Multiple Linear Regression

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Biol 520C: Statistical modelling for biological data

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Housekeeping

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- Datasets for papers: consider asking your supervisor or lab mates for data on your study system.

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Our verbal hypothesis in this case is 'Y is proportional to X_1, X_2, ... X_m'. With multiple factors, how do we approach the problem statistically?

Simple Linear Regression

Remembering that for simple linear regression our data is of the form:

$$d = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

X	Y
x_1	y_1
x_2	y_2
$\ldots$	$\ldots$
x_n	y_n

And our relationship is described by an intercept (β_0) and a slope (β_1):

$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$

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For m explanatory variables, the data are of the form:

Y	X_1	X_2	...	X_m
y_1	X_{11}	X_{21}	...	X_m1
y_1	X_{12}	X_{22}	...	X_{m2}
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A linear regression with multiple explanatory variables is described by an intercept (β_0) and a regression coefficients ($\beta_1, \beta_2, \ldots, \beta_m$):
$y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_m x_{mi} + \varepsilon_i$

Linear regression and matrix notation

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Notice how the dimensions of the matrices all line up, so even though we've added new parameters we can still estimate their values using our old friend $\beta = (\mathrm{x}^\mathsf{T}\mathrm{x})^{-1}\mathrm{x}^\mathsf{T}\mathrm{y}$.

Multiple Linear Regression
Pre-analysis

Mult. Parameters, Mult. Interactions

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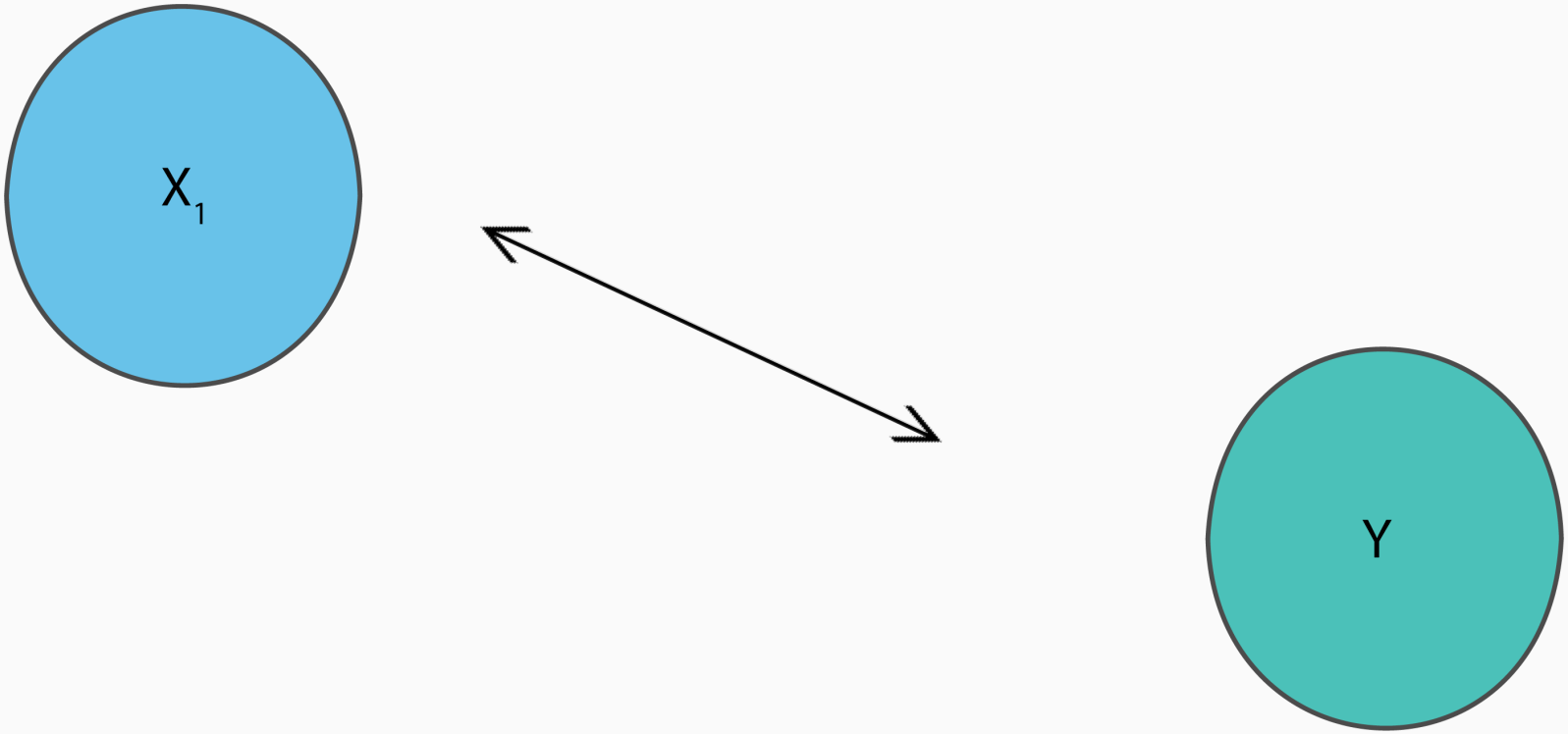
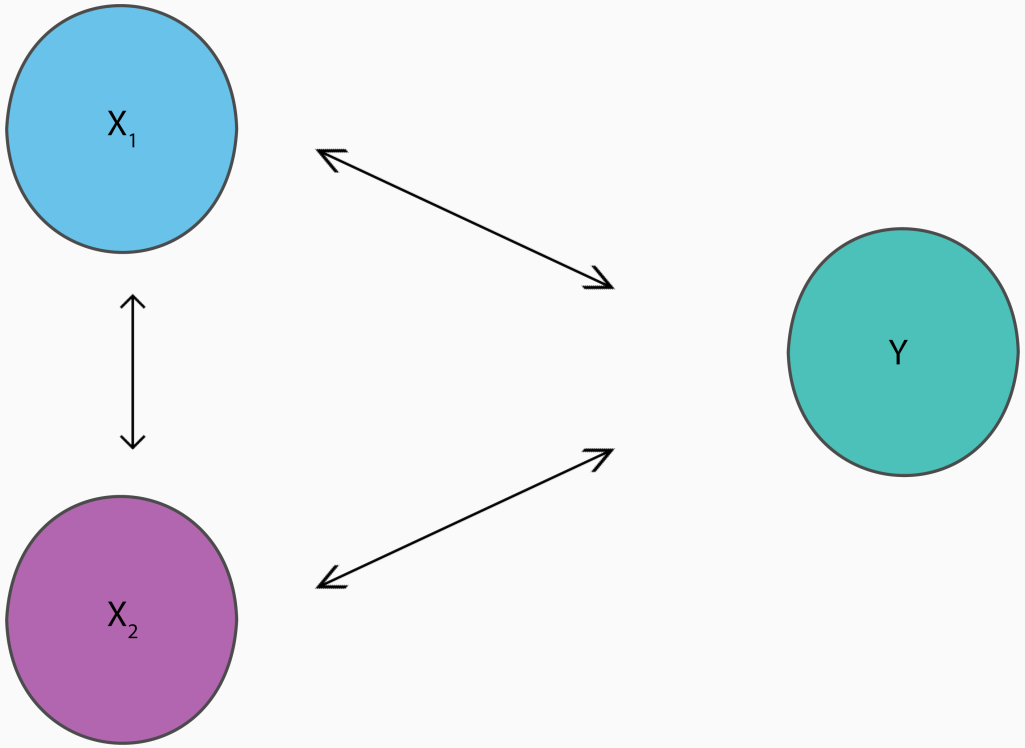


