Solution to Problem Sheet 2

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Solve for problem no. 1

Given

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \tag{1}$$

From L.H.S.

$$\begin{aligned}
\left[\gamma^{\kappa}\gamma^{\lambda}, \gamma^{\mu}\gamma^{\nu}\right] &= \gamma^{\kappa}\left[\gamma^{\lambda}, \gamma^{\mu}\gamma^{\nu}\right] + \left[\gamma^{\kappa}, \gamma^{\mu}\gamma^{\nu}\right]\gamma^{\lambda} \\
&= \gamma^{\kappa}\left(\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\right) + \left(\gamma^{\kappa}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\nu}\gamma^{\kappa}\right)\gamma^{\lambda} \\
&= \gamma^{\kappa}\left(\left(\gamma^{\lambda}\gamma^{\mu} + \gamma^{\mu}\gamma^{\lambda}\right)\gamma^{\nu} - \gamma^{\mu}\left(\gamma^{\lambda}\gamma^{\nu} + \gamma^{\nu}\gamma^{\lambda}\right) + \left(\left(\gamma^{\kappa}\gamma^{\mu} + \gamma^{\mu}\gamma^{\kappa}\right)\gamma^{\nu} - \gamma^{\mu}\left(\gamma^{\nu}\gamma^{\kappa} + \gamma^{\kappa}\gamma^{\nu}\right)\right)\gamma^{\lambda}\right) \\
&= \gamma^{\kappa}\left(\left(\left\{\gamma^{\lambda}, \gamma^{\mu}\right\}\gamma^{\nu} - \gamma^{\mu}\left\{\gamma^{\lambda}, \gamma^{\nu}\right\}\right) + \left(\left\{\gamma^{\kappa}, \gamma^{\mu}\right\}\gamma^{\nu} - \gamma^{\mu}\left\{\gamma^{\nu}, \gamma^{\kappa}\right\}\right)\gamma^{\lambda}\right) \\
\left[\gamma^{\kappa}\gamma^{\lambda}, \gamma^{\mu}\gamma^{\nu}\right] &= 2\eta^{\lambda\mu}\gamma^{\kappa}\gamma^{\nu} - 2\eta^{\lambda\nu}\gamma^{\kappa}\gamma\mu + 2\eta^{\kappa\mu}\gamma^{\nu}\gamma^{\lambda} - 2\eta^{\nu\kappa}\gamma^{\mu}\gamma^{\lambda} \qquad \text{(Showed)}
\end{aligned}$$

$$\begin{aligned}
(a)\; Tr(\gamma^{\mu}) &= Tr(\gamma^{\mu}\gamma_5\gamma_5) \qquad \text{([because } (\gamma_5)^2 = 1\text{])} \\
&= -Tr(\gamma_5\gamma^{\mu}\gamma_5) = -Tr(\gamma_5\gamma^{\mu}\gamma_5) \\
\implies Tr(\gamma^{\mu}\gamma_5\gamma_5) &= -Tr(\gamma_5\gamma^{\mu}\gamma_5) = Tr(\gamma_5\gamma^{\mu}\gamma_5) \\
Tr(\gamma^{\mu}\gamma_5\gamma_5) &= 0 \\
\implies Tr(\gamma^{\mu}) &= 0
\end{aligned}$$

