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Game Theory Q&A

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Cheat sheet

This paper is a summary of Game Theory course of UniPD, a.y. 2023/24.
It's not intended to substitute slides or book study.

Static games

Supposing you know how to write the normal form of a game and find NE, here are some of the most important formulas:

Mixed strategies

			Player B	
			q	*1-q*
			T	N
Player A	*p*	T	3,3	0,0
	1-p	N	0,0	6,1

If you have three players you can use the same method, but you need n matrixes, one for each move of one player (write as "title" of every matrix the move of the player left out).

Small tip: if a player has always the same payoff in a matrix, you can write it as a single number, ignoring probabilities (since, once you collect the payoff, the sum of all probabilities is 1).

If it's a zero-sum game, you can just write payoff of one player, usually the first one, and the other player will have the opposite payoff.

Find all additional Nash equilibria of this game in mixed strategies

$$u_A(T,T)q + u_A(T,N)(1-q) = u_A(N,T)q + u_A(N,N)(1-q)$$

$$u_B(T,T)p + u_B(N,T)(1-p) = u_B(T,N)p + u_B(N,N)(1-p)$$

Find p and q

If you have more than two players, probabilities would be $p_1, p_2, 1-p_1-p_2$.

Multistage games

Grim trigger strategy

The Grim Trigger strategy goal is to punish the other player if he deviates from the chosen strategy. The goal is to find a δ s.t. the punishment is enough to avoid deviation.

Here the procedure for **infinitely repeated games**:

- Chose the player who gains more from deviation
- Call p^* the payoff of the chosen strategy
- Call p the payoff of the other strategy (usually NE)
- Apply the formula below

- Solve for δ

$$\frac{p^*}{1-\delta} \geq p + \frac{p\delta}{1-\delta}$$

Stick and carrot

Same as before, but for **finitely repeated games**:

- Chose the player who gains more from deviation
- Let p the payoff of the chosen strategy (usually cooperation, not necessary a NE)
- Let p_s the payoff of deviating
- Let the payoff of the other strategy, usually stick and carrot, p_s and p_c respectively
- Apply the formula below
- Solve for δ

The intuition is to calculate the payoff of following cooperation plus the reward, then compare it with the payoff of deviating and receiving the stick.

$$p + \delta p_c \geq p_d + \delta p_s$$

Baeysian games

For the extensive form of the game, you can use the same method as before, but you have to consider **Nature** as a player who makes the first move (typically, with probability p and *1-*p).

The tree will have, then, Nature as root, and all possibile strategies as chilren, repeted for each Nature's move.

When it comes to write the normal form, you have to consider the probability of each strategy, which is the payoff of the stretegy times the probability of the path that leads to that strategy. For example, if one cell of the bi-matrix is AA, it means that player X will play A in any case: payoff is payoff of A in that leaf times probability of the path that leads to A.

1 Introduction, background concepts, decision problems

1. Q - What is a game in game theory?

   A - **Game theory** is a mathematical framework that allows to model specific type of problems. The problems studied by game theory are called games.

2. Q - What is a game?

   A - **A game** is a multi-person multi-objective problem.

   Multi-person = There are multiple agents (players) involved in a game
   Multi-objective = Players have, in general, different goals
   The **outcome** of a game depends on the choices made by all players
   The **purpose** of game theory is to find the "best choice" for each of them according to their objectives

3. Q - What does it mean for a preference to be rational?

   A - **A preference** is rational when it's complete and transitive

   Preference is a binary relationship $\succeq$ between elements of A (set of possible actions)
   If $a, b \in A, a \succeq b$ means that a is preferred to b
   A **preference** is always: reflexive, anti-symmetric A preference can also be:

   - **complete** if $\forall\ a, b \in A$ either $a \succeq b$ or $b \succeq a$
   - **transitive** if $\forall\ a, b \in A, a \succeq b \wedge b \succeq c \Rightarrow a \succeq c$

4. Q - What does it mean for a player to be rational?

   A - **A player is rational** when he always maximize their utility function, i.e., choose the action that leads to his preferred outcome. In other words, rational player act for their own good.

5. Q - What are the elements of a decision problem?

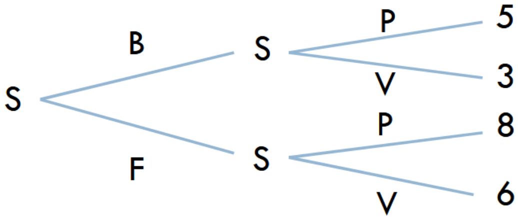
   A - **A decision problem** has three elements: actions, outcomes and preferences.

   Action a is selected from a set of possible actions A
   Action a results in a certain **outcome** (For 1-player problems actions = outcomes)
   Preferences describe the relationship between different outcomes (i.e., which one is preferred)

6. Q - How to represent a decision tree?

   A - **A decision tree** has players on nodes, actions on branches and payoffs on leaves.

- u(B)=2, u(F)=5, u(P)=3, u(V)=1



2 Lotteries, VNM utility theorem, backward induction

7. Q - When is it possible to model preferences between lotteries using average payoffs?

   A - When radmness is involved, i.e. when some events are called as nature (N) events, not in the hand of the player.

8. Q - Which utility function can we use to model a risk- averse player? 1) $u(x) = x^2$ or 2) $u(x) = log(x)$?

   A - **2)**Log funct models risk-averse: he prefers lower results with higher probability.

9. Q - How can we solve a decision problem involving sequential choices made by both a player and Nature?

   A - Having the decision tree, starting with nodes at the end of the tree, substitute N node with expected utility and player's move with payoff of the best move.

Expected utilty is calculated as probability expected value:
$\mathbb{E}_{x\sim p}[u(x)] = \sum_{k=1}^{n} p(x_k) \cdot u(x_k)$ where p is the payoff and u is the utilty (u don't say)

3 Static games of complete information, normal-form representation, strictly dominated strategies, IESDS

(No questions in slides)

10. Q - When a strategy is called Pareto dominated?

A - **A joint strategy** s **is Pareto-dominated** by another strategy $s^{'}$ if

- $u_i(s^{'}) \geq u_i(s)$ for each player i
- $u_i(s^{'}) > u_i(s)$ for some player i

11. Q - What is IESDS?

A - **IESDS** stands for "iterated elimination of strictly dominated strategies". Since a rational player will never play a Pareto-dominated strategy, with IESDS is possible to delete one Pareto-dominated strategy after the other, in order to get a set of just one strategy or at least an easier rapresentation of the game.

4 Nash equilibrium, best response, weak dominance, price of anarchy

12. Q - What is a NE

A - **Nash equilibrium** is what is played if believes of both players match.

A **belief** of player i is a possible profile of other player's strategy.

A **best strategy** is the response of a rational player to a believe. i.e. strategy $s_i \in S_i$ is a BS to moves $(s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ if

$$(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) \geq (s_1, ..., s_{i-1}, s_i', s_{i+1}, ..., s_n)$$
$$\forall\ s_i' \in S_i$$

13. Q - Consider NE $(s_1, ..., s_n)$. Suppose player i replaces the current strategy s_i with s_i'. Can this still be a NE?

A - **No.** In Nash equilibrium no player can improve his payoff unilaterally.

14. Q - If a strategy is ruled out by IESDS, can it be a NE?

A - **Yes.** A joint strategy $(s_1^*, ..., s_n^*)$ in which everyone plays a dominant strategy is a Nash equilibrium.

15. Q - Compute the PoA for the Prisoner's dilemma using

$$C(s) = -\sum_j u_j(s)$$

A -

Player A		Player B	
		M	F
	M	-1,-1	-9,0
	F	0,-9	-6,-6

Table 1: Prisoners' dilemma

Worst (and, in this case, unique) NE: MM. Best Pareto: MF or FM.

$$PoA = \frac{-[-(-1) + (-1)]}{-[(0) + (-6)]} = \frac{2}{6} = \frac{1}{3}$$

The price of anarchy (PoA) is the ratio between the social costs in the worst NE s^* and in the best Pareto efficient strategy (i.e., social optimum).

$$PoA = \frac{C(s^*)}{min_s\ C(s)}$$

16. Q - Solve this hw.

A - For a generic R

- A (crazy) professor decides your grade in the exam he teaches will be decided by a game:
  - You are paired with a random classmate
  - You secretly choose an integer between 18 and 30, and so does the classmate
  - If you choose the same number, that is the score that you both get
  - If the numbers are different, who proposes the lowest score L gets a grade of $L + R$, while the other gets $L - R$ (score $<$ 18 means the exam is failed, >30 means 30L and gives payoff 31)
- Play the game with $R = 1$, $R = 2$, and $R = 10$.
- How do the NE change?

Player B

		18	**19**	**. . .**	**29**	**30**
Player A	**18**	18,18	18+R,0	. . .	18+R,0	18+R,0
	19	0,18+R	19,19	. . .	19+R,21-R	19+R,19-R
	. . .	. . .	. . .	⋱	. . .	. . .
	29	0,18+R	19-R,19+R	. . .	29,29	29-R,29+R
	30	0,18+R	19-R,19+R	. . .	29-R,29+R	30,30

Table 2: HW1: Strange exam

For $R = 1$ all couples with same grades are Nash equilibria.
For $R > 1$ the only NE is a Pareto-dominated strategy (i.e., 18,18).

5 Exercise on NE, constitutions, electoral systems

(No questions in slides)

17. Q - What is a constitution?

A - **Constitution**, or **social welfare function**, is a map

$$f : R(A)^n \longrightarrow R(A)$$

$$(\succeq_1, \dots, \succeq_n) \xrightarrow{f} f(\succeq_1, \dots, \succeq_n)$$

which maps a profile of n rational preferences $\succeq_{(i)} = (\succeq_i, \dots, \succeq_n)$ into a unique rational social preference $\succeq = f(\succeq_{(i)})$.

18. Q - What are constitution properties?

A - **Properties:**

- **Independence of Irrelevant Alternatives** (IIA) if for all pairs $(\succeq_{(i)}), (\succeq'_{(i)})$

$$\forall\ i, \succeq_i \mid a, b = \ \forall\ i, \succeq'_i \mid a, b \Longrightarrow f(\succeq_{(i)})|\ a, b = f(\succeq'_{(i)})|\ a, b$$

i.e., adding or removing elements to the set of alternatives does not change the output of a constitution for the pair $\{a, b\}$

- **Pareto-efficiency**: constitution f is Pareto-efficient if $\forall$ profiles $(\succeq_{(i)})$, for all $a, b \in A$

$$\forall\ i, a \succeq_i b \Longrightarrow a \succeq b$$

where $\succeq \ = f((\succeq_{(i)})$. i.e., if everyone prefers a over b, that also becomes the preference of the constitution

- **Dictatorship** if $\exists i$ s.t.

$$a \succeq_i b \Longrightarrow a \succeq b$$

where $\succeq \ = f((\succeq_{(i)})$. i.e., if the constitution simply mimics i's preferences.

- **Monotonic** if a single individual ranking higher $a \in A$ never causes a rank lower in the constitution
- **Non-imposition** is satisfied if all rational preferences can be outputs

A - **Theorems:**

- Arrow, 1951: there is no constitution f for which all there properties hold at the same time: f is not a dictatorship, f is monotonic, f satisfies IIA and non-imposition
- Arrow, 1963 (Arrow's impossibility theorem): if constitution f is Pareto-efficient and satisfies IIA $\Longrightarrow f$ is a dictatorship

19. Q - Game theory and elections

A - In an election a candidate the beats (by majority) all the others is called the **Condorcet winner**. If there is no winner, then there is a cycle (i.e., A > B, B > C, C > A) called **Condorcet cycle**. With more than 3 candidates there can be a winner and a cycle. With preferences sufficiently randomized and a large $(n \longrightarrow \infty)$ numbers of candidates, Condorcet cycles are sure to occur

voters→					
choices↓	3	5	7	9	∞
3	5.6%	6.9%	7.5%	7.8%	8.8%
5	16.0%	20.0%	21.5%	23.0%	25.1%
7	23.9%	29.9%	30.5%	34.2%	36.9%
∞	100.0%	100.0%	100.0%	100.0%	100.0%

20. Q - What electoral methods we have?

A - We have these methods, all with their strengths and weaknesses

- **Plurality voting**: each voter sort candidates in order of personal preference. The candidate with most first places, wins.