Mult. Parameters, Mult. Interactions

With two parameters, there are 3 relationship that are possible



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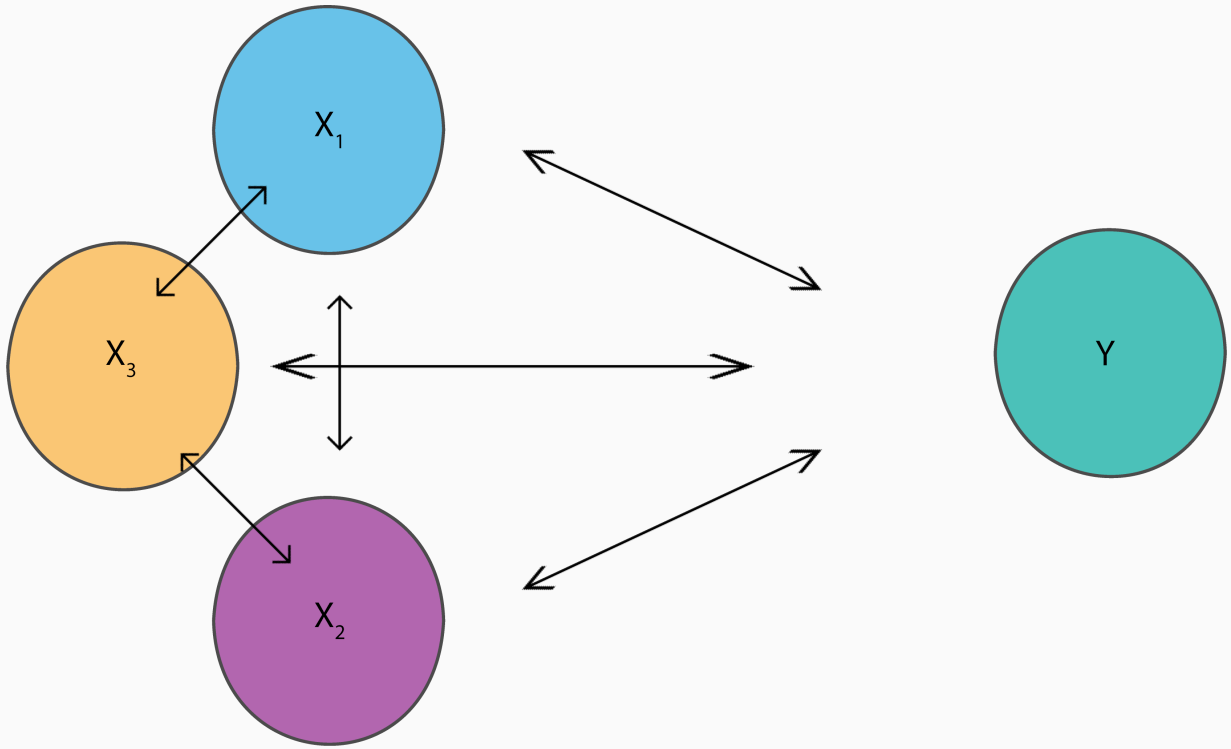
With three parameters, there are 6 relationship that are possible



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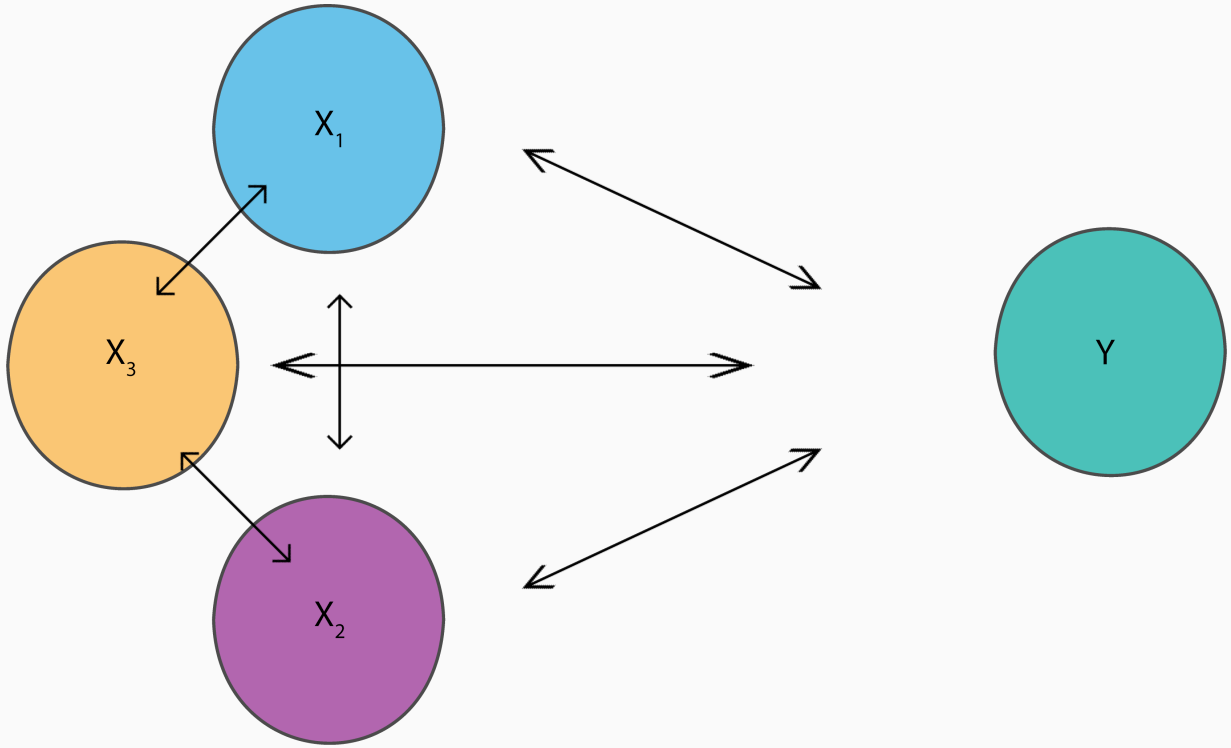
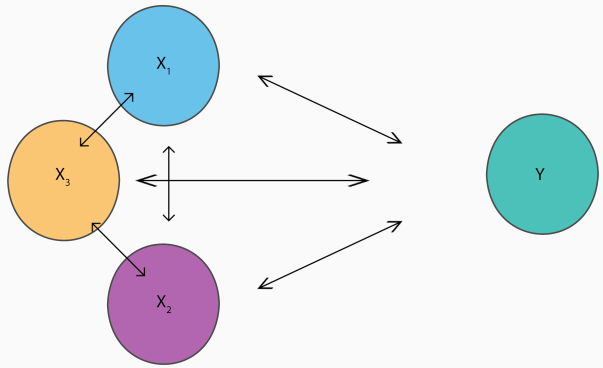


... and it only gets worse as you add more parameters

Collinearity



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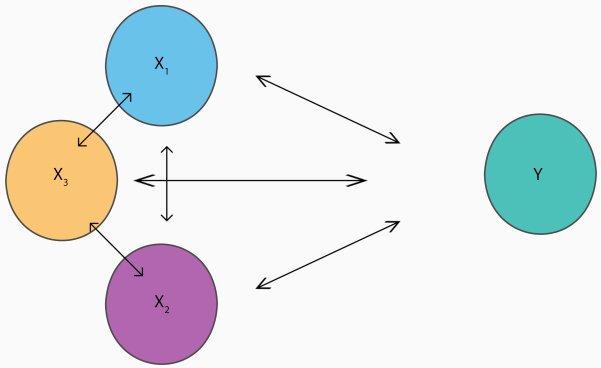
The challenge is that only some of the relationships are of interest, whereas others can introduce identifiability issues



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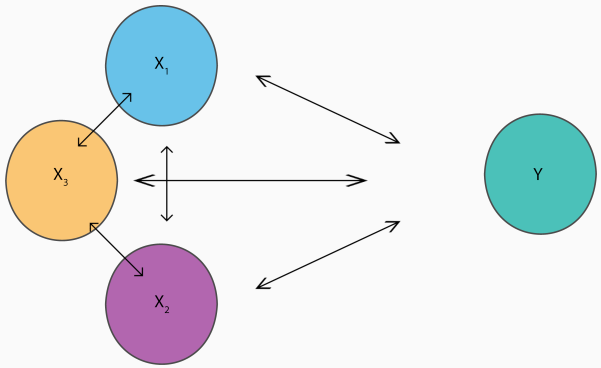

