

Solution to Problem Sheet 2

Noor E Mustafa Ferdous
email: nooremf@gmail.com

Solve for problem no. 1

Given

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu \\ &= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - j^\mu A_\mu \\ &= -\frac{1}{4}(\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) - \partial_\nu A_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)) - j^\mu A_\mu \\ &= -\frac{1}{2}(\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)) - j^\mu A_\mu\end{aligned}$$

(a)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_\nu} &= -j^\nu \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}\partial^\mu A^\nu + \frac{1}{2}\partial^\nu A^\mu \\ &= -(\partial^\mu A^\nu - \partial^\nu A^\mu) = F^{\mu\nu}\end{aligned}$$

Therefore Euler-Lagrangian is

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial A_\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -j^\nu + \partial_\mu F^{\mu\nu} = 0 \\ \partial_\mu F^{\mu\nu} &= j^\nu\end{aligned}$$

(b) taking $\mu = i$ and $\nu = 0$

$$\begin{aligned}\partial_i F^{i0} &= j^0 \\ \implies \vec{\nabla} \cdot \vec{E} &= \rho\end{aligned}$$

And taking $\nu = j$

$$\begin{aligned}\partial_\mu F^{\mu j} &= j^0 \\ \implies \partial_0 F^{0j} + \partial_i F^{ij} &= 0 \\ \implies \partial^0 F_{0j} + \partial^i F_{ij} &= 0 \\ \implies \frac{\partial E_j}{\partial t} + \partial^i (-\epsilon_{ijk} B_k) &= 0 \\ \implies \frac{\partial E_j}{\partial t} &= \partial^i (-\epsilon_{ijk} B_k) \\ \implies \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B}\end{aligned}$$

Solve for problem no. 2

(a) Given

$$\begin{aligned}\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \\ &= \partial_\alpha (\partial_\beta A_\gamma - \partial_\gamma A_\beta) + \partial_\beta (\partial_\gamma A_\alpha - \partial_\alpha A_\gamma) + \partial_\gamma (\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= \partial_\alpha \partial_\beta A_\gamma - \partial_\alpha \partial_\gamma A_\beta + \partial_\beta \partial_\gamma A_\alpha - \partial_\beta \partial_\alpha A_\gamma + \partial_\gamma \partial_\alpha A_\beta - \partial_\gamma \partial_\beta A_\alpha \\ &= 0\end{aligned}$$

Given

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \quad (1)$$

taking $\alpha = 1, \beta = 2, \gamma = 3$

$$\begin{aligned}\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0 \\ \implies \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0 \\ \implies \partial^1 B^1 + \partial^2 B^2 + \partial^3 B^3 &= 0 \\ \implies \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

Now taking $\alpha = 0, \beta = i, \gamma = j (i \neq j)$

$$\begin{aligned}
& \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \\
\Rightarrow & \partial^0 F^{ij} + \partial^j F^{0i} + \partial^i F^{j0} = 0 \\
\Rightarrow & -\frac{\partial \mathbf{B}}{\partial t} + \partial^j E^i - \partial^i E^j = 0 \\
& \Rightarrow -\frac{\partial \mathbf{B}}{\partial t} = \epsilon_{ijk} E^k \\
& \Rightarrow -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}
\end{aligned}$$

Solve for problem no. 4

$$\begin{aligned}
& \Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_\mu)} \\
\Rightarrow & \frac{\partial \mathcal{L}}{\partial (\partial_0 A_\mu)} = -F^{0\mu} \\
& \Rightarrow \Pi^0 = -F^{00} = 0 \\
& \Rightarrow A_0(x) = 0
\end{aligned}$$