  With this method, a candidate with a minority high preference (i.e., being in first place by less than 50% of voters) wins over candidates in lower places by majority (i.e., if A is in first place for 4/10 voters, in last place for others, B is in second place for all voters and other first places are distributed by other candidates, A wins even if B is preferred by the majority over A)

- **Two-phase run-off**: first round voting: select two candidates with highest amount of votes; second round voting: run-off between those candidates.

  With this method, a candidate being the "second choice" for a large majority will never be in the second phase, even if he is preferred to the others by majority (i.e. candidate C has few first place votes but it's preferred to A by every B's voter and it's preferred to B by every A's voter. It will not pass to the second phase, even if it's the Condorcet winner)

- **Borda counting**: suppose we have M candidates, each person gave M-1 points to his favourite candidate, M-2 to the second and so on till the last-favourite, who receive 0 points.

  With this method, a strong candidate voted by the majority and being in last position by the others loses against a candidate being mediocre (i.e., A is voted in first place by half voters and last place by the other half. Considering first two position of the other half equally distributed by B and C, A is the Condorcet winner. But B or C will win elections thanks to all the second place points). Borda counting has also an huge issue with dropouts: a single contestant withdrawn can totally change the outcome. (examples in slides)

- **Approval voting**: each voter can give more than one preference, every preference is a vote, the number N of preferences goes from 1 to M, with M the number of candidates (for N=1 we are in plurality voting).

  Depending on N, less favourite candidates has less or more chances to win.

- **Instant run-off**: asking voters to place candidates in order of preferences, only first placed goes counts in order to second round voting. Iteratively, we remove candidates with lowest amount of top preferences, till we get a majority.

  Also in this scenario, a small change (even an increase in preferences) can lead to a different outcome.

6 NE applications (Duopolies, tragedy of the commons, selfish routing)

To be better understand

7 Mixed strategies, Nash theorem

21. Q - What is a mixed strategy?

    A - **A mixed strategy** for player i is, in a game $G = (S_1, \dots, S_n; u_1, \dots, u_n)$ a probability distribution p_i over set S_i.

    i.e. choosing strategies $S_i = (s_i^{(1)}, \dots, s_i^{(k)})$with probabilities$(p_i(s_i^{(1)}), \dots, p_i(s_i^{(k)}))$

22. Q - What is expected utility?

    A - **Expected utility** is (as expansion of utility) a real function over $\Delta S_1 \times \cdots \times \Delta S_n$. For a chosen mixed strategy, payoff can be calculated as a weighted average over p_i.

    $$u_i(p_i, \dots, p_n) = \sum_{(s_1, \dots, s_n) \in S} p_1(s_1) \cdots p_n(s_n) \cdot u_i(s_1, \dots, s_n)$$

    with $S = S_1 \times \cdots \times S_n$.

    Strict and weak dominance are, as always, $>$ or $\geq$. Nash equiibrium is based on expected utility.

23. Q - What is the support of a mixed strategy?

    A - Given a mixed strategy $p_i \in \Delta S_i$ we define the support of p_i as $supp(p_i) = s_i \in S_i : p_i(s_i) > 0$ (if $p_i(s_i) = 1$, s_i is a pure strategy).

24. Q - Nash theorem

    A - Every game with finite pure-strategy sets S_i has at least one Nash equilibrium, possibly involving mixed strategies. More formally: for game $G = (S_1, \dots, S_n; u_n, \dots, u_n)$, define

    $$BR_i : \Delta S_1 \times \cdots \times \Delta S_{i-1} \times S_{i+1} \longrightarrow 2^{\Delta S_i}$$

    $$BR_i(p_{-i}) = p_i \in \Delta S_i : u_i(p_i, p_{-i}) \text{ is maximized}$$

    . Then define $\mathbf{BR} : \Delta S \longrightarrow 2^{\Delta S}$ as

    $$\mathbf{BR}(p) = BR_1(p_{-1}) \times \cdots \times BR_n(p_{-n})$$

    p is a NE if $p \in BR(p)$

8 Potential games, congestion games, coordination games; computational complexity of Nash Equilibrium search

25. Q - What is a potential game?

A - **A potential game** is a fictitious game that converges to NE. A fictitious game is a game where regrets become actual changes of moves.

Function $\Omega : S \to \mathbb{R}$ is an exact potential for $\mathbb{G}$ if:

$$\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = u_i(s_i', s_{-i}) - u_i(s_i, s_{-i} = \Delta U_i$$

Function $\Omega : S \to \mathbb{R}$ is a weighted potential for $\mathbb{G}$ with weight $w = w_i > 0$ if:

$$\Omega(s_i', s_{-i}) - \Omega(s_i, s_{-i}) = w_i \Delta u_i$$

Function $\Omega : S \to \mathbb{R}$ is an ordinal potential for $\mathbb{G}$ if:

$$\Omega(s_i', s_{-i}) > \Omega(s_i, s_{-i}) \Longleftrightarrow u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

26. Q - Th: Every potential game has (at least) one NE in pure strategies

27. Q - Which types of potential games does exist?

A - Congestion game: a potential game where players choose the ”least congest” resource.

Coordination game: models situations where players have incentive to coordinate their actions.

Dummy (or pure-externality) game: game in such that $\forall\ s_{-i}, u_i(s_i, s_{-i}) = u_i(s_i', s_{-i})$, i.e. players payoff depends only on s_{-i}.

N.B. every potential game is a sum of coordination game and dummy game.

28. Q - What is the computational complexity of finding NE?

A - **It's PPAD.** NE theorem states that a solution must exist, so it cannot be NP-complete. On the other hand, finding a NE in some case can be very difficult. Let's say (with the nth abuse of notation) that $P < PPAD < NP$. Wich means that $PPAD$ is hard till we demonstrate $P = NP$. In this class we find the ”end-of-line problem”.

9 Only for projects. LOL

10 Exercise set #1 (static games of complete information)

See the end of lectures for exercises and solutions.

11 Dynamic games

29. Q - What is a dynamic game?

A - **A dynamic game** is a game in which players moves are sequential and not simultaneous.

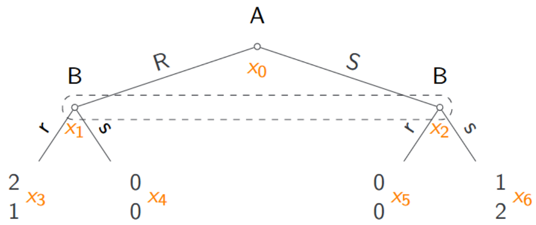
They can be of **perfect information** (meaning that every player can do every decision with full awareness) or **imperfect information** (meaning some decisions are "simultaneous" or Nature moves).

We have two scenarios for the former information:

- **endogenous uncertainty**: information sets contain multiple nodes (simultaneous moves)
- **exogenous uncertainty**: there is a choice of Nature (lotteries)

30. Q - How can we represent dynamic games?

A - Graphically, they can be represented as a tree, where each level is a move and each node is the response to a specific opponent' move. Dotted circle (or just a dotted arch) shows node from the same information set, i.e. the game stage a player can have.



A **information set** h_i is a mathematical representation of player's possible moves. If $h_i = \{x_j\}$ (i.e. it has just one node), then the node is fully aware of previous moves.

31. Q - How to define a strategy in dynamic games?

A - A strategy in dynamic games need to account for the history of play.

Imagine a two-move game, B has strategy $s_B = (a_1, a_2)$ where a_1 is the response to A playing move 1. If the match is played twice, B has strategy

$$s_B = (a_1, a_{11}, a_{12}, a_{21}, a_{22})$$

, where $a_1 1$ is the answer of A first move, B playing 1 and A answering with 1 (for example, $s_b = (1, 1, 1, 1, 1)$ means B playing 1 no matter what. Instead, $s_b = (1, 2, 1, 1, 1)$ means B playing 2 if A's second move is 1, otherwise playing 1.

12 Mixed strategies and Nash equilirbia in dynamic games, subgame-perfect Nash equilibria

32. Q - What is the difference between mixed and behavioral strategies? Under what condition are the two concepts equivalent?

> A - **Mixed strategies** are decided before the game starts; **behavioral**, on the other hand, are more "realistic" and every move is decided after opponent makes his (more formally: at each information set, select a random move from your set of available moves). They are equivalent under the perfect **recall assumption**: players do not forget the information they have acquired.

33. Q - How do you find the solution(s) of a sequential game?

> A - **By backward induction:** starting from the leaf with maximum payoff for the player, replace the parent node with the payoff, repeat until root. More formally:

- Start from the leaves of the tree
- Partition them according to their parent node (which is a move of some player i)
- For each set in the partition, find the leaf x_j with maximum payoff for player i
- Add the corresponding branch to player i's strategy
- Replace the parent node with the payoff of x_j
- Repeat until the root is reached

Solutions can be unique as can be more than one. This way, we find the **equilibrium path**.

A (proper) **subgame** $\mathbb{G}'$ of a game $\mathbb{G}$ contains a single node of the tree and all of its descendants, with the requirement that

$$x_j \in \mathbb{G}', x_j \in h_i \hookrightarrow x_k \in \mathbb{G}' \ \forall \ x_k \in h_i$$

An **equilibrium path** is a path that contains all nodes of a NE alont the tree rapresentation of the game.

34. Q - What is a subgame-perfect NE? In which cases is a NE not subgame-perfect?

> A - **A subgame-perfect NE (SPE)** is the strategy that yield a NE in every subgame. Every SPE is a NE in the parent game and every finite dynamic game has at least one SPE. This means that every sequential game (tic tac toe, chess etc ...) has a NE.

35. Q - **Exercise**

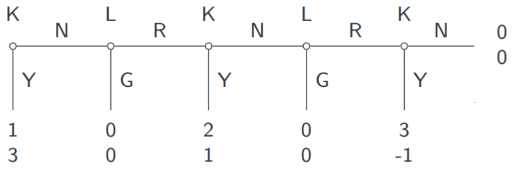
Solution can be found in Lecture13 pdf

It is the discount sales season, Karen (K) and Lou (L) wants to go shopping. He thinks it is best to wait until the last three days of the discount sales, because prices are cheapest. On day 1, Lou asks Karen (K) to go with him. If Karen says yes (Y), they go shopping and the game ends. If Karen says no (N), Lou can either give up (G) or request (R) again the following day. On day 2, the same happens. On day 3, if K still says N, the game ends as well and does not go into a further day. If in the end K and L do not go shopping, both of their payoffs are 0. If they do, their payoffs are computed as $u_K = d$ and $u_L = 5 - 2d$, respectively, where d is the day in which they go. All of this information about the game is common knowledge among the players.

1. Represent the game in its extensive form.

2. How many (pure) strategies do K and L have, respectively?
3. Find the SPE of this game.
4. Find one NE that is not subgame-perfect.
5. Represent the game in its normal form.
6. Find all the NE of this game (both SPE and non-SPE).

A - **Solution**



■ K's payoff are on top, L's payoff are on the bottom

1.
2. K plays in 3 information sets and has 2 possible actions in each set: $2^3 = 8$ possible strategies. Her strategies are triplets: $s_K = (a_0, a_2, a_4), a_j \in Y, N$

   L plays in 2 information sets has 2 possible actions in each set: $2^2 = 4$ possible strategies. His strategies are triplets: $s_L = (a_1, a_3), a_j \in G, R$
3. The SPE can be found via backward induction. On the last day, K prefers Y (payoff 3) to N (payoff 0): $a_4 = Y$. Before that, L prefers G (payoff 0) to R (payoff -1, knowing K's next choice): $a_3 = G$. Before that, K prefers Y (payoff 1) to N (payoff 0, knowing how the rest of the game will play out): $a_2 = Y$. Only SPE: $s_K = (N, Y, Y)$, $s_L = (R, G)$
4. Just chose and irrational move on SPE (outside equilibrium path)
5. Just draw it
6. Trivial and left to the reader

13 Multistage games pt. 1

36. Q - What is a multistage game?

A - **A multistage game** is a game which consist in a sequence of smaller games, called **stages**, where total payoff is the sum of every payoff.