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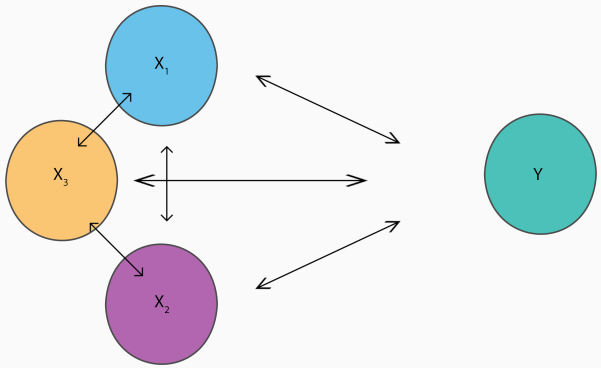

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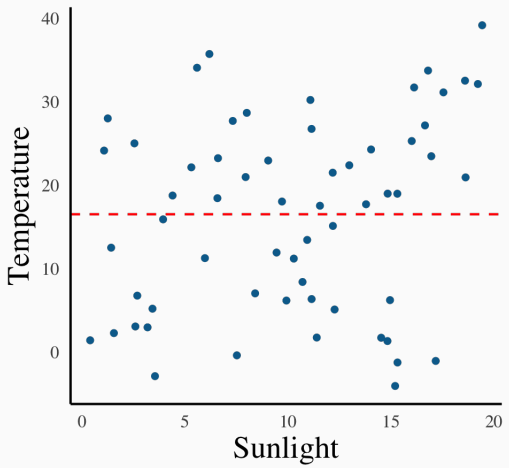

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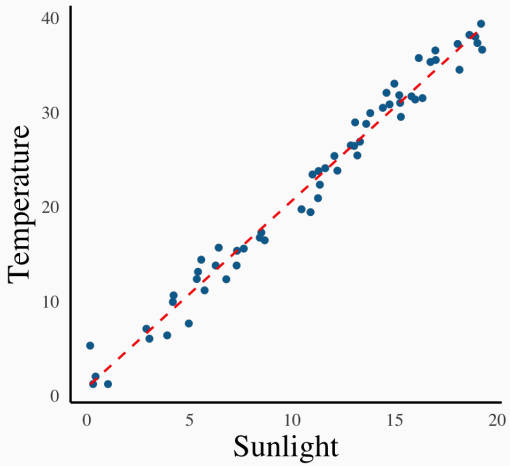

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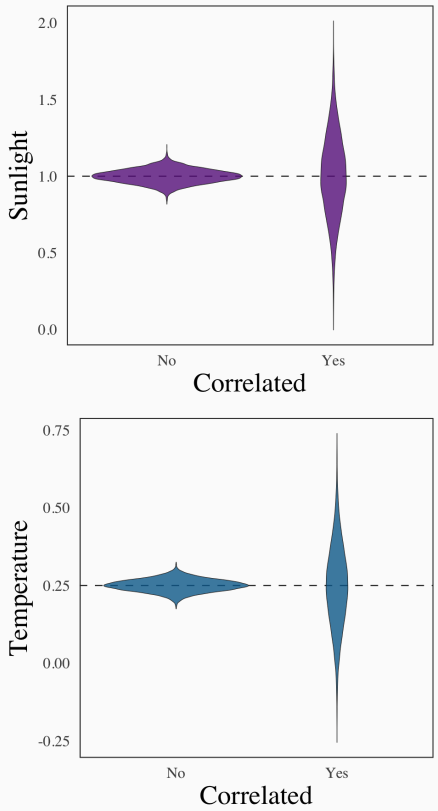

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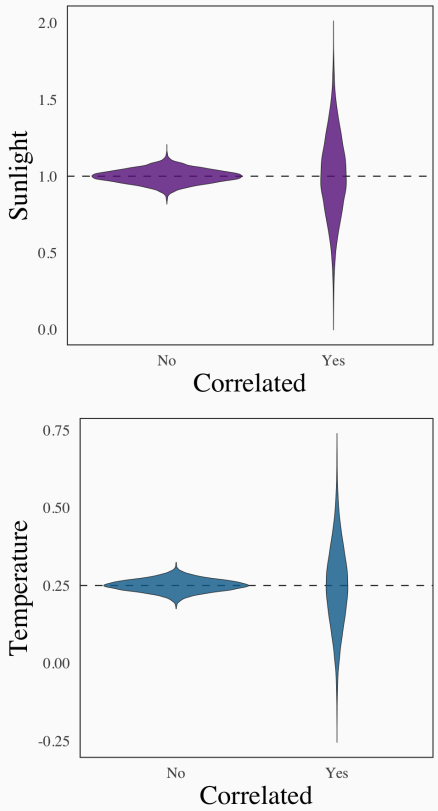

Note how some parameter estimates can even change signs!

Collinearity and Standard Errors

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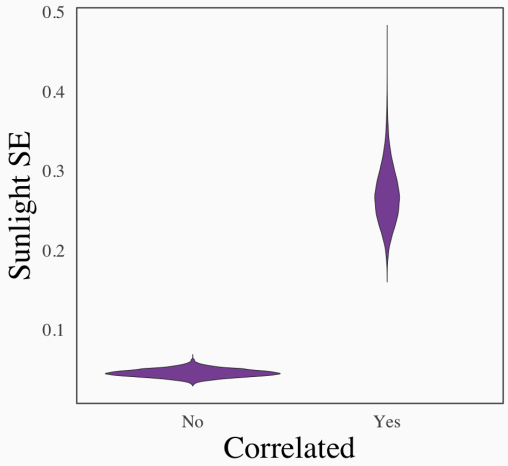
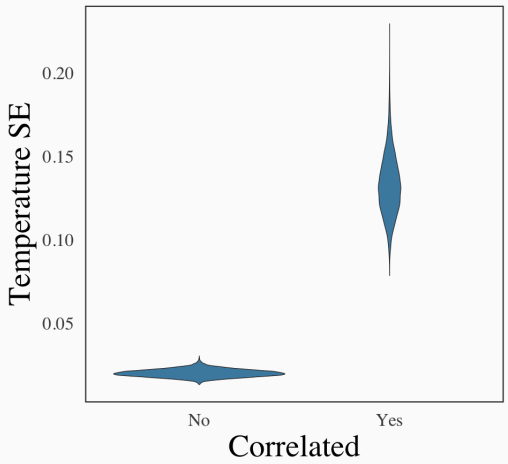
Collinearity also inflates standard errors





