EDP en finance - TP3 - Mean Field Games

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This whole work has been done in Python.

Contents

1 A short introduction to Mean Field Games Theory

We fix a finite time horizon $T > 0$. We will work on the state space $\mathbb{R}^d$. Let

$$f : \begin{cases} \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ (x, m, \gamma) \mapsto f(x, m, \gamma) \end{cases} \quad \text{and} \quad \phi : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, m) \mapsto \phi(x, m) \end{cases} \tag{1}$$

be respectively a running cost and a terminal cost, on which assumptions will be made later on.

We consider the following Mean Field Game: find a flow of probability densities $\hat{m} : [0, T] \times \mathbb{R}$ (denoted $\hat{m} \in \mathcal{M}$) and a feedback control $\hat{v} : [0, T] \times \mathbb{R} \to \mathbb{R}$ (denoted $\hat{v} \in \mathcal{V}$) satisfying the following two conditions:

- $\hat{v}$ minimizes

$$J_{\hat{m}} : v \mapsto J_{\hat{m}}(v) := \mathbb{E}\left[\int_0^T f\big(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v)\big)\mathrm{d}t + \phi\big(X_T^v, \hat{m}(T, X_T^v)\big)\right] \tag{2}$$

  subject to the constraint that $X^v = (X_t^v)_{t\geq 0}$ solves the stochastic differential equation (SDE):

$$\mathrm{d}X_t^v = b\big(X_t^v, \hat{m}(t, X_t^v), v(t, X_t^v)\big)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \geq 0 \tag{3}$$

  where $\sigma > 0$ is the volatility, $b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a given function, and X_0^v is an independent random variable in $\mathbb{R}^d$, distributed according to law m_0.

- For all $t \in [0, T]$, $\hat{m}(t, \cdot)$ is the law of X_t^v.

Proposition 1. It is useful to note that for a given feedback control v, the density m_t^v of the law of X_t^v following equation 3 solves the Kolmogorov-Fokker-Planck (KFP) equation:

$$\begin{cases} \partial_t m^v(t, x) - \nu \Delta m^v(t, x) + \operatorname{div}\Big(m^v(t, \cdot) b\big(\cdot, m^v(t, \cdot), v(t, \cdot)\big)\Big)(x) = 0, & \text{in } (0, T] \times \mathbb{R}^d \\ m^v(0, x) = m_0(x) & \text{in } \mathbb{R}^d \end{cases} \tag{4}$$

where $\nu = \sigma^2/2$.

Proof. See Appendix. □

We recall the definition of the Laplacian Δ and Divergence div operators for smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$ and $V : \mathbb{R}^d \to \mathbb{R}^d$:

$$\Delta\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R} \\ x \mapsto \sum_{i=1}^d \partial_{x_i}^2 \psi(x) \end{cases} \quad \text{and} \quad \operatorname{div}(V) : \begin{cases} \mathbb{R}^d \to \mathbb{R} \\ x \mapsto \sum_{i=1}^d \partial_{x_i} V_i(x) \end{cases} \tag{5}$$

where V_i denotes the i-th coordinate of V. Indeed, we also have the following definitions for the Gradient ∇ and the Hessian ∇^2 (of which the Laplacian is the trace):

$$\nabla\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R}^d \\ x \mapsto (\partial_{x_1}\psi(x), \ldots, \partial_{x_d}\psi(x))^T \end{cases} \quad \text{and} \quad \nabla^2\psi : \begin{cases} \mathbb{R}^d \to \mathbb{R}^{d\times d} \\ x \mapsto (\partial_{x_i, x_j}^2 \psi(x))_{1\leq i,j\leq d} \end{cases} \tag{6}$$

The control problem faced by an infinitesimal player can be addressed using standard optimal control theory. Let u be the value function of the above optimal control problem for a typical player, namely:

$$u(t,x) = \inf_{v \in \mathcal{V}} J_{\hat{m}}(t,x,v) \tag{7}$$

where

$$J_{\hat{m}}(t,x,v) = \mathbb{E}\left[\int_t^T f\big(X_s^{t,x,v}, \hat{m}(s, X_s^{t,x,v}), v(s, X_s^{t,x,v})\big)\mathrm{d}s + \phi\big(X_T^{t,x,v}, \hat{m}(T, X_T^{t,x,v})\big)\right] \tag{8}$$

with $X_s^{t,x,v}$ denoting the value at $t \leq s \leq T$ of the process diffusing from $X_t = x$ given equation 3.

Proposition 2. The value function u satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} \partial_t u(t,x) + \nu \Delta u(t,x) - H(x, m(t,x), \nabla u(t,x)) = 0 & \text{in } [0,T) \times \mathbb{R}^d \\ u(T,x) = \phi(x, m(T,x)) & \text{in } \mathbb{R}^d \end{cases} \tag{9}$$

where $\nu = \sigma^2/2$ and H is the Hamiltonian of the control problem, defined as:

$$H : \begin{cases} \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ (x,m,p) \mapsto H(x,m,p) := \sup\limits_{\gamma \in \mathbb{R}^d} \Big(- f(x,m,\gamma) - \langle b(x,m,\gamma), p \rangle \Big) \end{cases} \tag{10}$$

To get the HJB equation, we assume that the running cost f and the drift b are such that H is well-defined, $\mathcal{C}^1$ with respect to (x,p), and strictly convex with respect to p.

Proof. See Appendix. □

Finally, (u,m) solves the following forward-backward PDE system:

$$\begin{cases} -\partial_t u(t,x) - \nu \Delta u(t,x) + H\big(x, m(t,x), \nabla u(t,x)\big) = 0 & \text{in } [0,T) \times \mathbb{R}^d \\ \partial_t m(t,x) - \nu \Delta m(t,x) - \operatorname{div}\Big(m(t,\cdot)\, \partial_p H\big(\cdot, m(t,\cdot), \nabla u(t,\cdot)\big)\Big)(x) = 0 & \text{in } (0,T] \times \mathbb{R}^d \\ u(T,x) = \phi(x, m(T,x)) & \text{in } \mathbb{R}^d \\ m(0,x) = m_0(x) & \text{in } \mathbb{R}^d \end{cases} \tag{11}$$

2 Further assumptions

In the sequel, we shall consider the special case of H in the form $H(x,m,p) = H_0(x,p) - f_0(x,m)$. This way, H depends separately on m and p. For simplicity, we will even more specify the Halmitonian, by restricting ourselves to:

$$H_0(x,p) = \frac{1}{\beta}|p|^\beta - g(x) \quad \text{and} \quad f_0(x,m) = \tilde{f}_0(m(x)) \tag{12}$$

for a given real number $\beta > 0$, a function $g : \mathbb{R}^d \to \mathbb{R}$ and $\tilde{f}_0 : \mathbb{R} \to \mathbb{R}$.

