Personal project

Longstaff & Schwartz Algorithm for valuing American Options

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1 Introduction

One of the hottest topic in financial engineering concerns the pricing of derivatives. From the simple european call option on a stock to the cancelable index amortizing swap in a multifactor term structure model, the variety of derivatives that modern pricing methods need to address is...

2 Finite Differences Pricer

In this section, we leverage the Stochastic Differential Equations (SDE) framework to price non-path dependent options over a one-dimensional underlying.

Let us consider a derivative $\mathcal{D}$ which delivers a payoff $\varphi(X_T)$ at maturity $T > 0$, where (X_t) follows a Dupire's local volatility diffusion:

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma(t, X_t)X_t\,\mathrm{d}W_t, \quad X_0 = x_0 > 0 \tag{1}$$

$(\pi_t)_{t\in[0,T]}$ will denote the pricing process in the sequel. We wish to find the price today π_0 of this derivative in a complete market with constant interest rate $r > 0$. Note that taking a constant function $\sigma(t, X_t) = \sigma > 0$ in (1) directly produces the Black-Scholes model.
In section 2.1, we assume that $\mathcal{D}$ is an European option. We relax this assumption in section 2.2, where $\mathcal{D}$ is assumed to be an American option.

2.1 European option

Establishing the PDE The No Arbitrage Principle in this market ensures that the process of discounted price $(e^{-rt}\pi_t)_{t\in[0,T]}$ is a martingale. By Itô's formula, one gets:

$$\begin{aligned}
e^{rt}\,\mathrm{d}\big(e^{-rt}\pi_t\big) &= e^{rt}\cdot\big(-re^{-rt}\pi_t\,\mathrm{d}t + e^{-rt}\,\mathrm{d}\pi_t\big) = -r\pi_t\,\mathrm{d}t + \mathrm{d}\pi_t \\
&= -r\pi_t\,\mathrm{d}t + \partial_t\pi_t\,\mathrm{d}t + \partial_x\pi_t\,\mathrm{d}X_t + \frac{1}{2}\sigma^2(t, X_t)\partial^2_{x,x}\pi_t\,\mathrm{d}t \\
&= \left\{\partial_t\pi_t + rX_t\partial_x\pi_t + \frac{1}{2}\sigma^2(t, X_t)\partial^2_{x,x}\pi_t - r\pi_t\right\}\mathrm{d}t \\
&\quad + \sigma(t, X_t)\partial_x\pi_t\,\mathrm{d}W_t
\end{aligned}$$

The drift being null, the pricing function $\pi_t := u(t, X_t)$ satisfies the following Partial Differential Equation (PDE):

$$\begin{cases} \partial_t u + rx\partial_x u + \frac{1}{2}\sigma^2(t,x)x^2\partial^2_{x,x}u - ru = 0, & \forall t \in [0,T],\ \forall x \\ u(T,x) = \varphi(x), & \forall x \end{cases} \tag{2}$$

Notice that (2) is a backward PDE. To ease the computations we shall make it forward, by setting $v(t,x) := u(T-t,x)$:

$$\begin{cases} \partial_t v - rx\partial_x v - \frac{1}{2}\sigma^2(T-t,x)x^2\partial^2_{x,x}v + rv = 0, & \forall t \in [0,T],\ \forall x \\ v(0,x) = \varphi(x), & \forall x \end{cases} \tag{3}$$

Finding π_0 (with $X_0 = x$ known) therefore amounts to solving (3) to find $v(T,x)$.

Finite Scheme To solve (3), we will consider that x takes its value in a finite interval $I := [x_{\min}, x_{\max}]$. Plus, we introduce boundary conditions v_ℓ and v_r, so that (3) becomes:

$$\begin{cases} \partial_t v + \mathcal{A}v = 0, & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}) \\ v(t, x_{\min}) = v_\ell(t), & \forall t \in (0,T] \\ v(t, x_{\max}) = v_r(t), & \forall t \in (0,T] \\ v(0,x) = \varphi(x), & \forall x \in (x_{\min}, x_{\max}) \end{cases} \tag{4}$$

where

$$\mathcal{A}v(t,x) := -rx\partial_x v(t,x) - \frac{1}{2}\sigma^2(T-t,x)x^2\partial^2_{x,x}v(t,x) + rv(t,x)$$

