

Solution to Problem Sheet 2

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Solve for problem no. 1

Given

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (1)$$

From L.H.S.

$$\begin{aligned} [\gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu] &= \gamma^\kappa [\gamma^\lambda, \gamma^\mu \gamma^\nu] + [\gamma^\kappa, \gamma^\mu \gamma^\nu] \gamma^\lambda \\ &= \gamma^\kappa (\gamma^\lambda \gamma^\mu \gamma^\nu - \gamma^\mu \gamma^\nu \gamma^\lambda) + (\gamma^\kappa \gamma^\mu \gamma^\nu - \gamma^\mu \gamma^\nu \gamma^\kappa) \gamma^\lambda \\ &= \gamma^\kappa ((\gamma^\lambda \gamma^\mu + \gamma^\mu \gamma^\lambda) \gamma^\nu - \gamma^\mu (\gamma^\lambda \gamma^\nu + \gamma^\nu \gamma^\lambda) + ((\gamma^\kappa \gamma^\mu + \gamma^\mu \gamma^\kappa) \gamma^\nu - \gamma^\mu (\gamma^\nu \gamma^\kappa + \gamma^\kappa \gamma^\nu)) \gamma^\lambda) \\ &= \gamma^\kappa ((\{\gamma^\lambda, \gamma^\mu\} \gamma^\nu - \gamma^\mu \{\gamma^\lambda, \gamma^\nu\}) + (\{\gamma^\kappa, \gamma^\mu\} \gamma^\nu - \gamma^\mu \{\gamma^\nu, \gamma^\kappa\}) \gamma^\lambda) \\ [\gamma^\kappa \gamma^\lambda, \gamma^\mu \gamma^\nu] &= 2\eta^{\lambda\mu} \gamma^\kappa \gamma^\nu - 2\eta^{\lambda\nu} \gamma^\kappa \gamma^\mu + 2\eta^{\kappa\mu} \gamma^\nu \gamma^\lambda - 2\eta^{\nu\kappa} \gamma^\mu \gamma^\lambda \quad (\text{Showed}) \end{aligned}$$

$$\begin{aligned} (a) \operatorname{Tr}(\gamma^\mu) &= \operatorname{Tr}(\gamma^\mu \gamma_5 \gamma_5) && ([\text{because } (\gamma_5)^2 = 1]) \\ &= -\operatorname{Tr}(\gamma_5 \gamma^\mu \gamma_5) = -\operatorname{Tr}(\gamma_5 \gamma^\mu \gamma_5) \\ \implies \operatorname{Tr}(\gamma^\mu \gamma_5 \gamma_5) &= -\operatorname{Tr}(\gamma_5 \gamma^\mu \gamma_5) = \operatorname{Tr}(\gamma_5 \gamma^\mu \gamma_5) \\ \operatorname{Tr}(\gamma^\mu \gamma_5 \gamma_5) &= 0 \\ \implies \operatorname{Tr}(\gamma^\mu) &= 0 \end{aligned}$$

$$\begin{aligned}
(b) \operatorname{Tr}(\gamma^\mu \gamma^\nu) &= \frac{1}{2} (\operatorname{Tr}(\gamma^\mu \gamma^\nu) + \operatorname{Tr}(\gamma^\mu \gamma^\nu)) \\
&= \frac{1}{2} (\operatorname{Tr}(\gamma^\mu \gamma^\nu) + \operatorname{Tr}(\gamma^\nu \gamma^\mu)) \\
&= \frac{1}{2} (\operatorname{Tr}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)) \\
&= \frac{1}{2} \operatorname{Tr} \{ \gamma^\mu, \gamma^\nu \} \\
&= \frac{1}{2} \operatorname{Tr} 2\eta^{\mu\nu} \\
&= \frac{1}{2} 2\eta^{\mu\nu} \operatorname{Tr}(\mathbb{1}) \\
&= 4\eta^{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
(d) (\gamma^5)^2 &= i\gamma_0\gamma_1\gamma_2\gamma_3 \cdot i\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= -(-1)\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_1\gamma_2\gamma_3\gamma_2\gamma_3 &= -\gamma_2\gamma_3\gamma_2\gamma_3 \\
&= \gamma_2\gamma_2\gamma_3\gamma_3 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
(e) \operatorname{Tr}(\gamma^5) &= \operatorname{Tr}(\gamma^5 \gamma^0 \gamma^0) & ((\gamma^0)^2 = 1) \\
&= -\operatorname{Tr}(\gamma^0 \gamma^5 \gamma^0) = \operatorname{Tr}(\gamma^0 \gamma^5 \gamma^0) = 0
\end{aligned}$$

Solution to problem 2

Given,

$$\psi(\vec{x}) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^s u^s(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + c_{\vec{p}}^{s\dagger} v^s(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right] \quad (2)$$

And

$$\psi^\dagger(\vec{x}) = \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^{s\dagger} u^{s\dagger}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^s v^{s\dagger}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right] \quad (3)$$

