TABLE OF CONTENTS

1 Derivatives

1.1 Finding Partial Derivatives

$$f(x,y) = 5x^4 - 2y^2$$

$$\text{Find } \frac{\partial f}{\partial y}$$

So in order to do this problem you have to treat x as a constant.

$$\frac{d}{dy}(5x^4 - 2y^2)$$

$$\frac{d}{dy}5x^4 - \frac{d}{dy}2y^2.$$

$$-4y$$

So we can conclude

$$\boxed{\frac{\partial f}{\partial y} = -4y}.$$

$$f(x,y) = y^2\sin(x).$$

$$\text{Find } \frac{\partial f}{\partial x}$$

$$\frac{d}{dx}(y^2 \sin(y)).$$

Because there is no x term in the problem, this derivate evaluates to :

$$\boxed{0}.$$

$$f(x, y) = \sin(x) - 3\cos(y).$$

$$\text{Find } \frac{\partial f}{\partial y}$$

$$\frac{d}{dy}(\sin(x) - 3\cos(y)).$$

$$\frac{d}{dy}\sin(x) - 3\frac{d}{dy}\cos(y).$$

$$\boxed{3\sin(y)}.$$

$$f(x, y) = \frac{x^3}{3} + y^2.$$

$$\text{Finds } \frac{\partial f}{\partial x}$$

$$\frac{d}{dx}\left(\frac{x^3}{3} + y^2\right).$$

$$\frac{d}{dx}\frac{x^3}{3} + \frac{d}{dx}y^2.$$

$$\boxed{x^2}.$$

$$f(x, y) = (x + 1)\tan(y).$$

What is the partial derivative of f with respect to x?

Rewriting :

$$f(x) = x\tan(y) + \tan(y).$$

$$\frac{d}{dx}x\tan(y) + \frac{d}{dx}\tan(y)$$

$$\boxed{\tan(y)}$$

$$f(x, y) = \frac{y^2}{x}$$

What is the partial derivative of f with respect to x?

Treat y as a consent :

$$\frac{\partial f}{\partial x}\left(\frac{y^2}{x}\right)$$

$$\boxed{-\frac{y^2}{x^2}}$$

$$f(x,y) = \sin(x) + \cos(y^2)$$

What is $\frac{\partial f}{\partial y}$?

$$\frac{\partial f}{\partial y}\sin(x) + \cos(y^2)$$

$$0 - 2y\sin(y)$$

$$\boxed{-2y\sin(y)}$$

$$f(x,y) = \cos^2(x) - x$$

What is $\frac{\partial f}{\partial y}$
Since we have no y component in the equation, we can say that the derivative has to be

$$\boxed{0}$$

$$f(x,y) = x^2\sin(x)$$

What is $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y}x^2\sin(y)$$

x held constant

$$\boxed{x^2\cos(y)}$$

$$f(x,y) = \tan(x)y^{2.5}$$

Find $\frac{\partial^3 f}{\partial y^3}$

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]\right]$$

$$= \tan(x)\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left[\frac{5}{2}y^{1.5}\right]\right]$$

$$\tan(x)\frac{5}{2}\frac{\partial}{\partial y}\left[\frac{3}{2}y^{0.5}\right]$$

$$\frac{15}{4}\tan(x)\frac{\partial}{\partial y}y^{0.5}$$

$$\frac{15}{4}\tan(x)\frac{1}{2y^{0.5}}$$

$$\boxed{\frac{\partial^3 f}{\partial y^3} = \frac{15\tan(x)}{8\sqrt{y}}}$$

$$f(x,y) = x\cos(y) + y$$

What is $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} x\cos(y) + y$$

y is held constant

$$\frac{d}{dx} x\cos(y) + \frac{d}{dx} y$$

$$\cos(y) + 0$$

$$\boxed{\frac{\partial f}{\partial x} = \cos(y)}$$

$$f(x, y) = x(y - \sin(y))$$

Find $\frac{\partial^2 f}{\partial y^2}$
Rewriting :

$$f(x, y) = xy - x\sin(y)$$

Treating x as a constant we can say :

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} xy - x\sin(y)\right)$$

$$\frac{\partial}{\partial y}(x - x\cos(y))$$

$$0 + x\sin(y)$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = x\sin(y)}$$

1.2 Gradients

$$f(x, y, z) = -x + 3xy + 5xyz$$

Find the gradient of $f(x)$ at $(-1, 2, 1)$

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = -1 + 3y + 5yz$$

$$\frac{\partial f}{\partial y} = 0 + 3x + 5xz$$

$$\frac{\partial f}{\partial z} = 0 + 0 + 5xy$$

Since we want to find the slope at : $(-1, 2, 1)$
we can substitute :

$$\boxed{\nabla f(x, y, z) = 15, -8, -10}$$

$$f(x,y) = \cos^2(y) - \sin^2(x)$$

Find the gradient of $f(x,y)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = -2\cos(x)\sin(x)$$

$$\frac{\partial f}{\partial y} = -2\sin(y)\cos(y)$$

$$\boxed{\nabla f(x,y) = -2\cos(x)\sin(x), -2\cos(x)\sin(x)}$$

$$f(x,y,z) = z^\pi - xy$$

Find the gradient of $f(x,y,z)$

$$\frac{\partial f}{\partial x} = -y$$

$$\frac{\partial f}{\partial y} = -x$$

$$\frac{\partial f}{\partial z} = \pi z^{\pi-1}$$

So we can conclude :

$$\nabla f(x,y,z) = -y, -x, \pi z^{\pi-1}$$

$$f(x,y) = x^2 - xy$$

Find the gradient of $f(x,y)$ at $(1,-1)$

$$\frac{\partial f}{\partial x} = 2x - y$$

$$\frac{\partial f}{\partial y} = -x$$

$$\nabla f(x,y) = 3, -1$$

1.3 Finding Directional Derivatives

When you take a nudge in the direction of a vector, how is the output changed?

The directional derivative can be defined as the dot between the vector and the function in question, or more concisely :

Directional Derivative Definition

$$\vec{v} \cdot \nabla f(\vec{a})$$

Should also include the definition of a dot product :

Dot Product

$$\vec{v} \cdot \vec{w} = \vec{v}_1\vec{w}_1 + \ldots + \vec{v}_n\vec{w}_n$$

Let $f(y) = \ln(y)$. Suppose $\vec{a} = (3, 4)$ and $\vec{v} = \left(\sqrt{3}/2, 1/2\right)$
Find the directional derivative of $f(x, y)$ at $\vec{a}$ in the direction of $\vec{v}$
Because we know

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(0, \frac{1}{y}\right)$$

Since we know that $y = 4$

$$\vec{v} \cdot \nabla = \left(0, \frac{1}{4}\right) \cdot \left(\frac{\sqrt{3}}{2}\right)$$

Remember for the dot product you multiply the terms in the same position.

$$\boxed{\nabla_{\vec{v}} f(\vec{a}) = \frac{1}{8}}$$

Let $f(x, y, z) = \frac{xy}{z}$ Suppose $\vec{a} = (-3, 2, 1)$ and $\vec{v} = (1, 1, 1)$
Find the directional derivative of $f(x, y, z)$ at $\vec{a}$ in the direction of $\vec{v}$.

$$\vec{v} \cdot \nabla f(\vec{a})$$

So we just need to find the dot between the gradient of the $f(x, y, z)$ function in the direction of $\vec{v}$.

$$\nabla f(x, y, z) = \left(\frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2}\right)$$

Substiting the values given :

$$f(\vec{a}) = (2, -3, 6)$$

$$\vec{v} \cdot f(\vec{a}) = (1, 1, 1) \cdot (2, -3, 6) = 5$$

Let $f(x, y) = \sin(xy)$. Suppose $\vec{a} = (4, 0)$ and $\vec{v} = (0, 1)$
Find the directional derivative of $f(x, y)$ at $\vec{a}$ in the direction of $\vec{v}$.
Finding the gradient of $f(x, y, z)$:

$$\nabla f(x, y) = (y \cos(xy), x \cos(xy))$$

$$\nabla f(\vec{a}) = (0, 4)$$

$$\nabla f(\vec{a}) \cdot \vec{v} = 4$$

Let $f(x, y) = e^y + xy$ Suppose $\vec{a} = (1, 2)$ and $\vec{v} = (-3, 2)$.
Find the directional derivative of $f(x, y)$ at $\vec{a}$ in the direction of $\vec{v}$.

$$\nabla f(x, y) = (y, e^y + x)$$

$$\nabla f(\vec{a}) = \left(2, e^2 + 1\right)$$

$$\nabla f(\vec{a}) \cdot \vec{v}) = (2, e^2 + 1) \cdot (-3, 2) = -6 + 2e^2 + 2 = 2e^2 - 4$$

Define $f(x, y, z) = \sin(x) + y - z^2$ Let $\vec{a} = (-1, 1, -1)$ and $\vec{v} = (0, 0, 1)$.
Find the directional derivative of $f(x, y)$ at $\vec{a}$ in the direction of $\vec{v}$.

$$\nabla f(x, y, z) = \cos(x), 1, -2z$$
$$\nabla f(\vec{a}) = \cos(1), 1, 2$$
$$\nabla f(\vec{a}) \cdot \vec{v} = (\cos(1), 1, 2) \cdot (0, 0, 1)$$

$$f(x, y, z) = x^2 \ln(z)$$

Let $\vec{a} = (-1, 0, e)$ and $\vec{v} = (1, 0, 0)$
Calculate

$$\lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

So we have to take a derivative with respect to the given vector :
Since the function doesn't have a y component we can ignore that.

$$\vec{v} \cdot \nabla f(\vec{a})$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial z}\right) = 2x \ln(z), \frac{x^2}{z}$$

