

1 Out of Time Order Correlator of $H = xp$ model

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to $1/2$. The classical zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (1)$$

for $\text{Re } s > 1$. By the fundamental theorem of arithmetic, which is also equivalent to the Euler product over primes

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (2)$$

where p are all the prime numbers.

Zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negative integers (for $s = -2, -4, -6, \dots$). For complex $s (= \sigma + it)$ (with real part between zero and one), zeta function becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - it$ zeta function becomes zero $\zeta(\frac{1}{2} - it) = 0$. Hilbert-Pólya conjecture suggests that the imaginary parts of the nontrivial zeros are the eigenvalues of a self-adjoint hamiltonian operator \hat{H} . It is also one of the approach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-Pólya conjecture should take the form[1]

$$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \quad (3)$$

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[1] Berry proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obtained by quantization of a classical chaotic hamiltonian, whose periodic orbits are labeled by the prime numbers. Connes took the adelic approach to introduce $H = xp$ [2]. He showed that using different semiclassical regularization, Riemann zeros appear as missing spectral lines in a continuum.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is [1, 2]

$$H_0^{cl} = xp \quad (4)$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \quad p(t) = p_0 e^{-t} \quad (5)$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of regularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number $\mathcal{N}(E)$ with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \quad (6)$$

Where A is the area of the allowed phase space region below the curve $E = xp$. So the the number of semiclassical states will be for Berry-Keating

regularization

$$\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h} \int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + \dots\dots \\
&= \frac{1}{h} \left[\int_{l_x}^{\frac{E}{l_p}} dx \left[\frac{E}{x} - l_p \right] \right] \\
&= \frac{1}{h} \left[E [\ln x]_{l_x}^{\frac{E}{l_p}} - l_p \left[\frac{E}{l_p} - l_x \right] \right] \\
&= \frac{1}{h} \left[E \ln \frac{E}{l_x l_p} - E - l_x l_p \right] \\
&= \frac{1}{h} \left[E \ln \frac{E}{l_x l_p} - E - h \right] \\
&= \frac{E}{h} \left[\ln \frac{E}{l_x l_p} - 1 \right] + 1 \\
&= \frac{E}{2\pi\hbar} \left[\ln \frac{E}{2\pi\hbar} - 1 \right] + 1
\end{aligned} \tag{7}$$

adding Maslov phase $(-\frac{1}{8})$ and $\hbar = 1$, it becomes

$$\mathcal{N}_{BK}(E) = \frac{E}{2\pi} \left[\ln \frac{E}{2\pi} - 1 \right] + \frac{7}{8} + \dots\dots, \quad E \gg 1 \tag{8}$$

The exact formula for the Riemann zeros, $\mathcal{N}_R(E)$ contains a fluctuation term which depends on the zeta function.[3]

$$\begin{aligned}
\mathcal{N}_R(E) &= \langle \mathcal{N} \rangle + \mathcal{N}_{fl}(E) \\
\langle \mathcal{N}(E) \rangle &= \frac{1}{\pi} \text{Im} \ln \left[\Gamma \frac{1}{2} \left(\frac{1}{2} - iE \right) \right] - \frac{E}{2\pi} \ln \pi + 1 \\
\mathcal{N}_{fl}(E) &= \frac{1}{\pi} \text{Im} \ln \left[\zeta \left(\frac{1}{2} - iE \right) \right]
\end{aligned} \tag{9}$$

Bery-Keatin took this result and analogies between formulae in Number Theory and Quantum Chaos, they pointed the quantization of classical chaotic Hamiltonian give rise to the zeros as point like spectra.[1, 4] Whereas Connes found the number of semiclassical states diverges in the limit where the cutoff Λ goes to infinity, and that there is a finite size correction given by minus the average position of the Riemann zeros.

