CWRU EMAE 485 – Lecture Notes

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Lecture 3 and 4: Existence, Uniqueness, and Stability

Existence and Uniqueness

Consider the system:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{18}$$

If not for existence and uniqueness, small differences in data will result in very different solutions. We use the **Lipschitz condition**:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \tag{19}$$

Theorem 3.1: Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition for all $x, y \in B = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ and $t \in [t_0, t_f]$. Then there exists some $\delta > 0$ such that the system has a unique solution over $[t_0, t_0 + \delta]$.

Local vs. Global: For a 1D case, the condition holds if:

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L \tag{20}$$

What does the left side of this function look like to you (hint: think back to your basic calculus). Thus, as it turns out, if the derivative is bounded, the function is Lipschitz.

For vector-valued functions, this implies the norm of the Jacobian is bounded: $\|J\| \leq L$. This is a stronger condition than continuity (it implies continuity).

xStability and Lyapunov Theory

Comparison Lemma (3.4)

If $\dot{v}(t) \leq f(t, v(t))$ and $\dot{u}(t) = f(t, u(t))$, then $v(t) \leq u(t)$. This allows us to use known functions to bound unknown ones.

Stability Definitions

An equilibrium point is **stable** (i.s.L. - in the sense of Lyapunov) if all solutions starting nearby stay nearby.

Lyapunov Stability: This is a **sufficient**, but not necessary, condition (a system may be stable even if we can't show stability i.s.L.). Essentially, we use it to show the boundedness of a solution, even without equilibrium points.

We will begin with analysis of Autonomous Systems: $\dot{x} = f(x)$, where f is a locally Lipschitz map $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ and $\overline{x} = 0 \in D$ is an equilibrium point (we choose $\overline{x} = 0$ for convenience).

Definition 4.1:

- **Stable**: If for each $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ s.t. $\|x(0)\| < \delta \implies \|x(t)\| < \epsilon, \forall t > 0$.

- **Asymptotically Stable (A.S.)**: If it is stable and δ can be chosen s.t. $\|x(0)\| < \delta \implies \lim_{t\to\infty} x(t) = 0$.

Think about these definitions in the context of our pendulum example. Recall that the phase portrait took the form of a center or a stable focus, depending on the presence of damping. Which form of stability applies to which of these cases? What would δ be for this system?

Let's try to come up with an analytical way to say this.

Recall our simple pendulum dynamics:

$$ml^2\ddot{\theta} + mgl\sin\theta = \tau_p. \tag{21}$$

In state-space, let $x_1 = \theta, x_2 = \dot{\theta}$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 - bx_2. \tag{22}$$

The energies are: KE: $T = \frac{1}{2}ml^2\dot{\theta}^2$; PE: $U = -mgl\cos\theta$.

Thus, energy with $E(0) = 0$ can be written as:

$$E(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2, \tag{23}$$

where $a = g/\ell$.

Let's see how energy changes with time.

$$\dot{E} = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = ax_2\sin x_1 + x_2(-a\sin x_1 - bx_2) = -bx_2^2 \tag{24}$$

- $b = 0$: E is conserved and thus we can infer that this is a closed orbit and $x = 0$ is **stable**.
- $b > 0$: With friction, E must eventually go to 0, which implies $x \to 0$ as $t \to \infty$.

Other functions can play the same role.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function defined in D, which contains the origin.

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial f}{\partial t} = \begin{bmatrix}\frac{\partial V}{\partial x_1} \dots \frac{\partial V}{\partial x_n}\end{bmatrix}\begin{bmatrix}f_1(x)\\ \vdots \\ f_n(x)\end{bmatrix} = \frac{\partial V}{\partial x}f(x) \tag{25}$$

Note that if $\dot{V} < 0$, then V decreases along $x(t)$ (the solutions of the system).

Theorem 4.1

(1) Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$ in $D \subset \mathbb{R}^n$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that:

(2) $V(0) = 0$ and $V(x) > 0$ in $D \setminus \{0\}$ (**Positive Definite**)

(3) $\dot{V}(x) \leq 0$ in D (**Negative Semi-Definite**)

Then $x = 0$ is **stable**. Further, if $\dot{V}(x) < 0$ in $D \setminus \{0\}$, then $x = 0$ is **asymptotically stable**.

Proof of Theorem 4.1

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ s.t. $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subseteq D$ (the "Ball" with radius r).

Let $\alpha = \min_{\|x\|=r} V(x)$ (this is the smallest value of V on the edge of the ball).

Then $\alpha > 0$ (by **(2)**). Take $\beta \in (0, \alpha)$ and let $\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$.

Thus Ω_β is in the interior of B_r.

If it were not, it must have a point p on the boundary of B_r. This implies $V(p) \geq \alpha > \beta$, but for $x \in \Omega_\beta, V(x) \leq \beta$, a contradiction $\square$.

Any trajectory starting in Ω_β stays in $\Omega_\beta, \forall t \geq 0$ (from **(3)**):

$$\dot{V} \leq 0 \implies V(x(t)) \leq V(x(0)) \leq \beta, \forall t \geq 0 \tag{26}$$

(Because V does not increase).