Let also $d = 1$ for simplicity. We will finally consider that $\forall t \in [0,T], X_t \in \Omega := (0,1)$ almost surely. The boundary value problem becomes:

$$\begin{cases} -\partial_t u(t,x) - \nu \partial_x^2 u(t,x) + \frac{1}{\beta}\left|\partial_x u(t,x)\right|^\beta = g(x) + \tilde{f}_0(m(t,x)) & \text{in } [0,T) \times \Omega \\ \partial_t m(t,x) - \nu \partial_x^2 m(t,x) - \partial_x \Big(m(t,\cdot) \big|\partial_x u(t,\cdot)\big|^{\beta-2} \partial_x u(t,\cdot) \Big)(x) = 0 & \text{in } (0,T] \times \Omega \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 & \text{on } (0,T) \\ \partial_x m(t,0) = \partial_x m(t,1) = 0 & \text{on } (0,T) \\ u(T,x) = \phi(x, m(T,x)), \quad m(0,x) = m_0(x) & \text{in } \Omega \end{cases} \tag{13}$$

3 Finite Differences Schemes

Let N_T and N_h be two positive integers. We consider $N_T + 1$ and N_h points in time and space respectively. Set $\Delta t = T/N_T, h = 1/(N_h - 1)$, and $t_n = n \cdot \Delta t, x_i = i \cdot h$ for $(n,i) \in \{0,\ldots,N_T\} \times \{0,\ldots,N_h - 1\}$. We approximate u and m respectively by vectors U and M in $\mathbb{R}^{(N_T+1)\times N_h}$, that is, $u(t_n,x_i) \approx U_i^n$ and $m(t_n,x_i) \approx M_i^n$ for each (n,i) in $\{0,\ldots,N_T\} \times \{0,\ldots,N_h - 1\}$.
To take into account Neumann boundary conditions, we introduce ghost nodes $x_{-1} = -h, x_{N_h} = 1 + h$ and set:

$$U_{-1}^n = U_0^n, \quad U_{N_h}^n = U_{N_h-1}^n, \quad M_{-1}^n = M_0^n, \quad M_{N_h}^n = M_{N_h-1}^n \tag{14}$$

3.1 Finite differences operators

We introduce the following finite differences operators:

$$\begin{aligned} &\partial_t w(t_n,x) \longleftrightarrow (D_t W)^n = \frac{W^{n+1} - W^n}{\Delta t}, && n \in \{0,\ldots,N_T - 1\}, \quad W \in \mathbb{R}^{N_T+1} \\ &\partial_x w(t,x) \longleftrightarrow (DW)_i = \frac{W_{i+1} - W_i}{h}, && i \in \{0,\ldots,N_h - 1\}, \quad W \in \mathbb{R}^{N_h} \\ &\partial_x^2 w(t,x_i) \longleftrightarrow (\Delta_h W)_i = \frac{W_{i+1} - 2W_i + W_{i-1}}{h^2}, && i \in \{0,\ldots,N_h - 1\}, \quad W \in \mathbb{R}^{N_h} \\ &[\nabla_h W]_i = ((DW)_i, (DW)_{i-1}), && i \in \{0,\ldots,N_h - 1\}, \quad W \in \mathbb{R}^{N_h} \end{aligned}$$

In fact, we can set these operators under matrix forms. Considering a matrix $W \in \mathbb{R}^{(N_T+1)\times N_h}$, we have:

$$\big(\partial_x(t_n,x_i)\big)_{\substack{0\le n\le N_T \\ 0\le i\le N_h-1}} \longleftrightarrow \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} W_0^0 & W_0^1 & \cdots & W_0^{N_T} \\ W_1^0 & W_1^1 & \cdots & W_1^{N_T} \\ \vdots & \vdots & \ddots & \vdots \\ W_{N_h-1}^0 & W_{N_h-1}^1 & \cdots & W_{N_h-1}^{N_T} \end{pmatrix}$$

In the last row, we took into account Neumann conditions, considering the fact that $U_{N_h} = U_{N_h-1}$ and $M_{N_h} = M_{N_h-1}$. Let D_x be the matrix above.
Again, taking into account Neumann conditions:

$$\left(\partial_x^2(t_n, x_i)\right)_{\substack{0\le n\le N_T\\ 0\le i\le N_h-1}} \longleftrightarrow \frac{1}{h^2}\begin{pmatrix} -1 & 1 & 0 & 0 & \cdots & 0\\ 1 & -2 & 1 & 0 & \cdots & 0\\ 0 & 1 & -2 & 1 & \cdots & 0\\ & \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & -2 & 1\\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} W_0^0 & W_0^1 & \cdots & W_0^{N_T}\\ W_1^0 & W_1^1 & \cdots & W_1^{N_T}\\ \vdots & \vdots & \ddots & \vdots\\ W_{N_h-1}^0 & W_{N_h-1}^1 & \cdots & W_{N_h-1}^{N_T} \end{pmatrix}$$

Denote by D_x^2 the matrix above.
Notice that the matrix of $((DW)_{i-1} = \frac{1}{h}(W_i - W_{i-1}))_{0\le i<N_h}$ is (because of Neumann conditions):

$$\frac{1}{h}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0\\ -1 & 1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 1 & 0 & \cdots & 0\\ & \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 1 & 0\\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

3.2 Solving HJB equation

To elaborate a discrete version of the HJB equation, we need to have a discrete version of the Hamiltonian. We will model $H_0(x,p) = \frac{1}{\beta}|p|^\beta - g(x)$ by

$$\tilde{H}(x, p_1, p_2) = \frac{1}{\beta}\left((p_1)_-^2 + (p_2)_+^2\right)^{\beta/2} - g(x) \tag{15}$$

where $x_+ = \max(0,x)$ and $x_- = \max(0,-x)$. Note that $\tilde{H}$ takes three arguments (instead of two).
We can now consider the following discrete version of the HJB equation, supplemented with the Neumann conditions and the terminal condition:

$$\begin{cases} -(D_tU_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(x_i, [\nabla_h U^n]_i) = \tilde{f}_0(M_i^{n+1}), & 0\le i<N_h,\ 0\le n<N_T\\ U_{-1}^n = U_0^n, & 0\le n<N_T\\ U_{N_h}^n = U_{N_h-1}^n, & 0\le n<N_T\\ U_i^{N_T} = \phi(x_i, M_i^{N_T}), & 0\le i<N_h \end{cases} \tag{16}$$

This scheme is an implicit Euler scheme since the equation is backward in time. Given M^{n+1} and U^{n+1}, we will solve equation 16 for U^n. For this purpose, let us rewrite the above equation. We introduce

$$\mathcal{F}(U^n, U^{n+1}, M^{n+1}) := \begin{pmatrix} -(D_t U_0)^n - \nu(\Delta_h U^n)_0 + \tilde{H}(x_0, [\nabla_h U^n]_0) - \tilde{f}_0(M_0^{n+1}) \\ \vdots \\ -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(x_i, [\nabla_h U^n]_i) - \tilde{f}_0(M_i^{n+1}) \\ \vdots \\ -(D_t U_{N_h-1})^n - \nu(\Delta_h U^n)_{N_h-1} + \tilde{H}(x_{N_h-1}, [\nabla_h U^n]_{N_h-1}) - \tilde{f}_0(M_{N_h-1}^{n+1}) \end{pmatrix}$$