$$\begin{aligned}
\mathcal{N}_c(E) &= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \int_{\frac{E}{x}}^{\frac{E}{\Lambda}} dp \right] \\
&= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \left[\frac{E}{x} - \frac{E}{\Lambda} \right] \right] \\
&= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + E [\ln x]_{\frac{E}{\Lambda}}^{\Lambda} - \frac{E}{\Lambda} \left[\Lambda - \frac{E}{\Lambda} \right] \right] \\
&= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + E \left[\ln \frac{\Lambda^2}{E} \right] - E + \left(\frac{E}{\Lambda} \right)^2 \right] \tag{10} \\
&= \frac{1}{h} \left[E + E \left[\ln \frac{\Lambda^2}{E} \right] \right] \\
&= \frac{1}{h} \left[E + E \left[\ln \frac{\Lambda^2}{E} \frac{2\pi}{2\pi} \right] \right] \\
&= \frac{E}{h} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{h} \left[\ln \frac{E}{2\pi} - 1 \right] \\
&= \frac{E}{2\pi} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left[\ln \frac{E}{2\pi} - 1 \right] \quad [taking \hbar = 1]
\end{aligned}$$

This result les to the missing spectral interpretation of the Riemann zeros, according to which there is a continuum of eginstates (represented by the term $\frac{E}{\pi} \ln \Lambda$ in $\mathcal{N}(E)$) where states assoicated with Riemann zeros are missing.

Finally, in the S-regularization the number of semiclassical states diverges as $\frac{E}{2\pi} \ln \frac{\Lambda}{l_x}$ suggesting a continuum spectrum, ike in Connes's approach. But there is no finite size correction to that formula, and cosequently the possible connection to the Riemann zeros is lost.

Table 1:

Three different regularizations of $H = xp$ and the corresponding number of semiclassical states in units $\hbar = 1$ [5]

Type	Regularization	$\mathcal{N}(E)$
BK	$ x > l_x, p > l_p$	$\frac{E}{2\pi} \left(\ln \frac{E}{2\pi} - 1 \right) + 1$
C	$ x < \Lambda, p < \Lambda$	$\frac{E}{2\pi} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left(\ln \frac{E}{2\pi} - 1 \right)$
S	$l_x < x < \Lambda$	$\frac{E}{2\pi} \ln \frac{\Lambda}{l_x}$

3 Quantization of xp and $\frac{1}{xp}$

3.1 The Hamiltonian $H_0 = xp$

Here we construst a self adjoint operator H_0 which acts on a Hilbert space $L^2(a, b)$ of square integrable function in the interval (a, b) . Taking $x \geq 0$, there are four possible intervals: $a = 0, l_x$ and $b = \Lambda, \infty$ where l_x and Λ were introduced (we shall take l_x and $\Lambda = N > 1$). Berry-Keating defined the quantum Hamiltonian H_0 as the normal ordered expresion

$$H_0 = \frac{1}{2}(xp + px) \quad (11)$$

where $p = -i\hbar \frac{d}{dx}$. If $x \geq 0$, Eq. (11) is equivalent to

$$\begin{aligned} H_0 &= \sqrt{x}p\sqrt{x} = -i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x} \\ &= -i\hbar \left(x\frac{d}{dx} + \frac{d}{dx}x \right) \end{aligned} \quad (12)$$

We know that the canonical commutation relation is

$$[\hat{x}, \hat{p}] = i\hbar \quad (13)$$

or, (dropping hat cause we're dealing with quantum system and opera-

tors)

$$\begin{aligned}
[x, p] &= i\hbar \\
\Rightarrow \left[x, -i\hbar \frac{d}{dx} \right] &= i\hbar \\
\Rightarrow -i\hbar \left[x, \frac{d}{dx} \right] &= i\hbar \\
\Rightarrow \left[x, \frac{d}{dx} \right] &= -1 \\
\Rightarrow x \frac{d}{dx} - \frac{d}{dx} x &= -1 \\
\Rightarrow \frac{d}{dx} x &= x \frac{d}{dx} + 1
\end{aligned} \tag{14}$$

Taking this value to R.H.S of Eq(12)

$$\begin{aligned}
-i\hbar \left(x \frac{d}{dx} + \frac{d}{dx} \right) f &= -i\hbar \left[2x \frac{d}{dx} + 1 \right] f \\
&= -i\hbar 2x \left[\frac{d}{dx} + \frac{1}{x} \frac{1}{2} \right] f \\
&= -i\hbar \frac{1}{\sqrt{x}} \frac{d}{dx} (\sqrt{x} f) \\
&= -i\hbar \frac{1}{\sqrt{x}} \frac{d}{dx} (\sqrt{x} f) \\
&= \frac{1}{\sqrt{x}} \left(-i\hbar \frac{d}{dx} \right) \sqrt{x} f
\end{aligned} \tag{15}$$

so

$$H_0 = \frac{1}{2} (xp + px) = -i\hbar \sqrt{x} \frac{d}{dx} \sqrt{x} \tag{16}$$

