A Course on Quantum Field Theory

Noor E Mustafa Ferdous

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Time Ordered Product

Purpose of a picture is initial condition of a system From scalar quantization

$$[\ \phi_i,\ \pi_j\ ] = i\hbar\delta_{ij} \tag{1}$$

$\phi(x)$ in 3D

$$\implies [\ \phi(x),\ \pi(x')\ ] = i\delta^3(x - x') \tag{2}$$

which is more fundamental apporach for quantization. But in 1958, R. Haggs gave us a theorem that "Interaction picture does not exists in QFT".

In Dyson picture, the wave function will be

$$|\psi(t)\rangle_I = U(t, t_0)\, |\psi(t_0)\rangle \tag{3}$$

The expression of U is

$$U(t, t_0) = \begin{cases} e^{iH(t-t_0)}, & \text{if H is time independent} \\ Te^{-i\int_{t_0}^{t} H(t')\, dt'}, & \text{if H is time dependent} \end{cases}$$

if H is time dependent, We need time ordered products

$$\begin{aligned} T[A(t)B(t')] &= \begin{cases} A(t)B(t'), & \text{if } t > t' \\ B(t')A(t), & \text{if } t' > t \end{cases} \\ \implies T[A(t)B(t)'] &= \theta(t - t')A(t)B(t') + \theta B(t')A(t) \end{aligned} \tag{4}$$

For a real scalar field:

$$\pi(t) = \dot{\phi(t)}$$

$$\begin{aligned}
\frac{d^2}{dt^2} T[\phi(t)\phi(t')] &= \frac{d^2}{dt^2}[\theta(t-t')\phi(t)\phi(t') + \theta(t'-t)\phi(t')\phi(t)] \\
&= \frac{d}{dt}[\frac{d\theta}{dt}\phi(t)\phi(t') + \theta(t-t')\frac{d\phi(t)}{dt}\phi(t') \\
&+ \frac{d\theta(t'-t)}{dt}\phi(t)\phi(t') + \theta(t'-t)\frac{d\phi(t)}{dt}\phi(t')] \\
&= \frac{d}{dt}[\delta(t-t')\phi(t)\phi(t') + \theta(t-t')\dot{\phi}(t)\phi(t') \\
&- \delta(t'-t)\phi(t)\phi(t') + \theta(t'-t)\dot{\phi}(t)\phi(t')] \\
&= [\delta'(t-t')\phi(t)\phi(t') + \delta(t-t')\dot{\phi}(t)\phi(t') \\
&+ \delta(t'-t)\dot{\phi}(t)\phi(t') + \theta(t'-t)\ddot{\phi}(t)\phi(t')] \\
&+ \delta'(t'-t)\phi(t)\phi(t') - \delta(t'-t)\dot{\phi}(t)\phi(t')] \\
&- \delta(t'-t)\dot{\phi}(t)\phi(t') - \theta(t'-t)\ddot{\phi}(t)\phi(t')] \\
&= \delta(t-t')[\dot{\phi}(t), \phi(t)] \\
&= i\delta^4(t-t')
\end{aligned} \tag{5}$$

Consider a real scalar field whose Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial\phi\partial\phi - m^2\phi^2) \tag{6}$$

Equation of motion will be

$$`(\Box + m^2) = 0 \tag{7}$$

The time ordered product

$$\begin{aligned}
\langle 0| T[\phi_1, \phi_2] |0\rangle = \theta(t_1 - t_2) \langle\phi| (x_1, t_1)\phi(x_2, t_2) |0\rangle \\
+ \theta(t_2 - t_1) \langle\phi| (x_2, t_2)\phi(x_1, t_1) |0\rangle
\end{aligned} \tag{8}$$

Assigment 1: Show that

$$(\Box + m^2) \langle 0| T(\phi_1, \phi_2) |0\rangle = -i\delta^4(x_1 - x_2) \tag{9}$$

defination of Green's function from a linear operator $\hat{L}$

$$\begin{aligned} \hat{L}\psi &= 0 \\ \implies \hat{L}G_{ij} &= \delta_{ij} \\ \implies \hat{L}G(x-y) &= iG(x-y) \\ \therefore \langle 0| T(\phi_1, \phi_2) |0\rangle &= iG(x-y) \end{aligned} \tag{10}$$

Assignment 2:

$$(\Box + m^2) \langle 0| T[\phi_1, \phi_2, \phi_3] |0\rangle = \tag{11}$$

Find the RHS

$$\begin{aligned} T[ABC] = {} & \theta(t_2 - t_3)\theta(t_1 - t_2)A(t_1)B(t_2)C(t_3) + \theta(t_3 - t_1)\theta(t_2 - t_3)B(t_2)C(t_3)A(t_1) \\ & + \theta(t_1 - t_3)\theta(t_1 - t_2)C(t_3)A(t_1)B(t_2) - \theta(t_3 - t_2)\theta(t_1 - t_3)A(t_1)C(t_3)B(t_2) \\ & - \theta(t_2 - t_1)\theta(t_3 - t_2)C(t_3)B(t_2)A(t_1) - \theta(t_1 - t_3)\theta(t_2 - t_1)B(t_2)A(t_1)C(t_3) \end{aligned} \tag{12}$$

From Electrodynamics, there are two types of Green's funtion,

$$\begin{cases} G_{ret} = 0 & \text{if} x_0 < y_0 \\ G_{ret} = 0 & \text{if} x_0 > y_0 \end{cases} \tag{13}$$

y_0 is the location of cause, and x_0 is the obervers' position
Lightcone picture:
For free system

$$(\Box + m^2)\phi = 0 \tag{14}$$

For non free system

$$(\Box + m^2)\phi = J \tag{15}$$

Where the Lagrangian will be

$$\mathcal{L} = \mathcal{L}_0 + J\phi \tag{16}$$

Where

$$\mathcal{L}_{int} = -\mathcal{H}_{int} \tag{17}$$

$$|out\rangle = S\,|in\rangle \tag{18}$$

$$S = \lim_{t\to\infty} \lim_{t_0\to\infty} U(t, t_0) \tag{19}$$