1 Out of Time Order Correlator of $H = xp$ model

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to 1/2. The classical zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^s \tag{1}$$

for Re $s > 1$. By the fundamental theorem of arithmatic, which is also equivalent to the Euler product over primes

$$\zeta(s) = \prod_{p} (1-p)^{-1} \tag{2}$$

where p are all the prime numbers.

Zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negetive integers (for $s = -2, -4, -6, ...$). For complex s ($=\sigma + it$) (with real part between zero and one) , zeta fucntion becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - iE$ zeta funtion becomes zero $\zeta(\frac{1}{2} - it) = 0$. Hilbert-P$\acute{o}$lya conjecture suggests that the imaginary parts of the nontrivial zeros are the eiogenvalues of a self-adjoint hamiltonian operator $\hat{H}$. It is also one of the aprroach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-P$\acute{o}$lya cinjecture should take the form[1]

$$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \tag{3}$$

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[1] Berry proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obstained by quantization of a classical chaotic hamiltonian, whose periodic orbits are labeled by the prime numbers. Connes took the adelic approach to introduce $H = xp$ [2]. He showed that using different semiclassical regularization, Riemann zeros appear as missing spectral lines in a continuum.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is[1, 2]

$$H_0^{cl} = xp \tag{4}$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \qquad p(t) = p_0 e^{-t} \tag{5}$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of reularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number $\mathcal{N}(E)$ with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \tag{6}$$

Where A is the area of the allowed phase space region below the curve $E = xp$. So the the number of semiclassical states will be for Berry-Keating regularization

$$\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h}\int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + ... \\
&= \frac{1}{h}\left[\int_{l_x}^{\frac{E}{l_p}} dx \left[\frac{E}{x} - l_p\right]\right] \\
&= \frac{1}{h}\left[E\left[\ln x\right]_{l_x}^{\frac{E}{l_p}} - l_p\left[\frac{E}{l_p} - l_x\right]\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - l_x l_p\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - h\right] \\
&= \frac{E}{h}\left[\ln\frac{E}{l_x l_p} - 1\right] + 1 \\
&= \frac{E}{2\pi\hbar}\left[\ln\frac{E}{2\pi\hbar} - 1\right] + 1
\end{aligned} \tag{7}$$

adding Maslov phase $(-\frac{1}{8})$ and $\hbar = 1$, it becomes

$$\mathcal{N}_{BK}(E) = \frac{E}{2\pi}\left[\ln\frac{E}{2\pi} - 1\right] + \frac{7}{8} + ..., \qquad E >> 1 \tag{8}$$

The exact formula for the Riemann zeros, $\mathcal{N}_R(E)$ contains a fluctuation term which depends on the zeta funcion.[3]

$$\begin{aligned}
\mathcal{N}_R(E) &= \langle\mathcal{N}\rangle + \mathcal{N}_{fl}(E) \\
\langle\mathcal{N}(E)\rangle &= \frac{1}{\pi}Im\ ln\left[\Gamma\frac{1}{2}\left(\frac{1}{2} - iE\right)\right] - \frac{E}{2\pi}ln\pi + 1 \\
\mathcal{N}_{fl}(E) &= \frac{1}{\pi}Im\ ln\left[\zeta\left(\frac{1}{2} - iE\right)\right]
\end{aligned} \tag{9}$$

Bery-Keatin took this result and analogies between formulae in Nunber Theory and Quantum Chaos, they pointed the quantization of classical chaotic Hamiltonian give rise to the zeros as point like spectra.[1, 4] Whereas Connes found the number of semicassical states diverges in the limit where the cutoff Λ goes to infinity, and that therre us a finite size correction given by mins the average position of the Riemann zeros.

$$
\begin{aligned}
\mathcal{N}_c(E) &= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \int_{\frac{E}{x}}^{\frac{E}{\Lambda}} dp\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \left[\frac{E}{x} - \frac{E}{\Lambda}\right]\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + E\left[ln\ x\right]_{\frac{E}{\Lambda}}^{\Lambda} - \frac{E}{\Lambda}\left[\Lambda - \frac{E}{\Lambda}\right]\right] \\
&= \frac{1}{h}\left[2E - \left(\frac{E}{\Lambda}\right)^2 + E\left[ln\ \frac{\Lambda^2}{E}\right] - E + \left(\frac{E}{\Lambda}\right)^2\right] \\
&= \frac{1}{h}\left[E + E\left[ln\ \frac{\Lambda^2}{E}\right]\right] \\
&= \frac{1}{h}\left[E + E\left[ln\ \frac{\Lambda^2}{E}\frac{2\pi}{2\pi}\right]\right] \\
&= \frac{E}{h} ln\ \frac{\Lambda^2}{2\pi} - \frac{E}{h}\left[ln\ \frac{E}{2\pi} - 1\right] \\
&= \frac{E}{2\pi} ln\ \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\left[ln\ \frac{E}{2\pi} - 1\right] \qquad [taking\ \ \hbar = 1]
\end{aligned}
\tag{10}
$$

This result les to the missing spectral interpretation of the Riemann zeros, according to which there is a continuum of eginstates (represented by the term $\frac{E}{\pi} ln\ \Lambda$ in $\mathcal{N}(E)$) where states asscoiated with Riemann zeros are missing.

References

[1] Michael V Berry and Jonathan P Keating. H= xp and the riemann zeros. *Supersymmetry and trace formulae: chaos and disorder*, pages 355–367, 1999.

[2] Alain Connes. Trace formula in noncommutative geometry and the zeros of the riemann zeta function. *Selecta Mathematica*, 5(1):29, 1999.

[3] Edwards HM Riemanns Zeta Function. Academic press new york, 1974.

[4] Michael V Berry and Jonathan P Keating. The riemann zeros and eigenvalue asymptotics. *SIAM review*, 41(2):236–266, 1999.