Likelihood

Michael Noonan

January 18, 2021

Biol 520C: Statistical modelling for biological data

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Housekeeping

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- Practical 02 is now unlocked on canvas, and due next Tuesday.

Review

Clicker question: i

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$\Pr\{A|B\} = \Pr\{A,B\}/\Pr\{B\}$

A — The conditional probability of A given B occurred.

B — The conditional probability of B given A occurred.

C — Bayes' theorem.

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And I told you that this was important for fitting a model to data.

Likelihood

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The subscript i indicates that there are multiple possible outcomes, but only one parameter value p.

Probability distributions cont.

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Count data often follow a Poisson distribution, so $\Pr\{Y_i|p\}$ can be written as:

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In other words, we have a mathematical description for saying "What is the probability of observing our data (Y_i) given our hypothesis (rate parameter $= \lambda$)?"

Clicker question: ii

Let's say I'm conducting a bird survey in my back yard. What's the probability of counting 2 crows given a rate parameter of 6?

NB: The Poisson distribution has a probability mass function of $\frac{\lambda^k e^{-\lambda}}{k!}$

A — ≈ 0.27

B — ≈ 0.012

C — ≈ 0.16

D — ≈ 0.045

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Note how here the data are not subscripted (we only observed one outcome), but there are multiple possible parameter values p_m.

Likelihoods vs. probabilities cont.

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If we assume that the likelihood of the data given the hypothesis is *proportional* to the probability, we can write:

$$\mathcal{L}(p_m|Y) \quad = \quad c \quad \Pr\{Y|p_m\}$$

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Because we're typically interested in relative likelihoods (not exact values), the proportionality constant, c, can be set to 1

Likelihoods vs. probabilities cont.

Ok, cool, but so what?

Likelihoods vs. probabilities cont.

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If $\mathcal{L}(p_m|Y) = \Pr\{Y|p_m\}$, and we have some data, and if we make some distributional assumptions, we have a way of calculating the likelihood of specific parameter values

Likelihoods vs. probabilities cont.

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If $\mathcal{L}(p_m|Y) = \Pr\{Y|p_m\}$, and we have some data, and if we make some distributional assumptions, we have a way of calculating the likelihood of specific parameter values, and we can generalise this to any number of parameters and any distribution!

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If $\mathcal{L}(p_m|Y) = \Pr\{Y|p_m\}$, and we have some data, and if we make some distributional assumptions, we have a way of calculating the likelihood of specific parameter values, and we can generalise this to any number of parameters and any distribution!

For example, if we come back to those 2 crows I counted earlier we can now ask (and answer!) what's the more likely λ, 4, or 6?

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If $\mathcal{L}(p_m|Y) = \Pr\{Y|p_m\}$, and we have some data, and if we make some distributional assumptions, we have a way of calculating the likelihood of specific parameter values, and we can generalise this to any number of parameters and any distribution!

For example, if we come back to those 2 crows I counted earlier we can now ask (and answer!) what's the more likely λ, 4, or 6?

$$\mathcal{L}(\lambda = 4|2) = \frac{4^2 e^{-4}}{2!} \approx 0.15 \qquad \mathcal{L}(\lambda = 6|2) = \frac{6^2 e^{-6}}{2!} \approx 0.045$$

Remember that the Poisson distribution has a PMF given by $\frac{\lambda^k e^{-\lambda}}{k!}$

Maximum likelihood

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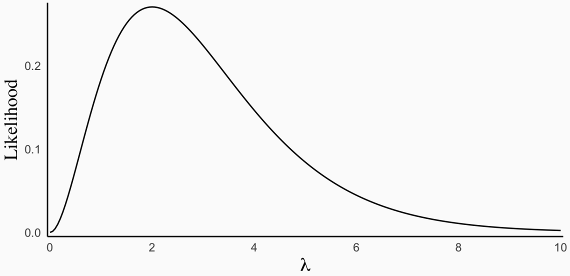
For our 2 crows, a plot of the likelihood as a function of λ looks like this:



