Out-of-Time-Order-Correlator of $H = xp$

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Abstract

Bery-Keating Hamiltonian $H_0 = \frac{1}{2}(xp + px)$ resembles the Hilber-Pólya cojectue of Riemnan Hypothesis, shows quantum chaotic behaviour. In this paper we first quantize xp and adding boundary condition evalute spectrum with proper eigenfucntion. Then we quantutize 1/xp and achieve same spectrum with minor phase addition. We observes OTOC of the hamiltonian, which gauge the quantum chaotic behaviour of it.

Contents

1 Introduction

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to $1/2$. The classical zeta function converges at Re $s > 1$. By the fundamental theorem of arithmatic, which is also equivalent to the Euler product over primes where p are all the prime numbers. So zeta can describe all the prime numbers.Also the zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negetive integers (for $s = -2, -4, -6, ...$). For complex s ($=\sigma + it$) (with real part between zero and one) , zeta fucntion becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - iE$ zeta funtion becomes zero. If the Riemann hypothesis is true the statistical distribution of the primes will be constrained in the most favourable way. Hilbert-P$\acute{o}$lya conjecture suggests that the imaginary parts of the nontrivial zeros are the eigenvalues of a self-adjoint hamiltonian operator $\hat{H}$. It is also one of the aprroach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-P$\acute{o}$lya conjecture should take the form[1]$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$. The classical versus quantum version of primes and the zeros is also at the heart of so-called quantum chaos approach to the RH.

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[1] In 1999, Berry and Keating[1, 2] on the one hand and Connes[3] on the other claimed that the classical Hamiltonian H = xp, where x and p are the location and momenta of a 1D particle, is strongly connected to the Riemann zeros. This startling hypothesis was based on a semiclassical analysis of H = xp, which led these authors to very different conclusions about the probable spectral interpretation of the Riemann zeros. The source of the dispute is the use of distinct regularizations of H = xp. Berry and Keating used a Planck cell regularization, in which the smooth component of the Riemann zeros is represented semiclassically as discrete energy levels. Connes, on the other hand, chose an upper cutoff for the position and momenta, resulting in a semiclassical continuous spectrum devoid of smooth zeros. All of these semiclassical solutions are heuristic, and there is no consistent quantum version as of yet.

The out-of-time-order correlator (OTOC) is typically defined by [4]

$$C_T \equiv -\left\langle [W(t), V(0)]^2 \right\rangle \tag{1}$$

where $< \cdots >$ represents the thermal average. W(t) and V(t) are operators as time in t in the Heisenberg representation. The OTOC, first introduced in a calculation of a vertex correction of a current for a superconductor[5], was recently turned out to be considered as a measure of the magnitude of quantum chaos. A naive argument for the relation between the OTOC and chaos is as follows[6]. Consider poistion and momentum operators, x(t) and p(t), in a quantum system. We can define an OTOC as $C_T = -\left\langle [x(t), p(0)]^2 \right\rangle$. Taking a naive semiclassical limit, we would be able to replace the commutator $[x(t), p(0)]$ by the Poissoin bracket $i\hbar\{x(t), p(0)\}_{PB} = i\hbar\frac{\delta x(t)}{\delta x(0)}$. For a classically chaotic system with a Lyapunov exponent λ, we have $\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda t}$ because of sensiitivity to initial condition. Thus, the OTOC should grow as $\sim \hbar^2 e^{2\lambda t}$ and we can read off the quatnum Lyapunov exponent λ form it. The quantization of a classically chaotic system may provide a positive quanutm Lyapunov exponent of the OTOC. Historically, the nearest neighbour distribution (NND) for the energy level spectrum has been used to quantify quantum chaos [7]. For integrable and non-integrable systems, it is considered that NNDs are given by Poission and Wigner distributions. The OTOC is expected to be another measure of qauntum chaos. A possible distinction from the classical chaotic system is that the OTOC does not grow eternally but saturates at the Ehrenfest time t_E. The Ehrenfest time is defined by the time scale beyond which the wave function spreads over the whole system. It is roughly characterized as a boundary between a particle-like behavior and a wave-like behavior of the wave function.

In the definition of OTOC, we consider the thermal average of $-\left\langle [W(t), V(0)]^2 \right\rangle$. When we take thermal average, we need to consider the four point operator $\left\langle [W(t), V(0)]^2 \right\rangle$ instead of two point operator $\langle [W(t), V(0)] \rangle$. The reason is as follows. Assuming that we can replace the commutator by Poission bracket by semiclassical limit, $\langle [W(t), V(0)] \rangle$ would also show the exponential growth $\sim e^{\lambda t}$. However, its coefficient can be both positive and negetive. By taking the thermal average, their contributions would be canceled. From the quantum theory point of view, $\langle [W(t), V(0)] \rangle$ measures the correlation between W(t) and V(0). Therefore, the two point function decays as $t \to \infty$ and cannot show the chaotic behavior.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is[1, 8]

$$H_0^{cl} = xp \tag{2}$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \qquad p(t) = p_0 e^{-t} \tag{3}$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of reularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number $\mathcal{N}(E)$ with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \tag{4}$$

Where A is the area of the allowed phase space region below the curve $E = xp$.

