

Out-of-Time-Order-Correlator of $H = xp$

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Abstract

Berry-Keating Hamiltonian $H_0 = \frac{1}{2}(xp + px)$ resembles the Hilbert-Pólya conjecture of Riemann Hypothesis, shows quantum chaotic behaviour. In this paper we first quantize xp and adding boundary condition evaluate spectrum with proper eigenfunction. Then we quantize $1/xp$ and achieve same spectrum with minor phase addition. We observe OTOC of the hamiltonian, which gauge the quantum chaotic behaviour of it.

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1 Introduction

The classical zeta/ Riemann zeta function defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ [1], where s is $s = \sigma + it$ and n is the integers. If the real part of s is greater than 1 ($\text{Re } s > 1$), the zeta function converges. But at real part equal to 1 ($\text{Re } s = 1$) it becomes the harmonic series $\zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ and diverges. So there begs the question where does the zeta function goes to zero. It was found that there are two types of condition that make the Riemann zeta function which is called Riemann zero or Zeta zero. For negative integers ($s = -2, -4, -6$) in the Riemann zeta function becomes zero. These types of zeros are called Trivial zeros. And the other kind of zeros, which are called Non-Trivial zeros, lie between real part of s in $\text{Re } 0 < s < 1$. Riemann hypothesize that all the non-trivial zeros lie at $s = \frac{1}{2}$. This is the famous Riemann hypothesis. Euler showed that the existing classical zeta function has a connection to the product of all prime numbers. He proved that $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$ where p is the primes. So if anyone can prove the Riemann hypothesis, they can also prove that all the prime numbers must be lies the line of $\text{Re } s = \frac{1}{2}$. The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to $1/2$. The classical zeta function converges at $\text{Re } s > 1$. By the fundamental theorem of arithmetic, which is also equivalent to the Euler product over primes where p are all the prime numbers. So zeta can describe all the prime numbers. Also the zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negative integers (for $s = -2, -4, -6, \dots$). For complex $s (= \sigma + it)$ (with real part between zero and one), zeta function becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - iE$ zeta function becomes zero. If the Riemann hypothesis is true the statistical distribution of the primes will be constrained in the most favourable way. Hilbert-Pólya conjecture suggests that the imaginary parts of the nontrivial zeros are the eigenvalues of a self-adjoint hamiltonian operator \hat{H} . It is also one of the approach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-Pólya conjecture should take the form [2] $\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})$. The classical versus quantum version of primes and the zeros is also at the heart of

so-called quantum chaos approach to the RH.

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[2] In 1999, Berry and Keating[2, 3] on the one hand and Connes[1] on the other claimed that the classical Hamiltonian $H = xp$, where x and p are the location and momenta of a 1D particle, is strongly connected to the Riemann zeros. This startling hypothesis was based on a semiclassical analysis of $H = xp$, which led these authors to very different conclusions about the probable spectral interpretation of the Riemann zeros. The source of the dispute is the use of distinct regularizations of $H = xp$. Berry and Keating used a Planck cell regularization, in which the smooth component of the Riemann zeros is represented semiclassically as discrete energy levels. Connes, on the other hand, chose an upper cutoff for the position and momenta, resulting in a semiclassical continuous spectrum devoid of smooth zeros. All of these semiclassical solutions are heuristic, and there is no consistent quantum version as of yet.

The out-of-time-order correlator (OTOC) is typically defined by [4]

$$C_T \equiv -\langle [W(t), V(0)]^2 \rangle \quad (1)$$

where $\langle \dots \rangle$ represents the thermal average. $W(t)$ and $V(t)$ are operators as time in t in the Heisenberg representation. The OTOC, first introduced in a calculation of a vertex correction of a current for a superconductor[5], was recently turned out to be considered as a measure of the magnitude of quantum chaos. A naive argument for the relation between the OTOC and chaos is as follows[6]. Consider position and momentum operators, $x(t)$ and $p(t)$, in a quantum system. We can define an OTOC as $C_T = -\langle [x(t), p(0)]^2 \rangle$. Taking a naive semiclassical limit, we would be able to replace the commutator $[x(t), p(0)]$ by the Poisson bracket $i\hbar\{x(t), p(0)\}_{PB} = i\hbar\frac{\delta x(t)}{\delta x(0)}$. For a classically chaotic system with a Lyapunov exponent λ , we have $\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda t}$ because of sensitivity to initial condition. Thus, the OTOC should grow as $\sim \hbar^2 e^{2\lambda t}$ and we can read off the quantum Lyapunov exponent λ from it. The quantization of a classically chaotic system may provide a positive quantum Lyapunov exponent of the OTOC. Historically, the nearest neighbour distribution (NND) for the energy level spectrum has been used to quantify quantum chaos [7]. For integrable and non-integrable systems, it is considered

that NNDs are given by Poisson and Wigner distributions. The OTOC is expected to be another measure of quantum chaos. A possible distinction from the classical chaotic system is that the OTOC does not grow eternally but saturates at the Ehrenfest time t_E . The Ehrenfest time is defined by the time scale beyond which the wave function spreads over the whole system. It is roughly characterized as a boundary between a particle-like behavior and a wave-like behavior of the wave function.

In the definition of OTOC, we consider the thermal average of $-\langle [W(t), V(0)]^2 \rangle$. When we take thermal average, we need to consider the four point operator $\langle [W(t), V(0)]^2 \rangle$ instead of two point operator $\langle [W(t), V(0)] \rangle$. The reason is as follows. Assuming that we can replace the commutator by Poisson bracket by semiclassical limit, $\langle [W(t), V(0)] \rangle$ would also show the exponential growth $\sim e^{\lambda t}$. However, its coefficient can be both positive and negative. By taking the thermal average, their contributions would be canceled. From the quantum theory point of view, $\langle [W(t), V(0)] \rangle$ measures the correlation between $W(t)$ and $V(0)$. Therefore, the two point function decays as $t \rightarrow \infty$ and cannot show the chaotic behavior.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is [2, 8]

$$H_0^{\text{cl}} = xp \tag{2}$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \quad p(t) = p_0 e^{-t} \tag{3}$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of regularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p .

