University of Padova – prof. Leonardo Badia
Game Theory exercises

Exercise 1 A gang of pirates has n ranks from 1 (the ship's boy) to n (the captain). After a raid, they share a treasure. Pirate with rank $k+1$ keeps an eye on pirate k to see whether he gets a bigger share than he should. The game starts when pirate $k = 1$ (the ship's boy) realizes that the treasure contains an extremely valuable pearl that has fallen far from the stash: he considers whether to take the pearl for himself (P) hiding it in his pocket or do nothing (N). Doing nothing ends the game with the pearl being unnoticed and unassigned. However, if pirate k takes the pearl, pirate $k+1$ will notice it; now, pirate $k+1$ may consider to kill him and keep the pearl for himself (P), or do nothing (N). If pirate $k+1$ does nothing, pirate k is left alive with the pearl – a very good outcome. If pirate $k+1$ kills pirate k and takes the pearl instead, this is spotted by pirate $k+2$ that now faces the same choice: whether to kill pirate $k+1$ and keep the pearl for himself (P), or to do nothing (N). This means that k is replaced with $k+1$ and the game continues up to the captain. For every pirate, the top preference is to stay alive and have the pearl; after that, they all prefer being alive without the pearl than to be killed.

1. Consider $n = 5$. Choose appropriate utility values for the outcomes and draw the extensive form of the game.
2. Consider $n = 5$. Solve the game by finding its subgame-perfect outcome.
3. Consider $n = 8$. Does the subgame-perfect outcome change, and why?

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Exercise 2 The mayor of a big city is to be selected among four candidates: (A)Amanda Amour; (B)Bruno Bravery; (C)Claire Constitution; (D)Dave Democracy. Using symbol $\succ$ to denote "preferred to", polls indicate that:

- (A) has 42% of supporters. Also, for them (B)$\succ$(C)$\succ$(D).
- (B) has 11% of supporters. Also, for them (A)$\succ$(C)$\succ$(D).
- (C) has 27% of supporters. Also, for them (B)$\succ$(D)$\succ$(A).
- (D) has 20% of supporters. Also, for them (C)$\succ$(B)$\succ$(A).

1. The election is being held as a two-round run-off (i.e., with a ballot). What is the outcome under *sincere voting*? Denote the winner as W.
2. Assume that the supporters of (D) can identify this outcome and plan a strategy. What is the best *strategic voting* that they can enact?
3. Discuss the identity of the winner W' under strategic vote of (D)'s supporters. What kind of choice is W'? Can the supporters of W prevent this outcome by counteracting strategic vote of (D)'s supporters, with a strategic vote of their own?

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Exercise 3 The football manager of Athletic United (A) wants to hire a player in the transfer market. His evaluation for the player is 2 million euros. The team for which the player is currently enrolled, Braves F.C., (B) is considering the player to be worth at least 1 million euros. These valuations are common knowledge. The negotiation opens, and during the transfer window there is time for four bouts of exchanges. First, A offers a selling price; if B accepts, deal is made; otherwise go to the second round. In the second round, roles are reversed, with B making their call. Third and fourth exchange are similar to the first and second respectively. If p is the selling price in the first round, profits are $2 - p$ for A and $p - 1$ for B. A discount factor of 90% is applied for any subsequent round.

1. What is the extensive form for this game?

2. Apply backward induction to find at which round the agreement is reached.

3. What are price paid and final payoffs for A and B?

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Exercise 4 Little Jimmy really hopes to receive a space train (T) for Christmas. His parents have already bought him a sweater (S) but they tell him that he might receive T if he behaves as a good boy (G) instead of a naughty one (N). Jimmy's behavior ultimately depends on whether Santa Claus exists or not, which he evaluates as having probability p. If Jimmy is naughty, he will not receive any gift, regardless of Santa Claus existing or not. Jimmy and the parents' utility will be 0 and -10, respectively. If Jimmy is good, he will either receive S or T, according to the decision made by his parents. If Santa exists, Jimmy can receive the space train at no cost for his parents. If Santa does not exist, Jimmy can receive T only if the parents pay for it. Jimmy's utility for S and T is -10 and $+40$, respectively. The parents' reward when giving S and T to their son is 0 and $+20$. However, if they have to buy the space train themselves, subtract 50 from their utility. Also note that, being adults, the parents know whether Santa exists or not; on the other hand, they also know the value of p estimated by their son (that is, the prior is common knowledge).

1. Represent this game in extensive form.
2. Represent this game in normal form, with a type-player representation of the Bayesian players.
3. What kind of equilibria would be enough to characterize this Bayesian game? Discuss for $p = 0.9$.

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Exercise 5 The city of Aurapolis is attacked by a horde of Barbarians. The city can be accessed by two gates, the Northern Gate and the Southern Gate. The commander of the city guard has 4 battalions, while the barbarians only have 3. However, the city guard cannot leave any of the gates undefended, while the barbarians can do as they please (they can send all battalions against a single gate). Two battles are fought at each gate. For each battle, denote with n_A and n_B the number of battalions sent by A and B, respectively. When the same number of battalions fight at one gate, i.e., $n_A = n_B$, the partial payoff of that gate is zero for both sides. If $n_A \neq n_B$, partial payoff is $\max(n_A, n_B)$ for the side with more battalions, and $-\max(n_A, n_B)$ for the other side. Total payoff is the sum of the partial payoffs at both gates.

1. Write this game in normal form.

2. Find the Nash equilibria in pure strategies.

3. Find all the Nash equilibria of the game.