Evaluating :

$$\nabla f(\vec{a}) = -2, \frac{1}{e}$$

$$\left[-2, \frac{1}{e}\right] \cdot [1, 0, 0]$$

$$\boxed{-2}$$

$$f(x, y) = 4x - y$$

Let $\vec{a} = (3, 1)$ and $\vec{v} = (1, 1)$

$$\text{Calculate } \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

We will take a derivative with respect to the given vector : $\vec{v}$

$$\vec{v} \cdot \nabla f(\vec{a})$$

$$\nabla f(\vec{a}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 4, -1$$

Evaluating (redundant because of no changing terms):

$$\nabla f(\vec{a}) = 4, -1$$

Now to take the dot product :

$$(1, 1) \cdot (4, -1)$$

$$\boxed{3}$$

$$f(x, y, z) = x^2 - 6y + 2z$$

Let $\vec{a} = (1, -1, 1)$ and $\vec{v} = (-1, 2, 2)$
Calculate

$$\lim_{h\to 0} \frac{f(\vec{a} - h\vec{v}) - f(\vec{a})}{h}$$

We will take a derivative with respect to the given vector : $\vec{v}$

$$\vec{v} \cdot \nabla f(\vec{a})$$

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, -6, 2)$$

Substituting $\vec{a} = (-1.2, 2)$ into $f(x, y, z)$

$$\nabla f(\vec{a}) = -2, -6, 2$$

Dotting

$$\nabla f(\vec{a}) \cdot \vec{v} = (-2, -6, 2) \cdot (-1, 2, 2)$$

$$2 + -12 + 4 = \boxed{-6}$$

1.3.1 Quiz

$$f(x, y, z) = \frac{xy}{z}$$

$$\text{Find } \frac{\partial f}{\partial v}$$

Take the gradient of $f(x)$ and dot it with e $\vec{v}$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2}\right)$$

Substituting $\vec{a} = -3, 2, 1$

$$f(\vec{a}) = (2, -3, 6)$$

Dotting :

$$\nabla_{\vec{v}} f(x) = \nabla f(x) \cdot \vec{v}$$

$$(2, -3, 6) \cdot (1, 1, 1) = 2 - 3 + 6$$

$$\boxed{5}$$

$$f(x, y, z) = xy + yz + zx$$

Find the gradient at $(-4, 3, 1)$

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y + z, z + x, y + x)$$

Substituting $(-4, 3, 1)$

$$\nabla f(-4, 3, 1) = 4, -3, -1$$

$$f(x, y) = x^2 + y^2$$

Find $\nabla f(x)$

$$\nabla f(x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$\boxed{2x, 2y}$$

$$f(x, y) = \sin(xy)$$

Suppose $\vec{a} = (4, 0)$ and $\vec{v} = (0, 1)$
Calculating Gradient :

$$\nabla f(x, y) = y\cos(xy), x\cos(xy)$$

Subsisting $\vec{a}$

$$\nabla f(\vec{a}) = 0, 4\cos(0) = 0, 4$$

Taking dot product :

$$\nabla f(\vec{a}) \cdot \vec{v} = (0, 4) \cdot (0, 1) = 4$$

1.4 Parametric Equations

1.4.1 Derivatives of Vector Valued Functions

$$g(t) = (-2\sin(t + 1), 5t^2 - 5t)$$

Find $g'(t)$
Applying the derivative to both sides :

$$g'(t) = -2\cos(t + 1), 10t - 2$$

$$h(t) = (\log(10t), \sin(t))$$

Find $h'(t)$
This is a little trick question, remember that $\ln(t) \neq \log(t)$

Log Rule

$$\frac{d}{dx}\log(cx) = \frac{1}{\ln(c)x}$$

$$h'(t) = \boxed{\frac{1}{\ln(10)c}, \cos(t)}$$

$$f(t) = (4t^3 + t, 3\ln(t))$$

Find $f'(t)$

$$\boxed{f'(t) = 12t^2 + 1, \frac{3}{t}}$$

$$g(t) = (5t + 4, 2^{t+1})$$

Find $g'(t)$

Constnant to an Exponential Rule

$$\frac{d}{dx}c^x = c^x \ln(c)\frac{d}{dx}(x)$$

$$\boxed{5, 2^{t+1}\ln(2)}$$

1.4.2 Parametric Velocity and Speed

$$p(t) = (4t - 3t^2, t + 5, t^8 - 4t^2)$$

Find $v(t)$

$$v(t) = p'(t) = \boxed{(4 - 6t, 1, 8t^7 - 8t)}$$

$$p(t) = (3\sin(2t), 3\cos(2t), t^2)$$

What is the speed of $p(t)$

$$v(t) = p'(t) = (6\cos(2t), -6\sin(2t), 2t)$$

$$s(t) = \langle v(t)\rangle = \sqrt{(6\cos(2t))^2 + (6(\sin(2t))^2 + (2t^2)^2)}$$

Solving on the right :

$$\sqrt{36\cos^2(2t) + 36\sin^2(2t) + 4t^4}$$

$$\sqrt{36(\sin^2(2t) + \cos^2(2t) + 4t^4)}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sqrt{36(1) + 4t^4}$$

Simplifying radicals

$$\boxed{2\sqrt{9 + t^2}}$$

$$f(t) = (22t, 33\sin(3t))$$

Find $\vec{v}$

$$\vec{v} = f'(t) = \boxed{22, 99\cos(3t)}$$

$$f(t) = (t^4, t^4)$$

What is the speed -$s(t)$- of $f(t)$

Finding $\vec{v}(t)$

$$\vec{v}(t) = f'(t) = (4t^3, 4t^3)$$

$$\vec{s}(t) = \langle v(t) \rangle = \sqrt{(4t^{3)})^2 + (4t^3)^2}$$

$$\sqrt{16t^6 + 16t^6}$$

$$4\sqrt{2t^6}$$

$$\boxed{4t^3\sqrt{2}}$$

1.5 Multivariable Chain Rule

Multivariable Chain Rule

$$\frac{dh}{dt} = \nabla f(g(t)) \cdot g'(t)$$

Suppose we have a vector-valued function g(t) and a scalar function f(x,y). Let h(t) = f(g(t)).
We know

$$g(3) = (-1, 4)$$

$$g'(3) = (2, 2)$$

$$\nabla f(-1, -4) = (3, 0)$$

Evaluate $\frac{dh}{dt}$ at $t = 3$.

We have to use the multivariable chain rule.

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

Since we have all of the direct values we can pretty easily substitute values :

$$h'(3) = \nabla f(g(3)) \cdot g'(3)$$

$$h'(3) = \nabla f(-1, -4) \cdot (2, 2)$$

$$h'(3) = (3, 0) \cdot (2, 2)$$

$$h'(3) = 6$$

Suppose we have a vector-valued function g(t) and a scalar function f(x,y). Let h(t) = f(g(t)).
We know :

$$g(-1) = (9, -4, 6)$$

$$g'(-1) = (4, 4 - 1)$$

$$\nabla f(9, -4, 6) = (1, 3, 2)$$

Evaluate $\frac{dh}{dt}$ at $t = -1$.

We have to use the multivariable chain rule.

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

Since we have all of the values we will substitute :

$$h'(-1) = \nabla f(g(-1)) \cdot g'(-1)$$
$$h'(-1) = \nabla f(9, -4, 6) \cdot (4, 4 - 1)$$
$$h'(-1) = (1, 3, 2) \cdot (4, 4, -1)$$
$$h'(-1) = 4 + 12 - 2$$
$$h'(-1) = 14$$

Let $f(x, y, z) = y^2 z$ and $g(t) = (t, t^2, t^3)$. $h(t) = f(g(t))$
What is $h'(2)$?
Here we will have to use the my chain rule.

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

We have to calculate a few things :

$$g'(t) = (1, 2t, 3t^2)$$
$$g'(2) = (1, 4, 12)$$
$$\nabla f(x, y, z) = (0, 2yz, y^2)$$
$$\nabla f(g(t)) = (0, 2y^5, t^4)$$
$$\nabla f(g(2)) = (0, 64, 16)$$
$$\nabla f(g(2)) \cdot g'(2) = (1, 4, 12) \cdot (0, 64, 16) = 448$$

Let $f(x, y, z) = xz + yz$ and $g(t) = (\cos(t), \sin(t), t)$. Assuming $h(t) = f(g(t))$, what is $h'(t)$?
Here we will once again use the my chain rule.

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

We're going to have to calculate a few things to use this.

$$g'(t) = (-\sin(t), \cos(t), 1)$$
$$\nabla f(x, y, z) = (z, z, x + y)$$
$$\nabla f(g(t)) = (t, t, \cos(t) + \sin(t))$$
$$\nabla f(g(t)) \cdot g'(t) = -t\sin(t) + t\cos(t) + \cos(t) + \sin(t)$$
$$\nabla f(g(t)) \cdot g'(t) = \cos(t) + \sin(t) + t(\cos(t) - \sin(t))$$

Let $f(x, y) = \cos(y) - \sin(x)$ and $g(t) = (-2t, 4t)$. Assuming $h(t) = f(g(t))$, what is $h'\left(\frac{\pi}{3}\right)$

$$g'(t) = (-2t, 4t)$$
$$g'\left(\frac{\pi}{3}\right) = \left(-\frac{2\pi}{3}, \frac{4\pi}{3}\right)$$
$$\nabla f(x, y) = (-\cos(x), \sin(y))$$
$$\nabla f\left(g\left(\frac{\pi}{3}\right)\right) = \frac{1}{2}, -\frac{\sqrt{3}}{2}$$

Suppose we have a vector-valued function g(t) and a scalar function f(x,y). Let h(t) = f(g(t)).