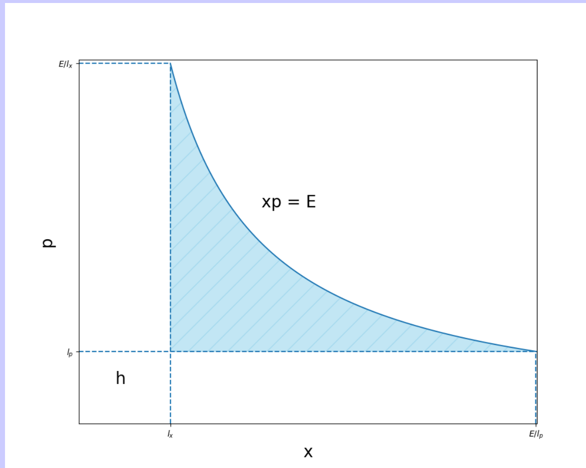

Figure 1: Berry-Keating Regularization

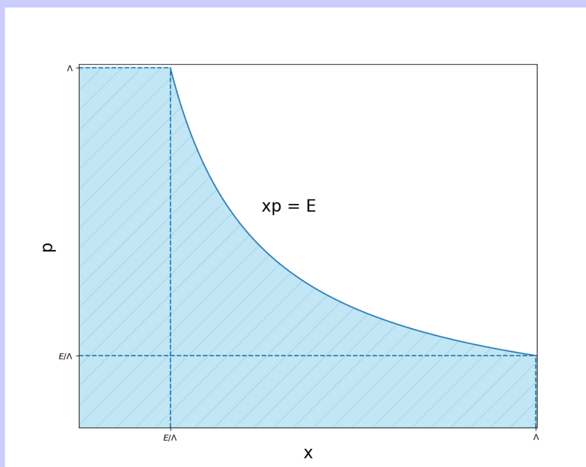

Figure 2: Connes Regularization

So the the number of semiclassical states will be for Berry-Keating regularization

$$
\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h}\int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + ... \\
&= \frac{1}{h}\left[\int_{l_x}^{\frac{E}{l_p}} dx \left[\frac{E}{x} - l_p\right]\right] \\
&= \frac{1}{h}\left[E\left[\ln x\right]_{l_x}^{\frac{E}{l_p}} - l_p\left[\frac{E}{l_p} - l_x\right]\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - l_x l_p\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - h\right] \\
&= \frac{E}{h}\left[\ln\frac{E}{l_x l_p} - 1\right] + 1 \\
&= \frac{E}{2\pi\hbar}\left[\ln\frac{E}{2\pi\hbar} - 1\right] + 1
\end{aligned}
\tag{5}
$$

adding Maslov phase $(-\frac{1}{8})$ and $\hbar = 1$, it becomes

$$
\mathcal{N}_{BK}(E) = \frac{E}{2\pi}\left[\ln\frac{E}{2\pi} - 1\right] + \frac{7}{8} + ..., \qquad E >> 1 \tag{6}
$$

The exact formula for the Riemann zeros, $\mathcal{N}_R(E)$ contains a fluctuation term which depends on the zeta funcion.[3]

$$
\begin{aligned}
\mathcal{N}_R(E) &= \langle\mathcal{N}\rangle + \mathcal{N}_{fl}(E) \\
\langle\mathcal{N}(E)\rangle &= \frac{1}{\pi} Im\ ln\left[\Gamma\frac{1}{2}\left(\frac{1}{2} - iE\right)\right] - \frac{E}{2\pi} ln\pi + 1 \\
\mathcal{N}_{fl}(E) &= \frac{1}{\pi} Im\ ln\left[\zeta\left(\frac{1}{2} - iE\right)\right]
\end{aligned}
\tag{7}
$$

Bery-Keatin took this result and analogies between formulae in Number Theory and Quantum Chaos, they pointed the quantization of classical chaotic Hamiltonian give rise to the zeros as point like spectra.[1, 9] Whereas Connes found the number of semicassical states diverges in the limit where the cutoff Λ goes to infinity, and that therre us a finite size correction given by mins the average position of the Riemann zeros.

$$
\begin{aligned}
\mathcal{N}_c(E) &= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \int_{\frac{E}{\Lambda}}^{\frac{E}{x}} dp\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \left[\frac{E}{x} - \frac{E}{\Lambda}\right]\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + E\left[ln\ x\right]_{\frac{E}{\Lambda}}^{\Lambda} - \frac{E}{\Lambda}\left[\Lambda - \frac{E}{\Lambda}\right]\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + E\left[ln\ \frac{\Lambda^2}{E}\right] - E + \left(\frac{E}{\Lambda}\right)^2\right] \qquad (8) \\
&= \frac{1}{h}\left[E + E\left[ln\ \frac{\Lambda^2}{E}\right]\right] \\
&= \frac{1}{h}\left[E + E\left[ln\ \frac{\Lambda^2}{E}\frac{2\pi}{2\pi}\right]\right] \\
&= \frac{E}{h} ln\ \frac{\Lambda^2}{2\pi} - \frac{E}{h}\left[ln\ \frac{E}{2\pi} - 1\right] \\
&= \frac{E}{2\pi} ln\ \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\left[ln\ \frac{E}{2\pi} - 1\right] \qquad [taking\ \ \hbar = 1]
\end{aligned}
$$

This result les to the missing spectral interpretation of the Riemann zeros, according to which there is a continuum of eginstates (represented by the term $\frac{E}{\pi} ln\ \Lambda$ in $\mathcal{N}(E)$) where states asscoiated with Riemann zeros are missing.

Finally, in the S-regularization the number of semiclasical states diverges as $\frac{E}{2\pi}\ ln\ \frac{\Lambda}{l_x}$ suggesting a continuum spectrum, ike in Connes's approach. But there is no finite size correction to that formula, and cosequently the possible connection to the Riemann zeros is lost.