Now

$$\begin{aligned} & \left\{ \psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y}) \right\} \\ &= \left\{ \sum_{\alpha=1}^4 \sum_{s=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^s u_\alpha^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}}^{s\dagger} v_\alpha^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right], \right. \\ & \quad \left. \sum_{\beta=1}^4 \sum_{r=1}^2 \int \frac{d^3 p'}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \left[b_{\vec{p}'}^{r\dagger} u_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}'\cdot\vec{y}} + c_{\vec{p}'}^r v_\beta^{r\dagger}(\vec{p}') e^{i\vec{p}'\cdot\vec{y}} \right] \right\} \\ &= \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \sum_{s=1}^2 \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \left[b_{\vec{p}}^s u_\alpha^s(\vec{p}) b_{\vec{p}'}^{r\dagger} u_\beta^{r\dagger}(\vec{p}') e^{i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} + b_{\vec{p}}^s u_\alpha^s(\vec{p}) c_{\vec{p}'}^r v_\beta^{r\dagger}(\vec{p}') e^{i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} \right. \\ & \quad + c_{\vec{p}}^{s\dagger} v_\alpha^s(\vec{p}) b_{\vec{p}'}^{r\dagger} u_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} + c_{\vec{p}}^{s\dagger} v_\alpha^s(\vec{p}) c_{\vec{p}'}^r v_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} + b_{\vec{p}'}^{r\dagger} u_\beta^{r\dagger}(\vec{p}') b_{\vec{p}}^s u_\alpha^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} \\ & \quad \left. + c_{\vec{p}'}^r v_\beta^{r\dagger}(\vec{p}') b_{\vec{p}}^s u_\alpha^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} + b_{\vec{p}'}^{r\dagger} u_\beta^{r\dagger}(\vec{p}') c_{\vec{p}}^{s\dagger} v_\alpha^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} + c_{\vec{p}}^{s\dagger} v_\alpha^s(\vec{p}) c_{\vec{p}'}^r v_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} \right] \\ &= \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \sum_{s=1}^2 \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \left[\{b_{\vec{p}}^s, b_{\vec{p}'}^{r\dagger}\} u_\alpha^s(\vec{p}) u_\beta^{r\dagger}(\vec{p}') e^{i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} + \{b_{\vec{p}}^s, c_{\vec{p}'}^{r\dagger}\} u_\alpha^s(\vec{p}) v_\beta^{r\dagger}(\vec{p}') e^{i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} \right. \\ & \quad \left. + \{c_{\vec{p}}^{s\dagger}, b_{\vec{p}'}^{r\dagger}\} v_\alpha^s(\vec{p}) u_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}\cdot\vec{x}-i\vec{p}'\cdot\vec{y}} + \{c_{\vec{p}}^{s\dagger}, c_{\vec{p}'}^{r\dagger}\} v_\alpha^s(\vec{p}) v_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{y}} \right] \end{aligned} \quad (4)$$

Given

$$\{b_{\vec{p}}^s, b_{\vec{p}'}^{r\dagger}\} = \{c_{\vec{p}}^s, c_{\vec{p}'}^{r\dagger}\} = \{c_{\vec{p}}^s, b_{\vec{p}'}^{r\dagger}\} = \{b_{\vec{p}}^s, c_{\vec{p}'}^{r\dagger}\} = \dots = 0 \quad (5)$$

$$\{b_{\vec{p}}^s, b_{\vec{p}'}^{r\dagger}\} = \{c_{\vec{p}}^s, c_{\vec{p}'}^{r\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}') \quad (6)$$

Therefore eqn(4) will be

$$\begin{aligned}
& \left\{ \psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y}) \right\} = \\
& \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \sum_{s=1}^2 \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \left[(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}') u_\alpha^s(\vec{p}) u_\beta^{r\dagger}(\vec{p}') e^{i\vec{p} \cdot \vec{x} - i\vec{p}' \cdot \vec{y}} \right. \\
& \left. + (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}') v_\alpha^s(\vec{p}) v_\beta^{r\dagger}(\vec{p}') e^{-i\vec{p} \cdot \vec{x} + i\vec{p}' \cdot \vec{y}} \right] \\
& = \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \int \frac{d^3 p}{(2\pi)^6} \frac{1}{2E_{\vec{p}}} \left[(2\pi)^3 u_\alpha(\vec{p}) u_\beta^\dagger(\vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{y})} + (2\pi)^3 v_\alpha(\vec{p}) v_\beta^\dagger(\vec{p}) e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} \right] \\
& = \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[u_\alpha(\vec{p}) u_\beta^\dagger(\vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{y})} + v_\alpha(\vec{p}) v_\beta^\dagger(\vec{p}) e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} \right] \\
& = \sum_{\alpha=1}^4 \sum_{\beta=1}^4 \left[u_\alpha(\vec{p}) u_\beta^\dagger(\vec{p}) \delta^{(3)}(\vec{x} - \vec{y}) + v_\alpha(\vec{p}) v_\beta^\dagger(\vec{p}) \delta^{(3)}(\vec{x} - \vec{y}) \right] \tag{7}
\end{aligned}$$

(8)

because

$$\int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} = \delta^{(3)}(\vec{x} - \vec{y}) \tag{9}$$

and for same spinors numbers

$$u_\alpha(\vec{p}) u_\beta^\dagger(\vec{p}) = 1 = v_\alpha(\vec{p}) v_\beta^\dagger(\vec{p}) \quad (\text{if } \alpha = \beta)$$

Therefore

$$\left\{ \psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y}) \right\} = \delta_{\alpha\beta} \delta^{(3)}(\vec{x} - \vec{y})$$