We know :

$$g(1) = (0, 2, 4)$$
$$g'(1) = (-1, 6, 2)$$
$$\nabla f(0, 2, 4) = (3, 5, 1)$$

Since we again have values we can just substitute :

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$
$$h'(1) = \nabla f(0, 2, 4) \cdot (-1, 6, 2)$$
$$h'(1) = (3, 5, 1) \cdot (-1, 6, 2)$$
$$h'(1) = 29$$

These will be a little harder :
Let $f(x, y, z)$ and $g(t)$ be :

$$f(x, y, z) = z - 3x^2 + 2y^2 \text{ and } g(t) = 2t, \sqrt{2t}, t$$
$$h(t) = f(g(t))$$

Find $h'(2)$
Since we know that the my chain rule is

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$

We have to solve for a few things :

$$g'(t) = \left(2, \frac{\sqrt{2}}{2\sqrt{t}}, 1\right)$$
$$g'(2) = \left(2, \frac{1}{2}, 1\right)$$

The gradient of $f(x)$ also has to be solved :

$$\nabla f(x, y, z) = (-6x, 4y, 1)$$
$$\nabla f(g(t)) = -12t, 4\sqrt{2t}, 1$$

Now we can substitute $t = 2$:

$$\nabla f(g(2)) = (-24, 8, 1)$$

Now we have to take the final dot product between $\nabla f(g(2))$ and $g'(2)$

$$\nabla f(g(2)) \cdot g'(2) = -43$$

Let $f(x, y, z)$ and $g(t)$ be defined as below :

$$f(x, y, z) = z\cos(y) + z^2 x \quad \text{and} \quad g(t) = (t, -t^2, -t)$$

$h(t) = f(g(t))$
Find $h'(t)$
We have to solve for a few values :

$$g'(t) = (1, -2t, -1)$$
$$\nabla f(x, y, z) = \left(z^2, -z\sin(y), \cos(y) + 2zx\right)$$
$$\nabla f(g(t)) = (t^2, t\sin(-t^2), \cos(-t^2) - 2t^2)$$

$$\nabla f(g(t)) \cdot g'(t) = t^2 + (-2t\sin(-t^2)) - (\cos(-t^2) - 2t^2)$$
$$h'(t) = -t^2 - 2t\sin(-t^2) - \cos(-t^2)$$
$$h'(t) = -t^2 + 2t\sin(t^2) - \cos(-t^2)$$

Let $f(x,y)$ and $g(t)$ be defined as below :

$$f(x,y) = 2x^2y \quad \text{and} \quad g(t) = (\cos(t), \sin(t))$$

$h(t) = f(g(t))$
Find $h'(\pi)$

$$g'(t) = (-\sin(t), \cos(t))$$
$$g'(\pi) = (0, 1)$$
$$\nabla f(x,y) = (4xy, 2x^2)$$
$$\nabla f(g(t)) = (-4\sin(t)\cos(t), 2\cos^2(t))$$
$$\nabla f(g(\pi)) = (0, -2)$$
$$\nabla f(g(t)) \cdot g'(t) = -2$$

Let $f(x,y)$ and $g(t)$ be defined as below :

$$f(x,y) = \ln(x^2) + y \quad \text{and} \quad g(t) = (\sin(t), -\cos(t))$$

$h(t) = f(g(t))$
Find $h'(t)$

$$g'(t) = (\cos(t), \sin(t))$$
$$\nabla f(x,y) = \left(\frac{2}{x}, 1\right)$$
$$\nabla f(g(t)) = \left(\frac{2}{\sin(t)}, 1\right)$$

Now to take the dot product :

$$\nabla f(g(t)) \cdot g'(t) = 2\cot(t) + \sin(t)$$

Quiz

$$v(t) = (10\cos(t), 10\sin(t), 100 - t)$$

What is the velocity of $v(t)$?
To get the velocity of this velocity you just take a gradient :

$$v'(t) = \nabla v(t) = (-10\sin(t), 10\cos(t), -1)$$

Suppose we have a vector-values function $g(t)$ and a scalar function $f(x,y)$. Let $h(t) = f(g(t))$
We are given :

$$g(1) = (-2, 3)$$
$$g'(1) = (4, 2)$$
$$\nabla f(-2, 3) = (6, 1)$$

Find $h'(1)$

Once again we will be using the my chain rule.

$$h'(t) = \nabla f(g(x)) \cdot g'(x)$$

Since we have all these values already we can just directly substitute :

$$h'(1) = \nabla f(-2, 3) \cdot (4, 2)$$

$$h'(1) = (6, 1) \cdot (4, 2)$$

$$h'(1) = 24 + 2 = 26$$

Let g be a vector-values function defined by

$$g(t) = (-2\sin(t + 1), 5t^2 - 2t)$$

Find $g'(t)$

$$g'(t) = (-2\cos(t + 1), 10t - 2)$$

Let $f(x, y, z)$ and $g(t)$ be defined below :

$$f(x, y, z) = y^2 z \quad \text{and} \quad g(t) = (t, t^2, t^3)$$

$h(t) = f(g(t))$
Find $h'(2)$
We will use chain rule for this.

$$g'(t) = (1, 2t, 3t^2)$$

$$g'(2) = (1, 4, 12)$$

$$\nabla f(x, y, z) = (0, 2yz, z)$$

$$\nabla f(g(t)) = (0, 2t^5, t^3)$$

$$\nabla f(g(2)) = (0, 64, 8)$$

$$\nabla f(g(2)) \cdot g'(2) = 0 + 256 + 96 = 352$$

Suppose we have a vector-values function $g(t)$ and a scalar function $f(x, y, z)$. Let $h(t) = f(g(t))$
We know :

$$g(-2) = (1, 2, 0)$$

$$g'(-2) = (0, 4, 3)$$

$$\nabla f(1, 2, 0) = (-10, 2, -3)$$

Find $h'(-2)$

$$(-10, 2, -3) \cdot (0, 4, 3) = -1$$

1.6 Partial Derivatives of Vector-Valued Functions

$$f(x, y, z) = (xyz, \cos(z), \sin(z))$$

Find $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial z} = (xy, -\sin(z), \cos(z))$$

$$f(v, w) = (v, v^2, w^3)$$

Find $\frac{\partial f}{\partial v}$

$$\frac{\partial f}{\partial v} = (1, 2v, 0)$$

$$f(x, y) = (-xy^2, yx^2)$$

Find $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = (-2xy, x^2)$$

$$f(x, y, z) = (\cos(z), 9xy, x^3)$$

Find $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = (0, 9y, 3x^2)$$

$$p(t) = (t, e^t, t)$$

What is the speed of $p(t)$

So we have to find the velocity of this position function :

$$v(t) = (1, e^t, 1)$$

Then to find the net speed :

$$s(t) = \sqrt{1 + e^{2t} + 1}$$
$$s(t) = \sqrt{2 + e^{2t}}$$

Let h be a vector-valued function defined by $h(t) = (\log(10t), \sin(t))$. Find $h'(t)$.

$$h'(t) = (\frac{1}{\ln(10)t}, \cos(t))$$

Suppose we have a vector-valued function $g(t)$ and a scalar function $f(x, y)$. Let $h(t) = f(g(t))$. We're given, $g(\pi) = (2, 0)$, $g'(\pi) = (3, 4)$, and $\nabla f(2, 0) = (-1, 3)$.

Evaluate $\frac{dh}{dt}$ at $t = \pi$.

This is pretty simple direct substitution :

$$h'(t) = \nabla f(g(t)) \cdot g'(t)$$
$$h'(t) = \nabla f(2, 0) \cdot (3, 4)$$
$$h'(t) = (-1, 3) \cdot (3, 4)$$
$$h'(t) = -3 + 12 = 9$$

Let $f(x,y,z) = zx^2 + y^3$ and $g(t) = (t, \sin(3t), \sin(2t))$. Assuming $h(t) = f(g(t))$, find $h'(t)$
In order to find $h'(t)$, we have to apply the multivariable chain rule.

$$h'(t) = f(g(t)) \cdot g'(t)$$

$$g'(t) = (1, 3\cos(3t), 2\cos(2t))$$

$$\nabla f(x,y,z) = (2xz, 3y^2, x^2)$$

$$\nabla f(g(t)) = (2t\sin(2t), 3\sin^2(3t), t^2)$$

$$h'(t) = \nabla f(g(t)) \cdot g'(t) = 2t\sin(2t) + 9\cos(3t)\sin^2(3t) + 2t^2\cos(2t)$$

Let f be a vector-valued function defined by $f(t) = (4t^3 + t, 3\ln(t))$
Find $f'(t)$.

$$f'(t) = \left(12t^2 + 1, \frac{3}{t}\right)$$

1.7 Divergence

The formula for divergence in three dimensions is

$$\text{div}(f) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Where P, Q, R are all the respective x, y, z components of the original $f(x,y,z)$ function.

What is the divergence of f at $(2, -1, 3)$

$$\text{div}(f) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial a} + \frac{\partial R}{\partial z}$$

$$\text{div}(f) = yz + 6y + 4z$$

$$\text{div}(f(2,-1,3)) = -4$$

What is $\text{div}(f)$

$$f(x,y) = (-\cos(x), -y(\sin(x)))$$

$$\text{div}(f) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$$\text{div}(f) = \sin(x) - \sin(x)$$

$$\text{div}(f) = 0$$

$$f(x,y,z) = \left(\frac{1}{z} + e^x, \sin(xy), 5\right)$$

Find $\text{div}(f)$

$$\text{div}(f) = \left(\frac{\frac{1}{z} + e^x}{dx} + \frac{\sin(xy)}{dy} + 5dz\right)$$

$$\text{div}(f) = e^x + x\cos(xy) + 0$$

$$f(x,y) = (-x^2, y)$$

Find the divergence of f at $(3, -1)$.

$$\text{div}(f) = \left(-\frac{d}{dx}x^2 + y\frac{d}{dy}\right)$$

$$\text{div}(f) = (-2x + 1)$$

Now to find it at $(3, -1)$

$$-6 + 1 = -5$$

$$f(x, y, z) = (-y + z, -z + x, -x + y)$$

What is the divergence of f at $(1, 2, 3)$?
This is just 0, looking at the components. You can see the divergence $= 0$.

$$f(v, w) = (ve^w, wv, 4)$$

What is $\frac{\partial f}{\partial w}$

$$\frac{\partial f}{\partial w} = ve^w, v, 0$$

$$f(x, y) = \left(y, \frac{x}{y}\right)$$

What is $\frac{\partial f}{\partial x}$?

$$\frac{\partial f}{\partial x} = \left(0, \frac{1}{y}\right)$$

1.8 Curl

Counterclockwise rotation is positive curl, Clockwise rotation is negative curl. Right hand rule. Up is positive, down is negative.

A curl can be generally defined as :

$$\text{curl}(F) = \nabla \times F$$

Curl takes in a vector field and outputs another vector field.
A divergence can then be defined as :

$$\text{div}(F) = \nabla \cdot F$$

Which takes a vector field but then outputs a scalar field
A gradient can be defined as :

$$\text{grad}(f) = \nabla f$$

A gradient takes a scalar and then gives a vector field
The formula for curl in two dimensions :

$$\text{curl}(f(x, y)) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

The formula for curl in three dimensions :

$$\text{curl}(f(x, y, z)) = \hat{\imath}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

$$\text{curl}\,\mathbf{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

Where P is the $x-$component of f and Q is the $y-$component.

$$f(x, y) = (4yx^2, x + 3xy^2)$$

What is the curl of f at $(-2, 3)$?

