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Lecture 20
Dynamic Bayesian games

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Previously on game theory

- Bayesian Nash equilibria
  - BNE can change if payoffs or priors ϕ are changed
- Signaling can affect single-person decision problems
- Specifically, in single-person decision, the decision-maker is inclined to "follow the signal"
- What about multi-person decision problems (i.e., games)?

Committee voting: recap

- Suppose a single jury member (juror) decides the fate of a defendant
- He starts with a prior estimate of the defendant being guilty $q > 1/2$
- He receives a signal (e.g., evidence) saying that the defendant is guilty (t_G) or innocent (t_I)
- The received signal matches the truth with probability $p > 1/2$
  - If he receives t_G that the defendant is guilty, his posterior probability is $\Pr[G|t_G] > q$ (he is even surer)
  - If he receives t_I, his posterior probability is $\Pr[G|t_G] < q$, and may even be less than $1/2$
- In case of single-person decision, the juror tends to "follow the signal"

Committee voting: 2-person decision

- Now, we would like to check whether $p > q$ implies that (CA, CA) is a BNE in the original problem (2-person decision)
  - That would correspond to "following the signal"
- First, draw the probability of each type pair

Member 2

Member 1	t_G	t_I
t_G	$qp^2 + (1-q)(1-p)^2$	$p(1-p)$
t_I	$p(1-p)$	$q(1-p)^2 + (1-q)p^2$

- **Note**: This is not a payoff matrix, it is just a table displaying the values of probabilities $\Pr[t_1 = t_x, t_2 = t_y]$

Committee voting: 2-person decision

- To check whether (CA,CA) is BNE we need to ask “Is CA a best response to CA?”
  - Assume member 2 plays CA, and check if CA is best for member 1
- We do not want to write down the whole table, let us try to see if we can draw conclusions just by looking at posteriors
- With the rules of the jury, a player’s choice is decisive (“**pivotal**”) only if the other juror chooses C
- If 2 chooses A, that is the result regardless of the 1’s choice
  - If 1 believes that 2 is playing CA, *any* strategy of 1 is always a best response if the 2’s type is t_I
  - In other words, if 1 thinks that 2 received signal t_I, then everything 1 does is a best response
  - So we need to check only the case $t_2 = t_G$

Committee voting: 2-person decision

- Again, check the posterior to see the signal effect

$$\Pr[G|t_1 = t_G, t_2 = t_G] = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

- **Meaning**: if both $t_1 = t_G$ and $t_2 = t_G$: conviction is even more certain
- as before, $p > 1/2$ implies
  $qp^2 + (1-q)(1-p)^2 < qp^2 + (1-q)p^2 = p^2$

$$\Pr[G|t_1 = t_I, t_2 = t_G] = \frac{qp(1-p)}{p(1-p)} = q$$

- **Meaning**: if they receive opposite signals, the received signal t_I is useless $\rightarrow$ posterior=prior

Committee voting: 2-person decision

- Recap:
  - If player 2 is of type $t_2 = t_I$, player 1 believes that 2's move is A $\rightarrow$ 1's move does not matter
  - If player 2 is of type $t_2 = t_G$ $\rightarrow$ player 1's posterior is either q or higher
- Therefore, CA is not a best response to CA
- Actually you can prove that (CC, CC) is a BNE

Dynamic Bayesian games

Refinement of NE concepts

- In static games of complete information, NE are enough
- In "static" Bayesian games, BNE are enough
  - The caveat in static Bayesian games is that strategies are type-dependent
- In dynamic games of complete information, we introduce the concept of SPE
  - Sequential rationality leads to "more rational" equilibria
  - E.g., avoid non-credible threats or irrational behaviors outside the equilibrium path
- Can we find a counterpart for dynamic Bayesian games?

