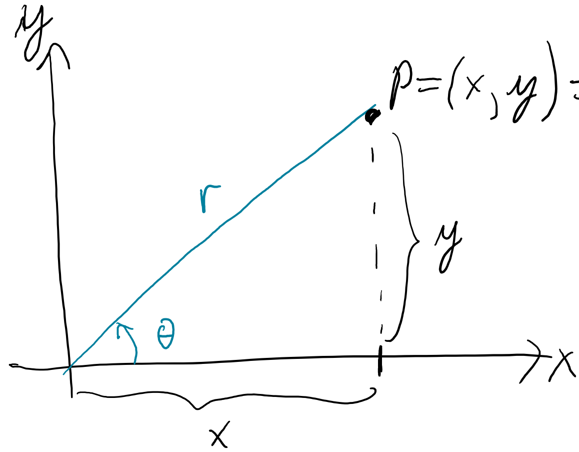
12.6: Triple Integrals in Cylindrical Coordinates

Thursday, October 8, 2020 10:16 AM



$P = (x, y) = (r, \theta)$ in polar

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$r^2 = x^2 + y^2$

$\tan(\theta) = \frac{y}{x}$

(In 2D: polar)

Cylindrical Coord. (3-D)

$(x, y, z) \longrightarrow (r, \theta, z)$

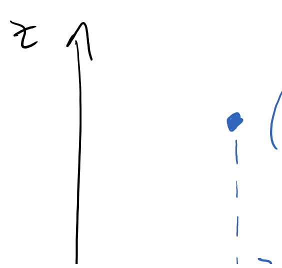
cartesian → cylindrical

$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$

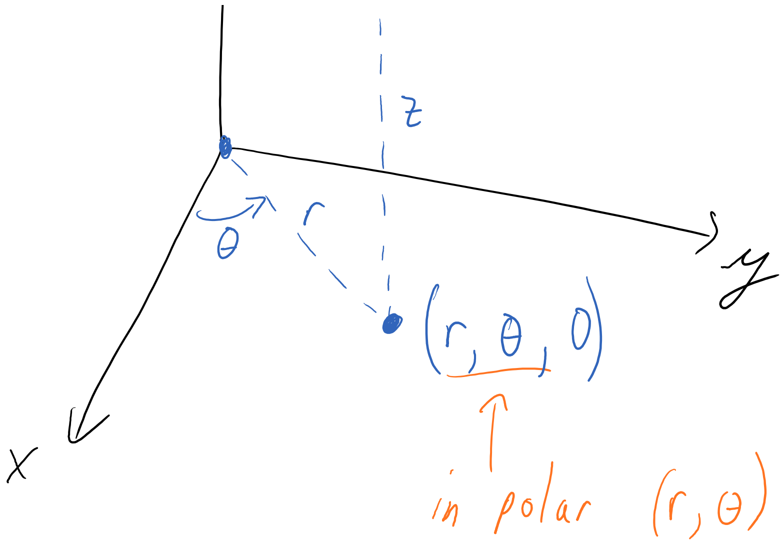
(from cyl. to cartesian)

$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$

(Cart. to cyl.)



(r, θ, z) in cyl. coord.

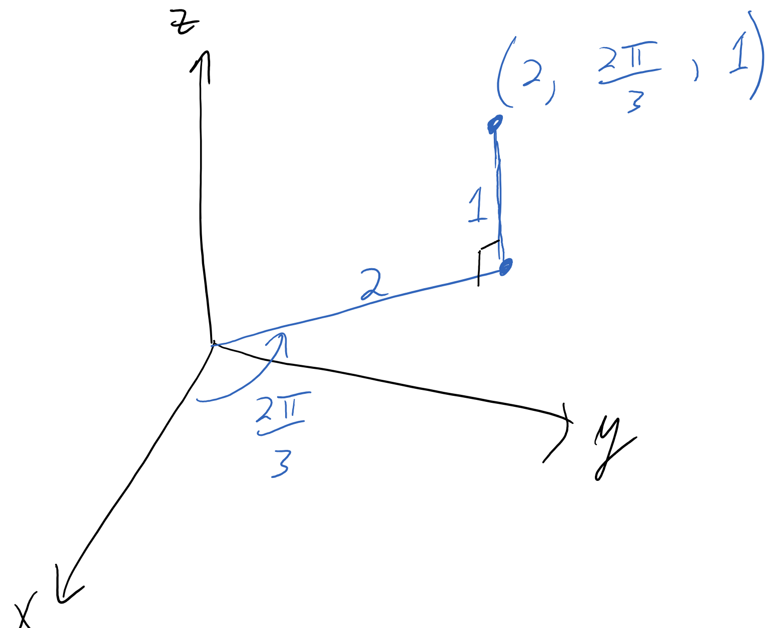


Ex. 1

(a) Plot the point w/cyl. coord. $\left(2, \frac{2\pi}{3}, 1\right)$ and find its rectangular coord.

