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Lecture 18
Bayesian games

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Previously on game theory

- Static games **of complete information**:
  - Players move simultaneously/without knowing each other's move
  - Game ends after a single interaction
- Dynamic games **of complete information**:
  - Sequential games: perfect information, each player knows exactly where they are in the extensive form tree
  - Multistage/repeated games: sequences of games (static or dynamic)

Today on game theory

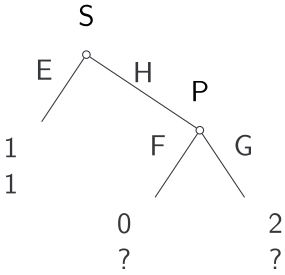
- Remember: Complete information means
  - Every player knows who are the other players
  - Every player knows the possible moves available to other players
  - Every player has full knowledge of other players' utility function → We drop this assumption
    - today, we see the case where some players might just have an estimate of the "**type**" of the other player
  - Everybody knows that everybody knows all this information
- **Bayesian games**: we do not know exactly the utility function of other players, we consider different possibilities
- Drop the assumption of complete information

Example: Lazy student's problem

- Suppose you decided to make a project for your Game Theory course, and you need to decide whether to put some actual effort (E), or to make a halfhearted project (H)
- If you put effort, you are guaranteed to get a good score on the project, which gives you utility $+1$
- However, ideally you would like to slack off and still get a good grade, which gives you utility $+2$; on the other hand, there is the possibility that your project gets a bad grade (payoff -1)
- It is up to your professor whether to give a good grade (G) to your halfhearted project or to flunk you (F)

Example: Lazy student's problem

- Problem is: you know your professor's possible moves (F or G), but you do not know his utility function
- It is reasonable to assume that the professor is always happy to see a good project, so his payoff if you put effort is $+1$

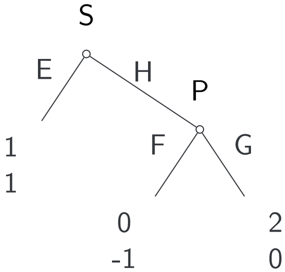


Example: Lazy student's problem

- Consider two possible scenarios:
    - *Scenario 1*: your professor is a **lenient** professor and always prefers to avoid flunking students (grading is a very boring activity, he does not want to do that twice). Playing F gives him a payoff of -1, whereas G gives him a payoff of 0
    - *Scenario 2*: your professor is a **strict** professor who believes that students who are slacking off deserve to get flunked. Playing F gives him a payoff of $+1$, and G still gives him a payoff of 0

Example: Lazy student's problem

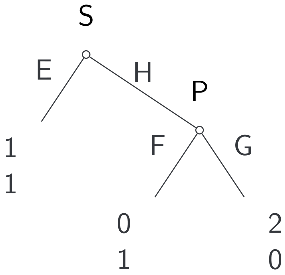
- *Scenario 1*: **lenient** professor case



- SPE (H, G) found via backward induction
- (E, F) is a NE but not a SPE: non-credible threat

Example: Lazy student's problem

- *Scenario 2*: **strict** professor case



- SPE (E, F) found via backward induction (also, only NE of the game)

Example: Lazy student's problem

- Same problem as before: you do not know whether your professor is lenient or strict
- Does that mean that you cannot make a rational decision?
- You may think of using a conservative approach and always assume the worst-case scenario (in this case, assuming the strict-professor scenario)
- However, that is not how people make decisions: how they behave depends on their estimate of each scenario
  - If you are 99% confident that your professor is lenient, you will be more inclined to make a halfhearted project
  - If you are 99% confident that your professor is strict, you are more likely to put effort
  - Can we generalize this for any estimate p?

Bayesian games

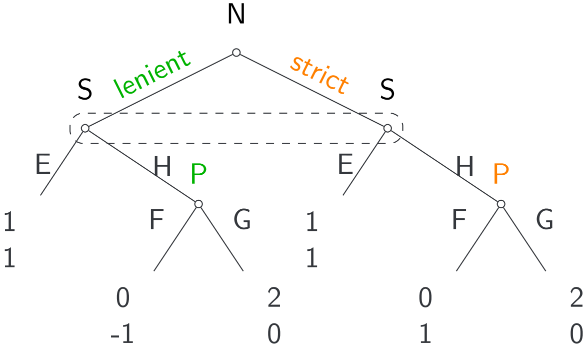
- Bayesian games are games of **incomplete information**, where some players do not have complete knowledge on the other players' utility
- This incomplete information, however, still needs to be modeled somehow
- Bayesian game assumptions:
  - Each player *i* has a set of possible types $T_i = \{\text{type } 1, \text{type } 2, \dots\}$ (e.g., $T_P = \{\text{lenient}, \text{strict}\}$)
    - This set can be a singleton for some players (if all players can have only one type → back to complete information)
  - The type of each player $t_1, \dots, t_n$ is determined by Nature's move *at the beginning of the game*
  - Each player knows only his/her own type
  - Other player's type is unknown