$$
\begin{aligned}
(b)\ Tr(\gamma^\mu\gamma^\nu) &= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu) + Tr(\gamma^\mu\gamma^\nu)\right) \\
&= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu) + Tr(\gamma^\nu\gamma^\mu)\right) \\
&= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\right) \\
&= \frac{1}{2} Tr\left\{\gamma^\mu, \gamma^\nu\right\} \\
&= \frac{1}{2} Tr 2\eta^{\mu\nu} \\
&= \frac{1}{2} 2\eta^{\mu\nu} Tr(\mathbb{1}) \\
&= 4\eta^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
(d)\ (\gamma^5)^2 &= i\gamma_0\gamma_1\gamma_2\gamma_3 \cdot i\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= -(-1)\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_1\gamma_2\gamma_3\gamma_2\gamma_3 \qquad\qquad = -\gamma_2\gamma_3\gamma_2\gamma_3 \\
&= \gamma_2\gamma_2\gamma_3\gamma_3 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
(e)\ Tr(\gamma^5) &= Tr(\gamma^5\gamma^0\gamma^0) \qquad\qquad ((\gamma^0)^2 = 1) \\
&= -Tr(\gamma^0\gamma^5\gamma 0) = Tr(\gamma^0\gamma^5\gamma 0) = 0
\end{aligned}
$$

Solution to problem 2

Given,

$$
\psi(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right] \tag{2}
$$

And

$$\psi^{\dagger}(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s\dagger}_{\vec{p}} u^{s\dagger}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c^{s}_{\vec{p}} v^{s\dagger}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right] \tag{3}$$

Now

$$\left\{ \psi_{\alpha}(\vec{x}), \psi^{\dagger}_{\beta}(\vec{y}) \right\}$$

$$= \left\{ \sum_{\alpha=1}^{4} \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right], \right.$$

$$\left. \sum_{\beta=1}^{4} \sum_{r=1}^{2} \int \frac{d^3p'}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p'}\cdot\vec{y}} + c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p'}\cdot\vec{y}} \right] \right\}$$

$$= \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \sum_{s=1}^{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}} + b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} \right.$$

$$+ c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} + b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}}$$

$$\left. + c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} + b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) e^{-i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} \right]$$

$$= \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \sum_{s=1}^{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[\left\{ b^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} u^{s}_{\alpha}(\vec{p}) u^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}} \right.$$

$$+ \left\{ b^{s}_{\vec{p}}, c^{r\dagger}_{\vec{p'}} \right\} u^{s}_{\alpha}(\vec{p}) v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} + \left\{ c^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} v^{s}_{\alpha}(\vec{p}) u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}}$$

$$\left. + \left\{ c^{s}_{\vec{p}}, c^{r\dagger}_{\vec{p'}} \right\} v^{s}_{\alpha}(\vec{p}) v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}} \right] \tag{4}$$

Given

$$\left\{ b^{s}_{\vec{p}}, b^{r}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, c^{r}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, b^{r}_{\vec{p'}} \right\} = ... = 0 \tag{5}$$

$$\left\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{p'}}\right\} = \left\{c^s_{\vec{p}}, c^{r\dagger}_{\vec{p'}}\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p} - \vec{p'}) \tag{6}$$

Therefore eqn(4) will be

$$\left\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\} =$$

$$\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{s=1}^{2}\sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\frac{1}{\sqrt{2E_{\vec{p'}}}}\Big[(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p} - \vec{p'})u^s_\alpha(\vec{p})u^{r\dagger}_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x} - i\vec{p'}\cdot\vec{y}}$$

$$+ (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p} - \vec{p'})v^s_\alpha(\vec{p})v^{r\dagger}_\beta(\vec{p'})e^{-i\vec{p}\cdot\vec{x} + i\vec{p'}\cdot\vec{y}}\Big]$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^6}\frac{1}{2E_{\vec{p}}}\left[(2\pi)^3u_\alpha(\vec{p})u^\dagger_\beta(\vec{p})e^{i\vec{p}\cdot(\vec{x} - \vec{y})} + (2\pi)^3v_\alpha(\vec{p})v^\dagger_\beta(\vec{p})e^{-i\vec{p}\cdot(\vec{x} - \vec{y})}\right]$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[u_\alpha(\vec{p})\overline{u}^\dagger_\beta\gamma^0_{\alpha\beta}(\vec{p})e^{i\vec{p}\cdot(\vec{x} - \vec{y})} + v_\alpha(\vec{p})\overline{v}^\dagger_\beta\gamma^0_{\alpha\beta}(\vec{p})e^{-i\vec{p}\cdot(\vec{x} - \vec{y})}\right]$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[[(\gamma\cdot p + m)\gamma^0]_{\alpha\beta}e^{i\vec{p}\cdot(\vec{x} - \vec{y})} + [(\gamma\cdot p - m)\gamma^0]_{\alpha\beta}e^{-i\vec{p}\cdot(\vec{x} - \vec{y})}\right]$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[[(\gamma^0E_p - \gamma\cdot p + m)\gamma^0]_{\alpha\beta}e^{i\vec{p}\cdot(\vec{x} - \vec{y})} + [(\gamma^0E_p - \gamma\cdot p - m)\gamma^0]_{\alpha\beta}e^{-i\vec{p}\cdot(\vec{x} - \vec{y})}\right]$$