(b) Find cyl. coord. of the rect. point $(3, -3, -7)$.

Sol'n. (a)



$x^{\checkmark}$

$$x = 2\cos\left(\frac{2\pi}{3}\right) = 2\left(\frac{-1}{2}\right) = -1$$

$$y = 2\sin\left(\frac{2\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

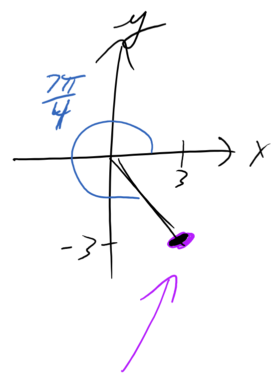
$$\therefore \boxed{\left(-1,\ \sqrt{3},\ 1\right)}$$

(b) $r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$

$(3, -3, -7)$

$$= \sqrt{3^2 + (-3)^2}$$

$$= 3\sqrt{2}$$



$$\tan(\theta) = \frac{-3}{3} = -1$$

$$\Rightarrow \theta = \frac{7\pi}{4}$$

$$z = -7$$

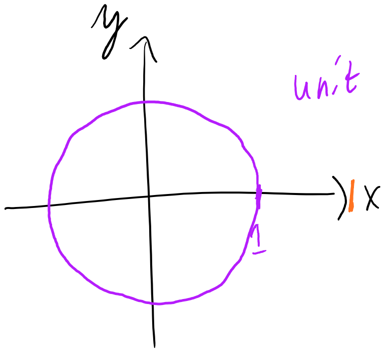
$$\therefore \boxed{\left(3\sqrt{2},\ \frac{7\pi}{4},\ -7\right)}$$

<u>Ex.</u> Describe surface w/ cyl. eq'n $z = r$.

