



Tree 1

Kiatnarong Tongprasert

1. Tree Definitions

2. Binary Tree

- Traversals
- Binary Search Tree
- Representations
- Application : Expression Tree



3. AVL Tree

4. Which Representations ?

5. n-ary Tree

6. Generic Tree

7. Multiway Search Tree

8. B-Trees

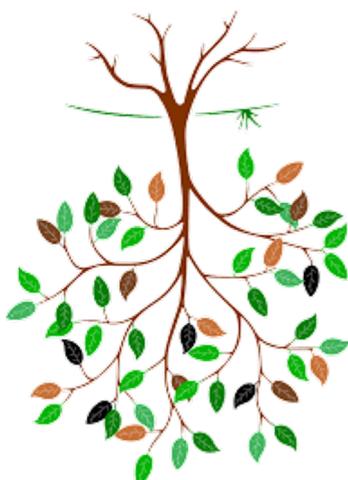
Tree Definitions

A **tree** คือ

- empty ไม่มี nodes เรียก **null tree / empty tree**
- ประกอบด้วย
 - root node**
 - ≥ 0 subtrees**

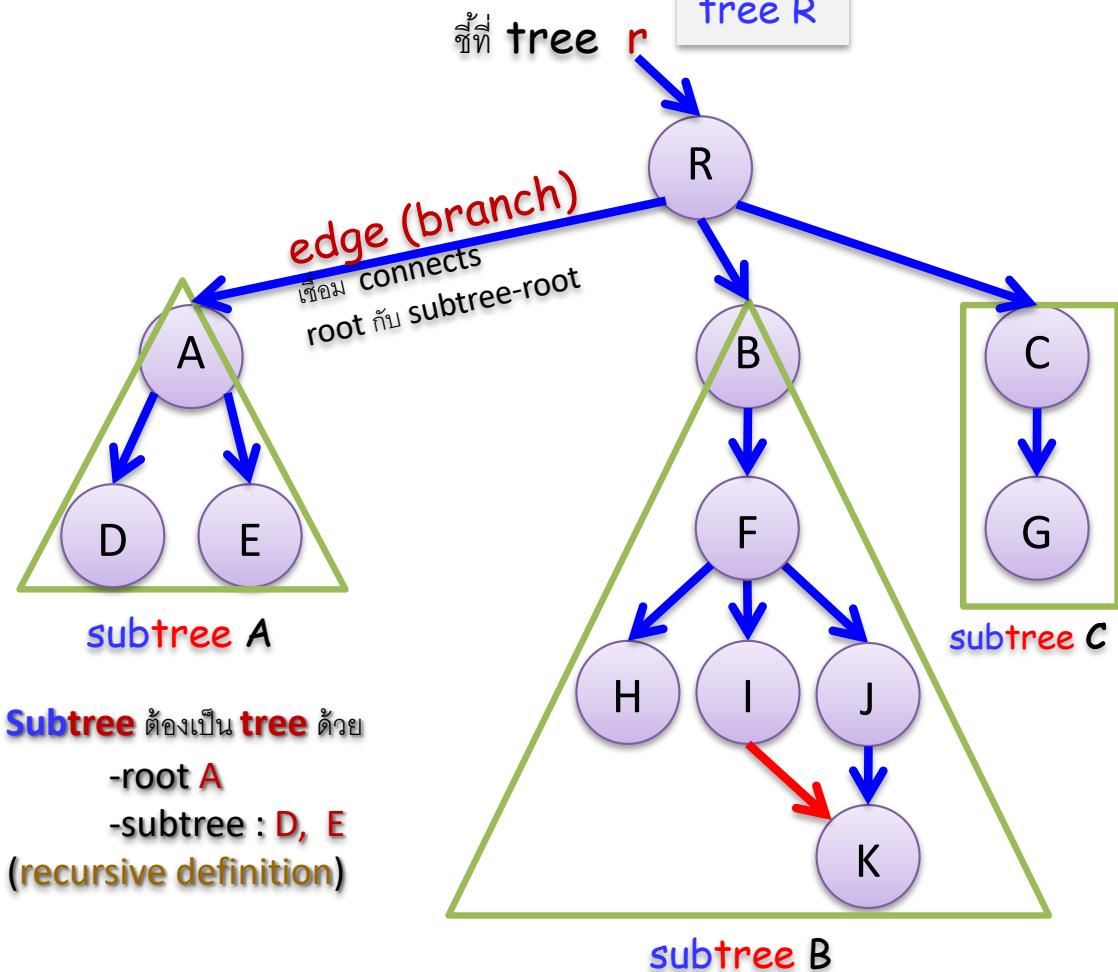
Node ใน tree ต้อง **disjoint** กัน

ต้องไม่มี node ร่วมกันใน root หรือ ใน subtrees



ชื่อของ tree นิยมเรียกตาม **root**

tree R



K ไม่ **disjoint** (อยู่ในทั้ง subtree I & subtree J) \rightarrow R ไม่ใช่ **tree**
ใน **tree : branches** ไม่เชื่อมเป็นวง

Tree Definitions

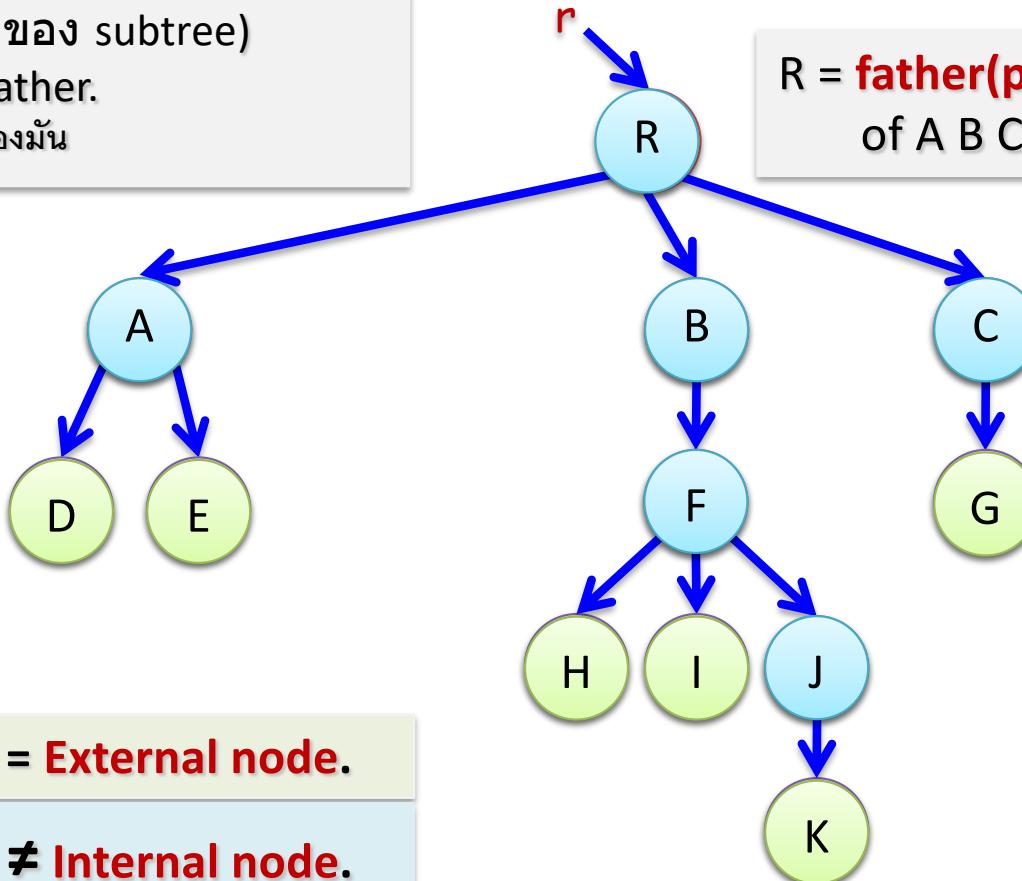
Root = **father (parent)** of subtree's root.
= **father** ของ root ของ subtree

Subtree's root (root ของ subtree)
= **son (child)** of his father.
= **son (child)** ของพ่อของมัน

Root ไม่มีพ่อ

R = **father(parent)**
of A B C

A B C = **sons(childs)**
of R



Leaf (leave) = External node.

Leaf (leave) ≠ Internal node.

Leaf (leave)
= node with no son.
= node ที่ไม่มีลูก



Father of F ?
Sons of A ?
Father of R ?
Sons of H ?

	Leaf	Internal
Siblings	Grand Parent	Grand Child

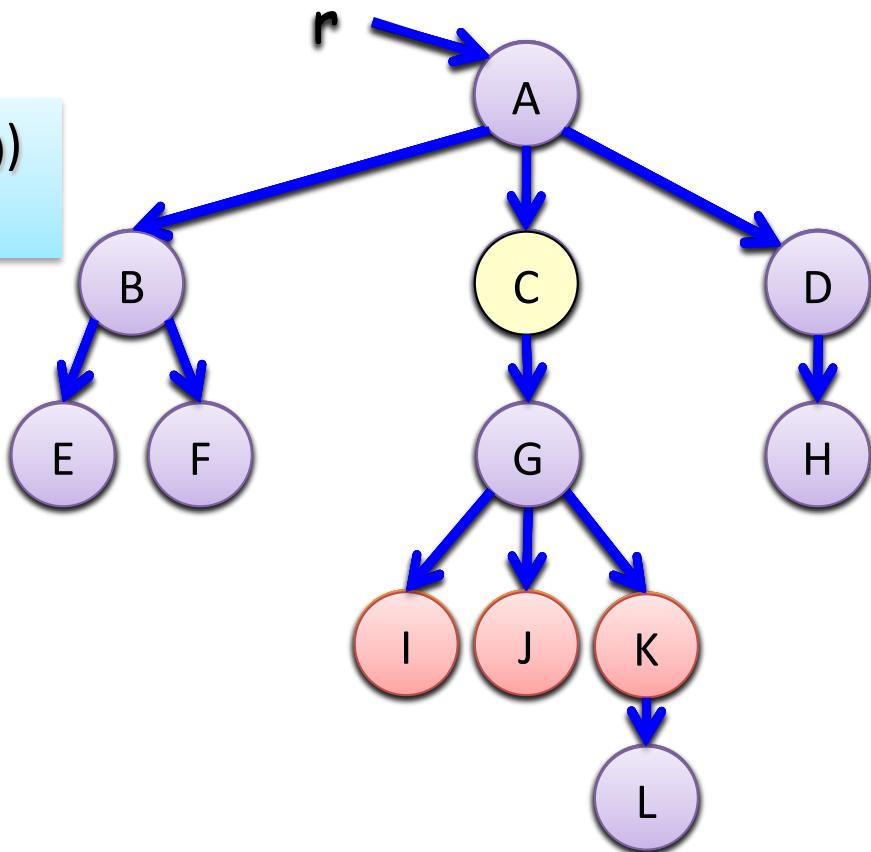
Leaf node (external, outer, terminal)
node with no son