This is a symmetric operator acting on a certain domain of the Hilbert space $L^2(a, b)$, By definition, if an operator is symmetric (or Hermitian)[6]

$$\langle \psi | H_0 \phi \rangle = \langle \psi H_0 | \phi \rangle \tag{17}$$

or with limit,

$$\langle \psi | H_0 \phi \rangle - \langle \psi H_0 | \phi \rangle = i\hbar [a\psi^*(a)\phi(a) - b\phi^*(b)\psi(b)] = 0 \tag{18}$$

which is satisfied if both $\psi(x)$ and $\phi(x)$ vanish at the points a, b. von Neumann Theorem of deficiency indices states that, an operator is symmetric if its deficiency indices n_{\pm} are equal.[7]. Deficiency indices (or the defect numbers) of a closable symmetric operator T are cardinal number S

$$\begin{aligned} n_+ &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda > 0 \\ n_- &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda < 0 \end{aligned} \quad (19)$$

If T is densely defined and symmetric, then T is closable, and by formula $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}$

$$\begin{aligned} n_+ &:= \dim \mathcal{N}(T^* - i\mathbb{1}) = \dim \mathcal{N}(T^* - \lambda\mathbb{1}) & \text{Im } \lambda > 0 \\ n_- &:= \dim \mathcal{N}(T^* + i\mathbb{1}) = \dim \mathcal{N}(T^* + \lambda\mathbb{1}) & \text{Im } \lambda < 0 \end{aligned} \quad (20)$$

By definition $n_{\pm}(T) = \dim \mathcal{N}(T^* \mp iT)$

Again if T is a symmetric operator, then

$$\begin{aligned} K_+ &= \ker (i\mathbb{1} - T^*) = \text{Ran } (i\mathbb{1} - T)^{\perp} \\ K_- &= \ker (i\mathbb{1} + T^*) = \text{Ran } (-i\mathbb{1} + T)^{\perp} \end{aligned} \quad (21)$$

K_+ and K_- are called the deficiency subspaces of T, The pair of numbers n_+, n_- given by $n_+(T) = \dim[K_+], n_-(T) = \dim[K_-]$ are called deficiency indices of T.

von Neumann Theorem for deficiency indices states that if T an closed operator with deficiency indices n_+ and n_- . Then

(1) T is symmetric if and only if $n_+ = n_- = 0$ and self adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$

(2) T is symmetric and self adjoint and also has many self adjoint extensions if and only if $n_+ = n_- \neq 0$ and $\mathcal{D}(T) = \mathcal{D}(T^*)$. There is one-one correspondence between self adjoint extensions of T and unitary maps from K_+ onto K_-

(3) If either $n_+ = 0 \neq n_-$ or $n_- = 0 \neq n_+$ then T is not symmetric and has no nontrivial self adjoint extension (such operators are called maximal symmetric operator).

So this indices counts the number of solutions of the equation, which comes from the deficiency spaces for subsystem T

$$K_{\pm} = \ker \left(-H_0^{\dagger} - \mp i\mathbb{1} \right) \quad (22)$$

which leads to find the solution of the equation.

$$H_0^{\dagger} \psi_{\pm} = \pm i\hbar\lambda \psi_{\pm} \quad (23)$$

belonging to the domain of $H_0^{\dagger}(\lambda > 0)$. If $n = n_+ = n_- > 0$, there are infinitely many self-adjoint extensions of H_0 parameterized by a unitary $n \times n$ matrix. Stone's theorem states that if $U(t)$ be a strongly continuous one parameter unitary group on a Hilbert space \mathcal{H} . Then, there is a self-adjoint operator A on \mathcal{H} so that $U(t) = e^{itA}$. The solution of the equation () is