Some games can have independent stages (like partial exam), other can have a discount **factor** $\delta \in [0, 1]$ when stages are far apart in time. If so, the total payoff of player i is

$$u_i = u_i^{(1)} + \delta u_i^{(2)} + \delta^2 u_i^{(3)} + \cdots + \delta^T u_i^{(T)} +$$

Discount is en exponential function of time t. It has to be exponential (not linear) because of time consistency.

14 Multistage games pt 2, stick-and-carrot strategies

37. Q - What is a strategy in a multistage game?

    A - Each player has to specify what to do in the first stage and what to do in subsequent games depending on the outcome of previous games.

38. Q - Theorem for independent stages

    A - Suppose $s^{(t)} = (s_1^{(1)}, \dots, s_n^{(t)})$ is a NE strategy for stage t of multistage game $\mathbb{G}$; then there is a SPE of $\mathbb{G}$ with equilibrium path $(s_1^{(1)}, \dots, s_n^{(t)})$. In other words, a strategy where each player always plays a NE, is a SPE.

39. Q - Theorem for linked stages 1

    A - Every NE s^* of multistage game $\mathbb{G} = (\mathbb{G}_1, \dots, \mathbb{G}_T)$ requires that a NE is played in the last stage. In other words, in order for a strategy to be a NE, it has to have a NE in the last stage, since players already know all previous outcomes.

40. Q - Theorem for linked stages 2

    A - If stage games $\mathbb{G}_1, \dots, \mathbb{G}_T$ has all unique NE, then $\mathbb{G} = (\mathbb{G}_1, \dots, \mathbb{G}_T)$ has unique SPE. In other words, if every stage has only one NE, this is the equilibrium path.

    There is a counter intuitive consequence of these theorems: if the last stage has multiple NE, that enable non-NE strategies to be played in previous stages.

41. Q - What is the meaning of δ?

    A - δ is the measure of how much we care about the future. A larger δ means we care more about future payoffs.

42. Q - One-stage deviation principle

    A - A one stage unimprovable strategy must be optimal. *(proof in Lecture14)*

Optimal strategy: a strategy s_i is optimal for player i i $\forall$ information set h_i there is no way to improve it (more formally: $\nexists s_i'/u_i(s_i'|\{h_i\} > u_i(s_i|\{h_i\})$.

One-stage unimprovable strategy: a strategy s_i is one-stage unimprovable if there is no s_i' that differes in one single stage s.t. $u_i(s_i'|\{h_i\} > u_i(s_i|\{h_i\}$

43. Q - Exercise 1

    A - Consider multistage game $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$, with $\mathbb{G}_1$ being the first stage, and $\mathbb{G}_2$ being the second (and last) stage:

- Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a non-NE is played in G2? **No**
- Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a NE for $\mathbb{G}$ where a non-NE is played in G2? **No**
- Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a non-NE is played in G1? **Yes**
- Is it possible (for some $\mathbb{G}_1$ and $\mathbb{G}_2$) to find a SPE for $\mathbb{G}$ where a strictly dominated strategy is played in G1? **Yes**

- What is the minimum number of NE in stage game $\mathbb{G}_2$ to enable a carrot-and-stick SPE in G? What characteristics should these NE have? **At least two NE (stick and carrot)**

  Answers are mine, so feel free to mail me if they are wrong

44. Q - Exercise 2

Ashley and Brook live together. During the winter break they contemplate giving each other a nice gift (G) for Christmas or not (N). They know each other's preferences so they are able to buy a gift for 10 euros that is worth like 100 euros for the other. They make this decision independently and without telling each other. After Christmas, they also consider whether to celebrate New Year's eve downtown (D) or stay home (H). For the New Year's eve celebration, they decide independently of each other in a coordination-game fashion. Staying home has utility of 0 for both. Going downtown has utility of 50. However, spending New Year's eve apart from each other has utility of -100 for both. The total payoff of the players is the sum of the partial payoffs in each stage with a discount factor of δ for the second stage.

1. Write down the normal form of both stages of the multi-stage game.
2. Find a trivial subgame-perfect equilibrium of the game where the players just play a Nash equilibrium in all stages, without any strategic connection.
3. Is there a strategically connected SPE of the whole game where Ashley and Brook give gifts to each other? If so, show the minimum required discount factor value δmin for that to hold.

A -

1. Normal form of stage game Only NE in Stage 1 is (N, N)

	G	N
G	90 90	-10 100
N	100 -10	0 0

	H	D
H	0 0	-100 -100
D	-100 -100	50 50

   In stage two NE are: (H, H) (stick) and (D, D) (carrot)

2. Trivial SPE is playing stages independently: $(NHHHH, NHHHH)$ (or with the carrot)
3. Cooperative SPE: "play G at stage 1. If (G, G) at stage 1, play D, otherwise play H". Sustainable if $u(\text{cooperative}) + \delta u(\text{carrot}) \geq u(\text{unilateral deviant}) + \delta u(\text{stick}) \longrightarrow 90 + 50\delta \geq 100 + 0\delta \longrightarrow \delta \geq \frac{1}{5}$

15 Repeated games, grim trigger, tit for tat, Friedman theorem

45. Q - What is a repeated game?

A - **A repeated game** $\mathbb{G}(T, \delta)$ is a dynamic game where a static game $\mathbb{G}$ is played as a stage game for t times with discount δ. We distinguish between:

- finitely repeated games ($T = 1, 2, \cdots < \infty$);
- infinitely repeated games ($T = \infty$). Here we must have $\delta < 1$, otherwise payoff will diverge.

46. Q - Theorems

A -

1. The outcome of the last stage is a NE
2. If stage game $\mathbb{G}$ only has NE p^*, then $\mathbb{G}(T, \delta)$ has a unique subgame-perfect equilibrium, where every player play p^* in every stage (boring)

47. Q - Cooperation in finitely repeated games

A - **Cooperation** is usually incentived in repeated games but in the last stage, which is played "egoistically". As for multistage games, cooperation is possible if there are multiple NE.

48. Q - Infinitely repeated game

A - Since these games has no last stage, there could be a SPE of $\mathbb{G}(\infty, \delta)$ in which no stage outcome is a NE of $\mathbb{G}$.

We call a **grim trigger strategy** (GrT) a strategy where: start playing M at stage 1, at stage $t > 1$, play M only if outcome of all $t - 1$ previous stages was (M, m), otherwise play F.
With $\delta = 1 - \epsilon$, joint strategy "both play GrT" is a SPE.

49. Q - How can we show that a strategy is a NE in overall game?

A - We only need to compare who options:

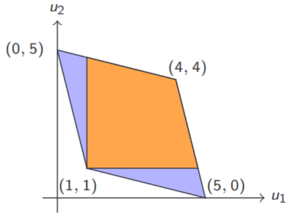
- Cooperate and keep playing the chosen strategy (payoff p^*) forever ($u_T = p^* + \delta p^* + \delta^2 p^* + \cdots = \frac{p^*}{1-\delta}$)
- Defect at stage 1 and keep playing other option (payoff p) forever ($u_T = p + \delta p + \delta^2 p + \cdots = p^* + \frac{p\delta}{1-\delta}$)

In order to find minimum δ s.t. cooperation is best choice, solve

$$\frac{p^*}{1-\delta} \geq p + \frac{p\delta}{1-\delta}$$

50. Q - Friedman theorem (a.k.a. "folk theorem")

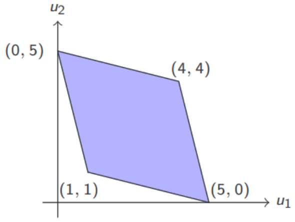
A - **Theorem** Let $\mathbb{G}$ be a finite static game of complete information. Let $(e_1, e_2, \dots, e_n)$ be the payoffs of a NE of $\mathbb{G}$ and $(x_1, x_2, \dots, x_n)$ be a feasible payoffs for $\mathbb{G}$. Suppose $\forall$ NE and $\forall$, $x_j > e_j$. Then, for δ close enough to 1, $\mathbb{G}(\infty, \delta)$ has a SPE with payoffs $(x_1, x_2, \dots, x_j)$.



A **feasible payoff** for game $\mathbb{G}$ is any convex combination

$$\alpha u(s_1) + \alpha u(s_2) + \cdots + \alpha_L(s_L), \text{with} \sum_{i=1}^{L} \alpha_i = 1$$

of pure-strategy payoffs ($L = |S_1| \cdot |S_2| \cdots |S_n|$ total numbers of pure strategy)



51. Q - Tit for tat (TFT)

    A - **TFT** is a replacement of GrT where "at stage t, play what the other player chose at stage $t-1$", it's a way to avoid keep punishment forever. It has immediately punish deviation but forgiveness after 1-step.

    Keep doing TFT forever is called "death spiral" and it's not a NE. generally, NE achieved by TFT is not subgame-perfect.

52. Q - Exercise 1
    Consider $\mathbb{G}$:

Player B

Player A		g	w
	G	5, 3	0, 4
	N	6, 0	1,1

1. Is it possible to find a SPE for $\mathbb{G}(4)$ that involves playing (G,g) at each stage?
2. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a grim-trigger strategy? If so, for what δ? Is that a SPE?
3. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a tit-for-tat strategy? If so, for what δ? Is that a SPE?

A - TODO:

53. Q - Exercise 2

Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(S, M) = u_D(M, S) = 0$. Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}$(T). Individual payoffs are summed with discount factor δ.

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. Consider an extended game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for both players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}$' . If you see a SPE of $\mathbb{G}$'(2) where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

A - TODO:

16 Repeated games (part 2), zero-sum games, minimax theorem

54. Q - What are maximin and minmax?

A - **Maximin and minmax** are strategis s.t.:

- **Maximin**

$$w_i = \max_{s_i \in S_i} \min_{s_{-1} \in S_{-i}} u_i(s_i, s_{-i})$$

In other words, it's about finding a security strategy (a conservative approach in order for i to achieve the highest payoff against $-i$ the worst move) $s_i^* = \arg\max_{s_i} f_i(s_i)$ with

$$f_i : S_i \to \mathbb{R}, f_i(s_i) = \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

Then evaluate $w_i = f_i(s_i^*)$

- **Minimax**

$$z_i = \min_{s_{-1} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

In other words, it's about finding a strategy for i to calculate the minimum payoff against $-i$ move: $s_{-i}^* = \arg\min_{s_i} F_i(s_i)$ with

$$F_i : S_i \to \mathbb{R}, F_i(s_i) = \max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Then evaluate $z_i = F_i(s_{-i}^*)$

Example

		Player B			
		L	C	R	f_A(min)
Player A	T	5, -	3, -	4, -	3
	D	2, -	6, -	1, -	1
	F_A(max)	5	6	4	

Maxmin of A : 3
Minimax of A: 4

55. Q - Consequences of maximin and minimax

A - We can prove that, if joint strategy is a NE

$$maxmin_i \leq minmax_i \leq u_i(NE)$$

If joint strategy is not a NE then just first disequation

56. Q - Zero-sum games

A - **Zero sum game** has the property

$$u_i(s) = -u_i(s)$$

(in every cell you have $x, -x$)

57. Q - Adversarial games

A - **Adversarial games** are a more general type of games where two players are adversaries and have utilities s.t.

$$u_i \uparrow \iff u_{-i} \downarrow$$

Many adversarial game can be framed as a zero-sum game with this trick: $u_A = \text{points}_A - \text{points}_B$ and $u_B = -\text{points}_A$

58. Q - Theorem

A - **For zero-sum game** Let $\mathbb{G}$ be a zero-sum game with finite number of strategies. Then,

1. $\mathbb{G}$ has a pure NE $\iff$ $maximin_i = minimax_i$ for each player i
2. All NE yield the same payoffs $(minimax_i, -minimax_i)$
3. In all NE, every player is playing a security strategy

59. Q - Consider a generic game $\mathbb{G}$

- Is $maximin_i$ always equal to $minimax_i$ for each player i?
- Is $maximin_i^p$ always equal to $minimax_i^p$ for each player i?
- If $maximin_i = minimax_i$ for each player i, does that mean that the game has a pure NE? Does your answer change if $\mathbb{G}$ is zero-sum?
- If $\mathbb{G}$ is a zero-sum game between i and $-i$, is there a relationship between the minimax of i and the maximin $-i$?