Semiclassical states number $\mathcal{N}(E)$ with an energy between 0 to E is given by

$$\begin{aligned}\mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h}\end{aligned}\tag{4}$$

Where A is the area of the allowed phase space region below the curve $E = xp$.

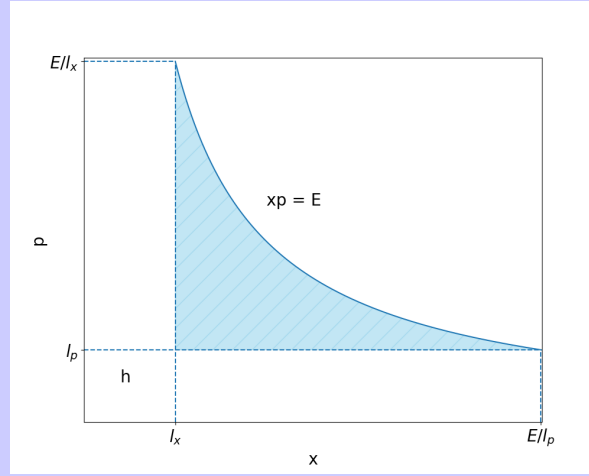


Figure 1: Berry-Keating Regularization

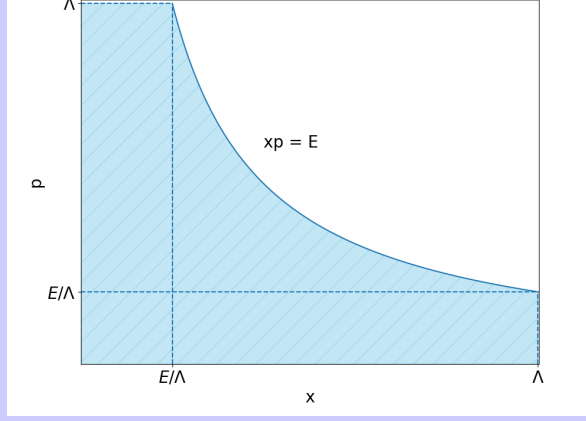


Figure 2: Connes Regularization

So the the number of semiclassical states will be for Berry-Keating regularization

$$\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h} \int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + \dots\dots\dots \\
&= \frac{1}{h} \left[\int_{l_x}^{\frac{E}{l_p}} dx \left[\frac{E}{x} - l_p \right] \right] \\
&= \frac{1}{h} \left[E \left[\ln x \right]_{l_x}^{\frac{E}{l_p}} - l_p \left[\frac{E}{l_p} - l_x \right] \right] \\
&= \frac{1}{h} \left[E \ln \frac{E}{l_x l_p} - E - l_x l_p \right] \\
&= \frac{1}{h} \left[E \ln \frac{E}{l_x l_p} - E - h \right] \\
&= \frac{E}{h} \left[\ln \frac{E}{l_x l_p} - 1 \right] + 1 \\
&= \frac{E}{2\pi\hbar} \left[\ln \frac{E}{2\pi\hbar} - 1 \right] + 1
\end{aligned} \tag{5}$$

adding Maslov phase $(-\frac{1}{8})$ and $\hbar = 1$, it becomes

$$\mathcal{N}_{BK}(E) = \frac{E}{2\pi} \left[\ln \frac{E}{2\pi} - 1 \right] + \frac{7}{8} + \dots\dots\dots, \quad E \gg 1 \tag{6}$$

The exact formula for the Riemann zeros, $\mathcal{N}_R(E)$ contains a fluctuation term which depends on the zeta function.[1]

$$\begin{aligned}\mathcal{N}_R(E) &= \langle \mathcal{N} \rangle + \mathcal{N}_{fl}(E) \\ \langle \mathcal{N}(E) \rangle &= \frac{1}{\pi} \text{Im} \ln \left[\Gamma \frac{1}{2} \left(\frac{1}{2} - iE \right) \right] - \frac{E}{2\pi} \ln \pi + 1 \\ \mathcal{N}_{fl}(E) &= \frac{1}{\pi} \text{Im} \ln \left[\zeta \left(\frac{1}{2} - iE \right) \right]\end{aligned}\tag{7}$$

Bery-Keatin took this result and analogies between formulae in Number Theory and Quantum Chaos, they pointed the quantization of classical chaotic Hamiltonian give rise to the zeros as point like spectra.[2, 9] Whereas Connes found the number of semiclassical states diverges in the limit where the cutoff Λ goes to infinity, and that there is a finite size correction given by minus the average position of the Riemann zeros.

