Solution to Problem Sheet 5

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Solve for problem no. 1

Given

$$\int \prod_{i=1}^{N} dx^i exp\left(-\frac{1}{2}x^T Ax + J^T x \right) = \frac{(2\pi)^{N/2}}{\sqrt{det\ A}} exp\left(\frac{1}{2} J^T A^{-1} J \right)$$

L.H.S

$$\int \prod_{i=1}^{N} dx^i exp\left(-\frac{1}{2}x^T Ax + J^T x \right)$$

where

$$\begin{aligned} x &= (x_1, ..., x_n)\ \epsilon\ \mathbb{R}^n \\ J &= (J_1, ..., J_n)\ \epsilon\ \mathbb{C}^n \\ A &= \begin{pmatrix} A_{11} & A_{12} & . & . & . \\ A_{21} & A_{22} & . & . & . \\ . & . & . & & \\ . & . & & . & \\ . & . & & & A_{nn} \end{pmatrix} \epsilon\ \mathbb{R}^{n\times n} \end{aligned}$$

where A is symmetric and + ve definite. From L.H.S.

$$\int \prod_{i=1}^{N} dx^i exp\left(-\frac{1}{2}x^T Ax + J^T x \right)$$

$x^T Ax$ is

$$\sum_{i,j=1}^{n} x_i A_{ij} x_j$$

which is

$$\sum_{i=1}^{n} \sum_{i<j} (x_i A_{ij} x_j + x_j A_{ji} x_i) + \sum_{i=1}^{n} A_{ii} x_i x_i$$

Any anti symmetric term will drop out. Symmetric matrices are diagonalisable. that is

$$\exists O \in O(n) = A \in \mathbb{R}^{n\times n} | O^T O = I_n$$
$$D = diag(\lambda_1, ..., \lambda_n) \;\; \lambda_i \; are \; the \; eigenvalues \; of \;\; A$$

s.t.

$$A = O^T DO$$
$$x^T Ax = x^t O^T DOx$$

and

$$J^T x = J^T O^T (Ox)$$
$$= (OJ)^T (Ox)$$

Let

$$y = Ox$$
$$y_i = O_{ij} x_j$$

And

$$d^n y = \left|\frac{\partial y}{\partial x}\right| d^n x$$
$$= \left|det \; \frac{\partial y_i}{\partial x_j}\right|$$
$$= |det \; O| \qquad\qquad (\tfrac{\partial y_i}{\partial x_j} = O_{ij})$$
$$= 1$$

and

$$D_{ij} = \lambda_i \delta_{ij}$$

Now

$$\int_{\mathbb{R}} d^n y e^{-\frac{1}{2}y^T D y + J' y} \qquad ((OJ)^T = J')$$
$$= \int_{\mathbb{R}} d^n y e^{-\frac{1}{2}\sum_i \sim j y_i \lambda_i \delta_{ij} y_j + \sum_i J'_i y_i}$$
$$= \int_{\mathbb{R}} d^n y e^{\sum_i (-\frac{1}{2} \sim j y_i^2 \lambda_i + J'_i y_i)}$$
$$= \prod_i \int_{\mathbb{R}} d^n y e^{-\frac{1}{2} \sim j y_i^2 \lambda_i + J'_i y_i}$$
$$= \prod_i \sqrt{\frac{2\pi}{\lambda_i}} e^{\frac{1}{2}\frac{J'_i}{\lambda_i}}$$

Now

$$D^{-1}_{ij} = \frac{\delta_{ij}}{\lambda_i}$$
$$\frac{J'_i}{\lambda_i} = J'^T D^{-1} J'$$
$$= J O^T D^{-1} O J$$
$$= M^{-1}$$

therefore

$$\int \prod_{i=1}^{N} dx^i exp\left(-\frac{1}{2}x^T A x + J^T x \right) = \sqrt{\frac{(2\pi)^n}{det\ M}} e^{\frac{1}{2}J^T M^{-1} J}$$

Solve for problem no. 3

We know

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = \frac{1}{Z_0}\frac{\partial}{\partial J(x_1)}\frac{\partial}{\partial J(x_2)} exp\left[-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right] Z[J]\bigg|_{J=0}$$