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\text{curl}(f) = 6xy - 8xy$$

$$\text{curl}(f) = -2xy$$

Testing at point $(-2, 3)$

$$\text{curl}(-2, 3) = 12$$

$$f(x, y) = (\cos(3x - y), x\sin(y))$$

What is the curl of f?

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\text{curl}(f) = \sin(y) - \sin(3x - y)$$

$$f(x, y) = (\sin(xy), x^2)$$

What is the curl of f at $3, \pi$
Finding curl :

$$\text{curl}(f) = 2x - x\cos(xy)$$

$$\text{curl}(3, \pi) = 6 - 3\cos(3\pi)$$

Cos is periodic so :

$$\text{curl}(3, \pi) = 6 + 3$$

$$\text{curl}(3, \pi) = 9$$

$$f(x, y, z) = (2\sin(y), \sin(z)\sin(x), z^2)$$

$$\text{curl}(f) = ?\hat{i} + ?\hat{j} + ?\hat{k}$$

Here we will have to find 3d curl, which is a bit more involved than the other examples of 2d curl.

$$\text{curl}(f(x, y, z)) = \hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) - \hat{j}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

What is the curl of $f(x, y) = (xy - 1, y(\cos(x) + x))$

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\text{curl}(f) = y(-\sin(x) + 1) - x$$

$$\text{curl}(f) = -y\sin(x) + y - x$$

What is the curl of $f(-3, 1)$ when $f(x, y) = (3x^2y, 3x)$ We know

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\text{curl}(f) = 3 - 3x^2$$

$$\text{curl}(f(-3,1)) = 3 - 3(-3)^2 = 3 - 3(9) = -3(8) = -24$$

$$f(x,y) = (y + \sin(y), x - \cos(x))$$

What is the curl of f?

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\text{curl}(f) = 1 + \sin(x) - (1 + \cos(y))$$

$$\text{curl}(f) = \sin(x) - \cos(y)$$

$$f(x,y) =)(5 + x, y\ln(x+y))$$

What is the curl of $f(4,8)$

$$\text{curl}(f) = \frac{y}{x+y} - 0$$

$$\text{curl}(f) = \frac{y}{x+y}$$

$$\text{curl}(f(4,8)) = \frac{8}{4+8} = \frac{8}{12} = \frac{2}{3}$$

$$f(x,y) = (1, 2x\ln(xy))$$

What is curl(f)

$$\text{curl}(f) = \frac{\partial 2x\ln(xy)}{\partial x} - 0$$

$$\text{curl}(f) = 2\ln(xy) + 2$$

$$f(x,y,z) = (4x + z, y, -x)$$

1.9 Laplacian

The Laplacian of a scalar field f is the sum of each of its second partial derivatives :

Laplacian Definition

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta f = \nabla^2 f = \text{div}\nabla f = \text{div}(\text{grad}(f))$$

Fields are harmonic if the Laplacian is zero

$$f(x,y) = xy - e^{xy}$$

What is the Laplacian of $f(x,y)$

We need to find the two partial derivatives :

$$f_{xx} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}[y - ye^{xy}] = -y^2 e^{xy}$$

$$f_{yy} = \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}[x - xe^{xy}] = -x^2e^{xy}$$

When adding these together, you can find the Laplacian

$$\nabla f(x,y) = f_{xx} + f_{yy} = -e^{xy}(x^2 + y^2)$$

$$f(x,y,z) = zx^2 - zy^2 + e^y\cos(z)$$

Find if $f(x,y,z)$ is harmonic.

So we have to take the derivatives :

$$f_{xx} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(2xz) = 2z$$

$$f_{yy} = \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(-2yz + e^y\cos(z)) = -2z + e^y\cos(z)$$

$$f_{zz} = \frac{\partial f}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}(x^2 - e^y\sin(z)) = -e^y\cos(z)$$

$$\Delta f = 2z - 2z + e^y\cos(z) - e^y\cos(z) = 0$$

This means that the function is harmonic.

$$f(x,y,z) = x^2z^4 - z^2y^3 + x$$

What is the Laplacian of $f(x,y,z)$?

Finding the derivatives :

$$f_{xx} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}2xz^4 + 1 = 2z^4$$

$$f_{yy} = \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(-3z^2y^2) = -6z^2y$$

$$f_{zz} = \frac{\partial f}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}4x^2z^3 - 2zy^3 = 12x^2z^2 - 2y^3$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = 2z^4 - 6z^2y + 12x^2z^2 - 2y^3$$

$$f(x,y) = 2e^x\sin(y)$$

Is f harmonic?

Finding derivatives :

$$f_{xx} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} = 2e^x\sin(y)$$

$$f_{yy} = \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}2e^x\cos(y) = -2e^x\sin(y)$$

$$\Delta f = f_{xx} + f_{yy} = 2e^x\sin(y) - 2e^x\sin(y) = 0$$

Yes, f is harmonic

$$f(x,y,z) = \sin(z) - xe^y$$

What is the Laplacian of $f(x,y,z)$

$$f_{xx} = 0$$

$$f_{yy} = -xe^y$$

$$f_{zz} = -\sin(z)$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = -xe^y - \sin(z)$$

$$f(x,y) = x^3 + x^2y + xy^2 + y^3$$

What is $\nabla \cdot (\nabla f)$?
This is the same thing as Δf

$$f_{xx} = \frac{\partial f}{\partial x} 3x^2 + 2xy + y^2 = 6x + 2y$$

$$f_{yy} = \frac{\partial f}{\partial y} 3y^2 + 2xy + x^2 = 6y + 2x$$

$$\nabla f = f_{xx} + f_{yy} = 6x + 2y + 6y + 2x = 8x + 8y$$

$$f(x,y,z) = xy - x^2z + y^2z$$

Is f harmonic?

$$f_{xx} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} y - 2xz = -2z$$

$$f_{yy} = \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y} x + 2yz = 2z$$

$$f_{zz} = \frac{\partial f}{\partial z}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} - x^2 + y^2 = 0$$

$$\nabla f = f_{xx} + f_{yy} + f_{zz} = -2z + 2z + 0 = 0$$

$$f(x,y) = xy - e^{xy}$$

What is the Laplacian of $f(x,y)$?

$$f_{xx} = \frac{\partial f}{\partial x} y - ye^{xy} = -y^2e^{xy}$$

$$f_{yy} = \frac{\partial f}{\partial y} x - xe^{xy} = -x^2e^{xy}$$

$$\nabla f = f_{xx} + f_{yy} = -y^2e^{xy} - x^2e^{xy}$$

1.10 Jacobian Matrix and Determinant

$$\text{Suppose } f(x,y,z) = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \text{ then the Jacobian is}$$

$$f(J) = \begin{bmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} & \frac{\partial f_0}{\partial z} \\ \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} & \frac{\partial f_0}{\partial z} \\ \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} & \frac{\partial f_0}{\partial z} \end{bmatrix}$$

Basically you just take a gradient across the rows.

$$f(J) = \begin{bmatrix} \nabla f_0 \\ \nabla f_1 \\ \nabla f_2 \end{bmatrix}$$

This is super tedious to write work out for, just know that you can go across the matrix, doing one column at a time taking your derivatives of the three functions x, then the next column taking the derivatives of y, then the next taking it for z.

You're also able to take the determinant of this function, which is the exact thing being ripped from those linear algebra classes with Senia.

As a reminder :

$$\det(f) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

. So specifically for this case, with a 2x2 matrix :

$$\det J(f) = \det \begin{bmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} \\ \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \end{bmatrix} = \frac{\partial f_0}{\partial x}\frac{\partial f_1}{\partial y} - \frac{\partial f_1}{\partial x}\frac{\partial f_0}{\partial y}$$

Problems :

Find the determinant of the transformation :

$$J(f) = \begin{bmatrix} \frac{1}{x} & 0 \\ \\ 1 & 1 \end{bmatrix}$$

Since this is already a Jacobian function, we can determine that the determinant is quite simple to solve for

$$\det\ J(f) = \frac{1}{x}(1) - 0(1) = \frac{1}{x}$$

How will f expand or contract space around the point $\left(\frac{1}{9}, \frac{1}{3}\right)$

If we evaluate $|J(f)|$ at $\left(\frac{1}{9}, \frac{1}{3}\right)$, we get 9. Because the Jacobian determinant here has a value greater than 1, we can conclude that f will finitely expand the space around $\left(\frac{1}{9}, \frac{1}{3}\right)$.

Let f be a transformation. Its Jacobian matrix is given below :

$$J(f) = \begin{bmatrix} \sin(y) & x\cos(y) \\ -y\sin(x) & \cos(x) \end{bmatrix}$$

Find the Jacobian determinant of f

$$\sin(y)\cos(x) - (x\cos(y))(-y\sin(x))$$

How will f expand or contract space around the point :

$$\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)?$$

Evaluating $|J(f)|$ at $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$ gives us

$$\sin\left(\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\cos(-\frac{\pi}{2})\right)\left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right)$$

$$1(0) - (\frac{\pi}{2}(0))(\frac{\pi}{2}(1))$$

$$0 - 0 = 0$$

Because the determinant at that point is 0, we can conclude that the space will contract infinitely at this point.

Consider f as a transformation. Its Jacobian matrix is given below :

$$J(f) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Find it's Jacobian determinant

$$\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\frac{3}{4} + \frac{1}{4} = 1$$

How will it expand or contract space around (1,0) Because at the point (1,0) the Jacobian determinant is equal to 1, we can determine that the space around point won't be warped.

Let f be a transformation in the real space. Its Jacobian matrix is given below :

$$J(f) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

Find the Jacobian determinant of f

$$\cos(\theta)r\cos(\theta) + r\sin(\theta)\sin(\theta)$$

How will f expand or contract space around the point

$$r = \frac{1}{2}, \theta = \frac{\pi}{2}$$

Substituting:

$$\cos\left(\frac{\pi}{2}\right)\frac{1}{2}\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$$

$$\frac{1}{2}\left(\cos^2(\theta) + \sin^2(\theta)\right)$$

$$\frac{1}{2}$$

Because the evaluation of the Jacobian Determinant is less than 1 but not 0, we can determine that the space around this point will be finitely contracted.