Branch node (internal, inner, inode (for short))
node \neq leaf

Siblings (brothers)
node with same father

Grand Parent
father of father

Grand Child
son of son



Path, Path Length, Depth, Height

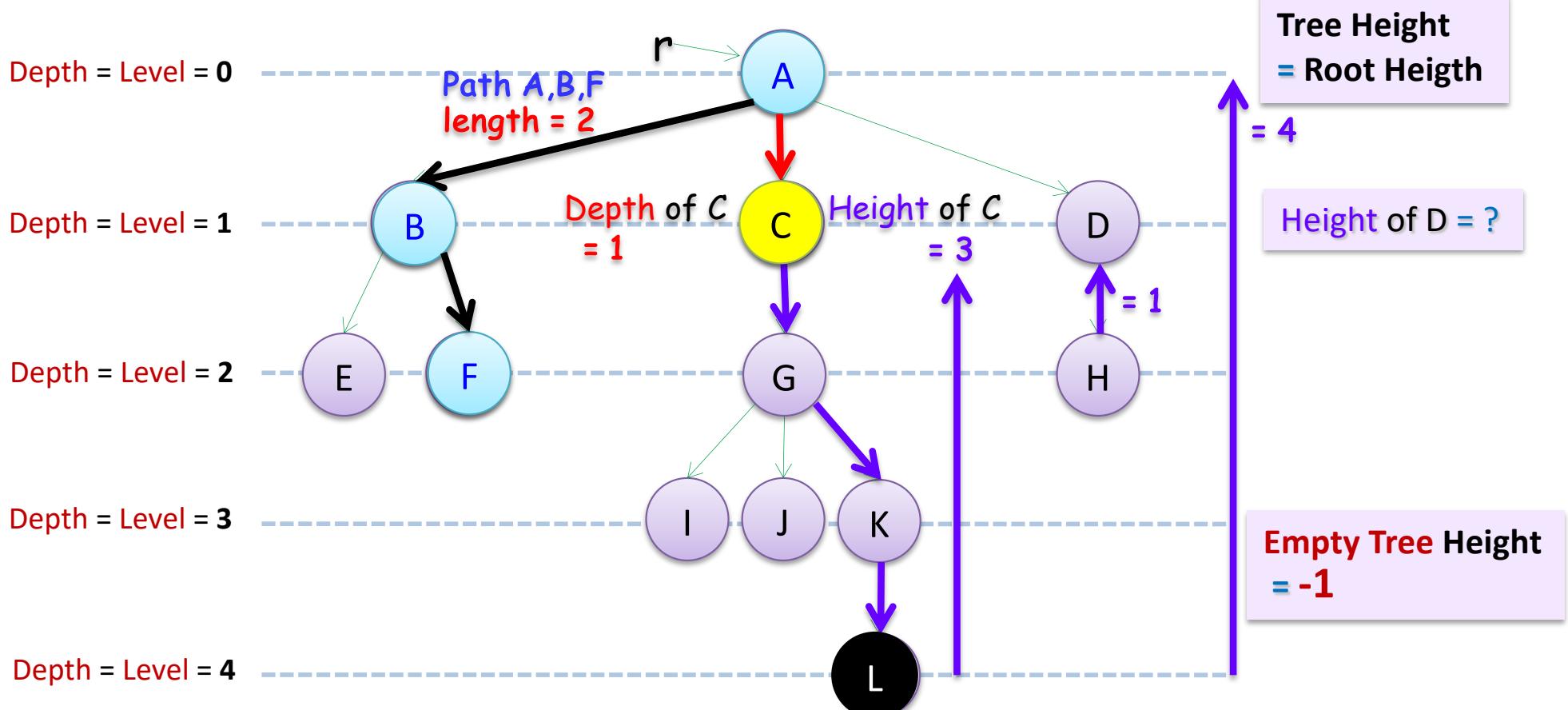
Path (from n to d) sequence of nodes and edges connecting a node n with a descendant d

In tree, only 1 path from node to node

Depth (level) of node = path length from root to node

Height of node = longest path length from node to leaf

Path length = # edge in path



Ancestor & Descendent

Ancestor បរាបុគ្នមេ

father of ancestor

A = ancestor of D

if has path from A to D

Decendent តួកអាណាព

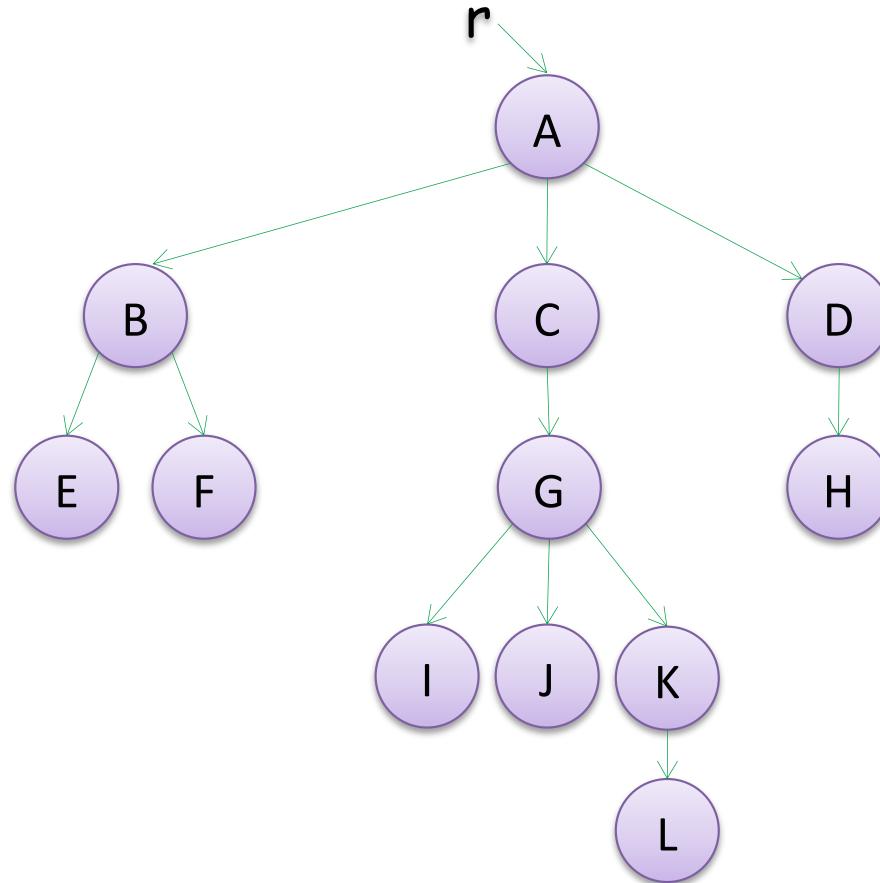
son of decendent

A = Proper Ancestor of D

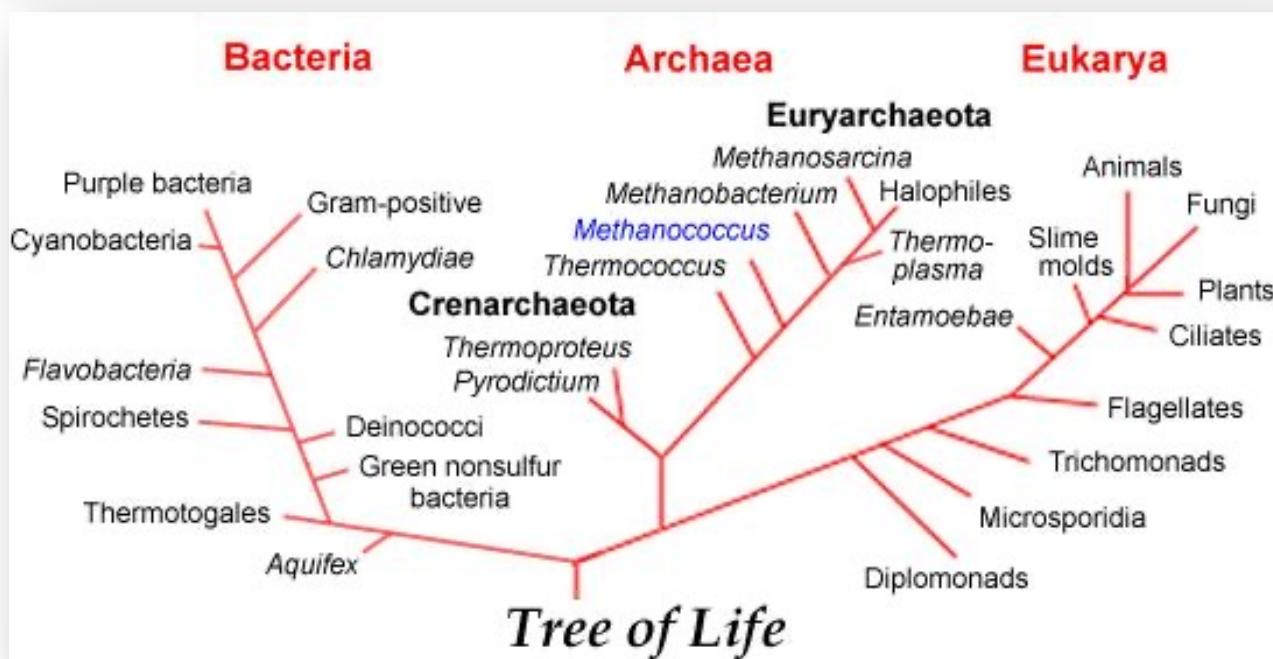
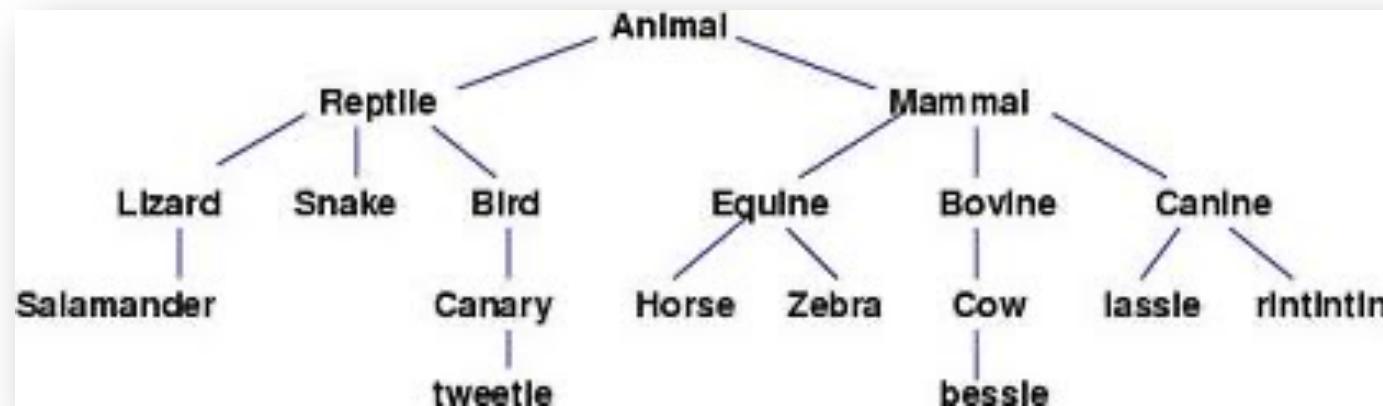
if $A \neq D$

