1 Out of Time Order Correlator of $H = xp$ model

The Riemann hypothesis states that non-trivial zeros of the classical zeta function have real part equal to 1/2. The classical zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^s \tag{1}$$

for Re $s > 1$. By the fundamental theorem of arithmatic, which is also equivalent to the Euler product over primes

$$\zeta(s) = \prod_{p} (1 - p)^{-1} \tag{2}$$

where p are all the prime numbers.

Zeros of Riemann zeta function are two different types. Trivial zeros of zeta / Riemann zeta function occurs at all negetive integers (for $s = -2, -4, -6, ...$). For complex s ($=\sigma + it$) (with real part between zero and one) , zeta fucntion becomes nontrivial ones. And the Riemann hypothesis is for $s = \frac{1}{2} - iE$ zeta funtion becomes zero $\zeta(\frac{1}{2} - it) = 0$. Hilbert-P$\acute{o}$lya conjecture suggests that the imaginary parts of the nontrivial zeros are the eiogenvalues of a self-adjoint hamiltonian operator $\hat{H}$. It is also one of the aprroach to proving the Riemann hypothesis. Berry-Keating conjectured that the hamiltonian operator of the Hilbert-P$\acute{o}$lya cinjecture should take the form[1]

$$\hat{H}_{BK} = \frac{1}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \tag{3}$$

Here x and p are position and momentum operators. This 1d classical Hamiltonian ($H = xp$) related to the Riemann zeros.[**?**] Berry proposed the Quantum Chaos conjecture, according to which the Riemann zeros are the spectrum of a Hamiltonian obstained by quantization of a classical chaotic hamiltonian, whose periodic orbits are labeled by the prime numbers. Connes took the adelic approach to introduce $H = xp$ [2]. He showed that using different semiclassical regularization, Riemann zeros appear as missing spectral lines in a continuum.

Now we look into the Berry-Keating and Connes semiclassical approaches to $H = xp$

2 Semiclassical approach

The classical Berry-Keating-Connes (BKC) Hamiltonian is[1, 2]

$$H_0^{cl} = xp \tag{4}$$

which has hyperbolic trajectories

$$x(t) = x_0 e^t \quad p(t) = p_0 e^{-t} \tag{5}$$

So the dynamics is unbounded. There is a continuous spectrum as the quantum level. Berry-Keating and Connes introduced two different types of reularizations and counted the semiclassical states. Berry-Keating introduced Plank cell in a phase space: $|x| > l_x$ and $|p| > l_p$, with $l_x l_p = 2\pi\hbar$. Connes choosed $|x| < \Lambda$ and $|p| < \Lambda$, where Λ is a cutoff. German Sierra introduced us a third regularization, $l_x < x < \Lambda$ combines the Berry-Keating and Connes regularization position, not taking assumptions for the momenta p.

Semiclassical states number $\mathcal{N}(E)$ with an enery between 0 to E is given by

$$\begin{aligned} \mathcal{N}(E) &= \frac{A}{2\pi\hbar} \\ &= \frac{A}{h} \end{aligned} \tag{6}$$

Where A is the area of the allowed phase space region below the curve $E = xp$. So the the number of semiclassical states will be for Berry-Keating regularization

$$
\begin{aligned}
\mathcal{N}_{BK}(E) &= \frac{1}{h}\int_{l_x}^{\frac{E}{l_p}} dx \int_{l_p}^{\frac{E}{x}} dp + ... \\
&= \frac{1}{h}\left[\int_{l_x}^{\frac{E}{l_p}} dx \left[\frac{E}{x} - l_p\right]\right] \\
&= \frac{1}{h}\left[E\left[\ln x\right]_{l_x}^{\frac{E}{l_p}} - l_p\left[\frac{E}{l_p} - l_x\right]\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - l_x l_p\right] \\
&= \frac{1}{h}\left[E\ln\frac{E}{l_x l_p} - E - h\right] \\
&= \frac{E}{h}\left[\ln\frac{E}{l_x l_p} - 1\right] + 1
\end{aligned} \tag{7}
$$

adding Maslov phase $(-\frac{1}{8})$, it becomes

References

[1] Michael V Berry and Jonathan P Keating. H= xp and the riemann zeros. *Supersymmetry and trace formulae: chaos and disorder*, pages 355–367, 1999.

[2] Alain Connes. Trace formula in noncommutative geometry and the zeros of the riemann zeta function. *Selecta Mathematica*, 5(1):29, 1999.