1.11 Unit Test and Various Practice

$$f(x, y) = 4yx - x + e^y$$

$$\text{Find } \frac{\partial^2 f}{\partial x \partial y}$$

This is solved by realizing that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

$$\frac{\partial}{\partial x} 4x + e^y$$

$$4$$

$$f(x, y) = (x, y^2)$$

Find div(f) The definition of div is

$$\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}$$

So for this problem :

$$x\frac{d}{dx} + y^2\frac{d}{dy} = 1 + 2y$$

Let $f(x, y) = 2x + 3y - 1$ and $g(t) = (t^2, 2t)$

$$h(t) = f(g(t))$$

Find $h'(4)$

This is a pretty clear example of where the standard product rule would be used :

$$\frac{dh}{dt} = \nabla f(g(t)) \cdot g'(t)$$

Finding $g'(4)$

$$g'(t) = \frac{d}{dt}g(t) = (2t, 2)$$

$$g'(4) = (8, 2)$$

Finding $\nabla f(g(t))$

$$\nabla f(x, y) = (2, 3, 0)$$

$$\nabla f(g(t)) = (2, 3, 0)$$

$$\nabla f(g(t)) \cdot g'(t) = (8, 2) \cdot (2, 3) = 16 + 6 = \boxed{22}$$

$\square$

$$f(x, y) = (\cos(x + y), -\sin(x - y))$$

What is the curl of f at $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$

Curl is generally the cross of the gradient and the vector field.

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x} - \sin(x - y) = -\cos(x - y)$$

$$\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y} \cos(x + y) = -\sin(x + y)$$

$$\text{curl}(f) = -\cos(x - y) + \sin(x + y)$$

Evaluating this curl at $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$

$$-\cos\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right)$$

$$-\cos(\pi) + \sin(2\pi)$$

$$1$$

Let $f(x, y, z) = z + x^2 - y^2$ suppose

Finding $\nabla f(g(t))$

$$\nabla f(x, y) = (2, 3, 0)$$

$$\nabla f(g(t)) = (2, 3, 0)$$

$$\nabla f(g(t)) \cdot g'(t) = (8, 2) \cdot (2, 3) = 16 + 6 = \boxed{22}$$

$\square$

$$f(x, y) = (\cos(x + y), -\sin(x - y))$$

What is the curl of f at $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$

Curl is generally the cross of the gradient and the vector field.

$$\text{curl}(f) = \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}$$

$$\frac{\partial f_y}{\partial x} = \frac{\partial}{\partial x} - \sin(x-y) = -\cos(x-y)$$
$$\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial y}\cos(x+y) = -\sin(x+y)$$
$$\text{curl}(f) = -\cos(x-y) + \sin(x+y)$$

Evaluating this curl at $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$

$$-\cos\left(\frac{3\pi}{2} - \frac{\pi}{2}\right) + \sin\left(\frac{3\pi}{2} + \frac{\pi}{2}\right)$$
$$-\cos(\pi) + \sin(2\pi)$$
$$1$$

Let $f(x,y,z) = z + x^2 - y^2$ Suppose $\vec{a} = (-1,2,0)$ and $\vec{v} = (1,2,1)$

Find the directional derivative of $f(x,y,z)$ at $\vec{a}$ in the direction of $\vec{v}$

Directional derivatives are found using the traditional :

$$\vec{v} \cdot \nabla f(\vec{a})$$

$$\nabla f(x,y,z) = (2x, -2y, 1)$$
$$\nabla f(\vec{a}) = (-2, -4, 1)$$
$$\nabla f(\vec{a}) \cdot \vec{v} = (-2,-4,1) \cdot (1,2,1) = -2 - 8 + 1 = -9$$

Let g be a vector-valued function defined by $g(t) = (5t+4, 2^{t+1})$

Find $g'(t)$

$$g'(t) = 5, 2^{t+1}\ln(2)$$

$$f(x,y,z) = (\cos(z), \sin(x+y), x)$$

Find the curl of f.

$$\text{curl}(f(x,y,z)) = \hat{i}\left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{j}\left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) + \hat{k}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

$$\hat{i}\,(0-0) - \hat{j}\,(-\sin(z) - 1) + (\cos(x+y) - 0)$$
$$\hat{i}0 + \hat{j}(-\sin(z) - 1) + \hat{k}\cos(x+y)$$

Suppose we have a vector-valued function $g(t)$ and a scalar function $f(x,y,z)$. Let $h(t) = f(g(t))$

We know :

$$g(8) = (0,4,1) \quad g'(8) = (5,-4,0) \quad \nabla f(0,4,1) = (3,3,3)$$

Evaluate $\frac{dh}{dt}$ at $t = 8$

This is a pretty simple application of a previous problem's formula.

$$\frac{dh}{dt} = \nabla f(g(t)) \cdot g'(t)$$

Because of what we were given we can just simply do the direct substation and a dot product.

$$\frac{dh}{dt} = (3,3,3) \cdot (5,-4,0) = 15 - 12 = 3$$

$$g(t) = (e^{2t}, t\sin(t))$$

What is the velocity of $g(t)$

$$g'(t) = (2e^{2t}, \sin(t) + t\cos(t))$$

$$f(x,y,z) = \cos(x+y) - \sin(z-y)$$

What is the Laplacian of $f(x,y,z)$?

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

So all we have to do here is take a bunch of double derivatives and see where that takes us.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x} - \sin(x+y) = -\cos(x+y)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y} - \sin(x+y) + \cos(z-y) = -\cos(x+y) + \sin(z-y)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial f}{\partial z} - \cos(z-y) = \sin(z-y)$$

$$-\cos(x+y) - \cos(x+y) + \sin(z-y)\sin(z-y)$$

$$-2\cos(x+y) + 2\sin(z-y)$$

$$f(x,y) = \frac{y^2}{2} + x^3$$

What is $\frac{\partial f}{\partial x}$?

$$3x^2$$

$$h(a,b) = 2^a(1+b)$$

What is the partial derivative of h with respect to b?

$$h(a,b) = 2^a + 2^a b$$

$$\frac{\partial h}{\partial b} = 2^a$$

Find the gradient of $f(x,y,z) = zxy - x + \ln(y)$

$$\frac{\partial f}{\partial x} = zy - 1$$

$$\frac{\partial f}{\partial y} = zx + \frac{1}{y}$$

$$\frac{\partial f}{\partial z} = xy$$

$$f(x,y,z) = (y^2 + xz, zx^3, 2x + y)$$

What is $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = (2y, 0, 1)$$

2 Applications of Derivatives

2.1 Tangential Planes

2D Planes :

The equation for a tangential plane of an explicitly defined surface $z = f(x,y)$ at the point (a,b) is :

$$S = f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)$$

3D Planes:2

The equation for a tangential plane of an explicitly defined surface $z = f(x,y,z)$ at the point (a,b,z) is :

$$S = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) + \frac{\partial f}{\partial z}(z-c)$$

Let S be a surface in $3D$ described by the equation $\sqrt{x^2+y^2} - z^2 = 0$
What is the equation of the plane tangent to S at $(3, 4, \sqrt{5})$?
In order to do this we have to take a 3D gradient:

$$\nabla f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -2z\right)$$

Substituting the values from the tangent point :

$$\nabla f(x,y,z) = \left(\frac{3}{\sqrt{9+16}}, \frac{4}{\sqrt{9+16}}, -2\sqrt{5}\right)$$

$$\nabla f(x,y,z) = \left(\frac{3}{5}, \frac{4}{5}, -2\sqrt{5}\right)$$

And you can now just take these points and substitute them and get your plane equation.

$$S : \frac{3}{5}(x-3) + \frac{4}{5}(y-4) - 2\sqrt{5}(z - \sqrt{5}3)$$

Let S be a surface in $3D$ described by the equation $z = ye^{2x} - y^2$
What is the equation of the plane tangent to S at $(1, 5)$.
We must take the 2D gradient :

$$\nabla f(x,y) = \left(2ye^{2x}, e^{2x} - 2y\right)$$

Now evaluating at the points $(1, 5)$

$$\nabla f(1,5) = \left(10e^2, e^2 - 10\right)$$

Now substituting the points to get plane equations :

$$S : 10e^2(x-1) + (e^2 - 10)(x-5)$$

Let S be a surface in 3D described by the equation :

$$x^2 + y^2 + z^2 = 4$$

$$x^2 + y^2 + z^2 - 4 = 0$$

Find the equation of the plane tangent to S at $(1, 1, \sqrt{2})$

First step, taking a gradient :

$$\nabla f(x, y, z) = (2x, 2y, 2z)$$

$$\nabla f(1, 1, \sqrt{2}) = (2, 2, 2\sqrt{2})$$

Then just straight substitution:

$$S : 2(2 - 1) + 2(y - 1) + 2\sqrt{2}(z - \sqrt{2}) = 0$$

Let S be a surface in 3D described by the equation $z = \sin(xy)$

Find the equation of the plane tangent to S at $(0, \pi)$.

Taking a gradient :

$$\nabla f(x, y) = (y\cos(xy), x\cos(xy))$$

$$\nabla f(0, \pi) = (\pi, 0)$$

Substituting :

$$z = 0 + \pi x + 0$$

$$z = x\pi$$

Let S be a surface in 3D described by the equation

$$3x + \sin(y) + z^2 = 0$$

What is the equation of the plane tangent to S at $(-3, \pi, 3)$?

