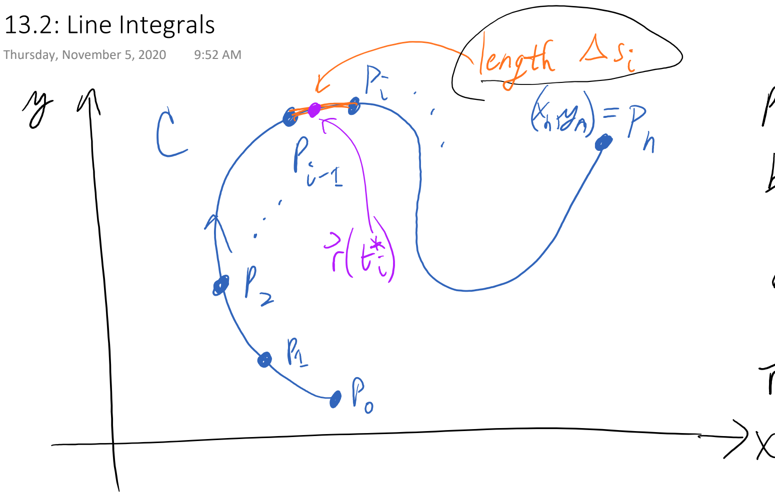
13.2: Line Integrals

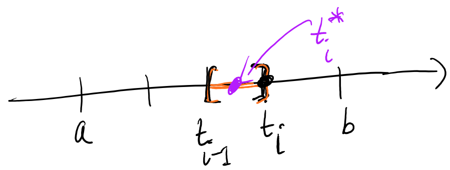
Thursday, November 5, 2020 9:52 AM



Plane curve given by $x = x(t)$, $y = y(t)$, $a \le t \le b$, i.e.

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

- Assume C smooth (i.e. $\vec{r}\,'$ continuous, and $\vec{r}\,' \neq \vec{0}$)
- Divide (partition) $[a, b]$ into n subintervals $[t_{i-1}, t_i]$,



where $x_i = x(t_i)$,
$y_i = y(t_i)$

- The points $P_i = (x_i, y_i)$ divide C into n sub-arcs w/ lengths $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.
- Choose any point $P_i^* = (x_i^*, y_i^*) = \vec{r}(t_i^*)$,

$t_{i-1} \le t_i^* \le t_i$.

- f a func. of 2 var. whose domain includes C, evaluate $f(x_i^*, y_i^*)$, then multiply by length Δs_i of subarc.
- Form sum $\displaystyle\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$ (like Riemann),

then take limit...

Def. Let f be defined on a smooth curve C, where C is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. The line integral of f along C is

$$\int_C f(x,y)\, ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i,$$

if the limit exists.

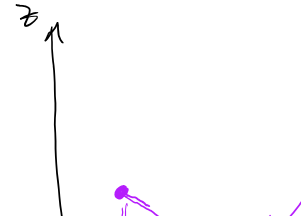
Note In Sec. 9.2, $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$

$$= \int_a^b |\vec{r}\,'(t)|\, dt \quad = ds$$

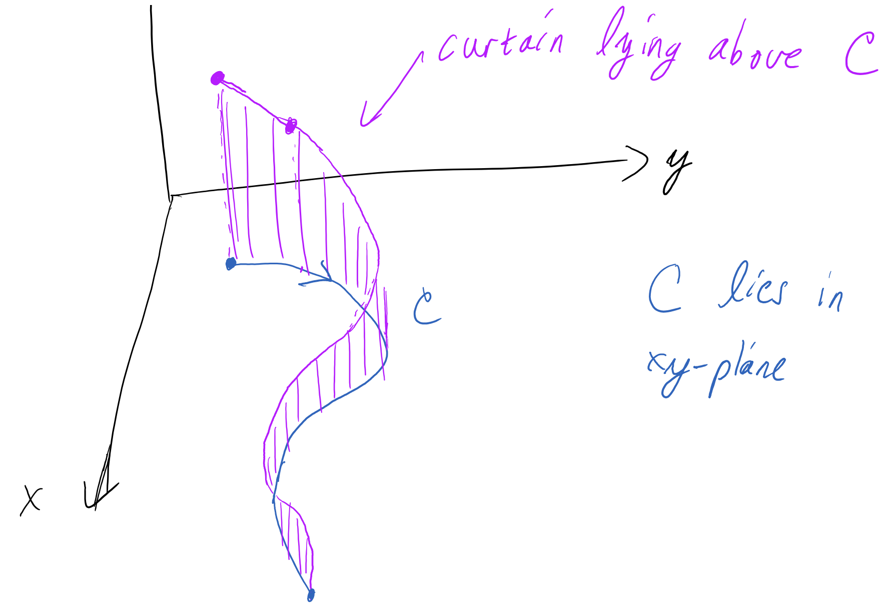
$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Fact Value of $\int_C f\, ds$ is independent of parametrization of C, as long as C traversed exactly once as t increases.