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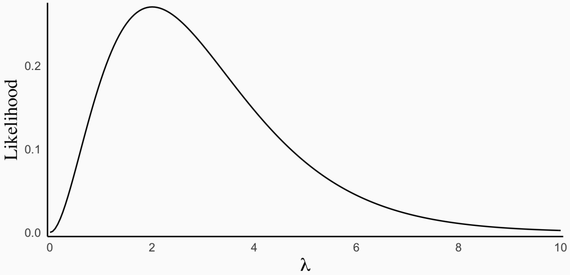

The value of λ that maximises the likelihood is the maximum likelihood estimate (MLE) of our parameter value

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If we assume our observations are independent, then
$\Pr\{A,B|p_m\} = \Pr\{A|p_m\} \times \Pr\{B|p_m\}$

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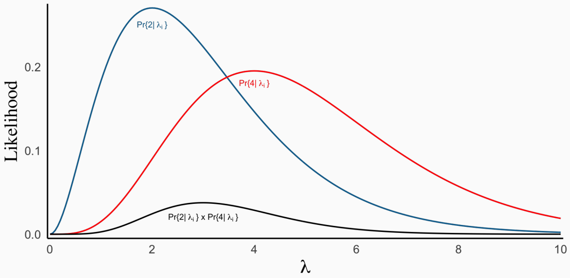

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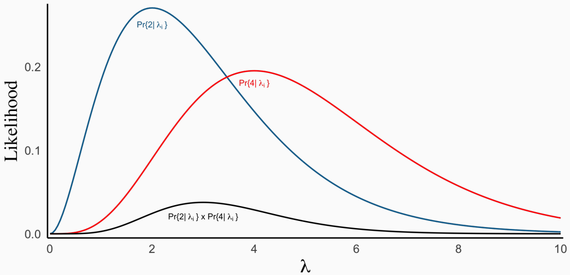

Here, $\hat{\lambda} = 3$ is the maximum likelihood estimate (MLE)

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$$\mathcal{L}(\lambda|x_i) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

Log-likelihoods

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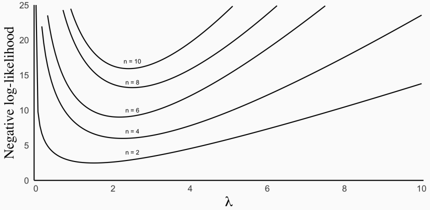
In most cases, as the sample size increases, the negative log-likelihood function becomes increasingly peaked around its maximum



MLE in action

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Let's say I go out and measured the height of ten people. I come back with $\{171,168,180,190,169,172,162,181,177,181\}$ (in cm).

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Let's say I go out and measured the height of ten people. I come back with $\{171,168,180,190,169,172,162,181,177,181\}$ (in cm). If we assume heights are normally distributed with a σ of 10 cm, we can estimate the mean height, μ, using MLE and the general form of the Gaussian distribution:

$$\mathcal{L}(\mu|x_i) = \prod_{i=1}^{10} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}(\frac{x_i-\mu}{\sigma})^2}$$

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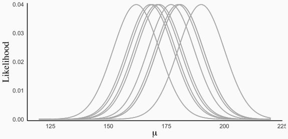
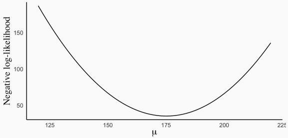
Plugging in data the resulting likelihood functions look like this:





MLE in action cont.

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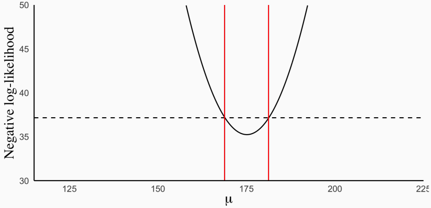
In the likelihood framework, we can use the likelihood profile to identify the 95% confidence intervals on our estimated parameter. E.g., a simple rule is to place the bounds within 1.92 of the minimum log-likelihood.



So we can say that $\mu = 175.1$ with 95% CIs of ~ 169 — 181

MLE and CIs.