$$\begin{aligned}\mathcal{N}_c(E) &= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \int_{\frac{E}{\Lambda}}^{\frac{E}{x}} dp \right] \\ &= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + \int_{\frac{E}{\Lambda}}^{\Lambda} dx \left[\frac{E}{x} - \frac{E}{\Lambda} \right] \right] \\ &= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + E [\ln x]_{\frac{E}{\Lambda}}^{\Lambda} - \frac{E}{\Lambda} \left[\Lambda - \frac{E}{\Lambda} \right] \right] \\ &= \frac{1}{h} \left[2E - \left(\frac{E}{\Lambda} \right)^2 + E \left[\ln \frac{\Lambda^2}{E} \right] - E + \left(\frac{E}{\Lambda} \right)^2 \right] \\ &= \frac{1}{h} \left[E + E \left[\ln \frac{\Lambda^2}{E} \right] \right] \\ &= \frac{1}{h} \left[E + E \left[\ln \frac{\Lambda^2}{E} \frac{2\pi}{2\pi} \right] \right] \\ &= \frac{E}{h} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{h} \left[\ln \frac{E}{2\pi} - 1 \right] \\ &= \frac{E}{2\pi} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left[\ln \frac{E}{2\pi} - 1 \right] \quad [taking \hbar = 1]\end{aligned}\tag{8}$$

This result leads to the missing spectral interpretation of the Riemann zeros, according to which there is a continuum of eigenstates (represented by

the term $\frac{E}{\pi} \ln \Lambda$ in $\mathcal{N}(E)$ where states associated with Riemann zeros are missing.

Finally, in the S-regularization the number of semiclassical states diverges as $\frac{E}{2\pi} \ln \frac{\Lambda}{l_x}$ suggesting a continuum spectrum, like in Connes's approach. But there is no finite size correction to that formula, and consequently the possible connection to the Riemann zeros is lost.

Table 1:

Three different regularizations of $H = xp$ and the corresponding number of semiclassical states in units $\hbar = 1$ [10]

Type	Regularization	$\mathcal{N}(E)$
BK	$ x > l_x, p > l_p$	$\frac{E}{2\pi} \left(\ln \frac{E}{2\pi} - 1 \right) + 1$
C	$ x < \Lambda, p < \Lambda$	$\frac{E}{2\pi} \ln \frac{\Lambda^2}{2\pi} - \frac{E}{2\pi} \left(\ln \frac{E}{2\pi} - 1 \right)$
S	$l_x < x < \Lambda$	$\frac{E}{2\pi} \ln \frac{\Lambda}{l_x}$

3 Quantization of xp and $\frac{1}{xp}$

3.1 The Hamiltonian $H_0 = xp$

Here we construct a self adjoint operator H_0 which acts on a Hilbert space $L^2(a, b)$ of square integrable function in the interval (a, b) . Taking $x \geq 0$, there are four possible intervals: $a = 0, l_x$ and $b = \Lambda, \infty$ where l_x and Λ were introduced (we shall take l_x and $\Lambda = N > 1$). Berry-Keating defined the quantum Hamiltonian H_0 as the normal ordered expression

$$H_0 = \frac{1}{2}(xp + px) \quad (9)$$

where $p = -i\hbar \frac{d}{dx}$. If $x \geq 0$, Eq. (11) is equivalent to

$$\begin{aligned} H_0 &= \sqrt{x}p\sqrt{x} = -i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x} \\ &= -i\hbar\left(x\frac{d}{dx} + \frac{d}{dx}x\right) \end{aligned} \quad (10)$$

We know that the canonical commutation relation is

$$[\hat{x}, \hat{p}] = i\hbar \quad (11)$$

or, (dropping hat cause we're dealing with quantum system and operators)

$$\begin{aligned} [x, p] &= i\hbar \\ \implies \left[x, -i\hbar \frac{d}{dx} \right] &= i\hbar \\ \implies -i\hbar \left[x, \frac{d}{dx} \right] &= i\hbar \\ \implies \left[x, \frac{d}{dx} \right] &= -1 \\ \implies x \frac{d}{dx} - \frac{d}{dx} x &= -1 \\ \implies \frac{d}{dx} x &= x \frac{d}{dx} + 1 \end{aligned} \quad (12)$$

Taking this value to R.H.S of Eq(12)

$$\begin{aligned} -i\hbar \left(x \frac{d}{dx} + \frac{d}{dx} \right) f &= -i\hbar \left[2x \frac{d}{dx} + 1 \right] f \\ &= -i\hbar 2x \left[\frac{d}{dx} + \frac{1}{x} \frac{1}{2} \right] f \\ &= -i\hbar \frac{1}{\sqrt{x}} \frac{d}{dx} (\sqrt{x} f) \\ &= -i\hbar \frac{1}{\sqrt{x}} \frac{d}{dx} (\sqrt{x} f) \\ &= \frac{1}{\sqrt{x}} \left(-i\hbar \frac{d}{dx} \right) \sqrt{x} f \end{aligned} \quad (13)$$

so

$$H_0 = \frac{1}{2} (xp + px) = -i\hbar \sqrt{x} \frac{d}{dx} \sqrt{x} \quad (14)$$

This is a symmetric operator acting on a certain domain of the Hilbert space $L^2(a, b)$, By definition, if an operator is symmetric (or Hermitian)[11]

$$\langle \psi | H_0 \phi \rangle = \langle \psi H_0 | \phi \rangle \quad (15)$$

or with limit,

$$\langle \psi | H_0 \phi \rangle - \langle \psi H_0 | \phi \rangle = i\hbar [a\psi^*(a)\phi(a) - b\phi^*(b)\psi(b)] = 0 \quad (16)$$

which is satisfied if both $\psi(x)$ and $\phi(x)$ vanish at the points a, b. von Neumann Theorem of deficiency indices states that, an operator is symmetric if its deficiency indices n_{\pm} are equal.[12]. Deficiency indices (or the defect numbers) of a closable symmetric operator T are cardinal number S

$$\begin{aligned} n_+ &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda > 0 \\ n_- &:= d_{\lambda} = \dim \mathcal{R}(T - \bar{\lambda}\mathbb{1})^{\perp} & \text{Im } \lambda < 0 \end{aligned} \quad (17)$$

