Assignment

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Ans for (a)

The Partiton fucntion is

$$Z[J] = \mathcal{N}_0 e^{V[\frac{\partial}{\partial J_i}]} e^{\frac{1}{2}J_m \Delta_{mn} J_n} \tag{1}$$

Taylor expanding $e^{V[\frac{\partial}{\partial J_i}]}$ to 2nd order of λ

$$Z[J] = [1 + \frac{\lambda}{4!}(\frac{\partial}{\partial J_i})^4 + \frac{\lambda}{4!}^2(\frac{\partial}{\partial J_i})^4(\frac{\partial}{\partial J_i})^4]\mathcal{N}_0 e^{\frac{1}{2}J_m \Delta_{mn} J_n} \tag{2}$$

Ans for (b)

Partition functionn for two points is

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z[J]} \frac{\partial^2 Z[J]}{\partial J_i \partial J_j} \tag{3}$$

expanding to first order of lambda

$$Z[J] = [1 + \frac{\lambda}{4!}(\frac{\partial}{\partial J_i})^4]\mathcal{N}_0 e^{\frac{1}{2}J_m \Delta_{mn} J_n} \tag{4}$$

Then

$$\frac{\partial^2 Z[J]}{\partial J_i \partial J_j} = \frac{\partial^2}{\partial J_i \partial J_j}[e^{\frac{1}{2}J_m \Delta_{mn} J_n} + \frac{\lambda}{4!}(\frac{\partial}{\partial J_i})^4 e^{\frac{1}{2}J_m \Delta_{mn} J_n}]\mathcal{N}_0 \Bigg|_{J=0} \tag{5}$$

First term of RHS of eqn(5)

$$\begin{aligned} \frac{\partial^2}{\partial J_i \partial J_j}[e^{\frac{1}{2}J_m \Delta_{mn} J_n}] &= \frac{\partial}{\partial J_j} \Delta_{im} J_m e^{\frac{1}{2}J_m \Delta_{mn} J_n} && (6) \\ &= \Delta_{ij} e^{\frac{1}{2}J_m \Delta_{mn} J_n} + \Delta_{im}\delta_{jm}\Delta_{ik}J_{ik}e^{\frac{1}{2}J_m \Delta_{mn} J_n} && (7) \\ &\quad + \Delta_{im}J_m \Delta_{ik}\delta_{jk}e^{\frac{1}{2}J_m \Delta_{mn} J_n} && (8) \\ & && (9) \end{aligned}$$

Therefore

$$\frac{\partial^2}{\partial J_i \partial J_j}[e^{\frac{1}{2}J_m \Delta_{mn} J_n}] \Bigg|_{J=0} = \Delta_{ij} \tag{10}$$

Second term of RHS of eqn(5) [taking $e^{\frac{1}{2}J_m \Delta_{mn} J_n} = U$]

$$\begin{aligned} \frac{\partial^2}{\partial J_i \partial J_j}(\frac{\partial}{\partial J_i})^4[e^{\frac{1}{2}J_m \Delta_{mn} J_n}] = \frac{\partial^2}{\partial J_i \partial J_j}\Delta_{ii}[\frac{\partial^2 U}{\partial J_i^2}] + 2\Delta_{im}\Delta ii \frac{\partial}{\partial J_i}(J_m U) \\ + \Delta_{im}\Delta_{ik}\frac{\partial}{\partial J_i}(J_m J_k \frac{\partial U}{\partial J_i}) \end{aligned} \tag{11}$$

First term of eqn(8)

$$\begin{aligned}
\frac{\partial^2}{\partial J_i \partial J_j}\Delta_{ii}[\frac{\partial^2 U}{\partial {J_i}^2}] &= \frac{\partial}{\partial J_j}\Delta_{ii}[\frac{\partial^3 U}{\partial {J_i}^3}] \\
&= \Delta_{ii}(\Delta_{ii}\frac{\partial U}{\partial J_i} + 2\Delta_{im}J_m\Delta_{ii}U + \Delta_{im}\Delta_{ik}J_kJ_m\frac{\partial U}{\partial J_i}) \\
&= \frac{\partial}{\partial J_j}\Delta_{ii}(\Delta_{ii}\Delta_{im}J_mU + 2\Delta_{im}J_m\Delta_{ii}U) \\
&\quad + \Delta_{im}\Delta_{ik}J_kJ_m\Delta_{im}J_mU \\
&= \Delta_{ii}(\Delta_{im}\delta_{mj}\Delta_{ii} + 2\Delta_{im}\Delta_{ii}\delta_{mj}) \qquad \text{[putting J=0]} \\
&= 3\Delta_{ii}\Delta_{im}\delta_{mj\Delta ii}
\end{aligned}$$

$$\therefore \frac{\partial^2}{\partial J_i \partial J_j}\Delta_{ii}[\frac{\partial^2 U}{\partial {J_i}^2}] = 3\Delta_{ii}\Delta_{im}\delta_{mj}\Delta ii$$