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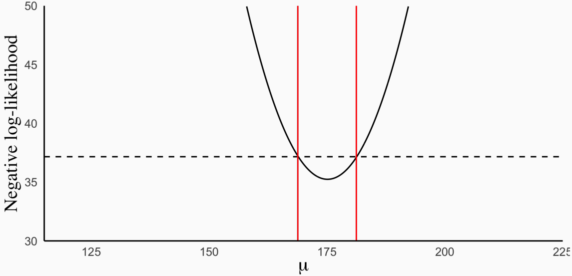

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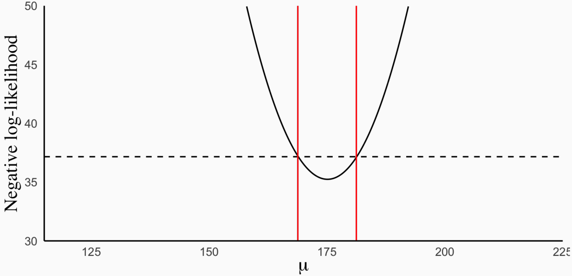

More data narrows the CIs (and vice versa, less data increases the amount of uncertainty in the MLE)

Regression as a problem of MLE

MLE and fitting a line to data

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We've now picked up enough of the basics to go back to our linear regression problem and fit a line to some data as probalists.

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and ε_i is our Gaussian distributed error with a mean of 0 and variance of σ^2, whose PDF is given by:

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Regression as a problem of MLE cont.

Given our dataset $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

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$$\mathcal{L}(x_i; \beta_0, \beta_1, \sigma^2 | y_i) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{y_i - \beta_0 + \beta_1 x_i}{\sigma})^2}$$

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For convenience, we work with negative log-likelihoods, which changes this to:

$$\mathcal{L}(x_i; \beta_0, \beta_1, \sigma^2 | y_i) = n(\log(\sigma) + \tfrac{1}{2}\log(2\pi)) + \tfrac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - (\beta_0 + \beta_1 x_i))^2$$

Regression as a problem of MLE cont.

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After some math that we won't go over, we get the following 3 estimators:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2$$

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So what did we gain by making a Gaussian-noise assumption and estimating the parameters via maximum likelihood?

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ii) A framework for placing confidence intervals on our parameter estimates using the likelihood profiles (which allows us to make formal statistical inference on the parameters)

Stochasticity and predictions

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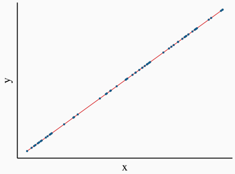
With a purely deterministic model, we can understand the mechanisms underlying a system, but our predictions inherently lack stochasticity and the outcome is always the same.



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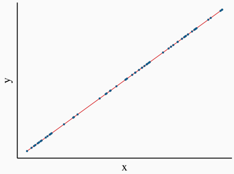

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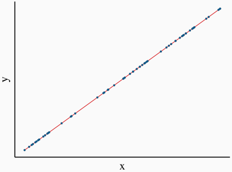

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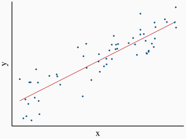
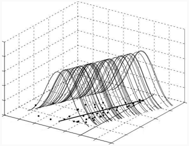
With a stochastic component, outcomes are variable and models provide a distribution of the values that y_i can be expected to take





Source: Zuur et al. 2009

Stochasticity and predictions example

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How informative is this? What if we measure an insect with weight $= 2$ and wingspan $= 21$? What does this tell us about our model?

Stochasticity and predic. examp. cont.

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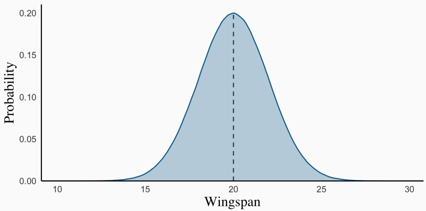

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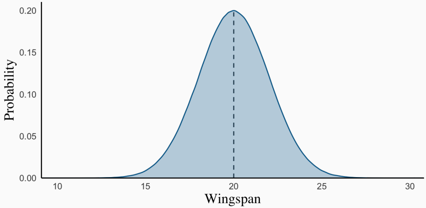