curtain lying above C



If $f(x,y) > 0$ on all $(x,y) \in C$, then

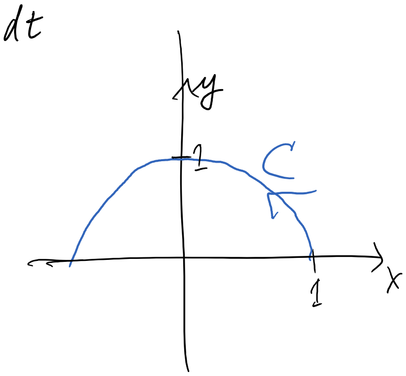
$\int_C f(x,y)\, ds$ = area of one side of curtain

Ex. $\int_C 2 + x^2 y \, ds$, C = upper half unit circle, param. counter-clockwise.

Sol'n: $\vec{r}'(t) = \langle \cos(t), \sin(t) \rangle$, $0 \le t \le \pi$

$$ds = |\vec{r}\,'(t)|\, dt = |\langle -\sin(t), \cos(t) \rangle|\, dt = dt$$

$$\int_C 2 + x^2 y \, ds = \int_0^{\pi} 2 + \cos^2(t)\sin(t)\, dt$$



$$= \int_0^{\pi} 2\, dt + \int_0^{\pi} \cos^2(t)\sin(t)\, dt$$

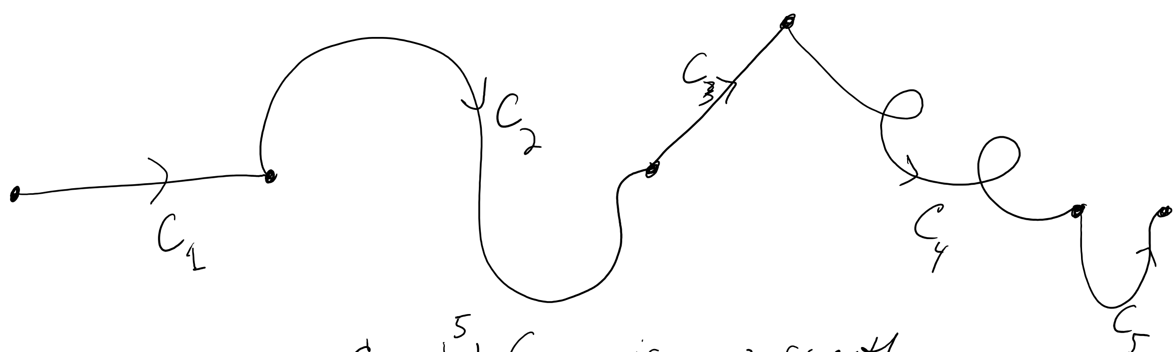
$$= 2\pi + \int_1^{-1} u^2 (-du)$$

$$= 2\pi + \int_{-1}^{1} u^2\, du = 2\pi + 2\int_0^1 u^2\, du = \underline{2\pi + \frac{2}{3}}$$

$$= 2\pi + \int_{-1}^{1} u^2\,du = 2\pi + 2\int_0^1 u^2\,du = 2\pi + \frac{2}{3}$$

$$= \boxed{\frac{6\pi + 2}{3}}$$

<u>Def.</u> A <u>piecewise-smooth curve</u> is a union of a finite # of smooth curves $C_1, C_2, \ldots, C_n$; where the initial point of C_{i+1} is the terminal point of C_i, for $i = 1, \ldots, n$. (all same direction)



