Personal project

Longstaff & Schwartz Algorithm for valuing American Options

MASSIN Keryann

May 7, 2024

Contents

1 Introduction

The need for efficient pricing methods for highly exotic derivatives has led to the development of a number of computation methods, among which the Longstaff&Schwartz method stands as a widely used one.

The goal of this project is to reproduce the results from the article.

2 Finite Differences Pricer

2.1 European 1D use-case

Establishing the PDE Let us consider a derivative $\mathcal{D}$ which delivers a payoff $\varphi(X_T)$ at maturity $T > 0$, where (X_t) follows a Black Scholes diffusion:

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad X_0 = x_0 > 0 \tag{1}$$

We wish to find the price today π_0 of this derivative in a complete market with constant interest rate $r > 0$. The No Arbitrage Principle in this market ensures that the process of discounted price $(e^{-rt}\pi_t)_{t\in[0,T]}$ is a martingale.

$$\begin{aligned} e^{rt}\,\mathrm{d}\big(e^{-rt}\pi_t\big) &= e^{rt}\cdot\big(-re^{-rt}\pi_t\,\mathrm{d}t + e^{-rt}\,\mathrm{d}\pi_t\big) = -r\pi_t\,\mathrm{d}t + \mathrm{d}\pi_t \\ &= -r\pi_t\,\mathrm{d}t + \partial_t\pi_t\,\mathrm{d}t + \partial_x\pi_t\,\mathrm{d}X_t + \frac{1}{2}\sigma^2\partial^2_{x,x}\pi_t\,\mathrm{d}t \\ &= \left\{\partial_t\pi_t + rX_t\partial_x\pi_t + \frac{1}{2}\sigma^2\partial^2_{x,x}\pi_t - r\pi_t\right\}\mathrm{d}t + \sigma\partial_x\pi_t\,\mathrm{d}W_t \end{aligned}$$

The drift being null, the pricing function $\pi_t := u(t, X_t)$ satisfies the following Partial Differential Equation (PDE):

$$\begin{cases} \partial_t u + rx\partial_x u + \frac{1}{2}\sigma^2x^2\partial^2_{x,x}u - ru = 0, & \forall t\in[0,T],\ \forall x \\ u(T,x) = \varphi(x), & \forall x \end{cases} \tag{2}$$

Notice that (2) is a backward PDE. To ease the computations we shall make it forward, by setting $v(t,x) := u(T-t,x)$:

$$\begin{cases} \partial_t v - rx\partial_x v - \frac{1}{2}\sigma^2x^2\partial^2_{x,x}v + rv = 0, & \forall t\in[0,T],\ \forall x \\ v(0,x) = \varphi(x), & \forall x \end{cases} \tag{3}$$

Finding π_0 (with $X_0 = x$ known) therefore amounts to solving (3) to find $v(T,x)$.

Finite Scheme To solve (3), we will consider that x takes its value in a finite interval $I := [x_{\min}, x_{\max}]$. Plus, we introduce boundary conditions v_ℓ and v_r, so that (3) becomes:

$$\begin{cases} \partial_t v + \mathcal{A}v = 0, & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}) \\ v(t, x_{\min}) = v_\ell(t), & \forall t \in (0,T] \\ v(t, x_{\max}) = v_r(t), & \forall t \in (0,T] \\ v(0,x) = \varphi(x), & \forall x \in (x_{\min}, x_{\max}) \end{cases} \tag{4}$$

where

$$\mathcal{A}v := -rx\partial_x v - \frac{1}{2}\sigma^2 x^2 \partial^2_{x,x} v + rv$$

To discretize (4), we consider a space mesh of size $I+1 > 0$ and a time mesh of size $N+1 > 0$:

$$\begin{cases} h & := (x_{\max} - x_{\min})/I \\ \Delta t & := N/T \\ x_i & := x_{\min} + i \cdot h, \quad \forall i = 0 \ldots, I \\ t_n & := n \cdot \Delta t, \quad \forall n = 0, \ldots, N \end{cases}$$

