

# 1 Out of Time Order Correlator of $H = xp$ model

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to  $1/2$ . The classical zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (1)$$

for  $\text{Re } s > 1$ . By the fundamental theorem of arithmetic, which is also equivalent to the Euler product over primes

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (2)$$

where  $p$  are all the prime numbers.

Zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negative integers (for  $s = -2, -4, -6, \dots$ ). For complex  $s (= \sigma + it)$  (with real part between zero and one), zeta function becomes nontrivial ones. And the Riemann hypothesis is for  $s = \frac{1}{2} - it$  zeta function becomes zero  $\zeta(\frac{1}{2} - it) = 0$ . Hilbert-Pólya conjecture suggests that the imaginary parts of the nontrivial zeros are the eigenvalues of a self-adjoint hamiltonian operator  $\hat{H}$ . It is also one of the approaches to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-Pólya conjecture should take the form[1]

$$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \quad (3)$$

Here  $x$  and  $p$  are position and momentum operators. This 1d classical Hamiltonian ( $H = xp$ ) related to the Riemann zeros.[1] Berry proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obtained by quantization of a classical chaotic hamiltonian, whose periodic orbits are labeled by the prime numbers. Connes took the adelic approach to introduce  $H = xp$  [2]. He showed that using different semiclassical regularization, Riemann zeros appear as missing spectral lines in a continuum.

Now we look into the Berry-Keating and Connes semiclassical approaches to  $H = xp$

## 2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is [1, 2]

$$H_0^{cl} = xp \quad (4)$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \quad p(t) = p_0 e^{-t} \quad (5)$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of regularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space:  $|x| > l_x$  and  $|p| > l_p$ , with  $l_x l_p = 2\pi\hbar$ . Connes choosed  $|x| < \Lambda$  and  $|p| < \Lambda$ , where  $\Lambda$  is a cutoff. German Sierra introduced us a third regularization,  $l_x < x < \Lambda$  combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number  $\mathcal{N}(E)$  with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \quad (6)$$

Where A is the area of the allowed phase space region below the curve  $E = xp$ . So the the number of semiclassical states will be for Berry-Keating

regularization

$$\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h} \int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + \dots\dots \\
&= \frac{1}{h} \left[ \int_{l_x}^{\frac{E}{l_p}} dx \left[ \frac{E}{x} - l_p \right] \right] \\
&= \frac{1}{h} \left[ E [\ln x]_{l_x}^{\frac{E}{l_p}} - l_p \left[ \frac{E}{l_p} - l_x \right] \right] \\
&= \frac{1}{h} \left[ E \ln \frac{E}{l_x l_p} - E - l_x l_p \right] \\
&= \frac{1}{h} \left[ E \ln \frac{E}{l_x l_p} - E - h \right] \\
&= \frac{E}{h} \left[ \ln \frac{E}{l_x l_p} - 1 \right] + 1 \\
&= \frac{E}{2\pi\hbar} \left[ \ln \frac{E}{2\pi\hbar} - 1 \right] + 1
\end{aligned} \tag{7}$$

adding Maslov phase  $(-\frac{1}{8})$  and  $\hbar = 1$ , it becomes

$$\mathcal{N}_{BK}(E) = \frac{E}{2\pi} \left[ \ln \frac{E}{2\pi} - 1 \right] + \frac{7}{8} + \dots\dots, \quad E \gg 1 \tag{8}$$

The exact formula for the Riemann zeros,  $\mathcal{N}_R(E)$  contains a fluctuation term which depends on the zeta function.[3]

$$\begin{aligned}
\mathcal{N}_R(E) &= \langle \mathcal{N} \rangle + \mathcal{N}_{fl}(E) \\
\langle \mathcal{N}(E) \rangle &= \frac{1}{\pi} \text{Im} \ln \left[ \Gamma \frac{1}{2} \left( \frac{1}{2} - iE \right) \right] - \frac{E}{2\pi} \ln \pi + 1 \\
\mathcal{N}_{fl}(E) &= \frac{1}{\pi} \text{Im} \ln \left[ \zeta \left( \frac{1}{2} - iE \right) \right]
\end{aligned} \tag{9}$$

Bery-Keatin took this result and analogies between formulae in Number Theory and Quantum Chaos, they pointed the quantization of classical chaotic Hamiltonian give rise to the zeros as point like spectra.[1, 4] Whereas Connes found the number of semiclassical states diverges in the limit where the cutoff  $\Lambda$  goes to infinity, and that there is a finite size correction given by minus the average position of the Riemann zeros.

$$\begin{aligned}
\mathcal{N}_c(E) &= \frac{1}{h} \left[ 2E - \left( \frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \int_{\frac{E}{x}}^{\frac{E}{\Lambda}} dp \right] \\
&= \frac{1}{h} \left[ 2E - \left( \frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \left[ \frac{E}{x} - \frac{E}{\Lambda} \right] \right] \\
&= \frac{1}{h} \left[ 2E - \left( \frac{E}{\Lambda} \right)^2 + E [\ln x]_{\frac{E}{\Lambda}}^{\Lambda} - \frac{E}{\Lambda} \left[ \Lambda - \frac{E}{\Lambda} \right] \right] \\
&= \frac{1}{h} \left[ 2E - \left( \frac{E}{\Lambda} \right)^2 + E \left[ \ln \frac{\Lambda^2}{E} \right] - E + \left( \frac{E}{\Lambda} \right)^2 \right] \\
&= \frac{1}{h} \left[ E + E \left[ \ln \frac{\Lambda^2}{E} \right] \right] \\
&= \frac{1}{h} \left[ E + E \left[ \ln \frac{\Lambda^2}{E} \frac{2\pi}{2\pi} \right] \right] \\
&= \frac{E}{h} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{h} \left[ \ln \frac{E}{2\pi} - 1 \right] \\
&= \frac{E}{2\pi} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left[ \ln \frac{E}{2\pi} - 1 \right] \quad [taking \quad \hbar = 1]
\end{aligned} \tag{10}$$

This result les to the missing spectral interpretation of the Riemann zeros, according to which there is a continuum of eginstates (represented by the term  $\frac{E}{\pi} \ln \Lambda$  in  $\mathcal{N}(E)$ ) where states assoicated with Riemann zeros are missing.

