

1 Out of Time Order Correlator of $H = xp$ model

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to $1/2$. The classical zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (1)$$

for $\text{Re } s > 1$. By the fundamental theorem of arithmetic, which is also equivalent to the Euler product over primes

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \quad (2)$$

where p are all the prime numbers.

Zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negative integers (for $s = -2, -4, -6, \dots$). For complex $s (= \sigma + it)$ (with real part between zero and one), zeta function becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - it$ zeta function becomes zero $\zeta(\frac{1}{2} - it) = 0$. Hilbert-Pólya conjecture suggests that the imaginary parts of the nontrivial zeros are the eigenvalues of a self-adjoint hamiltonian operator \hat{H} . It is also one of the approach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-Pólya conjecture should take the form[1]

$$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \quad (3)$$

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[?] Berry proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obtained by quantization of a classical chaotic hamiltonian, whose periodic orbits are labeled by the prime numbers. Connes took the adelic approach to introduce $H = xp$ [2]. He showed that using different semiclassical regularization, Riemann zeros appear as missing spectral lines in a continuum.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is [1, 2]

$$H_0^{cl} = xp \quad (4)$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \quad p(t) = p_0 e^{-t} \quad (5)$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of regularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number $\mathcal{N}(E)$ with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \quad (6)$$

Where A is the area of the allowed phase space region below the curve $E = xp$. So the number of semiclassical states will be for Berry-Keating regularization

$$\mathcal{N}_{BK}(E) = \int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + \quad (7)$$

References

- [1] Michael V Berry and Jonathan P Keating. $H = xp$ and the riemann zeros. *Supersymmetry and trace formulae: chaos and disorder*, pages 355–367, 1999.
- [2] Alain Connes. Trace formula in noncommutative geometry and the zeros of the riemann zeta function. *Selecta Mathematica*, 5(1):29, 1999.