CWRU EMAE 485 – Lecture Notes

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Lecture 2:

First Order Nonlinear Systems

Although first order systems won't be the focus of the class, they are helpful for building intuition and terminology. Example from Strogatz Chapter 2:

$$\dot{x} = \sin(x). \tag{12}$$

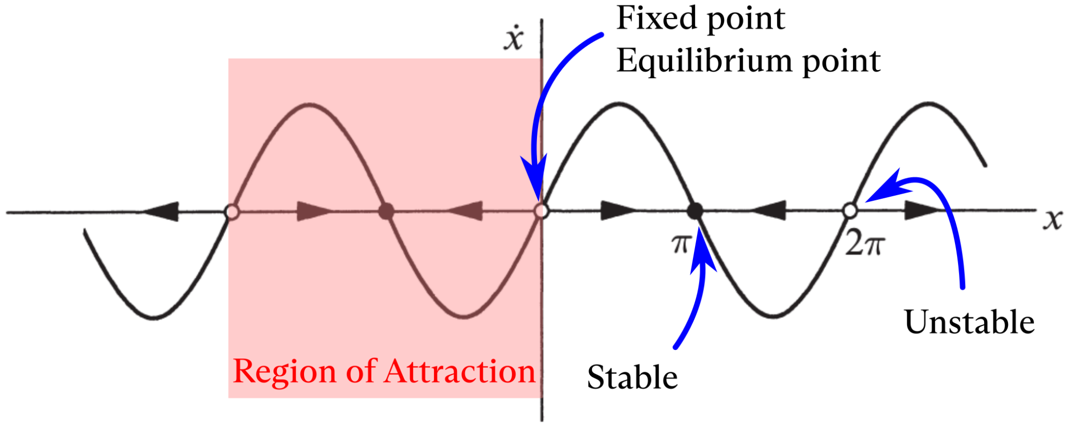

Figure 1: Dynamics of $\dot{x} = \sin(x)$. Adapted from Strogatz.

Second Order Nonlinear Systems

See demo code for an interactive version of these notes.

For a simple pendulum, we define Kinetic Energy (T) and Potential Energy (U):

- $T = \frac{1}{2}ml^2\dot{\theta}^2$
- $U = -mgl\cos\theta$
- Lagrangian: $L = T - U$

Using the Euler-Lagrange equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau_p$:

$$ml^2\ddot{\theta} + mgl\sin\theta = \tau_p \tag{13}$$

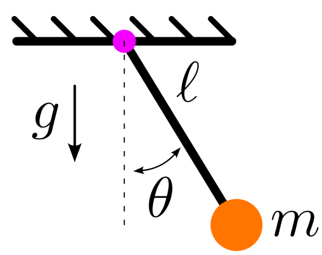

Figure 2: Pendulum system.

In state-space, let $x_1 = \theta, x_2 = \dot{\theta}$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{g}{l}\sin x_1 \tag{14}$$

There is no solution to this in elementary functions.

Phase Portraits

We can visualize behavior on the $x_1 - x_2$ plane by plotting the **vector field**:

$$\frac{dx_2}{dx_1} = \frac{-(g/l)\sin x_1}{x_2} \tag{15}$$

Any solution $x(t)$ starting at x_0 is a **trajectory**.

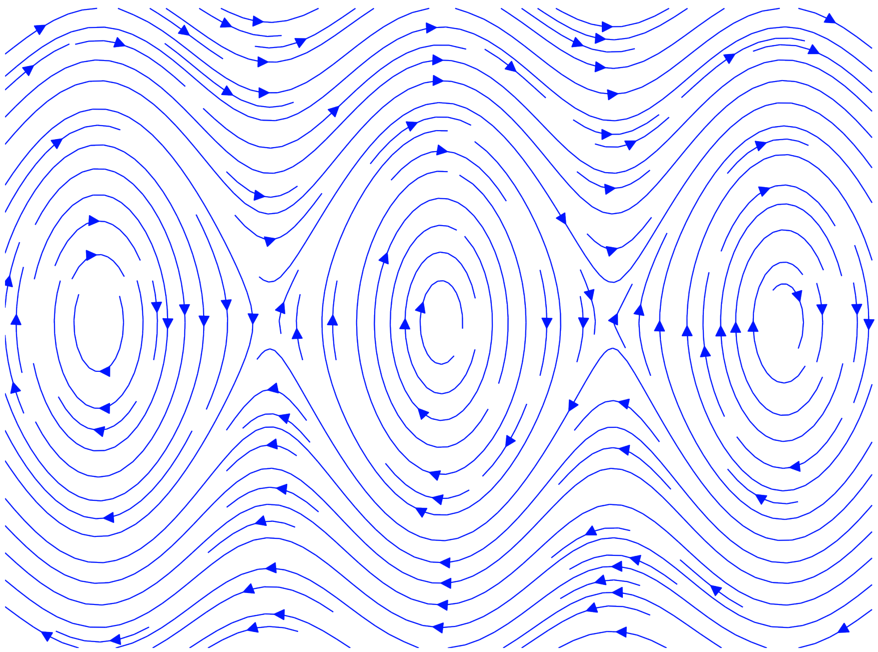

Figure 3: Phase portrait of the pendulum dynamics.

That's about as far as we can take graphical analysis. To get further, we need to proceed to mathematical analysis.

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Linearization

For an equilibrium point p, we use a Taylor series expansion:

$$\dot{x} = f(p) + \left.\frac{\partial f}{\partial x}\right|_{x=p} (x - p) + \text{H.O.T.} \tag{16}$$

The matrix $J = \left.\frac{\partial f}{\partial x}\right|_{x=p}$ is called the **Jacobian**:

$$J = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} \tag{17}$$

Near p, we approximate the system as $\dot{x} \approx Ax$ where $A = J$. This tells us about the stability of a fixed point based on **eigenvalues** (λ):

- $\lambda_1 \neq \lambda_2 \in \mathbb{R} < 0 \implies$ **stable node**
- $\lambda_1, \lambda_2 > 0 \implies$ **unstable node**
- $\lambda_1 > 0, \lambda_2 < 0 \implies$ **saddle**
- $\lambda_i = 0 \implies$ **inconclusive**