D = Proper Decendent of A

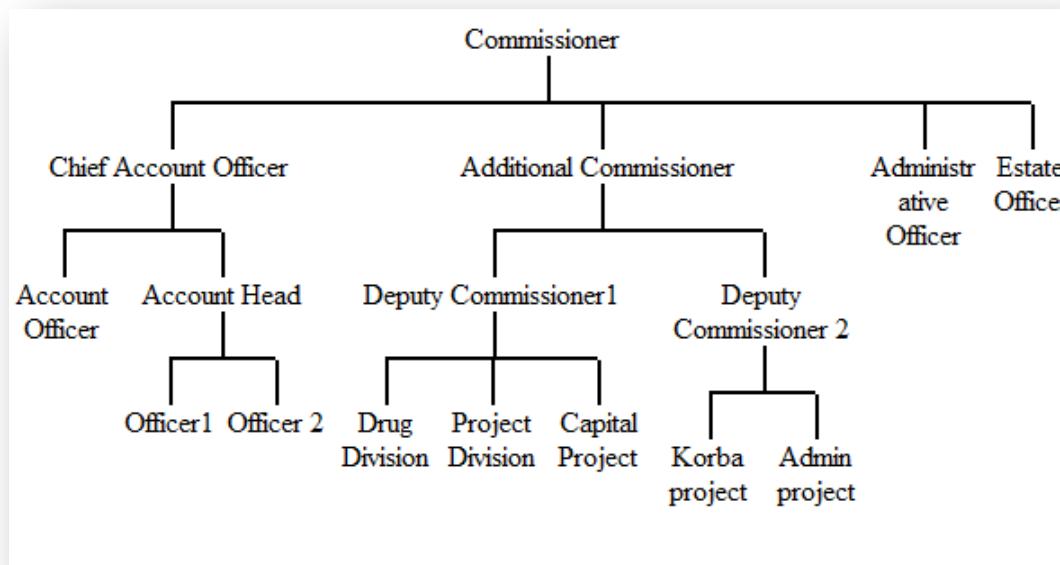
if $D \neq A$



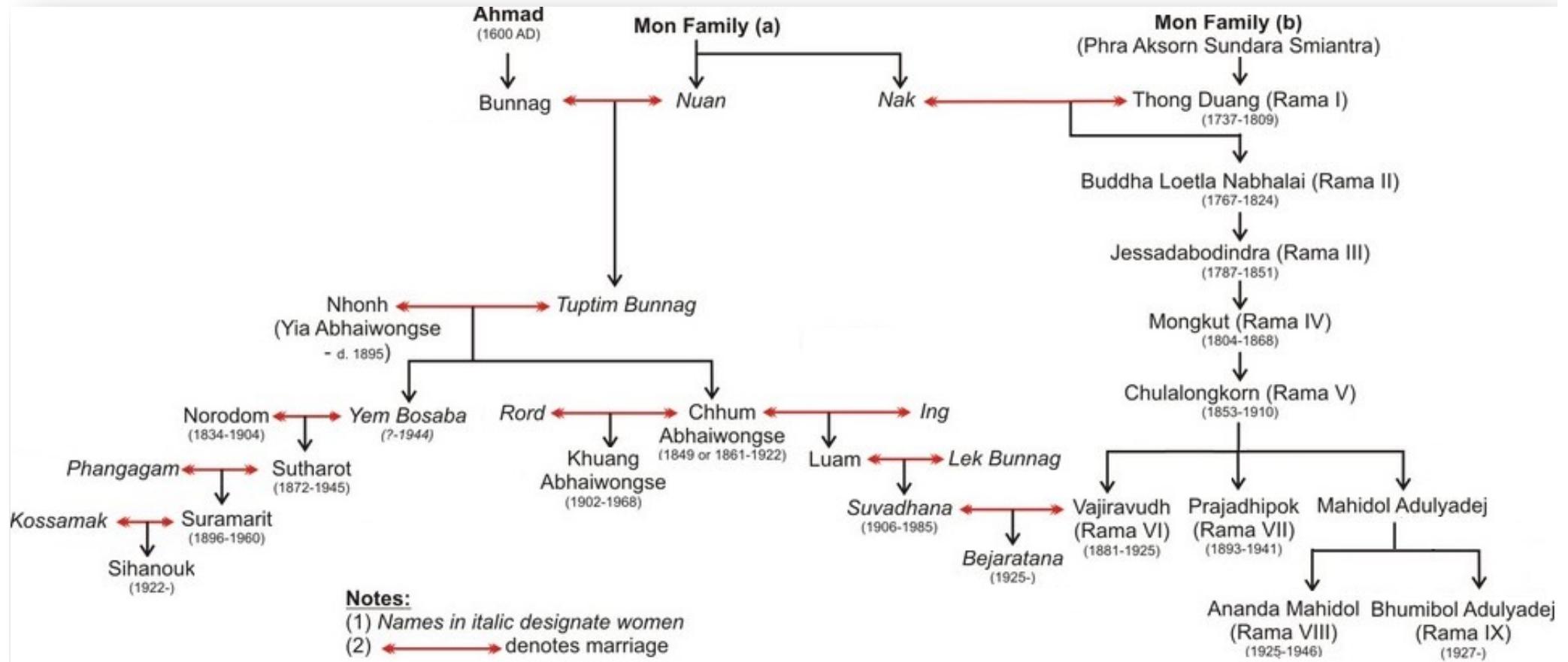
Tree Examples



Tree Examples



Thai Royal Family Tree



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- Representations
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3. AVL Tree

4. Which Representations ?

5. n-ary Tree

6. Generic Tree

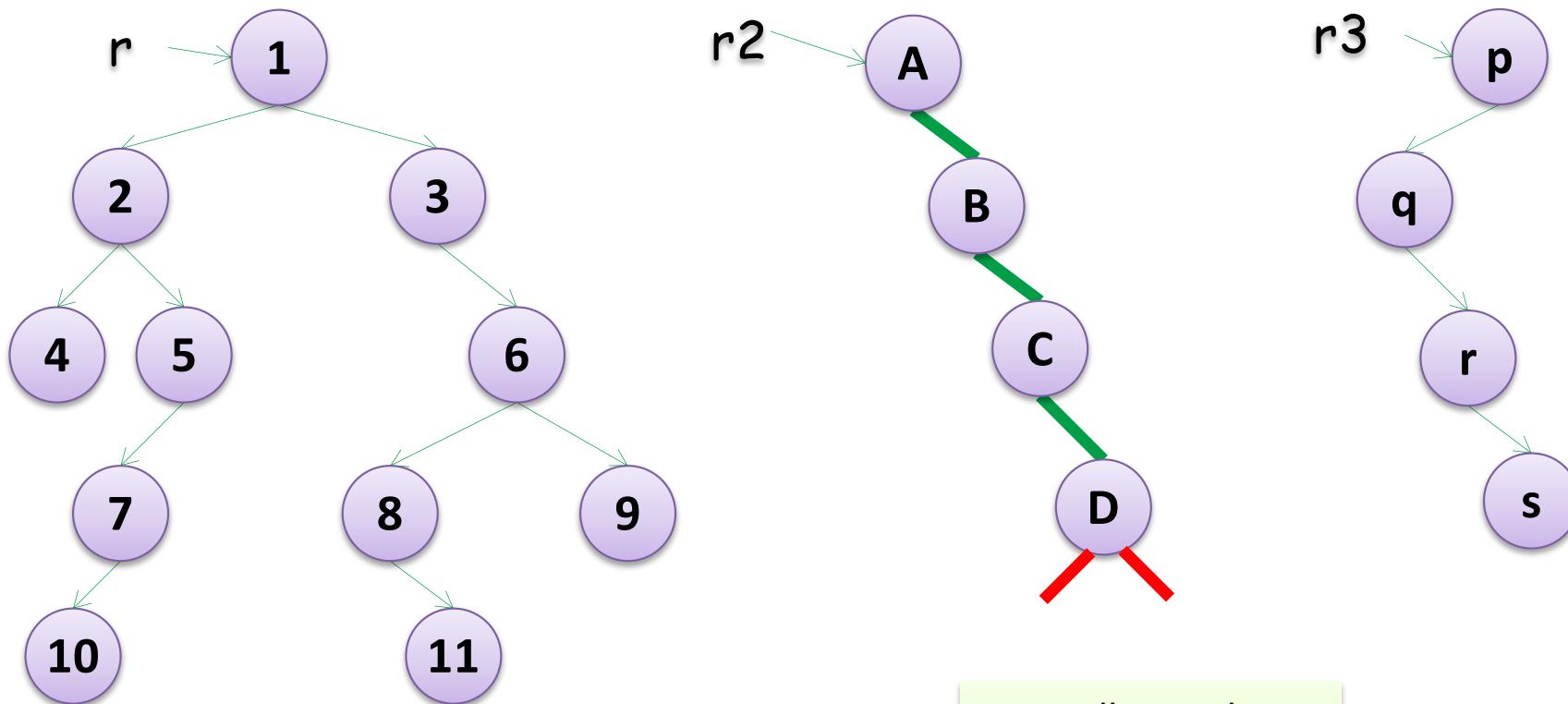
7. Multiway Search Tree

8. B-Trees

Binary Tree

bi = 2

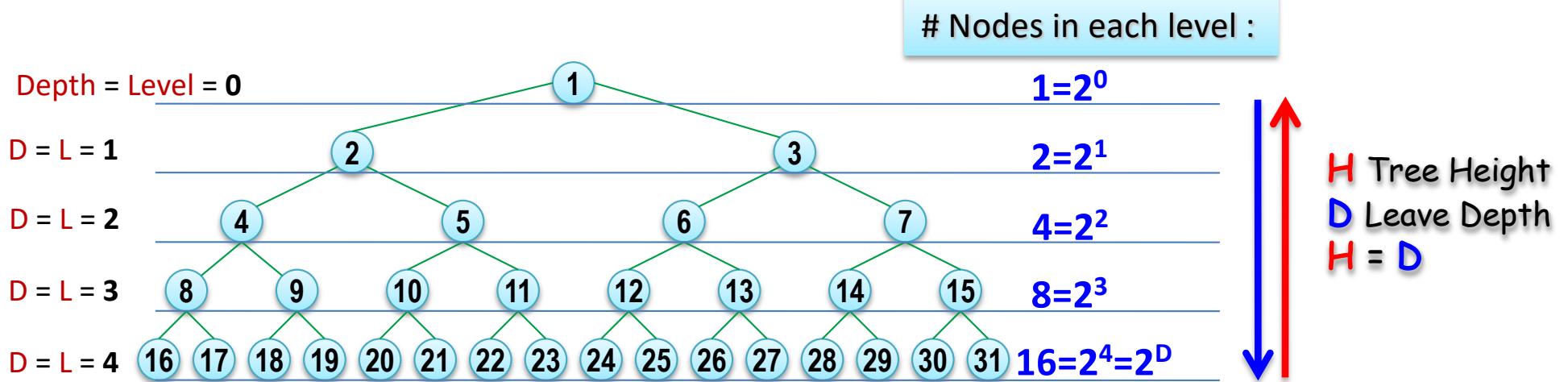
Binary Tree : มีอย่างมากที่สุด 2 subtrees (0, 1 or 2 subtrees)



Normally not draw
• branch direction
• null ptr

Perfect Binary Tree

Perfect Binary Tree (Ambiguously also called Complete Binary Tree) Every level is completely filled



N = จำนวน node ใน perfect binary tree

$$N = 2^0 + 2^1 + \dots + 2^H \quad \sum_{i=0}^N r^i = \frac{1 - r^{n+1}}{1 - r}, r <> 1$$
$$N = 2^{H+1} - 1$$

$$H = D = \log_2 (N+1) - 1$$

O(log₂(N))

$$N = (2^H \text{ external}) + (2^H - 1 \text{ internal})$$

Perfect Binary Tree of N nodes height H :

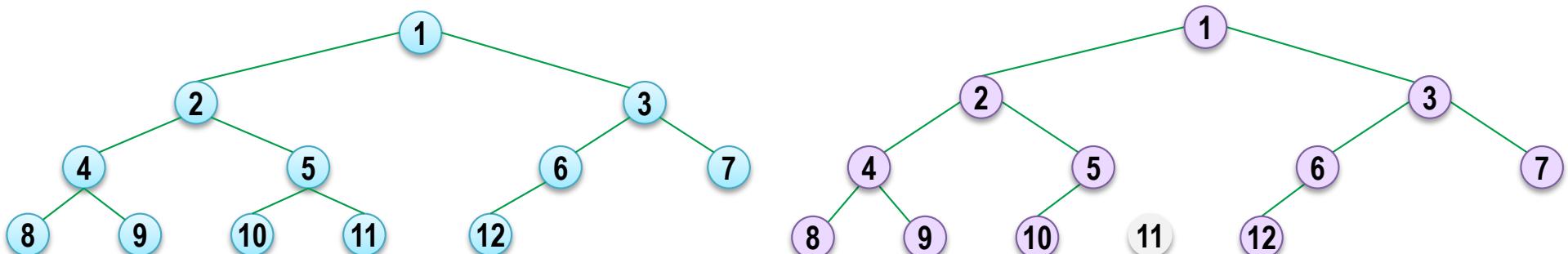
$$H = \log_2 (N+1) - 1$$

$$N = 2^{H+1} - 1$$

Complete Binary Tree

Complete Binary Tree

Every level, except possibly the last, is completely filled &
All nodes are as far left as possible.



Complete binary tree

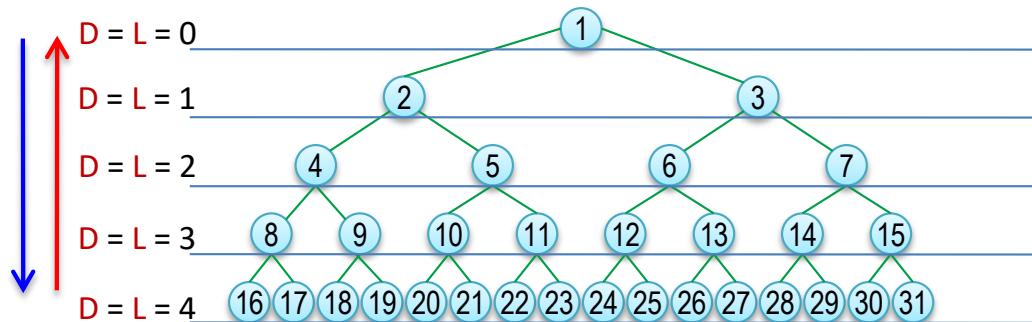
Without node 11 :
Not a complete binary tree

Perfect Binary Tree & Complete Binary Tree

N จำนวน node และ H ความสูง

Perfect Binary Tree

$$\begin{aligned} H &= \log_2(N+1) - 1 \\ &= \lfloor \log_2 N \rfloor \rightarrow O(\log_2(N)) \\ N &= 2^{H+1} - 1 \end{aligned}$$



H Tree Height

D Leaf Depth

$$H = D$$

Perfect Binary Tree

Every level is completely filled

(Ambiguously also called Complete Binary Tree)

Complete Binary Tree

$$\begin{aligned} H &= \log_2(N+1) - 1 \\ &= \lfloor \log_2 N \rfloor \rightarrow O(\log_2(N)) \\ N &\text{ ระหว่าง } [2^H, 2^{H+1} - 1] \end{aligned}$$

Nodes
in each level

$$1 = 2^0$$

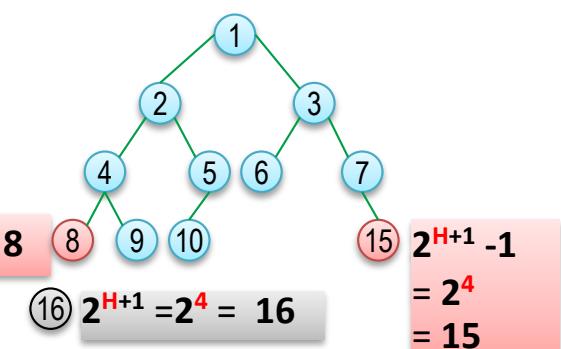
$$2 = 2^1$$

$$4 = 2^2$$

$$8 = 2^3$$

$$16 = 2^4 = 2^D$$

$$2^H = 2^3 = 8$$



N total # of nodes

$$= 2^0 + 2^1 + \dots + 2^H$$

$$= 2^{H+1} - 1$$

เป็น external 2^H

internal $2^H - 1$

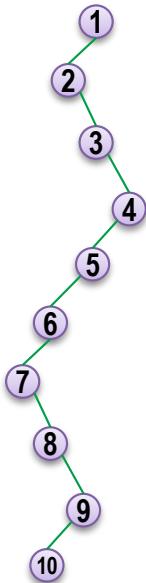
Complete Binary Tree

Every level is completely filled,
except possibly the last, &
All nodes are as far left as possible

Complete Binary Tree : H and N

How can 10 nodes in a binary tree has

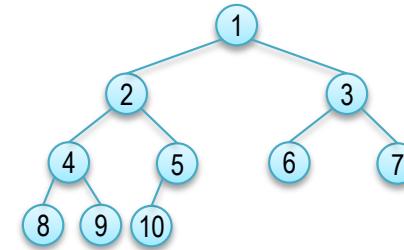
- longest search path ?
- shortest search path ?