If T is densely defined and symmetric, then T is closable, and by formula $\mathcal{N}(T^*) = \mathcal{R}(T)^{\perp}$

$$\begin{aligned} n_+ &:= \dim \mathcal{N}(T^* - i\mathbb{1}) = \dim \mathcal{N}(T^* - \lambda\mathbb{1}) & \text{Im } \lambda > 0 \\ n_- &:= \dim \mathcal{N}(T^* + i\mathbb{1}) = \dim \mathcal{N}(T^* + \lambda\mathbb{1}) & \text{Im } \lambda < 0 \end{aligned} \quad (18)$$

By definition $n_{\pm}(T) = \dim \mathcal{N}(T^* \mp iT)$

Again if T is a symmetric operator, then

$$\begin{aligned} K_+ &= \ker (i\mathbb{1} - T^*) = \text{Ran } (i\mathbb{1} - T)^{\perp} \\ K_- &= \ker (i\mathbb{1} + T^*) = \text{Ran } (-i\mathbb{1} + T)^{\perp} \end{aligned} \quad (19)$$

K_+ and K_- are called the deficiency subspaces of T, The pair of numbers n_+, n_- given by $n_+(T) = \dim[K_+], n_-(T) = \dim[K_-]$ are called deficiency indices of T.

von Neumann Theorem for deficiency indices states that if T an closed operator with deficiency indices n_+ and n_- . Then

(1) T is symmetric if and only if $n_+ = n_- = 0$ and self adjoint if $\mathcal{D}(T) = \mathcal{D}(T^*)$

(2) T is symmetric and self adjoint and also has many self adjoint extensions if and only if $n_+ = n_- \neq 0$ and $\mathcal{D}(T) = \mathcal{D}(T^*)$. There is one-one correspondence between self adjoint extensions of T and unitary maps from

K_+ onto K_-

(3) If either $n_+ = 0 \neq n_-$ or $n_- = 0 \neq n_+$ then T is not symmetric and has no nontrivial self adjoint extension (such operators are called maximal symmetric operator).

So this indices counts the number of solutions of the equation, which comes from the deficiency spaces for subsystem T

$$K_{\pm} = \ker \left(-H_0^{\dagger} - \mp i\mathbb{1} \right) \quad (20)$$

which leads to find the solution of the equation.

$$H_0^{\dagger} \psi_{\pm} = \pm i\hbar\lambda \psi_{\pm} \quad (21)$$

belonging to the domain of $H_0^{\dagger}(\lambda > 0)$. If $n = n_+ = n_- > 0$, there are infinitely many self-adjoint extensions of H_0 parameterized by a unitary $n \times n$ matrix. Stone's theorem states that if $U(t)$ be a strongly continuous one parameter unitary group on a Hilbert space \mathcal{H} . Then, there is a self-adjoint operator A on \mathcal{H} so that $U(t) = e^{itA}$. The solution of the equation () is

$$\begin{aligned}
H_0^\dagger \psi_\pm &= \pm i\hbar\lambda\psi_\pm \\
\implies H_0 \psi_\pm &= \pm i\hbar\lambda\psi_\pm \quad [\text{becuase } H_0 \text{ is self - adjoint}] \\
\implies \left(-i\hbar\sqrt{x}\frac{d}{dx}\sqrt{x}\right)\psi_\pm &= \pm i\hbar\lambda\psi_\pm \\
\implies -i\hbar\sqrt{x}\frac{d}{dx}(\sqrt{x}\psi_\pm) &= \pm i\hbar\lambda\psi_\pm \\
\implies -\sqrt{x}\frac{d}{dx}(\sqrt{x}\psi_\pm) &= \pm\lambda\psi_\pm \\
\implies -x\frac{d}{dx}\psi_\pm - \sqrt{x}\frac{1}{2\sqrt{x}}\frac{d}{dx}\psi_\pm &= \pm\lambda\psi_\pm \\
\implies -x\frac{d}{dx}\psi_\pm &= \left(\pm\lambda + \frac{1}{2}\right)\psi_\pm \\
\implies \frac{d}{dx}\psi_\pm &= -\frac{1}{x}\left(\pm\lambda + \frac{1}{2}\right)\psi_\pm \\
\implies \frac{d\psi_\pm}{\psi_\pm} &= -\frac{dx}{x}\left(\pm\lambda + \frac{1}{2}\right) \\
\implies \ln \psi_\pm &= -(\ln x)\left(\pm\lambda + \frac{1}{2}\right) + \ln C \\
\implies \psi_\pm &= Cx^{-\frac{1}{2}\mp\lambda}
\end{aligned}$$

whose norm in the interval (a,b) is

$$\begin{aligned}
\langle \psi_\pm | \psi_\pm \rangle &= \int_a^b C^2 x^{-1\mp\lambda} dx \\
&= \mp \frac{C^2}{2\lambda} (b^{\mp 2\lambda} - a^{\mp 2\lambda}) \\
&= \pm \frac{C^2}{2\lambda} (a^{\mp 2\lambda} - b^{\mp 2\lambda})
\end{aligned} \tag{22}$$

The deficiency indices corresponding to the four intervals cosidered above are collecte in Table[13]. We find the deficency indices by observing different intervals. For BK intervals(1, ∞) only ψ_+ belongs to Hilbert space (ψ_- blows out. or putting intervals in Eq. (20) and testing whether it belongs to the Hilbert space)[10]. And the rest are given below

Table 2: Deficiency indices of H_0 . The corresponding intervals are associated to the semiclassical regularizations of section 2 (BK, C ,S). The last one T, describes the case with no constraints on x except positivity (i.e. $x>0$)

