CWRU EMAE 485 – Lecture Notes

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Lecture 7: Underactuated systems and PFL

Underactuated Systems

Acrobot & Cartpole

The general dynamics are given by:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu \tag{55}$$

Where for the Cartpole:

- $M = \begin{bmatrix} m_c + m_p & m_p l \cos\theta \\ m_p l \cos\theta & m_p l^2 \end{bmatrix}$
- $C = \begin{bmatrix} 0 & -m_p l \dot{\theta} \sin\theta \\ 0 & 0 \end{bmatrix}$
- $G = \begin{bmatrix} 0 \\ m_p g l \sin\theta \end{bmatrix}$
- $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Convert to state space representation:

$$x = [q, \dot{q}]^T \tag{56}$$

$$\dot{x} = [\dot{q}, \ddot{q}]^T = \begin{bmatrix} \dot{q} \\ M^{-1}(Bu - C\dot{q} - G) \end{bmatrix} \tag{57}$$

Linearize about x^*, u^*

Using the Taylor series expansion:

$$\dot{x} = f(x,u) \approx f(x^*, u^*) + \left.\frac{\partial f}{\partial x}\right|_{x=x^*, u=u^*} (x - x^*) + \left.\frac{\partial f}{\partial u}\right|_{x=x^*, u=u^*} (u - u^*) \tag{58}$$

Manipulating about a fixed point where $f(x^*, u^*) = 0$:

$$\dot{\bar{x}} \approx A\bar{x} + B\bar{u}, \quad \bar{x} = x - x^* \tag{59}$$

$$A = \left.\begin{bmatrix} 0 & I \\ -M^{-1}\frac{\partial G}{\partial q} + \sum_j M^{-1}\frac{\partial B_j}{\partial q} u_j & 0 \end{bmatrix}\right|_{x=x^*, u=u^*} \tag{60}$$

$$B = \left.\begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}\right|_{x=x^*, u=u^*} \tag{61}$$

PFL: Partial Feedback Linearization

Cartpole w/ constants = 1

The simplified equations of motion are:

$$\ddot{\theta} = -\ddot{x}c - s \tag{62}$$

$$\ddot{x}(2 - c^2) - sc - \dot{\theta}^2 s = f \tag{63}$$

If we set the control input $f = (2 - c^2)\ddot{x}_d - sc - \dot{\theta}^2 s$, then:

$$\ddot{x} = \ddot{x}_d \tag{64}$$

$$\ddot{\theta} = -\ddot{x}_d c - s \tag{65}$$

This allows us to make the cart do what we want.

Now, focusing on the pole:

We can rearrange for $\ddot{x}$:

$$\ddot{x} = -\frac{\ddot{\theta} + s}{c} \tag{66}$$

Substituting this into the force equation:

$$-\frac{\ddot{\theta} + s}{c}(2 - c^2) - sc - \dot{\theta}^2 s = f \tag{67}$$

$$\ddot{\theta}(c - \frac{2}{c}) - 2\tan\theta - \dot{\theta}^2 s = f \quad (\text{assuming } \cos\theta \neq 0) \tag{68}$$

If we choose $f = (c - \frac{2}{c})\ddot{\theta}_d - 2\tan\theta - \dot{\theta}^2 s$, then:

$$\ddot{\theta} = \ddot{\theta}_d \tag{69}$$

$$\ddot{x} = -\frac{1}{c}\ddot{\theta}_d - \tan\theta \tag{70}$$

Notice, we are using this control input in HW1 for the swing-up problem.

General Form: There is a general form of this procedure for systems where some Degrees of Freedom (DOFs) are directly actuated and some are not (see equations 34-37 in Chapter 2 of Russ Tedrake's Underactuated Robotics Notes).

$$M_{aa}\ddot{q}_a + M_{au}\ddot{q}_u + \tau_a = u \tag{71}$$

$$M_{uu}\ddot{q}_u + M_{ua}\ddot{q}_a + \tau_u = 0 \tag{72}$$

We can use PFL if our system can be written in the form above, which is called a collocated form, where for states $q = [q_a, q_u]^T \in R^n$ and inputs $u \in R^m$, we have m equations that contain those inputs and $n - m$ that do not have any inputs. The terminology comes from the idea that the inputs are "co-located" with the actuated DOFs.