Ω_β is a compact set (closed and bounded) and therefore we know $\dot{x} = f(x)$ has a unique solution (Lipschitz) $\forall t \geq 0$ when $x(0) \in \Omega_\beta$.

As V is continuous and $V(0) = 0, \exists \delta > 0$ s.t. $\|x\| \leq \delta \implies V < \beta$.

Then, $B_\delta \subset \Omega_\beta \subset B_r$ and:

$$x(0) \in B_\delta \implies x(0) \in \Omega_\beta \implies x(t) \in \Omega_\beta \implies x(t) \in B_r, \forall t \geq 0 \tag{27}$$

Therefore $\|x(0)\| < \delta \implies \|x(t)\| < r \leq \epsilon, \forall t \geq 0$.

This concludes the proof $\square$.

If $\dot{V} < 0$, then it is **Asymptotically Stable (A.S.)** (see Khalil for the proof).

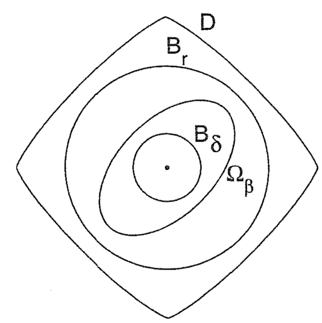

Figure 4: Figure 4.1 from Khalil's Nonlinear Systems. It is a geometric representation of the proof of Thm. 4.1.

Limitations of Lyapunov

Lyapunov functions are almost like magic, but they are hard to find. For electrical and mechanical systems, energy usually works.

Example: Pendulum

Consider the pendulum with:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 \tag{28}$$

Using energy as a candidate Lyapunov function V:

$$V = E = a(1 - \cos x_1) + \frac{1}{2}x_2^2, \quad V(0) = 0 \tag{29}$$

Checking the derivative $\dot{V}$:

$$V \geq 0, \quad \dot{V} = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0 \leq 0 \tag{30}$$

Thus, the system is **stable** (but not convergent).

With Damping

If we add damping, $\dot{x}_2 = -a \sin x_1 - bx_2$. Then:

$$\dot{V} = -bx_2^2 \leq 0 \text{ but not } < 0 \tag{31}$$

But we know this is A.S. (symptotically stable). What's up?

Let's try a new V

Consider a candidate of the form:

$$V = \frac{1}{2}x^T P x + a(1 - \cos x_1) \tag{32}$$

$$\dot{V} = x^T P\dot{x} + ax_2 \sin x_1 \tag{33}$$

Now we can choose elements of P s.t. $V \geq 0$ and $\dot{V} < 0$.

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \implies p_{11} > 0, p_{11}p_{22} - p_{12}^2 > 0 \tag{34}$$

Full example on pg 119-120. This illustrates **sufficiency**.

Global Asymptotic Stability (G.A.S.)

Region of Attraction

If we have a Lyapunov surface defined as $V(x) = c, c > 0$, and if $\dot{V} \leq 0$, this implies that when a trajectory crosses this surface, it enters and cannot leave the set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$. If $\dot{V} < 0$, the it eventually converges to the origin. In this case, the **region of attraction** (ROA) is Ω_c, which is a conservative estimate.

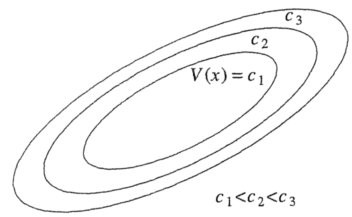

Figure 5: Figure 4.2 from Khalil's Nonlinear Systems. It shows the "level surfaces" of a Lyapunov function.

How about global?

If any possible $x \in \mathbb{R}^n$ is in Ω_c then $D = \mathbb{R}^n$ but this isn't quite enough.

Example:

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2. \tag{35}$$

- For small c, Ω_c is closed and bounded.
- For large c, it becomes unbounded.

For Ω_c to be in B_r, we must have $c < \inf_{\|x\| \geq r} V(x)$.

If $l = \lim_{r\to\infty} \inf_{\|x\|\geq r} V(x) < \infty$, then Ω_c is bounded for $c < l$.

In the example:

$$l = \lim_{r\to\infty} \min_{\|x\|=r} \left[\frac{x_1^2}{1 + x_1^2} + x_2^2 \right] = \lim_{|x_1|\to\infty} \frac{x_1^2}{1 + x_1^2} = 1 \tag{36}$$

So the region is bounded for $c < 1$.

Thus, we require that $V(x) \to \infty$ as $\|x\| \to \infty$. A function satisfying this condition is **radially unbounded**.

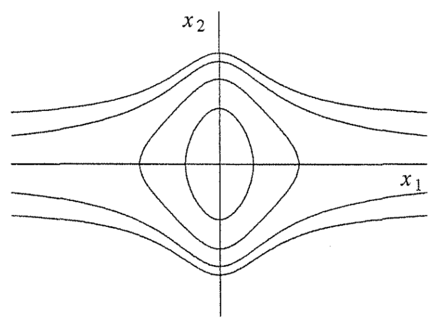

Figure 6: Figure 4.3 from Khalil's Nonlinear Systems showing the level surfaces of (35).