Worst Case Height

Like Linked list : $H = N - 1$

$O(N)$



Best Case Height

Complete Binary Tree of N nodes height H :

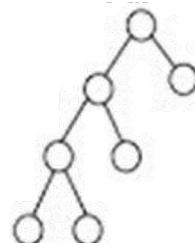
$$H = D = \lceil \log_2 (N+1) - 1 \rceil = \lfloor \log_2 N \rfloor$$

$O(\log N)$

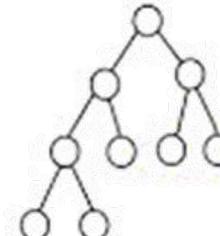
How much different N vs $\log_2 N$? $\log_2 1,000,000 < 20$

So
Complete Binary Tree
is an ideal tree

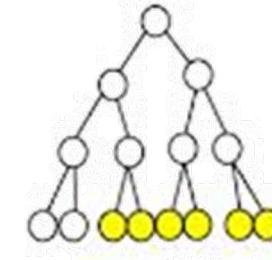
Full Complete Perfect



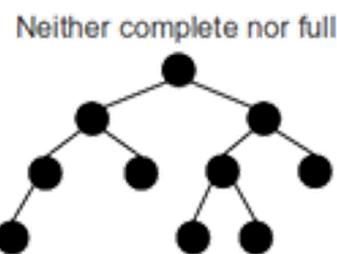
full



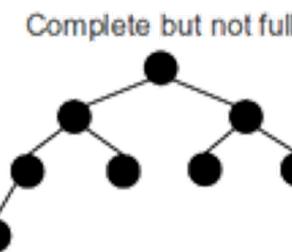
complete



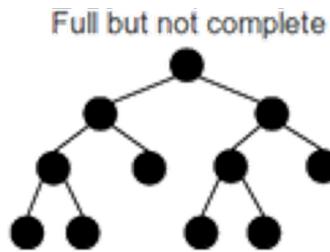
perfect



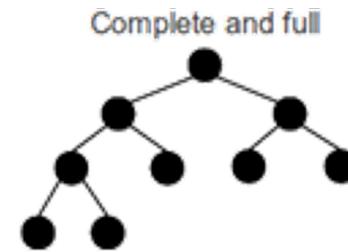
Neither complete nor full



Complete but not full



Full but not complete



Complete and full

Full Binary Tree: A Binary Tree is full if every node has 0 or 2 children. Following are examples of a full binary tree.

Complete Binary Tree: A Binary Tree is complete Binary Tree if all levels are completely filled except possibly the last level and the last level has all keys as left as possible.

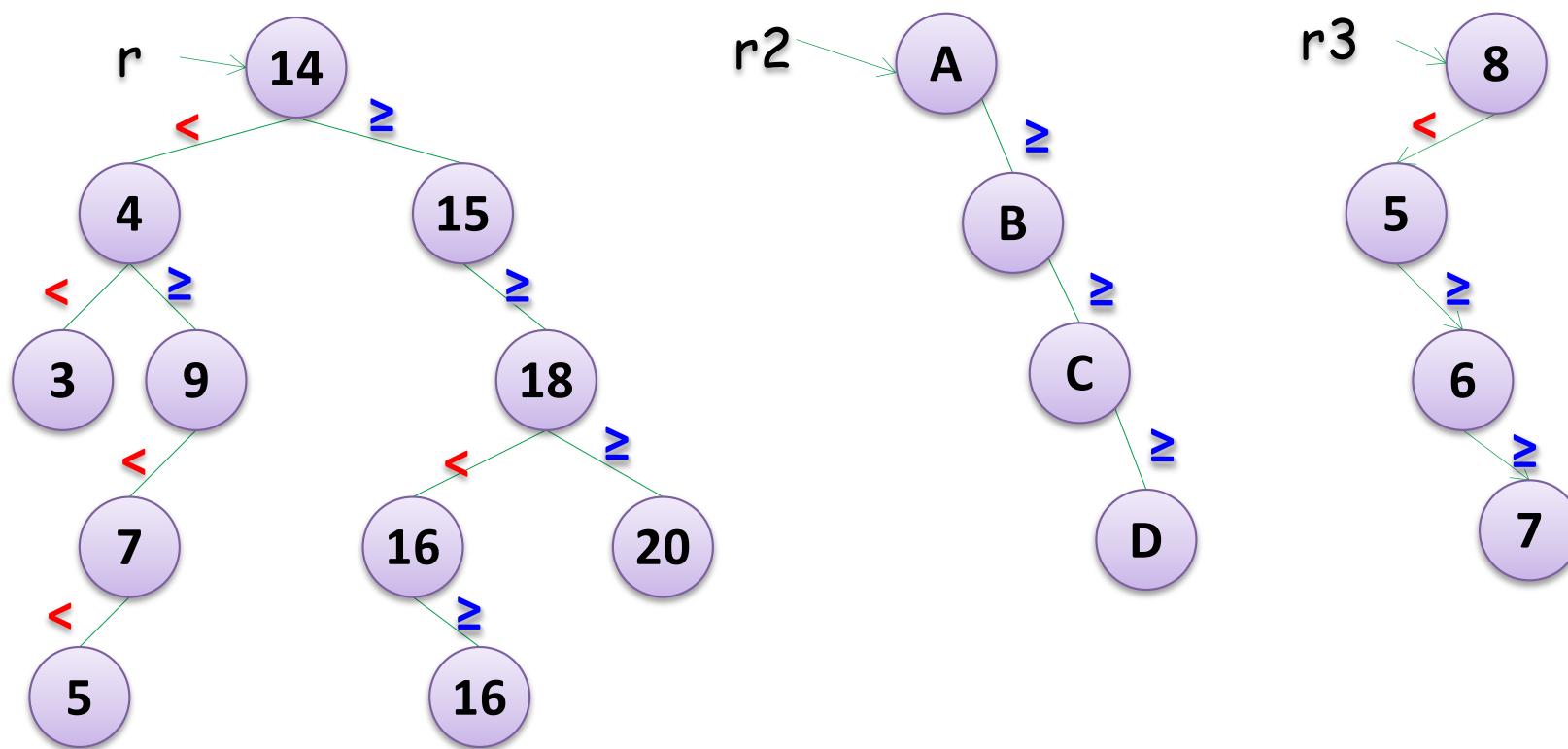
Perfect Binary Tree: A Binary tree is Perfect Binary Tree in which all internal nodes have two children and all leaves are at same level.

Binary Search Tree

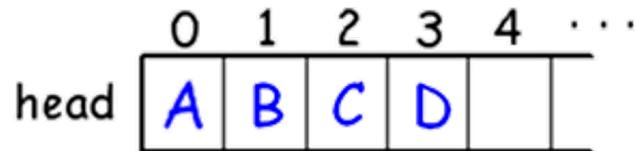
Binary **Search Tree** : for every node A

left descendants $< A$

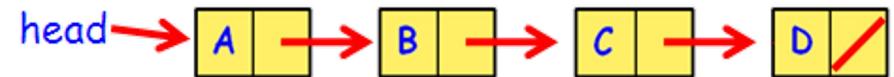
right descendants $\geq A$



Why Tree ?



Implicit (Sequential) Array
Insertion – Deletion Problems



Linear Linked list
Linear Search

$O(n)$



Tree

Tree Search

$O(\log_2 n)$

UNIX

File system of several popular OS

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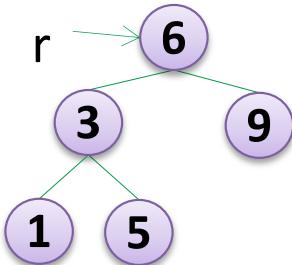


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Binary Tree Representations

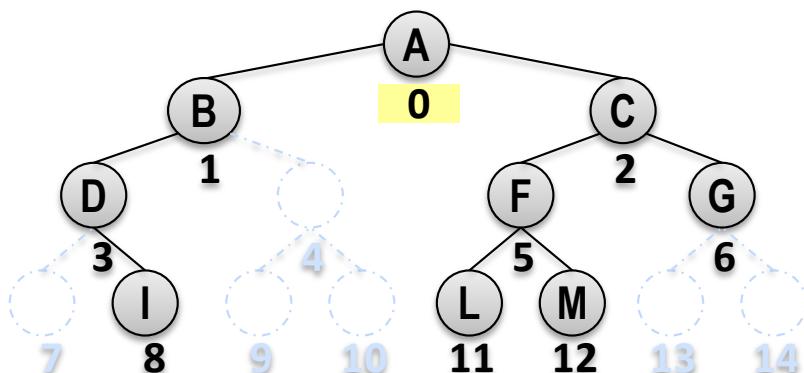
1. Dynamic



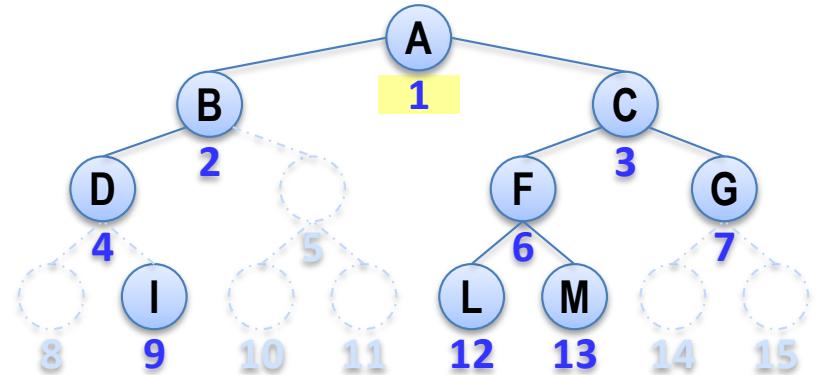
class Node:

```
def __init__(self, data, left = None, right = None):  
    self.data = data  
    self.left = None if left is None else left  
    self.right = None if right is None else right
```