beacause of antiparticle second half of the eqn wil be $p \to -p$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[[(\gamma^0E_p - \gamma\cdot p + m)\gamma^0]_{\alpha\beta} + [(\gamma^0E_p + \gamma\cdot p - m)\gamma^0]_{\alpha\beta}\right]e^{i\vec{p}\cdot(\vec{x} - \vec{y})}$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[[\gamma^0E_p\gamma^0]_{\alpha\beta} + [\gamma^0E_p\gamma^0]_{\alpha\beta}\right]e^{i\vec{p}\cdot(\vec{x} - \vec{y})}$$

$$= \int \frac{d^3p}{(2\pi)^3}\delta_{\alpha\beta}e^{i\vec{p}\cdot(\vec{x} - \vec{y})}$$

$$\therefore \left\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y})$$

$$\left\{\psi_\alpha(\vec{x}),\psi_\beta(\vec{y})\right\} =$$

$$= \left\{ \sum_{\alpha=1}^{4}\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s_\alpha(\vec{p})e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s_\alpha(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right],\right.$$

$$\left.\sum_{\beta=1}^{4}\sum_{r=1}^{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p'}}}}\left[b^r_{\vec{p'}}u^r_\beta(\vec{p'})e^{i\vec{p'}\cdot\vec{y}} + c^{r\dagger}_{\vec{p'}}v^r_\beta(\vec{p'})e^{-i\vec{p'}\cdot\vec{y}}\right]\right\}$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{s=1}^{2}\sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\frac{1}{\sqrt{2E_{\vec{p'}}}}\left[\left\{b^s_{\vec{p}},b^r_{\vec{p'}}\right\}u^s_\alpha(\vec{p})u^r_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}}\right.$$

$$+\left\{b^s_{\vec{p}},c^{r\dagger}_{\vec{p'}}\right\}u^s_\alpha(\vec{p})v^r_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + \left\{c^{s\dagger}_{\vec{p}},b^r_{\vec{p'}}\right\}v^s_\alpha(\vec{p})u^r_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}$$

$$\left.+\left\{c^s_{\vec{p}},c^r_{\vec{p'}}\right\}v^s_\alpha(\vec{p})v^r_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}\right]$$

because of eqn(5) every term vanishes.
Therefore

$$\left\{\psi_\alpha(\vec{x}),\psi_\beta(\vec{y})\right\} = 0 \tag{7}$$

And

$$\left\{\psi^\dagger_\alpha(\vec{x}),\psi^\dagger_\beta(\vec{y})\right\} =$$

$$= \left\{ \sum_{\alpha=1}^{4}\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^{s\dagger}_\alpha(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^{s\dagger}_\alpha(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right],\right.$$

$$\left.\sum_{\beta=1}^{4}\sum_{r=1}^{2}\int \frac{d^3p'}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p'}}}}\left[b^{r\dagger}_{\vec{p'}}u^{r\dagger}_\beta(\vec{p'})e^{-i\vec{p'}\cdot\vec{y}} + c^r_{\vec{p'}}v^{r\dagger}_\beta(\vec{p'})e^{i\vec{p'}\cdot\vec{y}}\right]\right\}$$