When solving the HJB equation, our goal is to find U^n knowing U^{n+1} and M^{n+1}. The condition $U_i^{N_T} = \phi(M_i^{N_T})$ allows initialization for $n = N_T$. For $n < N_T$, we use Newton-Raphson iterations, which consists in estimating U^n as the limit of a sequence $(U^{n,k})_k$ defined by

$$U^{n,k+1} = U^{n,k} - \mathcal{J}^{-1}(U^{n,k}, U^{n+1}, M^{n+1})\mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1}) \tag{17}$$

with $\mathcal{J}^{-1}(V, U^{n+1}, M^{n+1})$ is the Jacobian of the map $V \mapsto \mathcal{F}(V, U^{n+1}, M^{n+1})$. We may initialize $U^{n,0} = U^{n+1}$. The Newton iterations are stopped when $\|\mathcal{F}(U^{n,k}, U^{n+1}, M^{n+1})\|$ is below a given threshold, say 10^{-12}.

Closed form of the Jacobian Let $\mathcal{F}_i$ be the i-th coordinate of $\mathcal{F}(U^n, U^{n+1}, M^{n+1})$. The Jacobian is defined as:

$$\mathcal{J}(V, U^{n+1}, M^{n+1}) = \begin{pmatrix} \frac{\partial \mathcal{F}_0}{\partial V_0} & \frac{\partial \mathcal{F}_0}{\partial V_1} & \cdots & \frac{\partial \mathcal{F}_0}{\partial V_{N_h-1}} \\ \frac{\partial \mathcal{F}_1}{\partial V_0} & \frac{\partial \mathcal{F}_1}{\partial V_1} & \cdots & \frac{\partial \mathcal{F}_1}{\partial V_{N_h-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{F}_{N_h-1}}{\partial V_0} & \frac{\partial \mathcal{F}_{N_h-1}}{\partial V_1} & \cdots & \frac{\partial \mathcal{F}_{N_h-1}}{\partial V_{N_h-1}} \end{pmatrix} \tag{18}$$

Notice that (setting $A = -g(x_i) - \tilde{f}_0(M_i^{n+1})$):

$$\begin{aligned} \mathcal{F}_i &= -(D_t U_i)^n - \nu(\Delta_h U^n)_i + \tilde{H}(x_i, [\nabla_h U^n]_i) - \tilde{f}_0(M_i^{n+1}) \\ &= -\frac{U_i^{n+1} - U_i^n}{\Delta t} - \nu \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2} + \frac{1}{\beta h^\beta}\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2} + A \end{aligned}$$

This term only depends on $U_{i-1}^n, U_i^n, U_{i+1}^n$, so the Jacobian is a tridiagonal. Moreover:

$$\begin{aligned} \frac{\partial \mathcal{F}_i}{\partial U_{i-1}^n} &= -\frac{\nu}{h^2} - \frac{1}{h^\beta}(U_i^n - U_{i-1}^n)_+ \Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} \\ \frac{\partial \mathcal{F}_i}{\partial U_i^n} &= \frac{1}{\Delta t} + \frac{2\nu}{h^2} + \frac{1}{h^\beta}\Big((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\Big)\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} \\ \frac{\partial \mathcal{F}_i}{\partial U_{i+1}^n} &= -\frac{\nu}{h^2} - \frac{1}{h^\beta}(U_{i+1}^n - U_i^n)_- \Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} \end{aligned}$$

Let us denote J_H the Jacobian of $U^n \mapsto (\tilde{H}(x_i, [\nabla_h U^n]_i))_{0 \le i < N_h}$ evaluated in U^n, in the equations above. It will be useful in the sequel. From what we have above, its coefficients verify:

$$(J_H)_{i,i-1} = -\frac{1}{h^\beta}(U_i^n - U_{i-1}^n)_+ \Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2 \Big)^{\beta/2-1} \tag{19}$$

$$(J_H)_{i,i} = \frac{1}{h^\beta}\Big((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\Big) \Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2 \Big)^{\beta/2-1} \tag{20}$$

$$(J_H)_{i,i+1} = -\frac{1}{h^\beta}(U_{i+1}^n - U_i^n)_- \Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2 \Big)^{\beta/2-1} \tag{21}$$

3.3 Solving KFP equation

To define an appropriate discretization of the KFP equation, we first discuss how to discretize $\partial_x\Big(m(t,\cdot)\big|\partial_x u(t,\cdot)\big|^{\beta-2}\partial_x u(t,\cdot)\Big)(x)$. Recall that

$$\partial_x\Big(m(t,\cdot)\big|\partial_x u(t,\cdot)\big|^{\beta-2}\partial_x u(t,\cdot)\Big)(x) = \partial_x\Big(m(t,x)\,\partial_p H_0(x,\partial_x u(t,x))\Big) \tag{22}$$

Let us consider a function $w \in \mathcal{C}^\infty([0,T]\times\Omega)$. Using integration by parts and recalling Neumann boundary conditions, assuming $\partial_p H(x,0) = 0$, we get:

$$-\int_\Omega \partial_x\Big(m(t,x)\,\partial_p H_0(x,\partial_x u(t,x))\Big) w(t,x)\mathrm{d}x = \int_\Omega m(t,x)\,\partial_p H_0(x,\partial_x u(t,x))\partial_x w(t,x)\mathrm{d}x$$

It is natural to propose the following approximation of the right-hand side above:

$$h\sum_{i=0}^{N_h-1} M_i^{n+1}\left(\partial_{p_1}\tilde{H}(x_i,[\nabla_h U^n]_i)\frac{W_{i+1}^n - W_i^n}{h} + \partial_{p_2}\tilde{H}(x_i,[\nabla_h U^n]_i)\frac{W_i^n - W_{i-1}^n}{h}\right) \tag{23}$$

Performing discrete integrartion by parts, we obtain the discrete counterpart of the left-hand side as $-h\sum_{i=0}^{N_h-1}\mathcal{T}_i(U^n, M^{n+1})W_i^n$, where

$$\mathcal{T}_i(U,M) = \frac{1}{h}\Big(M_i\partial_{p_1}\tilde{H}(x_i,[\nabla_h U^n]_i) - M_{i-1}\partial_{p_1}\tilde{H}(x_{i-1},[\nabla_h U^n]_{i-1})\Big) \tag{24}$$

$$+\frac{1}{h}\Big(M_{i+1}\partial_{p_2}\tilde{H}(x_{i+1},[\nabla_h U^n]_{i+1}) - M_i\partial_{p_2}\tilde{H}(x_i,[\nabla_h U^n]_i)\Big) \tag{25}$$

We can now consider the following discrete version of the KFP equation, supplemented with the Neumann conditions and the terminal condition:

$$\left\{\begin{array}{ll} (D_t M_i)^n - \nu(\Delta_h M^{n+1})_i - \mathcal{T}_i(U^n, M^{n+1}) = 0, & 0 \le i < N_h,\ 0 \le n < N_T \\ M_{-1}^n = M_0^n, & 0 < n \le N_T \\ M_{N_h}^n = M_{N_h-1}^n, & 0 < n \le N_T \\ M_i^0 = \bar{m}_0(x_i), & 0 \le i < N_h \end{array}\right. \tag{26}$$

where for example:

$$\bar{m}_0(x_i) = \int_{|x-x_i|\le h/2} m_0(x)\mathrm{d}x \quad \text{or} \quad \bar{m}_0(x_i) = m_0(x_i) \tag{27}$$

This scheme is also implicit, but contrary to the HJB scheme, it consists in a forward loop. Starting from time step 0, $M_i^0 = \bar{m}_0(x_i)$ provides an explicit formula for M^0. The n-th step consists in computing M^{n+1} given U^n and M^n. The KFP system 26 being linear, it can be solved by basic linear algebra methods.

