Assignment

$$\mathcal{L} = \psi^*(i\frac{\partial\psi}{\partial t} - \mathcal{H}\psi) \tag{1}$$

Or equivalent Lagrangian density

$$\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2m}\nabla\psi^*\nabla\psi - \psi^*V\psi \tag{2}$$

Show that equation (1) and (2) differ by a divergent and lead to the same Euler Lagrangian equation.

Ans

Given

$$\mathcal{L} = \psi^*(i\frac{\partial \psi}{\partial t} - \mathcal{H}\psi) \tag{3}$$

$$\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2m}\nabla\psi^*\nabla\psi - \psi^* V\psi \tag{4}$$

here from equation (3)

$$\begin{aligned}
\mathcal{L} &= \psi^*(i\frac{\partial \psi}{\partial t} - \mathcal{H}\psi) \\
&= \psi^* i\frac{\partial \psi}{\partial t} - \frac{1}{2m}\psi^*\nabla^2\psi - \psi^* V\psi \\
&= i(\frac{\partial}{\partial t}(\psi^*\psi) - \frac{\partial \psi}{\partial t}\psi) + \frac{1}{2m}\nabla\cdot(\psi^*\nabla\psi) - \frac{1}{2m}(\nabla\psi^*)\cdot(\nabla\psi) - \psi^* V\psi
\end{aligned}$$

Comparing with equation (4) only 2nd term differs.
Again taking equation (3)

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\frac{\partial \psi}{\partial t} - \mathcal{H}\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} = 0$$

Euler Lagrangian equation

$$\begin{gathered}
\frac{\partial \mathcal{L}}{\partial \psi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} = 0 \\
\implies i\frac{\partial \psi}{\partial t} - \mathcal{H}\psi = 0
\end{gathered}$$

and from equation (4)

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\frac{\partial \psi}{\partial t} - V\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} = -\frac{i}{2}\psi + \frac{\hbar}{2m}\nabla^2\psi$$

Now

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial \psi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)} = 0 \\
&\implies \frac{i}{2}\frac{\partial \psi}{\partial t} - V\psi + \frac{i}{2}\frac{\partial \psi}{\partial t} - \frac{\hbar}{2m}\nabla^2\psi = 0 \\
&\implies i\frac{\partial \psi}{\partial t} - \frac{\hbar}{2m}\nabla^2\psi - V\psi = 0 \\
&\implies i\frac{\partial \psi}{\partial t} - \mathcal{H}\psi = 0
\end{aligned}$$

Equation (3) and (4) show us same Euler Lagrangian equation.