To discretize (4), we consider a space mesh of size $I+1 > 0$ and a time mesh of size $N+1 > 0$:

$$\begin{cases} h & := (x_{\max} - x_{\min})/I \\ \Delta t & := N/T \\ x_i & := x_{\min} + i \cdot h, \quad \forall i = 0 \dots, I \\ t_n & := n \cdot \Delta t, \quad \forall n = 0, \dots, N \end{cases}$$

We associate to v the matrix $U := (U_j^n)$ where $U_j^n = v(t_n, x_j)$. Let us denote by $A(t)$ the discretization matrix associated to the operator $v \mapsto \mathcal{A}v(t,\cdot)$, such that for any vector V:

$$\begin{aligned} (A(t)V + q(t))_j &= -rx_j\frac{V_{j+1} - V_{j-1}}{2h} - \frac{1}{2}\sigma^2(T-t,x_j)x_j^2\frac{V_{j-1} - 2V_j + V_{j+1}}{h^2} + rV_j \\ &= (-\alpha_j + \beta_j)V_{j-1} + (2\alpha_j + r)V_j + (-\alpha_j - \beta_j)V_{j+1} \end{aligned}$$

where $\alpha_j = \frac{\sigma^2(T-t,x_j)x_j^2}{2h^2}$ and $\beta_j = \frac{rx_j}{2h}$. The term $q(t)$ takes into account boundary conditions (as V_{-1} and V_{I+1} are not defined). Thus, $A(t)$ is the tridiagonal matrix tridiag$(-\alpha_j+\beta_j, 2\alpha_j+r, -\alpha_j-\beta_j)$ and

$$q(t) = \begin{pmatrix} (-\alpha_1+\beta_1)v_\ell(t) \\ 0 \\ \vdots \\ 0 \\ (-\alpha_I-\beta_I)v_r(t) \end{pmatrix}$$

Let g be the vector of components $g_j := \varphi(x_j)$ (the initial condition).

With all notations defined above, we can finally discretize (4) with the Euler Implicit Scheme, that consists in finding U^{n+1} such that

$$\frac{U^{n+1}-U^n}{\Delta t} + A(t_{n+1})U^{n+1} + q(t_{n+1}) = 0, \quad \forall n = 0, \dots, N-1 \tag{5}$$

The scheme is initialized by setting $U^0 = g$. At last, the vector U^N will contain today price of derivative $\mathcal{D}$ for each starting price x in the space mesh. A linear interpolation may provide prices of $\mathcal{D}$ for in-between x:

$$\pi_0 \mid \{X_0 = x\} \approx \frac{x_{i+1}-x}{h} \cdot U_i^N + \frac{x-x_i}{h} \cdot U_{i+1}^N$$

where $x_i \leq x < x_{i+1}$.

2.2 American option

Stochastic control problem Assume that the derivative $\mathcal{D}$ could deliver its payoff $\varphi(X_t)$ at $t \geq 0$ (not only at $t = T$). This derivative is said to be american, it can be exercised at any time $t \in [0,T]$. This means that its today price can be expressed as

$$\pi_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}[e^{-r\tau}\varphi(X_\tau)] \tag{6}$$

where $\mathcal{T}_{0,T}$ is the set of stopping times with values in $[0,T]$. We are faced with a problem of Stochastic Control. The value function is defined by

$$v(t,x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\varphi(X_\tau^{t,x})\right] \tag{7}$$

with $(X_s^{t,x})_{s\in[t,T]}$ a solution to (1) with initial condition $X_t = x$.
Our goal is to find $v(0,x) = \pi_0$. From (7), we get $v(t,x) \geq \varphi(x)$ for all $t \in [0,T]$ and all x. From the dynamic programming principle (DPP), the value function writes

$$v(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau\leq\theta} + e^{-r(\theta-t)}v(\theta, X_\theta^{t,x})\mathbf{1}_{\tau>\theta}\right] \tag{8}$$

for any stopping time $\theta \in \mathcal{T}_{t,T}$.
From (8), one can deduce that $\tilde{v}(t,x) := v(T-t,x)$ is a solution to

$$\begin{cases} \min\left(\partial_t\tilde{v} - rx\cdot\partial_x\tilde{v} - \frac{1}{2}\sigma^2(T-t,x)x^2\cdot\partial_x^2\tilde{v} + r\tilde{v}, \tilde{v}-\varphi\right) = 0 \\ \tilde{v}(0,x) = \varphi(x) \end{cases} \tag{9}$$

Proof of (9) can be found in annexe.

