UNIK4660 - Assignment 1

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1 Problem 1

$$u = -y$$
$$v = x$$
$$w = zt$$

We are given the velocity field above, in the right handed Cartesian coordinate system. Using this field, we are to show that, at $t = 0$, the stream lines are circles centered on the origin.
This can be done by using the dot products algebraic definition,

$$\bar{V} \bullet \bar{x} = \bar{V}_1\bar{x}_1 + \bar{V}_2\bar{x}_2 + \bar{V}_3\bar{x}_3$$

where $\bar{V}$ is the velocity vector and $\bar{x}$ is the position vector from the origin.

As u and v are independent of time and $w = 0$ when $t = 0$, we can ignore the third dimension in this problem.

$$\bar{V} \bullet \bar{x} = \bar{V}_1\bar{x}_1 + \bar{V}_2\bar{x}_2$$
$$\bar{V} \bullet \bar{x} = -yx + xy$$
$$\bar{V} \bullet \bar{x} = 0$$

Since for any point the velocity field is orthogonal to the position vector any particle following this flow will never increase or decrease its distance to the z-axis. Thus, since we have a constant radius we can conclude that at $t = 0$, and for any t, the streamline is a circle around the z-axis.

2 Problem 2

We are to show that the streamlines are circles with center at the z-axis going through the point $(r, 0, z_0)$. We are given the definition of r as $r^2 = x^2 + y^2$, and given $tan\theta = \frac{y}{x}$. The streamlines can then be expressed by:

$$\frac{dx}{d\theta} = -y$$
$$\frac{dy}{d\theta} = x$$
$$\frac{dz}{d\theta} = zt$$

As this problem is considered at $t = 0$, the z dimension is always 0 and can be ignored. This then becomes a two dimensional problem and we can use the definition of r to find the position.

$$x = \sqrt{r^2 - y^2} \qquad\qquad y = \sqrt{r^2 - x^2}$$

This can then be inserted into the streamline equations, and since we are only concerned with the point $(r, 0, z_0)$, we get:

$$\frac{dx}{d\theta} = -\sqrt{r^2 - x^2} \qquad = -\sqrt{r^2 - r^2} = 0$$
$$\frac{dy}{d\theta} = \sqrt{r^2 - y^2} \qquad = -\sqrt{r^2 - 0^2} = r$$
$$\frac{dz}{d\theta} = 0$$

As the algebraic definition of a circle, using the Pythagorean theorem, is $radius^2 = ankathete^2 + gegenthete^2$ which is the exact same definition we are given for r, which would imply that the streamlines are a circle around z.

No streamline that starts at $z > 0$ can cross $z = 0$. If $t = 0$ all streamlines are parallel and will not cross the $xy0$-plane. And if $t \neq 0$ then any streamline will point away from the plane, even for negative time.

3 Problem 3

We are to find the formula for the streamline, SL, going through the point $P = (r, 0, w_0)$, when the time $t > 0$.

Using the result from problem 3, we see that the x- and y-components are independent of time and so the only change is that the z-dimension is no longer always 0. We can write the formula as:

$$SL(x,y,z) = (-\sqrt{r^2 - x^2})\bar{i} + (\sqrt{r^2 - y^2})\bar{j} + (zt)\bar{k}$$

and for the point P at $t > 0$ this becomes:

$$SL(r,0,w_0) = (-\sqrt{r^2 - r^2})\bar{i} + (\sqrt{r^2 - 0^2})\bar{j} + (w_0 t)\bar{k}$$
$$SL(r,0,w_0) = 0\bar{i} + r\bar{j} + (w_0 t)\bar{k}$$

where r is the radius of the circle around on which z lies.
The streamlines going through $(r,0,0)$ are the same as the ones found in problem 3, since when $z = 0$ the z-component is on the origin.

