Road to Curvature

Recall the covariant derivative

$$\begin{aligned}
\nabla_\mu \phi &\equiv \partial_\mu \phi \\
&\downarrow \\
\nabla_\mu V^\nu &\equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \\
&\downarrow \\
\nabla_\mu W_\nu &\equiv \partial_\mu W_\nu - \Gamma^\lambda_{\mu\nu} W_\lambda
\end{aligned} \tag{1}$$

∇_μ is made unique by demanding
(a) Torsion free
(b) Metric compatible

Point a: Torsion Free)

$$[\nabla_\mu, \nabla_\nu]\,\phi = 0 \qquad \text{As } [\partial_\mu, \partial_\nu]\,\phi = 0$$

L.H.S.=

$$\begin{aligned}
&[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu]\,\phi \\
&= \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi \\
&= \nabla_\mu(\partial_\nu \phi) - \nabla_\nu(\partial_\mu \phi) \\
&= \partial_\mu(\partial_\nu \phi) - \Gamma^\lambda_{\mu\nu}\phi - \nabla_\nu(\partial_\mu \phi) + \Gamma^\lambda_{\nu\mu}\phi \\
&= \partial_\mu(\partial_\nu \phi) - \Gamma^\lambda_{\mu\nu}\partial_\lambda\phi - \nabla_\nu(\partial_\mu \phi) + \Gamma^\lambda_{\nu\mu}\partial_\lambda\phi \\
&= \left(\Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu}\right)\partial_\lambda \phi \\
&= T^\lambda_{\mu\nu}\partial_\lambda \phi
\end{aligned} \tag{2}$$

where

$$T^\lambda_{\mu\nu} \equiv \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) \tag{3}$$

is the Torsion tensor.

For Einstein theory, this torsion is required to be zero.
$\therefore$ In GR, Thus $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ is symmetric.

Point b: Metricity)

$$\nabla_\mu g_{\nu\lambda} = 0 \qquad \text{(Why?)}$$

Let

$$g_{\mu\nu} U^\mu V^\nu = X \tag{4}$$

$$\begin{aligned} \nabla_\lambda X &= \nabla_\lambda (g_{\mu\nu} U^\mu V^\nu) \\ &= (\nabla_\lambda g_{\mu\nu}) U^\mu V^\nu + g_{\mu\nu} \nabla_\lambda [U^\mu V^\nu] \end{aligned} \tag{5}$$

To have a uniform rule for $g_{\mu\nu}$, it is consistant to have

$$\nabla_\lambda g_{\mu\nu} = 0 \tag{6}$$

Consequence of a) +b) is that
Γ is completely determined by $g_{\mu\nu}$ (rather it's derivative)

Comment:

Unlike GR there is no analog of metricity condition in Yang-Mills's gauge theories.
Now, metriciry implies

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} - \Gamma^k_{\mu\nu} g_{k\lambda} - \Gamma^k_{\mu\lambda} g_{\nu k} = 0 \tag{7}$$

$$\nabla_\nu g_{\lambda\mu} = \partial_\nu g_{\lambda\mu} - \Gamma^k_{\nu\lambda} g_{k\mu} - \Gamma^k_{\nu\mu} g_{\lambda k} = 0 \tag{8}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^k_{\lambda\mu} g_{k\nu} - \Gamma^k_{\lambda\nu} g_{\mu k} = 0 \tag{9}$$

(7)+(8)-(9)

$$\begin{aligned} 0 = (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) - (\Gamma^k_{\mu\nu} + \Gamma^k_{\nu\mu}) g_{\lambda k} \\ g_{k\nu}(\Gamma^k_{\mu\lambda} - \Gamma^k_{\lambda\mu}) + g_{\mu k}(\Gamma^k_{\lambda\nu} - \Gamma^k_{\nu\lambda}) \end{aligned} \tag{10}$$

If the torsion free condition is imposed third and fourth term will be zero.$\left(i.e.\ \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \Leftrightarrow T^\lambda_{\mu\nu} = 0\right)$