2. Sequential (Implicit) Array

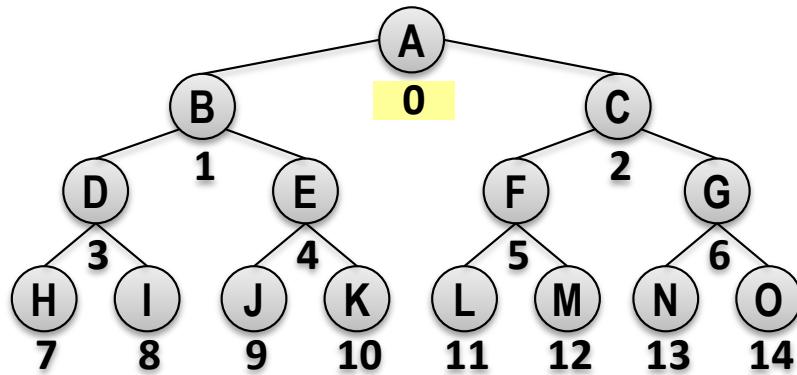


A	B	C	D	F	G	I	L	M		
0	1	2	3	4	5	6	7	8	9	10

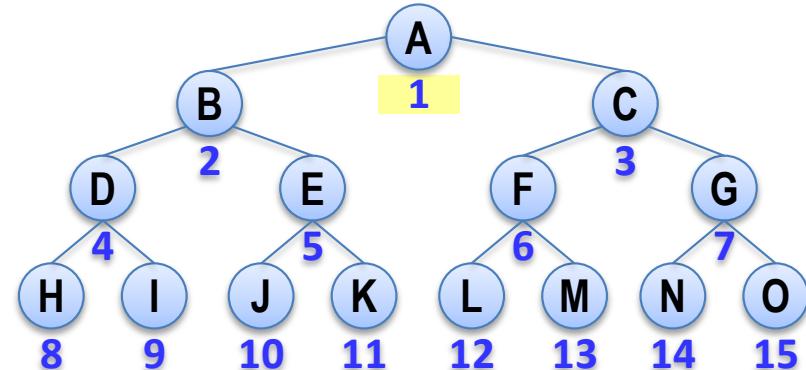


	A	B	C	D	F	G	I	L	M		
0	1	2	3	4	5	6	7	8	9	10	11

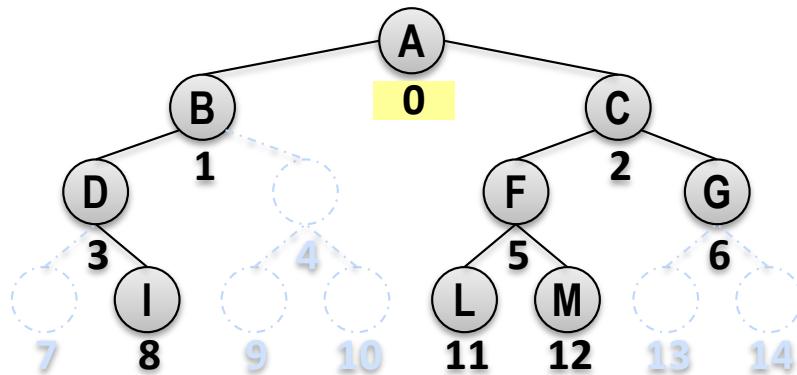
(Sequential) Implicit Array



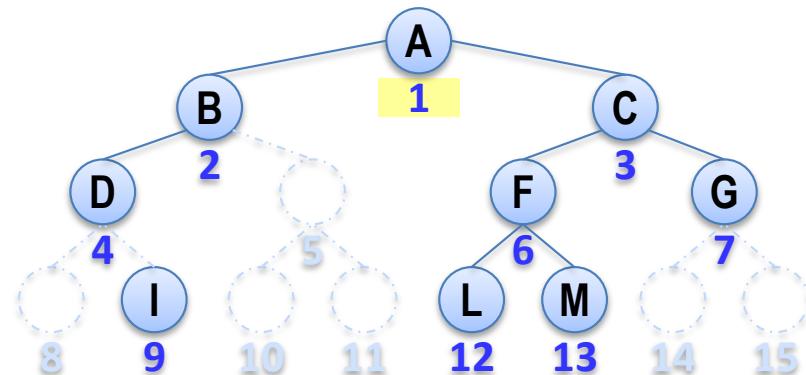
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

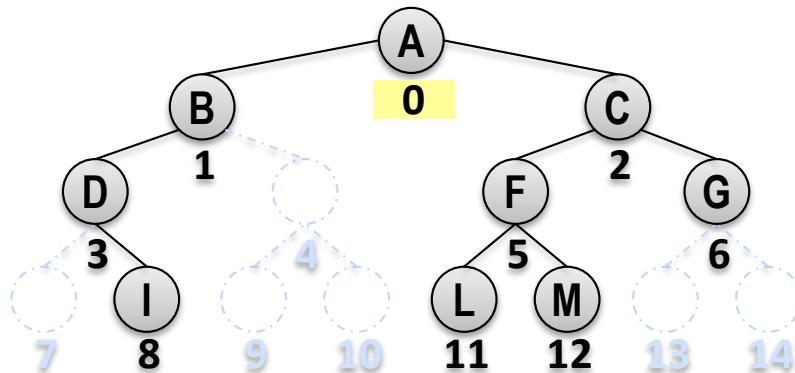


A	B	C	D		F	G		I		L	M			
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

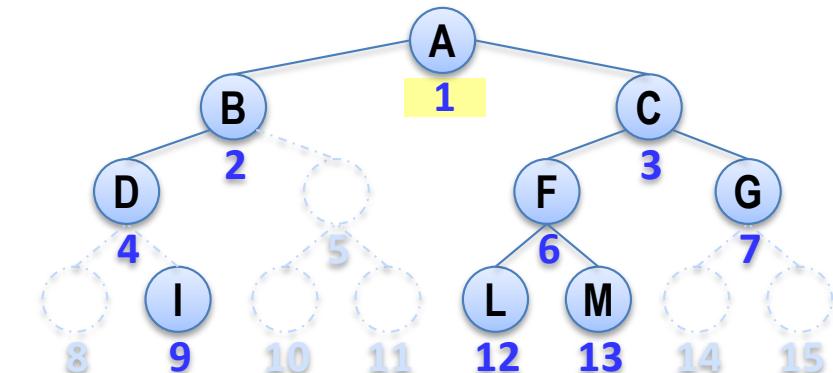


	A	B	C	D		F	G		I		L	M		
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

(Sequential) Implicit Array



A	B	C	D		F	G		I		L	M			
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14



	A	B	C	D		F	G	I		L	M			
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Where is the **root** ?

The node at index i 's	left son ?
	right son ?
	father ?

Start at index 0

0
$2i + 1$
$2i + 2$
$(i - 1) \text{ div } 2$

Start at index 1

1
$2i$
$2i + 1$
$i \text{ div } 2$

So good ! No memory for link ! Easy to calculate !

What happen if we have **only** node at indices : 0, 2, 6, 14 ? ie. **sparse**

What shape of tree should be best for sequential array?

Binary Tree Traversals

Tree Traversal

การไปเยี่ยม (visit เช่นการพิมพ์ การ update ข้อมูล) ทุก node node ละ 1 ครั้ง อย่างมีระบบ
แบ่งตามลำดับของการ visit

Breadth First (Level Order) จาก root ไปด้านข้างก่อน

1. ลูกคนแรก ลูกคนที่ 2 ลูกคนที่ 3 ...จนหมดลูกทุกคน
2. ทำข้อ 1. กับทุกคนที่ไปเยี่ยมมาตามลำดับ

Depth-First Order จาก root ไปด้านลึกก่อนไปด้านข้าง

ลูกคนแรก และไปหลานคนแรกก่อนไปที่ลูกคนที่ 2

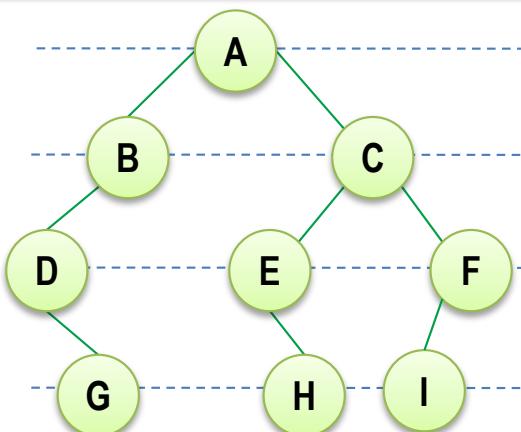
Breadth - First (Level Order)

Breadth First จาก root ไปด้านข้างก่อน ใช้ queue ช่วยในการหา

- ลูกคนแรก ลูกคนที่ 2 ลูกคนที่ 3 ...จนหมดลูกทุกคน
- ทำข้อ 1. กับทุกคนที่ไปเยี่ยมมาตามลำดับ

A B C D E F G H I

ไล่ไปทีละ level จึงเรียกอีกอย่างหนึ่งว่า Level Order



queue



deQ & visit

```
enQ( root )
while ( notEmptyQ ) {
    n = deQ( )
    visit(n)
    if ( n->left )
        enQ( n->left )
    if ( n->right )
        enQ( n-> right )
}
```

BST

```
def leverOrder(self):  
    q = Queue()  
    q.enQ(self.root)  
    while q.isEmpty() is not True :  
        n = q.deQ()  
        print(n.getData(), end = ' ')  
        if n.getLeft() is not None:  
            q.enQ(n.getLeft())  
        if n.getRight() is not None:  
            q.enQ(n.getRight())
```

Depth - First Order

Depth-First Order จาก root ไปด้านลึกก่อนไปด้านข้าง ใช้ stack ช่วยในการหาไปลูกคนแรก และไปหานคนแรกก่อนไปที่ลูกคนที่ 2
แบ่งเป็น 3 แบบ ขึ้นกับการวางแผนการ visit root ไว้ที่ใด

Inorder (Symmetric Order)

1. inOrder(leftSubtree)
2. **visit_root**
3. inOrder(rightSubtree)

Preorder

1. **visit_root**
2. preOrder(leftSubtree)
3. preOrder(rightSubtree)

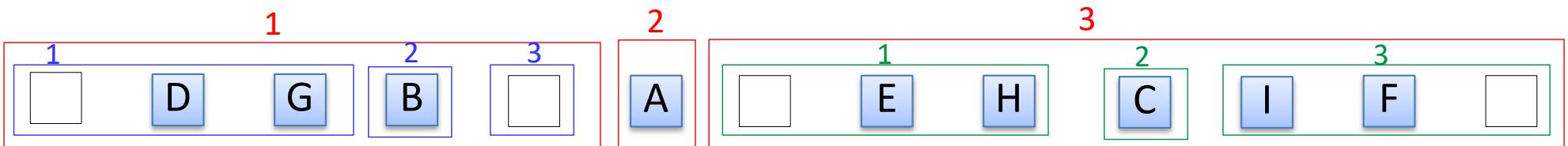
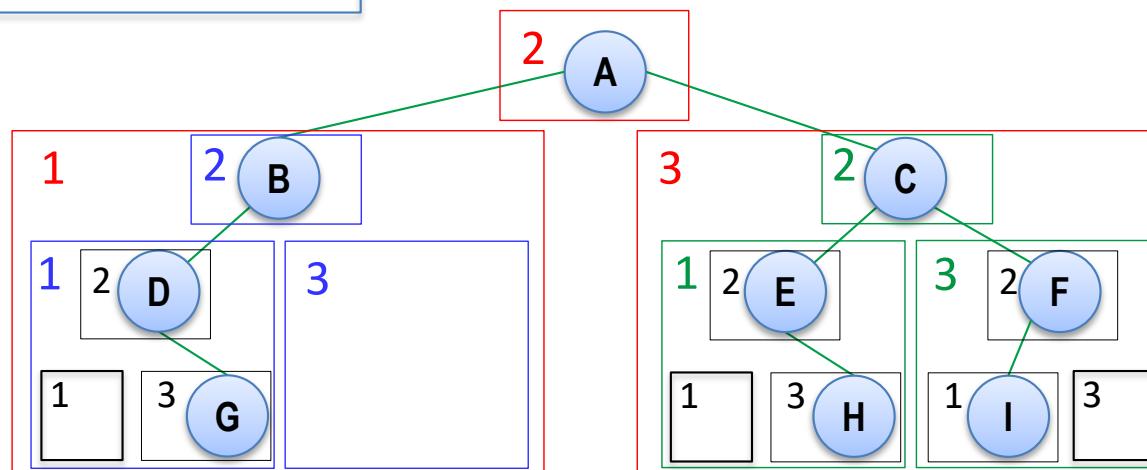
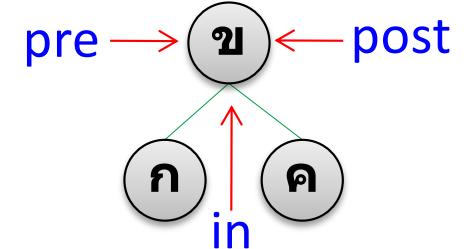
Postorder

1. postOrder(leftSubtree)
2. postOrder(rightSubtree)
3. **visit_root**

Inorder (Symmetric Order)

Algorithm:

```
inOrder( root ) {  
    if (root != null){  
        inOrder(left subtree)  
        visit(root)  
        inOrder(right subtree)  
    }  
}
```



