

Exponential Stability

$$\dot{V} \leq -\alpha V \implies V(x(t)) \leq V(x(0)) e^{-\alpha t} \tag{46}$$