

Game Theory Q&A

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a.y. 2023/24



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Disclaimer

This paper is a "flashcard style" summary of Game Theory course of UniPD, a.y. 2023/24. It's not intended to substitute slides or book study.

1 Introduction, background concepts, decision problems

1. Q - What is a game in game theory?

A - **Game theory** is a mathematical framework that allows to model specific type of problems. The problems studied by game theory are called games.

2. Q - What is a game?

A - **A game** is a multi-person multi-objective problem.

Multi-person = There are multiple agents (players) involved in a game

Multi-objective = Players have, in general, different goals

The **outcome** of a game depends on the choices made by all players

The **purpose** of game theory is to find the "best choice" for each of them according to their objectives

3. Q - What does it mean for a preference to be rational?

A - **A preference** is rational when it's complete and transitive

Preference is a binary relationship \succeq between elements of A (set of possible actions)

If $a, b \in A$, $a \succeq b$ means that a is preferred to b

A **preference** is always: reflexive, anti-symmetric A preference can also be:

- **complete** if $\forall a, b \in A$ either $a \succeq b$ or $b \succeq a$
- **transitive** if $\forall a, b \in A, a \succeq b \wedge b \succeq c \Rightarrow a \succeq c$

4. Q - What does it mean for a player to be rational?

A - **A player is rational** when he always maximize their utility function, i.e., choose the action that leads to his preferred outcome. In other words, rational player act for their own good.

5. Q - What are the elements of a decision problem?

A - **A decision problem** has three elements: actions, outcomes and preferences.

Action a is selected from a set of possible actions A

Action a results in a certain **outcome** (For 1-player problems actions = outcomes)

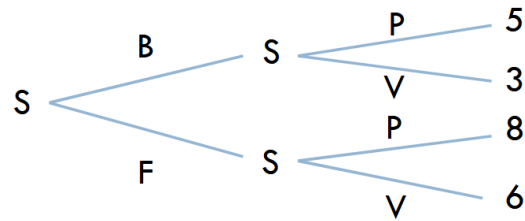
Preferences describe the relationship between different outcomes (i.e., which one is preferred)

6. Q - How to represent a decision tree?

A - **A decision tree** has players on nodes, actions on branches and payoffs on leaves.



■ $u(B)=2, u(F)=5, u(P)=3, u(V)=1$





2 Lotteries, VNM utility theorem, backward induction

7. Q - When is it possible to model preferences between lotteries using average payoffs?

A - When randomness is involved, i.e. when some events are called as nature (N) events, not in the hand of the player.

8. Q - Which utility function can we use to model a risk-averse player? 1) $u(x) = x^2$ or 2) $u(x) = \log(x)$?

A - 2) Log function models risk-averse: he prefers lower results with higher probability.

9. Q - How can we solve a decision problem involving sequential choices made by both a player and Nature?

A - Having the decision tree, starting with nodes at the end of the tree, substitute N node with expected utility and player's move with payoff of the best move.

Expected utility is calculated as probability expected value:

$\mathbb{E}_{x \sim p}[u(x)] = \sum_{k=1}^n p(x_k) \cdot u(x_k)$ where p is the payoff and u is the utility (u don't say)



3 Static games of complete information, normal-form representation, strictly dominated strategies, IESDS

(No questions in slides)

10. Q - When a strategy is called Pareto dominated?

A - A **joint strategy** s is **Pareto-dominated** by another strategy s' if

- $u_i(s') \geq u_i(s)$ for each player i
- $u_i(s') > u_i(s)$ for some player i

11. Q - What is IESDS?

A - **IESDS** stands for "iterated elimination of strictly dominated strategies". Since a rational player will never play a Pareto-dominated strategy, with IESDS is possible to delete one Pareto-dominated strategy after the other, in order to get a set of just one strategy or at least an easier representation of the game.



4 Nash equilibrium, best response, weak dominance, price of anarchy

12. Q - What is a NE

A - **Nash equilibrium** is what is played if believes of both players match.

A **belief** of player i is a possible profile of other player's strategy.

A **best strategy** is the response of a rational player to a believe. i.e. strategy $s_i \in S_i$ is a BS to moves $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ if

$$(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_n) \geq (s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n) \\ \forall s'_i \in S_i$$

13. Q - Consider NE (s_1, \dots, s_n) . Suppose player i replaces the current strategy s_i with s'_i . Can this still be a NE?

A - **No**. In Nash equilibrium no player can improve his payoff unilaterally.

14. Q - If a strategy is ruled out by IESDS, can it be a NE?

A - **Yes**. A joint strategy (s_1^*, \dots, s_n^*) in which everyone plays a dominant strategy is a Nash equilibrium.

15. Q - Compute the PoA for the Prisoner's dilemma using

$$C(s) = - \sum_j u_j(s)$$

A -

		Player B	
		M	F
Player A	M	-1,-1	-9,0
	F	0,-9	-6,-6

Table 1: Prisoners' dilemma

Worst (and, in this case, unique) NE: MM. Best Pareto: MF or FM.

$$PoA = \frac{-[(-1) + (-1)]}{-[(0) + (-6)]} = \frac{2}{6} = \frac{1}{3}$$

The price of anarchy (PoA) is the ratio between the social costs in the worst NE s^* and in the best Pareto efficient strategy (i.e., social optimum).

$$PoA = \frac{C(s^*)}{\min_s C(s)}$$

16. Q - Solve this hw.

A - For a generic R



- A (crazy) professor decides your grade in the exam he teaches will be decided by a game:
 - You are paired with a random classmate
 - You secretly choose an integer between 18 and 30, and so does the classmate
 - If you choose the same number, that is the score that you both get
 - If the numbers are different, who proposes the lowest score L gets a grade of $L + R$, while the other gets $L - R$ (score < 18 means the exam is failed, > 30 means 30L and gives payoff 31)
- Play the game with $R = 1$, $R = 2$, and $R = 10$.
- How do the NE change?

		Player B				
		18	19	...	29	30
Player A	18	18,18	18+R,0	...	18+R,0	18+R,0
	19	0,18+R	19,19	...	19+R,21-R	19+R,19-R

	29	0,18+R	19-R,19+R	...	29,29	29-R,29+R
	30	0,18+R	19-R,19+R	...	29-R,29+R	30,30

Table 2: HW1: Strange exam

For $R = 1$ all couples with same grades are Nash equilibria.

For $R > 1$ the only NE is a Pareto-dominated strategy (i.e., 18,18).



5 Exercise on NE, constitutions, electoral systems

(No questions in slides)

17. Q - What is a constitution?

A - **Constitution**, or **social welfare function**, is a map

$$f : R(A)^n \longrightarrow R(A)$$

$$(\succeq_1, \dots, \succeq_n) \xrightarrow{f} f(\succeq_1, \dots, \succeq_n)$$

which maps a profile of n rational preferences $\succeq_{(i)} = (\succeq_i, \dots, \succeq_n)$ into a unique rational social preference $\succeq = f(\succeq_{(i)})$.

18. Q - What are constitution properties?

A - **Properties:**

- **Independence of Irrelevant Alternatives (IIA)** if for all pairs $(\succeq_{(i)}), (\succeq'_{(i)})$

$$\forall i, \succeq_i \mid a, b = \forall i, \succeq'_i \mid a, b \implies f(\succeq_{(i)}) \mid a, b = f(\succeq'_{(i)}) \mid a, b$$

i.e., adding or removing elements to the set of alternatives does not change the output of a constitution for the pair $\{a, b\}$

- **Pareto-efficiency:** constitution f is Pareto-efficient if \forall profiles $(\succeq_{(i)})$, for all $a, b \in A$

$$\forall i, a \succeq_i b \implies a \succeq b$$

where $\succeq = f((\succeq_{(i)}))$. i.e., if everyone prefers a over b , that also becomes the preference of the constitution

- **Dictatorship** if $\exists i$ s.t.

$$a \succeq_i b \implies a \succeq b$$

where $\succeq = f((\succeq_{(i)}))$. i.e., if the constitution simply mimics i 's preferences.