Example: Entry game

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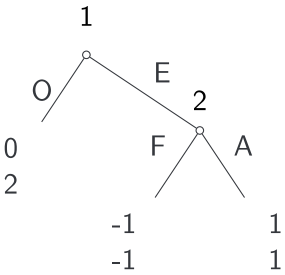
- Player 1 is a newcomer (e.g., in a market or network); 1 may enter (E) or stay out (O)
- Player 2 in an incumbent; 2 may fight (F) or accept (A) 1's entrance



- SPE: (E, A)
- non-SPE NE: (O, F)

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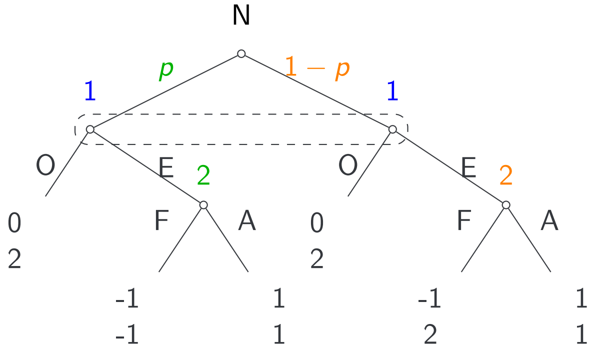
- Player 2 can be "reasonable" or "crazy" with probabilities p and $1-p$



Bayesian entry game, NE

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- For $p = 2/3$

Player 2

Player 1	AA	AF	FA	FF
O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
E	**1**, 1	**1/3**, **4/3**	-1/3, -1/3	-1, 0

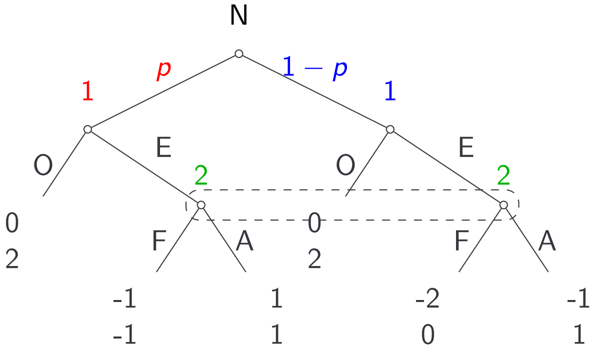
- NE: (E, AF), (O, FA), (O, FF)
- Here, we also have a SPE: (E, AF)
  - It is a NE in the overall game, and also in the two subgames where 2 plays as "reasonable" (choosing A), and 1 plays as "crazy" (choosing F).
- However, in most cases SPE is not be a sufficient concept for dynamic Bayesian games

Bayesian entry game 2.0

- Consider a different version of the Bayesian entry game
- This time it is the type of the newcomer (player 1) that is unknown
  - The newcomer can be either "competitive" or "weak"
  - the incumbent is always reasonable
- In case player 1 is a competitive newcomer, payoffs are the same as in the original entry game
- A weak newcomer, instead, does not have the resources to compete with the incumbent; in this case, the newcomer does not want to enter (always gets negative payoff)

Bayesian entry game 2.0

- Player 1 can be "competitive" or "weak" with probabilities p and $1 - p$



Bayesian entry game 2.0

- This time, the situation is reversed
  - Player 1 can have multiple types, while we have complete information on player 2
  - Player 1 is first to move $\rightarrow$ we need a way to account for the game dynamics
- Player 1 has two types: 1's pure strategies are OO, OE, EO, EE
- Player 2 has only one type: 2's pure strategies are F, A (2 moves without knowing 1's type)

Bayesian entry game 2.0

- *Note*: We cannot apply backward induction as the last player (player 2) does not know 1's type
- We can reduce the extensive form to yet another normal (static) form
- This time we need to average payoffs over 1's type, e.g.
  - $u_1(\mathrm{OE}, \mathrm{A}) = p \cdot 0 + (1-p) \cdot -1 = p - 1$
  - $u_2(\mathrm{OE}, \mathrm{A}) = p \cdot 2 + (1-p) \cdot 1 = p + 1$

Bayesian entry game 2.0

- Let us find NE for $p = 1/2$

Player 2

Player 1	F	A
OO	**0**, **2**	0, **2**
OE	-1, 1	-1/2, **3/2**
EO	-1/2, 1/2	**1/2**, **3/2**
EE	-1/2, -1/2	0, **1**

- Two NE:
  - (OO,F): equilibrium where the incumbent threatens to fight
  - (EO,A): equilibrium where the incumbent accepts but only a competitive outsider enters (a weak one just stays out from the beginning)