Type	(a,b)	(n_+, n_-)	Self-adjoint
BK	$(1, \infty)$	$(1,0)$	-
C	$(0,N)$	$(0,1)$	-
S	$(1,N)$	$(1,1)$	\checkmark
T	$(0, \infty)$	$(0,0)$	\checkmark

From the von Neumann theorem we see that H_0 is essentially self-adjoint on the half line $\mathbb{R}_+ = (0, \infty)$. This was studied by Twamley and Milbrn, who defined quantum Mellin transform using the eigenstates of H_0 [14] On the other hand, in the interval $(1,N)$ the operator H_0 admits infinitely many self-adjoint extensions parameterized by a phase $e^{i\theta}$. This phase defines the boundary condition of the functions belonging to the self-adjoint domain.[10]

$$\mathcal{D}(H_{0,\theta}) = \left\{ \psi, H_0\psi \in L^2(1, N), e^{i\theta}\psi(1) = \sqrt{N}\psi(N) \right\} \quad (23)$$

The eigenfunction of H_0

$$H_0\psi_E = E\psi_E, \quad (24)$$

are given by [2]

$$\psi_E(x) = \frac{C}{x^{\frac{1}{2}-iEh}}, \quad E \in \mathbb{R} \quad (25)$$

where C is a normalization constant. In the half line \mathbb{R}_+ there are no further restriction on E, hence the spectrum of H_0 is continuous and covers the whole real line \mathbb{R} . In this case the normalization constant is chosen as $C = \frac{1}{\sqrt{2\pi h}}$ which guarantees the standard normalization

$$\langle \psi_E | \psi_{E'} \rangle = C^2 \int_0^\infty \frac{dx}{x} x^{-i(E-E')/h} = \delta(E - E') \quad (26)$$

In the case where H_0 is defined in ther interval, the boundary condition (21) yeilds the quantization condition for E, namely

$$\begin{aligned}
e^{i\theta}\psi(1) &= \sqrt{N}\psi(N) \\
\Rightarrow e^{i\theta} \frac{C}{1^{\frac{1}{2}-iE\hbar}} &= \sqrt{N} \frac{C}{N^{\frac{1}{2}-iE\hbar}} \\
\Rightarrow e^{i\theta} &= N^{-iE\hbar} \\
\Rightarrow i\theta &= \left(\frac{-iE}{\hbar}\right) \ln N \\
\Rightarrow E &= \frac{\hbar\theta}{\ln N} \\
\Rightarrow E &= \frac{2\pi\hbar}{\ln N} \left(\frac{\theta}{2\pi}\right) \\
\Rightarrow E_n &= \frac{2\pi\hbar}{\ln N} \left(n + \frac{\theta}{2\pi}\right) \quad n \in \mathbb{N}
\end{aligned} \tag{27}$$

Hence the spectrum of H_0 is discrete, with a level spacing decreasing for largerm values of N . The normalization constant of the wave function is now $C = \frac{1}{\sqrt{\ln N}}$ which gives,

$$\langle \psi_{E_n} | \psi_{E_{n'}} \rangle = C^2 \int_1^N \frac{dx}{x} x^{-i(E_n - E_{n'})/\hbar} = \delta_{n,n'} \tag{28}$$

The spectrum (16) agrees with the semiclassical result given in Table 1 for the S-regularization (recall that $l_x = 1, \Lambda = N, \hbar = 1$) For the particular case where $\theta = \pi$, one observes that the energy spectrum is symmetric around zero, i.e, if E_n is an eigenenergy so is $-E_n$. This result is obtained in working[15] with the inverse Hamiltonian $\frac{1}{H_0}$. We are reviewing that construction in next section.

3.2 The inverse Hamiltonain $1/H_0$

First, we take the expression at Eq(16) and take the formal inverse, i.e., $H_0^{-1} = x^{1/2} p^{-1} x^{-1/2}$. The operator p_{-1} is the one-dimensional Green's function with matrix elements (definition of Green's function: Green's function is the kernel of and integral operator that represents the inverse of a differential operator. Let

$$Lu = f \tag{29}$$

Here u and f are vectors and L is a square, invertible matrix. The inverse matrix exists if $\lambda = 0$ is not an eigenvalue of L , or when $\det L \neq 0$. Now

$$u = L^{-1}f \quad (30)$$

where L^{-1} is the inverse operator of L . The inverse operator to be an integral operator of the form.

$$(L^{-1}f)(x) = \int_a^b g(x, \xi) f(\xi) d\xi \quad (31)$$

with kernel G . If L exists, then the kernel function $g(x, \xi)$ is called the Green's function associated with L .

So p^{-1} operator will be

$$\begin{aligned} \langle x | p^{-1} | x' \rangle &= \left\langle x \left| \frac{1}{-i\hbar \frac{d}{dx}} \right| x' \right\rangle \\ &= -i\hbar \left\langle x \left| \frac{1}{\frac{d}{dx}} \right| x' \right\rangle \\ &= \frac{\hbar}{i} G(x, x') \\ &= \frac{\hbar}{2i} \text{sign}(x - x') \end{aligned} \quad (32)$$

Here $\text{sign}(x - x')$ is the sign function.[15]. The operator H_0^{-1} is defined in the interval $(1, N)$ by the continuous matrix,

$$H_0^{-1}(x, x') = \frac{i}{2\hbar} \frac{\text{sign}(x - x')}{\sqrt{xx'}}, \quad 1 \leq x, x' \leq N. \quad (33)$$

It's spectrum is found solving the Schrödinger equation.