Table 1:
Three different regularizations of $H = xp$ and the corresponding number of semiclassical states in units $\hbar = 1$[10]

Type	Regularization	$\mathcal{N}(E)$
BK	$\lvert x\rvert > l_x, \lvert p\rvert > l_p$	$\frac{E}{2\pi}\left(ln\ \frac{E}{2\pi} - 1\right) + 1$
C	$\lvert x\rvert < \Lambda, \lvert p\rvert < \Lambda$	$\frac{E}{2\pi}ln\ \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\left(\ln\ \frac{E}{2\pi} - 1\right)$
S	$l_x < x < \Lambda$	$\frac{E}{2\pi}ln\ \frac{\Lambda}{l_x}$

3 Quantization of xp and $\frac{1}{xp}$

3.1 The Hamitoninan $H_0 = xp$

Here we construst a self adjoint operator H_0 which acts on a Hilbert space $L^2(a, b)$ of square integrable function in the interval (a, b). Taking $x \geqslant 0$, there are four possible intervals: $a = 0, l_x$ and $b = \Lambda, \infty$ where l_x and Λ were introduced (we shall take l_x and $\Lambda = N > 1$). Berry-Keating defined the quantnum Hamiltonian H_0 as the normal ordered expresion

$$H_0 = \frac{1}{2}(xp + px) \tag{9}$$

where $p = -i\hbar\frac{d}{dx}$. If $x \geqslant 0$, Eq. (11) is equivalent to

$$\begin{aligned} H_0 &= \sqrt{x}p\sqrt{x} = -i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x} \\ &= -i\hbar\left(x\frac{d}{dx} + \frac{d}{dx}x\right) \end{aligned} \tag{10}$$

We know that the canonical commutation relation is

$$[\hat{x}, \hat{p}] = i\hbar \tag{11}$$

or, (dropping hat cause we're dealing with quantum system and operators)

$$
\begin{aligned}
[x,p] &= i\hbar \\
\implies \left[x, -i\hbar\frac{d}{dx}\right] &= i\hbar \\
\implies -i\hbar\left[x, \frac{d}{dx}\right] &= i\hbar \\
\implies \left[x, \frac{d}{dx}\right] &= -1 \\
\implies x\frac{d}{dx} - \frac{d}{dx}x &= -1 \\
\implies \frac{d}{dx}x &= x\frac{d}{dx} + 1
\end{aligned}
\tag{12}
$$

Taking this value to R.H.S of Eq(12)

$$
\begin{aligned}
-i\hbar\left(x\frac{d}{dx} + \frac{d}{dx}\right)f &= -i\hbar\left[2x\frac{d}{dx} + 1\right] \\
&= -i\hbar 2x\left[\frac{d}{dx} + \frac{1}{x}\frac{1}{2}\right]f \\
&= -i\hbar\frac{1}{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}f\right) \\
&= -i\hbar\frac{1}{\sqrt{x}}\frac{d}{dx}\left(\sqrt{x}f\right) \\
&= \frac{1}{\sqrt{x}}\left(-i\hbar\frac{d}{dx}\right)\sqrt{x}f
\end{aligned}
\tag{13}
$$

so

$$
H_0 = \frac{1}{2}\left(xp + px\right) = -i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x} \tag{14}
$$

This is a symmetric operator acting on a certain domain of the Hilert space $L^2(a,b)$, By definition, if an operator is symmetric (or Hermitian)[11]

$$
\langle\psi|H_0\phi\rangle = \langle\psi H_0|\phi\rangle \tag{15}
$$

or with limit,

$$
\langle\psi|H_0\phi\rangle - \langle\psi H_0|\phi\rangle = i\hbar\left[a\psi^*(a)\phi(a) - b\phi^*(b)\psi(b)\right] = 0 \tag{16}
$$

which is satisfied if both $\psi(x)$ and $\phi(x)$ vanish at the points a, b. von Neumann Theorem of deficiancy indices states that, an operator in symmetric if its deficiency indices $n_{\pm}$ are equal.[12]. Deficiency indices (or the defect numbers) of a closable symmetric operator T are cardinal number S

$$\begin{aligned} n_+ &:= d_\lambda = dim\ \mathcal{R}(T - \overline{\lambda}\mathbb{1})^\perp \quad Im\ \lambda > 0 \\ n_- &:= d_\lambda = dim\ \mathcal{R}(T - \overline{\lambda}\mathbb{1})^\perp \quad Im\ \lambda < 0 \end{aligned} \tag{17}$$

If T is densly defined and symmetric, then T is closable, and by formula $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$

$$\begin{aligned} n_+ &:= dim\ \mathcal{N}(T^* - i\mathbb{1}) = dim\ \mathcal{N}(T^* - \lambda\mathbb{1}) \quad Im\ \lambda > 0 \\ n_- &:= dim\ \mathcal{N}(T^* + i\mathbb{1}) = dim\ \mathcal{N}(T^* + \lambda\mathbb{1}) \quad Im\ \lambda < 0 \end{aligned} \tag{18}$$

By definition $n_{\pm}(T) = dim\ \mathcal{N}(T^* \mp iT)$
Again if T is a symmetric operator, then

$$\begin{aligned} K_+ &= ker\ (i\mathbb{1} - T^*) = Ran\ (i\mathbb{1} - T)^\perp \\ K_- &= ker\ (i\mathbb{1} + T^*) = Ran\ (-i\mathbb{1} + T)^\perp \end{aligned} \tag{19}$$