Bayesian entry game 2.0

- (OO, F) seems to be a non-credible threat
  - Player 2 always plays F even when it would be more logical to yield (i.e. play A)
- The problem is: this game has only one subgame
- (OO, F) is technically a SPE, even though its "perfection" is questionable

  $\rightarrow$ We need to introduce a new type of equilibrium to distinguish decisions that are "perfectly rational" in dynamic Bayesian game

Perfect Bayesian equilibrium

Bayesian equilibrium path

- If we have a Bayesian NE $s^* = (s_1^*, \ldots, s_n^*)$, we say that an information set is **"on" the equilibrium path** if, given the distribution ϕ of types, it is reached with probability > 0
  - This definition applies to **Bayesian NE**
  - In the BNE given by (OO,F) the information set of node 2 is never reached $\rightarrow$ it is **"off" the equilibrium path**

System of beliefs

- In an extensive-form Bayesian game, a **system of beliefs** μ is a probability distribution over *decision nodes* for every information set
  - In other words, it is an estimate of being at a specific node, given an information set (possibly spanning over multiple nodes)
  - It is a conditional probability Pr(node|information set)

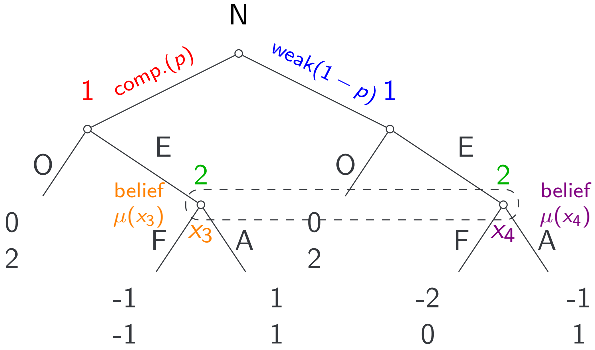
    Clearly, this is equal to Pr(node,information set)/Pr(information set), which in turn is Pr(node)/Pr(information set)
  - In our entry game, the system of beliefs of player 1 is sure, while that of player 2 depends on the types of player 1 (specifically, on its prior of 1 being competitive or weak)

Perfect Bayesian equilibrium

- We define the following requirements for sequential rationality in Bayesian games:
    1. Players must have a system of beliefs
    2. On the equilibrium path they must follow Bayes' rule on conditional probabilities
    3. Off the equilibrium path: arbitrary
    4. Given their beliefs, players are sequentially rational: i.e., they play a best response to their belief
- **Definition**: A **perfect Bayesian equilibrium** (PBE) is a pair (s^*, μ), where s^* is a Bayesian Nash equilibrium and μ is a system of beliefs satisfying 1–4.

PBE in Bayesian entry game 2.0

- Always remember that a PBE is not just a pair of strategies: there must be an associated system of beliefs μ



PBE in Bayesian entry game 2.0

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- A strategy pair must be sustained by a system of beliefs: $\mu(x_3)$ and $\mu(x_4) = 1 - \mu(x_3)$ for player 2
  - e.g., if 2 believes that 1 plays EO, then $\mu(x_3) = 1$ (in other words, if 1 enters, then 2 if fully convinced that 1 is competitive)
  - this reasoning can also be applied to mixed strategies
  - consider strategy $q_C q_W$, i.e.,
    - a competitive player 1 chooses E with probability q_C (and O with $1 - q_C$)
    - a weak player 1 chooses E with probability q_W (and O with $1 - q_W$)
    - In this case, the belief of x_3 given E is

$$\mu(x_3) = \frac{\Pr(\text{node})}{\Pr(\text{information set})} = \frac{p q_C}{p q_C + (1-p) q_W}$$

PBE in Bayesian entry game 2.0

- $s^* =$(EO, A) and μ form a PBE:
  - 2 believes that only "competitive" 1 chooses to enter, so $\mu(x_3) = 1$
  - 2 playing A is a sequentially-rational response to 2's belief
- $s^* =$(OO, F) cannot form a PBE with any system of beliefs μ:
  - Bayes' rule cannot be applied since playing OO means $q_C = q_W = 0$

$$\mu(x_3) = \frac{pq_C}{pq_C + (1-p)q_W} = \frac{0}{0}$$

  - x_3 and x_4 are off-path in this case, so the beliefs are arbitrary
  - However, F is irrational in both x_3 and x_4 (A is always better for 2) and it must be either $\mu(x_3) > 0$ or $\mu(x_4) > 0$
  - That means requirement 4 is violated $\rightarrow$ not a PBE