- **Monotonic** if a single individual ranking higher $a \in A$ never causes a rank lower in the constitution
- **Non-imposition** is satisfied if all rational preferences can be outputs

A - **Theorems:**

- Arrow, 1951: there is no constitution f for which all there properties hold at the same time: f is not a dictatorship, f is monotonic, f satisfies IIA and non-imposition
- Arrow, 1963 (Arrow's impossibility theorem): if constitution f is Pareto-efficient and satisfies IIA $\implies f$ is a dictatorship

19. Q - Game theory and elections

A - In an election a candidate the beats (by majority) all the others is called the **Condorcet winner**. If there is no winner, then there is a cycle (i.e., $A > B$, $B > C$, $C > A$) called **Condorcet cycle**. With more than 3 candidates there can be a winner and a cycle. With preferences sufficiently randomized and a large ($n \longrightarrow \infty$) numbers of candidates, Condorcet cycles are sure to occur



voters→ choices↓	3	5	7	9	∞
3	5.6%	6.9%	7.5%	7.8%	8.8%
5	16.0%	20.0%	21.5%	23.0%	25.1%
7	23.9%	29.9%	30.5%	34.2%	36.9%
∞	100.0%	100.0%	100.0%	100.0%	100.0%

20. Q - What electoral methods we have?

A - We have these methods, all with their strengths and weaknesses

- **Plurality voting:** each voter sort candidates in order of personal preference. The candidate with most first places, wins.

With this method, a candidate with a minority high preference (i.e., being in first place by less than 50% of voters) wins over candidates in lower places by majority (i.e., if A is in first place for 4/10 voters, in last place for others, B is in second place for all voters and other first places are distributed by other candidates, A wins even if B is preferred by the majority over A)

- **Two-phase run-off:** first round voting: select two candidates with highest amount of votes; second round voting: run-off between those candidates.

With this method, a candidate being the "second choice" for a large majority will never be in the second phase, even if he is preferred to the others by majority (i.e. candidate C has few first place votes but it's preferred to A by every B's voter and it's preferred to B by every A's voter. It will not pass to the second phase, even if it's the Condorcet winner)

- **Borda counting:** suppose we have M candidates, each person gave M-1 points to his favourite candidate, M-2 to the second and so on till the last-favourite, who receive 0 points.

With this method, a strong candidate voted by the majority and being in last position by the others loses against a candidate being mediocre (i.e., A is voted in first place by half voters and last place by the other half. Considering first two position of the other half equally distributed by B and C, A is the Condorcet winner. But B or C will win elections thanks to all the second place points). Borda counting has also an huge issue with dropouts: a single contestant withdrawn can totally change the outcome. (examples in slides)

- **Approval voting:** each voter can give more than one preference, every preference is a vote, the number N of preferences goes from 1 to M, with M the number of candidates (for N=1 we are in plurality voting).

Depending on N, less favourite candidates has less or more chances to win.

- **Instant run-off:** asking voters to place candidates in order of preferences, only first placed goes counts in order to second round voting. Iteratively, we remove candidates with lowest amount of top preferences, till we get a majority.

Also in this scenario, a small change (even an increase in preferences) can lead to a different outcome.



6 NE applications (Duopolies, tragedy of the commons, selfish routing)

To be better understand



7 Mixed strategies, Nash theorem

21. Q - What is a mixed strategy?

A - A **mixed strategy** for player i is, in a game $G = (S_1, \dots, S_n; u_1, \dots, u_n)$ a probability distribution p_i over set S_i .

i.e. choosing strategies $S_i = (s_i^{(1)}, \dots, s_i^{(k)})$ with probabilities $(p_i(s_i^{(1)}), \dots, p_i(s_i^{(k)}))$

22. Q - What is expected utility?

A - **Expected utility** is (as expansion of utility) a real function over $\Delta S_1 \times \dots \times \Delta S_n$. For a chosen mixed strategy, payoff can be calculated as a weighted average over p_i .

$$u_i(p_i, \dots, p_n) = \sum_{(s_1, \dots, s_n) \in S} p_1(s_1) \cdots p_n(s_n) \cdot u_i(s_1, \dots, s_n)$$

with $S = S_1 \times \dots \times S_n$.

Strict and weak dominance are, as always, $>$ or \geq . Nash equilibrium is based on expected utility.

23. Q - What is the support of a mixed strategy?

A - Given a mixed strategy $p_i \in \Delta S_i$ we define the support of p_i as $\text{supp}(p_i) = \{s_i \in S_i : p_i(s_i) > 0\}$ (if $p_i(s_i) = 1$, s_i is a pure strategy).

24. Q - Nash theorem

A - Every game with finite pure-strategy sets S_i has at least one Nash equilibrium, possibly involving mixed strategies. More formally: for game $G = (S_1, \dots, S_n; u_1, \dots, u_n)$, define

$$BR_i : \Delta S_1 \times \dots \times \Delta S_{i-1} \times \Delta S_{i+1} \times \dots \times \Delta S_n \longrightarrow \Delta S_i$$

$$BR_i(p_{-i}) = \{p_i \in \Delta S_i : u_i(p_i, p_{-i}) \text{ is maximized}\}$$

. Then define $\mathbf{BR} : \Delta S \longrightarrow \Delta S$ as

$$\mathbf{BR}(p) = BR_1(p_{-1}) \times \dots \times BR_n(p_{-n})$$

p is a NE if $p \in \mathbf{BR}(p)$



8 Potential games, congestion games, coordination games; computational complexity of Nash Equilibrium search

25. Q - What is a potential game?

A - **A potential game** is a fictitious game that converges to NE. A fictitious game is a game where regrets become actual changes of moves.

Function $\Omega : S \rightarrow \mathbb{R}$ is an exact potential for \mathbb{G} if:

$$\Omega(s'_i, s_{-i}) - \Omega(s_i, s_{-i}) = u_i(s'_i, s_{-i}) - u_i(s_i, s_{-i}) = \Delta U_i$$

Function $\Omega : S \rightarrow \mathbb{R}$ is a weighted potential for \mathbb{G} with weight $w = w_i > 0$ if:

$$\Omega(s'_i, s_{-i}) - \Omega(s_i, s_{-i}) = w_i \Delta u_i$$

Function $\Omega : S \rightarrow \mathbb{R}$ is an ordinal potential for \mathbb{G} if:

$$\Omega(s'_i, s_{-i}) > \Omega(s_i, s_{-i}) \iff u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$$

26. Q - Th: Every potential game has (at least) one NE in pure strategies

27. Q - Which types of potential games does exist?

A - Congestion game: a potential game where players choose the "least congest" resource.

Coordination game: models situations where players have incentive to coordinate their actions.

Dummy (or pure-externality) game: game in such that $\forall s_{-i}, u_i(s_i, s_{-i}) = u_i(s'_i, s_{-i})$, i.e. players payoff depends only on s_{-i} .

N.B. every potential game is a sum of coordination game and dummy game.