Finite Scheme In view of similarities between (3) and (9), we define

$$\begin{cases} \min\left(\partial_t v + \mathcal{A}v, v-\varphi\right) = 0, & \forall t\in(0,T],\ \forall x\in(x_{\min},x_{\max}) \\ v(t,x_{\min}) = v_\ell(t), & \forall t\in(0,T] \\ v(t,x_{\max}) = v_r(t), & \forall t\in(0,T] \\ v(0,x) = \varphi(x), & \forall x\in(x_{\min},x_{\max}) \end{cases} \tag{10}$$

We then consider the backward differentiation formula (BDF) scheme, defined as follow: we initialize $U^0 = g$ and compute U^1 using the Euler Implicit scheme. Finally, for $n = 1,\ldots,N-1$, we compute U^{n+1} such that

$$\min\left(\frac{3U^{n+1} - 4U^n + U^{n-1}}{2\Delta t} + A(t_{n+1})U^{n+1} + q(t_{n+1}), U^{n+1} - g\right) = 0 \tag{11}$$

3 Longstaff & Schwartz Pricer

The SDE framework demonstrates high efficiency in the computation of vanilla option prices. However, as soon as the considered derivative becomes more complex (path-dependent, or even just with more than one underlying), finite differences methods are easily outclassed by Monte Carlo based techniques. In this context, [1] presents a Least-square Monte Carlo algorithm aimed at valuing highly exotic american options. We dedicate this section to the presentation of this algorithm and its implementation.

3.1 How it works

Unlike a European option, an American option can be exercised at any time before its maturity. Recalling (6), the price of an American option $\mathcal{D}$ that pays $\varphi(X_s)$ for $s \in [0, T]$ can be expressed at time t as

$$\pi_t = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}[e^{-r(\tau - t)}\varphi(X_\tau) \mid X_t] \tag{12}$$

where $(X_t)_{t\in[0,T]}$ is assumed to follow a markovian diffusion (which allows us to replace the Brownian filtration $\mathcal{F}_t$ by X_t in the conditioning).

π_t can be approximated by solving a discretized dynamic programming problem defined recursively as

$$\begin{cases} \bar{\pi}_{t_{N_{\text{steps}}}} = e^{-rT}\varphi(\bar{X}_T) \\ \bar{\pi}_{t_k} = \max\left\{e^{-rt_k}\varphi(\bar{X}_{t_k}),\ \mathbb{E}[\bar{\pi}_{t_{k+1}} \mid \bar{X}_{t_k}]\right\} \end{cases} \tag{13}$$

given a time grid $\{0 = t_0 < t_1 < \cdots < t_{N_{\text{steps}}} = T\}$.

The objective of the Longstaff & Schwartz (L&S) algorithm is to model the continuation value $C_{t_k} := \mathbb{E}[\bar{\pi}_{t_{k+1}} \mid \bar{X}_{t_k}]$ efficiently. This is done by approximating C_{t_k} with a linear combination of a finite set of real functions $(\psi_m)_{1\leq m\leq M}$:

$$C_{t_k} \approx \sum_{m=1}^{M} \alpha_{k,m}\psi_m(\bar{X}_{t_k}), \quad \forall k = 0, 1, \ldots, N_{\text{steps}}$$