PBE in Bayesian entry game 2.0

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- Perfect Bayesian NE: (EO,A)
  - sustained by system of belief $\mu(x_1) = 1$
  - all players play in a sequentially-rational way
- Imperfect Bayesian NE: (OO,F)
  - Bayes' rule cannot be applied: $q_C = q_W = 0$
  - Whatever choice of $\mu(x_1), \mu(x_2)$ makes the choice of F irrational

Self-assessment

- What does it mean for a node x_i to be on the Bayesian equilibrium path, given BNE s^*?
- What elements do you need to characterize a PBE?
- How do you determine sustainable belief values $\mu(x_i)$ for nodes that are on the Bayesian equilibrium path?
- What values can $\mu(x_i)$ have if x_i is off the equilibrium path?

Signaling games

Signaling games

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- We saw 2 Bayesian versions of the entry game where **1**=outsider/entrant and **2**=incumbent
- This can be generalized as follows:
  - **2** has multiple types, **1** has only one type: **1** moves before **2**, without any hint about **2**'s type besides the prior ϕ
    $\rightarrow$ This is called a **screening game**, SPE is enough
  - **1** has multiple types, **2** has only one type: **2**'s first move may give a hint (*signal*) about **2**'s type
    $\rightarrow$ This is called a **signaling game**, and requires PBE to achieve sequential rationality

Signaling game: definition

- A signaling game is a 2-player dynamic Bayesian game: **1** (first to move) and **2** (second to move)
  - **1**'s type is chosen among many possible types (by Nature)
  - **2** has only one type
  - **2**'s beliefs are updated after **1**'s move

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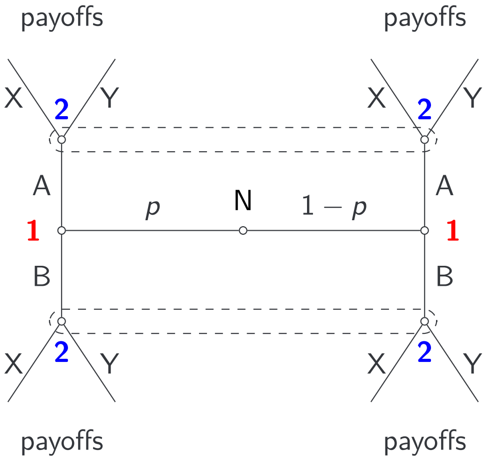
- Binary case is often shown as a "butterfly"



Equilibria of signaling games

- **Separating equilibria**: each type of **1** chooses a different action; thus revealing the type to **2**
- **Pooling equilibria**: all types of **1** choose the same action; thus, **2** gets no signal about **1**'s type
- **Intermediate cases**: **1**'s action does not fully define **1**'s type, but still provides some information
  - Beliefs are updated according to Bayes' rule
  - This type of equilibria is also called "semi-separating" or "partially pooling"

Example: a coffee for Brooke

- Ann and Brooke are dating; Brooke is invited by a colleague, Zoe, to get a coffee
- Ann is a typed player: her types are
  - Jealous with probability 0.8
  - Easygoing with probability 0.2

  (all this information is common knowledge)
- Ann can send a signal to either stay silent (S) about this proposal or to trash Zoe out (T)
- Brooke observes the signal, and decides whether to accept the coffee invitation (C) or to politely decline (D)