Proposition 3. We introduce $\mathcal{T}(U,M) := \big(\mathcal{T}_0(U,M),\ldots,\mathcal{T}_{N_h-1}(U,M)\big)^T$. Notice that $M \mapsto \mathcal{T}(U^n, M)$ is a linear map. Let A be the associated matrix. Then $A = (-J_H)^T$.

Proof. See Appendix. □

Considering the fact that $(D_t M)^n = \frac{1}{\Delta t}(M^{n+1} - M^n)$, we can finally rewrite our system:

$$\frac{M^{n+1} - M^n}{\Delta t} - \nu D_x^2 M^{n+1} + (J_H)^T M^{n+1} = 0 \tag{28}$$

Finding M^{n+1} then amounts to solving:

$$\Big(I_{N_h} - \nu \Delta t D_x^2 + \Delta t (J_H)^T\Big) M^{n+1} = M^n \tag{29}$$

3.4 Solving the whole forward-backward system

The idea will be to use Picard fixed points iterations to compute $\mathcal{M} := (M^n)_{0\le n\le N_T}$ and $\mathcal{U} := (U^n)_{0\le n\le N_T}$.
Let $0 < \theta < 1$ be a parameter (for instance, $\theta = 0.01$). Let $(\mathcal{M}^{(k)},\mathcal{U}^{(k)})$ be the running approximation of $(\mathcal{M},\mathcal{U})$. The next approximation $(\mathcal{M}^{(k+1)},\mathcal{U}^{(k+1)})$ is computed as follows:

- Solve the discrete HJB equation given $(\mathcal{M}^{(k)},\mathcal{U}^{(k)})$. The solution is named $\hat{\mathcal{U}}^{(k+1)}$.
- Solve the discrete KFP equation given $(\mathcal{M}^{(k)},\hat{\mathcal{U}}^{(k+1)})$. The solution is named $\hat{\mathcal{M}}^{(k+1)}$.
- Set $(\mathcal{M}^{(k+1)},\mathcal{U}^{(k+1)}) = (1-\theta)(\mathcal{M}^{(k)},\mathcal{U}^{(k)}) + \theta(\hat{\mathcal{M}}^{(k+1)},\hat{\mathcal{U}}^{(k+1)})$.

The iterations are stopped when the norm of the increment $(\mathcal{M}^{(k+1)},\mathcal{U}^{(k+1)}) - (\mathcal{M}^{(k)},\mathcal{U}^{(k)})$ becomes smaller than a given threshold, say 10^{-7}.

To initialize the loop, we set $M_i^{n,(0)} = \bar{m}_0(x_i)$ for all $0 \le i < N_h$ and $0 \le n \le N_T$. The matrix $U^{(0)}$ initial value has minimum consequence on the convergence of the algorithm. We set $U_i^{n,(0)} = 0$ for all i, n.

4 Results and answers to the questions

This section explores the results we came up with after performing simulations with different parameters.
The choice of parameters we made involves the following settings:

- $\Omega = (0,1)$ and $T = 1$
- $\sigma = 0.2$

- $\beta \in \{1.1, 2, 4\}$
- $g(x) = -\exp(-40(x-1/2)^2)$ and $g(x) = -\exp(-40(x-1/3)^2) - \exp(-40(x-2/3)^2)$
- $H_0(x,p) = \frac{1}{\beta}|p|^\beta - g(x)$
- $\tilde{f}(m(x)) = m(x)/10$ and $\tilde{f}(m(x)) = 10 \cdot m(x)$
- $\phi(x,m) = -\exp(-40(x-0.7)^2)$ and $\phi(x,m) = -\exp(-40(x-0.3)^2) - \exp(-40(x-0.9)^2)$
- $m_0(x) = \exp(-3000(x-0.2)^2)$
- $N_h = 201$, $N_T = 100$, and $\theta \in \{0.01, 0.1, 0.5\}$ (depending on which value of θ is more convenient)
- Stopping criterion in the Newton method: 10^{-11}
- Stopping criterion in the Picard Fixed Points method: 10^{-6} or 10^{-2}

Question 1. Simulate the Mean-Field Game corresponding to $\beta \in \{1.1, 2, 4\}$, $g(x) = -\exp(-40(x-1/2)^2)$ and $\tilde{f}(m(x)) = m(x)/10$.

Answer 1. Figure 1 shows the contour lines of M. The first thing to notice is that setting the running cost function $\tilde{f}(m(x)) = m(x) \cdot 10$ (instead of $\tilde{f}(m(x)) = m(x)/10$) induces the contour lines to be wider. This can be explained by the fact that $\tilde{f}$ corresponds to a cost in the HJB equation, and its form directly relates to the width of the final contour lines. Notice that points $(0.2, 0)$ and $(0.7, 1)$ are highlighted on the figures. This comes from the fact that the initial law $m_0(x)$ admits a maximum at $x = 0.2$, and the final cost $\phi(x,m)$ admits a minimum at $x = 0.7$. Indeed, we can change these settings. For instance, figure 2 displays what happens if we set $\phi(x) = -\exp(-40(x-0.3)^2) - \exp(-40(x-0.9)^2)$ and either set $m(x) = e^{-3000(x-0.2)^2}$ or $m(x) = e^{-3000(x-0.2)^2} + e^{-3000(x-0.6)^2} + e^{-3000(x-0.8)^2}$ (with $g(x) = -e^{-40(x-1/3)^2} - e^{-40(x-2/3)^2}$ in that case).
Figure 3 shows the impact of either decreasing or increasing β on the contour lines of M.

