Exercise 1

Show that,

$$[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$$

L.H.S.

$$
\begin{aligned}
&[\mathcal{L}_u, \mathcal{L}_v] \\
=&[\mathcal{L}_u, \mathcal{L}_v]\,\phi \\
=&\left[u^a\partial_a, v^b\partial_b\right] \\
=&u^a\partial_a\left\{v^b\left(\partial_b\phi\right)\right\} - v^a\partial_a\left\{u^b\left(\partial_b\phi\right)\right\} \\
=&u^a\left(\partial_a v^b\right)\left(\partial_b\phi\right) + u^a v^b\partial_a\partial_b\phi - v^a\left(\partial_a u^b\right)\left(\partial^b\phi\right) - v^a u^b\partial_a\partial_b\phi \\
=&u^a\left(\partial_a v^b\right)\left(\partial_b\phi\right) - v^a\left(\partial_a u^b\right)\left(\partial^b\phi\right) \\
=&u^a\partial_a\left(v^b\partial_b\phi\right) - v^a\partial_a\left(u^b\partial^b\phi\right) \\
=&\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_u\mathcal{L}_v \\
=&\mathcal{L}_{[u,v]} \\
=&R.H.S.
\end{aligned}
$$

Exercise 2

Find the rule for

$$\mathcal{L}_v T^a_c = ?$$

$$
\begin{aligned}
&\mathcal{L}_v T^a_c \\
=&\mathcal{L}_v\left(\mathcal{L}_v u^a\right)\omega_c + u^a\left(\mathcal{L}_v\omega_c\right) \\
=&\left(v^b\partial_b u^a - u^b\partial_b v^a\right)\omega_c + u^b\left(v^b\partial_b\omega_c + \omega_b\partial_c v^b\right) \\
=&v^b\partial_b\left(u^a\omega_c\right) - u^b\partial_b\left(v^a\omega_c\right) + u^a v^b\partial_b\omega_c + u^a\omega_b\partial_c v^b \\
=&v^b\partial_b\left(u^a\omega_c\right) - u^b\partial_b\left(v^a\right)\omega_c - u^b v^a\left(\partial_b\omega_c\right) + u^a v^b\partial_b\omega_c + u^a\omega_b\partial_c v^b \\
=&v^b\partial_b\left(u^a\omega_c\right) - u^b\omega_c\partial_b\left(v^a\right) + u^a\omega_b\partial_c v^b \\
=&v^b\partial_b T^a_c - T^b_c\partial_b\left(v^a\right) + T^a_b\partial_c v^b
\end{aligned}
$$