Example: a coffee for Brooke

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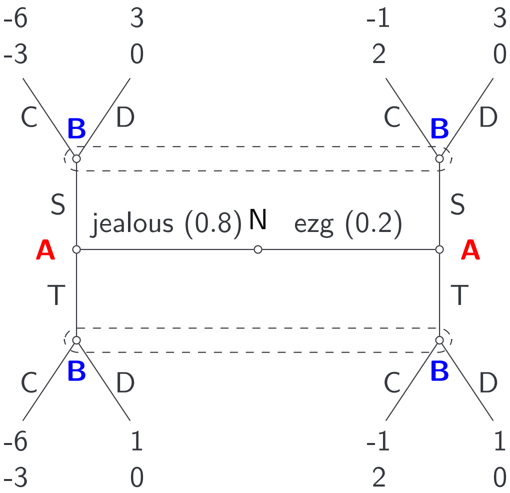
- Payoffs:
  - Jealous Ann is deeply hurt if Brooke accepts ($u_A = -6$)
  - Easygoing Ann is just not-so-angry, but still not fond of the idea ($u_A = -1$)
  - Ann prefers to stay silent ($u_A = 3$) rather than trash Zoe out ($u_A = 1$), only in case Brook declines
  - Brooke likes to go to the coffee if that is okay for Ann ($u_B = 2$)
  - If Ann is hurt, Brook prefers declining the invitation ($u_B = 0$) rather than accepting it ($u_B = -3$)

Example: a coffee for Brooke

- Extensive form



Example: a coffee for Brooke

- Both players have 4 strategies but for different reasons
  - Ann because of her type: strategy is (what to do if jealous, what to do if easygoing)
  - Brooke does not have a type but observes Ann's move: strategy is (what to do if Ann plays S, what to do if Ann plays T)
  - e.g., (TS,CD) means that Ann trashes Zoe if she is jealous and remains silent if she is easygoing (separating); Brooke just "follows the signal", going to the coffee if Ann stays silent, and declining if Ann starts trashing Zoe

Example: a coffee for Brooke

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- **Warning!** A's pair is left/right, but B's pair is *B's reaction to A's move*
- First row (SS): only consider B's 1st move (reaction to S)
- Last row (CC): only consider B's 2nd move (reaction to T)

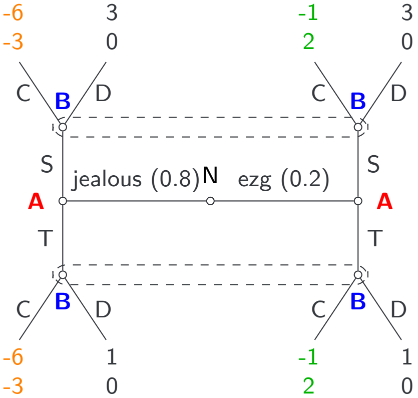
Brooke

Ann	CC	CD	DC	DD
SS	B plays C	B plays C	B plays D	B plays D
ST	B plays C			B plays D
TS	B plays C			B plays D
TT	B plays C	B plays D	B plays C	B plays D

Example: a coffee for Brooke

- If B plays C, utility is always

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (-1) = -5, \quad u_B = 0.8 \cdot (-3) + 0.2 \cdot (2) = -2$$



- When B plays D, we need to distinguish between Ann's 4 possible moves (her payoff changes, B's is always 0)

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	B plays D	B plays D
ST	-5, -2			B plays D
TS	-5, -2			B plays D
TT	-5, -2	B plays D	-5, -2	B plays D

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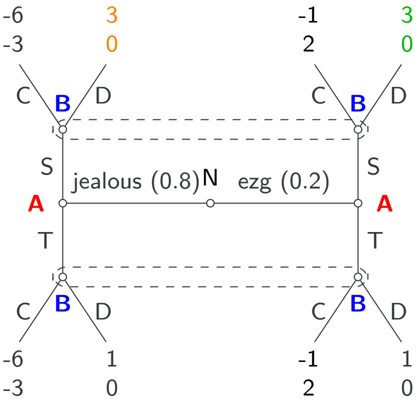
- If B plays D and A plays S, i.e., (SS, D*)

$$u_A = 0.8 \cdot (3) + 0.2 \cdot (3) = 3$$



Example: a coffee for Brooke

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- Likewise, if B plays D and A plays T, i.e., (TT,*D), then $u_A = 1$

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2			B plays D
TS	-5, -2			B plays D
TT	-5, -2	1, 0	-5, 2	1, 0

- What about intermediate cases (ST,DD) and (TS, DD)?