K_+ and K_- are called the deficiency subspaces of T, The pair of numbers n_+, n_- given by $n_+(T) = dim[K_+]$,$n_-(T) = dim[K_-]$ arre called deficiency indices of T.

von Neumann Theorem for deficiency indices states that if T an closed operator woth deficiency indices n_+ and n_-. Then

(1) T is symmetric if and only if $n_+ = n_- = 0$ ann self adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$

(2) T is symmetric adn self adjoint and also has many self adjoint extensions if and only if $n_+ = n_- \neq 0$ and $\mathcal{D}(T) = \mathcal{D}(T^*)$. There is one-one correspondence between self adjoint extensions of T and unitary maps from K_+ onto K_-

(3) If either $n_+ = 0 \neq n_-$ or $n_- = 0 \neq n_+$ then T is not symmetric and has no nontrivial self adjoint extension (such operators are called maximal symmetric operator).

So this indices counts the number of solutions of the equation, which comes from the deficiency spaces for subsystem T

$$K_{\pm} = ker\left(-H_0^{\dagger} - \mp i\mathbb{1}\right) \tag{20}$$

which leads to find the solution of the equation.

$$H_0^{\dagger}\psi_{\pm} = \pm i\hbar\lambda\psi_{\pm} \tag{21}$$

belonging to the domain og $H_0^{\dagger}(\lambda > 0)$. If $n = n_+ = n_- > 0$, there are infinitely many self-adoint extensions of H_0 parameterized by a unitary $n \times n$ matrix. Stone's theorem states that if $U(t)$ be a strongly continuous one parameter unitary group on a Hilbert space $\mathcal{H}$. Then, there is a self-adjoint operator A on $\mathcal{H}$ so that $U(t) = e^{itA}$. The solution of the equation () is

$$\begin{aligned}
H_0^{\dagger}\psi_{\pm} &= \pm i\hbar\lambda\psi_{\pm} \\
\implies H_0\psi_{\pm} &= \pm i\hbar\lambda\psi_{\pm} \qquad [becuase\ H_0\ is\ self-adjoint] \\
\implies \left(-i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x}\right)\psi_{\pm} &= \pm i\hbar\lambda\psi_{\pm} \\
\implies -i\hbar\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\psi_{\pm}\right) &= \pm i\hbar\lambda\psi_{\pm} \\
\implies -\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\psi_{\pm}\right) &= \pm\lambda\psi_{\pm} \\
\implies -x\frac{d}{dx}\psi_{\pm} - \sqrt{x}\frac{1}{2\sqrt{x}}\frac{d}{dx}\psi_{\pm} &= \pm\lambda\psi_{\pm} \\
\implies -x\frac{d}{dx}\psi_{\pm} &= \left(\pm\lambda + \frac{1}{2}\right)\psi_{\pm} \\
\implies \frac{d}{dx}\psi_{\pm} &= -\frac{1}{x}\left(\pm\lambda + \frac{1}{2}\right)\psi_{\pm} \\
\implies \frac{d\psi_{\pm}}{\psi_{\pm}} &= -\frac{dx}{x}\left(\pm\lambda + \frac{1}{2}\right) \\
\implies ln\ \psi_{\pm} &= -(ln\ x)\left(\pm\lambda + \frac{1}{2}\right) + ln\ C \\
\implies \psi_{\pm} &= Cx^{-\frac{1}{2}\mp\lambda}
\end{aligned}$$

whose norm in the interval (a,b) is

$$\begin{aligned}\langle\psi_\pm|\psi_\pm\rangle &= \int_a^b C^2 x^{-1\mp\lambda}\,dx \\ &= \mp\frac{C^2}{2\lambda}\left(b^{\mp 2\lambda} - a^{\mp 2\lambda}\right) \\ &= \pm\frac{C^2}{2\lambda}\left(a^{\mp 2\lambda} - b^{\mp 2\lambda}\right)\end{aligned} \tag{22}$$

The deficiency indices correponding to the four intervals cosidered above are collecte in Table[13]. We find the deficency indices by observing different intervals. For BK intervals$(1,\infty)$ only ψ_+ belongs to Hilbert space (ψ_- blows out. or putting intervals in Eq. (20) and testing whether it belongs to the Hilbert space)[10]. And the rest are given below

Table 2: Deficiency indices of H_0. The corresponding intervals are associated to the semiclassical regularizations of section 2 (BK, C ,S). The last one T, describes the case with no constraints on x except positivity (i.e. x>0)

Type	(a,b)	(n_+, n_-)	Self-adjoint
BK	$(1,\infty)$	(1,0)	-
C	(0,N)	(0,1)	-
S	(1,N)	(1,1)	$\surd$
T	$(0,\infty)$	(0,0)	$\surd$

From the von Neumann theorem we see that H_0 is essentially self-adjoint on the half line $\mathbb{R}_+ = (0,\infty)$. This was studied by Twamley and Milbrn, who defined quantum Mellin transform using the eigenstates of H_0[14] On the other hand, in the interval (1,N) the operator H_0 admits infinitely many self-adjoint extensions parameterrized y a phase $e^{i\theta}$. This phase defines the boundary condition of the functions belonging to the self-adjoint domain.[10]

$$\mathcal{D}(H_{0,\theta}) = \left\{\psi, H_0\psi \in L^2(1,N), e^{i\theta}\psi(1) = \sqrt{N}\psi(N)\right\} \tag{23}$$