$$\begin{aligned} H_0^{\dagger} \psi_{\pm} &= \pm i\hbar\lambda \psi_{\pm} \\ \implies H_0 \psi_{\pm} &= \pm i\hbar\lambda \psi_{\pm} \quad [\text{because } H_0 \text{ is self-adjoint}] \\ \implies \left(-i\hbar\sqrt{x} \frac{d}{dx} \sqrt{x} \right) \psi_{\pm} &= \pm i\hbar\lambda \psi_{\pm} \\ \implies -i\hbar\sqrt{x} \frac{d}{dx} (\sqrt{x} \psi_{\pm}) &= \pm i\hbar\lambda \psi_{\pm} \\ \implies -\sqrt{x} \frac{d}{dx} (\sqrt{x} \psi_{\pm}) &= \pm \lambda \psi_{\pm} \\ \implies -x \frac{d}{dx} \psi_{\pm} - \sqrt{x} \frac{1}{2\sqrt{x}} \frac{d}{dx} \psi_{\pm} &= \pm \lambda \psi_{\pm} \\ \implies -x \frac{d}{dx} \psi_{\pm} &= \left(\pm \lambda + \frac{1}{2} \right) \psi_{\pm} \\ \implies \frac{d}{dx} \psi_{\pm} &= -\frac{1}{x} \left(\pm \lambda + \frac{1}{2} \right) \psi_{\pm} \\ \implies \frac{d\psi_{\pm}}{\psi_{\pm}} &= -\frac{dx}{x} \left(\pm \lambda + \frac{1}{2} \right) \\ \implies \ln \psi_{\pm} &= -(\ln x) \left(\pm \lambda + \frac{1}{2} \right) + \ln C \\ \implies \psi_{\pm} &= Cx^{-\frac{1}{2} \mp \lambda} \end{aligned}$$

whose norm in the interval (a,b) is

$$\begin{aligned}
\langle \psi_{\pm} | \psi_{\pm} \rangle &= \int_a^b C^2 x^{-1 \mp \lambda} dx \\
&= \mp \frac{C^2}{2\lambda} (b^{\mp 2\lambda} - a^{\mp 2\lambda}) \\
&= \pm \frac{C^2}{2\lambda} (a^{\mp 2\lambda} - b^{\mp 2\lambda})
\end{aligned} \tag{24}$$

The deficiency indices corresponding to the four intervals considered above are collected in Table[8]. We find the deficiency indices by observing different intervals. For BK intervals $(1, \infty)$ only ψ_+ belongs to Hilbert space (ψ_- blows out. or putting intervals in Eq. (20) and testing whether it belongs to the Hilbert space)[5]. And the rest are given below

Table 2: Deficiency indices of H_0 . The corresponding intervals are associated to the semiclassical regularizations of section 2 (BK, C, S). The last one T, describes the case with no constraints on x except positivity (i.e. $x > 0$)

Type	(a,b)	(n_+, n_-)	Self-adjoint
BK	$(1, \infty)$	$(1, 0)$	-
C	$(0, N)$	$(0, 1)$	-
S	$(1, N)$	$(1, 1)$	\checkmark
T	$(0, \infty)$	$(0, 0)$	\checkmark

From the von Neumann theorem we see that H_0 is essentially self-adjoint on the half line $\mathbb{R}_+ = (0, \infty)$. This was studied by Twamley and Milbrn, who defined quantum Mellin transform using the eigenstates of H_0 [9]. On the other hand, in the interval $(1, N)$ the operator H_0 admits infinitely many self-adjoint extensions parameterized by a phase $e^{i\theta}$. This phase defines the boundary condition of the functions belonging to the self-adjoint domain.[5]

$$\mathcal{D}(H_{0,\theta}) = \left\{ \psi, H_0 \psi \in L^2(1, N), e^{i\theta} \psi(1) = \sqrt{N} \psi(N) \right\} \tag{25}$$