First you must take a gradient of the surface's equation :

$$\nabla f(x, y, z) = (3, \cos(y), 2z)$$

$$\nabla f(-3, \pi, 3) = (3, -1, 6)$$

Now just substitute :

$$S : 3(x + 3) - 1(y - \pi) + 6(z - 3)$$

Let S be a surface in 3D described by the equation :

$$z = x^4 - y^3 + x^2y - x + 2$$

What is the equation of the plane tangent to S at (-1,-1)?

$$f(-1, -1) = 1 + 1 - 1 + 1 + 2 = 4$$

$$\nabla f(x, y) = 4x^3 + 2xy - 1, -3y^2 + x^2$$

$$\nabla f(-1, -1) = -4 + 2 - 1, -3 + 1$$

$$\nabla f(-1, -1) = -3, -2$$

Combining everything :

$$S : 4 - 3(x + 1) - 2(y + 1)$$

2.2 Hessian Matrix

The Hessian matrix of a scalar field f is the matrix that contains all the 2nd order partial derivatives of a function :

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ \\ f_{yx} & f_{yy} \end{pmatrix}$$

BIGWIG HESSIAN MATRIX:

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

Let $f(x, y) = \ln(x)\ln(y)$
What is the Hessian of f?

$$\begin{aligned} f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x}\ln(x)\ln(y) \\ &= \frac{\partial f}{\partial x}\frac{\ln(y)}{x} \\ &= -\frac{\ln(y)}{x^2} \\ f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\ln(x)\ln(y) \\ &= \frac{\partial f}{\partial x}\frac{\ln(x)}{y} \\ &= \frac{1}{xy} \\ f_{yx} &= f_{xy} = \frac{1}{xy} \\ f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\ln(x)\ln(y) \\ &= -\frac{\ln(x)}{y^2} \end{aligned}$$

Let

$$f(x, y) = \cos(2y) - xy - y^2$$

What is the Hessian of f?

$$\begin{aligned}
f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x}(\cos(2y) - xy - y^2) \\
&= \frac{\partial f}{\partial x} y \\
&= 0 \\
f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(\cos(2y) - xy - y^2) \\
&= \frac{\partial f}{\partial x} 2\sin(2y) - x - 2y \\
&= -1 \\
f_{yx} &= f_{xy} = -1 \\
f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}(\cos(2y) - xy - y^2) \\
&= \frac{\partial f}{\partial y} 2\sin(2y) - x - 2y \\
&= -4\cos(2y) - 2
\end{aligned}$$

This leaves us with a final matrix of :

$$\begin{pmatrix} 0 & -1 \\ -1 & -4\cos(2y) - 2 \end{pmatrix}$$

Let $f(x, y) = e^x + 5y^3x$
What is the Hessian of f?

$$\begin{aligned}
f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x}(e^x + 5y^3x) \\
&= \frac{\partial f}{\partial x}(e^x + 6y^3) \\
&= e^x \\
f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}(e^x + 5y^3x) \\
&= \frac{\partial f}{\partial x}(15y^2x) \\
&= 15y^2 \\
f_{yx} &= f_{xy} = 15y^2x \\
f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}(e^x + 5y^3x) \\
&= \frac{\partial f}{\partial y} 15y^2x \\
&= 30yx
\end{aligned}$$

Putting this all together, this leaves us with the final matrix :

$$\begin{pmatrix} e^x & 15y^2 \\ 15y^2 & 30yx \end{pmatrix}$$

2.3 Critical points

$$f(x,y) = (-x^3 + 2y^2 + x)$$
$$\nabla f(x,y) = (-3x^2 + 1, 4y)$$

Now to set these to 0,

$$(0,0) = (-3x^2 + 1, 4y)$$

X coordinate,

$$0 = -3x^2 + 1$$
$$1 = 3x^2$$
$$\frac{1}{3} = x^2$$
$$\pm\frac{1}{\sqrt{3}} = x$$

Y coordinate,

$$0 = 4y$$
$$0 = y$$

$$f(x,y) = 4x - y + \cos(2x)$$

What are the critical points of f?

Find a gradient :

$$\nabla f(x,y) = (4 - 2\sin(2x), -1)$$

There isn't a critical point because you can see the -1 in the y is unable to equal 0 at any point, meaning this function has no troughs or peaks in the y axis.

$$f(x,y) = \cos(x+y) - \sin(y)$$

Find the critical points of f

First we find a gradient :

$$\nabla f(x,y) = (-\sin(x+y), -\sin(x+y) - \cos(y))$$

Now we must set these to coordinates to 0

$$(0,0) = -\sin(x+y), -\sin(x+y) - \cos(y)$$

X coordinate,

$$0 = -\sin(x+y)$$

Y coordinate,

$$0 = -\sin(x+y) - \cos(y)$$

Now we can use an interesting trick and subtract these two functions from each other:

$$0 = -\cos(y)$$

So we just need situation in which $y = \frac{\pi}{2}k$ where $k = \dots, -1, 0, 1, \dots$

$$f(x,y) = 9 - x^3y - 3xy^3$$

Find the critical points of f.

First you gotta find the gradient :

$$\nabla f(x,y) = (-9x^2y - 3y^3, x^3 - 9xy^2)$$

Now setting these to 0

$$(0,0) = (-9x^2y - 3y^3, x^3 - 9xy^2)$$

X coordinate :

$$\begin{aligned} 0 &= -9x^2y - 3y^3 \\ 3y^3 &= -9x^2y \\ y^2 &= -3x^2 \end{aligned}$$

Y coordinate :

$$\begin{aligned} 0 &= x^3 - 9xy^2 \\ x^3 &= 9xy^2 \\ x^2 &= 9y^2 \end{aligned}$$

There is no solution where x and y are nonzero. So the solution must be $(0,0)$.

2.4 2nd Derivative testing to classify critical points

You can derive the qualities of a critical point quite easily using this simple derivation :

$$H = f_{xx}f_{yy} - f_{xy}f_{yx}$$

- $H < 0$ implies a saddle point
- $H > 0$ and $f_{xx} > 0$ implies a local minimum
- $H > 0$ and $f_{xx} < 0$ implies a local maximum
- $H = 0$ means the test is inconclusive.

The scalar field $f(x,y) = 2x^2 - y^2 + 4x + 2y - 1$ has a critical point at $(-1,1)$.
How does the second partial derivative test classify this critical point?
We will need to take a few double derivatives :

$$\begin{aligned} f_{xx} &= \frac{\partial f}{\partial x \partial x} 2x^2 - y^2 + 4x + 2y - 1 \\ &= \frac{\partial f}{\partial x} 4x + 4 \\ &= 4 \\ f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} 2x^2 - y^2 + 4x + 2y - 1 \\ &= 0 \\ f_{yx} &= f_{xy} = 0 \\ f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} 2x^2 - y^2 + 4x + 2y - 1 \\ &= \frac{\partial f}{\partial y} - 2y + 2 \\ &= -2 \end{aligned}$$

Putting this all together :

$$\begin{aligned} H &= f_{xx}f_{yy} - f_{xy}f_{yx} \\ H &= (4)(-2) - (0)(0) \end{aligned}$$

$$H = -8$$

So we can conclude that this is a saddle point as H is negative.

The scalar field $f(x, y) = x\ln(y^2) - x$ has a critical point at $(0, \sqrt{e})$. How does the second partial derivative test classify this critical point?

In order to do this we will need to take a few derivatives :

$$\begin{aligned}
f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} x\ln(y^2) - x \\
&= \ln(y^2) - 1 \\
&= 0 \\
f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} x\ln(y^2) - x \\
&= \frac{\partial f}{\partial x}\frac{2x}{y} \\
&= \frac{2}{y} \\
f_{yx} &= f_{xy} = \frac{2}{y} \\
f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} x\ln(y^2) - x \\
&= \frac{\partial f}{\partial y}\frac{2x}{y} \\
&= -\frac{2x}{y^2} \\
H &= f_{xx}f_{yy} - f_{xy}f_{yx} \\
H &= 0\left(\frac{2x}{y^2}\right) - \frac{4}{y^2} \\
H &= -\frac{4}{e}
\end{aligned}$$

The scalar field $f(x, y) = x^2 y$ has a critical point at $(0, 0)$ How does the second partial derivative test classify this critical point?

Typical gradient :

$$\begin{aligned}
f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} x^2 y \\
&= \frac{\partial f}{\partial x} 2xy \\
&= 2y \\
f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} x^2 y \\
&= x^2 \\
&= 2x \\
f_{yx} &= f_{xy} = 2x \\
f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} x^2 y \\
&= \frac{\partial f}{\partial y} x^2 \\
&= 0
\end{aligned}$$

So we can use the standard definition :

$$H = f_{xx}f_{yy} - f_{xy}f_{yx}$$
$$H = 2y(0) - 4x^2$$
$$H = -4x^2$$
$$H = 0$$

The scalar field $f(x, y) = \sin(x) + \sin(y)$ has a critical point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
How does the second partial derivative test classify this critical point?

$$\begin{aligned}
f_{xx} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x}\sin(x) + \sin(y) \\
&= \frac{\partial f}{\partial x}\cos(x) \\
&= -\sin(x) \\
f_{xy} &= \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sin(x) + \sin(y) \\
&= \frac{\partial f}{\partial x}\cos(y) \\
&= 0 \\
f_{yx} &= f_{xy} = 0 \\
f_{yy} &= \frac{\partial f}{\partial y}\frac{\partial f}{\partial y}\sin(x) + \sin(y) \\
&= \frac{\partial f}{\partial y}\cos(y) \\
&= -\sin(y)
\end{aligned}$$

Now we can use the Hessian determinant :

$$H = f_{xx}f_{yy} - f_{xy}f_{yx}$$
$$H = (-\sin(x) - \sin(y)) - 0$$
$$H = -1 \cdot -1 = 1$$
$$f_{xx} < 0$$

This means that the point is a local minimum.

2.5 Jacobian Matrix

The Jacobian Matrix is quite simple, it is essentially just a series of gradients with respect to different variables and functions taken across the rows of a matrix.

$$\text{Suppposef}(x, y, z) = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}$$

Then the Jacobian would be :

$$J(f) = \begin{bmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_0}{\partial y} & \frac{\partial f_0}{\partial z} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \end{bmatrix}$$

This is an annoying problem to format with LaTeX largely because duh. It's tedious and annoying. So I won't be typing any of these problems out. Will look to change up the snippet manager so that it's better at working with the tables tho. Gonna be important later.

2.6 Unit Test

The scalar field $f(x,y) = x^3 + y^3 - x - y$ has a critical point at $\left(\sqrt{3}/\right)$

3 Integrals

3.1 Line Integrals in Scalar Fields

Given a scalar field f, a parameterization α, and bounds t_0 and t_1, we can calculate the line integral as :

Parametric Integral

$$\int_C f\,ds = \int_{t_0}^{t_1} f(\alpha(t))\,||\alpha'(t)||\,dt$$

Example :

Suppose we have a scalar field

$$f(x,y) = \sin(x) + \sin(y)$$

and a curve C that is parameterized by

$$\alpha(t) = (t,t)$$

for $0 < t < \pi$. What is the line integral of f along C?