The eigenfunction of H_0

$$H_0\psi_E = E\psi_E, \tag{24}$$

are given by [1]

$$\psi_E(x) = \frac{C}{x^{\frac{1}{2}-iE\hbar}}, \qquad E \in \mathbb{R} \tag{25}$$

where C is a normalization constant. In the half line $\mathbb{R}_+$ there are no further restriction on E, hence the spectrum of H_0 is continuous and covers the whole real line $\mathbb{R}$. In this case the normalization constant is chosen as $C = \frac{1}{\sqrt{2\pi\hbar}}$ which gurantees the standard normalization

$$\langle \psi_E | \psi_{E'} \rangle = C^2 \int_0^\infty \frac{dx}{x} x^{-i(E-E')/\hbar} = \delta(E - E') \tag{26}$$

In the case where H_0 is defined in ther interval, the boundary condition (21) yeilds the quantization condition for E, namely

$$\begin{aligned}
e^{i\theta}\psi(1) &= \sqrt{N}\psi(N) \\
\implies e^{i\theta}\frac{C}{1^{\frac{1}{2}-iE\hbar}} &= \sqrt{N}\frac{C}{N^{\frac{1}{2}-iE\hbar}} \\
\implies e^{i\theta} &= N^{-iE\hbar} \\
\implies i\theta &= (\frac{-iE}{\hbar})ln\ N \\
\implies E &= \frac{\hbar\theta}{ln\ N} \\
\implies E &= \frac{2\pi\hbar}{ln\ N}(\frac{\theta}{2\pi}) \\
\implies E_n &= \frac{2\pi\hbar}{ln\ N}(n + \frac{\theta}{2\pi}) \qquad n \in \mathbb{N}
\end{aligned} \tag{27}$$

Hence the spectrum of H_0 is dscrete, with a level spacing decreasing for largerm values of N. The normalization constant of the wave function is now $C = \frac{1}{\sqrt{ln\ N}}$ which gives,

$$\left\langle \psi_{E_n} | \psi_{E_{n'}} \right\rangle = C^2 \int_1^N \frac{dx}{x} x^{-i(E_n - E_{n'})/\hbar} = \delta_{n,n'} \tag{28}$$

The spectrum (16) agrees with the semiclassical result given in Table 1 for the S-regularization(recall that $l_x = 1, \Lambda = N, \hbar = 1$) For the particular case where $\theta = \pi$, one observes that the energy spectrum is symmetric around zero, i.e, if E_n is an eigenenergy so is $-E_n$. This result is obtained in working[15] with the inverse Hamiltonian $\frac{1}{H_0}$. We are reviewing that construction in next section.

3.2 The inverse Hamiltonain $1/H_0$

First, we take the expression at Eq(16) and take the formal inverse, i.e., $H_0^{-1} = x^{1/2} p^{-1} x^{-1/2}$. The operator p_{-1} is the one-dimensional Green's function with matrix elements(definition of Green's function: Green's function is the kernel of and integral operator that represents the inverse of a differential operator. Let

$$Lu = f \tag{29}$$

Here u and f are vectors and L is a square, invertible matrix. The inverse matrix exsits if $\lambda = 0$ is not an eigenvalue of L, or when det $detL \neq 0$. Now

$$u = L^{-1} f \tag{30}$$

where L^{-1} is the inverse operator of L. The inverse operator to be an integral operator of the form.

$$\left(L^{-1} f\right)(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi \tag{31}$$

with kernel G. If L exists, then the kernel fucntion $g(x,\xi)$ is called the Green's function associated with L.

So p^{-1} operator will be

$$\begin{aligned} \left\langle x \middle| p^{-1} \middle| x' \right\rangle &= \left\langle x \middle| \frac{1}{-i\hbar \frac{d}{dx}} \middle| x' \right\rangle \\ &= -i\hbar \left\langle x \middle| \frac{1}{\frac{d}{dx}} \middle| x' \right\rangle \\ &= \frac{\hbar}{i} G(x,x') \\ &= \frac{\hbar}{2i} sign(x-x') \end{aligned} \tag{32}$$

Here $sign(x-x')$ is the sign function.[15]. The operator H_0^{-1} is defined in the interval (1,N) by the continuous matrix,

$$H_0^{-1}(x,x') = \frac{i}{2\hbar} \frac{sign(x-x')}{\sqrt{xx'}}, \qquad 1 \leqslant x, x' \leqslant N. \tag{33}$$

It's spectrum is found solving the Schrӧdinger equation.

$$
\begin{aligned}
&H_0(x,x^{'})\psi(x^{'}) = E\psi(x)\\
&\implies H_0^{-1}(x,x^{'})\psi(x^{'}) = E^{-1}\psi(x)\\
&\implies \frac{i}{2\hbar}\int_1^N dx^{'}\frac{sign(x-x^{'})}{\sqrt{xx^{'}}}\psi(x^{'}) = E^{-1}\psi(x)
\end{aligned}
\tag{34}
$$

for the eigenvalue E^{-1}, which must not be singular for H_0^{-1} to be invertible. Define a new wave function

$$
\phi(x) = \frac{\psi(x)}{\sqrt{x}} \tag{35}
$$

which satisfies

$$
\frac{iE}{2\hbar}\int_1^N dx^{'}sign(x-x^{'})\phi(x^{'}) = x\phi(x) \tag{36}
$$