4 Problem 4

We are to find the path line, PL, for the velocity field, $\bar{u}$, given in problem 1, that goes through the point $P = (r, 0, z_0)$ at $t = 0$.
The path lines are defined to be the trajectories of an individual particle in a fluid.

$$PL = \begin{cases} \frac{d\bar{x}}{dt} & = \bar{u}(\bar{x}, t) \\ \bar{x_{t_0}} & = \bar{\xi}(t_0) \end{cases}$$

From the velocity field we can derive the spatial coordinate formula, which is based on the initial position and time.

$$\frac{dx}{dt} = \bar{u_1} = -y$$
$$dx = -ydt$$
$$\int_0^x dx = \int_0^t -ydt$$
$$x = x_0 - yt$$

$$\frac{dy}{dt} = \bar{u_2} = x$$
$$dy = xdt$$
$$\int_0^y dy = \int_0^t xdt$$
$$y = xt + y_0$$

$$
\begin{aligned}
\frac{dz}{dt} &= \bar{u_3} = zt \\
\frac{dz}{z} &= tdt \\
\int_0^y \frac{1}{z} dz &= \int_0^t tdt \\
ln(\frac{z}{z_0}) &= \frac{1}{2}t^2 \\
z &= z_0 e^{\frac{1}{2}t^2}
\end{aligned}
$$

The path line can then be written as:

$$PL(x, y, z, t) = (x_0 - yt)\hat{\mathbf{i}} + (xt + y_0)\hat{\mathbf{j}} + (z_0 e^{\frac{1}{2}t^2})\hat{\mathbf{k}}$$

The path line on point P at As $t = 0$ becomes:

$$PL(r, 0, z_0, 0) = (x_0)\hat{\mathbf{i}} + (y_0)\hat{\mathbf{j}} + (z_0)\hat{\mathbf{k}}$$

5 Problem 5

We are to find the expression for the streak lines emerging from the point $(r_0, 0, z_0)$.

Streak lines are the curve connecting all particles that pass through a given point, P, from time $t_0 \leq s \leq t$, and is defined as $x = x(\xi[P, s], t)$.

As we did in problem 4, we find the expression for the spatial coordinates from the given velocity field.

$$x(x, y, z, t) = (x_0 - yt)\hat{\mathbf{i}} + (y_0 + xt)\hat{\mathbf{j}} + (z_0 e^{\frac{1}{2}t^2})\hat{\mathbf{k}}$$

We can replace the initial coordinates, x_0, y_0 and z_0, with the material coordinates ξ_i, since ξ is equal to the initial spatial coordinates for each particle in the flow.

$$x(x, y, z, t) = (\xi_1 - yt)\hat{\mathbf{i}} + (\xi_2 + xt)\hat{\mathbf{j}} + (\xi_3 e^{\frac{1}{2}t^2})\hat{\mathbf{k}}$$

Reshuffling this gives us:

$$\xi_i(x, y, z, t) = (x + yt)\hat{\mathbf{i}} + (y - xt)\hat{\mathbf{j}} + (ze^{-\frac{1}{2}t^2})\hat{\mathbf{k}} \tag{5.1}$$

The expression for the material coordinate of the particle at position $P(x_P, y_P, z_P)$ and time s is then:

$$\xi_i(x_P, y_P, z_P, s) = (x_P + y_P s)\hat{\mathbf{i}} + (y_P - x_P s)\hat{\mathbf{j}} + (z_P e^{-\frac{1}{2}s^2})\hat{\mathbf{k}} \tag{5.2}$$

The streak line expression is then found by inserting (5.1) into (5.2) and using the coordinates for the given point, $(r_o, 0, z_0)$:

$$(x + yt)\hat{\mathbf{i}} + (y - xt)\hat{\mathbf{j}} + (ze^{-\frac{1}{2}t^2})\hat{\mathbf{k}} = (x_P + y_P s)\hat{\mathbf{i}} + (y_P - x_P s)\hat{\mathbf{j}} + (z_P e^{-\frac{1}{2}s^2})\hat{\mathbf{k}}$$

This gives us the equations below for the x, y, z-components respectively.

