CWRU EMAE 485 – Lecture Notes

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Lecture 4: Lyapunov Stability

Limitations of Lyapunov

Lyapunov functions are almost like magic, but they are hard to find. For electrical and mechanical systems, energy usually works.

Example: Pendulum

Consider the pendulum with:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 \tag{28}$$

Using energy as a candidate Lyapunov function V:

$$V = E = a(1 - \cos x_1) + \frac{1}{2}x_2^2, \quad V(0) = 0 \tag{29}$$

Checking the derivative $\dot{V}$:

$$V \geq 0, \quad \dot{V} = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = ax_2 \sin x_1 - ax_2 \sin x_1 = 0 \leq 0 \tag{30}$$

Thus, the system is **stable** (but not convergent).

With Damping

If we add damping, $\dot{x}_2 = -a \sin x_1 - bx_2$. Then:

$$\dot{V} = -bx_2^2 \leq 0 \text{ but not } < 0 \tag{31}$$

But we know this is A.S. (symptotically stable). What's up?

Let's try a new V

Consider a candidate of the form:

$$V = \frac{1}{2}x^T P x + a(1 - \cos x_1) \tag{32}$$

$$\dot{V} = x^T P \dot{x} + ax_2 \sin x_1 \tag{33}$$

Now we can choose elements of P s.t. $V \geq 0$ and $\dot{V} < 0$.

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \implies p_{11} > 0, p_{11}p_{22} - p_{12}^2 > 0 \tag{34}$$

Full example on pg 119-120. This illustrates **sufficiency**.

Global Asymptotic Stability (G.A.S.)

Region of Attraction

If we have a Lyapunov surface defined as $V(x) = c, c > 0$, and if $\dot{V} \leq 0$, this implies that when a trajectory crosses this surface, it enters and cannot leave the set $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$. If $\dot{V} < 0$, the it eventually converges to the origin. In this case, the **region of attraction** (ROA) is Ω_c, which is a conservative estimate.

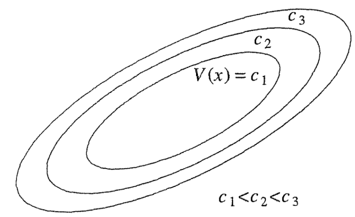

Figure 5: Figure 4.2 from Khalil's Nonlinear Systems. It shows the "level surfaces" of a Lyapunov function.

How about global?

If any possible $x \in \mathbb{R}^n$ is in Ω_c then $D = \mathbb{R}^n$ but this isn't quite enough. Recall from our proof of Theorem 4.1 that Ω_β needed to be bounded. Thus, any Ω_c must be bounded to verify Thm 4.1.

Example:

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2. \tag{35}$$

- For small c, Ω_c is closed and bounded (see the plot).
- For large c, it becomes unbounded.

We need Ω_c to be bounded and in the interior of B_r for Thm 4.1, so we must have $c < \inf_{\|x\| \geq r} V(x)$.
(c must be less than the lowest bound of V(x) for which the magnitude of x is greater than or equal to r).

If $l = \lim_{r\to\infty} \inf_{\|x\| \geq r} V(x) < \infty$, then Ω_c is bounded for $c < l$. Thus, if we can find this limit and it is finite, Ω_c becomes unbounded for $c \geq \ell$, and thus we cannot show G.A.S.

In the example:

$$l = \lim_{r\to\infty} \min_{\|x\|=r} \left[\frac{x_1^2}{1 + x_1^2} + x_2^2\right] = \lim_{|x_1|\to\infty} \frac{x_1^2}{1 + x_1^2} = 1 \tag{36}$$

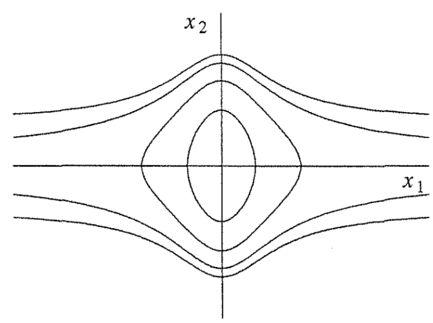

Figure 6: Figure 4.3 from Khalil's Nonlinear Systems showing the level surfaces of (35).

So the region is bounded for $c < 1$, which means that Ω_c becomes unbounded for higher values.

To ensure G.A.S, we thus require an additional condition to ensure that Ω_c remains bounded. We can ensure this by showing that $V(x) \to \infty$ as $\|x\| \to \infty$. A function satisfying this condition is **radially unbounded**.

Invariance

Recall our pendulum example

$$\dot{x}_1 = x_2 \tag{37}$$
$$\dot{x}_2 = -a \sin x_1 - b x_2 \tag{38}$$

with $V = E = \frac{1}{2}x_2^2 + a(1 - \cos x_1)$. With damping, $\dot{V} = -bx_2^2 \leq 0$, so it is negative semi-definite.

But, $\dot{V} < 0$ everywhere except $x_2 = 0$. And we can see that if we stay at $x_2 = 0$:

$$\dot{V} \implies x_2 = 0 \implies \dot{x}_2 = 0 \implies \sin x_1 = 0 \tag{39}$$

For $-\pi < x_1 < \pi$, this means $x_1 = 0$. And thus $x(t) \to 0$ as $t \to \infty$.

Thus, if $\dot{V} \leq 0$ in D, and if we can show that no trajectory stays at points s.t. $\dot{V} = 0$ except at $x = 0$, then A.S. (Asymptotically Stable).

Lasalle's Invariance Principle

Thm 4.4: Let $\Omega \subset D$ be a compact set, positively invariant with respect to $\dot{x} = f(x)$.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function s.t. $\dot{V} \leq 0$ in Ω.

Let $E = \{x \in \Omega \mid \dot{V} = 0\}$.

Let M be the largest invariant set in E. (M is an **invariant set** if $x(0) \in M \implies x(t) \in M, \forall t \in \mathbb{R}$.)

Then $x(0) \in \Omega \implies x(t)$ approaches M as $t \to \infty$.

Proof in Khalil.

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A simpler definition:

Given a system $\dot{\mathbf{x}} = f(\mathbf{x})$ with f continuous. If we can produce a scalar function $V(\mathbf{x})$ with continuous derivatives for which we have:

$$V(\mathbf{x}) > 0, \quad \dot{V}(\mathbf{x}) \leq 0, \tag{40}$$

and $V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$, then $\mathbf{x}$ will converge to the largest *invariant set* where $\dot{V}(\mathbf{x}) = 0$.

Corollary 4.1: Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$.

Let $V : D \to \mathbb{R}$ be a positive definite continuous differentiable function, s.t.

- $\{x = 0\} \subset D$.
- $\dot{V} \leq 0$.

Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and no solution can stay in S except $x(t) = 0$.

Then A.S. (Asymptotically Stable).

If $D = \mathbb{R}^n$, G.A.S. (Globally Asymptotically Stable).