3.3 Homogeneous Linear Equations with Constant Coefficients

Reduction is order is a useful technique if we happen to know (or can easily find) a solution. However, it is very rare that a starting solution is given to us. To start the exploration of how to find our solution(s), we will start by looking at homogeneous linear equations with constant coefficients.

Consider the second-order equation $ay'' + by' + cy = 0$, where a, b, and c are constants. We will start by attempting to find a solution of the form $\boldsymbol{y = e^{mx}}$. After substitution, we get

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \rightarrow \quad e^{mx}(am^2 + bm + c) = 0.$$

Since $e^{mx} \neq 0$ for all x, then the only way for $y = e^{mx}$ to be a solution is for m to be a root of the equation

$$\boldsymbol{am^2 + bm + c = 0}.$$

This equations is called the **auxiliary equation** of the differential equation. Solving for the roots m_1 and m_2 results in one of three cases.

Case I: Distinct Real Roots: In this case, we get two linearly independent solutions $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$. Since this produces a fundamental set of solutions, or general solution is

$$\boldsymbol{y = c_1e^{m_1x} + c_2e^{m_2x}}.$$

Case II: Repeated Real Roots: When $m_1 = m_2$, we only get one solution $y_1 = e^{m_1x}$. However, we need a second solution in order to have a fundamental set of solutions. Thankfully, if we have one solution, we can find a second solution of the form $y_2(x) = u(x)e^{m_1x}$ by using reduction of order. After going through the work (which is left as an exercise), we find that $y_2 = xe^{m_1x}$, giving us a general solution of

$$\boldsymbol{y = c_1e^{m_1x} + c_2xe^{m_1x}}.$$

Case III: Complex Conjugate Roots: If m_1 and m_2 are complex conjugates, then we can write them as $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\beta > 0$. Formally, this is the same as in case I, and we get the general solution

$$\boldsymbol{y = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}}.$$

However, in practice it is usually easier to work with real-valued functions. To accomplish this we can use Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$, giving us

$$e^{i\beta x} = \cos\beta x + i\sin\beta x\,, \qquad e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

By adding and subtracting these two equations, we get

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x\,, \qquad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Since $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$ produce a fundamental set of solutions, any linear combination of them would also be a solution. Thus,

$$y_1 = \frac{1}{2}e^{(\alpha+i\beta)x} + \frac{1}{2}e^{(\alpha-i\beta)x} = \frac{1}{2}e^{\alpha x}\left(e^{i\beta x} + e^{-i\beta x}\right) = e^{\alpha x}\cos\beta x$$

and

$$y_2 = -\frac{i}{2}e^{(\alpha+i\beta)x} + \frac{i}{2}e^{(\alpha-i\beta)x} = -\frac{i}{2}e^{\alpha x}\left(e^{i\beta x} - e^{-i\beta x}\right) = e^{\alpha x}\sin\beta x$$

are solutions and form a fundamental set of solutions. Therefore, the general solution is

$$\boldsymbol{y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)}.$$

**Since the process for solving homogeneous linear equations with constant coefficients results in solving the auxiliary equation, it is common practice to just jump straight to the auxiliary equation. However, do not forget what the original "guess" was to get to the auxiliary equation.*

- *Example*: Solve the following differential equations.

  (a) $2y'' - 7y' + 3y = 0$

  Solution: Solving the auxiliary equation we get

  $$2m^2 - 7m + 3 = 0 \quad \rightarrow \quad m_1 = 3, \qquad m_2 = \frac{1}{2},$$

  giving us a general solution of

  $$\boldsymbol{y = c_1 e^{3x} + c_2 e^{x/2}}.$$

  (b) $y'' + 6y' + 9y = 0$

  Solution: Solving the auxiliary equation we get

  $$m^2 + 6m + 9 = 0 \quad \rightarrow \quad m_1 = m_2 = -3,$$

  giving us a general solution of $\boldsymbol{y = c_1 e^{-3x} + c_2 x e^{-3x}}$.

$$\textbf{(c) } y'' + 4y' + 7y = 0$$

Solution: Solving the auxiliary equation we get

$$m^2 + 4m + 7 = 0 \quad \rightarrow \quad m_1 = -2 + i\sqrt{3}, \qquad m_2 = -2 - i\sqrt{3},$$

giving us a general solution of

$$\boldsymbol{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}.$$

In general, if we wish to solve an n^{th}-order linear equation with constant coefficients, we need to solve an n^{th}-degree polynomial equation

$$a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

If all of the roots are distinct, then $\{e^{m_1 x}, e^{m_1 x}, \dots, e^{m_n x}\}$ forms a fundamental set of solutions.

When there is a repeated root m_1, of multiplicity k, then the fundamental set should contain the linearly independent solutions

$$e^{m_1 x}, \;\; xe^{m_1 x}, \;\; x^2 e^{m_1 x}, \dots, x^{k-1} e^{m_1 x}.$$

When there is a (pair of) repeated complex roots $m_1, m_2 = \alpha \pm i\beta$, of multiplicity k, then the fundamental set should contain the linearly independent solutions

$$e^{\alpha x}\cos\beta x\,, \;\; xe^{\alpha x}\cos\beta x\,, \dots, x^{k-1}e^{\alpha x}\cos\beta x,$$

$$e^{\alpha x}\sin\beta x\,, \;\; xe^{\alpha x}\sin\beta x\,, \dots, x^{k-1}e^{\alpha x}\sin\beta x.$$

- *Example:* Solve the following equations

$$\textbf{(a) } y''' - y'' - y' + y = 0.$$

Solution: The auxiliary equation gives us

$$m^3 - m^2 - m + 1 = 0 \quad \rightarrow \quad (m+1)(m-1)^2 = 0 \quad \rightarrow \quad m_1 = -1, \qquad m_2 = m_3 = 1.$$

Therefore our fundamental set of solutions would be $\{e^{-x},\; e^x,\; xe^x\}$, giving a general solution

$$\boldsymbol{y = c_1 e^{-x} + c_2 e^x + c_3 x e^x}.$$

$$\textbf{(b) } y^{(4)} + 2y'' + y = 0.$$

Solution: The auxiliary equation gives us

$$m^4 + 2m^2 + 1 = 0 \quad \to \quad (m^2+1)^2 = 0 \quad \to \quad m_1, m_2 = m_3, m_4 \pm i.$$

Therefore our fundamental set of solutions would be $\{\cos x,\ x\cos x, \sin x,\ x \sin x\}$, giving a general solution

$$\boldsymbol{y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.}$$

**Note: the order of the solutions in the general solution does not matter.*