Coefficients $(\alpha_{k,m})_{1\leq m\leq M}$ are obtained using the least squares method (the basis functions are the regressors) on a set of N_{paths} Monte Carlo samples. Note that only paths where the option $\mathcal{D}$ is in the money at t_{k+1} (i.e $\varphi(\bar{X}_{t_{k+1}}) > 0$) are considered.
In practice, functions ψ_m often consist in a polynomial basis. This choice is backed up by the Weierstrass theorem that ensures convergence of a sequence of polynomials to any wanted function over a compact set. In [1], authors argue that the weighted Laguerre polynomials, defined as

$$L_n(X) = e^{-X/2}\frac{e^X}{n!}\frac{\mathrm{d}^n}{\mathrm{d}X^n}(X^n e^{-X}), \quad \forall n \geq 0 \tag{14}$$

are a suitable choice of basis. This comes from the fact that for $n \neq m$,

$$\int_0^{+\infty} L_n(x) L_m(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_0^{+\infty} L_n(x)^2\,\mathrm{d}x > 0 \tag{15}$$

Overall, the L& S algorithm consists in a two steps procedure: first generate N_{paths} Monte Carlo paths, each of them between $t = 0$ and $t = T$; second, compute for each time step t_k the value $\bar{\pi}_{t_k}$ via a Least Square minimization problem. At last, $\bar{\pi}_0$ should be a fair approximation of π_0. The accuracy of the algorithm depends on multiple factors, such as the number M of regressors used to approximate the continuation value C_{t_k}, or the number N_{paths} of Monte Carlo paths generated. Usually, $M \in \{3, 4, 5, 6\}$ and $N_{\text{paths}} = 1,000,000$ deliver accurate results.

3.2 Miscellaneous

About European options In the case of a Bermudean or an European option, only certain time steps are exercisable, hence it shall be necessary to distinguish two time grids: one for the Monte Carlo paths, and one for the exercisable times. In (13), only time steps from the exercisable mesh should be used. In the case of American options, these two grids are the same.

Non Markovian settings Although this would dramatically increase the time of computations, there is no theoretical constraints on whether the diffusion of $(X_t)_{t\in[0,T]}$ should be markovian or not. The algorithm adapts well even in the worst cases.

A numerical example We refer the reader to the **section 1, A Numerical Example** of [1].

4 Conclusion and perspectives

To come.

References

[1] Francis Longstaff and Eduardo Schwartz. Valuing american options by simulation: A simple least-squares approach. *Review of Financial Studies*, 14:113–47, 02 2001.

A Proofs

A.1 Variational inequality 1D

We wish to prove (9). Applying Itô's formula to v, we get:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \partial_t v(s, X_s^{t,x})\,\mathrm{d}s + \int_t^{t+h} D_x v(s, X_s^{t,x})\,\mathrm{d}X_s + \frac{1}{2}\int_t^{t+h} D_x^2 v(t, X_s^{t,x})\,\mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s$$

Let's assume general settings $\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t$ where (X_t) is n-dimensional and (W_t) is a m-dimensional Brownian motion. Notice that σ takes vectors $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ as input and outputs a matrix of size $n \times m$. Therefore, $\mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s = \sigma\sigma^T(s, X_s^{t,x})$ and we can rewrite the expression above:

$$\begin{aligned} v(t+h, X_{t+h}^{t,x}) &= v(t,x) + \int_t^{t+h} \left(\partial_t v + b \cdot D_x v + \frac{1}{2}\mathrm{tr}(D_x^2 v \cdot \sigma\sigma^T)\right)(s, X_s^{t,x})\,\mathrm{d}s \\ &\quad + \int_t^{t+h} (D_x v \cdot \sigma)(s, X_s^{t,x})\,\mathrm{d}W_s \\ &=: v(t,x) + \int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s + M \end{aligned}$$

where $\mathcal{L}$ is the drift differential operator and M is the local martingale part of the expression above.
From the DPP of equation (8) with $\theta = t+h$, we obtain

$$v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau-t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} + e^{-rh} v(t+h, X_{t+h}^{t,x})\mathbf{1}_{\tau > t+h}\right]$$

Thus:

$$\begin{aligned} v(t,x) &= \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\Big[e^{-r(\tau-t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} + e^{-rh} v(t,x)\mathbf{1}_{\tau > t+h} \\ &\quad + e^{-rh}\int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s\mathbf{1}_{\tau > t+h} + e^{-rh} M\mathbf{1}_{\tau > t+h}\Big] \\ &= \sup_{\tau \in \mathcal{T}_{t,T}} e^{-rh} v(t,x)\mathbf{1}_{\tau > t+h} + \mathbb{E}\Big[e^{-r(\tau-t)}\varphi(X_\tau^{t,x})\mathbf{1}_{\tau \le t+h} \\ &\quad + e^{-rh}\int_t^{t+h} \mathcal{L}v(s, X_s^{t,x})\,\mathrm{d}s\mathbf{1}_{\tau > t+h}\Big] \end{aligned}$$