So the first thing we can do is substituting $\alpha(t)$ into $f(x,y)$

$$f(\alpha(t)) = \sin(t) + \sin(t)$$

$$f(\alpha(t)) = 2\sin(t)$$

We now will need to find the magnitude of the parameterization vector.

$$||\alpha'(t)|| = ||(1,1)|| = \sqrt{1+1} = \sqrt{2}$$

$$\int_C f\,ds = \int_0^{\pi} 2\sin(t)\sqrt{2}\,dt$$

$$\begin{aligned}\int_0^{2} 2\sin(t)\sqrt{2}\,dt &= 2\sqrt{2}\int_0^{\pi}\sin(t)\,dt\\ &= 2\sqrt{2}[-\cos(t)]_0^{\pi}\\ &= -2\sqrt{2}(-2)\\ &= 4\sqrt{2}\end{aligned}$$

Suppose we have a scalar field

$$f(x,y) = \frac{x^2}{r}$$

And a curve C that is parameterized by

$$\alpha(t) = (r\cos(t), r\sin(t))$$

For $0 < t < 2\pi$, Find the line integral of f along C.

So first we have to reference the equation :

$$\int_C f\,ds = \int_{t_0}^{t_1} f(\alpha(t))\,||\alpha'(t)||\,dt$$

$$f(\alpha(t)) = \frac{r^2\cos^2(t)}{r} = r\cos^2(t)$$

$$||\alpha'(t)|| = ||(-r\sin(t), r\cos(t))|| = \sqrt{(-r\sin(t))^2 + (r\cos(t))^2}$$

$$= \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2(\sin^2(t) + \cos^2(t))}$$

$$\sqrt{r^2(1)} = r$$

$$\int_C f\,ds = \int_0^{2\pi} r\cos^2(t) r\,dt$$

$$= r^2\int_0^{2\pi}\cos^2(t)\,dt$$

We can use the fact that

$$\cos^2(t) = \frac{1+\cos(2t)}{2}$$

$$= r^2\int_0^{2\pi}\frac{1+\cos(2t)}{2}\,dt$$

$$= \frac{r^2}{2}\int_0^{2\pi} 1+\cos(2t)\,dt$$

$$= \frac{r^2}{2}\int_0^{2\pi} 1\,dt + \frac{r^2}{2}\int_0^{2\pi}\cos(2t)\,dt$$

$$= \frac{r^2}{2}2\pi + 0$$

$$= \pi r^2$$

(Gonna have to use the align environment next time loll)

Suppose we have a scalar field

$$f(x,y) = \frac{2x+3y}{x+y}$$

and a curve C that is parameterized by

$$\alpha(t) = (-t, 2t^2)$$

From $1 < t < 2$

What is the line integral of f along C?

$$f(\alpha(t)) = \frac{-2t+6t^2}{-t+2t^2}$$

$$||\alpha'(t)|| = ||(-1, 4t)|| = \sqrt{1+16t^2}$$

$$\int_1^2 \frac{-2t+6t^2}{-t+2t^2}\sqrt{1+16t^2}\,dt$$

3.2 Line Integral Across Vector Fields

Suppose we have a vector field

$$f(x,y) = (6x^2, -\sqrt{y})$$

and a curve C that is parameterized by

$$\alpha(t) = (t, 4t^2) \text{ from } -1 < t < 1$$

What is the line integer of f along C?

$$\alpha'(t) = (1, 8t)$$
$$f(\alpha(t)) = (6t^2, -2t)$$

$$\begin{aligned}\int_{-1}^{1} (1,8t)\cdot(6t^2,-2t)\,dt &= \int_{-1}^{1} 6t^2 - 16t^2\,dt \\ &= \int_{-1}^{1} -10t^2\,dt \\ &= \left[-\frac{10}{3}t^3\right]]_{-1}^{1} \\ &= \frac{-20}{3}\end{aligned}$$

Suppose we have a vector field

$$f(x,y) = (3, x+y)$$

And a curve C that is parameterized by

$$\alpha(t) = (t, 3t+3) \text{ from } 0 < t < 1$$

What is the line integral of f along C?

$$a'(t) = (1, 3)$$
$$f(a(t)) = (3, 4t+3)$$

$$\begin{aligned}\int_{0}^{1} (1,3)(3,4t+3)\,dt &= \int_{0}^{1} 3 + 12t + 9\,dt \\ &= \int_{0}^{1} 12t + 12\,dt \\ &= [6t^2 + 12t]_0^1 \\ &= 18\end{aligned}$$

Suppose we have a vector field

$$f(x,y) = (1, 2)$$

and a curve C that is parameterized by

$$\alpha(t) = (3\cos(t), 3\sin(t)) \text{ from } 0 < t < \pi$$

Find the line integral of f along C.

So we can solve this quite easily :

$$\alpha'(t) = (-3\sin(t), 3\cos(t))$$

$$f(a(t)) = (1, 2)$$

$$\begin{aligned}\int_0^\pi (1,2)\cdot(-3\sin(t), 3\cos(t))\,dt &= \int_0^\pi -3\sin(t) + 6\cos(t)\,dx \\ &= [3\cos(t) + 6\sin(t)]_0^\pi \\ &= 3(\cos(t) + 2\sin(t))_0^\pi \\ &= 3((-1) - (1)) \\ &= 3(-2) \\ &= -6\end{aligned}$$

Suppose we have a vector field

$$f(x, y) = (\sqrt{x}, 2y)$$

And a curve C that is parameterized by

$$\alpha(t) = (t^2, 3t) \text{ from } 0 < t < 2.$$

Find the line integral of f along C.

$$\alpha'(t) = (2t, 3)$$

$$f(a(t)) = (t, 6t)$$

$$\begin{aligned}\int_0^2 (2t,3)\cdot(t,6t)\,dt &= \int_0^2 2t^2 + 18t\,dt \\ &= \left[\frac{2}{3}t^3 + 9t^2\right]_0^2 \\ &= \frac{16}{3} + 36 \\ &= \frac{124}{3}\end{aligned}$$

Define a scalar field

$$\phi(x, y) = \sin(x^3 y)$$

Let C be the peremiter of the figure formed by the two parabolas :

$$y = x^2$$

and

$$y = 2 - x^2$$

from $-1 \leq x \leq 1$, traversed twice counterclockwise.

Find the line integral of the gradient of ϕ around the curve C.

So this is going to be less calculation based,

A vector field f is conservative if $f(x, y) = \nabla g(x, y)$ for some scalar field g.

The fundamental theorem of line integrals says that when calculating a line integral around a conservative vector field, you can just evaluate the original scalar field g at the start and end points of the curve.

So we can say this :

Assuming that $f(x, y)$ is conversative (can be tested by this :)

$$f(x, y) = \nabla g(x, y)$$

$$\int_C f \cdot ds = g(C_{\text{end}}) - g(C_{\text{start}})$$

This leads to the conclusion that the line integral around a closed curve is always zero.

So since this is a line integral along a closed loop, the line integral will equal zero.

Let $\phi(x, y) = 4x^3 - 2y^4$ and C be a curve along an ellipse centered at the origin. C starts at $(-2, 0)$, oasses through (0,1), and ends at $(2, 0)$. Find the line intgral of the gradient of ϕ around the curve C

$$\int_C \nabla\phi \cdot ds =?$$

$$\nabla\phi = (12x^2 - 8y^3)$$

$$\int_C \nabla\phi \cdot ds = \phi(2, 0) - \phi(-2, 0)$$

$$4(2)^3 - 0 - (4(-2)^3) = 4(8) + 4(8) = 64$$

Let $\phi(x, y) = x^2 + e$ and C be the curve (t^2, t^2), traversed from $t = -1$ to $t = 1$.

Find the line integral of the gradient of ϕ around the curve C.

This is just another case where it equals zero, because you have a t^2 term, so $t = -1$ and $t = 1$ means that you have a curve that starts and ends at $(1, 1)$. This means that your line integral is equal to 0.

Let

$$\phi(x, y) = x^2 + y^2$$

and

$$C = (t^4 - 2t^2, -t^3)$$

Traversed from $t = 1$ to $t = 2$.

Find the line integral of the gradient of ϕ around the curve

$$C_{\text{start}} = (-1, -1) || C_{\text{end}} = (8, -8)$$

$$\int_C \nabla\phi \cdot ds = \phi(8, -8) - \phi(-1, -1)$$

$$64 + 64 - (1 + 1) = 126$$

$$f(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath}$$

$$\text{Let } P(x, y) = \ln(y) \text{ and } Q(x, y) = 4x + 1$$

$$P_{\text{y}} =??$$

$$Q_{\text{x}} =??$$

Is f a conservative vector field?

Remember that vector fields are only conservative if there is a scalar field ϕ such that $\nabla\phi = f$. This is only possible when $\phi_{\text{xy}} = \phi_{\text{yx}}$

In this equation this essentially would mean that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

We can pretty clearly see here how, $\frac{\partial p}{\partial y} = \frac{1}{4}$ and $\frac{\partial Q}{\partial x} = 4$

$$f(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$
$$\text{Let } P(x,y) = e^{x-y} + 1 \text{ and } Q(x,y) = -e^{x-y}$$
$$P_y = -e^{x-y}$$
$$Q_x = -e^{x-y}$$

f is a conservative vector field.

$$f(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$
$$\text{Let } P(x,y) = e^{\cos(y)} \text{ and } Q(x,y) = e^{\cos(x)}$$
$$P_y = -\sin(y)e^{\cos(y)}$$
$$Q_x = -\sin(x)e^{\cos(x)}$$

f is not a conservative vector field.

$$f(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$
$$\text{Let } P(x,y) = x^3y^2 \text{ and } Q(x,y) = \frac{x^4y}{2}$$
$$P_y = 2x^3y$$
$$Q_x = 2x^3y$$

This vector field is conversative.

3.3 Finding potential functions

$$f(x,y) = e^y, -e^{-y} + xe^y$$

Find F such that $f = \nabla F$.