$$\begin{aligned} H_0(x, x') \psi(x') &= E \psi(x) \\ \implies H_0^{-1}(x, x') \psi(x') &= E^{-1} \psi(x) \\ \implies \frac{i}{2\hbar} \int_1^N dx' \frac{\text{sign}(x - x')}{\sqrt{xx'}} \psi(x') &= E^{-1} \psi(x) \end{aligned} \quad (34)$$

for the eigenvalue E^{-1} , which must not be singular for H_0^{-1} to be invertible. Define a new wave function

$$\phi(x) = \frac{\psi(x)}{\sqrt{x}} \quad (35)$$

which satisfies

$$\frac{iE}{2\hbar} \int_1^N dx' \text{sign}(x - x') \phi(x') = x\phi(x) \quad (36)$$

Taking derivative with respect to x

$$\begin{aligned} \frac{d}{dx} \left(\frac{iE}{2\hbar} \int_1^N dx' \text{sign}(x - x') \phi(x') \right) &= \frac{d}{dx} (x\phi(x)) \\ \implies \frac{iE}{2\hbar} \int_1^N dx' 2\delta(x - x') \phi(x') &= \phi(x) + x \frac{d}{dx} \phi(x) \\ \implies \frac{iE}{\hbar} \phi(x) &= \phi(x) + x \frac{d}{dx} \phi(x) \\ \implies x \frac{d}{dx} \phi(x) &= \left(1 - \frac{iE}{\hbar} \right) \phi(x) \\ \implies \frac{d\phi(x)}{\phi(x)} &= \left(1 - \frac{iE}{\hbar} \right) \frac{dx}{x} \\ \implies \ln \phi(x) &= \left(1 - \frac{iE}{\hbar} \right) \ln x + \ln C \\ \implies \phi(x) &= \frac{C}{x^{1 - \frac{iE}{\hbar}}} \end{aligned} \quad (37)$$

$$\psi(x) = \frac{C}{x^{1/2 - \frac{iE}{\hbar}}} \quad (38)$$

with $C = \frac{1}{\sqrt{\ln N}}$ as in Eq(30). Eq(40) fixes the functional form of $\psi(x)$. To find the spectrum we impose (38) at one point, say $x = 1$, obtaining,

$$\begin{aligned} \frac{iE}{2\hbar} \int_1^N dx' \text{sign}(1 - x') \phi(x') &= \phi(1) \\ \implies \frac{iE}{2\hbar} \int_1^N dx' \text{sign}(1 - x') \frac{C}{x'^{1 - iE/\hbar}} &= \frac{C}{1^{1 - iE/\hbar}} \\ \implies \frac{iE}{2\hbar} \int_1^N dx' \text{sign}(1 - x') \frac{1}{x'^{1 - iE/\hbar}} &= 1 \end{aligned} \quad (39)$$

we know that sign function

$$\text{sign}(1 - x') = \begin{cases} 1 & 1 - x' > 0 \implies 1 > x' \\ 0 & 1 - x' = 0 \implies 1 = x' \\ -1 & 1 - x' < 0 \implies 1 < x' \end{cases}$$

Therefore

$$\begin{aligned} \frac{-iE}{2\hbar} \int_1^N dx' x'^{-1+iE/\hbar} &= 1 \\ \frac{-iE}{2\hbar} \left[\frac{x'^{iE/\hbar}}{iE/\hbar} \right]_1^N &= 1 \\ \frac{1}{2} [N^{iE/\hbar} - 1^{iE/\hbar}] &= 1 \\ N^{iE/\hbar} - 1 &= -2 \\ N^{iE/\hbar} &= -1 \\ N^{iE/\hbar} &= e^{i\pi} \\ \frac{iE}{\hbar} \ln N &= i\pi \\ E &= \frac{\pi\hbar}{\ln N} \\ E &= \frac{2\pi\hbar}{\ln N} \frac{1}{2} \end{aligned}$$

$$\therefore E_n = \frac{2\pi\hbar}{\ln N} \left[n + \frac{1}{2} \right] \quad (40)$$

This spectrum coincides with (29) for $\theta = \pi$, so that the eigenstates come in pairs $\{E_n, -E_n\}$ as corresponds to an Hermitian antisymmetric operator. Including a BCS coupling in (35), related to θ yields the spectrum (29)[14]

We take this spectrum and wavefunction to calculate OTOC(out of time order correlator)

4 Out-of-Time-Order Correlator of $H = xp$

First we formulate how to calculate the OTOC for generic quantum mechanics. In particular, by the reason described above, we choose $W = x$ and $V = p$ to measure a possible indication of quantum chaos. We consider the out-of-time-order correlator (OTOC) defined by

$$C_T \equiv -\langle [x(t), p(0)]^2 \rangle \quad (41)$$

where $\langle \mathcal{O} \rangle \equiv \frac{\text{tr}[e^{-\beta H} \mathcal{O}]}{\text{tr} e^{-\beta H}}$. Here we define $\frac{1}{\beta}$ with the temperature of the system T . We will omit the argument of Heisenberg operators for $t = 0$; $\mathcal{O} \equiv \mathcal{O}(0)$. Taking energy eigenstates as the basis of the Hilbert space, we can rewrite the OTOC as

$$C_T(t) = \frac{1}{Z} \sum_n e^{-\beta E_n} c_n(t) \quad (42)$$

$$c_n \equiv -\langle n | [x(t), p(0)]^2 | n \rangle \quad (43)$$

where $H|n\rangle = E_n|n\rangle$. We will refer the OTOC for a fixed energy eigenstate, $c_n(t)$, as a microcanonical OTOC. On the other hand, we will refer $C_T(t)$ as a thermal OTOC. Once we compute microcanonical OTOCs, we can obtain the thermal OTOC by taking their thermal average. Let us rewrite the microcanonical OTOC using matrix elements of x and p for numerical calculations. Using the completeness relation $1 = \sum_m |m\rangle\langle m|$, we rewrite the microcanonical OTOC as