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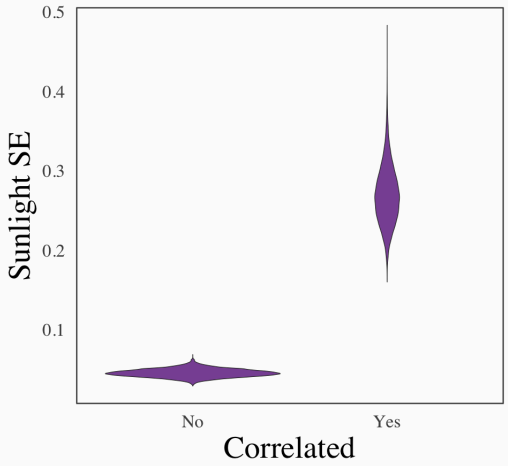
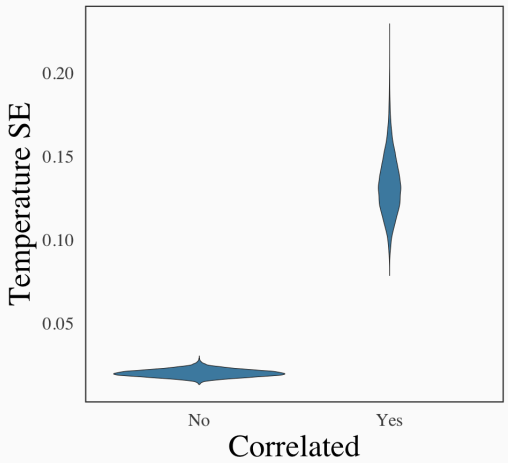
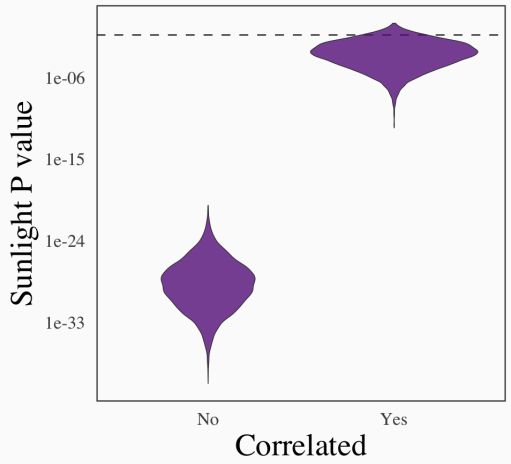
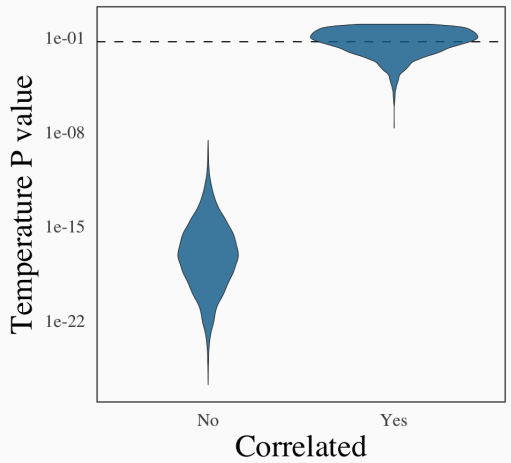




. . . which means that p values are untrustworthy





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- Perform a Principal Component Analysis (PCA) on the data to reduce dimensionality.

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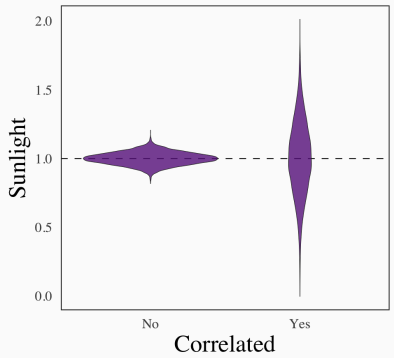
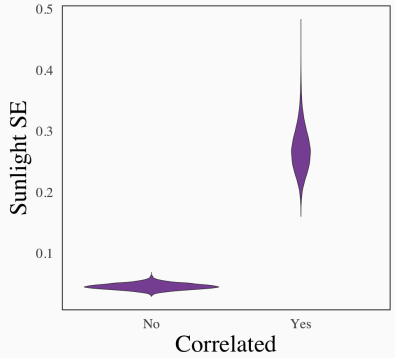
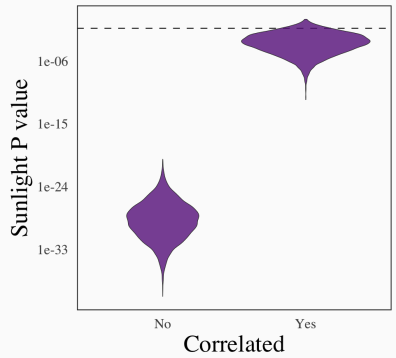
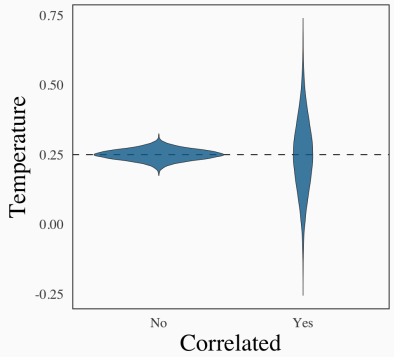
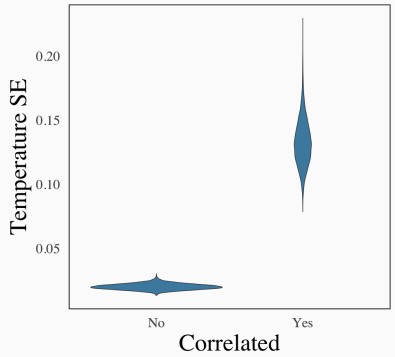
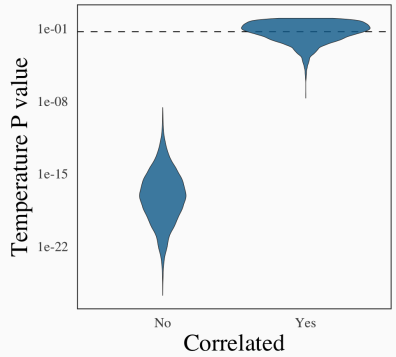






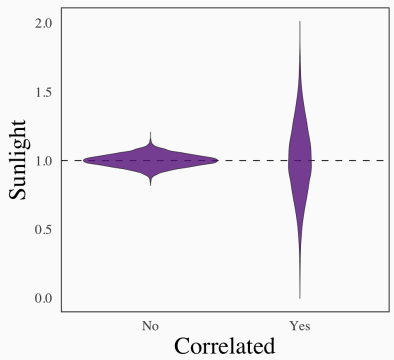


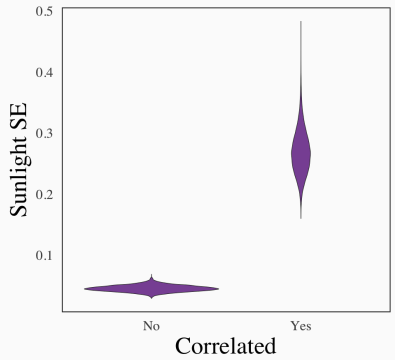


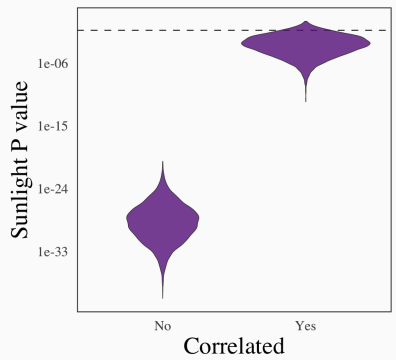


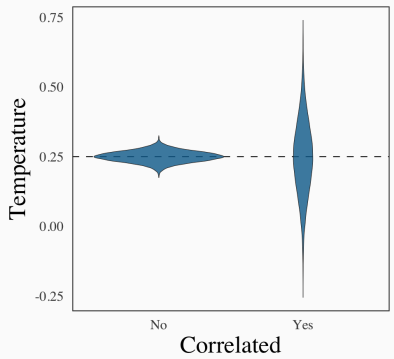
Correction: Do nothing

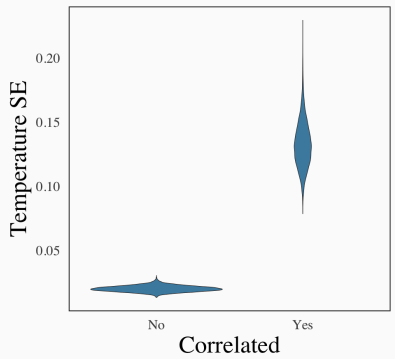
Collinearity doesn't necessarily mean your model is incorrect