28. Q - What is the computational complexity of finding NE?

A - **It's PPAD.** NE theorem states that a solution must exist, so it cannot be NP-complete. On the other hand, finding a NE in some case can be very difficult. Let's say (with the nth abuse of notation) that $P < PPAD < NP$. Which means that $PPAD$ is hard till we demonstrate $P = NP$. In this class we find the "end-of-line problem".



9 Only for projects. LOL

10 Exercise set #1 (static games of complete information)

See the end of lectures for exercises and solutions.

11 Dynamic games

29. Q - What is a dynamic game?

A - A **dynamic game** is a game in which players moves are sequential and not simultaneous.

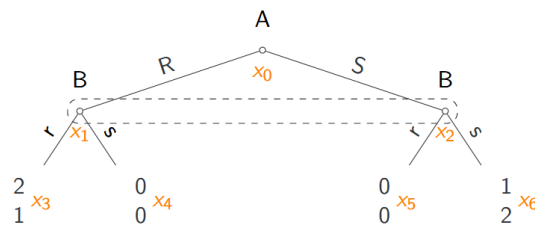
They can be of **perfect information** (meaning that every player can do every decision with full awareness) or **imperfect information** (meaning some decisions are “simultaneous” or Nature moves).

We have two scenarios for the former information:

- **endogenous uncertainty**: information sets contain multiple nodes (simultaneous moves)
- **exogenous uncertainty**: there is a choice of Nature (lotteries)

30. Q - How can we represent dynamic games?

A - Graphically, they can be represented as a tree, where each level is a move and each node is the response to a specific opponent' move. Dotted circle (or just a dotted arch) shows node from the



same information set, i.e. the game stage a player can have.

A **information set** h_i is a mathematical representation of player's possible moves. If $h_i = \{x_j\}$ (i.e. it has just one node), then the node is fully aware of previous moves.

31. Q - How to define a strategy in dynamic games?

A - A strategy in dynamic games need to account for the history of play.

Imagine a two-move game, B has strategy $s_B = (a_1, a_2)$ where a_1 is the response to A playing move 1. If the match is played twice, B has strategy

$$s_B = (a_1, a_{11}, a_{12}, a_{21}, a_{22})$$

, where a_{11} is the answer of A first move, B playing 1 and A answering with 1 (for example, $s_b = (1, 1, 1, 1, 1)$ means B playing 1 no matter what. Instead, $s_b = (1, 2, 1, 1, 1)$ means B playing 2 if A's second move is 1, otherwise playing 1.



12 Mixed strategies and Nash equilibria in dynamic games, subgame-perfect Nash equilibria

32. Q - What is the difference between mixed and behavioral strategies? Under what condition are the two concepts equivalent?

A - **Mixed strategies** are decided before the game starts; **behavioral**, on the other hand, are more "realistic" and every move is decided after opponent makes his (more formally: at each information set, select a random move from your set of available moves). They are equivalent under the perfect **recall assumption**: players do not forget the information they have acquired.

33. Q - How do you find the solution(s) of a sequential game?

A - **By backward induction**: starting from the leaf with maximum payoff for the player, replace the parent node with the payoff, repeat until root. More formally:

- Start from the leaves of the tree
- Partition them according to their parent node (which is a move of some player i)
- For each set in the partition, find the leaf x_j with maximum payoff for player i
- Add the corresponding branch to player i 's strategy
- Replace the parent node with the payoff of x_j
- Repeat until the root is reached

Solutions can be unique as can be more than one. This way, we find the **equilibrium path**.

A (proper) **subgame** \mathbb{G}' of a game \mathbb{G} contains a single node of the tree and all of its descendants, with the requirement that

$$x_j \in \mathbb{G}', x_j \in h_i \leftrightarrow x_k \in \mathbb{G}' \forall x_k \in h_i$$

An **equilibrium path** is a path that contains all nodes of a NE along the tree representation of the game.

34. Q - What is a subgame-perfect NE? In which cases is a NE not subgame-perfect?

A - A **subgame-perfect NE (SPE)** is the strategy that yields a NE in every subgame. Every SPE is a NE in the parent game and every finite dynamic game has at least one SPE. This means that every sequential game (tic tac toe, chess etc ...) has a NE.

35. Q - **Exercise**

Solution can be found in Lecture13 pdf

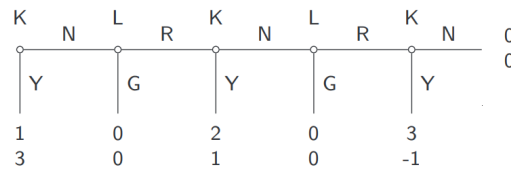
It is the discount sales season, Karen (K) and Lou (L) wants to go shopping. He thinks it is best to wait until the last three days of the discount sales, because prices are cheapest. On day 1, Lou asks Karen (K) to go with him. If Karen says yes (Y), they go shopping and the game ends. If Karen says no (N), Lou can either give up (G) or request (R) again the following day. On day 2, the same happens. On day 3, if K still says N, the game ends as well and does not go into a further day. If in the end K and L do not go shopping, both of their payoffs are 0. If they do, their payoffs are computed as $u_K = d$ and $u_L = 5 - 2d$, respectively, where d is the day in which they go. All of this information about the game is common knowledge among the players.

1. Represent the game in its extensive form.



2. How many (pure) strategies do K and L have, respectively?
3. Find the SPE of this game.
4. Find one NE that is not subgame-perfect.
5. Represent the game in its normal form.
6. Find all the NE of this game (both SPE and non-SPE).

A - Solution



■ K's payoff are on top, L's payoff are on the bottom

- 1.
2. K plays in 3 information sets and has 2 possible actions in each set: $2^3 = 8$ possible strategies. Her strategies are triplets: $s_K = (a_0, a_2, a_4), a_j \in Y, N$
L plays in 2 information sets has 2 possible actions in each set: $2^2 = 4$ possible strategies. His strategies are triplets: $s_L = (a_1, a_3), a_j \in G, R$
3. The SPE can be found via backward induction. On the last day, K prefers Y (payoff 3) to N (payoff 0): $a_4 = Y$. Before that, L prefers G (payoff 0) to R (payoff -1, knowing K's next choice): $a_3 = G$. Before that, K prefers Y (payoff 1) to N (payoff 0, knowing how the rest of the game will play out): $a_2 = Y$. Only SPE: $s_K = (N, Y, Y), s_L = (R, G)$.
4. Just chose and irrational move on SPE.
5. Just draw it
6. Trivial and left to the reader



13 Multistage games pt. 1

36. Q - What is a multistage game?

A - A **multistage game** is a game which consist in a sequence of smaller games, called **stages**, where total payoff is the sum of every payoff.

Some games can have independent stages (like partial exam), other can have a discount **factor** $\delta \in [0, 1]$ when stages are far apart in time. If so, the total payoff of player i is

$$u_i = u_i^{(1)} + \delta u_i^{(2)} + \delta^2 u_i^{(3)} + \dots + \delta^T u_i^{(T)} +$$

Discount is en exponential function of time t . It has to be exponential (not linear) because of time consistency.



14 Multistage games pt 2, stick-and-carrot strategies

37. Q - What is a strategy in a multistage game?

A - Each player has to specify what to do in the first stage and what to do in subsequent games depending on the outcome of previous games.

38. Q - Theorem for independent stages

A - Suppose $s^{(t)} = (s_1^{(t)}, \dots, s_n^{(t)})$ is a NE strategy for stage t of multistage game \mathbb{G} ; then there is a SPE of \mathbb{G} with equilibrium path $(s_1^{(1)}, \dots, s_n^{(t)})$. In other words, a strategy where each player always plays a NE, is a SPE.