$C = \bigcup_{i=1}^{5} C_i$ is <u>p-w smooth</u>

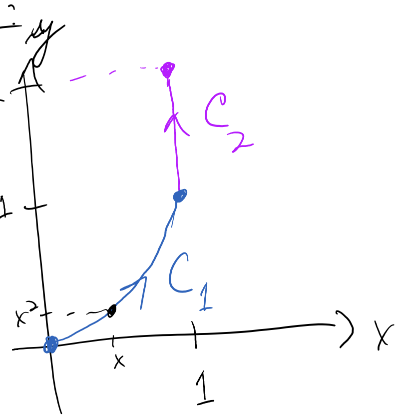
<u>Def.</u> If $C = \bigcup_{i=1}^{n} C_i$ is p-w smooth w/smooth pieces $C_1, \ldots, C_n$, the line integral of f along C is

$$\int_C f\,ds = \int_{C_1} f\,ds + \cdots + \int_{C_n} f\,ds.$$

<u>Ex.</u> Evaluate $\int_C 2x\,ds$, where C is the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$, followed by the vertical line seg. C_2 from $(1,1)$

to $(1, 2)$.

Sol'n:



$$\int_C 2x\,ds = \underbrace{\int_{C_1} 2x\,ds}_{(i)} + \underbrace{\int_{C_2} 2x\,ds}_{(ii)}$$

$ds = ?$

(i) C_1: Use x as parameter, so $\vec{r}(x) = \langle x, x^2 \rangle$, $0 \le x \le 1$.

$$ds = |\vec{r}\,'(x)|\,dx = \sqrt{1 + (2x)^2}\,dx = \sqrt{1+4x^2}\,dx$$

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1+4x^2}\,dx, \quad u = 1+4x^2, \quad \frac{du}{8} = \cancel{8}x\,dx$$

$$= \frac{1}{4}\int_1^5 \sqrt{u}\,du$$

$$= \frac{1}{4}\,\frac{2}{3}\left(5^{3/2} - 1^{3/2}\right) = \frac{1}{6}\left(5\sqrt{5} - 1\right)$$

(ii) C_2: Use y as param. $\rightsquigarrow \vec{r}(y) = \langle 1, y \rangle$, $1 \le y \le 2$.

$$ds = |\langle 0, 1 \rangle|\,dy = dy$$

$$\int_{C_2} 2x\,ds = \int_1^2 2(1)\,dy = 2$$

$$\int_{C_2} 2x \, ds = \int_1 2(1)\, dy$$

$$\therefore \int_C 2x \, ds = \frac{1}{6}\left(5\sqrt{5}-1\right) + 2 = \boxed{\frac{5\sqrt{5}+11}{6}}$$

Def. 1) Replace Δs_i by $\Delta x_i = x_i - x_{i-1}$

$$\int_C f(x,y)\, dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta x_i$$

2) Similarly $\int_C f(x,y)\, dy = \lim_{\max \Delta y_i \to 0} f(x_i^*, y_i^*)\, \Delta y_i$

Names $\int_C f(x,y)\, ds$ = line integral (w.r.t. arc length)
(or simply "line integral")

$\int_C f(x,y)\, dx$ = " " " x

$\int_C f(x,y)\, dy$ = " " " y

$x = x(t)$ $\quad$ $y = y(t)$

$dx = x'(t)\, dt$ $\quad$ $dy = y'(t)\, dt$

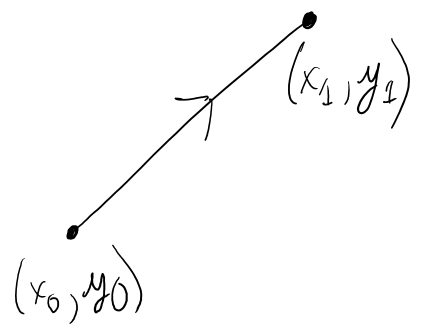
Note $\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$, $\quad a \le t \le b$

$\int^d$

$$\int_C f(x,y)\,dy = \int_c^d f(x(t),\,y(t))\,y'(t)\,dt \quad , \quad c \le t \le d$$

<u>Rmk</u> $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$

<u>Recall</u> Can parametrize line seg. from (x_0, y_0) to (x_1, y_1):



Let $\vec{r}_0 = \langle x_0, y_0 \rangle$ and
$\vec{r}_1 = \langle x_1, y_1 \rangle$.

Then $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \,,\; 0 \le t \le 1$

parametrizes the segment.

(Also works in 3-D)

<u>Ex.</u> Evaluate $\int_C y^2\,dx + x\,dy$, where

(a) $C = C_1$ is the line seg. from $(-5,-3)$ to $(0,2)$.