Preliminary Nature's move

- Bayesian games can be seen as a weird kind of dynamic games that is played as follows:
    - Nature draws a type vector $(t_1, \dots, t_n)$ among all the possible combinations of players' types
    - Nature reveals type t_i only to player i (this is captured by information sets)
    - Players choose their actions
    - Final payoffs are computed
- This is a dynamic game where the players do not know Nature's move in its entirety

Extensive form of Bayesian games

- Nature chooses at the beginning whether P is being “lenient” or “strict”
  - S does not know about Nature’s choice: only one information set
  - P knows about Nature’s choice: one information set for “lenient” and one for “strict”

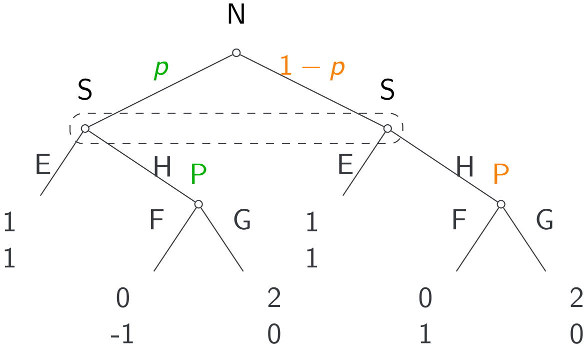


Beliefs in Bayesian games

- Players know their types but not their opponents'
- How can they find best responses?
    - They create beliefs about these types
    - *Assumption*: players do not precisely know the types of their opponents but they have an estimate of those
    - In other words, they know the **probability distribution** of the opponents' types (this is common knowledge!)
    - This means that all players know that player i is of type t_1 with probability $p_i(t_1)$, t_2 with probability $p_i(t_2)$, and so on
    - Player i knows about his/her true type but also knows the opponents' estimates $p_i(t_1), p_i(t_2), \ldots$
    - This is called the **common prior assumption**

Extensive form of Bayesian games

- Suppose your estimate is that the professor is lenient with probability p and strict with probability $1 - p$
- The **extensive form** of the Bayesian game is as follows

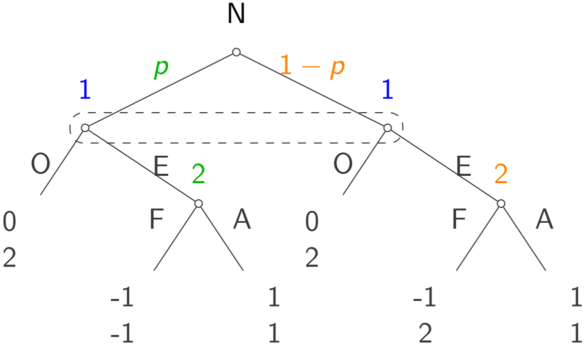


Bayesian entry game

- The lazy student's problem is a more general game called "**Bayesian entry game**"
- Original version: incumbent (player 1) vs. outsider (player 2)
- The outsider decides whether to enter the market (E) or stay out (O)
- The incumbent decides whether to accept the outsider (A) or fight (F)
- The incumbent could be of two types: "reasonable" (does not like to fight) or "crazy" (likes to fight)
- The incumbent knows its type, the outsider estimates reasonable/crazy with probabilities $(p, 1 - p)$

Bayesian entry game

- **Extensive form** of the Bayesian entry game:



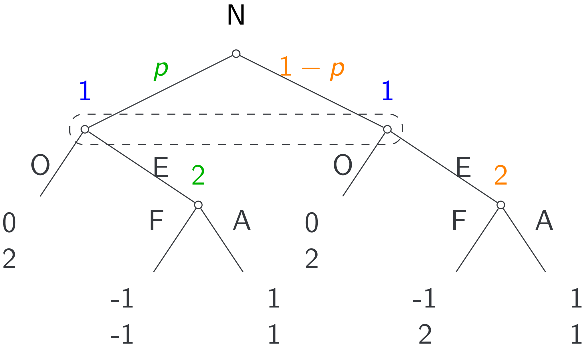
Normal form of Bayesian games

- The normal form of Bayesian games can be inferred from the extensive form as we did in dynamic games
- Let us start by listing strategies: this can be done by considering all combinations possible moves in each information set
- In the Bayesian Entry Game:
    - Player 1 only has one information set (does not know Nature's move, which decide player 2's type): only one move needed to define pure strategies
    - Player 2 has two information sets (knows his/her type): his/her pure strategies are all possible combination of moves in these 2 information sets

Strategies in Bayesian games

- In Bayesian games, the concept of strategy is further extended: a strategy specifies what the player does for each type
    - Think of it as a "Dr. Jekyll and Mr. Hyde" situation
    - E.g., strategy AF for player 2 means "If I am a "reasonable" type, I accept the outsider; if I am a "crazy" type, I fight them"
    - This might seem strange, since players know their actual type; however, they need to do this reasoning in order to predict the opponents' strategy
- *Example*: In the Bayesian Entry Game:
    - Player 1 has two strategies: {O, E}
    - Player 2 has four strategies: {AA, AF, FA, FF}
- Each pair of pure strategies determines how the game is played, which also depends on Nature's choice