39. Q - Theorem for linked stages 1

A - Every NE s^* of multistage game $\mathbb{G} = (\mathbb{G}_1, \dots, \mathbb{G}_T)$ requires that a NE is played in the last stage. In other words, in order for a strategy to be a NE, it has to have a NE in the last stage, since players already know all previous outcomes.

40. Q - Theorem for linked stages 2

A - If stage games $\mathbb{G}_1, \dots, \mathbb{G}_T$ has all unique NE, then $\mathbb{G} = (\mathbb{G}_1, \dots, \mathbb{G}_T)$ has unique SPE. In other words, if every stage has only one NE, this is the equilibrium path.

There is a counter intuitive consequence of these theorems: if the last stage has multiple NE, that enable non-NE strategies to be played in previous stages.

41. Q - What is the meaning of δ ?

A - δ is the measure of how much we care about the future. A larger δ means we care more about future payoffs.

42. Q - One-stage deviation principle

A - A one stage unimprovable strategy must be optimal. (*proof in Lecture14*)

Optimal strategy: a strategy s_i is optimal for player i i \forall information set h_i there is no way to improve it (more formally: $\nexists s'_i / u_i(s'_i | \{h_i\}) > u_i(s_i | \{h_i\})$).

One-stage unimprovable strategy: a strategy s_i is one-stage unimprovable if there is no s'_i that differs in one single stage s.t. $u_i(s'_i | \{h_i\}) > u_i(s_i | \{h_i\})$

43. Q - Exercise 1

A - Consider multistage game $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$, with \mathbb{G}_1 being the first stage, and \mathbb{G}_2 being the second (and last) stage:

- Is it possible (for some \mathbb{G}_1 and \mathbb{G}_2) to find a SPE for \mathbb{G} where a non-NE is played in \mathbb{G}_2 ? **No**
- Is it possible (for some \mathbb{G}_1 and \mathbb{G}_2) to find a NE for \mathbb{G} where a non-NE is played in \mathbb{G}_2 ? **No**
- Is it possible (for some \mathbb{G}_1 and \mathbb{G}_2) to find a SPE for \mathbb{G} where a non-NE is played in \mathbb{G}_1 ? **Yes**
- Is it possible (for some \mathbb{G}_1 and \mathbb{G}_2) to find a SPE for \mathbb{G} where a strictly dominated strategy is played in \mathbb{G}_1 ? **Yes**



- What is the minimum number of NE in stage game G_2 to enable a carrot-and-stick SPE in G ? What characteristics should these NE have? **At least two NE (stick and carrot)**

Answers are mine, so feel free to mail me if they are wrong

44. Q - Exercise 2

Ashley and Brook live together. During the winter break they contemplate giving each other a nice gift (G) for Christmas or not (N). They know each other's preferences so they are able to buy a gift for 10 euros that is worth like 100 euros for the other. They make this decision independently and without telling each other. After Christmas, they also consider whether to celebrate New Year's eve downtown (D) or stay home (H). For the New Year's eve celebration, they decide independently of each other in a coordination-game fashion. Staying home has utility of 0 for both. Going downtown has utility of 50. However, spending New Year's eve apart from each other has utility of -100 for both. The total payoff of the players is the sum of the partial payoffs in each stage with a discount factor of δ for the second stage.

1. Write down the normal form of both stages of the multi-stage game.
2. Find a trivial subgame-perfect equilibrium of the game where the players just play a Nash equilibrium in all stages, without any strategic connection.
3. Is there a strategically connected SPE of the whole game where Ashley and Brook give gifts to each other? If so, show the minimum required discount factor value δ_{\min} for that to hold.

A -

1. Normal form of stage game Only NE in Stage 1 is (N, N)

	G	N		H	D
G	90 90	-10 100	H	0 0	-100 -100
N	100 -10	0 0	D	-100 -100	50 50

In stage two NE are: (H, H) (stick) and (D, D) (carrot)

2. Trivial SPE is playing stages independently: $(NHHHH, NHHHH)$ (or with the carrot)
3. Cooperative SPE: "play G at stage 1. If (G, G) at stage 1, play D, otherwise play H". Sustainable if $u(\text{cooperative}) + \delta u(\text{carrot}) \geq u(\text{unilateral deviant}) + \delta u(\text{stick}) \rightarrow 90 + 50\delta \geq 100 + 0\delta \rightarrow \delta \geq \frac{1}{5}$



15 Repeated games, grim trigger, tit for tat, Friedman theorem

45. Q - What is a repeated game?

A - A **repeated game** $\mathbb{G}(T, \delta)$ is a dynamic game where a static game \mathbb{G} is played as a stage game for t times with discount δ . We distinguish between:

- finitely repeated games ($T = 1, 2, \dots < \infty$);
- infinitely repeated games ($T = \infty$). Here we must have $\delta < 1$, otherwise payoff will diverge.

46. Q - Theorems

A -

1. The outcome of the last stage is a NE
2. If stage game \mathbb{G} only has NE p^* , then $\mathbb{G}(T, \delta)$ has a unique subgame-perfect equilibrium, where every player play p^* in every stage (boring)

47. Q - Cooperation in finitely repeated games

A - **Cooperation** is usually incentivised in repeated games but in the last stage, which is played "egoistically". As for multistage games, cooperation is possible if there are multiple NE.

48. Q - Infinitely repeated game

A - Since these games have no last stage, there could be a SPE of $\mathbb{G}(\infty, \delta)$ in which no stage outcome is a NE of \mathbb{G} .

We call a **grim trigger strategy** (GrT) a strategy where: start playing M at stage 1, at stage $t > 1$, play M only if outcome of all $t - 1$ previous stages was (M, m) , otherwise play F.

With $\delta = 1 - \epsilon$, joint strategy "both play GrT" is a SPE.

49. Q - How can we show that a strategy is a NE in overall game?

A - We only need to compare who options:

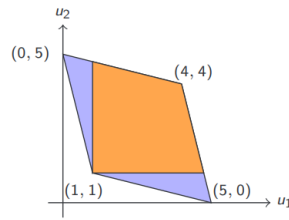
- Cooperate and keep playing the chosen strategy (payoff $*$) forever ($u_T = p^* + \delta p^* + \delta^2 p^* + \dots = \frac{p^*}{1-\delta}$)
- Defect at stage 1 and keep playing other option (payoff p) forever ($u_T = p + \delta p + \delta^2 p + \dots = p + \frac{\delta}{1-\delta}$)

In order to find minimum δ s.t. cooperation is best choice, solve

$$\frac{p^*}{1-\delta} \geq p + \frac{\delta}{1-\delta}$$

50. Q - Friedman theorem (a.k.a. "folk theorem")

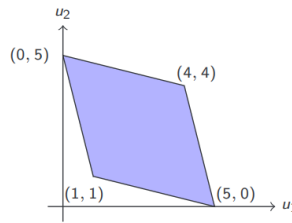
A - **Theorem** Let \mathbb{G} be a finite static game of complete information. Let (e_1, e_2, \dots, e_n) be the payoffs of a NE of \mathbb{G} and (x_1, x_2, \dots, x_n) be a feasible payoffs for \mathbb{G} . Suppose \forall NE and $\forall, x_j > e_j$. Then, for δ close enough to 1, $\mathbb{G}(\infty, \delta)$ has a SPE with payoffs (x_1, x_2, \dots, x_j) .