Sol'n

$z = 1$

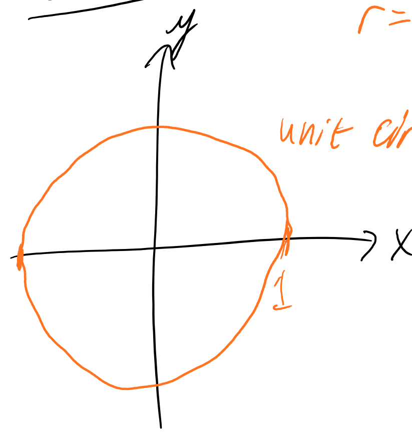
$r = 1$



unit circle

$z = -1$

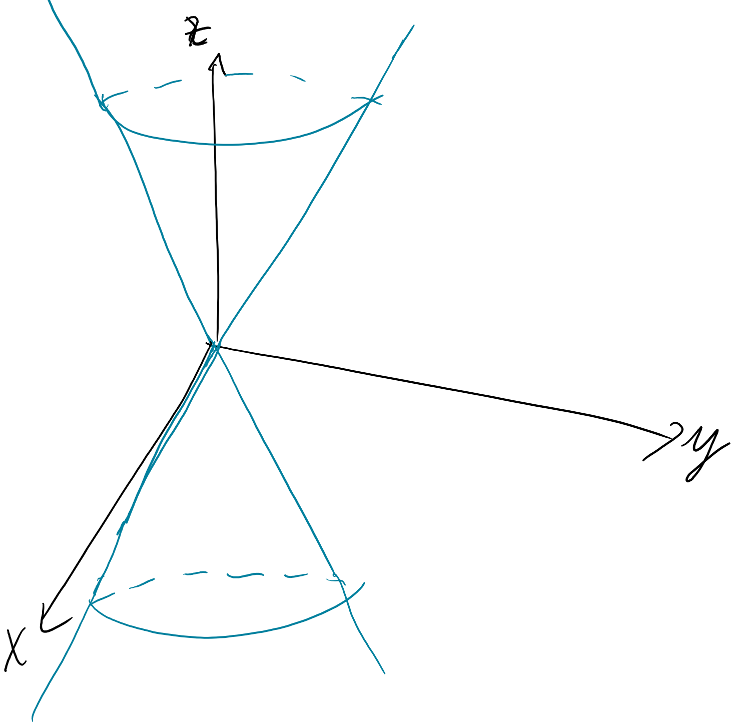
$r = -1$



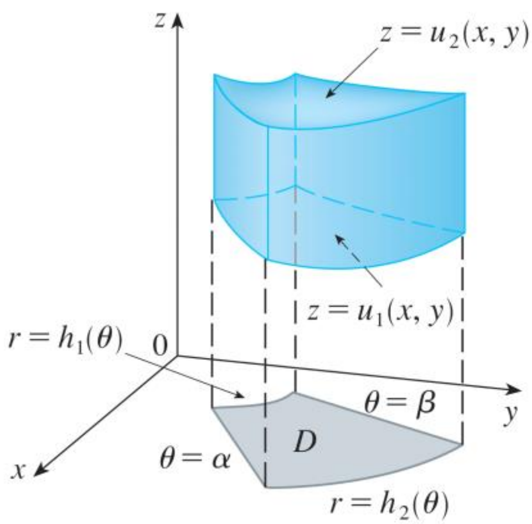
unit circle

$\Rightarrow$ cone

$$z^2 = r^2 = x^2 + y^2$$



Integration with Cyl. Coord.



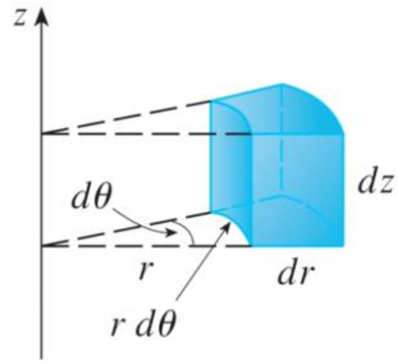


FIGURE 7
Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Say $E \subset \mathbb{R}^3$ a type I region easily described with cyl. coord. (i.e. D easily described in polar)

$$E = \{(x, y, z) : (x, y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

$$D = \{(r, \theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

Know $$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta),\, r\sin(\theta))}^{u_2(r\cos(\theta),\, r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\, r \,dz\,dr\,d\theta$$

from $\iint_D$ polar

$dV \approx r\,dz\,dr\,d\theta$

$dV \rightsquigarrow r\,dz\,dr\,d\theta$ from $\iint_D$ polar

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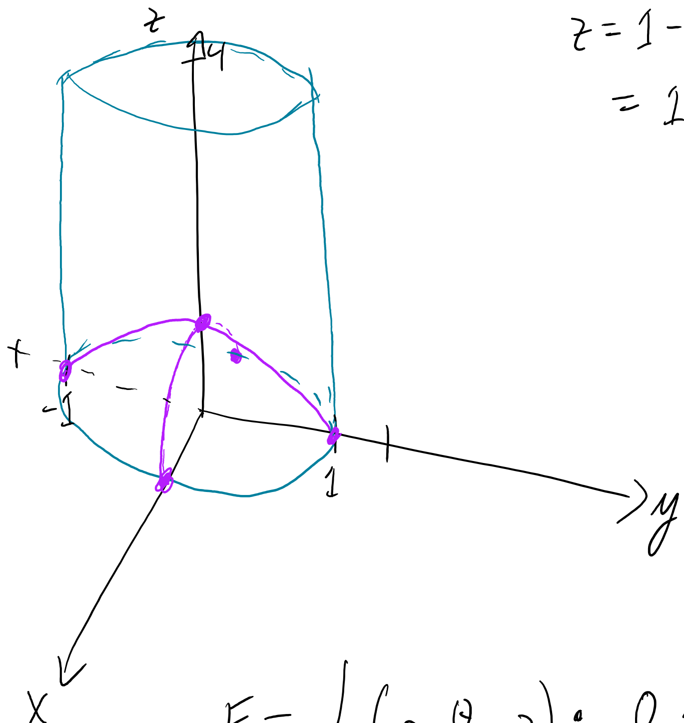
Ex. 3 A solid E lies within the cylinder $x^2+y^2=1$, below the plane $z=4$, and above paraboloid $z=1-x^2-y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Sol'n: In cyl. coord, $x^2+y^2=1 \longleftrightarrow r=1$,

$$z=1-x^2-y^2$$
$$=1-(x^2+y^2) \longleftrightarrow z=1-r^2$$
$$(z=0 \Rightarrow x^2+y^2=1)$$