A - TODO:

60. Q - What is the value of a zero-sum game? Does it always exist if the game has infinitely many strategies?

A - TODO:

17 Stackelberg games, dynamic bargaining

61. Q - Stackelberg game

    A - **A Stackelberg game** is a sequential version of a static game. Players move one after the other (first the leader, than he follower). Result, which can be obtained via backward induction, is called a Stackelberg equilibrium.

62. Q - How to find Stackelberg equilibrium

    A - First find best options of the follower, then among these options, the best one for the leader. In case of a tie, we can have

- generous follower
- generous leader

63. Q - Stackelberg game consequences

    A -

- leader payoff in Stackelberg equilibrium $\geq$ payoff in NE of the static game
- follower payoff, in general, $\geq$ minimax

In a rational game, follower knows leader's move, but leader has a full knowledge of the game: can anticipate follower's move.

64. Q - Dynamic bargain

    A - **Bargain** means negotiation of resources. (It's a game of negotiation).

    Assume two players get to split a given amount of resources, we can use **dynamic bargaining** where players switch proposer/responder at any stage. If they disagree at any stage till stage T, they both get payoff 0. If they agre at stage $0 < t < T$ they get a discount δ^{t-1}.

    Is the deadline is $T = 1$ the game is called **Ultimatum game**.

65. Q - SPE of dynamic bargain

    A - **Proposition:** Any SPE of the dynamic bargaining game must have the players reaching an agreement in the first round

- Simply a consequence of backward induction
- Iterating the game: (i) wastes reward because of the discount; (ii) sends the players to another round of proposer-responder, which rational players want to avoid

For $T \to \infty$ we have

$$u_1 = \frac{1}{1+\delta}$$

$$u_2 = \frac{\delta}{1+\delta}$$

18 Bayesian games

66. Q - Bayesian games

A - **Bayesian games** are games of incomplete information, where some players do not have complete knowledge on the other players' utility. Bayesian games have the following assumptions:

- Each player i has a set of possible types $T_I = \{\text{type } 1, ..., \}$
- The type of each player $t_1, ..., t_n$ is determined by Nature's move at the beginning of the game.
- Each player knows only his/her type
- Other player's type is unknown

67. Q - Preliminary Nature's move

A - Bayesian game scan be seen as a weird kind of dynamic games that is played as follows: Nature draws the type vector $(t_1, .., t_n)$ among all the possible combination of players' types; Nature reveals type t_i only to player i; player choose their action; Final payoffs are computed.

68. Q - Beliefs in Bayesian games

A - To find best responses player create beliefs about these types under the assumption that players do not precisely know the types of their opponents but they have an estimate of those. In other words, they know the probability distribution of the opponents' types. This is called the common prior assumption.

69. Q - Bayesian entry game

A - We have an incumbent (player 1) and and outsider (Player 2). The outsider decides whether to enter the market (E) or stay out (O). The incumbent decides whether to accept the outsider (A) or fight (F). The incumbent could be of two types: reasonable or crazy (in the first case it does not like to fight, in the latter it does). The incumbent knows its type, the outsider estimates reasonable/crazy with probabilities (p,1-p).

70. Q - Normal form of Bayesian games

A - The normal form of Bayesian games can be inferred from the extensive form as we did in dynamic games.

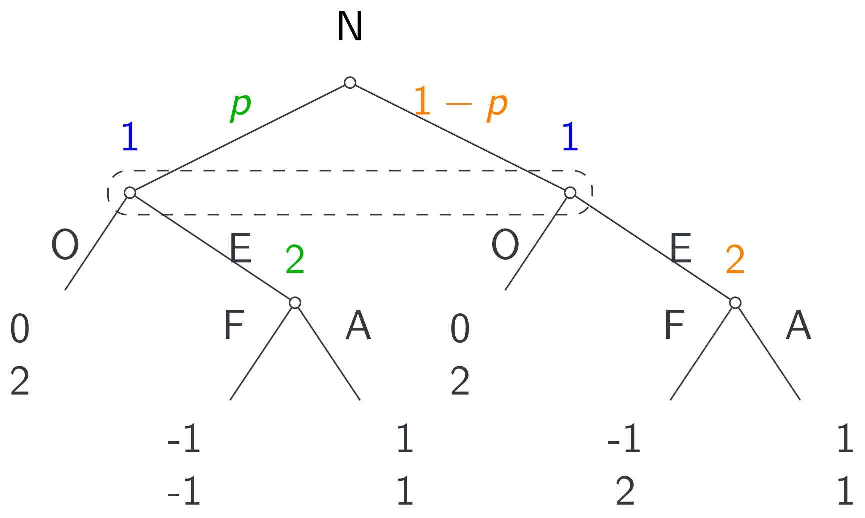
71. Q - Strategies in Bayesian games

A - A strategy specifies what the player does for each type. In the entry game: Player one has two strategies $\{O, E\}$, while player 2 has four strategies: $\{AA, AF, FA, FF\}$. For example if 2 plays AF it means play A if reasonable or play F if crazy.

72. Q - Example to understand the normal form:

Bayesian entry game

- **Extensive form** of the Bayesian entry game:



Normal form of Bayesian games

- The normal form of Bayesian games can be inferred from the extensive form as we did in dynamic games
- Let us start by listing strategies: this can be done by considering all combinations possible moves in each information set
- In the Bayesian Entry Game:
  - Player 1 only has one information set (does not know Nature's move, which decide player 2's type): only one move needed to define pure strategies
  - Player 2 has two information sets (knows his/her type): his/her pure strategies are all possible combination of moves in these 2 information sets

Strategies in Bayesian games

- In Bayesian games, the concept of strategy is further extended: a strategy specifies what the player does for each type
  - Think of it as a "Dr. Jekyll and Mr. Hyde" situation
  - E.g., strategy AF for player 2 means "If I am a "reasonable" type, I accept the outsider; if I am a "crazy" type, I fight them"
  - This might seem strange, since players know their actual type; however, they need to do this reasoning in order to predict the opponents' strategy
- *Example*: In the Bayesian Entry Game:
  - Player 1 has two strategies: {O, E}
  - Player 2 has four strategies: {AA, AF, FA, FF}
- Each pair of pure strategies determines how the game is played, which also depends on Nature's choice

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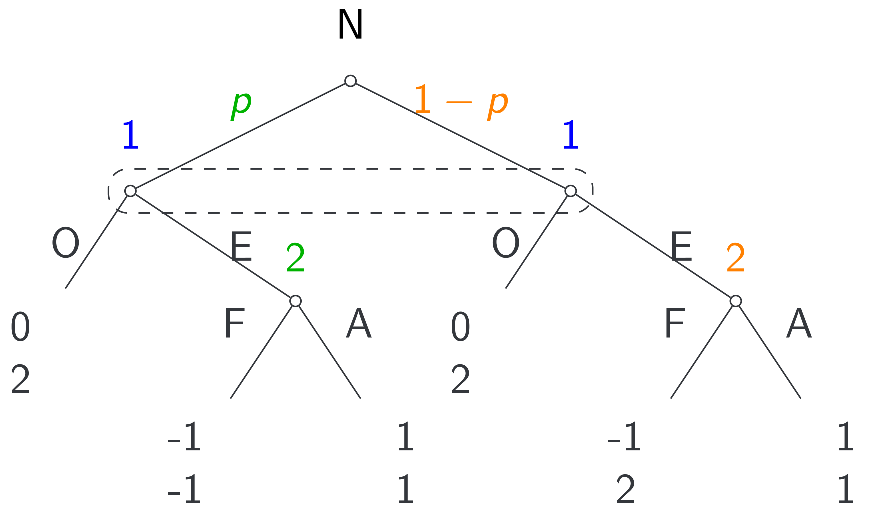
Bayesian entry game



- For example, if the game is played as (E, AF)
  - player 1 gets: $p \cdot 1 + (1-p) \cdot (-1) = 2p - 1$
  - player 2 gets: $p \cdot 1 + (1-p) \cdot 2 = 2 - p$

Bayesian entry game, normal form

- We can determine player 2's best responses against O (anything)
- We can determine player 1's best response against AA and FF (when 2 always plays the same regardless of its type)
- So we know that (O,FF) is a NE regardless of p
- However, we need to know p in order to find *all* NE

Player 2

		AA	AF	FA	FF
Player 1	O	0, **2**	0, **2**	0, **2**	**0**, **2**
	E	**1**, 1	2p-1, 2-p	1-2p, 1-2p	-1, 2-3p

Bayesian entry game, NE

- For $p = 0$ (The outsider is sure that the incumbent is crazy)

Player 2

		AA	AF	FA	FF
Player 1	O	0, **2**	**0**, **2**	0, **2**	**0**, **2**
	E	**1**, 1	-1, **2**	**1**, 1	-1, **2**

- (O, AF) and (O, FF) are NE
- Payoffs of player 2 only reflect the choice of the crazy type (choices of reasonable type are neglected)
- **Interpretation**: the outsider (player 1) is 100% sure that the incumbent is crazy, so the moves of a "reasonable" player 2 are ignored
- Therefore, the NE are all joint strategies (O, *F)

Bayesian entry game, NE

- For $p = 1$ (The outsider is sure that the incumbent is reasonable)

Player 2

Player 1

	AA	AF	FA	FF
O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
E	**1**, **1**	**1**, **1**	-1, -1	-1, -1

- (E, AA), (E, FA) and (E, AF) are Nash equilibria
- **Interpretation**: the outsider is 100% sure that the incumbent is reasonable, so the crazy-incumbent moves are ignored
- NE are (E, A*) and (O, F*)

Bayesian entry game, NE

- For $p = 2/3$

Player 2

Player 1

	AA	AF	FA	FF
O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
E	**1**, 1	**1/3**, **4/3**	-1/3, -1/3	-1, 0

- NE: (E, AF), (O, FA), (O, FF)
- **Interpretation**: The outsider thinks that the incumbent is likely to be reasonable, but there is still some probability of them being crazy. Therefore, joint strategies like (E, AA) where also the crazy type is accepting the outsider are not NE.

73. Q - Bayesian game: definition

A - It is described as follows:

- Set of players $\{1, .., n\}$
- Strategy sets $S_1, .., S_n$
- Utility functions $u_i : \S_1 \times ... \times S_n \rightarrow \mathbb{R}$(for i=1,...,n)
- **Type space** of each player T_i
- **Type-dependent utilities:** define $u_i(a_1, ..., a_n, t_i)$ for each $t_i \in T_i$
- **Beliefs about players types:** a probability distribution ϕ_i defined over types for each player

74. Q - Static Bayesian game

A - Consider a static Bayesian game: n players, each player's strategy is just an action:

- Player i's type is $t_i \in T_i$, chosen by Nature for each player from 1 to n through the joint prior probability distribution $\phi(t_1, ..., t_n)$ where $\phi : T_1 \times ... \times T_n \rightarrow [0, 1]$
- Player i knows his private values of his/her utility function $u_i(a_1, ..., a_n, t_i)$

75. Q - Type of a player

A - Suppose i can have two different payoff functions $u_{i,A}(a_i, a_{-i})$ and $u_{i,B}(a_I, a_{-i})$. We represent this by setting a type space $T_i = \{t_A, t_B\}$ and imposing $u_{i,j}(a_i, a_{-i}) = u_i(a_i, a_{-i}, t_j)$.
Types can be correlated; they are independent if

$$\phi(t_1, ..., t_n) = \phi(t_1)...\phi(t_n)$$

19 Bayesian Nash equilibria, signaling.

76. Q - Static Bayesian game

A - **A static Bayesian game** needs a set of player $\mathcal{N} = \{1, ..., n\}$, action space $A_1, ..., A_n$(pure strategy sets), type space T_i(for i=1,...,n), beliefs (on types) $\phi_1, ..., \phi_n$, time-dependent payoffs $u_i(a_1, ..., a_n, t_i)$.

$$\mathbb{G}(\mathcal{N}; A_1, ..., A_n; T_1, ..., T_n; \phi_1, ..., \phi_n; u_1, ..., u_n)$$

A **pure strategy** for i can be seen as a map $s_i : T_i \to A_i$,i.e., it tells what i plays as his/her type is known. A **Mixed strategy** for i is a probability distribution over pure strategies.