A **feasible payoff** for game \mathbb{G} is any convex combination

$$\alpha u(s_1) + \alpha u(s_2) + \dots + \alpha_L(s_L), \text{ with } \sum_{i=1}^L \alpha_i = 1$$

of pure-strategy payoffs ($L = |S_1| \cdot |S_2| \cdot \dots \cdot |S_n|$ total numbers of pure strategy)



51. Q - Tit for tat (TFT)

A - **TFT** is a replacement of GrT where "at stage t , play what the other player chose at stage $t-1$ ", it's a way to avoid keep punishment forever. It has immediately punish deviation but forgiveness after 1-step.

Keep doing TFT forever is called "death spiral" and it's not a NE. generally, NE achieved by TFT is not subgame-perfect.

52. Q - Exercise 1

Consider \mathbb{G} :

		Player B	
		g	w
Player A	G	5, 3	0, 4
	N	6, 0	1, 1

1. Is it possible to find a SPE for $\mathbb{G}(4)$ that involves playing (G,g) at each stage?
2. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a grim-trigger strategy? If so, for what δ ? Is that a SPE?
3. Is it possible to find a NE for $\mathbb{G}(\infty)$ where (G,g) is played at each stage using a tit-for-tat strategy? If so, for what δ ? Is that a SPE?



A - TODO:

53. Q - Exercise 2

Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(S, M) = u_D(M, S) = 0$. Call \mathbb{G} this game, and consider it in a repeated version $\mathbb{G}(T)$. Individual payoffs are summed with discount factor δ .

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a “Grim Trigger” strategy where each player ends up in always choosing S?
3. Consider an extended game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for both players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game \mathbb{G}' . If you see a SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

A - TODO:



Lecture 10

Exercise set #1

Thomas Marchioro

November 9, 2023

Invariance of NE to linear transf.



- **Theorem:** Consider game $\mathbb{G} = (S_1, \dots, S_n; u_1, \dots, u_i, \dots, u_n)$. If $p^* = (p_1^*, \dots, p_n^*)$ is a NE for \mathbb{G} , then it is also a NE for $\mathbb{G}' = (S_1, \dots, S_n; u_1, \dots, u'_i, \dots, u_n)$, with

$$u'_i(s) = a u_i(s) + c, \quad \forall s \in S_1 \times \dots \times S_n$$

and $a > 0, c \in \mathbb{R}$.

- In other words, you can apply any *positive* linear transformation (positive scaling + shift) to *all* payoffs of one or multiple players in the game, and preserve the NE.

- *Proof*: Suppose p^* is a NE for \mathbb{G} . Then, the following must hold for player i :

$$u_i(p_i^*, p_{-i}^*) \geq u_i(p_i, p_{-i}^*), \quad \forall p_i \forall p_{-i}^*$$

- We can expand both sides into linear combinations

$$\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \geq \sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*)$$

- We multiply both sides by $\alpha > 0$ and add $c \in \mathbb{R}$

$$\alpha \left(\sum_{s_i \in S_i} p_i^*(s_i) u_i(s_i, p_{-i}^*) \right) + c \geq \alpha \left(\sum_{s_i \in S_i} p_i(s_i) u_i(s_i, p_{-i}^*) \right) + c$$

- Finally, we apply the associative property

$$\sum_{s_i \in S_i} p_i^*(s_i) \underbrace{(\alpha u_i(s_i, p_{-i}^*) + c)}_{u'_i(s_i, p^*)} \geq \sum_{s_i \in S_i} p_i(s_i) \underbrace{(\alpha u_i(s_i, p_{-i}^*) + c)}_{u'_i(s_i, p^*)}$$

- Q.E.D. \square
- Actually, to have a “complete” proof, one should also expand p_{-i}^* so as to get $u'_i(s)$ in the sum; however, the procedure is still the same, since it is still a linear combination of values and associative property can be applied

- Find all NE (pure and mixed)

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

Exercise 1a (sol.)

- N is strictly dominated by M ($4 > 1$; $6 > 5$)

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

Exercise 1a (sol.)

- Now G is strictly dominated by F

		B	
		M	N
A	F	2, 4	0, 1
	G	1, 6	3, 5

- (F, M) is the only NE

Exercise 1b

- Find all NE (pure and mixed)

		B	
		M	N
A	F	0, 4	3, 0
	G	6, 0	0, 5

- No NE in pure strategies (Nash theorem: there must be one in mixed strategies)

		B	
		M	N
A	F	0, 4	3, 0
	G	6, 0	0, 5

- It must be that $\text{supp}(\mathbf{p}_A) = \{F, G\}$ and $\text{supp}(\mathbf{p}_B) = \{M, N\}$
- $\mathbf{p}_A = (q, 1 - q)$, $\mathbf{p}_B = (r, 1 - r)$

- A plays F with probability q and G with $1 - q$
- B plays M with probability r and N with $1 - r$
- Apply the characterization theorem:

$$\begin{cases} \underbrace{4q}_{u_B(q,M)} = \underbrace{5(1-q)}_{u_B(q,N)} \\ \underbrace{3(1-r)}_{u_A(F,r)} = \underbrace{6r}_{u_A(G,r)} \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (5/9, 4/9)$, $\mathbf{p}_B = (1/3, 2/3)$

- Find all NE (pure and mixed)

		B	
		M	N
A	F	9, 3	2, 2
	G	0, 0	3, 9

Exercise 1c (sol.)

- Battle-of-the-sexes-like game
- Two “opposite” NE in pure strategies
- There must be one “in the middle”

		B	
		M	N
A	F	9, 3	2, 2
	G	0, 0	3, 9

- A plays F with probability q and G with $1 - q$
- B plays M with probability r and N with $1 - r$
- Apply the characterization theorem:

$$\begin{cases} 3q = 2q + 9(1 - q) \\ 9r + 2(1 - r) = 3(1 - r) \end{cases}$$

- Mixed NE: $\mathbf{p}_A = (9/10, 1/10)$, $\mathbf{p}_B = (1/10, 9/10)$

Exercise 1d

- Find all NE (pure and mixed)

		B	
		M	N
A	F	2, 2	0, 6
	G	6, 0	1, 1

- Prisoner's-dilemma-like game

		B	
		M	N
A	F	2, 2	0, 6
	G	6, 0	1, 1

- Only NE is (G, N)
- If we try to find a mixed NE, we get

$$2q = 6q + (1 - q) \Rightarrow q = -1/5 \Rightarrow \text{Impossible!}$$

Exercise 2

		B			
		J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
	Y	4, 9	9, 2	0, 4	2, 3
	Z	8, 4	2, 8	4, 2	3, 6

- 1 Show that there is no NE in pure strategies
- 2 Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE
- 3 List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

- 1 Show that there is no NE in pure strategies

		B			
		J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
	Y	4, 9	9, 2	0, 4	2, 3
	Z	8, 4	2, 8	4, 2	3, 6

Exercise 2 (sol.)

