Solution to Problem Sheet 2

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Solve for problem no. 1

Given

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

From L.H.S.

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu\right] &= \gamma^\kappa\left[\gamma^\lambda, \gamma^\mu\gamma^\nu\right] + [\gamma^\kappa, \gamma^\mu\gamma^\nu]\,\gamma^\lambda \\
&= \gamma^\kappa\left(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\lambda\right) + (\gamma^\kappa\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\kappa)\,\gamma^\lambda \\
&= \gamma^\kappa\left(\left(\gamma^\lambda\gamma^\mu + \gamma^\mu\gamma^\lambda\right)\gamma^\nu - \gamma^\mu\left(\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\right) + ((\gamma^\kappa\gamma^\mu + \gamma^\mu\gamma^\kappa)\,\gamma^\nu - \gamma^\mu\,(\gamma^\nu\gamma^\kappa + \gamma^\kappa\gamma^\nu))\,\gamma^\lambda\right) \\
&= \gamma^\kappa\left(\left(\left\{\gamma^\lambda, \gamma^\mu\right\}\gamma^\nu - \gamma^\mu\left\{\gamma^\lambda, \gamma^\nu\right\}\right) + (\{\gamma^\kappa, \gamma^\mu\}\,\gamma^\nu - \gamma^\mu\,\{\gamma^\nu, \gamma^\kappa\})\,\gamma^\lambda\right) \\
\left[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu\right] &= 2\eta^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2\eta^{\lambda\nu}\gamma^\kappa\gamma\mu + 2\eta^{\kappa\mu}\gamma^\nu\gamma^\lambda - 2\eta^{\nu\kappa}\gamma^\mu\gamma^\lambda \qquad \text{(Showed)}
\end{aligned}$$

$$\begin{aligned}
(a)\; Tr(\gamma^\mu) &= Tr(\gamma^\mu\gamma_5\gamma_5) \qquad \text{([because } (\gamma_5)^2 = 1\text{])} \\
&= -Tr(\gamma_5\gamma^\mu\gamma_5) = -Tr(\gamma_5\gamma^\mu\gamma_5) \\
\implies Tr(\gamma^\mu\gamma_5\gamma_5) &= -Tr(\gamma_5\gamma^\mu\gamma_5) = Tr(\gamma_5\gamma^\mu\gamma_5) \\
Tr(\gamma^\mu\gamma_5\gamma_5) &= 0 \\
\implies Tr(\gamma^\mu) &= 0
\end{aligned}$$

$$
\begin{aligned}
(b)\ Tr(\gamma^\mu\gamma^\nu) &= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu) + Tr(\gamma^\mu\gamma^\nu)\right) \\
&= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu) + Tr(\gamma^\nu\gamma^\mu)\right) \\
&= \frac{1}{2}\left(Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\right) \\
&= \frac{1}{2}Tr\left\{\gamma^\mu, \gamma^\nu\right\} \\
&= \frac{1}{2}Tr 2\eta^{\mu\nu} \\
&= \frac{1}{2}2\eta^{\mu\nu}Tr(\mathbb{1}) \\
&= 4\eta^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
(d)\ (\gamma^5)^2 &= i\gamma_0\gamma_1\gamma_2\gamma_3 \cdot i\gamma_0\gamma_1\gamma_2\gamma_3 \\
&= -(-1)\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_2\gamma_3\gamma_1\gamma_2\gamma_3 \\
&= \gamma_1\gamma_1\gamma_2\gamma_3\gamma_2\gamma_3 && = -\gamma_2\gamma_3\gamma_2\gamma_3 \\
&= \gamma_2\gamma_2\gamma_3\gamma_3 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
(e)\ Tr(\gamma^5) &= Tr(\gamma^5\gamma^0\gamma^0) && ((\gamma^0)^2 = 1) \\
&= -Tr(\gamma^0\gamma^5\gamma 0) = Tr(\gamma^0\gamma^5\gamma 0) = 0
\end{aligned}
$$

Solution to problem 2

Given,

$$
\psi(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right] \tag{2}
$$

And

$$\psi^{\dagger}(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s\dagger}_{\vec{p}} u^{s\dagger}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + c^{s}_{\vec{p}} v^{s\dagger}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \right] \tag{3}$$