77. Q - Strategies of Bayesian games

A - We can think of a general strategy as being defined before the type of i is even set. Player i decides a strategy $s_i : T_I \to A_i$, then if their type is $t_i \in T_i$, they will play $s_i(t_i)$. It allows other players to create beliefs over the strategy of a player who can be of different types

78. Q - Bayesian Nash Equilibrium

A - **A Bayesian Nash Equilibrium** is a Nash equilibrium in Bayesian games. In $\mathbb{G}(\mathcal{N}; A_1, ..., A_n; T_1, ..., T_n; \phi_1, ..., \phi_n; u_1, ...,$ joint strategy $s^* = (s_1^*, ..., s_n^*)$ is a Bayesian Nash Equilibrium is, for each player i and type $t_i \in T_i, s_i^*$ maximizes the expected payoff against s_{-i}^*:

$$s_i^* = \arg \max_{s_i \in S_i} \sum_{t_{-i}} u_i(s_i, s_{-i}^*(t_{-i}), t_i)\phi(t_{-i})$$

that is the same as:

$$\mathbb{E}[u_i(s_i^*, s_{-i}^*(t_{-i}), t_i)|t_{-i}] \geq \mathbb{E}[u_i(s_i, s_{-i}^*(t_{-i}), t_i)|t_{-i}]$$

Look at slides for some examples

20 Dynamic Bayesian games, perfect Bayesian equilibrium, signaling games.

79. Q - Bayesian equilibrium path

A - If we have a Bayesian NE $s^* = (s_1^*, ..., s_n^*)$, we say that an information set is **"on" the equilibrium path** if, given the distribution ϕ of types, it is reached with probability > 0

80. Q - System of beliefs

A - **A system of beliefs** μ is a probability distribution over decision nodes for every information set. In other words, it is an estimate of being at a specific node, given an information set. It is a conditional probability $\mathbb{P}(\text{node}|\text{information set})$

81. Q - Perfect Bayesian Equilibrium

A - **A perfect Bayesian equilibrium (PBE)**is a pair (s^*, μ), where s^* is a Bayesian equilibrium and μ is a system of beliefs satisfying the following:

- Players must have a system of beliefs
- On the equilibrium path they must follow Bayes' rule on conditional probabilities
- Off the equilibrium path: arbitrary
- Given their beliefs, players are sequentially rational: i.e., they play a best response to their belief

82. Q - How do you determine sustainable beliefs values $\mu(x_i)$ for nodes that are on the Bayesian equilibrium path?

A - Apply the Bayesian rule to the node

83. Q - What values can $\mu(x_i)$ have if x_i is off the equilibrium path?

A -

84. Q - Signaling games

A - **A signaling game** is a 2-player dynamic Bayesian game, where 1 is the first to move and 2 is the second to move. 1's type is chosen among many possible types, 2 has only one type. 2's beliefs are updated after 1's move.

85. Q - Equilibria of signaling games

A - We have 3 kind of equilibria:

- **Separating equilibria:** each type of 1 chooses a different action; thus revealing the type to 2
- **Pooling equilibria:** all types of 1 choose the same action; thus, 2 gets no signal about 1's type
- **Intermediate cases:** 1's action does not fully define 1's type, but still provides some information

86. Q - Example on butterfly games

Example: a coffee for Brooke

- Ann and Brooke are dating; Brooke is invited by a colleague, Zoe, to get a coffee
- Ann is a typed player: her types are
  - Jealous with probability 0.8
  - Easygoing with probability 0.2

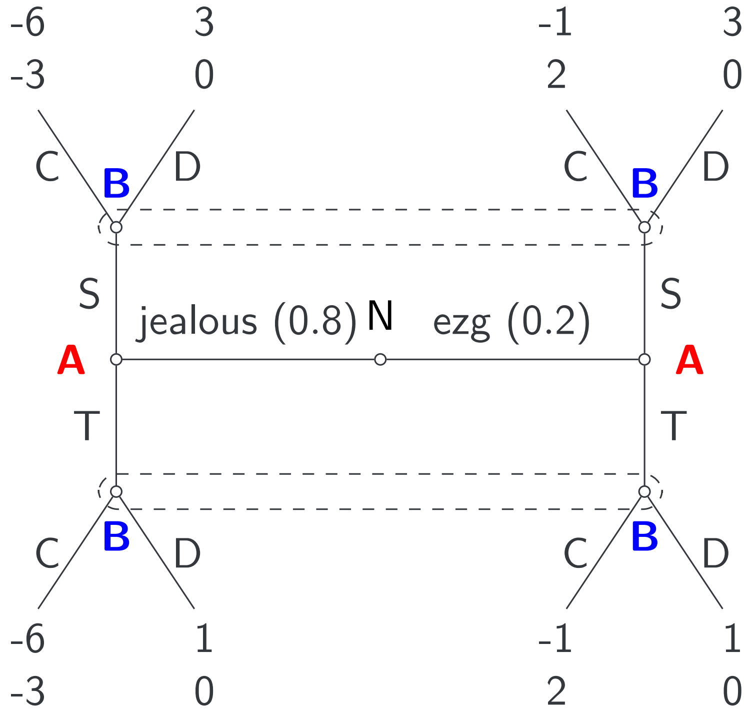
  (all this information is common knowledge)
- Ann can send a signal to either stay silent (S) about this proposal or to trash Zoe out (T)
- Brooke observes the signal, and decides whether to accept the coffee invitation (C) or to politely decline (D)

Example: a coffee for Brooke

- Payoffs:
  - Jealous Ann is deeply hurt if Brooke accepts ($u_A = -6$)
  - Easygoing Ann is just not-so-angry, but still not fond of the idea ($u_A = -1$)
  - Ann prefers to stay silent ($u_A = 3$) rather than trash Zoe out ($u_A = 1$), only in case Brook declines
  - Brooke likes to go to the coffee if that is okay for Ann ($u_B = 2$)
  - If Ann is hurt, Brook prefers declining the invitation ($u_B = 0$) rather than accepting it ($u_B = -3$)

Example: a coffee for Brooke

- Extensive form



Example: a coffee for Brooke

- Both players have 4 strategies but for different reasons
    - Ann because of her type: strategy is (what to do if jealous, what to do if easygoing)
    - Brooke does not have a type but observes Ann's move: strategy is (what to do if Ann plays S, what to do if Ann plays T)
    - e.g., (TS,CD) means that Ann trashes Zoe if she is jealous and remains silent if she is easygoing (separating); Brooke just "follows the signal", going to the coffee if Ann stays silent, and declining if Ann starts trashing Zoe

Example: a coffee for Brooke

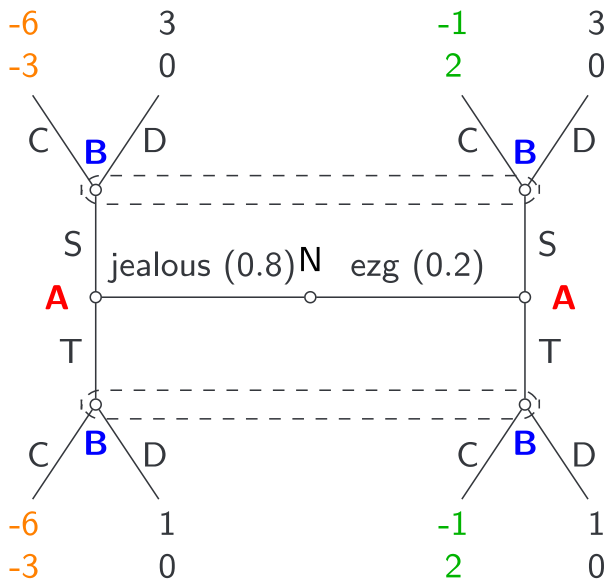
- **Warning!** A's pair is left/right, but B's pair is *B's reaction to A's move*
- First row (SS): only consider B's 1st move (reaction to S)
- Last row (CC): only consider B's 2nd move (reaction to T)

		Brooke			
		CC	CD	DC	DD
Ann	SS	B plays C	B plays C	B plays D	B plays D
	ST	B plays C			B plays D
	TS	B plays C			B plays D
	TT	B plays C	B plays D	B plays C	B plays D

Example: a coffee for Brooke

- If B plays C, utility is always

$$u_A = 0.8\cdot(-6)+0.2\cdot(-1) = -5, \quad u_B = 0.8\cdot(-3)+0.2\cdot(2) = -2$$



Example: a coffee for Brooke

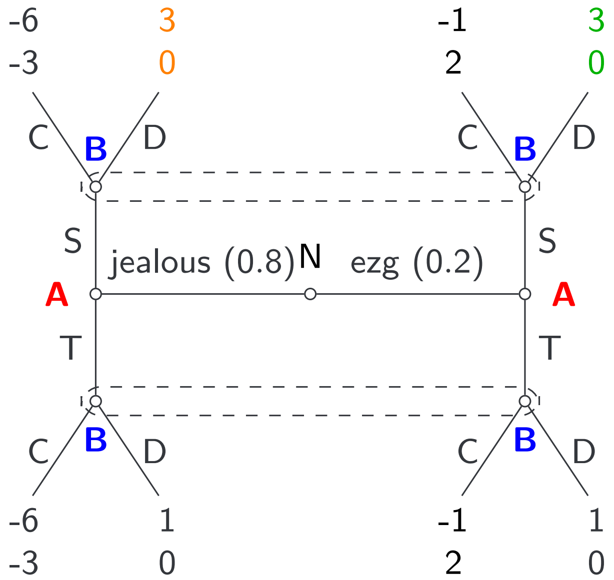
- When B plays D, we need to distinguish between Ann's 4 possible moves (her payoff changes, B's is always 0)

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	B plays D	B plays D
	ST	-5, -2			B plays D
	TS	-5, -2			B plays D
	TT	-5, -2	B plays D	-5, -2	B plays D

Example: a coffee for Brooke

- If B plays D and A plays S, i.e., (SS, D*)

$$u_A = 0.8 \cdot (3) + 0.2 \cdot (3) = 3$$



Example: a coffee for Brooke

- Likewise, if B plays D and A plays T, i.e., (TT,*D), then $u_A = 1$

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	3, 0	3, 0
	ST	-5, -2			B plays D
	TS	-5, -2			B plays D
	TT	-5, -2	1, 0	-5, 2	1, 0

- What about intermediate cases (ST,DD) and (TS, DD)?

