CLASS-2

Tuesday, August 05, 2025 9:49 PM

Starting from **<u>axiomatic approach</u>**:-

$$(\Omega, \mathcal{F}, P)$$

Sample space, event space, probability law

(Sample space is the set of all possible outcomes of an experiment)

-><u>Consider rolling a dice</u>:

Ω={1,2,3,4,5,6} (this can be sample space)

Ω={1 or 2,3,4,5,6}

This is **VALID** but non standard sample space,

Whereas the one on it's left is standard sample space

-<u>Can this set be Ω?</u>

{1 or 2, 2 ,3 ,4 ,5 ,6 } (Not mutually exclusive)

->no unique mapping if I get 2

Ω: **mutually exclusive and collectively exhaustive**

-<u>What about this?</u>

{1,3,5.6} not exhaustive

EXAMPLES-

I)<u>Finite sample space</u>:-

Roll a pair of dice

$$\Omega = \{(i,j) : i,j \in [1:6]\}$$

II)<u>Countably infinite sample space</u>:-

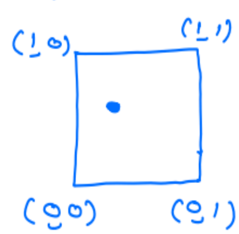
No. Of tosses to get a tail:

Ω={1,2,3,4,5,6.......}=N

III)<u>Uncountably infinite sample space:</u>-

Eg 1)

Consider throwing a dart on a square target.



$$\Omega = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

Event Space

An *event* is a subset of the sample space.
The collection of all *events* is called an *event space*. An *event space* should be a σ-field or σ-algebra.

(A σ-field Ǝ over a sample space Ω is a collection of subsets of Ω (i.e., events), satisfying three axioms:)

1. **Ω ∈ Ǝ**
2. If **A ∈ Ǝ** , then $\mathbf{A^c} \in$ Ǝ
   (closure under complements)
3. If $\mathbf{A_1, A_2, \ldots}$ ∈ Ǝ , then $\bigcup_{i=1}$**^∞** $\mathbf{A_i}$ ∈ Ǝ
   (closure under countable unions)

(**σ-field and Ǝ are synonymous**)

Say, A, B ∈ Ǝ, so A U B ∈ Ǝ ,but why?

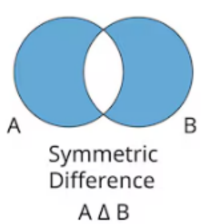
Because it's a trivial fact that all other sets A3,A4... are phi(empty).

Further propositions:

(i) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.
(closed under countable intersections)

Proof: (here you go :)
Ai ∈ Ǝ
∴ Ai^c ∈ Ǝ , i ∈ **N** (using property-2) **-->** $\bigcup_{i=1}$**^∞** $\mathbf{A_i^c}$ ∈ Ǝ (using property-3)

∴ ($\bigcup_{i=1}$**^∞** $\mathbf{A_i^c}$ $)^c$ ∈ Ǝ (using property-2)
then applying da-murugan law will give us the required result 😊



(remember this shit, as GRK will use it later)

And A\B=A ∩ B^c , similarly B\A= B∩ A^c
(and we can also prove that these two ∈ Ǝ, if A and B ∈ Ǝ)

And if these two ∈ Ǝ, then A\B U B\A ∈ Ǝ (which is the symmetric difference itself)

Examples of various σ-field:-

(i) $\mathcal{F} = \{\Omega, \phi\}$ (Smallest σ-Field)

Because it satisfies **all three σ-field axioms** while containing the **fewest possible subsets**:

1. **Contains** Ω ✅
2. **Closed under complement:**
   - $\Omega^c = \emptyset$, and $\emptyset^c = \Omega$ ✅
3. **Closed under countable unions:**
   - $\emptyset \cup \Omega = \Omega$, etc. ✅

ii) Smallest σ-field Ǝ that contains an event E:

Ǝ ={Φ, E , E^c ,Ω}

(Verify all three conditions yourself for better understanding)

iii)Smallest σ-field Ǝ that contains two events A and B:-

Method 1: Brute forcing

(basic elements)

1. ∅ 2. Ω

(basic events)

3. A 4. B

(complement of events)

5. A^c 6. B^c

(intersection of combinations of A,A^c,B,B^c)

7. $A \cap B$ 9. $A^c \cap B$

8. $A \cap B^c$ 10. $A^c \cap B^c$

(union of combinations of A,A^c,B,B^c)

11. $A \cup B$ 13. $A^c \cup B$

12. $A \cup B^c$ 14. $A^c \cup B^c$

(got this by $(A \cap B^c) \cup (B \cap A^c)$

15. $A \triangle B$

16. $(A \triangle B)^c$

(don't use this shitty method, as if A and B have some common events then some of these will be repeated)

Eg) for example 16 is just 8 U 9

(so to avoid such trivialities, let's use partitioning method)

Method 2: Partitioning

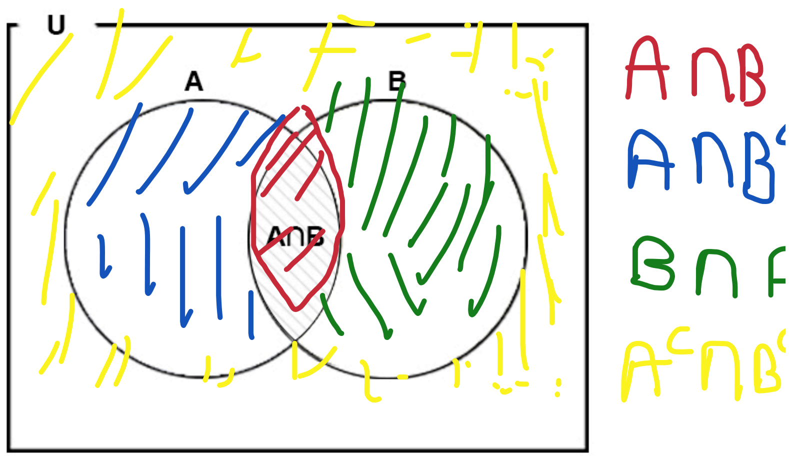
Let σ(c) denote smallest field that contains element c