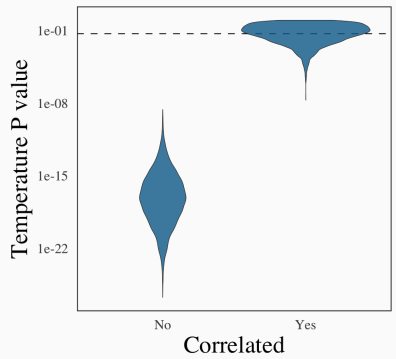












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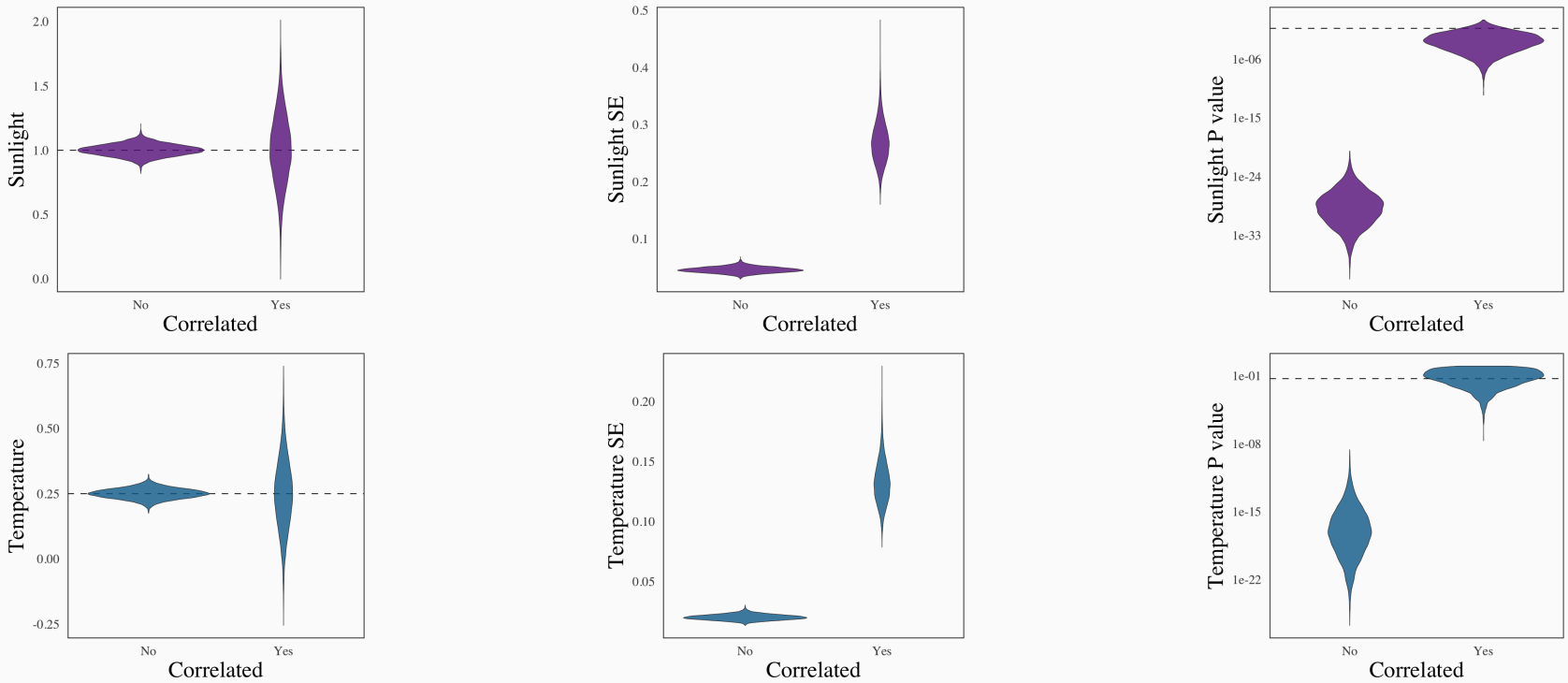

. . . but it does mean you need to be careful when making inferences. Methods like cross-validation or sensitivity analyses should be used to confirm model fit and performance.

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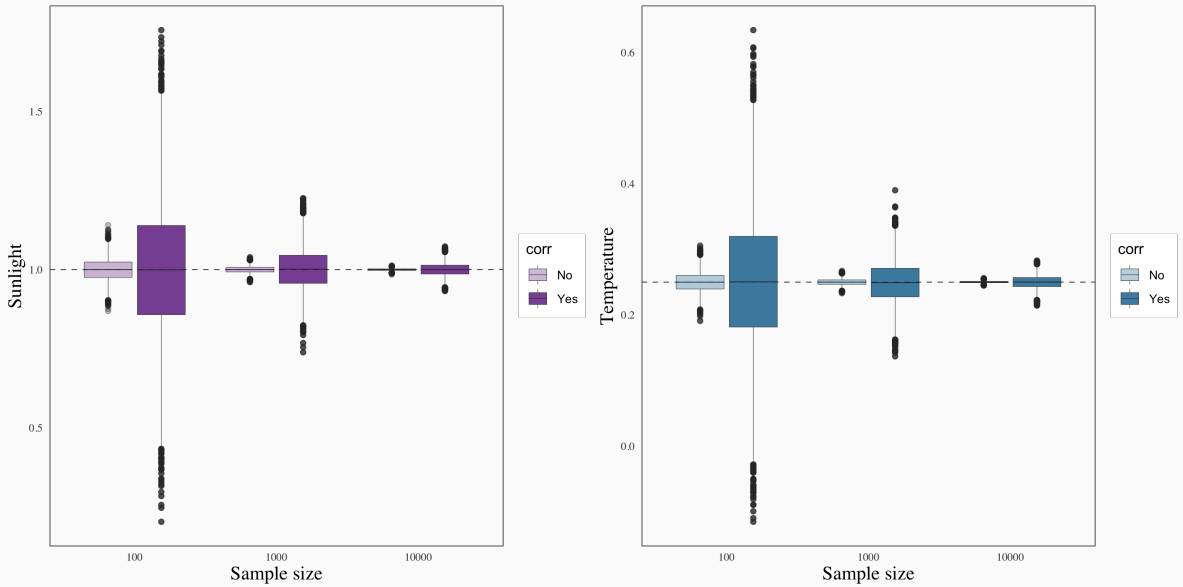
With enough data, the problem of collinearity eventually goes away (terms asymptotic convergence).



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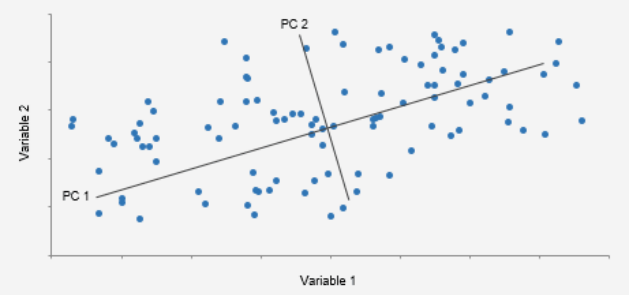

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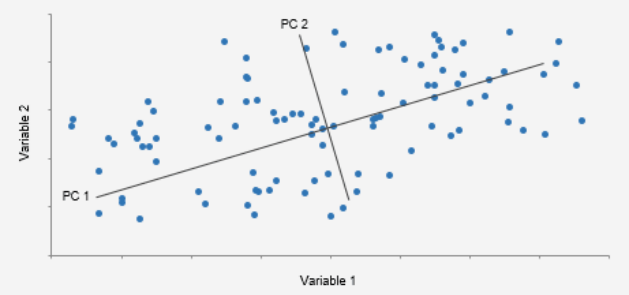

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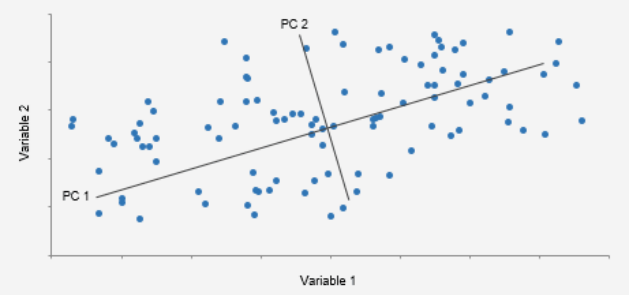

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Analyses are robust to collinearity in the original data, but results can be more difficult to interpret.