Second term of eqn(8)

$$\begin{aligned}
\frac{\partial}{\partial J_j}2\Delta_{im}\Delta_{ii}\frac{\partial^2}{\partial {J_i}^2}(J_mU) &= \frac{\partial}{\partial J_j}2\Delta_{im}\Delta_{ii}\frac{\partial}{\partial J_i}(\delta_{mi}U + J_m\frac{\partial U}{\partial J_i}) \\
&= \frac{\partial}{\partial J_j}2\Delta_{im}\Delta_{ii}(2\delta_{im}\frac{\partial U}{\partial J_i} + J_m\frac{\partial^2 U}{\partial {J_i}^2}) \\
&= \frac{\partial}{\partial J_j}2\Delta_{im}\Delta_{ii}(2\delta_{im}\Delta_{im}J_mU + J_m(\Delta_{ii}U + \Delta_{im}J_m\Delta_{ik}J_kU)) \\
&= 2\Delta_{im}\Delta_{ii}(2\delta_{im}\Delta im\delta_{mj} + \delta_{mj}\Delta_{ii}) \\
&\qquad \text{[putting J=0]} \\
&= 4\delta_{im}\delta_{mj}\Delta_{im}\Delta_{ii}\Delta_{im} + 2\Delta_{im}\Delta_{ii}\Delta_{ii}\delta_{mj} \\
&= 4\Delta_{ii}\Delta_{ii}\Delta_{ij} + 2\Delta_{ii}\Delta_{ii}\Delta_{ij}
\end{aligned}$$

$$\therefore \frac{\partial}{\partial J_j}2\Delta_{im}\Delta_{ii}\frac{\partial^2}{\partial {J_i}^2}(J_mU) = 6\Delta_{ii}\Delta_{ii}\Delta_{ij}$$

Third term of eqn(8)

$$
\begin{aligned}
\frac{\partial}{\partial J_j}\Delta_{im}\Delta_{ik}\frac{\partial^2}{\partial J_i^2}(J_m J_k\frac{\partial}{\partial J_i}) &= \frac{\partial}{\partial J_j}\Delta_{im}\Delta_{ik}\frac{\partial}{\partial J_i}(\delta_{im}J_k\frac{\partial U}{\partial J_i} + J_m\delta_{ik}\frac{\partial U}{\partial J_i} + J_m J_k\frac{\partial^2 U}{\partial {J_i}^2}) \\
&= \frac{\partial}{\partial J_j}\Delta_{im}\Delta_{ik}(\delta_{im}\delta_{ik}\frac{\partial U}{\partial J_i} + \delta_{im}J_k\frac{\partial^2 U}{\partial {J_i}^2} + \delta_{im}\delta_{ik}\frac{\partial U}{\partial J_i} \\
&\quad + J_m\delta_{ik}\frac{\partial^2 U}{\partial {J_i}^2} + \delta_{im}J_k\frac{\partial^2 U}{\partial {J_i}^2} + J_m\delta_{ik}\frac{\partial^2 U}{\partial {J_i}^2} + J_m J_k\frac{\partial^3 U}{\partial {J_i}^3}) \\
&= \frac{\partial}{\partial J_j}\Delta_{im}\Delta_{ik}(\delta_{im}\delta_{ik}J_m\Delta_{im}U + 2\delta_{im}J_k(\Delta_{ii}U + \Delta_{im}J_m\Delta_{ik}J_kU) \\
&\quad + \delta_{im}\delta_{ik}\Delta_{im}J_mU + 2J_m\delta_{ik}(\Delta_{ii}U + \Delta imJ_m\Delta ikJ_kU) \\
&\quad + J_mJ_k(\Delta_{ii}J_m\Delta_{im}U + 2\Delta_{im}\Delta_{ii}J_mU + \Delta_{im}\Delta_{ik}J_kJ_mJ_m\Delta_{im}U)) \\
&= \Delta_{im}\Delta_{ik}\delta_{im}\delta_{ik}\delta_{mj}\Delta_{im} + 2\Delta_{im}\Delta_{ik}\delta_{im}\delta_{kj}\Delta_{ii} \\
&\quad + \Delta_{im}\Delta_{ik}\Delta_{im}\delta_{im}\delta_{ik}\delta_{mj} + 2\Delta_{im}\Delta_{ik}\delta_{mj}\delta_{ik}\Delta_{ii} \\
&= \Delta_{ij}\Delta_{ii}\Delta_{ii} + 2\Delta_{ii}\Delta_{ii}\Delta_{ij} + \Delta_{ii}\Delta_{ii}\Delta_{ij} + 2\Delta_{ii}\Delta_{ij}\Delta_{ii} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[putting J=0]} \\
&= 6\Delta_{ii}\Delta_{ii}\Delta_{ij}
\end{aligned}
$$

Eqn(8) becomes

$$
\frac{\partial^2}{\partial J_i\partial J_j}(\frac{\partial}{\partial J_i})^4[e^{\frac{1}{2}J_m\Delta_{mn}J_n}] = 15\Delta_{ii}\Delta_{ii}\Delta_{ij} \tag{12}
$$

Eqn(5) will be

$$
\begin{aligned}
\frac{\partial^2 Z[J]}{\partial J_i\partial J_j} &= \frac{\partial^2}{\partial J_i\partial J_j}[e^{\frac{1}{2}J_m\Delta_{mn}J_n} + \frac{\lambda}{4!}(\frac{\partial}{\partial J_i})^4 e^{\frac{1}{2}J_m\Delta_{mn}J_n}]\mathcal{N}_0\Bigg|_{J=0} && (13) \\
&= \Delta_{ij} + 15\Delta_{ii}\Delta_{ii}\Delta_{ij}\mathcal{N}\prime && (14) \\
& && (15)
\end{aligned}
$$

$$
\langle\phi_i\phi_j\rangle = \frac{(\Delta_{ij} + 15\Delta_{ii}\Delta_{ii}\Delta_{ij}\frac{\lambda^4}{4!})\mathcal{N}\prime}{(1 + \frac{\lambda^4}{4!}3\Delta_{ii}\Delta_{ii})\mathcal{N}\prime} \tag{16}
$$