$$c_n(t) = \sum_m b_{nm}(t) b_{nm}^*(t) \quad (44)$$

$$b_{nm}(t) \equiv -i \langle n | [x(t), p(0)] | m \rangle \quad (45)$$

Note that $b_{nm}(t)$ is Hermitian: $b_{nm}(t) = b_{nm}^*(t)$. Substituting $x(t) = e^{iHt} x e^{-iHt}$ and inserting the completeness relation again, we obtain

$$\begin{aligned}
b_{nm}(t) &= -i \langle n | [e^{iHt} x e^{-iHt}, p] | m \rangle \\
&= -i \langle n | [e^{iHt} x e^{-iHt} p - p e^{iHt} x e^{-iHt}] | m \rangle \\
&= -i \left[\sum_k \langle n | e^{iHt} x e^{-iHt} | k \rangle \langle k | p | m \rangle - \sum_k \langle n | p | k \rangle \langle k | e^{iHt} x e^{-iHt} | m \rangle \right] \\
&= -i \left[\sum_k \langle n | e^{iE_n t} x e^{-iE_k t} | k \rangle \langle k | p | m \rangle - \sum_k \langle n | p | k \rangle \langle k | e^{iE_k t} x e^{-iE_m t} | m \rangle \right] \\
&= -i \sum_k [e^{iE_{nk} t} x_{nk} p_{km} - e^{iE_{km} t} p_{nk} x_{km}]
\end{aligned}$$

$$\therefore b_{nm}(t) = -i \sum_k [e^{iE_{nk} t} x_{nk} p_{km} - e^{iE_{km} t} p_{nk} x_{km}] \quad (46)$$

where $E = E_n - E_m$, $x_{nm} \equiv \langle n | x | m \rangle$, $p_{nm} \equiv \langle n | p | m \rangle$. In this expression, there are matrix components of p . They are not desirable since numerical derivatives of wave functions lose the numerical accuracy. For a natural Hamiltonian with the form,

$$H = \sum_{i=1}^N p_i^2 + U(x_1, \dots, x_N) \quad (47)$$

we can express p_{nm} using x_{nm} . We know from canonical commutation relation

$$[x, p] = i \quad (\hbar = 1)$$

Therefore

$$\begin{aligned}
[H, x] &= [p^2 + U(x), x] \\
&= [p^2, x] + [U(x), x] \\
&= p[p, x] + [p, x]p \\
&= -ip - ip
\end{aligned}$$

$$\therefore [H, x] = -2ip \quad (48)$$

Applying $\langle m | \cdots | n \rangle$ to the both sides of the equation, we obtain

$$\begin{aligned} \langle m | [H, x] | n \rangle &= \langle m | -2ip | n \rangle \\ \langle m | (Hx - xH) | n \rangle &= -2i \langle m | p | n \rangle \\ \langle m | (E_m x - x E_n) | n \rangle &= -2ip_{mn} \\ E_{mn} x_{mn} &= -2ip_{mn} \\ p_{mn} &= \frac{i}{2} E_{mn} x_{mn} \end{aligned}$$

Substituting this expression into (49) we have

$$\begin{aligned} b_{nm}(t) &= \frac{-i\hbar}{2} \sum_k (e^{iE_{nk}t} x_{nk} E_{km} x_{km} - e^{iE_{km}t} E_{nk} x_{nk} x_{km}) \\ b_{nm}(t) &= \frac{1}{2} \sum_k x_{nk} x_{km} (E_{km} e^{iE_{nk}t} - E_{nk} e^{iE_{km}t}) \end{aligned} \quad (49)$$

Now we can take this equation to calculate OTOC of Berry-Keating Hamiltonian from (40) and (42). The spectrum is

$$E_n = \frac{2\pi\hbar}{\ln N} \left[n + \frac{1}{2} \right] \quad (50)$$

and the wavefunction

$$\psi(x) = \frac{C}{x^{1/2 - \frac{iE}{\hbar}}} \quad (51)$$

from here we can evaluate $x_{nm} = \langle n | x | m \rangle$ value

$$\begin{aligned}
\langle \psi_E | x | \psi_{E'} \rangle &= C^2 \int_1^N \frac{dx}{x} x^{-i(E-E')/\hbar} x \\
&= C^2 \int_1^N dx x^{-i(E-E')/\hbar} \\
&= C^2 \left[\frac{x^{1-i(E-E')/\hbar}}{1-i(E-E')/\hbar} \right]_1^N \\
&= C^2 \frac{1}{1-i(E-E')/\hbar} \left[N^{1-i(E-E')/\hbar} - 1 \right] \\
&= C^2 \frac{1}{1-i(E-E')/\hbar} \left[N^{1-i(E-E')/\hbar} - 1 \right]
\end{aligned}$$

Therefore, $\langle n | x | m \rangle = x_{nm}$ is

$$\langle n | x | m \rangle = \frac{1}{\ln N} \frac{1}{1-i\frac{2\pi\hbar}{\ln N}(n+\frac{1}{2}-m-\frac{1}{2})/\hbar} \left[N^{1-i\frac{2\pi\hbar}{\ln N}(n+\frac{1}{2}-m-\frac{1}{2})/\hbar} - 1 \right] \quad (52)$$

$$x_{nm} = \frac{1}{\ln N} \frac{1}{1-i\frac{2\pi}{\ln N}(n+\frac{1}{2}-m-\frac{1}{2})} \left[N^{1-i\frac{2\pi}{\ln N}(n+\frac{1}{2}-m-\frac{1}{2})} - 1 \right] \quad (53)$$

putting these values on Eq(52)

$$b_{nm}(t) = \frac{1}{2} \sum_k x_{nk} x_{km} (E_{km} e^{iE_{nk}t} - E_{nk} e^{iE_{km}t})$$

and calculate OTOC analytically. cause carry out the summation of x_{nm} and energy eigenstates makes things little complicated. The graph and codes are given below.