Taking derivative with respect to x

$$
\begin{aligned}
&\frac{d}{dx}\left(\frac{iE}{2\hbar}\int_1^N dx^{'}sign(x-x^{'})\phi(x^{'})\right) = \frac{d}{dx}\left(x\phi(x)\right)\\
&\implies \frac{iE}{2\hbar}\int_1^N dx^{'}2\delta(x-x^{'})\phi(x^{'}) = \phi(x) + x\frac{d}{dx}\phi(x)\\
&\implies \frac{iE}{\hbar}\phi(x) = \phi(x) + x\frac{d}{dx}\phi(x)\\
&\implies x\frac{d}{dx}\phi(x) = \left(1-\frac{iE}{\hbar}\right)\phi(x)\\
&\implies \frac{d\phi(x)}{\phi(x)} = \left(1-\frac{iE}{\hbar}\right)\frac{dx}{x}\\
&\implies ln\ \phi(x) = \left(1-\frac{iE}{\hbar}\right) ln\ x + ln\ C\\
&\implies \phi(x) = \frac{C}{x^{1-\frac{iE}{\hbar}}}
\end{aligned}
\tag{37}
$$

$$
\psi(x) = \frac{C}{x^{1/2-\frac{iE}{\hbar}}} \tag{38}
$$

with $C = \frac{1}{\sqrt{ln\ N}}$ as in Eq(30). Eq(40) fixes the functional form of $\psi(x)$. To find the spectrum we impose (38) at one point , say $x = 1$, obtaining,

$$\begin{aligned}
&\frac{iE}{2\hbar}\int_1^N dx^{'} sign(1-x^{'})\phi(x^{'}) = \phi(1) \\
&\implies \frac{iE}{2\hbar}\int_1^N dx^{'} sign(1-x^{'})\frac{C}{x^{'1-iE/\hbar}} = \frac{C}{1^{1-iE/\hbar}} \\
&\implies \frac{iE}{2\hbar}\int_1^N dx^{'} sign(1-x^{'})\frac{1}{x^{'1-iE/\hbar}} = 1
\end{aligned} \tag{39}$$

we know that sign function

$$sign(1-x^{'}) = \begin{cases} 1 & 1-x^{'} > 0 \implies 1 > x^{'} \\ 0 & 1-x^{'} = 0 \implies 1 = x^{'} \\ -1 & 1-x^{'} < 0 \implies 1 < x^{'} \end{cases}$$

Therefore

$$\begin{aligned}
&\frac{-iE}{2\hbar}\int_1^N dx^{'} x^{'-1+iE/\hbar} = 1 \\
&\frac{-iE}{2\hbar}\left[\frac{x^{'iE/\hbar}}{iE/\hbar}\right]_1^N = 1 \\
&\frac{1}{2}\left[N^{iE/\hbar} - 1^{iE/\hbar}\right] = 1 \\
&N^{iE/\hbar} - 1 = -2 \\
&N^{iE/\hbar} = -1 \\
&N^{iE/\hbar} = e^{i\pi} \\
&\frac{iE}{\hbar} ln\ N = i\pi \\
&E = \frac{\pi\hbar}{ln\ N} \\
&E = \frac{2\pi\hbar}{ln\ N}\frac{1}{2}
\end{aligned}$$

$$\therefore E_n = \frac{2\pi\hbar}{ln\ N}\left[n + \frac{1}{2}\right] \tag{40}$$

This sprectrum coincides with (29) for $\theta = \pi$, so that the eigenstates come in pairs $\{E_n, -E_n\}$ as corresponds to an Hermitian antisymmetric operator. Including a BCS coupling in (35), related to θ yields the spectrum (29)[14]

We take this spectrum and wavefunction to calculate OTOC(out of time order correlator)

4 Out-of-Time-Order Correlator of $H = xp$

First we formulate how to calculate the OTOC for generic quauntum mechanics. In particular, by the reason described above, we choose $W = x$ and $V = p$ to measure a possible indication of quantum chaos. We consider the out-of-time-order correlator (OTOC) defined by

$$C_T \equiv -\left\langle [x(t), p(0)]^2 \right\rangle \tag{41}$$

where $\langle \mathcal{O} \rangle \equiv \frac{tr\left[e^{-\beta H}\mathcal{O}\right]}{tr\ e^{-\beta H}}$. Here we define $\frac{1}{\beta}$ with the temperature of the system T. We will omit the argument of Heisenberg operators for $t = 0$; $\mathcal{O} \equiv \mathcal{O}(0)$. Taking energy eigenstates as the basis of the Hilber space, we can rewrite the OTOC as

$$C_T(t) = \frac{1}{Z}\sum_n e^{-\beta E_n} c_n(T) \tag{42}$$

$$c_n \equiv -\left\langle n \middle| [x(t), p(0)]^2 \middle| n \right\rangle \tag{43}$$

where $H|n >= E_n|n >$. We will refer the OTOC for a fixed energy eigenstate, $c_n(t)$,as a microcanonical OTOC. On the other hand, we will refer $C_T(t)$ as a thermal OTOC. Once we compute microcanonical OTOCs, we can obtain the thermal OTOC bu taking their thermal average. Let us rewrite the microcanonical OTOC using matrix element of x and p for numerical calculations. Using the completeness relation $1 = \sum_m |m >< m|$, we rewrite the microcanonical OTOC as