The eigenfunction of H_0

$$H_0 \psi_E = E \psi_E, \tag{26}$$

are given by [1]

$$\psi_E(x) = \frac{C}{x^{\frac{1}{2}-iE\hbar}}, \quad E \in \mathbb{R} \quad (27)$$

where C is a normalization constant. In the half line \mathbb{R}_+ there are no further restriction on E , hence the spectrum of H_0 is continuous and covers the whole real line \mathbb{R} . In this case the normalization constant is chosen as $C = \frac{1}{\sqrt{2\pi\hbar}}$ which gurantees the standard normalization

$$\langle \psi_E | \psi_{E'} \rangle = C^2 \int_0^\infty \frac{dx}{x} x^{-i(E-E')/\hbar} = \delta(E - E') \quad (28)$$

In the case where H_0 is defined in ther interval, the boundary condition (21) yeilds the quantization condition for E , namely

$$\begin{aligned} e^{i\theta} \psi(1) &= \sqrt{N} \psi(N) \\ \implies e^{i\theta} \frac{C}{1^{\frac{1}{2}-iE\hbar}} &= \sqrt{N} \frac{C}{N^{\frac{1}{2}-iE\hbar}} \\ \implies e^{i\theta} &= N^{-iE\hbar} \\ \implies i\theta &= \left(\frac{-iE}{\hbar}\right) \ln N \\ \implies E &= \frac{\hbar\theta}{\ln N} \\ \implies E &= \frac{2\pi\hbar}{\ln N} \left(\frac{\theta}{2\pi}\right) \\ \implies E_n &= \frac{2\pi\hbar}{\ln N} \left(n + \frac{\theta}{2\pi}\right) \quad n \in \mathbb{N} \end{aligned} \quad (29)$$

Hence the spectrum of H_0 is dscrete, with a level spacing decreasing for largerm values of N . The normalization constant of the wave function is now $C = \frac{1}{\sqrt{\ln N}}$ which gives,

$$\langle \psi_{E_n} | \psi_{E_{n'}} \rangle = C^2 \int_1^N \frac{dx}{x} x^{-i(E_n-E_{n'})/\hbar} = \delta_{n,n'} \quad (30)$$

The spectrum (16) agrees with the semiclassical result given in Table 1 for the S-regularization(recall that $l_x = 1, \Lambda = N, \hbar = 1$) For the particular case where $\theta = \pi$, one observes that the energy spectrum is symmetric around zero, i.e, if E_n is an eigenenergy so is $-E_n$. This result is obtained in working[10] with the inverse Hamiltonian $\frac{1}{H_0}$. We are reviewing that construction in next section.

3.2 The inverse Hamiltonian $\frac{1}{H_0}$

First, we take the expression at Eq(16) and take the formal inverse, i.e., $H_0^{-1} = x^{1/2}p^{-1}x^{-1/2}$. The operator p_{-1} is the one-dimensional Green's function with matrix elements (definition of Green's function: Green's function is the kernel of an integral operator that represents the inverse of a differential operator. Let

$$Lu = f \quad (31)$$

Here u and f are vectors and L is a square, invertible matrix. The inverse matrix exists if $\lambda = 0$ is not an eigenvalue of L , or when $\det L \neq 0$. Now

$$u = L^{-1}f \quad (32)$$

where L^{-1} is the inverse operator of L . The inverse operator to be an integral operator of the form.

$$(L^{-1}f)(x) = \int_a^b g(x, \xi) f(\xi) d\xi \quad (33)$$

with kernel G . If L exists, then the kernel function $g(x, \xi)$ is called the Green's function associated with L .

So p_{-1} operator will be

$$\begin{aligned} \langle x | p^{-1} | x' \rangle &= \left\langle x \left| \frac{1}{-i\hbar \frac{d}{dx}} \right| x' \right\rangle \\ &= -i\hbar \left\langle x \left| \frac{1}{\frac{d}{dx}} \right| x' \right\rangle \\ &= \frac{\hbar}{i} G(x, x') \\ &= \frac{\hbar}{2i} \text{sign}(x - x') \end{aligned} \quad (34)$$

Here $\text{sign}(x - x')$ is the sign function.[10]. The operator H_0^{-1} is defined in the interval $(1, N)$ by the continuous matrix,

$$H_0^{-1}(x, x') = \frac{i}{2\hbar} \frac{\text{sign}(x - x')}{\sqrt{xx'}}, \quad 1 \leq x, x' \leq N. \quad (35)$$

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