Assuming the sup on the right-hand side is reached by τ^*, we have:

$$\begin{aligned}(1-e^{-rh}\mathbf{1}_{\tau^*>t+h})v(t,x) = \mathbb{E}\Big[&e^{-r(\tau^*-t)}\varphi(X^{t,x}_{\tau^*})\mathbf{1}_{\tau^*\leq t+h}\\ &+ e^{-rh}\int_t^{t+h}\mathcal{L}v(s,X^{t,x}_s)\,\mathrm{d}s\,\mathbf{1}_{\tau^*>t+h}\Big]\end{aligned}$$

At this point either $\tau^*=t$, and then $v(t,x)=\varphi(x)$ or $\tau^*>t$, so by sending h to 0 we eventually get $\tau^*>t+h>t$, and

$$\frac{1-e^{-rh}}{h}v(t,x) = \mathbb{E}\left[\frac{e^{-rh}}{h}\int_t^{t+h}\mathcal{L}v(s,X^{t,x}_s)\,\mathrm{d}s\right] \quad\Rightarrow\quad rv(t,x)=\mathcal{L}v(t,x)$$

Either way, on $\quad[0,T)\times\mathbb{R}^n$:

$$\begin{aligned}&\min\left(rv-\mathcal{L}v, v-\varphi\right)=0\\ \Leftrightarrow\quad&\min\left(-\partial_t v - b\cdot D_x v - \frac{1}{2}\mathrm{tr}(D^2_x v\cdot\sigma\sigma^T)+rv, v-\varphi\right)=0\end{aligned}$$

In the case of the Black Scholes diffusion of (**??**), this translates to

$$\min\left(-\partial_t v - rx\cdot\partial_x v - \frac{1}{2}\sigma^2x^2\cdot\partial^2_x v + rv, v-\varphi\right)=0$$

Performing the transformation $\tilde{v}(t,x):=v(T-t,x)$ leads to

$$\begin{cases}\min\left(\partial_t\tilde{v} - rx\cdot\partial_x\tilde{v} - \frac{1}{2}\sigma^2(T-t,x)x^2\cdot\partial^2_x\tilde{v} + r\tilde{v}, \tilde{v}-\varphi\right)=0\\ \tilde{v}(0,x)=\varphi(x)\end{cases}$$

B 2D use-case: Asian options

In section [REF], we developped the Finite Differences method for the pricing of vanilla non path-dependent options (such as call or put options). We will now consider an exotic option with path-dependent payoff: the Asian Call option.

B.1 European option

Estabilishing the PDE Let $\mathcal{D}$ be a derivative that pays

$$\varphi((X_s)_{0\le s\le T}) := \left(\frac{1}{T}\int_0^T X_s \,\mathrm{d}s - K\right)_+ =: (A_T - K)_+ \tag{16}$$

at maturity T, for a given strike $K > 0$.
Due to the path-dependency of the payoff of $\mathcal{D}$, the PDE-way of finding the price at $t = 0$ of $\mathcal{D}$ is hard. Fortunately, the problem can be made Markovian by introducing a new state variable: $A_t := \frac{1}{t}\int_0^t X_s \,\mathrm{d}s$. Assuming a Black-Scholes diffusion of $(X_t)_{t\in[0,T]}$, we get the following dependency for the price function π_t:

$$\begin{aligned} \pi_t &= \pi_t(t, X_t, A_t) \\ \mathrm{d}X_t &= rX_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t \\ \mathrm{d}A_t &= \frac{1}{t}(X_t - A_t)\,\mathrm{d}t \end{aligned} \tag{17}$$