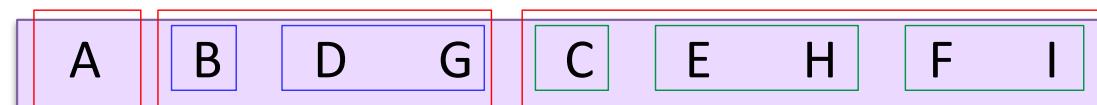
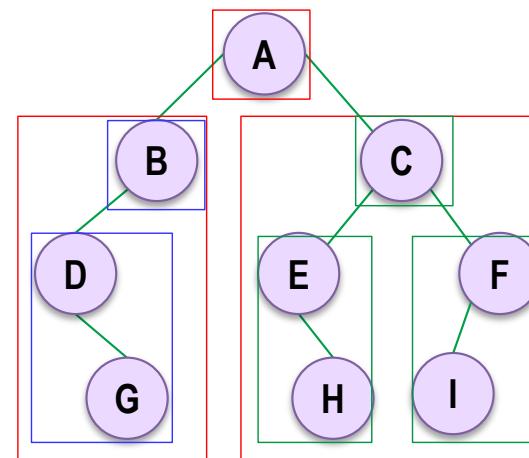
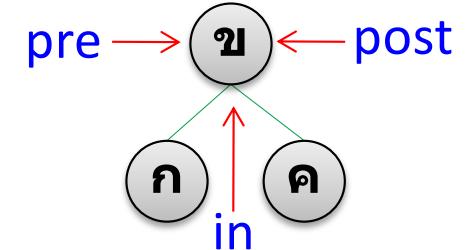
BST

```
def inOrder(self):  
    BST._inOrder(self.root)  
  
def _inOrder(root):  
    if root is not None:  
        BST._inOrder(root.getLeft())  
        print(root, end = ' ')  
        BST._inOrder(root.getRight())
```

Preorder

Algorithm:

```
preOrder( root ) {  
    if (root != null){  
        visit(root)  
        preOrder(left subtree)  
        preOrder(right subtree)  
    }  
}
```



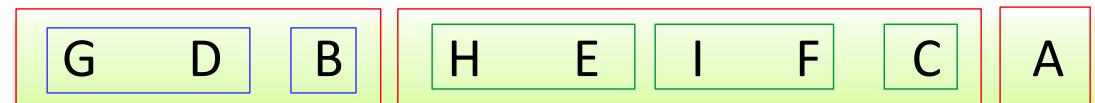
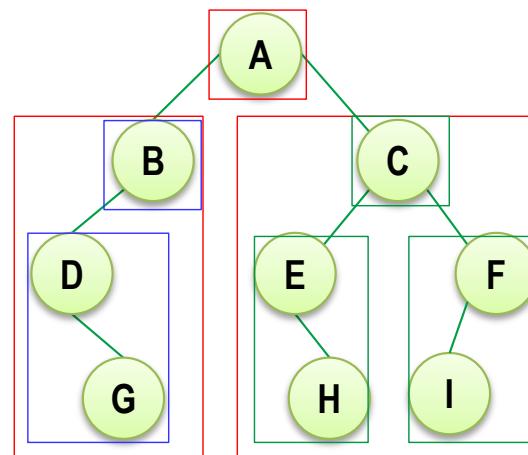
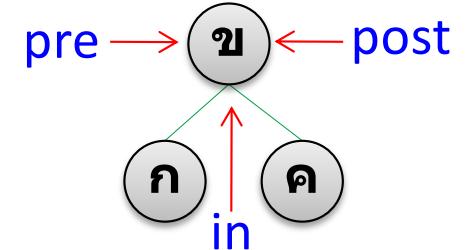
BST

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def preOrder(self):  
    BST._preOrder(self.root)  
  
def _preOrder(root):  
    if root is not None:  
        print(root, end = ' ')  
        BST._preOrder(root.getLeft())  
        BST._preOrder(root.getRight())
```

Postorder

Algorithm:

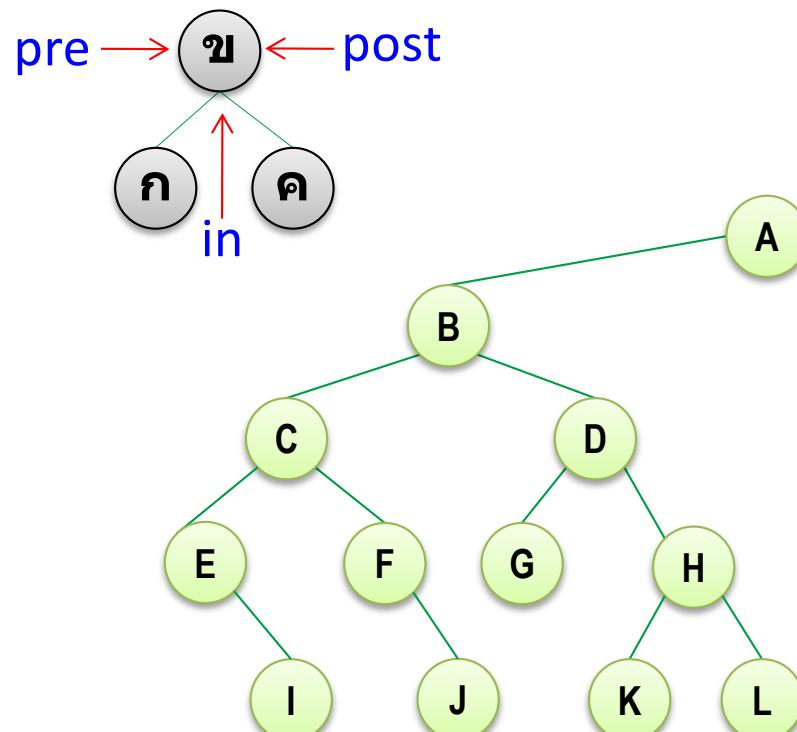
```
postOrder( root ) {  
    if (root != null){  
        postOrder(left subtree)  
        postOrder(right subtree)  
        visit(root)  
    }  
}
```



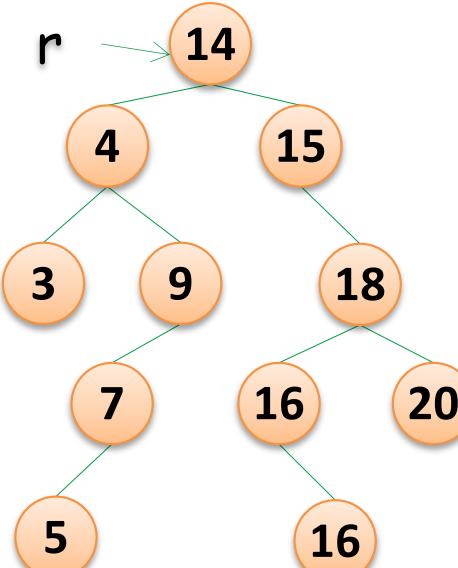
BST

```
def postOrder(self):  
    BST._postOrder(self.root)  
  
def _postOrder(root):  
    if root is not None:  
        BST._postOrder(root.getLeft())  
        BST._postOrder(root.getRight())  
        print(root, end = ' ')
```

Test Your Self



Binary Search Tree



Inorder : LrootR
 1. inOrder(L)
 2. visit_root
 3. inOrder(R)

Postorder : LRroot

Preorder : rootLR

Inorder

E I C F J B G D K H L A

3 4 5 7 9 14 15 16 16 18 20

Asscending Order !

Preorder

A B C E I F J D G H K L

14 4 3 9 7 5 15 18 16 16 20

Postorder

I E J F C G K L H D B A

3 5 7 9 4 16 16 20 18 15 14

1. Tree Definitions
2. Binary Tree
 - Traversals
 - **Binary Search Tree**
 - Representations
 - Application : Expression Tree
3. AVL Tree
4. Which Representations ?
5. n-ary Tree
6. Generic Tree
7. Multiway Search Tree
8. B-Trees

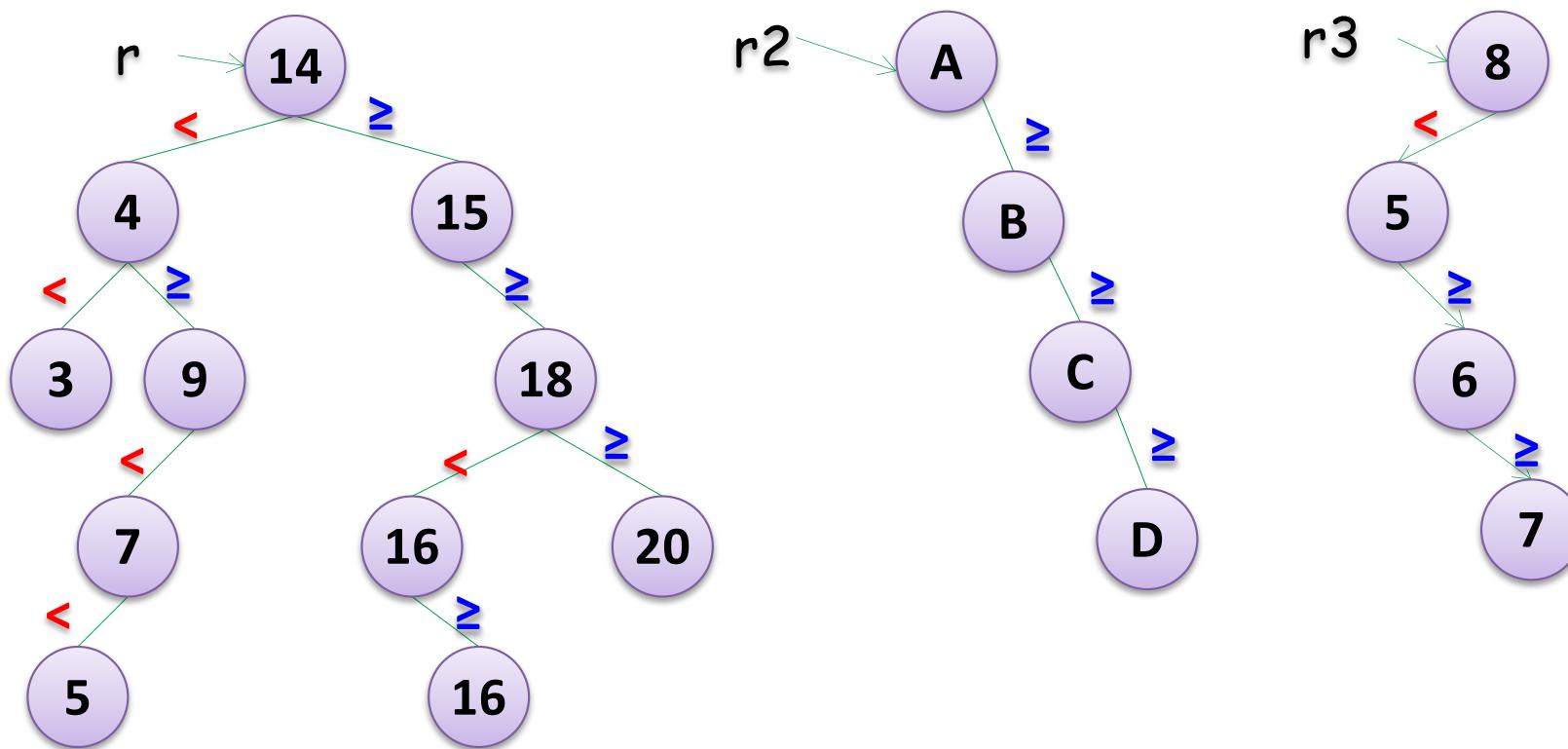


Binary Search Tree

Binary **Search** Tree : for every node A

left descendents $< A$

right descendents $\geq A$

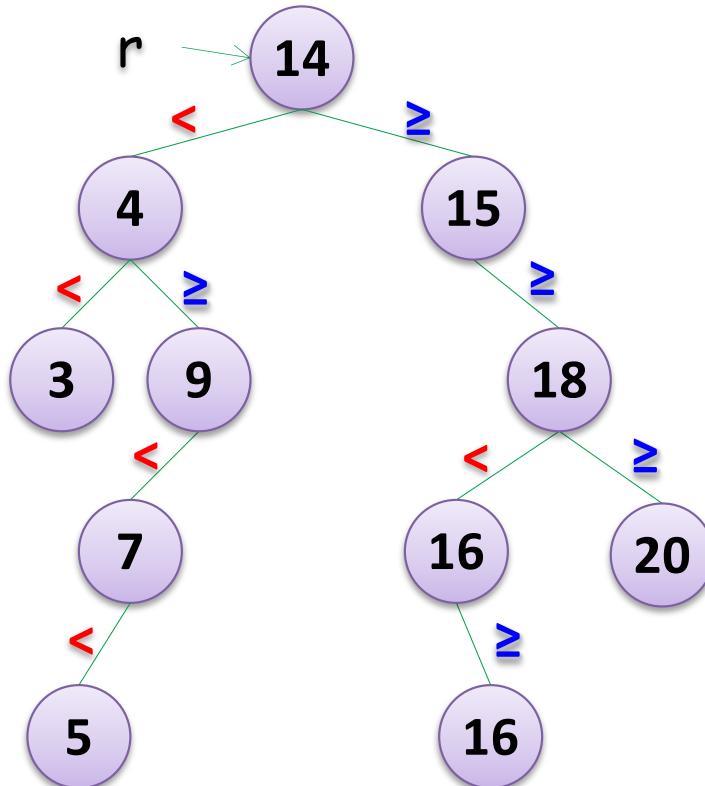


Node