(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

<u>Sol'n:</u> (a) $\vec{r}(t) = (1-t)\langle -5, -3 \rangle + t\langle 0, 2 \rangle \,,\; 0 \le t \le 1$

$$= \langle -5+5t, -3+3t \rangle + \langle 0, 2t \rangle$$

$$= \langle -5+5t, -3+5t \rangle$$

$$= \langle x(t), y(t) \rangle$$

$$dx = 5\,dt, \quad dy = 5\,dt$$

$$\int_C y^2\,dx + x\,dy = \int_0^1 (5t-3)^2 5\,dt + (5t-5)5\,dt$$

$$= 5\int_0^1 25t^2 - 30t + 9 + 5t - 5\,dt$$

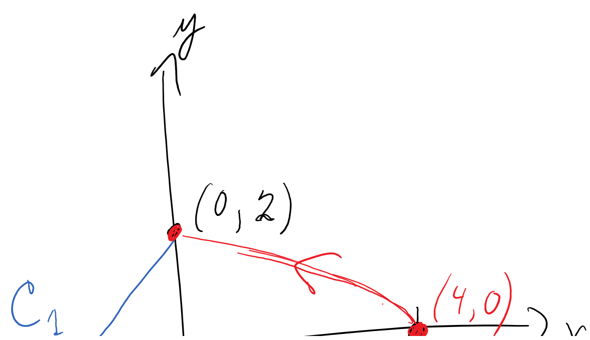
$$= 5\int_0^1 25t^2 - 25t + 4\,dt$$

$$= 5\left(\frac{25}{3} - \frac{25}{2} + 4\right)$$

$$= 5\left(\frac{50 - 75 + 24}{6}\right)$$

$$= \boxed{-\frac{5}{6}}$$

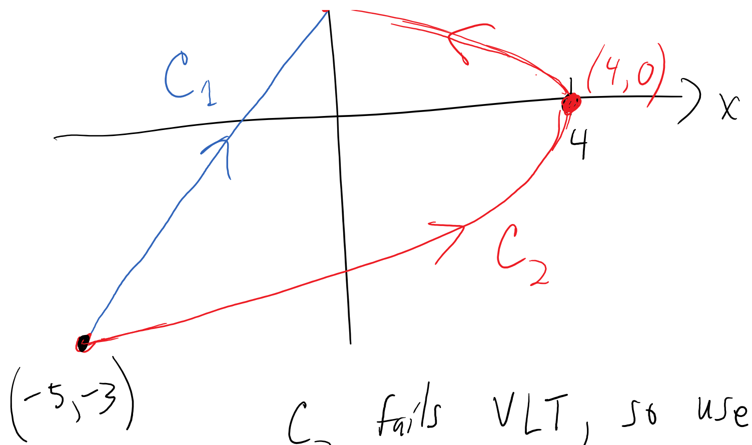
(b)



C_2:

$$x = 4 - y^2$$

$$= x(y)$$



C_2 fails VLT, so use y as param.:

$$\vec{r}(y) = \langle \underbrace{4-y^2}_{\overset{\shortparallel}{x}}, y \rangle, \quad -5 \le y \le 2, \quad dx = -2y\,dy$$

$$\int_{C_2} y^2\,dx + x\,dy = \int_{-5}^{2} y^2(-2y\,dy) + (4-y^2)\,dy$$

$$= \int_{-5}^{2} -2y^3 - y^2 + 4\,dy$$

$$= \left[-\frac{1}{2}y^4 - \frac{y^3}{3} + 4y\right]_{-5}^{2}$$

$$= -\frac{1}{2}2^4 - \frac{2^3}{3} + 4(2) - \left(-\frac{1}{2}(-5)^4 - \frac{(-5)^3}{3} + 4(-5)\right)$$

$$= \boxed{\frac{245}{6}}$$

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* Rmk 1) In general, the value of a line integral depends not only on endpoints, but on path as well.

2) Given a parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, this determines an orientation of the curve C, with positive direction corresp. to increasing values of t.