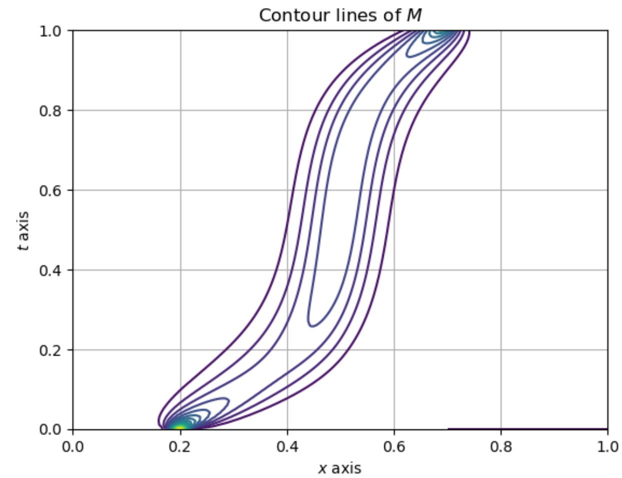
Question 2. Set $g(x) = -\exp(-40(x-1/3)^2) - \exp(-40(x-2/3)^2)$ and comment the changes.

Answer 2. In figure 4, we compare the impact of the fixed cost $g(x)$ over the contour lines of M. For $g(x) = -\exp(-40(x-1/2)^2)$, we can clearly see that these lines patterns is shaped around the vertical line $x = 1/2$. This impact is also present in figure 2.

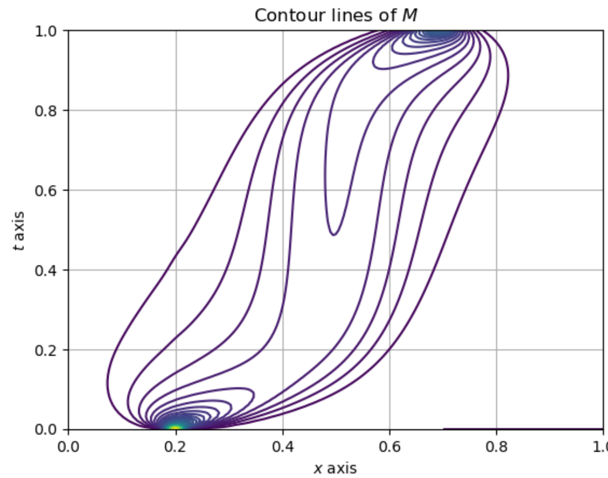
Question 3. Understand why the mass of M^n, i.e $\sum_{i=0}^{N_h-1} M_i^n$ does not depend on n.

Answer 3. In this question, the next ones, and whenever it is necessary, we consider that M^n is a column vector. We recall that:

$$\underbrace{\left(I_{N_h} - \nu \Delta t D_x^2 + \Delta t (J_H)^T\right)}_{=:B} M^{n+1} = M^n \tag{30}$$

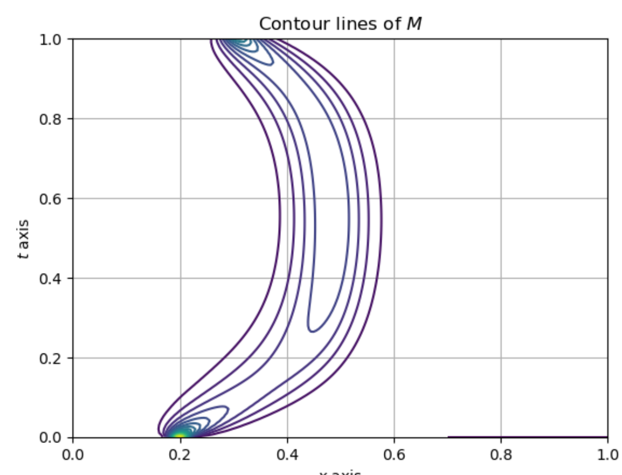


(a) Running cost $\tilde{f}(m(x)) = m(x)/10$

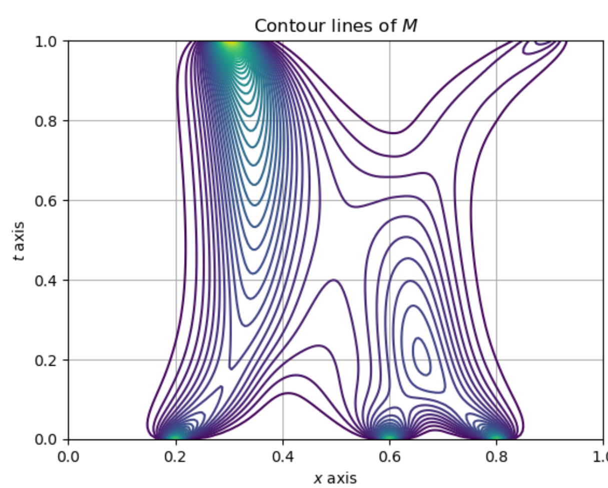


(b) Running cost $\tilde{f}(m(x)) = m(x) \cdot 10$

Figure 1: Contour lines of M for $\beta = 2$ and $g(x) = -\exp(-40(x-1/2)^2)$ and $\phi(x) = -\exp(-40(x-0.7)^2)$, ($x$ in abcissa, t in ordinate)

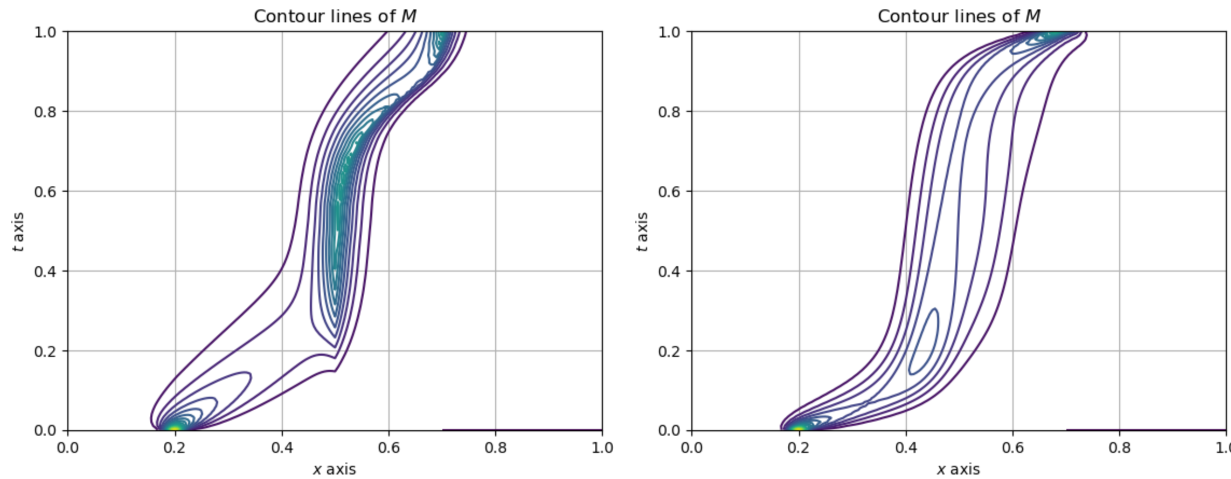


(a) First example of change in $\phi(x,m)$



(b) Second example, with $m_0(x) = e^{-3000(x-0.2)^2} + e^{-3000(x-0.6)^2} + e^{-3000(x-0.8)^2}$