Example: a coffee for Brooke

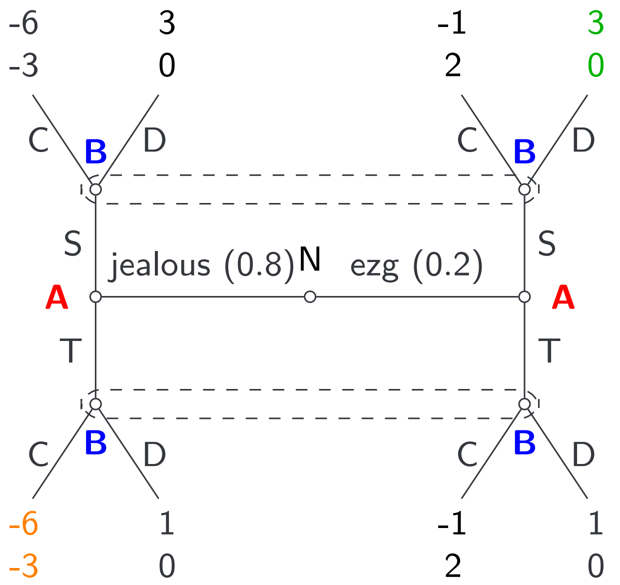
- For (ST,DD) and (TS, DD), you just average the payoffs: in the first case S is played with probability 0.8 and T with probability 0.2; the second case is the opposite

		Brooke			
		CC	CD	DC	DD
Ann	SS	-5, -2	-5, -2	3, 0	3, 0
	ST	-5, -2	?, ?	?, ?	2.6, 0
	TS	-5, -2	?, ?	?, ?	1.4, 0
	TT	-5, -2	1, 0	-5, 2	1, 0

Example: a coffee for Brooke

- E.g., for (TS,DC) (remember: D is answer to S and C is answer to T)

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (3) = -4.2$$



Example: a coffee for Brooke

Brooke

Ann	CC	CD	DC	DD
SS	**-5**, -2	-5, -2	**3**, **0**	**3**, **0**
ST	**-5**, -2	-4.6, -2.4	2.2, **0.4**	2.6, 0
TS	**-5**, -2	0.6, **1.6**	-4.2, -2.4	1.4, 0
TT	**-5**, -2	**1**, **0**	-5, 2	1, **0**

- 3 pure NE: (SS,DC), (SS,DD), (TT,CD)
- 2 mixed NE: (TT, 1/2CD+1/2DD), (1/6SS+5/6TS,2/9CD+7/9DD)

Example: a coffee for Brooke

- So far, we have only found NE, **now we need to classify them**! Are they PBE?
- To verify that, we need to construct systems of beliefs μ for Brooke
  - i.e., $\mu =$ is Brooke's belief that Ann is *jealous*
  - One belief for each possible observed move by Ann: μ_S if she stays silent; μ_T if she trashes Zoe out

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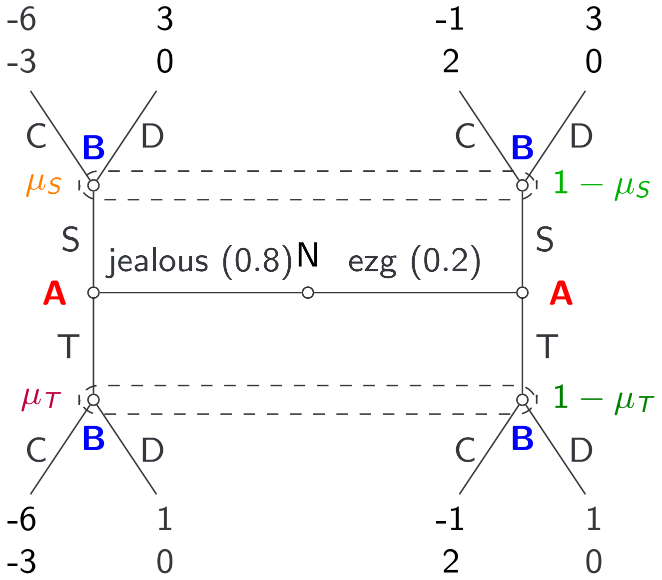
Example: a coffee for Brooke

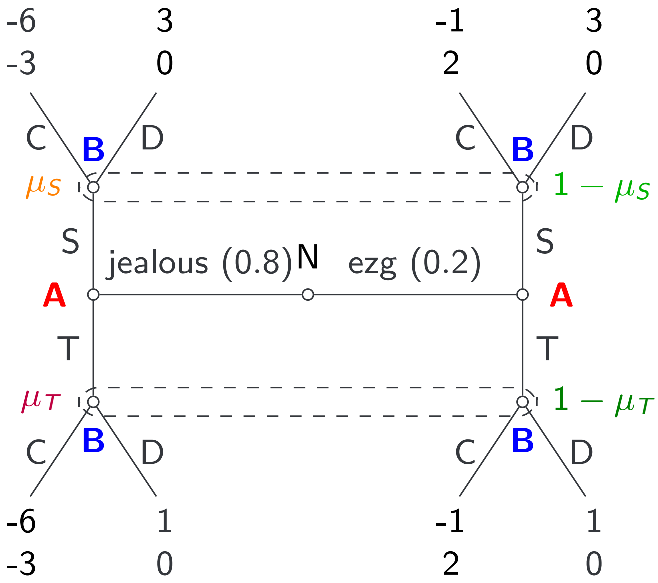
- Beliefs are easy to compute for separating strategies like ST
  - *Note*: We do not need to do that in this exercise, it is just an example



Brooke believes jealous Ann stays silent and easygoing Ann trashes Zoe out: $\mu_S = 1$ (100% chance Ann is jealous if S is observed); $\mu_T = 0$ (0% chance Ann is jealous if T is observed)

Example: a coffee for Brooke

- Unfortunately, in this game we only have pooling equilibria and intermediate cases
  - E.g., consider pooling strategy SS



Brooke believes jealous Ann stays silent regardless of whether she is jealous easygoing: $\mu_S = 0.8$ (same as the prior); What about μ_T?

Example: a coffee for Brooke

- We know that to sustain a PBE with pooling strategy SS, μ_S must stay 0.8
- Off the Bayesian equilibrium path, beliefs are arbitrary. However, they should still satisfy sequential rationality!
- E.g., to make ((SS,DC),(μ_S, μ_T)) as PBE, C must be a best response to T for Brooke
  - That happens if

$$\mu_T u_B(C|J)+(1-\mu_T)u_B(C|E) \geq \mu_T u_B(D|J)+(1-\mu_T)u_B(D|E)$$

$$\mu_T(-3)+(1-\mu_T)(2) \geq 0$$

  - Meaning that any $\mu_T \leq 2/5$ is sufficient to sustain a PBE with NE (SS,DC)
  - Conversely, any $\mu_T \geq 2/5$ sustains a PBE with NE (SS,DD)

Example: a coffee for Brooke

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- Summary so far:
- NE1: ((SS, DC),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \leq 0.4$)
- NE2: ((SS, DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \geq 0.4$)
- NE3: ((TT,CD),(μ_S,μ_T)) is a PBE for ($\mu_S \leq 0.4, \mu_T = 0.8$)
  - Analogous to NE1, same payoffs for Brooke
- NE4: ((TT, 1/2CD+1/2DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.4,\ \mu_T = 0.8$)
  - Same as above, but this time Brooke should be indifferent between C and D against S
- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T)) ?

Example: a coffee for Brooke

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- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- This can be a semi-separating PBE
  - Ann is always silent if easygoing but may start badmouthing Zoe if she is jealous
  - This is because she believes that Brooke may sometimes choose C if she stays 100% silent (if she stays silent, B chooses C with probability 2/9)
  - The description makes sense, but what about the system of beliefs? It is actually more complex and requires Bayes' rule to be used non-trivially

Example: a coffee for Brooke

- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- Easy part: $\mu_T = 1 \rightarrow$ Brooke believes Ann chooses to trash Zoe out only if she is jealous; if she is easygoing, Ann always plays S
- Harder part: $\mu_S =?$
- Depending on it, Brooke may prefer C or D. And to play a mixed strategy, Brooke must be indifferent between them (characterization theorem)
- We have already seen that this happens for $\mu_S = 0.4$

Example: a coffee for Brooke

- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- Denote with q the probability that jealous Ann plays S (the probability that she plays T is $1 - q$)
- Remember:

$$\mu_S = \frac{\Pr[\mathrm{S}, \mathrm{jealous}]}{\Pr[\mathrm{S}]} = \frac{pq}{pq + (1-p) \cdot 1} = \frac{0.8 \cdot 1/6}{0.8 \cdot 1/6 + 0.2} = 0.4$$

- If we already know μ_S, we can use this formula to find q

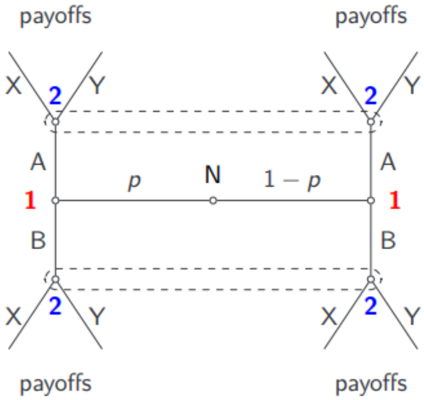

Figure 1: Representation of the signaling game(butterfly diagram)

A - Apply the Bayesian rule to the node

87. Q - **SA1:** consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has three available moves (A,B,C) and only one possible type; player 2 has two available moves (J,K) and two possible types.

- Draw the game in extensive form (without payoffs)
- How many moves specify one strategy of player 1? Why?
- How many moves specify one strategy of player 2? Why?
- Is SPE enough to characterize the equilibria of the game? Or do you need PBE?

88. Q - **SA2:** Consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has two available moves (C,D) and two possible types (t_L, t_R) with prior (p,1-p); player 2 has only one type and moves (M,N)

- Draw the game in extensive form (without payoffs)
- How many moves specify one strategy of player 1? Why?
- How many moves specify one strategy of player 2? Why?
- Is SPE enough to characterize the equilibria of the game (i.e., to determine wheter a BNE is sequentially rational)? Or do you need PBE?

89. Q - **SA3:** Consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has two available moves (C,D) and two possible types (t_L, t_R) with prior (p,1-p); player 2 has only one type and moves (M,N)

- Let h_D be the information set where player 2 moves after observing move D by player 1: what are 2's beliefs values μ in each node of h_D assuming separating strategy DC for 1?
- What are 2's belief values μ in each node of h_D assuming pooling strategy DD for 1?
- What are 2's belief value μ in each node of h_D assuming pooling strategy CC for 1?

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Lecture 10
Exercise set #1

Thomas Marchioro

November 9, 2023

Invariance of NE to linear transf.

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- **Theorem**: Consider game $\mathbb{G} = (S_1, \dots, S_n; u_1, \dots, u_i, \dots, u_n)$. If $p^* = (p_1^*, \dots, p_n^*)$ is a NE for $\mathbb{G}$, then it is also a NE for $\mathbb{G}' = (S_1, \dots, S_n; u_1, \dots, u_i', \dots, u_n)$, with

$$u_i'(s) = \alpha u_i(s) + c, \ \forall s \in S_1 \times \cdots \times S_n$$

and $a > 0$, $c \in \mathbb{R}$.

- In other words, you can apply any *positive* linear transformation (positive scaling + shift) to *all* payoffs of one or multiple players in the game, and preserve the NE.

Invariance of NE to linear transf.

- *Proof*: Suppose p^* is a NE for $\mathbb{G}$. Then, the following must hold for player i:

$$u_i(p_i^*, p_{-i}^*) \geq u_i(p_i, p_{-i}^*), \ \forall p_i \forall p_{-i}^*$$

- We can expand both sides into linear combinations

$$\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \geq \sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*)$$

- We multiply both sides by $\alpha > 0$ and add $c \in \mathbb{R}$

$$\alpha \left(\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \right) + c \geq \alpha \left(\sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*) \right) + c$$

Invariance of NE to linear transf.

- Finally, we apply the associative property

$$\sum_{s_i \in S_i} p_i^*(s_i) \underbrace{(\alpha u_i(s_i, p_{-i}^*) + c)}_{\color{red}{u_i'}(s_i, p^*)} \geq \sum_{s_i \in S_i} p_i(s_i) (\underbrace{\alpha u_i(s_i, p_{-i}^*)}_{\color{red}{u_i'}(s_i, p^*)} + c)$$

- Q.E.D. $\square$
- Actually, to have a "complete" proof, one should also expand p_{-i}^* so as to get $u_i'(s)$ in the sum; however, the procedure is still the same, since it is still a linear combination of values and associative property can be applied

Exercise 1a

- Find all NE (pure and mixed)

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

Exercise 1a (sol.)

- N is strictly dominated by M $(4 > 1; 6 > 5)$

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

Exercise 1a (sol.)

- Now G is strictly dominated by F

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

- (F, M) is the only NE

Exercise 1b

- Find all NE (pure and mixed)

		B	
		M	N
A	F	0, 4	3, 0
	G	6, 0	0, 5

Exercise 1b (sol.)

- No NE in pure strategies (Nash theorem: there must be one in mixed strategies)

		B	
		M	N
A	F	0, 4	3, 0
	G	6, 0	0, 5

- It must be that $\text{supp}(\mathbf{p}_A) = \{\text{F}, \text{G}\}$ and $\text{supp}(\mathbf{p}_B) = \{\text{M}, \text{N}\}$
- $\mathbf{p}_A = (q, 1-q)$, $\mathbf{p}_B = (r, 1-r)$

Exercise 1b (sol.)