Then this equation simplifies to :

$$
\begin{aligned}
0 &= (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) - 2\Gamma^k_{\mu\nu} g_{\lambda k} \\
\Gamma^k_{\mu\nu} g_{\lambda k} &= \frac{1}{2}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\
g^{\lambda\rho} g_{\lambda k} \Gamma^k_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\
\delta^\rho_k \Gamma^k_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)
\end{aligned} \tag{11}
$$

Covariant Divergence

$$\nabla_\mu V^\mu \equiv \partial_\mu V^\mu + \Gamma^\mu_{\mu\rho} V^\rho \tag{12}$$

Now

$$
\begin{aligned}
\Gamma^\mu_{\mu\rho} &= \frac{1}{2} g^{\mu\lambda}(\partial_\mu g_{\lambda\rho} + \partial_\rho g_{\lambda\mu} - \partial_\lambda g_{\mu\rho}) \\
&= \frac{1}{2} g^{\mu\lambda} \partial_\rho g_{\lambda\rho} \\
&= \frac{1}{2} Tr(g^{-1}\partial_\rho g)
\end{aligned} \tag{13}
$$

Recall for any non-singular finite matrix

$$
\begin{aligned}
ln(det\ A) &= Tr(ln\ A) \\
if \quad A &= S\Lambda S^{-1} \\
f(A) &= S f(\Lambda) S^{-1} \\
ln f(A) &= S f(\Lambda) S^{-1} \\
ln(det\ g) &= Tr(ln\ g) \\
\partial_\rho ln(det\ g) &= Tr(\partial_\rho ln\ g) \\
\partial_\rho ln(det\ g) &= Tr(g^{-1} ln\ g)
\end{aligned} \tag{14}
$$

$$\begin{aligned}
\Gamma^{\mu}_{\mu\rho} &= \frac{1}{2}Tr(g^{-1}\partial_{\rho}g) \\
&= \frac{1}{2}\partial_{\rho}ln(det\ g) \\
&= \partial_{\rho}ln\sqrt{(det\ g)} \\
&= \partial_{\rho}ln\sqrt{g} \\
&= \frac{1}{\sqrt{g}}\partial_{\rho}\sqrt{g}
\end{aligned} \tag{15}$$

$$\begin{gathered}
\therefore \nabla_{\mu}V^{\mu} = \partial_{\mu}V^{\mu} + \frac{1}{\sqrt{g}}\partial_{\rho}\sqrt{g} \\
\boxed{\nabla_{\mu}V^{\mu} = \frac{1}{\sqrt{g}}\partial_{\rho}\left(\sqrt{g}V^{\mu}\right)}
\end{gathered} \tag{16}$$

Recall :

$$\begin{aligned}
g_{\mu\nu} \longrightarrow g^{'}_{\mu\nu} &= \frac{\partial x^{a}}{\partial x'^{\mu}}\frac{\partial x^{b}}{\partial x'^{\nu}}g_{ab} \\
&= \frac{\partial x^{a}}{\partial x'^{\mu}}g_{ab}\frac{\partial x^{b}}{\partial x'^{\nu}} \\
&= MgM^{T} \\
\Longrightarrow\ det\ g^{'} &= (det\ M)^{2}\, det\ g \\
Under\ \ X \longrightarrow X^{'} & \\
g \ \Longrightarrow\ g^{'} &= (det\ M)^{2}\, g \\
\sqrt{g^{'}} &= (detM)\,\sqrt{(g)}
\end{aligned} \tag{17}$$

Jacobi teaches us : $x \rightarrow x^{'}$

$$d^{D}x \rightarrow d^{D}x^{'} = \left\| \frac{\partial x^{'}}{\partial x} \right\| d^{D}x = (det\ M)^{-1}d^{D}x \tag{18}$$

Hence, $\sqrt{g}\ d^{D}x$ is invariant.
We are thus immidiately led to the covarinat form of the Gauss's theorem.

$$\int d^{D}x\sqrt{g}\frac{1}{\sqrt{g}}\partial_{\mu}\left(\sqrt{g}\ V^{\mu}\right) = \int d\Sigma_{\mu}V^{\mu}\sqrt{g} \tag{19}$$