- 2 Show that $(\mathbf{p}_A, \mathbf{p}_B)$ with $\mathbf{p}_A = (2/3, 0, 1/3)$ and $\mathbf{p}_B = (5/11, 4/11, 2/11, 0)$ is a mixed NE
- Use characterization theorem: for each player i

$$u_i(s_i, \mathbf{p}_{-i}) = u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \in \text{supp}(\mathbf{p}_i)$$

$$u_i(s_i, \mathbf{p}_{-i}) \leq u_i(\mathbf{p}_i, \mathbf{p}_{-i}) \text{ for each } s_i \notin \text{supp}(\mathbf{p}_i)$$

- For player A

$$u_A(X, \mathbf{p}_B) = 6 \cdot \frac{5}{11} + 5 \cdot \frac{4}{11} + 3 \cdot \frac{2}{11} = \frac{56}{11}$$

$$u_A(Y, \mathbf{p}_B) = 4 \cdot \frac{5}{11} + 9 \cdot \frac{4}{11} + 0 \cdot \frac{2}{11} = \frac{56}{11}$$

$$u_A(Z, \mathbf{p}_B) = 8 \cdot \frac{5}{11} + 2 \cdot \frac{4}{11} + 4 \cdot \frac{2}{11} = \frac{56}{11}$$

- All pure strategies yield equal payoff, so \mathbf{p}_B is a sustainable NE strategy

- For player B

$$u_B(\mathbf{p}_A, J) = 7 \cdot \frac{2}{3} + 4 \cdot \frac{1}{3} = \frac{18}{3} = 6$$

$$u_B(\mathbf{p}_A, K) = 5 \cdot \frac{2}{3} + 8 \cdot \frac{1}{3} = \frac{18}{3} = 6$$

$$u_B(\mathbf{p}_A, L) = 8 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{18}{3} = 6$$

$$u_B(\mathbf{p}_A, M) = 1 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{8}{3} \leq 6$$

- M does not belong to the support and yields lower payoff, so \mathbf{p}_A is a sustainable NE strategy

- 3 List all joint pure strategies that are Pareto optimal (i.e., Pareto efficient)

- Easier to delete Pareto dominated strategies

		B			
		J	K	L	M
A	X	6, 7	5, 5	3, 8	8, 1
	Y	4, 9	9, 2	0, 4	2, 3
	Z	8, 4	2, 8	4, 2	3, 6

- Remaining ones are Pareto optimal: (X, J), (Y, J), (Z, J), (Y, K).

Exercise 3

- Two firms (F1 and F2) work on a joint project from the European Commission. They can allocate an integer number of employees on the project, from 0 to infinity. They decide independently and without consulting with each other. The outcome of the project is that, if the number of employees allocated by each firm is identical (even zero!), both firms get a funding of 290 k€ from the European Commission. If the two firms assign a different number of employees, the European Commission gives them different fundings: 700 k€ to the one with more employees, and 320 k€ to the one with fewer employees. However, assigning employees costs 200 k€ per employee. The *utility* of a firm is funding minus costs.

- 1 Show that no rational firm will allocate more than 2 employees
- 2 Draw the normal form of this game and find its pure NE
- 3 Find the additional NE in mixed strategies.

- Let us start by understanding the problem
- Consider $1\text{k€} = 1$ payoff unit
- The firm that allocates most employees receives more money (700) but needs to pay more (-200 payoff for each employee); the other firm receives 320 and also needs to pay the employees
- Worst case scenario is when they choose the number: they only receive 290 and still need to pay their employees
- Examples:
 - if F1 allocates 2 employees and F2 allocates 1, they get $700 - 400 = 300$ and $320 - 200 = 120$, respectively
 - if they both allocate 1 employee, they get $290 - 200 = 90$ each

Exercise 3 (sol.)

- 1 Show that no rational firm will allocate more than 2 employees
- By choosing $n_i = 0$, the firm can secure a payoff of at least 290
 - $u_i(0, 0) = 290$; $u_i(0, n_{-i}) = 320$ for $n_{-i} > 0$
 - with $n_i \geq 3$, a firm can get at most $u(3, n_{-i}) = 700 - n_i \cdot 200 < 290$ when $n_{-i} < n_i$
 - So all $n_i \geq 3$ are strictly dominated strategies (a rational firm would never choose them)
 - Notice that 2 is not strictly dominated, since a firm may play it and get $700 - 400 = 300 > 290$, in case the other chooses 1 or 0 employees

2 Draw the normal form of this game and find its pure NE

- The set of players is $\{F1, F2\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

		F2		
		0	1	2
F1	0	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

2 Draw the normal form of this game and find its pure NE

- The set of players is $\{F1, F2\}$
- The set of strategies for both is $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1, 2\}$

		F2		
		0	1	2
F1	0	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Even though $n_i = 2$ is not dominated by 0 or 1 separately, it is strictly dominated by a combination of them
- i.e., “choose 0 with probability 0.95 and 1 with probability 0.05”, which gives utility $29 \cdot 0.95 + 50 \cdot 0.05 = 300.5 > 300$ against 0 and $32 \cdot 0.95 + 9 \cdot 0.05 = 309.5 > 300$ and some positive value > -110 against 2.

		F2		
		0	1	2
F1	0	290, 290	320, 500	320, 300
	1	500, 320	90, 90	120, 300
	2	300, 320	300, 120	-110, -110

Exercise 3 (sol.)

- Strictly dominated strategies cannot be part of NE (pure or mixed)

		F2	
		0	1
F1	0	290, 290	320, 500
	1	500, 320	90, 90

- (0, 1) and (1, 0) are NE
- There must be a mixed NE as well

3 Find the additional NE in mixed strategies.

- Symmetric payoffs: $q = r$
- Characterization theorem:
 $290q + 320(1 - q) = 500q + 90(1 - q) \Rightarrow q = r = 23/44$
- Mixed: NE $\mathbf{p}_1 = \mathbf{p}_2 = (23/44, 21/44)$

Alternative sol.

- Q: “What if I do not realize that strategy 2 is dominated?”
- Pure NE can be found via best-response search
- To find mixed NE, you just apply the characterization theorem for mixed strategy $\mathbf{p}_i = (q_0, q_1, q_2)$, i.e.,
 $u_i(0, \mathbf{p}_i) = u_i(1, \mathbf{p}_i) = u_i(2, \mathbf{p}_i)$

$$\begin{cases} q_0 u_i(0, 0) + q_1 u_i(0, 1) + q_2 u_i(0, 2) = q_0 u_i(1, 0) + q_1 u_i(1, 1) + q_2 u_i(1, 2) \\ q_0 u_i(0, 0) + q_1 u_i(0, 1) + q_2 u_i(0, 2) = q_0 u_i(2, 0) + q_1 u_i(2, 1) + q_2 u_i(2, 2) \\ q_0 + q_1 + q_2 = 1 \end{cases}$$

- You get $q_2 < 0$ which is impossible \Rightarrow Strategy 2 does not belong to the support!

- A strategic interaction takes place between the taxpayer T and the tax inspector I . T is supposed to pay a share S of his income so as to have a net income equal to R after paying taxes. However, T is considering two alternatives: hide part of his income (**H**) so as to pay $S - L$ instead of S (thus getting a net income of $R + L$), or pay all due taxes in full (**P**). Player I also has two options: check T for tax fraud (**C**) or not (**N**). Performing a check has a cost equal to E . If the inspector finds out that T has hidden part of the income, then the taxpayer will have to pay all due taxes plus a fine of F , and an equal amount is collected by the inspector. The probability of being caught after a tax inspection is p . Formalize this conflict in the form of a static game of complete information and find the Nash equilibria.

Exercise 4 (sol.)

- Normal form representation

		I	
		C	N
T	H	$R + (1 - p)L - pF,$ $S - (1 - p)L - E + pF$	$R + L, S - L$
	P	$R, S - E$	R, S

Exercise 4 (sol.)

- To simplify the game we can apply the invariance to linear transformations and: subtract R from I's payoffs; subtract S from T's payoffs
- Remember: All NE are preserved

		I	
		C	N
T	H	$(1 - p)L - pF,$ $-(1 - p)L - E + pF$	$L, -L$
	P	$0, -E$	$0, 0$

Exercise 4 (sol.)