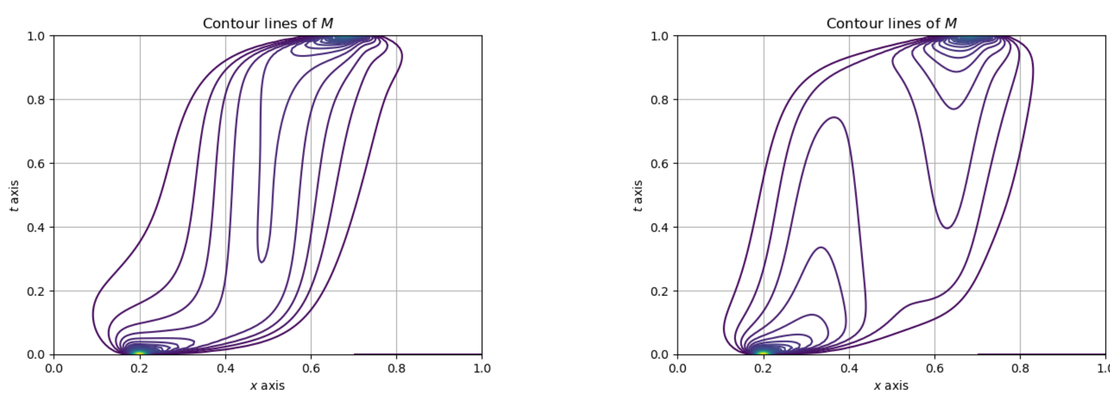
Figure 2: Contour lines of M for special settings of ϕ, m_0, g and $\tilde{f}$



(a) $\beta = 1.1$

(b) $\beta = 4$

Figure 3: Contour lines of M for $g(x) = -\exp(-40(x-1/2)^2)$, $\phi(x) = -\exp(-40(x-0.7)^2)$ and $\tilde{f}(m(x)) = m(x)/10$



(a) Constant cost g centered on $1/2$

(b) Constant cost g bicentered on $1/3$ and $2/3$

Figure 4: Contour lines of M for $\beta = 2$, final cost $\phi(x,m) = -\exp(-40(x-0.7)^2)$ and running cost $\tilde{f}(m(x)) = m(x) \cdot 10$

Let $v = (1\ 1\ \dots\ 1)^T \in \mathbb{R}^{N_h}$. If we can prove that $B^T v = v$, then we are done as this would imply $v^T B = v^T$ and then $v^T B M^{n+1} = v^T M^n \ \Rightarrow\ v^T M^{n+1} = v^T M^n$, which is basically what we are trying to show.
We have:

$$B^T v = \Big(I_{N_h} - \nu \Delta t (D_x^2)^T + \Delta t J_H \Big) v \tag{31}$$

From equations 19-21, we deduce that the sum of coefficients of J_H over any row is 0. The same goes for matrix D_x^2, so that:

$$(B^T v)_i = 1 - \nu \Delta t \cdot 0 + \Delta t \cdot 0 = 1 = v_i \tag{32}$$

Hence $B^T v = v$, and we can conclude.

Question 4. Try to prove uniqueness in the discrete HJB equation, i.e. that given $(M^n)_n$, $(U^n)_n$ is unique. *Hint*: take two solutions $(U^n)_n$ and $(V^n)_n$ and consider n_0, i_0 such that $\max_{(n,i)}(U_i^n - V_i^n)$ is achieved at (n_0, i_0), and use the monotonicity of the discrete Hamiltonian.

Answer 4. Let $(U^n)_n$ and $(V^n)_n$ be two solutions of the HJB equation (given $(M^n)_n$). We consider (n_0, i_0) such that $U_{i_0}^{n_0} - V_{i_0}^{n_0} = \max_{(n,i)}(U_i^n - V_i^n)$. Thus:

$$-(D_t U_{i_0})^{n_0} - \nu(\Delta_h U^{n_0})_{i_0} + \tilde{H}(x_{i_0}, [\nabla_h U^n]_{i_0}) = -(D_t V_{i_0})^{n_0} - \nu(\Delta_h V^{n_0})_{i_0} + \tilde{H}(x_{i_0}, [\nabla_h V^{n_0}]_{i_0})$$

We can rearrange the terms:

$$\tilde{H}(x_{i_0}, [\nabla_h U^n]_{i_0}) - \tilde{H}(x_{i_0}, [\nabla_h V^{n_0}]_{i_0}) = (D_t U_{i_0})^{n_0} + \nu(\Delta_h U^{n_0})_{i_0} - (D_t V_{i_0})^{n_0} - \nu(\Delta_h V^{n_0})_{i_0}$$

Knowing that $U_{i_0}^{n_0} - V_{i_0}^{n_0}$ is the largest difference possible, we have:

$$\begin{cases} (D_t U_{i_0})^{n_0} - (D_t V_{i_0})^{n_0} = \frac{1}{\Delta t}\Big(-(U_{i_0}^{n_0} - V_{i_0}^{n_0}) + (U_{i_0}^{n_0+1} - V_{i_0}^{n_0+1}) \Big) \le 0 \\ \nu(\Delta_h U^{n_0})_{i_0} - \nu(\Delta_h V^{n_0})_{i_0} = \frac{\nu}{h^2}\Big((U_{i_0+1}^{n_0} - V_{i_0+1}^{n_0}) - 2(U_{i_0}^{n_0} - V_{i_0}^{n_0}) + (U_{i_0-1}^{n_0} - V_{i_0-1}^{n_0}) \Big) \le 0 \end{cases} \tag{33}$$

Hence $\tilde{H}(x_{i_0}, (DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}) \le \tilde{H}(x_{i_0}, (DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1})$.
However:

$$\begin{cases} (DU^{n_0})_{i_0} - (DV^{n_0})_{i_0} = \frac{1}{h}\Big((U_{i_0+1}^{n_0} - V_{i_0+1}^{n_0}) - (U_{i_0}^{n_0} - V_{i_0}^{n_0}) \Big) \le 0 \\ (DU^{n_0})_{i_0-1} - (DV^{n_0})_{i_0-1} = \frac{1}{h}\Big((U_{i_0}^{n_0} - V_{i_0}^{n_0}) - (U_{i_0-1}^{n_0} - V_{i_0-1}^{n_0}) \Big) \ge 0 \end{cases}$$

As $\tilde{H}$ is non-increasing in its second argument and non-decreasing in its third argument, it follows that:

$$\begin{aligned} \tilde{H}(x_{i_0}, (DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}) &\le \tilde{H}(x_{i_0}, (DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}) \\ &\le \tilde{H}(x_{i_0}, (DU^{n_0})_{i_0}, (DV^{n_0})_{i_0-1}) \qquad \text{by non-increasing property} \\ &\le \tilde{H}(x_{i_0}, (DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}) \qquad \text{by non-decreasing property} \end{aligned}$$

We conclude that $\tilde{H}(x_{i_0}, (DU^{n_0})_{i_0}, (DU^{n_0})_{i_0-1}) = \tilde{H}(x_{i_0}, (DV^{n_0})_{i_0}, (DV^{n_0})_{i_0-1})$. A sum of negative numbers is equal to 0 whenever all of the terms are identically null. Equation 33 then translates to

$$\begin{cases} (D_t U_{i_0})^{n_0} - (D_t V_{i_0})^{n_0} = 0 \\ \nu(\Delta_h U^{n_0})_{i_0} - \nu(\Delta_h V^{n_0})_{i_0} = 0 \end{cases} \tag{34}$$

From what we said above, we conclude that the maximum of $U_i^n - V_i^n$ is also reached on the adjacents points of (i_0, n_0). The arguments we developped then propagate until we reach the boundary condition $U_i^{N_T} = V_i^{N_T} = \phi(M_i^{N_T})$ for all $0 \leq i < N_h$, which implies that $\max_{(i,n)} U_i^n - V_i^n = 0$. Hence $U = V$ and the solution to the HJB equation is unique.