$$c_n(t) = \sum_m b_{nm}(t) b^*_{nm}(t) \tag{44}$$

$$b_{nm}(t) \equiv -i \langle n|[x(t), p(0)]|m\rangle \tag{45}$$

Note that $b_{nm}(t)$ is Hermitian: $b_{nm}(t) = b^*_{nm}(t)$. Substituting $x(t) = e^{iHt}xe^{-iHt}$ and inserting the completeness relation again, we obtain

$$\begin{aligned}
b_{nm}(t) &= -i \langle n\big|\left[e^{iHt}xe^{-iHt}, p\right]\big|m\rangle \\
&= -i \langle n\big|\left[e^{iHt}xe^{-iHt}p - pe^{iHt}xe^{-iHt}\right]\big|m\rangle \\
&= -i \left[\sum_k \langle n\big|e^{iHt}xe^{-iHt}\big|k\rangle \langle k|p|m\rangle - \sum_k \langle n|p|k\rangle \langle k\big|e^{iHt}xe^{-iHt}\big|m\rangle\right] \\
&= -i \left[\sum_k \langle n\big|e^{iE_nt}xe^{-iE_kt}\big|k\rangle \langle k|p|m\rangle - \sum_k \langle n|p|k\rangle \langle k\big|e^{iE_kt}xe^{-iE_mt}\big|m\rangle\right] \\
&= -i \sum_k \left[e^{iE_{nk}t}x_{nk}p_{km} - e^{iE_{km}t}p_{nk}x_{km}\right]
\end{aligned}$$

$$\therefore b_{nm}(t) = -i \sum_k \left[e^{iE_{nk}t}x_{nk}p_{km} - e^{iE_{km}t}p_{nk}x_{km}\right] \tag{46}$$

where $E = E_n - E_m$, $x_{nm} \equiv < n|x|m >$ $p_{nm} \equiv < n|p|m >$. In this expression, there are matrix components of p. They are not desirable since numerical derivatives of wave fuincitons lose the numerical accuracy. For a natural Hamiltonian with the form,

$$H = \sum_{i=1}^{N} p_i^2 + U(x_1, \cdots, x_N) \tag{47}$$

we can express p_{nm}using x_{nm}. We know from canonical commutation relation

$$[x, p] = i \qquad (\hbar = 1)$$

Therefore

$$
\begin{aligned}
[H, x] &= [p^2 + U(x), x] \\
&= [p^2, x] + [U(x), x] \\
&= p[p, x] + [p, x]p \\
&= -ip - ip
\end{aligned}
$$

$$\therefore [H, x] = -2ip \tag{48}$$

Applying $< m| \cdots |n >$ to the both sides of the equation, we obtain

$$
\begin{aligned}
&< m|[H, x]|n >=< m| - 2ip|n > \\
&< m|(Hx - xH)|n >= -2i < m|p|n > \\
&< m|(E_m x - xE_n)|n >= -2ip_{mn} \\
&E_{mn}x_{mn} = -2ip_{mn} \\
&p_{mn} = \frac{i}{2}E_{mn}x_{mn}
\end{aligned}
$$

Substituting this expression into (49) we have

$$b_{nm}(t) = \frac{-ii}{2}\sum_k (e^{iE_{nk}t}x_{nk}E_{km}x_{km} - e^{iE_{km}t}E_{nk}x_{nk}x_{km})$$

$$b_{nm}(t) = \frac{1}{2}\sum_k x_{nk}x_{km}(E_{km}e^{iE_{nk}t} - E_{nk}e^{iE_{km}t}) \tag{49}$$

Now we can take this equation to calculate OTOC of Berry-Keating Hamiltonian from (40) and (42). The spectrum is

$$E_n = \frac{2\pi\hbar}{ln\ N}\left[n + \frac{1}{2}\right] \tag{50}$$

and the wavefunction

$$\psi(x) = \frac{C}{x^{1/2 - \frac{iE}{\hbar}}} \tag{51}$$

from here we can evaluate $x_{nm} =< n|x|m >$ value

$$\begin{aligned}
\langle\psi_E|x|\psi_{E'}\rangle &= C^2\int_1^N \frac{dx}{x}x^{-i(E-E^{'})/\hbar}x \\
&= C^2\int_1^N dx x^{-i(E-E^{'})/\hbar} \\
&= C^2\left[\frac{x^{1-i(E-E^{'})/\hbar}}{1-i(E-E')/\hbar}\right]_1^N \\
&= C^2\frac{1}{1-i(E-E')/\hbar}\left[N^{1-i(E-E^{'})/\hbar}-1\right] \\
&= C^2\frac{1}{1-i(E-E')/\hbar}\left[N^{1-i(E-E^{'})/\hbar}-1\right]
\end{aligned}$$

Therefore, $< n|x|m >= x_{nm}$ is

$$< n|x|m > = \frac{1}{ln\ N}\frac{1}{1-i\frac{2\pi\hbar}{\ln\ N}(n+\frac{1}{2}-m-\frac{1}{2})/\hbar}\left[N^{1-i\frac{2\pi\hbar}{\ln\ N}(n+\frac{1}{2}-m-\frac{1}{2})/\hbar}-1\right] \tag{52}$$

$$x_{nm} = \frac{1}{ln\ N}\frac{1}{1-i\frac{2\pi}{\ln\ N}(n+\frac{1}{2}-m-\frac{1}{2})}\left[N^{1-i\frac{2\pi}{\ln\ N}(n+\frac{1}{2}-m-\frac{1}{2})}-1\right] \tag{53}$$

putting these values on Eq(52)

$$b_{nm}(t) = \frac{1}{2}\sum_k x_{nk}x_{km}(E_{km}e^{iE_{nk}t}-E_{nk}e^{iE_{km}t})$$

and calculate OTOC analytically. cause carry out the summation of x_{nm} and energy eigenstates makes things little complicated. The graph and codes are given below.