- Since all quantities E, F, L are > 0 we know that:
 - if I plays N, T's best response is H ($L > 0$)
 - If T plays P, I's best response is N ($-E < 0$)

		I	
		C	N
T	H	$(1 - p)L - pF,$ $-(1 - p)L - E + pF$	$L, -L$
	P	$0, -E$	$0, 0$

- What if T plays H, or I plays C? It depends

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E > p(F + L)$

		I	
		C	N
T	H	$(1-p)L - pF,$ $-(1-p)L - E + pF$	$L, -L$
	P	$0, -E$	$0, 0$

- (H, N) is the only NE
- C is strictly dominated by N, so there cannot be NE involving C
- Interpretation: too much effort required to perform the check

Exercise 4 (sol.)

- If $-(1-p)L - E + pF > -L$, i.e., $E < p(F + L)$; and $(1-p)L - pF > 0$, i.e., $p < L/(L + F)$

		I	
		C	N
T	H	$(1-p)L - pF,$ $-E + pF - (1-p)L$	$L, -L$
	P	$0, -E$	$0, 0$

- Only NE is (H, C): T has low probability of being caught, so he always decides to hide his income; however, checking does not require much effort, so it for the inspector

Exercise 4 (sol.)

- If $-(1-p)L - E + pF < -L$, i.e., $E < p(F + L)$; and $(1-p)L - pF < 0$, i.e., $p > L/(L + F)$

		I	
		C	N
T	H	$(1-p)L - pF,$ $-E + pF - (1-p)L$	$L, -L$
	P	$0, -E$	$0, 0$

- No NE in pure strategies: there must be a mixed one.

Exercise 4 (sol.)

- Mixed NE is $(\mathbf{p}_T, \mathbf{p}_I)$, with $\mathbf{p}_T = (q^*, 1 - q^*)$ and $\mathbf{p}_I = (r^*, 1 - r^*)$
- The values of q^* and r^* can be found using the characterization theorem

$$r(1-p)L - rpF + (1-r)L = 0 \Rightarrow r^* = \frac{L}{p(L+F)}$$

$$-qE + qpF - q(1-p)L - (1-q)E = -qL \Rightarrow q^* = \frac{E}{p(L+F)}$$

- Interpretation: q^* could be a reasonable estimate for the prob. of T committing tax fraud: proportional to I's effort E ; inv. proportional to $L + F$ and to the prob. of being caught p
- r^* is the fraction of taxpayers that the inspector should check; this is proportional L

Questions?

Exercise set #2

Game Theory 2023/24

Exercise 1 It is the discount sales season, and Lou (L) wants to go shopping. He thinks it is best to wait until the last three days of the discount sales, because prices are cheapest. On day 1, Lou asks Karen (K) to go with him. If Karen says yes (Y), they go shopping and the game ends. If Karen says no (N), Lou can either give up (G) or request (R) again the following day. On day 2, the same happens. On day 3, if K still says N, the game ends as well and does not go into a further day. If in the end K and L do not go shopping, both of their payoffs are 0. If they do, their payoffs are computed as $u_K = d$ and $u_L = 5 - 2d$, respectively, where d is the day in which they go. All of this information about the game is common knowledge among the players.

1. Represent the game in its extensive form.
2. Find the SPE of this game.
3. Find *one* NE that is not subgame-perfect.

Exercise 2 Ashley and Brook live together. During the winter break they contemplate giving each other a nice gift (G) for Christmas or not (N). They know each other's preferences so they are able to buy a gift for 10 euros that is worth like 100 euros for the other. They make this decision independently and without telling each other. After Christmas, they also consider whether to celebrate New Year's eve downtown (D) or stay home (H). This means that receiving a gift gives utility of 100 to the receiver that is moved by the gesture, but also -10 to the utility of the buyer. Not giving gifts implies no variation of the utility for both.

For the New Year's eve celebration, they decide independently of each other in a coordination-game fashion. Staying home has utility of 0 for both. Going downtown has utility of 50. However, spending New Year's eve apart from each other has utility of -100 for both. The total payoff of the players is the sum of the partial payoffs in each stage with a discount factor of δ for the second stage.

1. Write down the normal form of both stages of the multi-stage game.
2. Find a trivial subgame-perfect equilibrium of the game where the players just play a Nash equilibrium in all stages, without any strategic connection.
3. Is there a strategically connected SPE of the whole game where Ashley and Brook give gifts to each other? If so, show the minimum required discount factor value δ_{\min} for that to hold.

Exercise 3 Carl (C) and Diana (D) are two university students. Every night they go to the department library, but they do not coordinate or plan any action together. Upon their arrival, they independently decide whether to: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(S,M) = u_D(M,S) = 0$. Call \mathbb{G} this game, and consider it in a repeated version $\mathbb{G}(T)$. Individual payoffs are summed with discount factor δ .

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a "Grim Trigger" strategy where each player ends up in always choosing S?
3. Consider an *extended* game where a punishment strategy P is also available to both players. When either player P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game \mathbb{G}' . If you see a SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

Exercise 4 Consider the following game in normal form. Players are denoted as 1 and 2 and their strategy sets are $\mathcal{S}_1 = \{A, B, C\}$ and $\mathcal{S}_2 = \{X, Y\}$. The payoff matrix is as follows:

		2	
		X	Y
1	A	9, -9	-5, 5
	B	-2, 2	7, -7
	C	8, -8	-1, 1

1. Describe what kind of game is that and name a specific solution concept you might use to find its Nash equilibria.
2. What is the support within \mathcal{S}_1 of the mixed strategy played at the Nash equilibrium by 1?
3. Find the mixed strategies played by 1 and 2 at the Nash equilibrium.

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Exercise 1 A gang of pirates has n ranks from 1 (the ship's boy) to n (the captain). After a raid, they share a treasure. Pirate with rank $k + 1$ keeps an eye on pirate k to see whether he gets a bigger share than he should. The game starts when pirate $k = 1$ (the ship's boy) realizes that the treasure contains an extremely valuable pearl that has fallen far from the stash: he considers whether to take the pearl for himself (P) hiding it in his pocket or do nothing (N). Doing nothing ends the game with the pearl being unnoticed and unassigned. However, if pirate k takes the pearl, pirate $k + 1$ will notice it; now, pirate $k + 1$ may consider to kill him and keep the pearl for himself (P), or do nothing (N). If pirate $k + 1$ does nothing, pirate k is left alive with the pearl – a very good outcome. If pirate $k + 1$ kills pirate k and takes the pearl instead, this is spotted by pirate $k + 2$ that now faces the same choice: whether to kill pirate $k + 1$ and keep the pearl for himself (P), or to do nothing (N). This means that k is replaced with $k + 1$ and the game continues up to the captain. For every pirate, the top preference is to stay alive and have the pearl; after that, they all prefer being alive without the pearl than to be killed.

1. Consider $n = 5$. Choose appropriate utility values for the outcomes and draw the extensive form of the game.
2. Consider $n = 5$. Solve the game by finding its subgame-perfect outcome.
3. Consider $n = 8$. Does the subgame-perfect outcome change, and why?