Question 5. Try to prove uniqueness in the discrete KFP equation, i.e. that given $(U^n)_n$, $(M^n)_n$ is unique. *Hint*: prove that the matrices of the linear systems arising in the discrete KFP are the conjugate of M-matrices.

Answer 5. We know that finding M^{n+1} given M^n and $(U^n)_n$ amounts to solving equation 29. We recall it:

$$\underbrace{\left(I_{N_h} - \nu\Delta t D_x^2 + \Delta t (J_H)^T\right)}_{=:B} M^{n+1} = M^n \tag{35}$$

We will show that B^T has the M-property, which implies that B has it too.

- First, $B_{i,i}^T = B_{i,i} > 0$ for all $0 \leq i < N_h$. Indeed:

  $$B_{i,i} = 1 - \nu\Delta t \cdot \frac{-1 \text{ or } -2}{h^2} + \Delta t \cdot (J_H)_{i,i}^T \geq 1 + \nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,i}^T$$

  which is strictly greater than 0 recalling the form of J_H in equations 19 - 21.

- Second, for $j \neq i$, we have on the other hand $B_{i,j}^T \leq 0$. Indeed:

  $$B_{i,j}^T = -\nu\Delta t \cdot \frac{1}{h^2} + \Delta t \cdot (J_H)_{i,j} \leq 0$$

  for the same reason.

- Finally, $\sum_{j=0}^{N_h-1} B_{i,j}^T > 0$ comes from the fact that the sum of coefficients over any row of J_H is null (this also holds for D_x^2), which leads to

  $$\sum_{j=0}^{N_h-1} B_{i,j}^T = 1 - \nu\Delta t \cdot 0 + \Delta t \cdot 0 > 0$$

As a result, B^T and B have the M-property, so B is invertible. Hence, the system $BM^{n+1} = M^n$ has a unique solution $M^{n+1} \in \mathbb{R}^{N_h}$. The KFP equation therefore has a unique solution.

Question 6. Try to prove that if M^0 is positive, then M^n is positive for all n.

Answer 6. From the previous question, we know that $(M^n)_n$ satisfies $BM^{n+1} = M^n$, with B an M-matrix. It follows that whenever $M^0 \geq 0$, then $M^1 \geq 0$, and then $M^2 \geq 0$ and so on. Hence, $M^0 \geq 0$ does imply that M^n is positive for all n.

5 Appendix

This section consists in proofs of the article's propositions.

Proof of proposition 1. Let (X_t) follow equation 3's diffusion. By Itô's formula, for any smooth function $f : \mathbb{R}^d \to \mathbb{R}$:

$$f(X_t^x) = f(x) + \int_0^t \langle \nabla f, \mathrm{d}X_s^x \rangle + \frac{1}{2}\mathrm{tr}\left(\nabla^2 f \, \mathrm{d}\langle X^x, X^x \rangle_s\right) \tag{36}$$

We can compute:

$$\begin{aligned} \langle \nabla f, \mathrm{d}X_s^x \rangle &= \langle \nabla f, b \rangle \mathrm{d}s + \langle \nabla f, \sigma \mathrm{d}W_s \rangle \\ \mathrm{tr}\left(\nabla^2 f \, \mathrm{d}\langle X^{t,x}, X^{t,x} \rangle_s\right) &= \sigma^2 \Delta f \, \mathrm{d}s \end{aligned}$$

Hence:

$$f(X_t^x) = f(x) + \int_0^t \langle \nabla f, b \rangle \mathrm{d}s + \langle \nabla f, \sigma \mathrm{d}W_s \rangle + \frac{\sigma^2}{2} \Delta f \, \mathrm{d}s \tag{37}$$

Taking the expectation at times t and $t+h$, we get:

$$\mathbb{E}[f(X_{t+h}^x)] - \mathbb{E}[f(X_t^x)] = \mathbb{E}\left[\int_t^{t+h} \left(\langle \nabla f, b \rangle + \frac{\sigma^2}{2}\Delta f\right) \mathrm{d}s\right] \tag{38}$$

Dividing by h and sending h to 0, we obtain:

$$\partial_t \mathbb{E}[f(X_t^x)] = \mathbb{E}[\langle \nabla f(X_t^x), b \rangle + \frac{\sigma^2}{2}\Delta f(X_t^x)] =: \mathbb{E}[\mathcal{L}f(X_t^x)] \tag{39}$$

By definition of the law $m(t, \cdot)$ of X_t^x, we have:

$$\mathbb{E}[f(X_t^x)] = \int_{\mathbb{R}^d} m(t, \mathrm{d}y) f(y) =: M_t f(x) \quad \text{and} \quad \mathbb{E}[\mathcal{L}f(X_t^x)] = M_t \mathcal{L} f(x) \tag{40}$$

From the equation above, we wish to obtain an equation in $m(t, \cdot)$. For this purpose, we introduce the adjoint operator $\mathcal{L}^* f$ of $\mathcal{L}f = \langle \nabla f, b \rangle + \frac{\sigma^2}{2}\Delta f$, defined as: $\mathcal{L}^* f := -\mathrm{div}(bf) + \frac{\sigma^2}{2}\Delta f$. One can check that this adjoint verifies

$$\int_{\mathbb{R}^d} \mathcal{L}f(y) \cdot g(y) \mathrm{d}y = \int_{\mathbb{R}^d} f(y) \cdot \mathcal{L}^* g(y) \mathrm{d}y \tag{41}$$

From equation 39, we then derive $\partial_t m(t, \cdot) = \mathcal{L}^* m(t, \cdot)$, i.e

$$\begin{cases} \partial_t m(t,x) - \frac{\sigma^2}{2}\Delta m(t,x) + \mathrm{div}\Big(m(t,\cdot) b\big(\cdot, m(t,\cdot), v(t,\cdot)\big)\Big)(x) = 0, & \text{in } (0,T] \times \mathbb{R}^d \\ m(0,x) = m_0(x) & \text{in } \mathbb{R}^d \end{cases} \tag{42}$$