Multiple linear regression in action

Example: Ant richness

The Question: Do latitude and elevation have a measurable effect on ant species richness in forest plots in New England?

Gotelli, N.J. & Ellison, A.M. (2002). Biogeography at a regional scale: determinants of ant species density in bogs and forests of New England. Ecology, 83, 1604–1609

Ant richness: The data

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Number of Species	Latitude	Elevation
1.8	41.97	389
2.8	42	8
2.9	42.03	152
2.8	42.05	1
2.2	42.05	210
2.7	42.17	78
1.9	42.19	47
2.5	42.23	491
2.6	42.27	121
2.2	42.31	95
2.3	42.56	274
2.3	42.57	335
1.4	42.58	543
1.6	42.69	323
1.9	43.33	158
1.9	44.06	313
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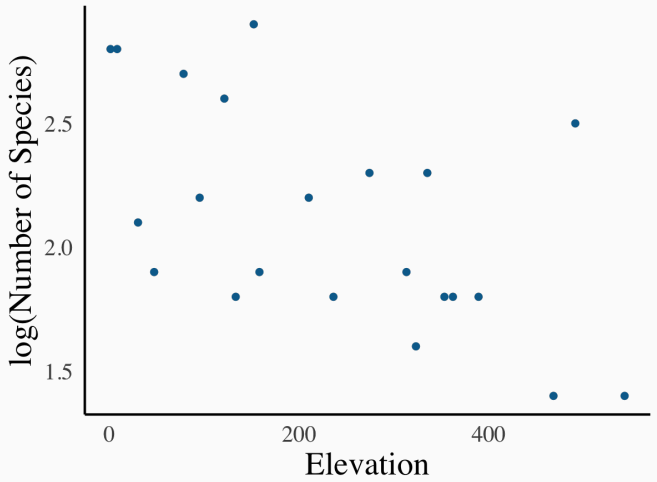

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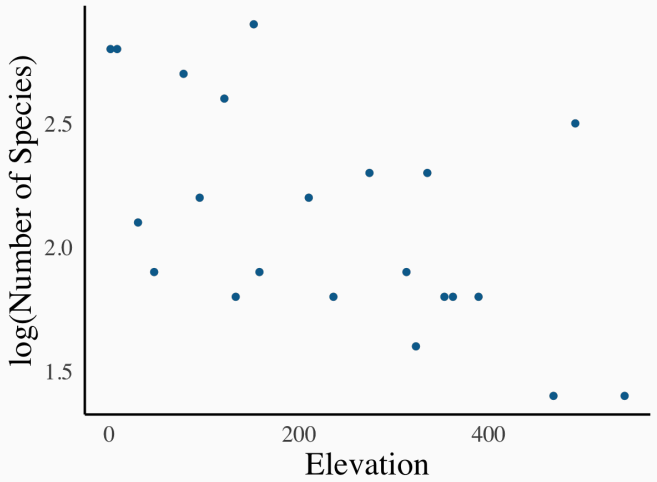

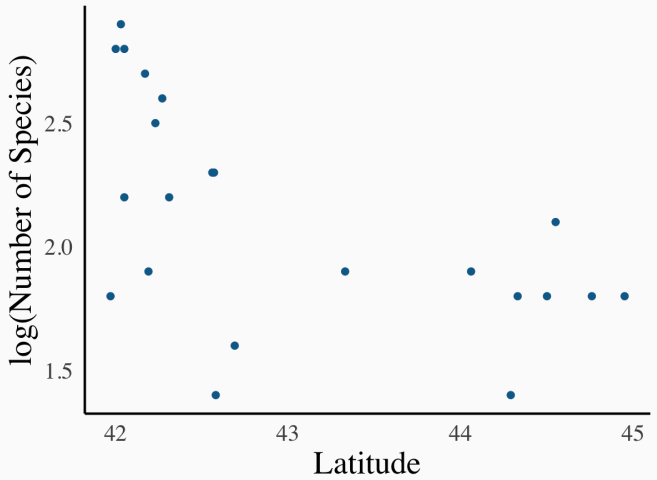

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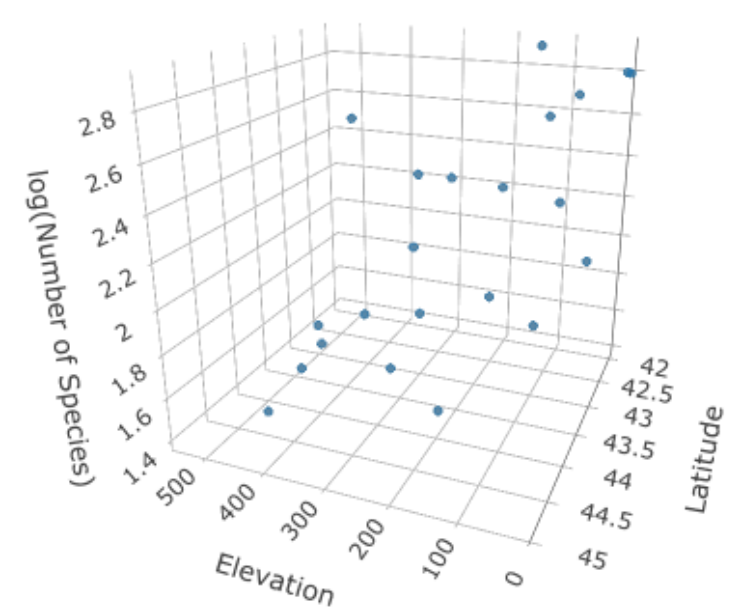
The relationships between richness, latitude, and elevation play out in multiple dimensions.



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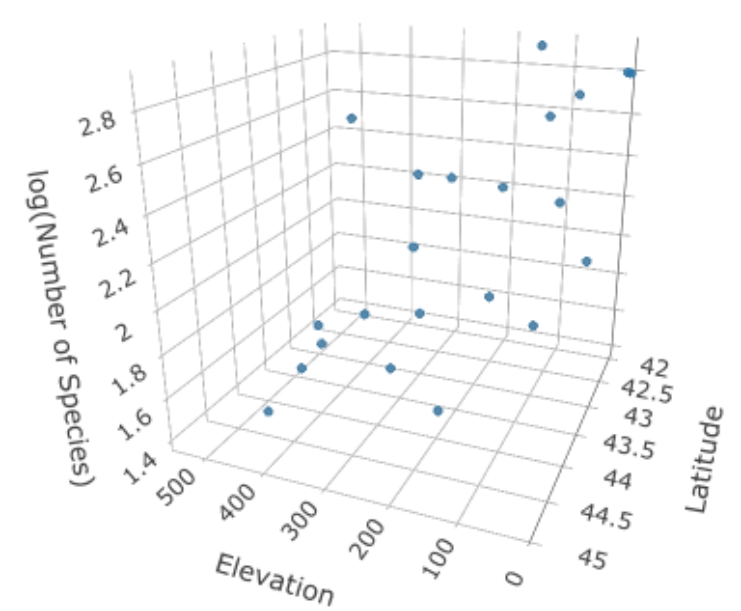

The first step is to check for any relationship between elevation and latitude

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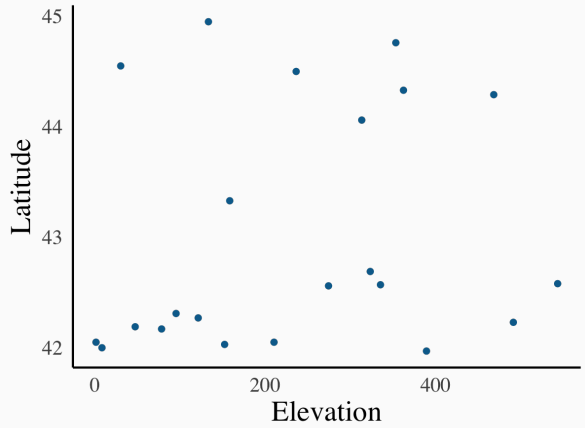
Call:
lm(formula = latitude ~ elevation, data = data)