We associate to v the matrix $U := (U_j^n)$ where $U_j^n = u(t_n, x_j)$. Let us denote by A the discretization matrix associated to the operator $\mathcal{A}$, such that for any vector V:

$$\begin{aligned} (AV + q)_j &= -rx_j \frac{V_{j+1} - V_{j-1}}{2h} - \frac{1}{2}\sigma^2 x_j^2 \frac{V_{j-1} - 2V_j + V_{j+1}}{h^2} + rV_j \\ &= (-\alpha_j + \beta_j)V_{j-1} + (2\alpha_j + r)V_j + (-\alpha_j - \beta_j)V_{j+1} \end{aligned}$$

where $\alpha_j = \frac{\sigma^2 x_j^2}{2h^2}$ and $\beta_j = \frac{rx_j}{2h}$. The term q takes into account boundary conditions (as V_{-1} and V_{I+1} are not defined). Thus, A is the tridiagonal matrix $\text{tridiag}(-\alpha_j + \beta_j, 2\alpha_j + r, -\alpha_j - \beta_j)$ and

$$q(t) = \begin{pmatrix} (-\alpha_1 + \beta_1)v_\ell(t) \\ 0 \\ \vdots \\ 0 \\ (-\alpha_I - \beta_I)v_r(t) \end{pmatrix}$$

Let g be the vector of components $g_j := \varphi(x_j)$ (the initial condition).

With all notations defined above, we can finally discretize (4) with the Euler Implicit Scheme, that consists in finding U^{n+1} such that

$$\frac{U^{n+1} - U^n}{\Delta t} + AU^{n+1} + q(t_{n+1}) = 0, \quad \forall n = 0, \ldots, N-1 \tag{5}$$

The scheme is initialized by setting $U^0 = g$. At last, the vector U^N will contain today price of derivative $\mathcal{D}$ for each starting price x in the space mesh.

2.2 American 1D use-case

Stochastic control problem Assume that a derivative $\mathcal{D}'$ delivers a payoff $\varphi(X_t)$ at time $t > 0$. This derivative is american, as it can be exercised at any time $t \in [0, T]$. This means that its today price can be expressed as

$$\pi'_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[e^{-r\tau}\varphi(X_\tau)]$$

where $\mathcal{T}_{0,T}$ is the set of stopping times with values in $[0, T]$. We are faced with a problem of Stochastic Control. The value function is defined by

$$v(t, x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)}\varphi(X_\tau^{t,x})\right] \tag{6}$$

with $(X_s^{t,x})_{s\in[t,T]}$ a solution to (1) with initial condition $X_t = x$.
Our goal is to find $v(0, x) = \pi'_0$. From (6), we get $v(t, x) \geq \varphi(x)$ for all $t \in [0, T]$ and all x. From the dynamic programming principle (DPP), the value function writes

$$v(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \leq \theta} + e^{-r(\theta - t)}v(\theta, X_\theta^{t,x})\mathbf{1}_{\tau > \theta}\right] \tag{7}$$

for any stopping time $\theta \in \mathcal{T}_{t,T}$.
From (7), one can deduce that $\tilde{v}(t, x) := v(T - t, x)$ is a solution to

$$\begin{cases} \min\left(\partial_t \tilde{v} - rx \cdot \partial_x \tilde{v} - \dfrac{1}{2}\sigma^2 x^2 \cdot \partial_x^2 \tilde{v} + r\tilde{v}, \tilde{v} - \varphi\right) = 0 \\ \tilde{v}(0, x) = \varphi(x) \end{cases} \tag{8}$$

Proof of (8) can be found in annexe.

Finite scheme In view of similarities between (3) and (8), we define

$$\begin{cases} \min\left(\partial_t v + \mathcal{A}v, v - \varphi\right) = 0, & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}) \\ v(t, x_{\min}) = v_\ell(t), & \forall t \in (0,T] \\ v(t, x_{\max}) = v_r(t), & \forall t \in (0,T] \\ v(0,x) = \varphi(x), & \forall x \in (x_{\min}, x_{\max}) \end{cases} \tag{9}$$

We then consider the backward differentiation formula (BDF) scheme, defined as follow: we initialize $U^0 = g$ and compute U^1 using the Euler Implicit scheme. Finally, for $n = 1, \dots, N-1$, we compute U^{n+1} such that

$$\min\left(\frac{3U^{n+1} - 4U^n + U^{n-1}}{2\Delta t} + AU^{n+1} + q(t_{n+1}), U^{n+1} - g\right) = 0 \tag{10}$$

3 Conclusion and perspectives

Bite.