$$x + yt = x_P + y_P s \qquad y - xt = y_P - x_P s \qquad ze^{-\frac{1}{2}t^2} = z_P e^{-\frac{1}{2}s^2}$$

The x and y component can be found by using substitution. Substituting x into y gives the following result:

$$x = r_0 - \frac{(r_0 t - r_0 s)t}{1 - t^2} \qquad y = \frac{r_0 t - r_0 s}{1 + t^2} \qquad z = z_0 e^{\frac{1}{2}t^2 - \frac{1}{2}s^2}$$

$$Streakline = (r_0 - \frac{(r_0 t - r_0 s)t}{1 - t^2})\hat{\mathbf{i}} + (\frac{r_0 t - r_0 s}{1 + t^2})\hat{\mathbf{j}} + (z_0 e^{\frac{1}{2}t^2 - \frac{1}{2}s^2})\hat{\mathbf{k}}$$

6 Problem 6

Using the velocity field given in problem 1, we are to calculate the vorticity field. The vorticity field is defined as $\bigtriangledown \times \bar{v}$. This gives us:

$$\begin{aligned}
\bigtriangledown \times \bar{v} &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ -y & x & zt \end{vmatrix} \\
&= (\frac{d(zt)}{dy} - \frac{dx}{dz})\bar{i} + (-\frac{dy}{dz} - \frac{dzt}{dx})\bar{j} + (\frac{dx}{dx} + \frac{dy}{dy})\bar{k} \\
Vorticity &= 0\bar{i} + 0\bar{j} + 2\bar{k}
\end{aligned}$$

7 Problem 7

We are to calculate the strain rate tensor and rotations tensor for the velocity field given in Problem 1.
The definition of the strain rate tensor, e, according to Kundu is:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

while his definition of the rotation tensor, ω, is:

$$\omega_{ij} = \bigtriangledown \times \bar{u} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}$$

We can find the strain rate tensor by simply expanding the tensor formula and solving for it using our two vectors given by the velocity field

$$\vec{x} =< x, y, z >$$

$$\vec{u} =< -y, x, zt >$$

$$e_{ij} = \begin{bmatrix} -\frac{\partial y}{\partial x} & \frac{1}{2}(-\frac{\partial y}{\partial y} + \frac{\partial x}{\partial x}) & \frac{1}{2}(-\frac{\partial y}{\partial z} + \frac{\partial (zt)}{\partial x}) \\ \frac{1}{2}(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}) & \frac{\partial x}{\partial y} & \frac{1}{2}(\frac{\partial x}{\partial z} + \frac{\partial (zt)}{\partial y}) \\ \frac{1}{2}(\frac{\partial (zt)}{\partial x} - \frac{\partial y}{\partial z}) & \frac{1}{2}(\frac{\partial (zt)}{\partial y} + \frac{\partial x}{\partial z}) & \frac{\partial (zt)}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix}$$

The rotation tensor can similarly be found using the same method.

$$\omega_{ij} = \begin{bmatrix} -\frac{\partial y}{\partial x} + \frac{\partial y}{\partial x} & -\frac{\partial y}{\partial y} - \frac{\partial x}{\partial x} & -\frac{\partial y}{\partial z} - \frac{\partial zt}{\partial x} \\ \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} & \frac{\partial x}{\partial y} - \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} - \frac{\partial zt}{\partial y} \\ \frac{\partial zt}{\partial x} + \frac{\partial y}{\partial z} & \frac{\partial zt}{\partial y} - \frac{\partial x}{\partial z} & \frac{\partial zt}{\partial z} - \frac{\partial zt}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8 Problem 8

We are to calculate the relative dilatation following a particle path.
The definition of dilatation is $\bigtriangledown \bullet \bar{u}$.
The calculation becomes:

$$\begin{aligned} \bigtriangledown \bullet \bar{u} &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \\ &= -\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial (zt)}{\partial z} \\ &= 0 + 0 + t \end{aligned}$$