$$= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{s=1}^{2}\sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\frac{1}{\sqrt{2E_{\vec{p'}}}}\left[\left\{b^{s\dagger}_{\vec{p}},b^{r\dagger}_{\vec{p'}}\right\}u^{s\dagger}_\alpha(\vec{p})u^{r\dagger}_\beta(\vec{p'})e^{-i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}\right.$$

$$+\left\{b^{s\dagger}_{\vec{p}},c^r_{\vec{p'}}\right\}u^{s\dagger}_\alpha(\vec{p})v^{r\dagger}_\beta(\vec{p'})e^{-i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}} + \left\{c^s_{\vec{p}},b^{r\dagger}_{\vec{p'}}\right\}v^{s\dagger}_\alpha(\vec{p})u^{r\dagger}_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}$$

$$\left.+\left\{c^s_{\vec{p}},c^r_{\vec{p'}}\right\}v^{s\dagger}_\alpha(\vec{p})v^{r\dagger}_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}}\right]$$

Also for eqn(5)

$$\left\{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\right\} = 0 \tag{8}$$

Now

$$
\begin{aligned}
H =& \\
& \int d^3x \overline{\psi}(-i\gamma^i\partial_i + m)\psi \\
& = \int \frac{d^3x}{(2\pi)^6} \sum_{s,s'=1}^{2} \int \frac{d^3p'}{\sqrt{2E^{\vec{p'}}}} \frac{d^3p}{\sqrt{2E^{\vec{p}}}} (\overline{u}^s(\vec{p}) b^s_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} \overline{v}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}})(-i\gamma^i\partial_i + m) \\
& \times (b^{s\prime\dagger}_{\vec{p'}} u^{s'\dagger}(\overrightarrow{p'}) e^{i\overrightarrow{p'}\cdot\vec{x}} + c^{s'}_{\vec{p'}} v^{s'\dagger}(\overrightarrow{p'}) e^{-i\overrightarrow{p'}\cdot\vec{x}}) \\
& = \int \frac{d^3x}{(2\pi)^6} \sum_{s,s'=1}^{2} \int \frac{d^3p'}{\sqrt{2E^{\vec{p'}}}} \frac{d^3p}{\sqrt{2E^{\vec{p}}}} (\overline{u}^s(\vec{p}) b^s_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} \overline{v}^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}}) \\
& \times (e^{i\overrightarrow{p'}\cdot\vec{x}} b^{s\prime\dagger}_{\vec{p'}} (\gamma \cdot p' + m) u^{s\prime\dagger}(\overrightarrow{p'}) + e^{-i\overrightarrow{p'}\cdot\vec{x}} c^{s'\dagger}_{\vec{p'}} (\gamma \cdot p' - m) v^{s'\dagger}(\overrightarrow{p'}))
\end{aligned}
$$

We know

$$
\begin{aligned}
(\not{p} - m)u^s(\vec{p}) &= (\gamma^0 p^0 - \gamma \cdot p - m)u^s(\vec{p}) = 0 \\
(\not{p} + m)v^s(\vec{p}) &= (\gamma^0 p^0 - \gamma \cdot p + m)v^s(\vec{p}) = 0
\end{aligned}
$$

Therefore

$$= \int \frac{d^3x}{(2\pi)^6} \sum_{s,s'=1}^{2} \int \frac{d^3p'}{\sqrt{2E^{\overrightarrow{p'}}}} \frac{d^3p}{\sqrt{2E^{\overrightarrow{p}}}} p^0(\overline{u}^s(\overrightarrow{p}) b^s_{\overrightarrow{p}} e^{i\overrightarrow{p}\cdot\overrightarrow{x}} + c^{s\dagger}_{\overrightarrow{p}} \overline{v}^s(\overrightarrow{p}) e^{i\overrightarrow{p}\cdot\overrightarrow{x}})$$