□

Proof of proposition 2. By the Dynamic Programming Principle (DPP), we know that

$$u(t,x) = \inf_{v \in \mathcal{V}} \mathbb{E}\left[\int_t^\theta f(X_s^{t,x,v}, \hat{m}(s, X_s^{t,x,v}), v(s, X_s^{t,x,v}))\mathrm{d}s + u(\theta, X_\theta^{t,x,v})\right] \tag{43}$$

for any stopping time $\theta \in \mathcal{T}_{t,T}$ (set of stopping times valued in $[t,T]$).
for any $v \in \mathcal{V}$ and controlled process $(X_s^{t,x})_{s \geq t}$ (starting from $X_t = x$), applying Itô's formula between t and $t+h > t$ gives off:

$$u(t+h, X_{t+h}^{t,x}) = u(t,x) + \int_t^{t+h} \partial_t u \mathrm{d}s + \langle \nabla u, \mathrm{d}X_s^{t,x}\rangle + \frac{1}{2}\mathrm{tr}\left(\nabla^2 u\, \mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s\right) \tag{44}$$

We can compute:

$$\begin{aligned}
\langle \nabla u, \mathrm{d}X_s^{t,x}\rangle &= \langle \nabla u, b\rangle \mathrm{d}s + \langle \nabla u, \sigma \mathrm{d}W_s\rangle \\
\mathrm{tr}\left(\nabla^2 u\, \mathrm{d}\langle X^{t,x}, X^{t,x}\rangle_s\right) &= \sigma^2 \Delta u\, \mathrm{d}s
\end{aligned}$$

Hence:

$$u(t+h, X_{t+h}^{t,x}) = u(t,x) + \int_t^{t+h} \left(\partial_t u + \langle \nabla u, b\rangle + \frac{\sigma^2}{2}\Delta u\right)\mathrm{d}s + \int_t^{t+h} \langle \nabla u, \mathrm{d}W_s\rangle \tag{45}$$

Plugging into the DPP (with $\theta = t+h$), and recording that the expectation of the local martingale term will vanish, we get:

$$0 = \inf_{v \in \mathcal{V}} \mathbb{E}\left[\int_t^{t+h} \left(f + \partial_t u + \langle \nabla u, b\rangle + \frac{\sigma^2}{2}\Delta u\right)\mathrm{d}s\right] \tag{46}$$

Dividing by h and sending h to 0, we get:

$$\partial_t u(t,x) + \frac{\sigma^2}{2}\Delta u(t,x) + \inf_{\gamma \in \mathbb{R}^d}\left\{f\big(x, \hat{m}(t,x), \gamma\big) + \big\langle \nabla u(t,x), b\big(x, m(t,x), \gamma\big)\big\rangle\right\} = 0$$

where γ has to be understood as the value $\gamma := v(t,x)$. The hamiltonian writes:

$$\begin{aligned}
H(x,m,p) &:= -\inf_{\gamma \in \mathbb{R}^d}\left\{f(x,m,\gamma) + \big\langle p, b(x,m,\gamma)\big\rangle\right\} \\
&= \sup_{\gamma \in \mathbb{R}^d}\left\{-f(x,m,\gamma) - \big\langle p, b(x,m,\gamma)\big\rangle\right\}
\end{aligned}$$

for $(x,m,p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.
The best response $\hat{v}(t,x)$ is such that:

$$\hat{v}(t,x) \in \arg\max_{\gamma \in \mathbb{R}^d}\left\{-f(x, m(t,x), \gamma) - \big\langle \nabla u(t,x), b(x, m(t,x), \gamma)\big\rangle\right\} \tag{47}$$

Under right hypothesis, this value is unique, and we have:

$$\partial_p H\big(x, m(t,x), \nabla u(t,x)\big) = -b(x, m(t,x), \hat{v}(t,x)) \tag{48}$$

Finally, putting all pieces together we obtain:

$$\begin{cases}
\partial_t u(t,x) + \frac{\sigma^2}{2}\Delta u(t,x) - H(x, m(t,x), \nabla u(t,x)) = 0 & \text{in } [0,T) \times \mathbb{R}^d \\
u(T,x) = \phi(x, m(T,x)) & \text{in } \mathbb{R}^d
\end{cases} \tag{49}$$

□

Proof of proposition 3. From equation 24, we can conclude that

$$\begin{cases} A_{i,i-1} & = -\frac{1}{h}\partial_{p_1}\tilde{H}(x_{i-1},[\nabla_h U^n]_{i-1}) \\ A_{i,i} & = \frac{1}{h}\Big(\partial_{p_1}\tilde{H}(x_i,[\nabla_h U^n]_i) - \partial_{p_2}\tilde{H}(x_i,[\nabla_h U^n]_i)\Big) \\ A_{i,i+1} & = \frac{1}{h}\partial_{p_2}\tilde{H}(x_{i+1},[\nabla_h U^n]_{i+1}) \end{cases}$$

and $A_{i,j} = 0$ for $j \notin \{i-1, i, i+1\}$. Notice that $\tilde{H}(x,p_1,p_2) = \frac{1}{\beta}\big((p_1)_-^2 + (p_2)_+^2\big)^{\beta/2} - g(x)$, so

$$\begin{cases} \partial_{p_1}\tilde{H}(x,p_1,p_2) = -(p_1)_-\big((p_1)_-^2 + (p_2)_+^2\big)^{\beta/2-1} \\ \partial_{p_2}\tilde{H}(x,p_1,p_2) = (p_2)_+\big((p_1)_-^2 + (p_2)_+^2\big)^{\beta/2-1} \end{cases} \tag{50}$$

As a result:

$$\begin{aligned} A_{i,i-1} &= \frac{1}{h^\beta}(U_i^n - U_{i-1}^n)_-\Big((U_i^n - U_{i-1}^n)_-^2 + (U_{i-1}^n - U_{i-2}^n)_+^2\Big)^{\beta/2-1} \\ A_{i,i} &= -\frac{1}{h^\beta}\Big((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\Big)\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} \\ A_{i,i+1} &= \frac{1}{h^\beta}(U_{i+2}^n - U_{i+1}^n)_+\Big((U_{i+2}^n - U_{i+1}^n)_-^2 + (U_{i+1}^n - U_i^n)_+^2\Big)^{\beta/2-1} \end{aligned}$$

In other word:

$$\begin{aligned} \big(-A^T\big)_{i,i} &= \frac{1}{h^\beta}\Big((U_{i+1}^n - U_i^n)_- + (U_i^n - U_{i-1}^n)_+\Big)\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} = (J_H)_{i,i} \\ \big(-A^T\big)_{i,i-1} &= -A_{i-1,i} = -\frac{1}{h^\beta}(U_{i+1}^n - U_i^n)_+\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} = (J_H)_{i,i-1} \\ \big(-A^T\big)_{i,i+1} &= -A_{i+1,i} = -\frac{1}{h^\beta}(U_{i+1}^n - U_i^n)_-\Big((U_{i+1}^n - U_i^n)_-^2 + (U_i^n - U_{i-1}^n)_+^2\Big)^{\beta/2-1} = (J_H)_{i,i+1} \end{aligned}$$

Hence $-A^T = J_H$. $\square$