$$z=4 \longleftrightarrow z=4$$



$$E=\{(r,\theta,z): 0\le\theta\le 2\pi,\ 0\le r\le 1,\ 1-r^2\le z\le 4\}.$$

density at (x,y,z) prop. to dist. from z-axis

$\Rightarrow$ density func. $f(x,y,z) = K\sqrt{x^2+y^2} = Kr$

$$\text{Mass} = \iiint_E f(x,y,z)\, dV = \iiint_E K\sqrt{x^2+y^2}\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 Kr^2(4-1+r^2)\, dr\, d\theta$$

$$= 2\pi K \int_0^1 3r^2 + r^4\, dr$$

$$= 2\pi K\left(1+\frac{1}{5}\right)$$

$$= \boxed{\frac{12\pi K}{5}}$$

Ex. 4 Evaluate $I = \displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$

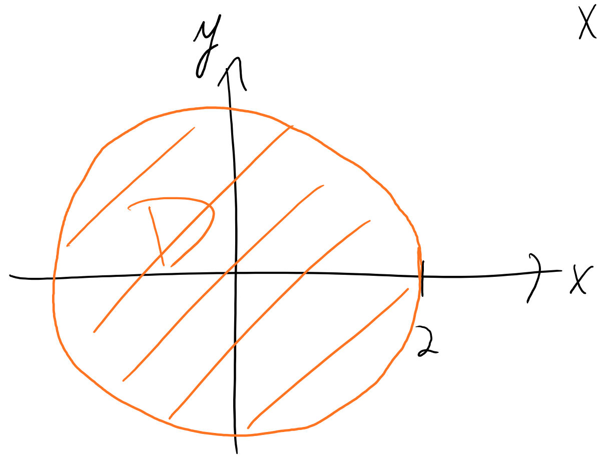
Sol'n: $E = \{(x,y,z) : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

Sol'n:

$$E = \left\{(x,y,z) : -2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ \sqrt{x^2+y^2} \le z \le 2\right\}$$

D: $-2 \le x \le 2$, $\quad y^2 \le 4 - x^2$

$x^2 + y^2 \le 4$

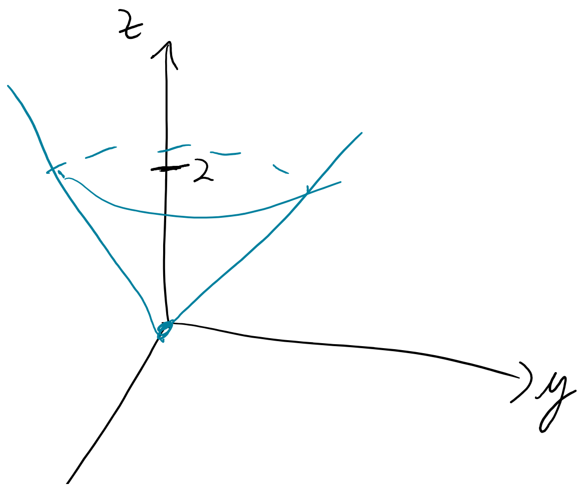


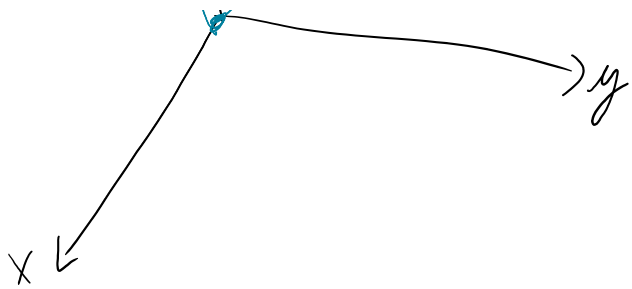
In E, bottom surface is $z = \sqrt{x^2+y^2}$

$\Rightarrow z^2 = x^2 + y^2 = r^2$

(upper half cone) $\Rightarrow z = r$

In E, top surf. is $z = 2$ (plane)





$$E = \left\{ (r, \theta, z) : 0 \le \theta \le 2\pi,\ 0 \le r \le 2, \right.$$
$$\left. r \le z \le 2 \right\}$$

$$\therefore\ I = \iiint_E x^2 + y^2 \, dV,$$

doesn't dep. on z

$$= \int_0^{2\pi} \int_0^2 \int_r^2 r^2 \, r \, dz \, dr \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^2 r^3 (2 - r) \, dr \right)$$

$$= 2\pi \left(\frac{8\pi}{5} \right)$$

$$= \boxed{\frac{16\pi}{5}}$$