3) If $-C$ has same points as C but opposite orientation, then

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx$$

$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

$$\int_{-C} f(x,y)\,\underline{ds} = \int_C f(x,y)\,\underline{ds}$$

Line Integrals in Space

- C smooth space curve def. by $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \le t \le b$ (i.e. $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$).
- If a func. $f(x,y,z)$ is continuous on a domain D

with $C \subset D$, the line integral of f along C

is $$\int_C f(x,y,z)\, ds = \lim_{\max \Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

$$= \int_a^b f(x(t), y(t), z(t)) \left| \vec{r}\,'(t) \right| dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\; dt$$

$$= \int_a^b f(\vec{r}(t)) \left| \vec{r}\,'(t) \right| dt$$

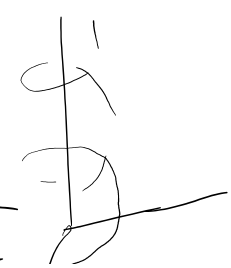
$z = z(t) \Rightarrow dz = z'(t)\, dt$

Def. $$\int_C f(x,y,z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

Common: $$\int_C P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz$$

Ex. Evaluate $\int_C y\sin(z)\, ds$, C = circular helix def. by

$x = \cos(t)$, $y = \sin(t)$, $z = t$, $0 \le t \le 2\pi$.

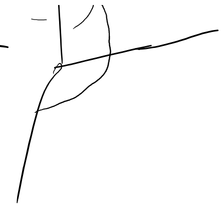


Sol'n: $z'(t) = 1$

$\int^{2\pi} \ldots \left| \vec{r}\,'(t) \right| dt$ $\qquad \left| \vec{r}\,'(t) \right| = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1}$

Soln.

$$\int_0^{2\pi} \sin(t)\sin(t)\,|\vec{r}\,'(t)|\,dt, \qquad |\vec{r}\,'(t)| = \sqrt{(-\sin(t))^2+\cos^2(t)+1}$$
$$= \sqrt{1+1}$$
$$= \sqrt{2}$$



$$= \int_0^{2\pi} \sin^2(t)\sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt$$

$$= \frac{\sqrt{2}}{2}(2\pi - 0)$$

$$= \boxed{\pi\sqrt{2}}$$

<u>Fact</u> $L = \int_C 1\,ds$ = length of C $= \int_a^b |\vec{r}\,'(t)|\,dt$

<u>Line Integrals of Vector Fields</u>

Before, integrated $f: \mathbb{R}^n \longrightarrow \mathbb{R}$

<u>Def.</u> Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a continuous force [vector] field on $\mathbb{R}^3$ (so $\vec{F}: \mathbb{R}^3 \longrightarrow V_3$).

The <u>work</u> done by $\vec{F}$ in moving a particle along a smooth curve C in $\mathbb{R}^3$ is

$$W = \int \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,ds = \int_C \vec{F}\cdot\vec{T}\,ds,$$

$$W = \int_C \vec{F}(x,y,z) \bullet \vec{T}(x,y,z)\, ds = \int_C F \bullet T\, ds,$$

where $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, $\vec{r}(t)$ parametrizes C, $a \le t \le b$.

(Note Comes from $W = \vec{F} \bullet \vec{D}$, $\vec{D}$ = displacement)

Def. Let $\vec{F}$ continuous vect. field on C, where C param. by $\vec{r}(t)$, $a \le t \le b$. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

Note $W = \int_C \vec{F} \bullet \vec{T}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \bullet \dfrac{\vec{r}\,'(t)}{\cancel{|r'(t)|}} \cancel{|\vec{r}\,'(t)|}\, dt$

$$= \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

Ex. Find $\int_C \vec{F} \bullet d\vec{r}$ if $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$ and $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

Sol'n: $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(t, t^2, t^3) \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_0^1 t^3 + 2t^6 + 3t^6 \, dt$$

$$= \int_0^1 t^3 + 5t^6 \, dt$$

$$= \frac{1}{4} + \frac{5}{7}$$

$$= \boxed{\frac{27}{28}}$$

Rmk $\vec{F}(x, y, z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ param. C

$a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a^b \langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_a \left\langle P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)) \right\rangle \cdot \left\langle x'(t), y'(t), z'(t) \right\rangle dt$$

$$= \int_a^b \left[P(x(t), y(t), z(t))\, x'(t) + Q(x(t), y(t), z(t))\, y'(t) + R(x(t), y(t), z(t))\, z'(t) \right] dt$$

$$= \int_a^b P(x(t), y(t), z(t))\, x'(t)\, dt + Q(x(t), y(t), z(t))\, y'(t)\, dt + R(x(t), y(t), z(t))\, z'(t)\, dt$$

$$= \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

Note these results also hold in 2-D.