Solution to Problem Sheet 2

Noor E Mustafa Ferdous
email: nooremf@gmail.com

Solve for problem no. 1

Given

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^{\mu}A^{\mu} \\
&= -\frac{1}{4}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial^{\mu}A^{\nu} - \partial^{\mu}A^{\nu}) - j^{\mu}A^{\mu} \\
&= -\frac{1}{4}(\partial_{\mu}A_{\nu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) - \partial_{\nu}A_{\mu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})) - j^{\mu}A^{\mu} \\
&= -\frac{1}{2}(\partial_{\mu}A_{\nu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})) - j^{\mu}A^{\mu}
\end{aligned}
$$

(a)

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A_{\nu}} &= -j^{\nu} \\
\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} &= -\frac{1}{2}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})) - \frac{1}{2}\partial^{\mu}A^{\nu} + \frac{1}{2}\partial^{\nu}A^{\mu} \\
&= -(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu})) = F^{\mu\nu}
\end{aligned}
$$

Therefore Euler-Langrangian is

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial A_{\nu}} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\nu})} &= -j^{\nu} + \partial_{\mu}F^{\mu}\nu = 0 \\
\partial_{\mu}F^{\mu\nu} &= j^{\nu}
\end{aligned}
$$

(b) taking $\mu = i$ and $\nu = 0$

$$
\begin{aligned}
\partial_i F^{i0} &= j^0 \\
\implies \vec{\nabla} \cdot \vec{E} &= \rho
\end{aligned}
$$

And taking $\nu = j$

$$
\begin{aligned}
\partial_\mu F^{\mu j} &= j^0 \\
\implies \partial_0 F^{0j} + \partial_i F^{ij} &= 0 \\
\implies \partial^0 F_{0j} + \partial^i F_{ij} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\implies \frac{\partial E_j}{\partial t} + \partial^i(-\epsilon_{ijk} B_k) = 0 \\
\implies \frac{\partial E_j}{\partial t} = \partial^i(-\epsilon_{ijk} B_k) \\
\implies \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B}
\end{aligned}
$$

Solve for problem no. 2

(a) Given

$$
\begin{aligned}
&\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \\
&= \partial_\alpha(\partial_\beta A_\gamma - \partial_\gamma A_\beta) + \partial_\beta(\partial_\gamma A_\alpha - \partial_\alpha A_\gamma) + \partial_\gamma(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\
&= \partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta + \partial_\beta \partial_\gamma A_\alpha - \partial_\beta \partial_\alpha A_\gamma + \partial_\gamma \partial_\alpha A_\beta - \partial_\gamma \partial_\beta A_\alpha \\
&= 0
\end{aligned}
$$

Given

$$
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{1}
$$

taking $\alpha = 1, \beta = 2, \gamma = 3$

$$
\begin{aligned}
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0 \\
\implies \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0 \\
\implies \partial^1 B^1 + \partial^2 B^2 + \partial^3 B^3 &= 0 \\
\implies \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0
\end{aligned}
$$

Now taking $\alpha = 0, \beta = i, \gamma = j (i \neq j)$

$$
\begin{aligned}
\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0 \\
\implies \partial^0 F^{ij} + \partial^j F^{0i} + \partial^i F^{j0} &= 0 \\
\implies -\frac{\partial \boldsymbol{B}}{\partial t} + \partial^j E^i - \partial^i E^j &= 0 \\
\implies -\frac{\partial \boldsymbol{B}}{\partial t} &= \epsilon_{ijk} E^k \\
\implies -\frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{\nabla} \times \boldsymbol{E}
\end{aligned}
$$