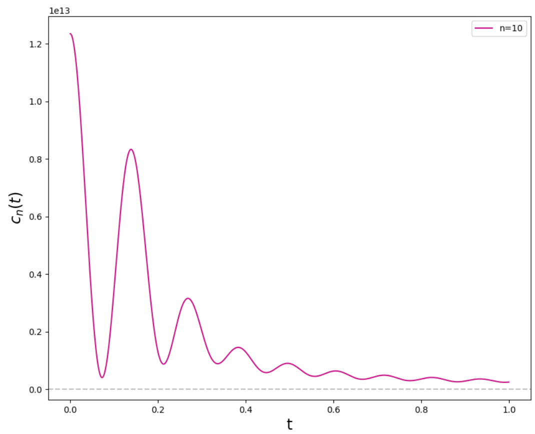

Figure 3

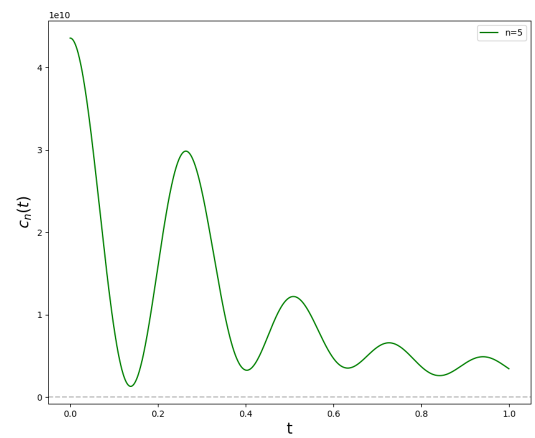

Figure 4

Here microcanonical OTOC and thermal OTOC are same with different temperatures.

```
import numpy as np
import matplotlib.pyplot as plt
import math
import cmath

a = [] #initialize some list to plot data
b = []
r=6    # matrix dimension number
```

```
N=10     #Higest interval in the spectrum
T=40     #temperature

#function for spetrum E_{nm}
def E(n, m):
    return (2*math.pi*(n-m))/(math.log(N))

#function for position x_{nm}
def x(n, m):
    a = (1/1-(2j*math.pi*(n-m)/math.log(N)))/(math.sqrt(math.log(N)))
    b = (N**(1-(2j*math.pi*(n-m)/math.log(N)))-1)
    return a*b

#calculating Thermal OTOC
for t in np.arange(0,1,0.001):
    Z=0
    C=0
    s3=0
    for n in range(0,r):
        s2 = 0
        for m in range(0,r):
            s1=0
            for k in range(0,r):
                s1 += (1/2.0)*x(n,k)*x(k,m)*(E(k,m)*cmath.exp(1j*E(n,k)*t) - E(n,k)*cmath.exp(1j*E(k,m)*t))
            s2 += s1*np.conjugate(s1)
    Z += cmath.exp(-E(n,0)/T)   #partition function
    C += (cmath.exp(-E(n,0)/T)*s2)  #e^{-\beta E_n} O
    s3=C/Z  #expectation value or Thermal OTOC
    a.append(t)
    b.append(s3)


#for plot
f = plt.figure()
f.set_figwidth(10)
f.set_figheight(8)
plt.plot(a, b, label='n=10', color='mediumvioletred')
plt.ylabel('$c_{n}(t)$',fontsize=30)
plt.xlabel('t',fontsize=30)
plt.legend()
plt.show()

f = plt.figure()
```

```
f.set_figwidth(10)
f.set_figheight(8)
plt.plot(a, b, label='n=5', color='green')
plt.ylabel('$c_{n}(t)$',fontsize=30)
plt.xlabel('t',fontsize=40)
plt.axhline(0,ls='--',alpha=0.5,c='grey')
plt.legend()
plt.savefig('test.png')
```

Listing 1: Python example

5 Conclusions

In this paper, we discussed what is Riemann hypothesis is, and why it is important to the field to Quantum mechanics. The imaginary part of Hamiltonian proposed by Berry-keating Hamiltonian $H_0 = \frac{1}{2}(xp+px)$ shows the self-adjoint and quanutm chaos. We look at Sierra regularization of Berry-Keating Hamiltonian, and their intervals $l_x = 1, \Lambda = 1$ and evaluated the spectrum $E_n = \frac{2\pi\hbar}{ln\ N}(n + \frac{\theta}{2\pi})$ of $\psi_E(x) = \frac{C}{x^{\frac{1}{2}-iE\hbar}}$ eigenfunction. Then we look at the quantization of inverse Hamiltonian $H_0^{-1} = x^{-1/2}p^{-1}x^{1/2}$ and obtained the spectrum $E_n = \frac{2\pi\hbar}{ln\ N}(n+\frac{1}{2})$ which is reckon to be $E_n = \frac{2\pi\hbar}{ln\ N}(n+\frac{\theta}{2\pi})$ with $\theta = \pi$

We discussed OTOC in quantum mehcanics and it thermal and microcanonical relation between them. We input the spccetrum and position operator of inverse Hamiltonaian and calculate its quantum chaos analytically. We see that microcanonical OTOC settles at value greater than zero while the thermal OTOC displays the same behaviour of microcanonical OTOC.

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