```
class node:  
    def __init__(self, data, left = None, right = None):  
        self.data = data  
        self.left = None if left is None else left  
        self.right = None if right is None else right  
  
    def __str__(self):  
        return str(self.data)
```

```
class BST:  
    def __init__(self, root = None):  
        self.root = None if root is None else root
```

Insertion



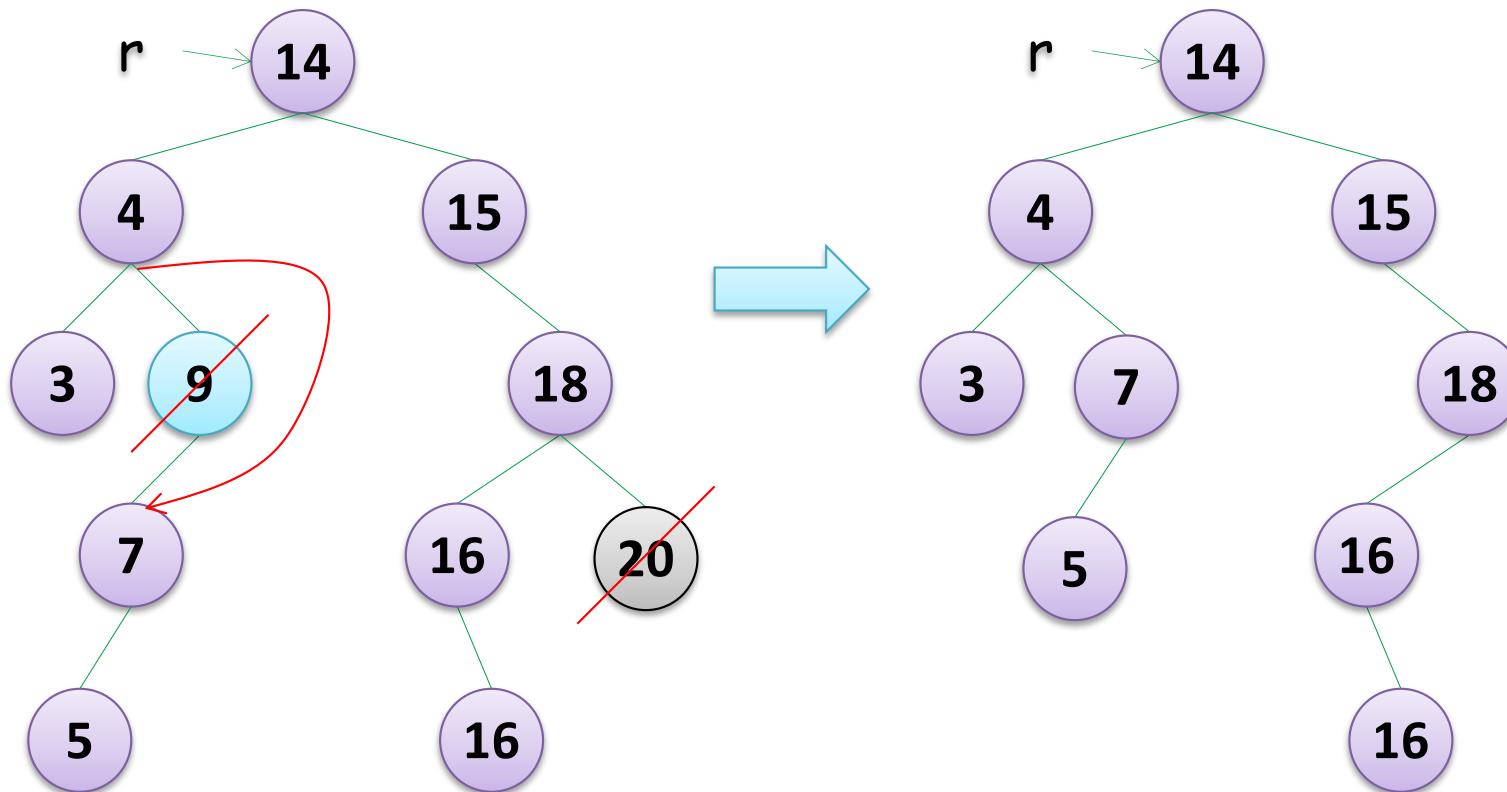
Inserts : 14 4 9 7 15 3 18 16 20 5 16

Node

```
class node:  
    def __init__(self, data, left = None, right = None):  
        self.data = data  
        self.left = None if left is None else left  
        self.right = None if right is None else right  
  
    def __str__(self):  
        return str(self.data)
```

```
class BST:  
    def __init__(self, root = None):  
        self.root = None if root is None else root  
  
  
    def add(self, data):  
        self.root = BST._add(self.root, data)  
  
