1 Road to Curvature

Recall the covariant derivative

$$\begin{aligned}
\nabla_\mu V^\nu &\equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \\
\nabla_\mu \phi &\equiv \partial_\mu \phi \\
\nabla_\mu W_\nu &\equiv \partial_\mu W_\nu - \Gamma^\lambda_{\mu\nu} W_\lambda
\end{aligned} \tag{1}$$

∇_μ is made unique by demanding
(a) Torsion free
(b) Metric compatible

Point a)

$$[\nabla_\mu, \nabla_\nu]\,\phi = 0 \qquad \text{As } [\partial_\mu, \partial_\nu]\,\phi = 0$$

L.H.S

$$\begin{aligned}
&[\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu]\,\phi \\
&= \nabla_\mu \nabla_\nu \phi - \nabla_\nu \nabla_\mu \phi \\
&= \nabla_\mu (\partial_\nu \phi) - \nabla_\nu (\partial_\mu \phi) \\
&= \partial_\mu (\partial_\nu \phi) - \Gamma^\lambda_{\mu\nu} \phi - \nabla_\nu (\partial_\mu \phi) \Gamma^\lambda_{\nu\mu} \phi \\
&= \partial_\mu (\partial_\nu \phi) - \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi - \nabla_\nu (\partial_\mu \phi) + \Gamma^\lambda_{\nu\mu} \partial_\lambda \phi \\
&= \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) \partial_\lambda \phi \\
&\equiv T^\lambda_{\mu\nu} \partial_\lambda \phi
\end{aligned} \tag{2}$$