- A plays F with probability q and G with $1-q$
- B plays M with probability r and N with $1-r$
- Apply the characterization theorem:

$$\begin{cases} \underbrace{4q}_{u_B(q,M)} = \underbrace{5(1-q)}_{u_B(q,N)} \\ \underbrace{3(1-r)}_{u_A(F,r)} = \underbrace{6r}_{u_A(G,r)} \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (5/9, 4/9)$, $\mathbf{p}_B = (1/3, 2/3)$

Exercise 1c

- Find all NE (pure and mixed)

		B	
		M	N
A	F	9, 3	2, 2
	G	0, 0	3, 9

Exercise 1c (sol.)

- Battle-of-the-sexes-like game
- Two "opposite" NE in pure strategies
- There must be one "in the middle"

		B	
		M	N
A	F	9, 3	2, 2
	G	0, 0	3, 9

Exercise 1c (sol.)

- A plays F with probability q and G with $1 - q$
- B plays M with probability r and N with $1 - r$
- Apply the characterization theorem:

$$\begin{cases} 3q = 2q + 9(1-q) \\ 9r + 2(1-r) = 3(1-r) \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (9/10, 1/10)$, $\mathbf{p}_B = (1/10, 9/10)$

Exercise 1d

- Find all NE (pure and mixed)

		B	
		M	N
A	F	2, 2	0, 6
	G	6, 0	1, 1

Exercise 1d (sol.)

- Prisoner's-dilemma-like game

		B	
		M	N
A	F	2, 2	0, 6
	G	6, 0	1, 1

- Only NE is (G, N)
- If we try to find a mixed NE, we get

$$2q = 6q + (1 - q) \Rightarrow q = -1/5 \Rightarrow \text{Impossible!}$$

Exercise 2

		B			
		J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
	Y	4, 9	9, 2	0, 4	2, 3
	Z	8, 4	2, 8	4, 2	3, 6

1. Show that there is no NE in pure strategies
2. Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE
3. List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

Exercise 2 (sol.)

1. Show that there is no NE in pure strategies

B

		J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
	Y	4, 9	9, 2	0, 4	2, 3
	Z	8, 4	2, 8	4, 2	3, 6

Exercise 2 (sol.)

2. Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE

- Use characterization theorem: for each player i

$$u_i(s_i, \mathbf{p}_{-i}) = u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \in \operatorname{supp}(\mathbf{p}_i)$$
$$u_i(s_i, \mathbf{p}_{-i}) \leq u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \notin \operatorname{supp}(\mathbf{p}_i)$$

Exercise 2 (sol.)

- For player A

$$u_A(X, \mathbf{p}_B) = 6 \cdot \frac{5}{11} + 5 \cdot \frac{4}{11} + 3 \cdot \frac{2}{11} = \frac{56}{11}$$
$$u_A(Y, \mathbf{p}_B) = 4 \cdot \frac{5}{11} + 9 \cdot \frac{4}{11} + 0 \cdot \frac{2}{11} = \frac{56}{11}$$
$$u_A(Z, \mathbf{p}_B) = 8 \cdot \frac{5}{11} + 2 \cdot \frac{4}{11} + 4 \cdot \frac{2}{11} = \frac{56}{11}$$

- All pure strategies yield equal payoff, so $\mathbf{p}_B$ is a sustainable NE strategy

Exercise 2 (sol.)

- For player B

$$u_B(\mathbf{p}_A, J) = 7 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{18}{3} = 6$$
$$u_B(\mathbf{p}_A, K) = 5 \cdot \frac{2}{3} + 8 \cdot \frac{1}{3} = \frac{18}{3} = 6$$
$$u_B(\mathbf{p}_A, L) = 8 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{18}{3} = 6$$
$$u_B(\mathbf{p}_A, M) = 1 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{8}{3} \leq 6$$

- M does not belong to the support and yields lower payoff, so $\mathbf{p}_A$ is a sustainable NE strategy

Exercise 2 (sol.)

3. List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

- Easier to delete Pareto dominated strategies

		B			
		J	K	L	M
	X	6, 7	~~5, 5~~	~~3, 8~~	~~8, 1~~
A	Y	4, 9	9, 2	~~0, 4~~	~~2, 3~~
	Z	8, 4	~~2, 8~~	~~4, 2~~	~~3, 6~~

- Remaining ones are Pareto optimal: (X, J), (Y, J), (Z, J), (Y, K).

Exercise 3

- Two firms (F1 and F2) work on a joint project from the European Commission. They can allocate an integer number of employees on the project, from 0 to infinity. They decide independently and without consulting with each other. The outcome of the project is that, if the number of employees allocated by each firm is identical (even zero!), both firms get a funding of 290 k€ from the European Commission. If the two firms assign a different number of employees, the European Commission gives them different fundings: 700 k€ to the one with more employees, and 320 k€ to the one with fewer employees. However, assigning employees costs 200 k€ per employee. The *utility* of a firm is funding minus costs.
  1. Show that no rational firm will allocate more than 2 employees
  2. Draw the normal form of this game and find its pure NE
  3. Find the additional NE in mixed strategies.

Exercise 3 (sol.)

- Let us start by understanding the problem
- Consider 1k€= 1 payoff unit
- The firm that allocates most employees receives more money (700) but needs to pay more (-200 payoff for each employee); the other firm receives 320 and also needs to pay the employees
- Worst case scenario is when they choose the number: they only receive 290 and still need to pay their employees
- Examples:
  - if F1 allocates 2 employees and F2 allocates 1, they get $700 - 400 = 300$ and $320 - 200 = 120$, respectively
  - if they both allocate 1 employee, they get $290 - 200 = 90$ each

---

Exercise 3 (sol.)

1. Show that no rational firm will allocate more than 2 employees

- By choosing $n_i = 0$, the firm can secure a payoff of at least 290
  - $u_i(0, 0) = 290$; $u_i(0, n_{-i}) = 320$ for $n_{-i} > 0$
- with $n_i \geq 3$, a firm can get at most $u(3, n_{-i}) = 700 - n_i \cdot 200 < 290$ when $n_{-i} < n_i$
- So all $n_i \geq 3$ are strictly dominated strategies (a rational firm would never choose them)
- Notice that 2 is not strictly dominated, since a firm may play it and get $700 - 400 = 300 > 290$, in case the other chooses 1 or 0 employees

Exercise 3 (sol.)

2. Draw the normal form of this game and find its pure NE

- The set of players is $\{\text{F1}, \text{F2}\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

		F2		
		0	**1**	**2**
F1	**0**	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

2. Draw the normal form of this game and find its pure NE

- The set of players is $\{\text{F1}, \text{F2}\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

		F2		
		0	**1**	**2**
F1	**0**	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Even though $n_i = 2$ is not dominated by 0 or 1 separately, it is strictly dominated by a combination of them
- i.e., "choose 0 with probability 0.95 and 1 with probability 0.05", which gives utility $29 \cdot 9.5 + 50 \cdot 0.5 = 300.5 > 300$ against 0 and $32 \cdot 9.5 + 9 \cdot 0.5 = 309.5 > 300$ and some positive value > -110 against 2.

			F2	
		0	**1**	**2**
F1	**0**	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Strictly dominated strategies cannot be part of NE (pure or mixed)

			F2
		0	**1**
F1	**0**	290, 290	320, 500
	1	500, 320	90, 90

- (0, 1) and (1, 0) are NE
- There must be a mixed NE as well

Exercise 3 (sol.)

3. Find the additional NE in mixed strategies.

- Symmetric payoffs: $q = r$
- Characterization theorem: $290q + 320(1-q) = 500q + 90(1-q) \Rightarrow q = r = 23/44$
- Mixed: NE $\mathbf{p}_1 = \mathbf{p}_2 = (23/44, 21/44)$

Alternative sol.

- Q: "What if I do not realize that strategy 2 is dominated?"
- Pure NE can be found via best-response search
- To find mixed NE, you just apply the characterization theorem for mixed strategy $\mathbf{p}_i = (q_0, q_1, q_2)$, i.e., $u_i(0, \mathbf{p}_i) = u_i(1, \mathbf{p}_i) = u_i(2, \mathbf{p}_i)$

$$\begin{cases} q_0 u_i(0,0) + q_1 u_i(0,1) + q_2 u_i(0,2) = q_0 u_i(1,0) + q_1 u_i(1,1) + q_2 u_i(1,2) \\ q_0 u_i(0,0) + q_1 u_i(0,1) + q_2 u_i(0,2) = q_0 u_i(2,0) + q_1 u_i(2,1) + q_2 u_i(2,2) \\ q_0 + q_1 + q_2 = 1 \end{cases}$$

- You get $q_2 < 0$ which is impossible $\Rightarrow$ Strategy 2 does not belong to the support!

Exercise 4

- A strategic interaction takes place between the taxpayer T and the tax inspector I. T is supposed to pay a share S of his income so as to have a net income equal to R after paying taxes. However, T is considering two alternatives: hide part of his income (**H**) so as to pay $S - L$ instead of S (thus getting a net income of $R + L$), or pay all due taxes in full (**P**). Player I also has two options: check T for tax fraud (**C**) or not (**N**). Performing a check has a cost equal to E. If the inspector finds out that T has hidden part of the income, then the taxpayer will have to pay all due taxes plus a fine of F, and an equal amount is collected by the inspector. The probability of being caught after a tax inspection is p. Formalize this conflict in the form of a static game of complete information and find the Nash equilibria.

Exercise 4 (sol.)

- Normal form representation

		I	
		C	N
T	H	$R + (1-p)L - pF$, $S - (1-p)L - E + pF$	$R + L$, $S - L$
	P	R, $S - E$	R, S

Exercise 4 (sol.)

- To simplify the game we can apply the invariance to linear transformations and: subtract R from I's payoffs; subtract S from T's payoffs
- Remember: All NE are preserved

		I	
		C	N
T	H	$(1-p)L-pF$, $-(1-p)L-E+pF$	L, $-L$
	P	0, $-E$	0, 0

Exercise 4 (sol.)

- Since all quantities E, F, L are > 0 we know that:
    - if I plays N, T's best response is H ($L > 0$)
    - If T plays P, I's best response is N ($-E < 0$)

		I	
		C	N
T	H	$(1-p)L-pF$, $-(1-p)L-E+pF$	L, $-L$
	P	0, $-E$	0, 0

- What if T plays H, or I plays C? It depends

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E > p(F+L)$

		I	
		C	N
T	H	$(1-p)L - pF$, $-(1-p)L - E + pF$	L, $-L$
	P	0, $-E$	0, 0

- (H, N) is the only NE
- C is strictly dominated by N, so there cannot be NE involving C
- Interpretation: too much effort required to perform the check

Exercise 4 (sol.)

- If $-(1-p)L - E + pF > -L$, i.e., $E < p(F+L)$; and $(1-p)L - pF > 0$, i.e., $p < L/(L+F)$

		I	
		C	N
T	H	$(1-p)L - pF$, $-E + pF - (1-p)L$	L, $-L$
	P	0, $-E$	0, 0

- Only NE is (H, C): T has low probability of being caught, so he always decides to hide his income; however, checking does not require much effort, so it for the inspector

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E < p(F+L)$; and $(1-p)L - pF < 0$, i.e., $p > L/(L+F)$

		I	
		C	N
T	H	$(1-p)L - pF$, $-E + pF - (1-p)L$	L, $-L$
	P	0, $-E$	0, 0

- No NE in pure strategies: there must be a mixed one.

Exercise 4 (sol.)

- Mixed NE is $(\mathbf{p}_T, \mathbf{p}_I$, with $\mathbf{p}_T = (q^*, 1-q^*)$ and $\mathbf{p}_I = (r^*, 1-r^*)$
- The values of q^* and r^* can be found using the characterization theorem

$$r(1-p)L - rpF + (1-r)L = 0 \Rightarrow r^* = \frac{L}{p(L+F)}$$

$$-qE + qpF - q(1-p)L - (1-q)E = -qL \Rightarrow q^* = \frac{E}{p(L+F)}$$

- Interpretation: q^* could be a reasonable estimate for the prob. of T committing tax fraud: proportional to I's effort E; inv. proportional to $L+F$ and to the prob. of being caught p
- r^* is the fraction of taxpayers that the inspector should check; this is proportional L

Questions?