From Itô's formula, we therefore get:

$$\begin{aligned} e^{rt}\,\mathrm{d}\big(e^{-rt}\pi_t\big) &= e^{rt}\cdot\big(-re^{-rt}\pi_t\,\mathrm{d}t + e^{-rt}\,\mathrm{d}\pi_t\big) = -r\pi_t\,\mathrm{d}t + \mathrm{d}\pi_t \\ &= -r\pi_t\,\mathrm{d}t + \partial_t\pi_t\,\mathrm{d}t + \partial_X\pi_t\,\mathrm{d}X_t + \partial_A\pi_t\,\mathrm{d}A_t + \frac{1}{2}\partial^2_{X,X}\pi_t\,\mathrm{d}\langle X, X\rangle_t \\ &= \left\{\partial_t\pi_t + rX_t\partial_X\pi_t - r\pi_t + \frac{1}{t}(X_t - A_t)\partial_A\pi_t + \frac{1}{2}\sigma^2x^2\partial^2_{X,X}\pi_t\right\}\mathrm{d}t \\ &+ \sigma X_t\partial_X\pi_t\,\mathrm{d}W_t \end{aligned}$$

As a result, the price function satisfies the following forward equation:

$$\begin{cases} \partial_t u + rx\partial_X u + \frac{1}{t}(x-a)\partial_A u + \frac{1}{2}\sigma^2x^2\partial^2_{X,X}u - ru = 0 \\ u(T, x, a) = (a-K)_+ \end{cases} \tag{18}$$

In the same manner as for the 1D case, we set $v(t, x, a) := u(T-t, x, a)$ and

$$\begin{cases} \partial_t v - rx\partial_X v - \frac{1}{T-t}(x-a)\partial_A v - \frac{1}{2}\sigma^2x^2\partial^2_{X,X}v + rv = 0 \\ v(0, x, a) = (a-K)_+ \end{cases} \tag{19}$$

Finite Scheme We introduce left, right, bottom and top conditions in (19) as follows:

$$\begin{cases} \partial_t v + \mathcal{A}_t v = 0, & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}),\ \forall a \in (a_{\min}, a_{\max}) \\ v(t, x_{\min}, a) = v_\ell(t,a), & \forall t \in (0,T],\ \forall a \in (a_{\min}, a_{\max}) \\ v(t, x_{\max}, a) = v_r(t), & \forall t \in (0,T],\ \forall a \in (a_{\min}, a_{\max}) \\ v(t, x, a_{\min}) = v_b(t,x), & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}) \\ v(t, x, a_{\max}) = v_\tau(t,x), & \forall t \in (0,T],\ \forall x \in (x_{\min}, x_{\max}) \\ v(0,x,a) = (a-K)_+, & \forall x \in (x_{\min}, x_{\max}) \end{cases} \tag{20}$$

where

$$\mathcal{A}_t v = -rx\partial_X v - \frac{1}{T-t}(x-a)\partial_A v - \frac{1}{2}\sigma^2 x^2 \partial^2_{X,X} v + rv$$

To discretize (20), we consider a space mesh of size $I_a + 1, I_x + 1 > 0$ and a time mesh of size $N + 1 > 0$:

$$\begin{cases} h_x & := (x_{\max} - x_{\min})/I_x \\ h_a & := (a_{\max} - a_{\min})/I_a \\ \Delta t & := N/T \\ x_i & := x_{\min} + i \cdot h_x, \quad \forall i = 0 \ldots, I_x \\ a_j & := a_{\min} + j \cdot h_a, \quad \forall j = 0 \ldots, I_a \\ t_n & := n \cdot \Delta t, \quad \forall n = 0, \ldots, N \end{cases}$$

We associate to v the tensor $U := (U_{i,j,n})$ where $U_{i,j,n} = v(t_n, x_i, a_j)$. Let us denote by A_t the discretization matrix associated to the operator $\mathcal{A}_t$, such that for any matrix V:

$$\begin{aligned} (A_t V)_{i,j} = & - rx_i \frac{V_{i+1,j} - V_{i-1,j}}{2h_x} - \frac{1}{T-t}(x_i - a_j)\frac{V_{i,j+1} - V_{i,j-1}}{2h_a} \\ & - \frac{1}{2}\sigma^2 x_i \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h_x^2} + rV_{i,j} \end{aligned}$$