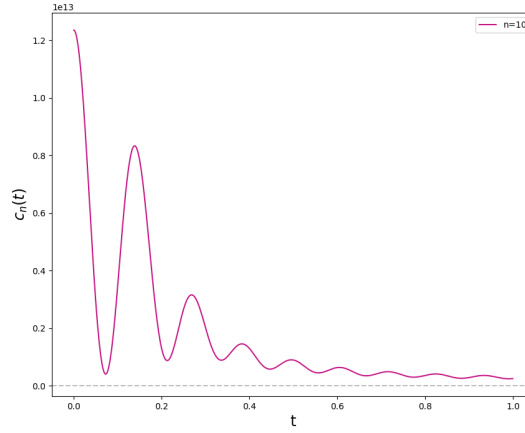


Figure 3

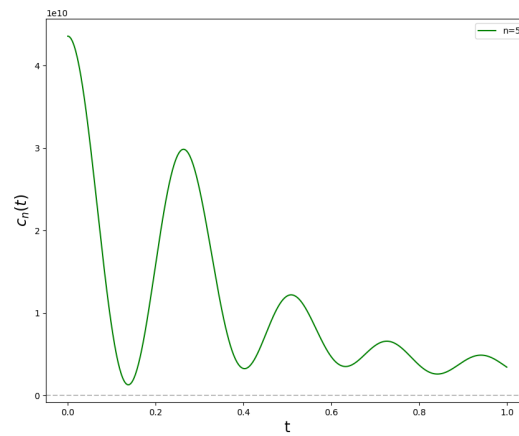


Figure 4

Here microcanonical OTOC and thermal OTOC are same with different temperatures.

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3 import math
4 import cmath
5
6 a = [] #initialize some list to plot data
7 b = []
8 r=6    # matrix dimension number

```



```

9 N=10      #Highest interval in the spectrum
10 T=40      #temperature
11
12 #function for spetrum E_{nm}
13 def E(n, m):
14     return (2*math.pi*(n-m))/(math.log(N))
15
16 #function for position x_{nm}
17 def x(n, m):
18     a = (1/1-(2j*math.pi*(n-m)/math.log(N)))/(math.sqrt(math.
19     log(N)))
20     b = (N**(1-(2j*math.pi*(n-m)/math.log(N)))-1)
21     return a*b
22
23 #calculating Thermal OTOC
24 for t in np.arange(0,1,0.001):
25     Z=0
26     C=0
27     s3=0
28     for n in range(0,r):
29         s2 = 0
30         for m in range(0,r):
31             s1=0
32             for k in range(0,r):
33                 s1 += (1/2.0)*x(n,k)*x(k,m)*(E(k,m)*cmath.exp
34                 (1j*E(n,k)*t) - E(n,k)*cmath.exp(1j*E(k,m)*t))
35                 s2 += s1*np.conjugate(s1)
36             Z += cmath.exp(-E(n,0)/T) #partition function
37             C += (cmath.exp(-E(n,0)/T)*s2) #e^{-\beta E_n} 0
38             s3=C/Z #expectation value or Thermal OTOC
39             a.append(t)
40             b.append(s3)
41
42 #for plot
43 f = plt.figure()
44 f.set_figwidth(10)
45 f.set_figheight(8)
46 plt.plot(a, b, label='n=10', color='mediumvioletred')
47 plt.ylabel('$c_{n}(t)$', fontsize=30)
48 plt.xlabel('$t$', fontsize=30)
49 plt.legend()
50 plt.show()
51 f = plt.figure()

```

```

52 f.set_figwidth(10)
53 f.set_figheight(8)
54 plt.plot(a, b, label='n=5', color='green')
55 plt.ylabel('$c_{\{n\}}(t)$', fontsize=30)
56 plt.xlabel('$t$', fontsize=40)
57 plt.axhline(0, ls='--', alpha=0.5, c='grey')
58 plt.legend()
59 plt.savefig('test.png')

```

Listing 1: Python example

5 Conclusions

In this paper, we discussed what is Riemann hypothesis is, and why it is important to the field to Quantum mechanics. The imaginary part of Hamiltonian proposed by Berry-keating Hamiltonian $H_0 = \frac{1}{2}(xp + px)$ shows the self-adjoint and quantum chaos. We look at Sierra regularization of Berry-Keating Hamiltonian, and their intervals $l_x = 1, \Lambda = 1$ and evaluated the spectrum $E_n = \frac{2\pi\hbar}{\ln N}(n + \frac{\theta}{2\pi})$ of $\psi_E(x) = \frac{C}{x^{\frac{1}{2}-iE\hbar}}$ eigenfunction. Then we look at the quantization of inverse Hamiltonian $H_0^{-1} = x^{-1/2}p^{-1}x^{1/2}$ and obtained the spectrum $E_n = \frac{2\pi\hbar}{\ln N}(n + \frac{1}{2})$ which is reckon to be $E_n = \frac{2\pi\hbar}{\ln N}(n + \frac{\theta}{2\pi})$ with $\theta = \pi$

We discussed OTOC in quantum mechanics and its thermal and microcanonical relation between them. We input the spectrum and position operator of inverse Hamiltonian and calculate its quantum chaos analytically. We see that microcanonical OTOC settles at value greater than zero while the thermal OTOC displays the same behaviour of microcanonical OTOC.

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