Exercise set #2

Game Theory 2023/24

Exercise 1 It is the discount sales season, and Lou (L) wants to go shopping. He thinks it is best to wait until the last three days of the discount sales, because prices are cheapest. On day 1, Lou asks Karen (K) to go with him. If Karen says yes (Y), they go shopping and the game ends. If Karen says no (N), Lou can either give up (G) or request (R) again the following day. On day 2, the same happens. On day 3, if K still says N, the game ends as well and does not go into a further day. If in the end K and L do not go shopping, both of their payoffs are 0. If they do, their payoffs are computed as $u_K = d$ and $u_L = 5 - 2d$, respectively, where d is the day in which they go. All of this information about the game is common knowledge among the players.

1. Represent the game in its extensive form.
2. Find the SPE of this game.
3. Find *one* NE that is not subgame-perfect.

Exercise 2 Ashley and Brook live together. During the winter break they contemplate giving each other a nice gift (G) for Christmas or not (N). They know each other's preferences so they are able to buy a gift for 10 euros that is worth like 100 euros for the other. They make this decision independently and without telling each other. After Christmas, they also consider whether to celebrate New Year's eve downtown (D) or stay home (H). This means that receiving a gift gives utility of 100 to the receiver that is moved by the gesture, but also -10 to the utility of the buyer. Not giving gifts implies no variation of the utility for both.

For the New Year's eve celebration, they decide independently of each other in a coordination-game fashion. Staying home has utility of 0 for both. Going downtown has utility of 50. However, spending New Year's eve apart from each other has utility of -100 for both. The total payoff of the players is the sum of the partial payoffs in each stage with a discount factor of δ for the second stage.

1. Write down the normal form of both stages of the multi-stage game.
2. Find a trivial subgame-perfect equilibrium of the game where the players just play a Nash equilibrium in all stages, without any strategic connection.
3. Is there a strategically connected SPE of the whole game where Ashley and Brook give gifts to each other? If so, show the minimum required discount factor value $\delta_{\min}$ for that to hold.

Exercise 3 Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(\text{S,M}) = u_D(\text{M,S}) = 0$. Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}(T)$. Individual payoffs are summed with discount factor δ.

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. Consider an *extended* game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}'$. If you see a SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

Exercise 4 Consider the following game in normal form. Players are denoted as 1 and 2 and their strategy sets are $\mathcal{S}_1 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and $\mathcal{S}_2 = \{\mathrm{X}, \mathrm{Y}\}$. The payoff matrix is as follows:

		2	
		X	Y
1	A	9, -9	-5, 5
	B	-2, 2	7, -7
	C	8, -8	-1, 1

1. Describe what kind of game is that and name a specific solution concept you might use to find its Nash equilibria.
2. What is the support within $\mathcal{S}_1$ of the mixed strategy played at the Nash equilibrium by 1?
3. Find the mixed strategies played by 1 and 2 at the Nash equilibrium.

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Game Theory exercises

Exercise 1 A gang of pirates has n ranks from 1 (the ship's boy) to n (the captain). After a raid, they share a treasure. Pirate with rank $k+1$ keeps an eye on pirate k to see whether he gets a bigger share than he should. The game starts when pirate $k=1$ (the ship's boy) realizes that the treasure contains an extremely valuable pearl that has fallen far from the stash: he considers whether to take the pearl for himself (P) hiding it in his pocket or do nothing (N). Doing nothing ends the game with the pearl being unnoticed and unassigned. However, if pirate k takes the pearl, pirate $k+1$ will notice it; now, pirate $k+1$ may consider to kill him and keep the pearl for himself (P), or do nothing (N). If pirate $k+1$ does nothing, pirate k is left alive with the pearl – a very good outcome. If pirate $k+1$ kills pirate k and takes the pearl instead, this is spotted by pirate $k+2$ that now faces the same choice: whether to kill pirate $k+1$ and keep the pearl for himself (P), or to do nothing (N). This means that k is replaced with $k+1$ and the game continues up to the captain. For every pirate, the top preference is to stay alive and have the pearl; after that, they all prefer being alive without the pearl than to be killed.

1. Consider $n=5$. Choose appropriate utility values for the outcomes and draw the extensive form of the game.

2. Consider $n=5$. Solve the game by finding its subgame-perfect outcome.

3. Consider $n=8$. Does the subgame-perfect outcome change, and why?

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Exercise 2 The mayor of a big city is to be selected among four candidates: (A)Amanda Amour; (B)Bruno Bravery; (C)Claire Constitution; (D)Dave Democracy. Using symbol $\succ$ to denote "preferred to", polls indicate that:

- (A) has 42% of supporters. Also, for them (B)$\succ$(C)$\succ$(D).
- (B) has 11% of supporters. Also, for them (A)$\succ$(C)$\succ$(D).
- (C) has 27% of supporters. Also, for them (B)$\succ$(D)$\succ$(A).
- (D) has 20% of supporters. Also, for them (C)$\succ$(B)$\succ$(A).

1. The election is being held as a two-round run-off (i.e., with a ballot). What is the outcome under *sincere voting*? Denote the winner as W.
2. Assume that the supporters of (D) can identify this outcome and plan a strategy. What is the best *strategic voting* that they can enact?
3. Discuss the identity of the winner W' under strategic vote of (D)'s supporters. What kind of choice is W'? Can the supporters of W prevent this outcome by counteracting strategic vote of (D)'s supporters, with a strategic vote of their own?

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Exercise 3 The football manager of Athletic United (A) wants to hire a player in the transfer market. His evaluation for the player is 2 million euros. The team for which the player is currently enrolled, Braves F.C., (B) is considering the player to be worth at least 1 million euros. These valuations are common knowledge. The negotiation opens, and during the transfer window there is time for four bouts of exchanges. First, A offers a selling price; if B accepts, deal is made; otherwise go to the second round. In the second round, roles are reversed, with B making their call. Third and fourth exchange are similar to the first and second respectively. If p is the selling price in the first round, profits are $2 - p$ for A and $p - 1$ for B. A discount factor of 90% is applied for any subsequent round.

1. What is the extensive form for this game?
2. Apply backward induction to find at which round the agreement is reached.
3. What are price paid and final payoffs for A and B?

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Exercise 4 Little Jimmy really hopes to receive a space train (T) for Christmas. His parents have already bought him a sweater (S) but they tell him that he might receive T if he behaves as a good boy (G) instead of a naughty one (N). Jimmy's behavior ultimately depends on whether Santa Claus exists or not, which he evaluates as having probability p. If Jimmy is naughty, he will not receive any gift, regardless of Santa Claus existing or not. Jimmy and the parents' utility will be 0 and -10, respectively. If Jimmy is good, he will either receive S or T, according to the decision made by his parents. If Santa exists, Jimmy can receive the space train at no cost for his parents. If Santa does not exist, Jimmy can receive T only if the parents pay for it. Jimmy's utility for S and T is -10 and $+40$, respectively. The parents' reward when giving S and T to their son is 0 and $+20$. However, if they have to buy the space train themselves, subtract 50 from their utility. Also note that, being adults, the parents know whether Santa exists or not; on the other hand, they also know the value of p estimated by their son (that is, the prior is common knowledge).

1. Represent this game in extensive form.
2. Represent this game in normal form, with a type-player representation of the Bayesian players.
3. What kind of equilibria would be enough to characterize this Bayesian game? Discuss for $p = 0.9$.

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Exercise 5 The city of Aurapolis is attacked by a horde of Barbarians. The city can be accessed by two gates, the Northern Gate and the Southern Gate. The commander of the city guard has 4 battalions, while the barbarians only have 3. However, the city guard cannot leave any of the gates undefended, while the barbarians can do as they please (they can send all battalions against a single gate). Two battles are fought at each gate. For each battle, denote with n_A and n_B the number of battalions sent by A and B, respectively. When the same number of battalions fight at one gate, i.e., $n_A = n_B$, the partial payoff of that gate is zero for both sides. If $n_A \neq n_B$, partial payoff is $\max(n_A, n_B)$ for the side with more battalions, and $-\max(n_A, n_B)$ for the other side. Total payoff is the sum of the partial payoffs at both gates.

1. Write this game in normal form.
2. Find the Nash equilibria in pure strategies.
3. Find all the Nash equilibria of the game.

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Game Theory exam – January 28, 2016

Exercise 1 Agatha (`A`) and Bruno (`B`) are two siblings. They own a model train set (`T`) and they usually play together with it. Their respective utilities when doing so is $u_A(\mathtt{T},\mathtt{T}) = u_B(\mathtt{T},\mathtt{T}) = 3$. Their mom (`M`) buys them a new toy (`N`), which can be either a dollhouse (`D`) or a set of small soldier figurines (`S`). The two kids must independently decide whether to keep playing with `T` or the new toy `N`. All the three players (`A`, `B`, `M`) involved decide their action simultaneously and unbeknownst to each other. The action set of the kids includes just `T` or `N`; the mom instead decides what new toy to buy. If the kids end up in not playing together, *all* the players (also `M`) get utility equal to 0. If they play together with the new toy, they get a positive utility that for `A` is equal to 6 and 1 for `D` and `S`, respectively; for `B`, the values are instead 1 and 4, respectively. The utility of `M` is the lump sum of the utility of the two kids.

1. Write down the normal form of this game (likely, you will need a 3D matrix: you can write two matrices instead, one per each move of player `M`).
2. Find all Nash equilibria of this game in pure strategies.
3. Find all additional Nash equilibria of this game in mixed strategies.

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Exercise 2 Carl (`C`) and Diana (`D`) are two university students that have found that the department library is unoccupied overnight. It is a really good place to study and has a very fast Internet connection. So, they go there every night, but they do not coordinate or plan any action together. Upon their arrival every night, they independently decide whether: (`S`) study or (`M`) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for `C` and 18 for `D`. However, if they both choose `M`, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(\mathtt{S},\mathtt{M}) = u_D(\mathtt{M},\mathtt{S}) = 0$ (`C` is written as the first player). Call $\mathbb{G}$ this game, and consider it in a repeated version $\mathbb{G}(T)$, where $\mathbb{G}$ is played every night for T nights. Individual payoffs are cumulated with discount factor δ. Finally, consider an *extended* game where a punishment strategy `P` is also available to both players. When either player chooses `P`, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game $\mathbb{G}'$. Note: despite `P` being weakly dominated, (`P`,`P`) is a NE for $\mathbb{G}'$.

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$.
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing `S`?
3. If you see an SPE of $\mathbb{G}'(2)$ where players may play `S`, state at which round do they play it, and what value of δ do you need to obtain it.

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Exercise 3 Jane's bank account contains 10000 euros. She can keep (`K`) the whole sum in the bank account, or buy a utility car (`C`) that costs 8000 euros. The bank `B` is also a player: they observe Jane's decision and after that can either (`L`) leave Jane's money in the account, or (`S`) buy some subprime bonds for Jane, regardless of Jane's desire. In case (`S`), they invest every euro that Jane has left in the account (either 2000 or 10000). After one year, Jane's account is increased with some interest $\mathcal{G}$. If the bank played `L`, $\mathcal{G}$ is equal to 5% of the initial amount. In case they played `S`, $\mathcal{G}$ depends on whether the bank is a "good" bank or a "bad" bank. Jane estimates the probability of "good" as p and this is a common prior. A good bank playing `S` gives Jane her money back, plus $\mathcal{G} = 15\%$ of yearly interest rate. A bad bank steals every euro that Jane put in the bank account ($\mathcal{G} = -100\%$). The final payoff are computed after one year as:
for Jane: the grand total of the euros in the account (i.e., the initial value increased by $\mathcal{G}$); plus the value of the car, if she owns one: a one-year old car is worth 5000 euros.
for the bank: if $\mathcal{G}$ is positive, then their payoff is $\mathcal{G}$; otherwise it is $-\mathcal{G}/5$.

1. Represent this Bayesian game with `J` and `B` in extensive form.
2. Represent this Bayesian game in normal form, with a type-agent representation of player `B` (i.e., its strategies are n-tuples of actions, one per type).
3. Find the Bayesian equilibria of the game if $p = 0.2$ and discuss whether they are (subgame) perfect.

21 Disclaimers

For 2023-24 exam, professor said he will not ask for:

- Dupolies
- Constitutions
- Potential games