Finally, in the S-regularization the number of semiclassical states diverges as  $\frac{E}{2\pi} \ln \frac{\Lambda}{l_x}$  suggesting a continuum spectrum, ike in Connes's approach. But there is no finite size correction to that formula, and cosequently the possible connection to the Riemann zeros is lost.

### 3 Quantization of $xp$ and $\frac{1}{xp}$

#### 3.1 The Hamitoninan $H_0 = xp$

Here we construst a self adjoint operator  $H_0$  which acts on a Hilbert space  $L^2(a, b)$  of square integrable function in the interval  $(a, b)$ . Taking  $x \geq 0$ ,

there are four possible intervals:  $a = 0, l_x$  and  $b = \Lambda, \infty$  where  $l_x$  and  $\Lambda$  were introduced (we shall take  $l_x$  and  $\Lambda = N > 1$ ). Berry-Keating defined the quantum Hamiltonian  $H_0$  as the normal ordered expression

$$H_0 = \frac{1}{2}(xp + px) \quad (11)$$

where  $p = -i\hbar \frac{d}{dx}$ . If  $x \geq 0$ , Eq. (11) is equivalent to

$$H_0 = \sqrt{x}p\sqrt{x} = -i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x} \quad (12)$$

This is a symmetric operator acting on a certain domain of the Hilbert space  $L^2(a, b)$ , By definition, if an operator is symmetric (or Hermitian)[5]

$$\langle \psi | H_0 \phi \rangle = \langle \psi H_0 | \phi \rangle \quad (13)$$

or with limit,

$$\langle \psi | H_0 \phi \rangle - \langle \psi H_0 | \phi \rangle = i\hbar [a\psi^*(a)\phi(a) - b\phi^*(b)\psi(b)] = 0 \quad (14)$$

which is satisfied if both  $\psi(x)$  and  $\phi(x)$  vanish at the points  $a, b$ . von Neumann Theorem of deficiency indices states that, an operator is symmetric if its deficiency indices  $n_{\pm}$  are equal.[6]. Deficiency indices (or the defect numbers) of a closable symmetric operator  $T$  are cardinal number  $S$

$$\begin{aligned} n_+ &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda > 0 \\ n_- &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda < 0 \end{aligned} \quad (15)$$

If  $T$  is densely defined and symmetric, then  $T$  is closable, and by formula  $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}$

$$\begin{aligned} n_+ &:= \dim \mathcal{N}(T^* - i\mathbb{1}) = \dim \mathcal{N}(T^* - \lambda\mathbb{1}) & \text{Im } \lambda > 0 \\ n_- &:= \dim \mathcal{N}(T^* + i\mathbb{1}) = \dim \mathcal{N}(T^* + \lambda\mathbb{1}) & \text{Im } \lambda < 0 \end{aligned} \quad (16)$$

By definition  $n_{\pm}(T) = \dim \mathcal{N}(T^* \mp iT)$

Again if  $T$  is a symmetric operator, then

$$\begin{aligned} K_+ &= \ker (i\mathbb{1} - T^*) = \text{Ran } (i\mathbb{1} + T)^{\perp} \\ K_- &= \ker (i\mathbb{1} + T^*) = \text{Ran } (-i\mathbb{1} + T)^{\perp} \end{aligned} \quad (17)$$

$K_+$  and  $K_-$  are called the deficiency subspaces of  $T$ , The pair of numbers  $n_+$ ,  $n_-$  given by  $n_+(T) = \dim[K_+]$ ,  $n_-(T) = \dim[K_-]$  are called deficiency indices of  $T$ .

von Neumann Theorem for deficiency indices states that if  $T$  is a closed operator with deficiency indices  $n_+$  and  $n_-$ . Then

(1)  $T$  is symmetric if and only if  $n_+ = n_- = 0$  and self adjoint if  $\mathcal{D}(T) = \mathcal{D}(T^*)$

(2)  $T$  is symmetric and self adjoint and also has many self adjoint extensions if and only if  $n_+ = n_- \neq 0$  and  $\mathcal{D}(T) = \mathcal{D}(T^*)$ . There is one-one correspondence between self adjoint extensions of  $T$  and unitary maps from  $K_+$  onto  $K_-$

(3) If either  $n_+ = 0 \neq n_-$  or  $n_- = 0 \neq n_+$  then  $T$  is not symmetric and has no nontrivial self adjoint extension (such operators are called maximal symmetric operator).

So this indices counts the number of solutions of the equation, which comes from the deficiency spaces for subsystem  $T$

$$n_{\pm} = \ker \left( -H_0^{\dagger} - \mp i1 \right) \quad (18)$$

$$H_0^{\dagger} \psi_{\pm} = \pm i \hbar \lambda \psi_{\pm} \quad (19)$$

## References

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