    def _add(root, data):  
        if root is None:  
            return node(data)  
        else:  
            if data < root.data:  
                root.left = BST._add(root.left, data)  
            else:  
                root.right = BST._add(root.right, data)  
        return root
```

Delete a : Leaf, Node with only 1 child



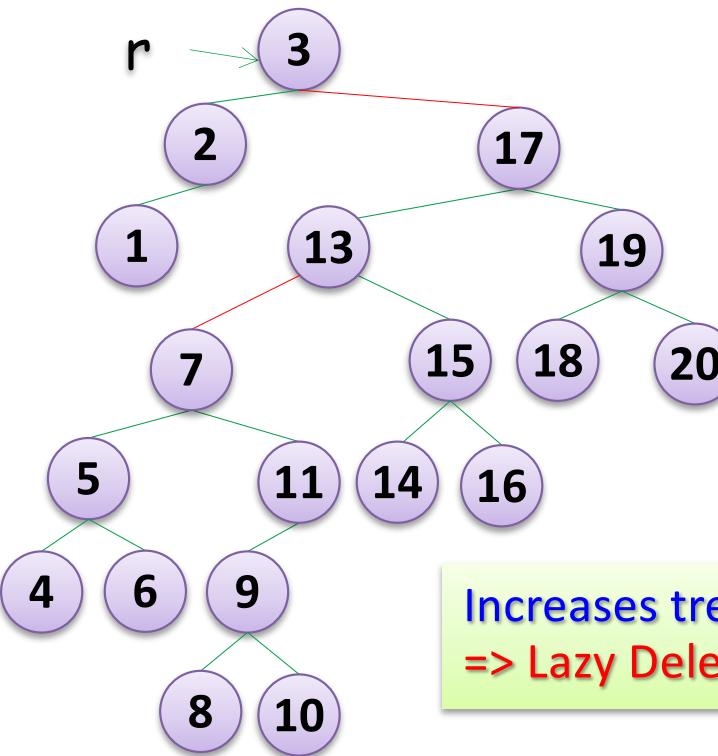
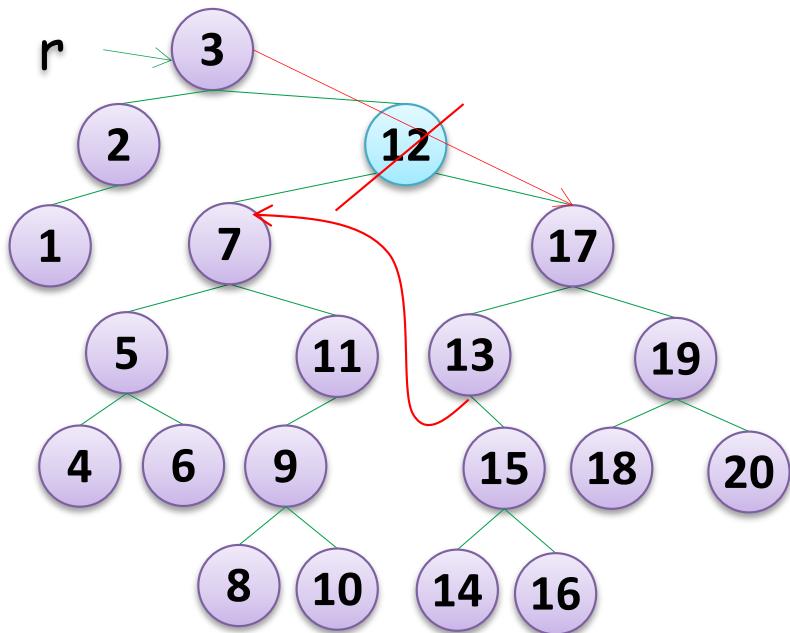
Delete Leaf

20 can be deleted right away, still be a binary search tree

Delete a node with only one child

Delete 9, replace a subtree at the deleted node

Delete a Node with Both Children : Lazy Deletion 1



Increases tree height.
=> Lazy Deletion

Delete 12

Where can we put tree 7, and tree 17 ?

Can replace only 1 at the deleted node.

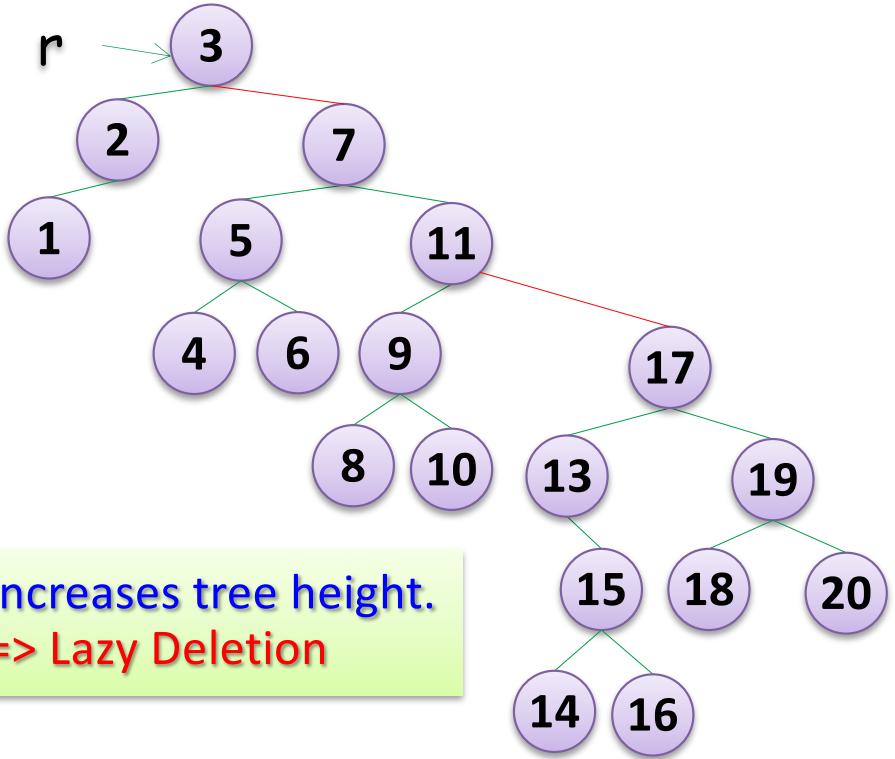
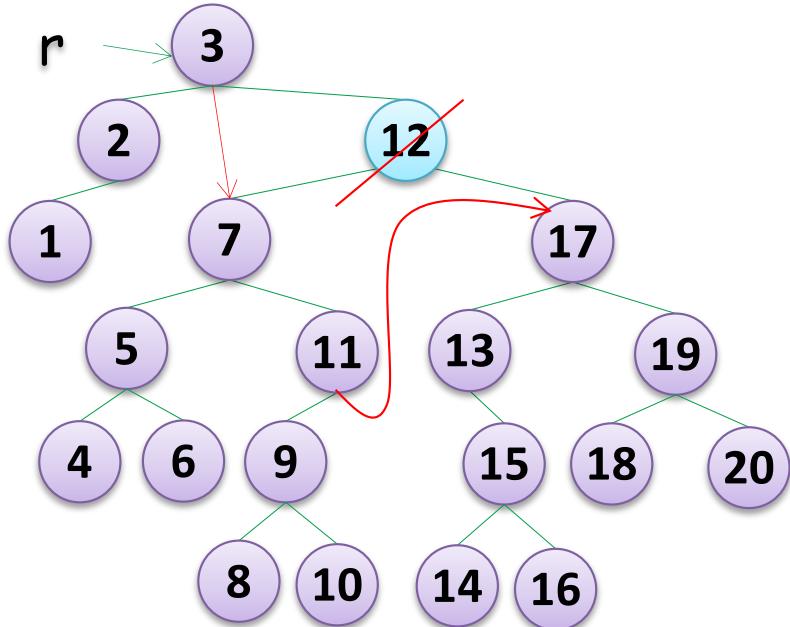
Let choose 17.

Where to put tree 7 ?

It's a binary search tree!

Where is its place ?

Delete a Node with Both Children : Lazy Deletion 2



Delete 12

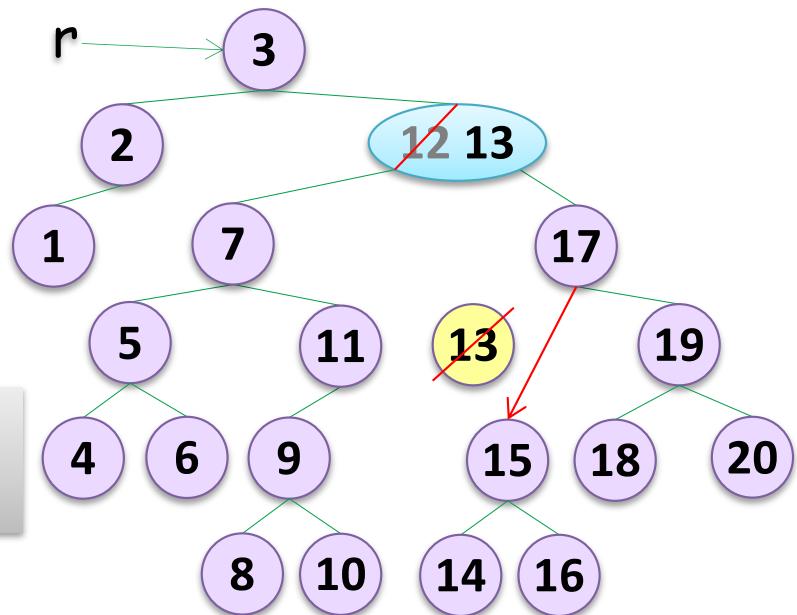
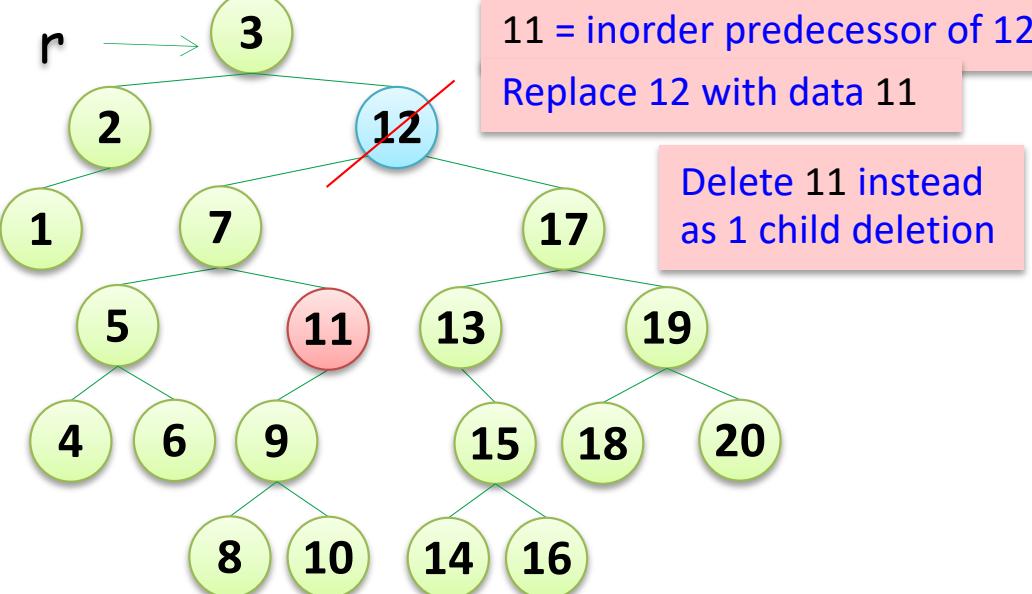
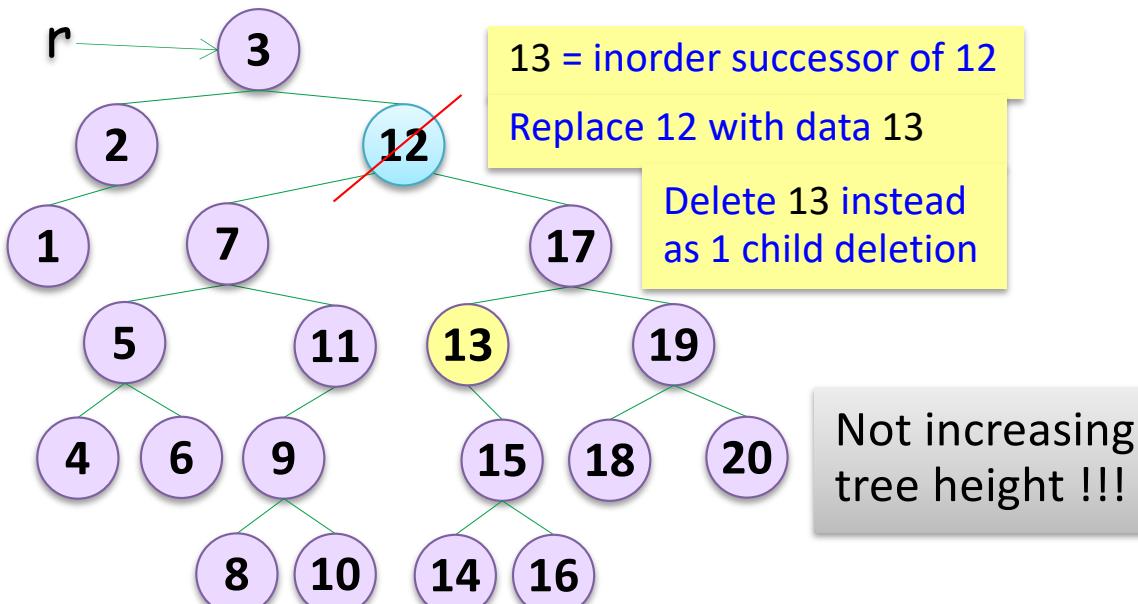
Can replace only 1 at the deleted node.

If we choose 7.

Where is 17's place ?

Deletion : Using Inorder Predecessor/ Successor

inorder successor → ค่าที่เล็กที่สุดใน subtree ขวา



inorder predecessor → ค่าที่มากที่สุดใน subtree ซ้าย

BST

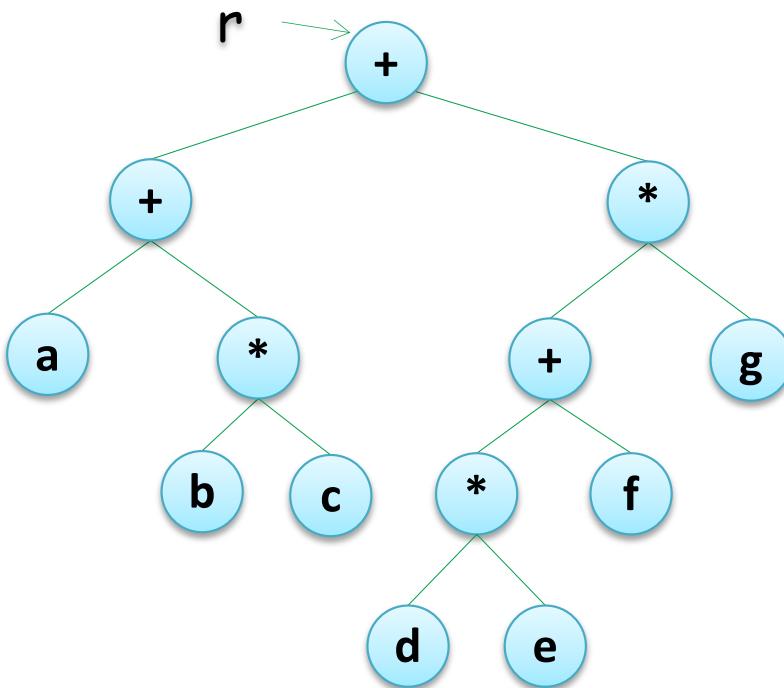
```
def delete_node(root, key):
    if root is None:
        return root
    # ค้นหาค่า key ที่จะลบ
    if key < root.key:
        root.left = delete_node(root.left, key)
    elif key > root.key:
        root.right = delete_node(root.right, key)
    else:
        # เจอ node ที่จะลบ
        # กรณีมีลูก 0 หรือ 1 คน
        if root.left is None:
            temp = root.right
            root = None
            return temp
        elif root.right is None:
            temp = root.left
            root = None
            return temp
        # กรณีมีลูก 2 คน: ใช้ inorder successor (ค่าที่เล็กสุดใน subtree ขวา)
        temp = find_min(root.right)
        root.key = temp.key
        root.right = delete_node(root.right, temp.key)
    return root
```

```
def find_min(node):
    while node.left:
        node = node.left
    return node
```

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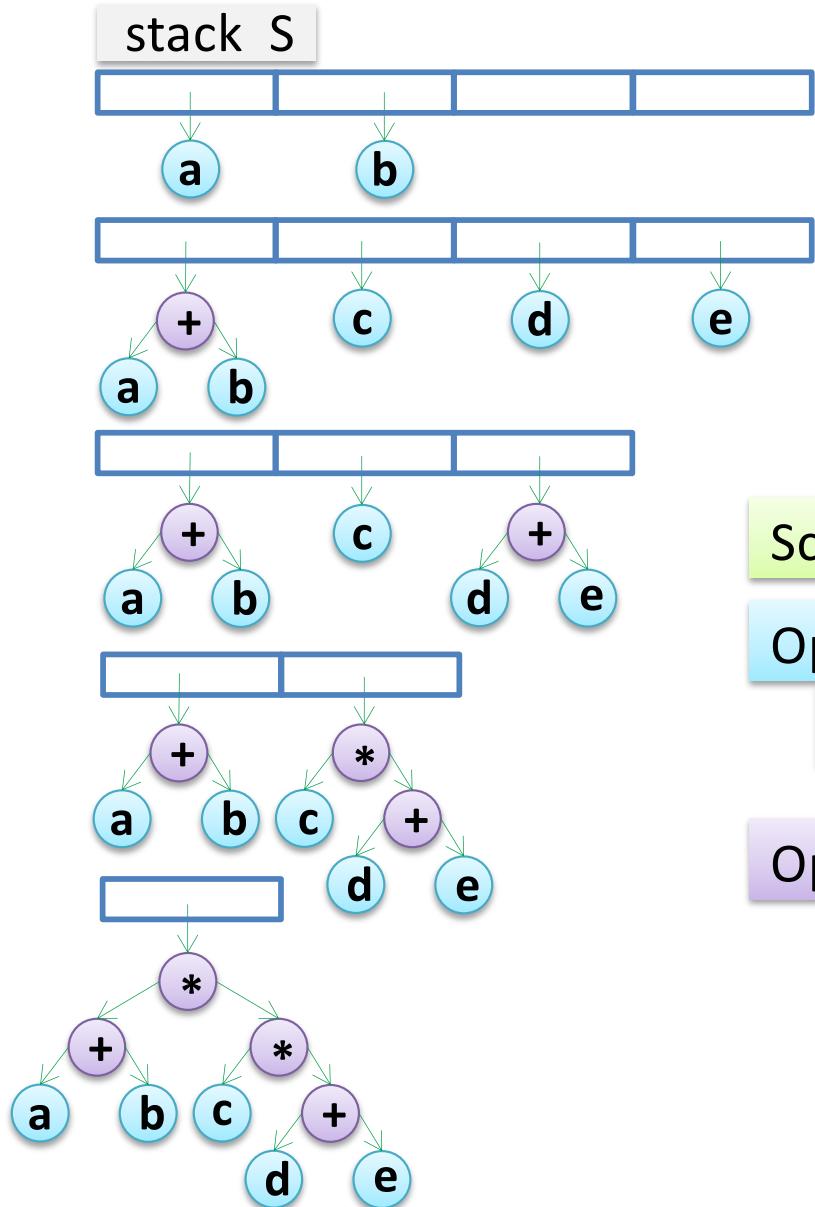


Example : Expression Tree



$$(a + b * c) + ((d * e + f) * g)$$

Constructing an Expression Tree



Scan input from left to right.

Operand

Create an operand node and push to the stack

Operator

Create an operator node

Pop 2 operands to be its children.

Push to the stack.