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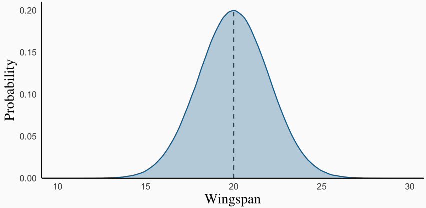

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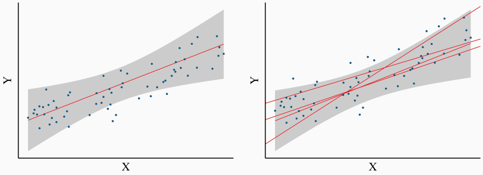

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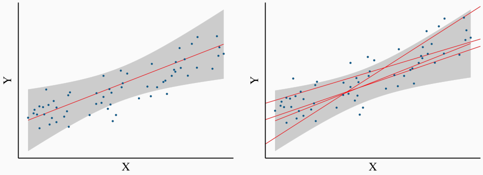

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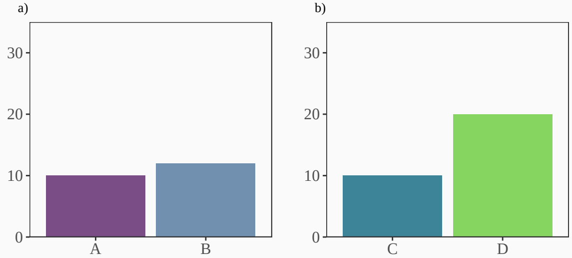
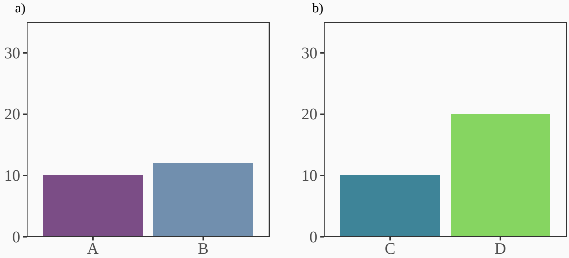
What happens to prediction intervals when $n \to \infty$? Which will be wider, confidence intervals, or prediction intervals?



Which panel depicts a pair of significantly different datasets.

A — a
B — b
C — Both a and b
D — I don't know.

Clicker question: iii



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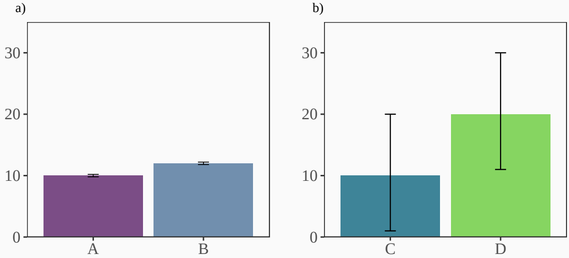

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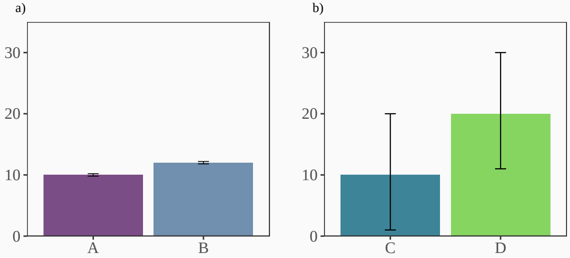

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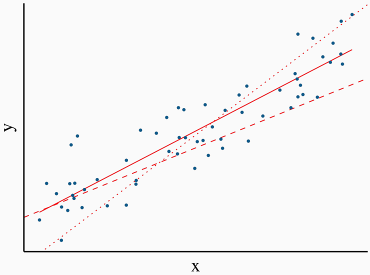

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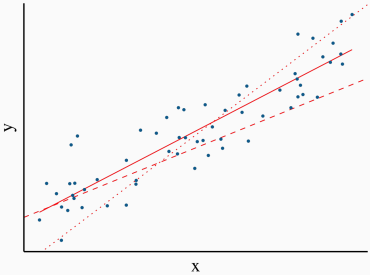

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The answer depends on the amount of information in our data, the (un)certainty in our estimate, and the shape of our likelihood profile.