Residuals:
    Min      1Q  Median      3Q     Max
-1.2357 -0.7383 -0.5589  0.9789  2.0485

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 42.743402   0.411929 103.764   <2e-16 ***
elevation    0.001189   0.001461   0.814    0.425
---

Residual standard error: 1.091 on 20 degrees of freedom
Multiple R-squared:  0.03205, Adjusted R-squared:  -0.01635
F-statistic: 0.6622 on 1 and 20 DF,  p-value: 0.4254
```



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The regression problem in matrix notation is:

$$\begin{pmatrix} 1.8 \\ 2.8 \\ \vdots \\ 1.8 \end{pmatrix} = \begin{pmatrix} 1 & 41.97 & 389 \\ 1 & 42.00 & 8 \\ \vdots & \vdots & \\ 1 & 44.95 & 133 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

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Because the dimensions of the matrices are all compatible we can estimate the parameters values using $\beta = (\mathrm{x}^\mathsf{T}\mathrm{x})^{-1}\mathrm{x}^\mathsf{T}\mathrm{y}$.

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              data$latitude,
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            nrow = nrow(data), ncol = 3)

y <- matrix(data$num_sp,
            nrow = nrow(data), ncol = 1)

xtx <- t(x) %*% x
xtx.inv <- solve(xtx)
xty <- t(x) %*% y

beta <- xtx.inv %*% xty

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Coefficients:
             Estimate Std. Error t value Pr(>t)
(Intercept) 11.1158189  2.7445783   4.050 0.000683 ***
latitude    -0.2018289  0.0641510  -3.146 0.005318 **
elevation   -0.0013729  0.0004259  -3.223 0.004473 **
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Residual standard error: 0.3131 on 19 degrees of freedom
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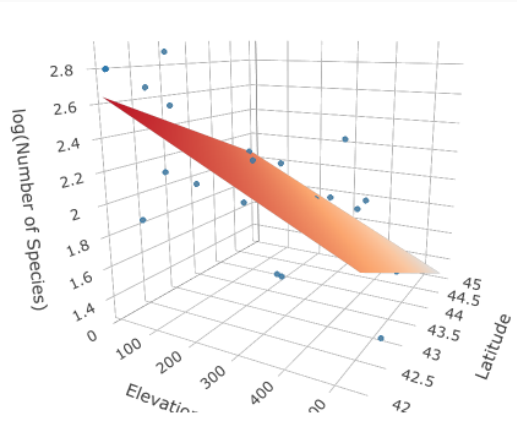

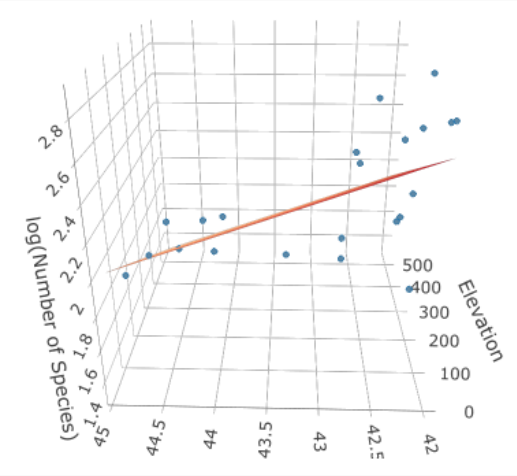

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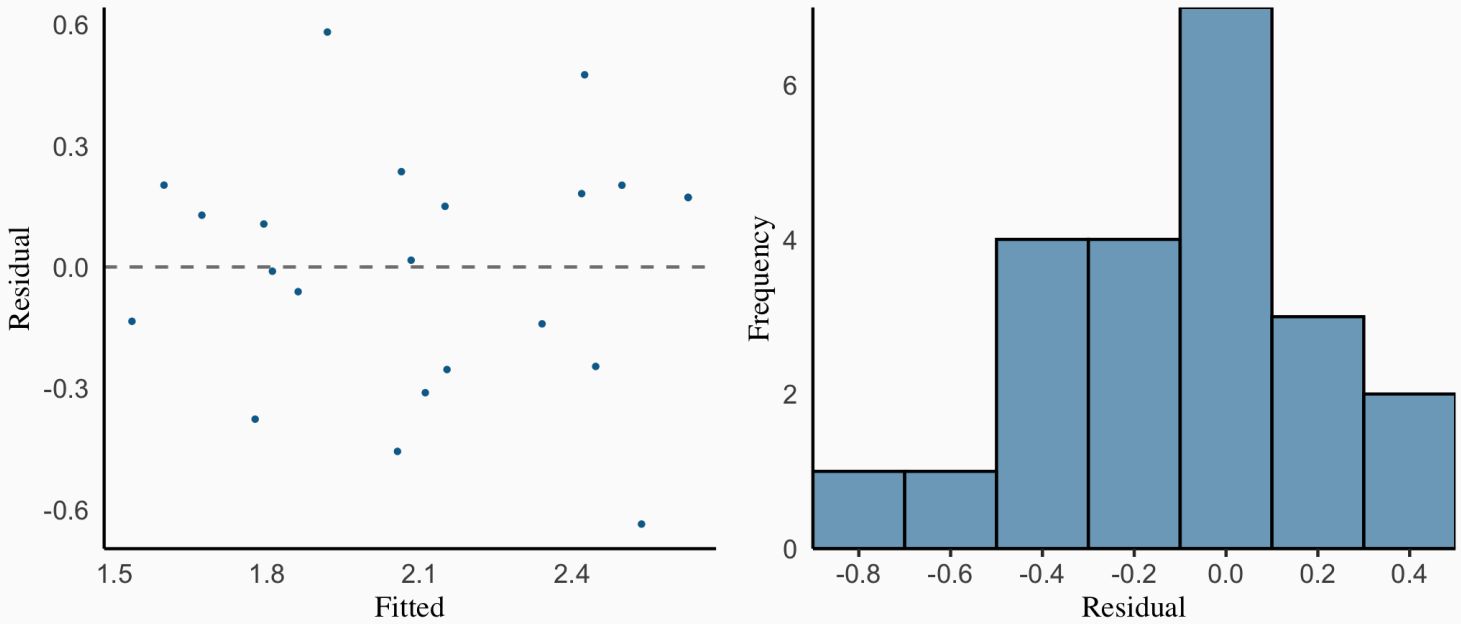

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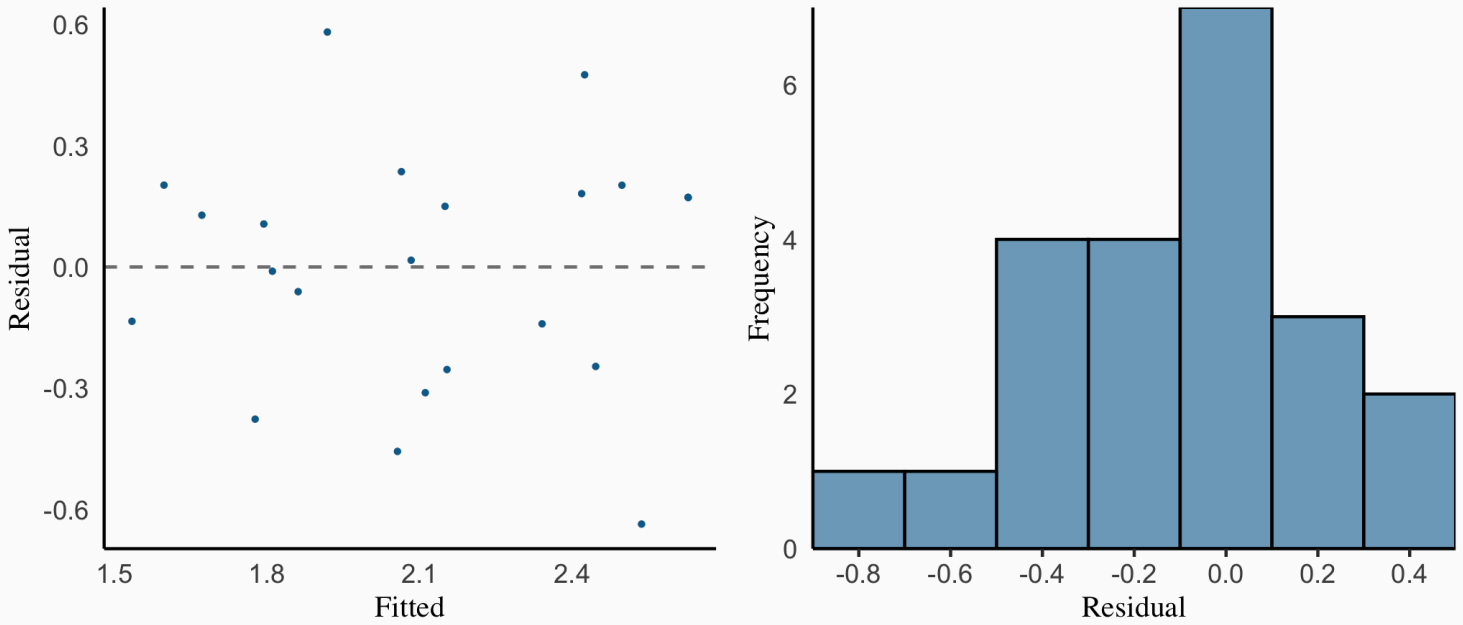