References

A Proofs

A.1 Variational inequality 1D

We wish to prove (8). Applying Itô's formula to v, we get:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \partial_t v(s, X_s^{t,x})\,\mathrm{d}s + \int_t^{t+h} D_x v(s, X_s^{t,x})\,\mathrm{d}X_s + \frac{1}{2}\int_t^{t+h} D_x^2 v(t, X_s^{t,x})\,\mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s$$

Let's assume general settings $\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$ where (X_t) is n-dimensional and (W_t) is a m-dimensional Brownian motion. Notice that σ takes vectors $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ as input and outputs a matrix of size $n \times m$. Therefore, $\mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s = \sigma\sigma^T(s, X_s^{t,x})$ and we can rewrite the expression above:

$$\begin{aligned} v(t+h, X_{t+h}^{t,x}) &= v(t,x) + \int_t^{t+h} \left(\partial_t v + b \cdot D_x v + \frac{1}{2}\mathrm{tr}(D_x^2 v \cdot \sigma\sigma^T)\right)(s, X_s^{t,x})\,\mathrm{d}s \\ &\quad + \int_t^{t+h} \left(D_x v \cdot \sigma\right)(s, X_s^{t,x})\,\mathrm{d}W_s \\ &=: v(t,x) + \int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s + M \end{aligned}$$

where $\mathcal{L}$ is the drift differential operator and M is the local martingale part of the expression above.
From the DPP of equation (7) with $\theta = t+h$, we obtain

$$v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} + e^{-rh} v(t+h, X_{t+h}^{t,x})\mathbf{1}_{\tau > t+h}\right]$$

Thus:

$$\begin{aligned} v(t,x) &= \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\Big[e^{-r(\tau - t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} + e^{-rh} v(t,x)\mathbf{1}_{\tau > t+h} \\ &\quad + e^{-rh}\int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s\mathbf{1}_{\tau > t+h} + e^{-rh} M\mathbf{1}_{\tau > t+h}\Big] \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} e^{-rh} v(t,x)\mathbf{1}_{\tau > t+h} + \mathbb{E}\Big[e^{-r(\tau - t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} \\ &\quad + e^{-rh}\int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s\mathbf{1}_{\tau > t+h}\Big] \end{aligned}$$

Assuming the sup on the right-hand side is reached by τ^*, we have:

$$\begin{aligned}(1-e^{-rh}\mathbf{1}_{\tau^*>t+h})v(t,x) = \mathbb{E}\Big[& e^{-r(\tau^*-t)}\varphi(X^{t,x}_{\tau^*})\mathbf{1}_{\tau^*\le t+h} \\ & + e^{-rh}\int_t^{t+h} \mathcal{L}v(s,X^{t,x}_s)\,\mathrm{d}s\,\mathbf{1}_{\tau^*>t+h}\Big]\end{aligned}$$

At this point either $\tau^*=t$, and then $v(t,x)=\varphi(x)$ or $\tau^*>t$, so by sending h to 0 we eventually get $\tau^*>t+h>t$, and

$$\frac{1-e^{-rh}}{h}v(t,x) = \mathbb{E}\left[\frac{e^{-rh}}{h}\int_t^{t+h}\mathcal{L}v(s,X^{t,x}_s)\,\mathrm{d}s\right] \quad \Rightarrow \quad rv(t,x) = \mathcal{L}v(t,x)$$

Either way, on $[0,T)\times\mathbb{R}^n$:

$$\begin{aligned} & \min\left(rv-\mathcal{L}v, v-\varphi\right) = 0 \\ \Leftrightarrow \quad & \min\left(-\partial_t v - b\cdot D_x v - \frac{1}{2}\mathrm{tr}(D^2_x v\cdot\sigma\sigma^T) + rv, v-\varphi\right) = 0\end{aligned}$$

In the case of the Black Scholes diffusion of (1), this translates to

$$\min\left(-\partial_t v - rx\cdot\partial_x v - \frac{1}{2}\sigma^2x^2\cdot\partial^2_x v + rv, v-\varphi\right) = 0$$

Performing the transformation $\tilde{v}(t,x) := v(T-t,x)$ leads to

$$\begin{cases} \min\left(\partial_t\tilde{v} - rx\cdot\partial_x\tilde{v} - \frac{1}{2}\sigma^2x^2\cdot\partial^2_x\tilde{v} + r\tilde{v}, \tilde{v}-\varphi\right) = 0 \\ \tilde{v}(0,x) = \varphi(x)\end{cases}$$