∴ we need σ(∅,Ω,A,B) hence, we will use partitioning method, to understand this I will also be using a venn diagram

IDEA: splitting the universal set into **non-overlapping sets.**

Hence, we can express Ω as a disjoint union of these four sets:-

(Here, we can divide it into 4 areas)



$\Omega=(A\cap B)\cup(A\cap B^c)\cup(A^c\cap B)\cup(A^c\cap B^c)$

So

$$\sigma(A, B, \Omega, \phi)$$

$$= \sigma(\underbrace{A\cap B,\ A\cap B^c,\ A^c\cap B,\ A^c\cap B^c}_{\text{collection of disjoint events}})$$

$$\sigma(E_1, E_2, E_3, E_4) = \left\{ \bigcup_{i\in I} E_i \ : \ I \subseteq \{1,2,3,4\} \right\}$$

"Thus the smallest σ-field can have at most 16 elements (no. of possible unions of the sets among $A \cap B$, $A \cap B^c$, $A^c \cap B$, $A^c \cap B^c$). It is at most 16 because not all of them are always distinct, e.g., $A \cap B$ can be $\varnothing$."

Simpler explanation on why 16 elements only:
If you know PNC Or basic combinatorics then it will easier:-

E_1 E_2 E_3 E_4

(any subset is formed by choosing combinations of either 4 sets,
So total possibilites= 2*2*2*2 =16
2 represents options of choosing and not choosing

III)Probability Law:-

A probability law or a probability measure is a set function $P: \mathcal{F} \to \mathbb{R}$ that satisfies the following axioms.

(1) (Non-negativity) $P(E) \geq 0$ for all $E \in \mathcal{F}$.

(2) (Normalization) $P(\Omega) = 1$.

(3) (Additivity) If $A_1, A_2, \ldots$ are disjoint events (i.e., mutually exclusive), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

--> Some redundant proofs:_
(iske baad I have just used the notes on moodle, this is the easy part and you will be able to understand)

Examples.

Just the power set of omega

(i) $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$.

$P(\{1\}) = 0.2$ $\qquad$ $P(\{4\}) = 0.3$

$P(\{2\}) = 0.04$ $\qquad$ $P(\{5\}) = 0.15$

$P(\{3\}) = 0.06$ $\qquad$ $P(\{6\}) = 0.25$

$$P(A) = \sum_{i \in A} P(\{i\}), \quad A \subseteq [1:6].$$

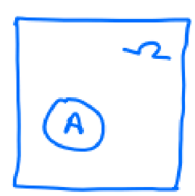
This defines a probability i.e., it satisfies all the axioms.

(ii) $\Omega = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$.

Let $\mathcal{F}$ contain all subsets of Ω.

$$P(A) = \frac{\text{area of } A}{\text{area of } \Omega}$$

$$= \text{area of } A.$$



Properties of Probability Law

(a) $P(A) + P(A^c) = 1$, $P(A) \le 1$, $P(\phi) = 0$.

(b) If $A \subseteq B$, then $P(A) \le P(B)$.

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. (a) $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$.

$P(\phi) = 0$.

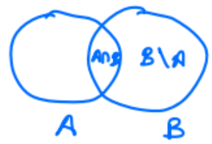
(b) $B = A \cup (B \setminus A) \implies P(B) = P(A) + P(B \setminus A) \geq P(A)$.

$A \subseteq \Omega \implies P(A) \leq P(\Omega) = 1$.

(c) $P(A \cup B) = P(A \cup (B \setminus A))$

$= P(A) + P(B \setminus A)$

$= P(A) + P(B) - P(A \cap B)$.



More generally if $A_1, A_2, \ldots$ are events then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{1 \leq i \leq n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k)$$

$$- \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

This is called the inclusion-exclusion principle, proof follows by induction.

One more doubt some may have: if we have 2 sets A and B ,
and we use partitioning method for giving smallest sigma field

Then what we are doing is we are dividing the sets into disjoint
union and then we are taking power set of that disjoint sets

We are not taking power set of omega
F ⊆ P(omega)

(F can only be equal to P(omega) only when a and b are already
disjoint)

✅ **Example 1:**

Let the sample space be:

$$\Omega = \{a, b, c\}$$

Then the **power set** $\mathcal{P}(\Omega)$ is:

$$\mathcal{P}(\Omega) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Now define a **σ-field** $\mathcal{F}$ as:

$$\mathcal{F} = \{\emptyset, \{a, b\}, \{c\}, \Omega\}$$

- Clearly:
  $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ ✅
- Also:
  $\mathcal{F}$ contains Ω, is closed under complementation, and countable union. ✅

✅ Example where σ-field = power set:

Let:

$$\Omega = \{a, b\}$$

Then, the **power set** of Ω is:

$$\mathcal{P}(\Omega) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Now, suppose we want the **smallest σ-field containing** the atoms:

Let's say our atoms are:

- $E_1 = \{a\}$
- $E_2 = \{b\}$

Since the atoms are **disjoint**, and their union is $\Omega = \{a, b\}$, the **σ-field** generated by them is:

$$\mathcal{F} = \text{all possible unions of } E_1 \text{ and } E_2 = \mathcal{P}(\{a, b\})$$

So explicitly:

$$\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} = \mathcal{P}(\Omega)$$