So we are going to have to find the two parts and concatenate them.

$$F_x = e^y || F_y = -e^{-y} + xe^y$$

$$\begin{aligned} F &= \int F_x \, dx \\ &= \int e^y \, dx \\ &= xe^y + C_1(y) \end{aligned}$$

$$\begin{aligned} F &= \int F_{\text{y}} \, dy \\ &= \int -e^{-y} + xe^y \, dy \\ &= e^{-y} + xe^y + C_2(x) \end{aligned}$$

Now because f is conservative, we can set these two integrals equal to each other :

$$xe^y + C_1(y) = e^{-y} + xe^y + C_2(x)$$

So we can now observe quite clearly that C_2 is just a numerical constant C, and that C_1 is $e^{-y} + C$. We can now put everything together and get :

$$F(x,y) = e^{-y} + ce^y + C$$

$$f(x,y) = \left(16x^3y^2 + \frac{1}{2}, 8x^4y\right)$$

Find F such that $f = \nabla F$

Here we will break the two parts up and integrate them seperately.

$$F_x = 16x^3y^2 + \frac{1}{2} || F_y = 8x^4y$$

$$\begin{aligned}
F &= \int F_x\, dx \\
&= \int 16x^3y^2 + \frac{1}{2}\, dx \\
&= 4x^4y^2 + \frac{x}{2} + C_1 \\
F &= \int F_y\, dy \\
&= \int 8x^4y\, dy \\
&= 4x^4y^2 + C_2
\end{aligned}$$

So we now need to equate these two functions :

$$4x^4y^2 + \frac{x}{2} + C_1 = 4x^4y^2 + C_2$$

We can see that C_1 is just a numerical constant C, and C_2 is $\frac{x}{2} + C$. Making these substitions we can get our final answer of :

$$F = 4x^4y^2 + \frac{x}{2} + C$$

$$f(x,y) = \left(5\sqrt{y}, \frac{5x}{2\sqrt{y}} + 1\right)$$

Find F such that $f = \nabla F$

$$\begin{aligned}
F &= \int F_x\, dx \\
&= \int 5\sqrt{y}\, dx \\
&= 5x\sqrt{y} + C_1 \\
F &= \int \frac{5x}{2\sqrt{y}} + 1\, dy \\
&= 5x\sqrt{y} + y + C_2
\end{aligned}$$

Setting these two values equal to each other :

$$5x\sqrt{y} + C_1 = 5x\sqrt{y} + y + C_2$$

You can see here that $C_2 = C$, and $C_1 = y + C$

$$f(x, y) = (2xy^{-2}, -2x^2y^{-3})$$

Find F such that $f = \nabla F$.

$$\begin{aligned}
F &= \int F_x\, dx \\
&= 2xy^{-2} \\
&= x^2y^{-2} \\
&= \frac{x^2}{y^2} \\
F &= \int F_y\, dy \\
&= \int -2x^2y^{-3}\, dy \\
&= x^2y^{-2} \\
&= \frac{x^2}{y^2}
\end{aligned}$$

3.3.1 Quiz

Suppose we have a scalar field $f(x, y) = x^2$ and a curve C that is parameterized by $\alpha(t) = (6t, t)$ for $-1 < t < 1$. What is the line integral of f along C?

What is the line integral of f along C?

$$f(\alpha(t)) = 36t^2$$

$$||\alpha'(t)|| = ||(6, 1)|| = \sqrt{37}$$

$$\begin{aligned}
\int_C f\, ds &= \int_{-1}^{1} \sqrt{37}(36t^2)\, dt \\
&= \sqrt{37} \int_{-1}^{1} 36t^2\, dt \\
&= \sqrt{37} \left[12t^3\right]_{-1}^{1} \\
&= 24\sqrt{37}
\end{aligned}$$

Suppose we have a vector field $f(x, y) = (-5, y\sin(x))$ and a curve C that is parameterized by $\alpha(t) = (3t, 1)$ for $-3 < t < 1$.

$$f(\alpha(t)) = (-5, \sin(3t))$$

$$||\alpha'(t)|| = ||(3, 0)|| = 3$$

$$\begin{aligned}\int_C f\,ds = \int_{-3}^{1} (-5, \sin(3t)) \cdot (3,0)\,dt &= \int_{-3}^{1} -15\,dt \\ &= [-15t]_{-3}^{1} \\ &= -15[t]_{-3}^{1} \\ &= -15(4) \\ &= -60\end{aligned}$$

$$f(x,y) = (y\cos(xy)+1, x\cos(xy)+1)$$

Find F such that $f = \nabla F$.

$$\begin{aligned}F_x &= \int y\cos(xy)+1\,dx \\ &= \sin(xy)+x+C_1 \\ F_y &= \int x\cos(xy)+1\,dy \\ &= \sin(xy)+y+C_2\end{aligned}$$

We can see here that that $C_1 = y + C$ and $C_2 = x + C$ Which means that we end up with the expression :

$$F(x,y) = x + y + \sin(xy) + C$$

$$f(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

$$\text{Let } P(x,y) = x^2y+3 \text{ and } Q(x,y) = xy^2$$

$$P_y = x^2$$

$$Q_x = y^2$$

This means that f is not a conservative vector field.

3.4 Double Integrals

Evaluate the below integral :

$$\int_{-1}^{2}\left(\int_{0}^{2} x^3 e^y\,dx\right)dy$$

$$\int_{-1}^{2} e^y \left(\left[\frac{x^4}{4}\right]_0^2\right)dy$$

$$\int_{-1}^{2} 4e^y\,dy$$

$$4\left[e^y\right]_{-1}^{2}$$

$$4e^2 - 4e^{-1} \text{ or } 4(e^2 - e^{-1})$$

Evaluate the iterated integral:

$$\int_0^1 \left(\int_e^{e^5} \frac{x^2}{y}\, dy \right) dx$$

$$\int_0^1 x^2 \left(\int_e^{e^5} \frac{1}{y}\, dy \right) dx$$

$$\int_0^1 x^2 \left(\ln(e^5) - \ln(e) \right) dx$$

$$4 \int_0^1 x^2\, dx$$

$$\frac{4}{3}$$

Evaluate the iterated integral :

$$\int_{-2}^1 \left(\int_0^3 2x^2 + y^2 - 8xy^3\, dx \right) dy$$

$$\int_{-2}^1 \left[\frac{2}{3}x^3 + y^2 x - 4x^2 y^3 \right]_{x=0}^{x=3} dy$$

$$\int_{-2}^1 \left(18 + 3y^2 - 36y^3 \right) dy$$

$$\left[18y + y^3 - 9y^4 \right]_{-2}^1$$

$$(18 + 1 - 9) - (-36 - 8 - 144)$$

$$198$$

Evaluate the iterated integral :

$$\int_0^\pi \left(\int_0^{\frac{\pi}{2}} \cos(x)\sin(y)\, dx \right) dy$$

$$\int_0^\pi \sin(y)[\sin(x)]_0^{\frac{\pi}{2}}\, dy$$

$$\int_0^\pi \sin(y)(1)\, dy$$

$$[-\cos(y)]_0^\pi$$

$$-(\cos(\pi) - \cos(0))$$

$$-(-1 - 1)$$

$$2$$

Evaluate the iterated integral:

$$\int_{-1}^1 \left(\int_{-2}^2 e^x + e^y\, dy \right) dx$$

$$\int_{-1}^{1} [ye^x + e^y]_{-2}^{2}\, dx$$
$$\int_{-1}^{1} (2e^x + e^2) - (-2e^x + e^{-2})\, dx$$
$$\int_{-1}^{1} 4e^x + e^2 - e^{-2}\, dx$$
$$\left[4e^x + xe^2 - xe^{-2}\right]_{-1}^{1}$$
$$(4e + e^2 - e^{-2}) - (4e^{-1} - e^2 + e^{-2})$$
$$4e - 4e^{-1} + 2e^2 - 2e^{-2}$$

Evaluate the double integral

$$\int_0^2 \int_{-y}^{y} x - 4y\, dx\, dy$$
$$\int_0^2 \left[\frac{x^2}{2} - 4xy\right]_{-y}^{y} dy$$
$$\int_0^2 \left(\frac{y^2}{2} - 4y^2\right) - \left(\frac{y^2}{2} + 4y^2\right) dy$$
$$\int_0^2 -8y^2\, dy$$
$$\left[-\frac{8}{3}y^3\right]_0^2$$
$$-\frac{64}{3}$$

3.5 Triple Integrals

Evaluate

$$\int_{-2}^{0} \int_{1}^{4} \int_{0}^{-x} 4z - 2y\, dz\, dx\, dy$$
$$\int_{-2}^{0} \int_{1}^{4} \left[2z^2 - 2yz\right]_{0}^{-x} dx\, dy$$
$$\int_{-2}^{0} \int_{1}^{4} 2x^2 + 2yx\, dx\, dy$$
$$\int_{-2}^{0} \left[\frac{2}{3}x^3 + yx^2\right]_{1}^{4} dy$$
$$\int_{-2}^{0} \left(\frac{128}{3} + 16y\right) - \left(\frac{2}{3} + y\right) dx$$
$$\int_{-2}^{0} (42 + 15y)\ dy$$
$$\left[42y + \frac{15}{2}y^2\right]_{-2}^{0}$$

$$0 - (-84 + 15 \cdot 2)$$

$$54$$

Suppose we have a function

$$f(x, y) = 2y - xy$$

We have a change of variables:

$$x = X_1(u, v) = -2u - 5v$$

$$y = X_2(u, v) = 3u - 2v$$

So what is $f(x, y)$ under the change of variables?

$$f(x, y) = 2(3u - 2v) - (-2u - 5v)(3u - 2v)$$

$$f(x, y) = 6u - 4v - (-6u^2 - 11uv + 10v^2)$$

$$f(x, y) = 6u^2 + 11uv - 10v^2 + 6u - 4v$$

Let C be the line defined by $y = -\frac{x}{2} + 1$
We have a change of variables:

$$x = X_1(u, v) = \frac{u}{3} + \frac{v}{2}$$

$$f(x, y) = \frac{u^2}{4} - \left(\frac{u^2}{4} - \frac{uv}{4} + \frac{v^2}{4}\right)$$

$$f(x, y) = \frac{uv}{4} - \frac{v^2}{4}$$