Example: a coffee for Brooke

- For (ST,DD) and (TS, DD), you just average the payoffs: in the first case S is played with probability 0.8 and T with probability 0.2; the second case is the opposite

Ann \ Brooke	CC	CD	DC	DD
SS	-5, -2	-5, -2	3, 0	3, 0
ST	-5, -2	?, ?	?, ?	2.6, 0
TS	-5, -2	?, ?	?, ?	1.4, 0
TT	-5, -2	1, 0	-5, 2	1, 0

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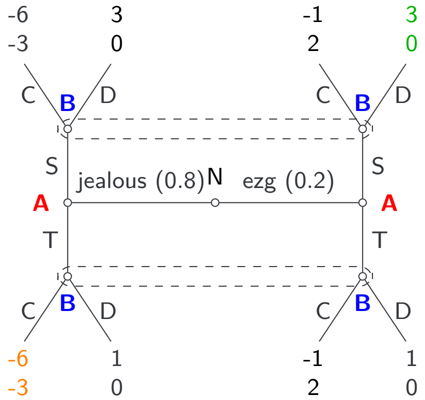
- E.g., for (TS,DC) (remember: D is answer to S and C is answer to T)

$$u_A = 0.8 \cdot (-6) + 0.2 \cdot (3) = -4.2$$



Example: a coffee for Brooke

Brooke

Ann	CC	CD	DC	DD
SS	-**5**, -2	-5, -2	**3**, **0**	**3**, **0**
ST	-**5**, -2	-4.6, -2.4	2.2, **0.4**	2.6, 0
TS	-**5**, -2	0.6, **1.6**	-4.2, -2.4	1.4, 0
TT	-**5**, -2	**1**, **0**	-5, 2	1, **0**

- 3 pure NE: (SS,DC), (SS,DD), (TT,CD)
- 2 mixed NE: (TT, 1/2CD+1/2DD), (1/6SS+5/6TS,2/9CD+7/9DD)

Example: a coffee for Brooke

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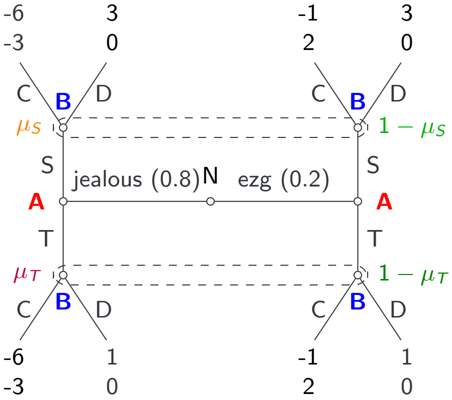
- So far, we have only found NE, **now we need to classify them**! Are they PBE?
- To verify that, we need to construct systems of beliefs μ for Brooke
  - i.e., $\mu =$ is Brooke's belief that Ann is *jealous*
  - One belief for each possible observed move by Ann: μ_S if she stays silent; μ_T if she trashes Zoe out

Example: a coffee for Brooke

- Beliefs are easy to compute for separating strategies like ST
  - *Note*: We do not need to do that in this exercise, it is just an example



Brooke believes jealous Ann stays silent and easygoing Ann trashes Zoe out: $\mu_S = 1$ (100% chance Ann is jealous if S is observed); $\mu_T = 0$ (0% chance Ann is jealous if T is observed)

Example: a coffee for Brooke

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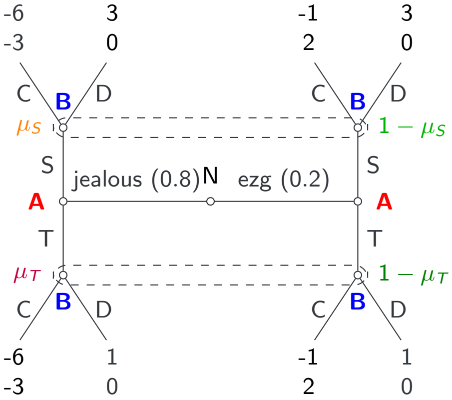
- Unfortunately, in this game we only have pooling equilibria and intermediate cases
  - E.g., consider pooling strategy SS



Brooke believes jealous Ann stays silent regardless of whether she is jealous easygoing: $\mu_S = 0.8$ (same as the prior); What about μ_T?

