Solution to Problem Sheet 1

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Solve for problem no. 1

Given

$$\begin{aligned}
\eta_{\mu\nu}x^\mu x^\nu &= \eta_{\mu\nu}x'^\mu x'^\nu \\
&= \eta_{\mu\nu}\Lambda^\mu{}_\sigma x^\sigma \Lambda^\nu{}_\tau x^\tau \\
&= \eta_{\mu\nu}\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau x^\sigma x^\tau \\
&= \eta_{\sigma\tau}\Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu x^\mu x^\nu \qquad\qquad (\sigma \to \mu, \tau \to \nu)
\end{aligned}$$

Therefore,

$$\eta_{\mu\nu} = \eta_{\sigma\tau}\Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu \tag{1}$$

Again, given

$$\Lambda^\mu_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{2}$$

putting this into eqn(1)

$$\begin{aligned}
\eta_{\mu\nu} &= \eta_{\sigma\tau}(\delta^\sigma{}_\mu + \omega^\sigma{}_\mu)(\delta^\tau{}_\nu + \omega^\tau{}_\nu) \\
&= \eta_{\sigma\tau}(\delta^\sigma{}_\mu \delta^\tau{}_\nu + \delta^\sigma{}_\mu \omega^\tau{}_\nu + \delta^\tau{}_\nu \omega^\sigma{}_\mu) \qquad [ignoring\ higher\ order\ \omega] \\
&= \eta_{\mu\nu}\eta_{\mu\tau}\omega^\tau{}_\nu + \eta_{\sigma\nu}\omega^\sigma{}_\mu \\
&= \eta_{\mu\nu} + \omega_{\mu\nu} + \omega_{\nu\mu} \\
&= \eta_{\mu\nu} \qquad [\omega_{\mu\nu} = -\omega_{\nu\mu}]
\end{aligned}$$

Therefore infinitesimal transformation around identity of the form of eqn(2) is a Lorentz transformation.

Solve for problem no. 2

For real scalar field ϕ the Euler-Langrange eqn will be

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{3}$$

putting

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{\lambda^2}{2}(\phi^* \phi)^2 \tag{4}$$

in eqn (3), we get,

$$\begin{aligned} -m^2\phi^* - \partial_\mu \partial^\mu \phi^* &= 0 \\ (\partial_\mu \partial^\mu + m^2 + \lambda^2 \phi^* \phi])\phi^* &= 0 \end{aligned}$$

And for complex scalar field ϕ^* the Euler-Langrange eqn will be

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = 0 \tag{5}$$

putting eqn (4) in eqn (5) wiil be

$$\begin{aligned} -m^2\phi - \partial_\mu \partial^\mu \phi &= 0 \\ (\partial_\mu \partial^\mu + m^2 + \lambda^2 \phi^* \phi)\phi &= 0 \end{aligned}$$

For deriving Noether theorem of given Langrangian. The infinitesimal tranformation of the fields are

$$\begin{aligned} \phi(x) \to \phi'(x) &= e^{i\alpha}\phi(x) \\ \phi^\dagger(x) \to \phi'^\dagger(x) &= \phi^\dagger(x) e^{-i\alpha} \end{aligned}$$

where θ is a real constant parameter of a transformation (a global transformation). Such a transformation is not a space-time symmetry transformation since the space-time coordinates are not changed by this transformation, such a transformation is known as an internal symmetry transformation, Infinitesimally, the transformation takes the form of

$$
\begin{aligned}
\delta\phi(x) &= \phi'(x) - \phi(x) = i\alpha\phi \\
\delta\phi^\dagger(x) &= \phi'^\dagger(x) - \phi^\dagger(x) = -i\alpha\phi^\dagger \\
x^\mu &= x'^\mu
\end{aligned}
$$

so the least action principle will be

$$
\begin{aligned}
&\delta S = 0 \\
\implies &\int d^4x' \mathcal{L}(\phi'(x'), \partial_\mu^\dagger \phi'(x'), \phi'^\dagger(x'), \partial_\mu^\dagger \phi'^\dagger(x')) - \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) = 0 \\
\implies &\int d^4x \mathcal{L}(\phi'(x), \partial_\mu \phi'(x), \phi'^\dagger(x), \partial_\mu^\dagger \phi'^\dagger(x)) - \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) = 0
\end{aligned}
$$

$$
\implies \mathcal{L}(\phi'(x), \partial_\mu \phi'(x), \phi'^\dagger(x), \partial_\mu^\dagger \phi'^\dagger(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) = K^\mu \tag{6}
$$

For internal symmetry $K^\mu = 0$ And

$$
\begin{aligned}
\delta(\partial_\mu \phi(x) &= \partial_\mu \phi'(x) - \partial_\mu \phi(x) \\
&= \partial_\mu \delta\phi(x) \\
\delta(\partial_\mu^\dagger \phi(x) &= \partial_\mu \phi'^\dagger(x) - \partial_\mu \phi(x) \\
&= \partial_\mu \delta\phi^\dagger(x)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{L}(\phi'(x), \partial_\mu \phi'(x), \phi'^\dagger(x), \partial_\mu^\dagger \phi'^\dagger(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) \\
&= \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) + \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \delta(\partial_\mu \phi(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} \\
&+ \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \phi^\dagger(x)} + \delta(\partial_\mu \phi^\dagger(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} - \mathcal{L}(\phi(x), \partial_\mu \phi(x), \phi^\dagger(x), \partial_\mu \phi^\dagger(x)) \\
&= \delta\phi(x) \frac{\partial \mathcal{L}}{\partial \phi(x)} + \partial_\mu(\delta\phi(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \phi^\dagger(x)} + \partial_\mu(\delta\phi^\dagger(x)) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)}
\end{aligned}
$$

$$= \partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} \right) \tag{7}$$