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Exercise 2 The mayor of a big city is to be selected among four candidates: (A)Amanda Amour; (B)Bruno Bravery; (C)Claire Constitution; (D)Dave Democracy. Using symbol \succ to denote “preferred to”, polls indicate that:

- (A) has 42% of supporters. Also, for them $(B) \succ (C) \succ (D)$.
 - (B) has 11% of supporters. Also, for them $(A) \succ (C) \succ (D)$.
 - (C) has 27% of supporters. Also, for them $(B) \succ (D) \succ (A)$.
 - (D) has 20% of supporters. Also, for them $(C) \succ (B) \succ (A)$.
1. The election is being held as a two-round run-off (i.e., with a ballot). What is the outcome under *sincere voting*? Denote the winner as W .
 2. Assume that the supporters of (D) can identify this outcome and plan a strategy. What is the best *strategic voting* that they can enact?
 3. Discuss the identity of the winner W' under strategic vote of (D)'s supporters. What kind of choice is W' ? Can the supporters of W prevent this outcome by counteracting strategic vote of (D)'s supporters, with a strategic vote of their own?

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Exercise 3 The football manager of Athletic United (A) wants to hire a player in the transfer market. His evaluation for the player is 2 million euros. The team for which the player is currently enrolled, Braves F.C., (B) is considering the player to be worth at least 1 million euros. These valuations are common knowledge. The negotiation opens, and during the transfer window there is time for four bouts of exchanges. First, A offers a selling price; if B accepts, deal is made; otherwise go to the second round. In the second round, roles are reversed, with B making their call. Third and fourth exchange are similar to the first and second respectively. If p is the selling price in the first round, profits are $2 - p$ for A and $p - 1$ for B. A discount factor of 90% is applied for any subsequent round.

1. What is the extensive form for this game?
2. Apply backward induction to find at which round the agreement is reached.
3. What are price paid and final payoffs for A and B?

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Exercise 4 Little Jimmy really hopes to receive a space train (T) for Christmas. His parents have already bought him a sweater (S) but they tell him that he might receive T if he behaves as a good boy (G) instead of a naughty one (N). Jimmy's behavior ultimately depends on whether Santa Claus exists or not, which he evaluates as having probability p . If Jimmy is naughty, he will not receive any gift, regardless of Santa Claus existing or not. Jimmy and the parents' utility will be 0 and -10 , respectively. If Jimmy is good, he will either receive S or T, according to the decision made by his parents. If Santa exists, Jimmy can receive the space train at no cost for his parents. If Santa does not exist, Jimmy can receive T only if the parents pay for it. Jimmy's utility for S and T is -10 and $+40$, respectively. The parents' reward when giving S and T to their son is 0 and $+20$. However, if they have to buy the space train themselves, subtract 50 from their utility. Also note that, being adults, the parents know whether Santa exists or not; on the other hand, they also know the value of p estimated by their son (that is, the prior is common knowledge).

1. Represent this game in extensive form.
2. Represent this game in normal form, with a type-player representation of the Bayesian players.
3. What kind of equilibria would be enough to characterize this Bayesian game? Discuss for $p = 0.9$.

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Exercise 5 The city of Aurapolis is attacked by a horde of Barbarians. The city can be accessed by two gates, the Northern Gate and the Southern Gate. The commander of the city guard has 4 battalions, while the barbarians only have 3. However, the city guard cannot leave any of the gates undefended, while the barbarians can do as they please (they can send all battalions against a single gate). Two battles are fought at each gate. For each battle, denote with n_A and n_B the number of battalions sent by A and B, respectively. When the same number of battalions fight at one gate, i.e., $n_A = n_B$, the partial payoff of that gate is zero for both sides. If $n_A \neq n_B$, partial payoff is $\max(n_A, n_B)$ for the side with more battalions, and $-\max(n_A, n_B)$ for the other side. Total payoff is the sum of the partial payoffs at both gates.

1. Write this game in normal form.
2. Find the Nash equilibria in pure strategies.
3. Find all the Nash equilibria of the game.

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Exercise 1 Agatha (A) and Bruno (B) are two siblings. They own a model train set (T) and they usually play together with it. Their respective utilities when doing so is $u_A(T, T) = u_B(T, T) = 3$. Their mom (M) buys them a new toy (N), which can be either a dollhouse (D) or a set of small soldier figurines (S). The two kids must independently decide whether to keep playing with T or the new toy N. All the three players (A, B, M) involved decide their action simultaneously and unbeknownst to each other. The action set of the kids includes just T or N; the mom instead decides what new toy to buy. If the kids end up in not playing together, *all* the players (also M) get utility equal to 0. If they play together with the new toy, they get a positive utility that for A is equal to 6 and 1 for D and S, respectively; for B, the values are instead 1 and 4, respectively. The utility of M is the lump sum of the utility of the two kids.

1. Write down the normal form of this game (likely, you will need a 3D matrix: you can write two matrices instead, one per each move of player M).
2. Find all Nash equilibria of this game in pure strategies.
3. Find all additional Nash equilibria of this game in mixed strategies.

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Exercise 2 Carl (C) and Diana (D) are two university students that have found that the department library is unoccupied overnight. It is a really good place to study and has a very fast Internet connection. So, they go there every night, but they do not coordinate or plan any action together. Upon their arrival every night, they independently decide whether: (S) study or (M) watch some movies on their laptop. If they both study, they both get utility 10. The individual benefit from watching a movie is instead 15 for C and 18 for D. However, if they both choose M, their individual benefit is halved (since they have half the connection speed). Also, trying studying while somebody else is playing a movie breaks the concentration, so $u_C(S, M) = u_D(M, S) = 0$ (C is written as the first player). Call \mathbb{G} this game, and consider it in a repeated version $\mathbb{G}(T)$, where \mathbb{G} is played every night for T nights. Individual payoffs are cumulated with discount factor δ . Finally, consider an *extended* game where a punishment strategy P is also available to both players. When either player chooses P, payoffs are -10 for *both* players (that would correspond, e.g., to do something really stupid in the library and get the library permanently closed). Call this game \mathbb{G}' . Note: despite P being weakly dominated, (P,P) is a NE for \mathbb{G}' .

1. Find the Nash equilibria of $\mathbb{G}(3)$, for $\delta = 1$.
2. What values of δ allow for sustaining a Nash equilibrium of $\mathbb{G}(\infty)$ via a “Grim Trigger” strategy where each player ends up in always choosing S?
3. If you see an SPE of $\mathbb{G}'(2)$ where players may play S, state at which round do they play it, and what value of δ do you need to obtain it.

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Exercise 3 Jane’s bank account contains 10000 euros. She can keep (K) the whole sum in the bank account, or buy a utility car (C) that costs 8000 euros. The bank B is also a player: they observe Jane’s decision and after that can either (L) leave Jane’s money in the account, or (S) buy some subprime bonds for Jane, regardless of Jane’s desire. In case (S), they invest every euro that Jane has left in the account (either 2000 or 10000). After one year, Jane’s account is increased with some interest \mathcal{G} . If the bank played L, \mathcal{G} is equal to 5% of the initial amount. In case they played S, \mathcal{G} depends on whether the bank is a “good” bank or a “bad” bank. Jane estimates the probability of “good” as p and this is a common prior. A good bank playing S gives Jane her money back, plus $\mathcal{G} = 15\%$ of yearly interest rate. A bad bank steals every euro that Jane put in the bank account ($\mathcal{G} = -100\%$). The final payoff are computed after one year as:

for Jane: the grand total of the euros in the account (i.e., the initial value increased by \mathcal{G}); plus the value of the car, if she owns one: a one-year old car is worth 5000 euros.

for the bank: if \mathcal{G} is positive, then their payoff is \mathcal{G} ; otherwise it is $-\mathcal{G}/5$.

1. Represent this Bayesian game with J and B in extensive form.
2. Represent this Bayesian game in normal form, with a type-agent representation of player B (i.e., its strategies are n-tuples of actions, one per type).
3. Find the Bayesian equilibria of the game if $p = 0.2$ and discuss whether they are (subgame) perfect.