Now

$$\left\{ \psi_{\alpha}(\vec{x}), \psi^{\dagger}_{\beta}(\vec{y}) \right\}$$

$$= \left\{ \sum_{\alpha=1}^{4} \sum_{s=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right], \right.$$

$$\left. \sum_{\beta=1}^{4} \sum_{r=1}^{2} \int \frac{d^3 p'}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p'}\cdot\vec{y}} + c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p'}\cdot\vec{y}} \right] \right\}$$

$$= \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \sum_{s=1}^{2} \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}} \right.$$

$$+ c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}} + b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}$$

$$\left. + c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) b^{s}_{\vec{p}} u^{s}_{\alpha}(\vec{p}) e^{i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}} + b^{r\dagger}_{\vec{p'}} u^{r\dagger}_{\beta}(\vec{p'}) c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + c^{s\dagger}_{\vec{p}} v^{s}_{\alpha}(\vec{p}) c^{r}_{\vec{p'}} v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}} \right]$$

$$= \sum_{\alpha=1}^{4} \sum_{\beta=1}^{4} \sum_{s=1}^{2} \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p'}}}} \left[\left\{ b^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} u^{s}_{\alpha}(\vec{p}) u^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + \left\{ b^{s}_{\vec{p}}, c^{r\dagger}_{\vec{p'}} \right\} u^{s}_{\alpha}(\vec{p}) v^{r\dagger}_{\beta}(\vec{p'}) e^{i\vec{p}\cdot\vec{x}+} \right.$$

$$+ \left\{ c^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} v^{s}_{\alpha}(\vec{p}) u^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}} + \left\{ c^{s}_{\vec{p}}, c^{r\dagger}_{\vec{p'}} \right\} v^{s}_{\alpha}(\vec{p}) v^{r\dagger}_{\beta}(\vec{p'}) e^{-i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}}$$

(4)

Given

$$\left\{ b^{s}_{\vec{p}}, b^{r}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, c^{r}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, b^{r}_{\vec{p'}} \right\} = ... = 0 \tag{5}$$

$$\left\{ b^{s}_{\vec{p}}, b^{r\dagger}_{\vec{p'}} \right\} = \left\{ c^{s}_{\vec{p}}, c^{r\dagger}_{\vec{p'}} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p'}) \tag{6}$$

Therefore eqn(4) will be

$$\left\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\} =$$
$$\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{s=1}^{2}\sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\frac{1}{\sqrt{2E_{\vec{p'}}}}\Bigg[(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{p'})u^s_\alpha(\vec{p})u^{r\dagger}_\beta(\vec{p'})e^{i\vec{p}\cdot\vec{x}-i\vec{p'}\cdot\vec{y}}$$
$$+(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{p'})v^s_\alpha(\vec{p})v^{r\dagger}_\beta(\vec{p'})e^{-i\vec{p}\cdot\vec{x}+i\vec{p'}\cdot\vec{y}}\Bigg]$$
$$=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^6}\frac{1}{4E_{\vec{p}}}\left[(2\pi)^3u_\alpha(\vec{p})u^\dagger_\beta(\vec{p})e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+(2\pi)^3v_\alpha(\vec{p})v^\dagger_\beta(\vec{p})e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right]$$
$$=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\int \frac{d^3p}{(2\pi)^3}\frac{1}{4E_{\vec{p}}}\left[u_\alpha(\vec{p})u^\dagger_\beta(\vec{p})e^{i\vec{p}\cdot(\vec{x}-\vec{y})}+v_\alpha(\vec{p})v^\dagger_\beta(\vec{p})e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right]$$

$$=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\frac{1}{2}\left[u_\alpha(\vec{p})u^\dagger_\beta(\vec{p})\delta^{(3)}(\vec{x}-\vec{y})+v_\alpha(\vec{p})v^\dagger_\beta(\vec{p})\delta^{(3)}(\vec{x}-\vec{y})\right] \tag{7}$$
$$\tag{8}$$

because
$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y}) \tag{9}$$

and for same spinors numbers

$$u_\alpha(\vec{p})u^\dagger_\beta(\vec{p}) = 1 = v_\alpha(\vec{p})v^\dagger_\beta(\vec{p}) \qquad (\text{if } \alpha=\beta)$$

Therefore

$$\left\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\right\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y})$$