$$\times \gamma^0 (e^{i\overrightarrow{p'}\cdot\overrightarrow{x}} b^{s'\dagger}_{\overrightarrow{p'}} u^{s'\dagger}(\overrightarrow{p'}) - e^{-i\overrightarrow{p'}\cdot\overrightarrow{x}} c^{s'}_{\overrightarrow{p'}} v^{s'\dagger}(\overrightarrow{p'}))$$

$$= \int \frac{d^3x}{(2\pi)^6} \sum_{s,s'=1}^{2} \int \frac{d^3p'}{\sqrt{2E^{\overrightarrow{p'}}}} \frac{d^3p}{\sqrt{2E^{\overrightarrow{p}}}} p^0(\overline{u}^s(\overrightarrow{p})\gamma^0 u^{s\prime}(\overrightarrow{p'}) b^{s\prime}_{\overrightarrow{p'}} b^{s\dagger}_{\overrightarrow{p}} \delta^3(p-p') - c^{s'\dagger}_{\overrightarrow{p'}} c^s_{\overrightarrow{p}} \overline{v}^s(\overrightarrow{p})\gamma^0 v^{s'}(\overrightarrow{p'}))\delta^3(p-p')$$

$$- \delta^3(p+p')(b^{s\dagger}_{\overrightarrow{p}} c^{s'\dagger}_{\overrightarrow{p'}} \overline{u}^s(\overrightarrow{p})\gamma^0 v^{s'}(\overrightarrow{p'}) - \overline{v}^s(\overrightarrow{p})\gamma^0 u^{s\prime}(\overrightarrow{p'}) b^{s\prime}_{\overrightarrow{p'}} c^{s'}_{\overrightarrow{p'}})$$

$$= \sum_{s,s'=1}^{2} \int d^3p (u^{s\dagger}(\overrightarrow{p}) u^{s\prime}(\overrightarrow{p}) b^{s\prime}_{\overrightarrow{p}} b^{s\dagger}_{\overrightarrow{p}} + c^{s\dagger}_{\overrightarrow{p}} c^s_{\overrightarrow{p}} v^{s\dagger}(\overrightarrow{p}) v^{s'}(\overrightarrow{p}) + b^{s\dagger}_{\overrightarrow{-p}} c^{s'\dagger}_{\overrightarrow{-p}} u^{s\dagger}(\overrightarrow{-p}) v^{s'}(\overrightarrow{-p}) + v^{s\dagger}(\overrightarrow{-p}) u^{s\prime}(\overrightarrow{-p}) b^{s\prime}_{\overrightarrow{-p}} c^{s'}_{\overrightarrow{-p}})$$

$$= \sum_{s,s'=1}^{2} \int d^3p \delta_{ss'} E_{\overrightarrow{p}} (b^{s'\dagger}_{\overrightarrow{p}} b^s_{\overrightarrow{p}} - c^{s'}_{\overrightarrow{p}} c^{s\dagger}_{\overrightarrow{p}})$$

$$= \sum_{s,s'=1}^{2} \int d^3p E_{\overrightarrow{p}} (b^{s\dagger}_{\overrightarrow{p}} b^s_{\overrightarrow{p}} - c^s_{\overrightarrow{p}} c^{s\dagger}_{\overrightarrow{p}})$$

for normal ordering in fermionic field

$$:\ c^s_{\overrightarrow{p}} c^{s\dagger}_{\overrightarrow{p}}\ := -c^{s\dagger}_{\overrightarrow{p}} c^s_{\overrightarrow{p}} \tag{9}$$

Therefore

$$H = \sum_{s,s'=1}^{2} \int d^3p E_{\overrightarrow{p}} (b^{s\dagger}_{\overrightarrow{p}} b^s_{\overrightarrow{p}} c^{s\dagger}_{\overrightarrow{p}} c^s_{\overrightarrow{p}}) \tag{10}$$