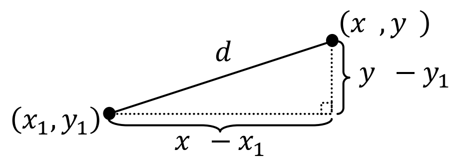
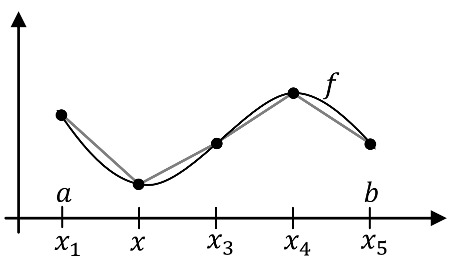
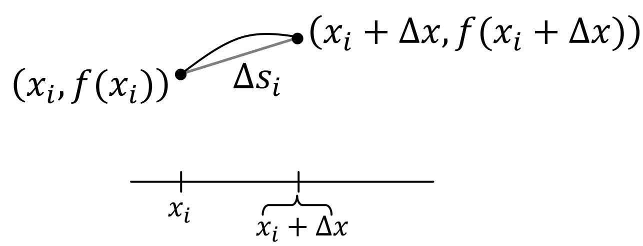
Unit 5.5: Arc Length

We now turn out attention away from volumes and look at the length of curves or arc length. We will restrict our attention to continuous functions on closed intervals. Our basic concept of length is based on straight lines. Consider the following two points joined by a line segment.



By the Pythagorean Theorem we have $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Because of this natural way of defining length, we turn to line segments in order to measure the length of curves. Over short distances, curves closely resemble line segments. Thus we can approximate the length of a curve by chopping it up into small segments, approximating the lengths of each segment and then summing those approximations. This can then be followed by introducing a limit to get the exact length. Of course this process of chopping something into lots of little pieces, approximating each piece, followed by a sum and then a limit is exactly what generates a definite integral. The following figure shows a curve approximated by four segments as well as a representative segment of length Δs_i.





We now wish to write an expression for the increment of arc length, Δs_i. This length can be obtained using the distance formula above. With this we obtain,

$$\Delta s_i = \sqrt{((x_i + \Delta x) - x_i) \;\; + (f(x_i + \Delta x) - f(x_i))} \;\; = \sqrt{(\Delta x) \;\; + (f(x_i + \Delta x) - f(x_i))}$$

The sum of these increments will give us an estimate of s, the arc length of the entire curve. Before doing this, let's think ahead. If we wish this to become a definite integral, we need a factor of Δx, which will then become the dx. To introduce the Δx, we will multiply by the expression $\frac{\Delta x}{\sqrt{(\Delta x)^2}}$, which is equivalent to 1. Note: since $\Delta x > 0$, the denominator $\sqrt{(\Delta x)}$ is equal to Δx. This allows us to combine the denominator with the radical to obtain a simpler expression.

$$\Delta s_i = \sqrt{(\Delta x) \;\; + (f(x_i + \Delta x) - f(x_i))} \cdot \frac{\Delta x}{\sqrt{(\Delta x)}} = \sqrt{\frac{(\Delta x)}{(\Delta x)} + \frac{(f(x_i + \Delta x) - f(x_i))}{(\Delta x)}} \cdot \Delta x$$

Finally we simplify to obtain

$$\Delta s_i = \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2} \cdot \Delta x$$

Take the time to make sure you understand the algebraic manipulations above. From this we obtain the following Riemann Sum approximating the arc length of the curve.

$$s \approx \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right)^2} \cdot \Delta x$$

Before we apply the limit to obtain our definite integral, notice that the expression inside the parentheses under the radical is the difference quotient for f. Thus when we let n approach infinity, this difference quotient approaches the derivative f'. Therefore our definite integral giving the exact length of the curve is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Unlike the formulas generated previous two sections for finding volumes of solids of revolution, I recommend that you memorize this formula.

Example Find the arc length of the curve $f(x) = x^{3/2}$ on $[1,4]$.

Solution: Let us first note that $f'(x) = \frac{3}{2}x^{1/2}$ and thus $[f'(x)]^2 = \frac{9}{4}x$. From this we obtain

$$s = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg]_1^4 = \boxed{\frac{8}{27}\left(10^{3/2} - (13/4)^{3/2}\right)} \approx 7.634$$

In closing, I would like to point out that the formula for arc length typically yields an integral that must be approximated. The example chosen above is one of the few that we can actually evaluate using antiderivatives. In the narrated examples, you will find an example where we use the calculator to approximate the integral.