Bayesian entry game



- For example, if the game is played as (E, AF)
  - player 1 gets: $p \cdot 1 + (1-p) \cdot (-1) = 2p - 1$
  - player 2 gets: $p \cdot 1 + (1-p) \cdot 2 = 2 - p$

Bayesian entry game, normal form

- We can determine player 2's best responses against O (anything)
- We can determine player 1's best response against AA and FF (when 2 always plays the same regardless of its type)
- So we know that (O,FF) is a NE regardless of *p*
- However, we need to know *p* in order to find *all* NE

		Player 2			
		AA	AF	FA	FF
Player 1	O	0, **2**	0, **2**	0, **2**	**0**, **2**
	E	**1**, 1	2p-1, 2-p	1-2p, 1-2p	-1, 2-3p

Bayesian entry game, NE

- For $p = 0$ (The outsider is sure that the incumbent is crazy)

		Player 2			
		AA	AF	FA	FF
Player 1	O	0, **2**	**0**, **2**	0, **2**	**0**, **2**
	E	**1**, 1	-1, **2**	**1**, 1	-1, **2**

- (O, AF) and (O, FF) are NE
- Payoffs of player 2 only reflect the choice of the crazy type (choices of reasonable type are neglected)
- **Interpretation**: the outsider (player 1) is 100% sure that the incumbent is crazy, so the moves of a "reasonable" player 2 are ignored
- Therefore, the NE are all joint strategies (O, *F)

Bayesian entry game, NE

- For $p = 1$ (The outsider is sure that the incumbent is reasonable)

		Player 2			
		AA	AF	FA	FF
Player 1	O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
	E	**1**, **1**	**1**, **1**	-1, -1	-1, -1

- (E, AA), (E, FA) and (E, AF) are Nash equilibria
- **Interpretation**: the outsider is 100% sure that the incumbent is reasonable, so the crazy-incumbent moves are ignored
- NE are (E, A*) and (O, F*)

Bayesian entry game, NE

- For $p = 2/3$

Player 2

Player 1		AA	AF	FA	FF
	O	0, **2**	0, **2**	**0**, **2**	**0**, **2**
	E	**1**, 1	**1/3**, **4/3**	-1/3, -1/3	-1, 0

- NE: (E, AF), (O, FA), (O, FF)
- **Interpretation**: The outsider thinks that the incumbent is likely to be reasonable, but there is still some probability of them being crazy. Therefore, joint strategies like (E, AA) where also the crazy type is accepting the outsider are not NE.

Bayesian games: definition

Bayesian game: definition

- Up to know, our normal-form representation of a game included:
    - Set of players $\{1, \ldots, n\}$
    - Strategy sets $S_1, \ldots, S_n$
    - Utility functions $u_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ (for $i = 1, \ldots, n$)
- In Bayesian games we need three more ingredients:
    1. **Type space** of each player T_i (for $i = 1, \ldots, n$)
    2. **Type-dependent utilities**: define $u_i(a_1, \ldots, a_n, t_i)$ for each $t_i \in T_i$
    3. **Beliefs about players' types**: a probability distribution ϕ_i defined over types for each player

Static Bayesian game

- Consider a static Bayesian game: n players, each player's (pure) strategy is just an action
  1. Player i's type is $t_i \in T_i$, chosen by Nature for each player from 1 to n through the joint **prior probability distribution** $\phi(t_1, \ldots, t_n)$, where $\phi : T_1 \times \cdots \times T_n \to [0, 1]$
     - This prior is **common knowledge** among players
  2. Player i knows his **private values** of his/her utility function $u_i(a_1, \ldots, a_n, t_i)$
     - Other players know **common values** of i's utility function

Type of a player

- For example, suppose player i can have two different payoff functions $u_{i,A}(a_i, a_{-i})$ and $u_{i,B}(a_i, a_{-i})$
  - We represent this by setting a type space $T_i = \{t_A, t_B\}$ and imposing
$$u_{i,j}(a_i, a_{-i}) = u_i(a_i, a_{-i}, t_j)$$
- Types can be used to limit available actions
  - If a player has actions {F,G,H}, but H is permitted only with probability q, we define types t_A and t_B
  - t_A and t_B have respective probabilities q and $1 - q$
  - In both cases {F,G,H} are feasible actions, but all payoffs of move H under type t_B are $-\infty$

Beliefs on types

3. Types can be correlated; they are independent if

$$\phi(t_1, \ldots, t_n) = \phi(t_1) \cdots \phi(t_n)$$

- *i* knows his/her own type, but not others' (t_{-i})
  - He/she estimates that as $\phi_i(t_{-i}|t_i)$, exploiting the correlation (**belief** is derived from conditional probability)
  - Our assumption of the prior being common knowledge equals to **perfect information**
  - In the case of incomplete and imperfect info., belief $\phi_i(t_{-i}|t_i)$ may even be wrong and have nothing to do with the true prior

Questions?