Example: a coffee for Brooke

- We know that to sustain a PBE with pooling strategy SS, μ_S must stay 0.8
- Off the Bayesian equilibrium path, beliefs are arbitrary. However, they should still satisfy sequential rationality!
- E.g., to make ((SS,DC),(μ_S, μ_T)) as PBE, C must be a best response to T for Brooke
  - That happens if

$$\mu_T u_B(C|J)+(1-\mu_T)u_B(C|E) \geq \mu_T u_B(D|J)+(1-\mu_T)u_B(D|E)$$

$$\mu_T(-3)+(1-\mu_T)(2) \geq 0$$

  - Meaning that any $\mu_T \leq 2/5$ is sufficient to sustain a PBE with NE (SS,DC)
  - Conversely, any $\mu_T \geq 2/5$ sustains a PBE with NE (SS,DD)

Example: a coffee for Brooke

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- Summary so far:
- NE1: ((SS, DC),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \leq 0.4$)
- NE2: ((SS, DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.8, \mu_T \geq 0.4$)
- NE3: ((TT,CD),(μ_S,μ_T)) is a PBE for ($\mu_S \leq 0.4, \mu_T = 0.8$)
  - Analogous to NE1, same payoffs for Brooke
- NE4: ((TT, 1/2CD+1/2DD),(μ_S,μ_T)) is a PBE for ($\mu_S = 0.4$, $\mu_T = 0.8$)
  - Same as above, but this time Brooke should be indifferent between C and D against S
- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T)) ?

Example: a coffee for Brooke

- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- This can be a semi-separating PBE
  - Ann is always silent if easygoing but may start badmouthing Zoe if she is jealous
  - This is because she believes that Brooke may sometimes choose C if she stays 100% silent (if she stays silent, B chooses C with probability 2/9)
  - The description makes sense, but what about the system of beliefs? It is actually more complex and requires Bayes' rule to be used non-trivially

Example: a coffee for Brooke

- NE5: ((1/6SS+5/6TS,2/9CD+7/9DD),(μ_S,μ_T))
- Easy part: $\mu_T = 1 \rightarrow$ Brooke believes Ann chooses to trash Zoe out only if she is jealous; if she is easygoing, Ann always plays S
- Harder part: $\mu_S =$?
- Depending on it, Brooke may prefer C or D. And to play a mixed strategy, Brooke must be indifferent between them (characterization theorem)
- We have already seen that this happens for $\mu_S = 0.4$

Example: a coffee for Brooke

- NE5: $((1/6\text{SS}+5/6\text{TS},2/9\text{CD}+7/9\text{DD}),(\mu_S,\mu_T))$
- Denote with q the probability that jealous Ann plays S (the probability that she plays T is $1-q$)
- Remember:

$$\mu_S = \frac{\Pr[\text{S, jealous}]}{\Pr[\text{S}]} = \frac{pq}{pq + (1-p)\cdot 1} = \frac{0.8 \cdot 1/6}{0.8 \cdot 1/6 + 0.2} = 0.4$$

- If we already know μ_S, we can use this formula to find q

Self-assessment

- **SA1**: Consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has three available moves (A,B,C) and only one possible type; player 2 has two available moves (J,K) and two possible types
  - Draw the game in extensive form (without payoffs)
  - How many moves specify one strategy of player 1? Why?
  - How many moves specify one strategy of player 2? Why?
  - Is SPE enough to characterize the equilibria of the game? Or do you need PBE?

Self-assessment

- **SA2**: Consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has two available moves (C,D) and two possible types (t_L,t_R) with prior $(p, 1-p)$; player 2 has only one type and moves (M,N)
  - Draw the game in extensive form (without payoffs)
  - How many moves specify one strategy of player 1? Why?
  - How many moves specify one strategy of player 2? Why?
  - Is SPE enough to characterize the equilibria of the game (i.e., to determine whether a BNE is sequentially rational)? Or do you need PBE?

Self-assessment

- **SA2 (cont'd)**: Consider a dynamic Bayesian game where player 1 moves first and player 2 moves second. Player 1 has two available moves (C,D) and two possible types (t_L, t_R) with prior $(p, 1-p)$; player 2 has only one type and moves (M,N)
  - Let h_D be the information set where player 2 moves after observing move D by player 1: what are 2's belief values μ in each node of h_D assuming separating strategy DC for 1?
  - What are 2's belief values μ in each node of h_D assuming pooling strategy DD for 1?
  - What are 2's belief values μ in each node of h_D assuming pooling strategy CC for 1?

Questions?