Comparing eqn (6) eqn (7) we get,

$$\partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} \right) = \partial K_\mu \tag{8}$$

$$\partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} - K_\mu \right) = 0 = \partial_\mu \boldsymbol{J}^\mu \tag{9}$$

Which is Noether current. Because $K_\mu = 0$, we get

$$\partial_\mu \left(\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} \right) = \partial_\mu \boldsymbol{J}^\mu \tag{10}$$

$$\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} = \boldsymbol{J}^\mu \tag{11}$$

Now

$$\begin{aligned} &\delta\phi(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi(x)} + \delta\phi^\dagger(x) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger(x)} \\ &\qquad = i\alpha\phi(x)\partial^\mu\phi^\dagger(x) - i\alpha\phi^\dagger(x)\partial^\mu\phi(x) \\ &\qquad = i\alpha \left(\phi(x)\partial^\mu\phi^\dagger(x) - \phi^\dagger(x)\partial^\mu\phi(x) \right) \end{aligned}$$

$$\boldsymbol{J}^\mu = i\alpha\phi(x)\overleftrightarrow{\partial_\mu}\phi^\dagger \tag{12}$$

Given Langrangian is

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda^2}{2}(\phi^*\phi)^2 \tag{13}$$

where

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{14}$$

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x)) \tag{15}$$

putting these in eqn (13), we get

$$\mathcal{L} = \frac{1}{2}\partial_\mu \left(\phi_1 + i\phi_2\right) \frac{1}{2}\partial^\mu \left(\phi_1 - i\phi_2\right) - \frac{m^2}{2} \left(\phi_1 + i\phi_2\right) \left(\phi_1 - i\phi_2\right) - \frac{\lambda^2}{4}(\left(\phi_1 + i\phi_2\right) \left(\phi_1 - i\phi_2\right))^2$$

Therefore

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}\phi_1^2 - \frac{\lambda^2}{4}\phi_1^4 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}\phi_2^2 - \frac{\lambda^2}{4}\phi_2^4 \qquad (16)$$

Solve for problem no. 3

From Wick's theory we know that

$$T(\phi_1\phi_2.......\phi_n) = N(\phi_1\phi_2.......\phi_n) + \; all\ possible\ contractions \qquad (17)$$

Wick's thoery for three scalar product will be

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) =: \phi(x_1)\phi(x_2)\phi(x_3) : + \overbrace{\phi(x_1)\phi(x_2)} : \phi(x_3) : + \overbrace{\phi(x_2)\phi(x_3)} : \phi(x_1) : + \overbrace{\phi(x_3)\phi(x_1)} : \phi(x_2) :$$

And we know

$$\overbrace{\phi(x_1)\phi(x_2)} = \Delta_F(x_1 - x_2) = \overbrace{\phi(x_2)\phi(x_1)} \qquad (18)$$

Therefore

$$T(\phi(x_1)\phi(x_2)\phi(x_3)) =: \phi(x_1)\phi(x_2)\phi(x_3) : + \Delta_F(x_1 - x_2) : \phi(x_3) : + \Delta_F(x_2 - x_3) : \phi(x_1) : + \Delta_F(x_3 - x_1) : \phi(x_2) : \qquad (19)$$

Wick's theorem for four scalar product

$$
\begin{aligned}
T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =\, &: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : + \overbracket{\phi(x_1)\phi}(x_2) : \phi(x_3)\phi(x_4) : \\
&+ \overbracket{\phi(x_2)\phi}(x_3) : \phi(x_1)\phi(x_4) : + \overbracket{\phi(x_3)\phi}(x_4) : \phi(x_2)\phi(x_1) : \\
&+ \overbracket{\phi(x_4)\phi}(x_1) : \phi(x_2)\phi(x_3) : + \overbracket{\phi(x_1)\phi}(x_3) : \phi(x_2)\phi(x_4) : \\
&+ \overbracket{\phi(x_2)\phi}(x_4) : \phi(x_1)\phi(x_3) : + \overbracket{\phi(x_1)\phi}(x_2)\, \overbracket{\phi(x_3)\phi}(x_4) \\
&+ \overbracket{\phi(x_1)\phi}(x_3)\, \overbracket{\phi(x_2)\phi}(x_4) + \overbracket{\phi(x_1)\phi}(x_4)\, \overbracket{\phi(x_2)\phi}(x_3)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =\, &: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : + \Delta_F(x_1 - x_2) : \phi(x_3)\phi(x_4) : \\
&+ \Delta_F(x_2 - x_3) : \phi(x_1)\phi(x_4) : + \Delta_F(x_3 - x_4) : \phi(x_1)\phi(x_2) : \\
&+ \Delta_F(x_4 - x_1) : \phi(x_2)\phi(x_3) : + \Delta_F(x_1 - x_3) : \phi(x_2)\phi(x_4) : \\
&+ \Delta_F(x_2 - x_4) : \phi(x_1)\phi(x_3) : + \Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) \\
&+ \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)
\end{aligned} \tag{20}
$$