We'll cover residuals in more detail next lecture, but for now suffice it to say these look pretty good.

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It's important to be very careful when adding parameters to a model.

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set.seed(84)
NOISE <- rnorm(22)
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Call:
lm(formula = num_sp ~ latitude + elevation + NOISE, data = data)

Residuals:
     Min       1Q   Median       3Q      Max
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Coefficients:
             Estimate Std. Error t value Pr(>t)
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Residual standard error: 0.2592 on 18 degrees of freedom
Multiple R-squared:  0.7179,  Adjusted R-squared:  0.6708
F-statistic: 15.27 on 3 and 18 DF,  p-value: 3.448e-05
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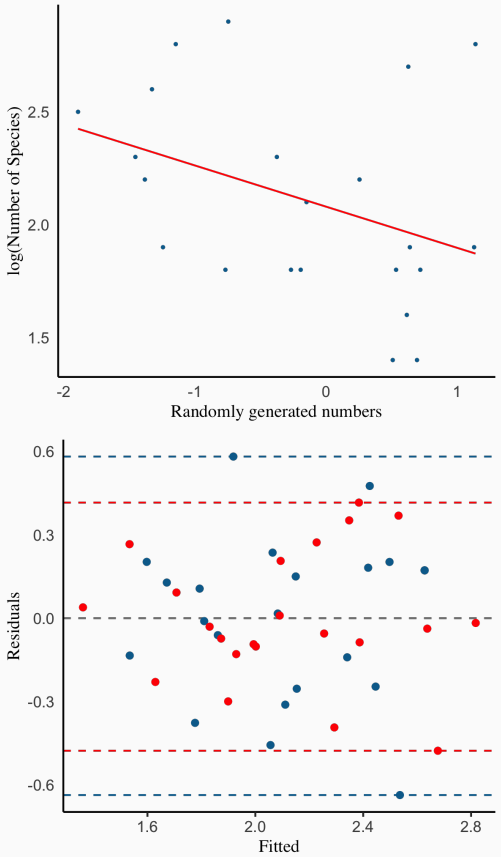

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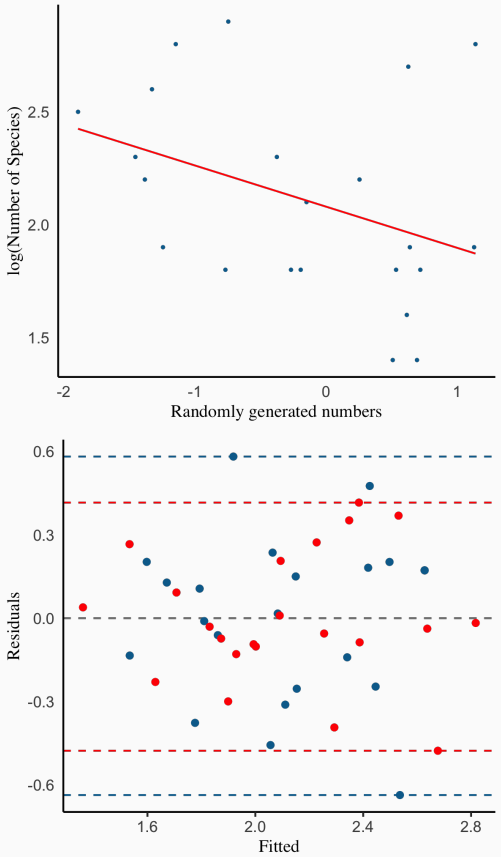

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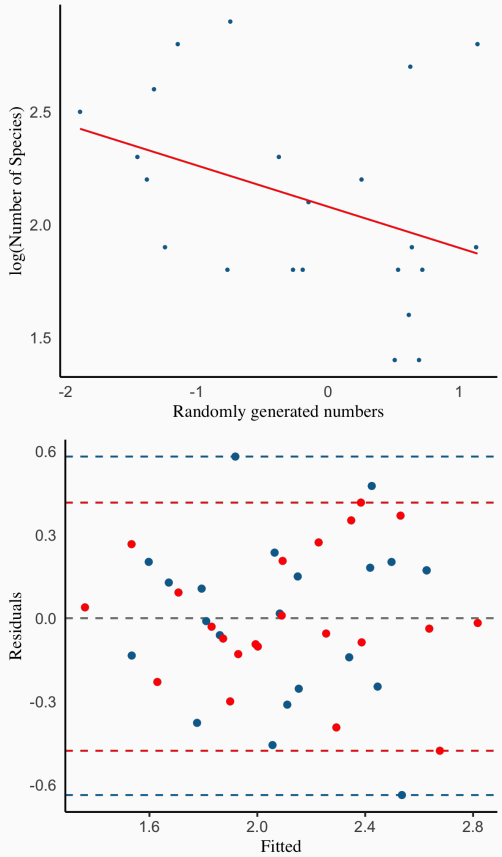

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Interactions between explanatory variables are expressed as a product of the X's: $y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{1i} x_{2i} + \varepsilon_i$

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For a model with two predictor variables and their interaction,

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Parameter Interactions in Action

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It's conceivable that high elevation at high latitude is worse for ants than high elevation at lower latitudes.

Source: WallpaperAccess

Source: Wikipedia

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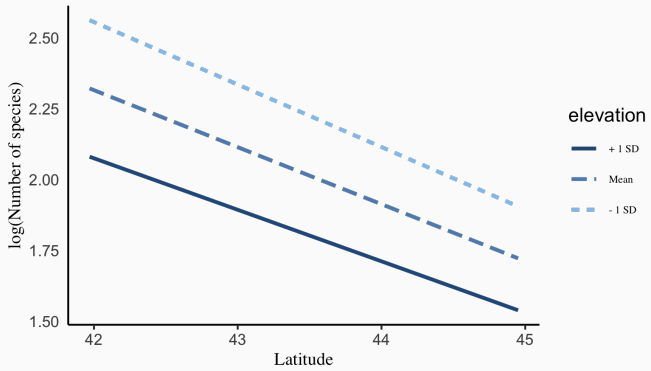

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. . . and always use objective measures to decide on what parameters should be included/discarded.