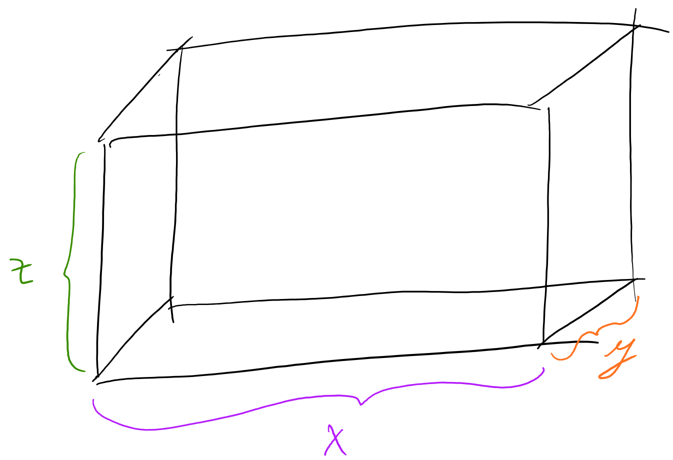
11.8: Lagrange Multipliers

Tuesday, September 22, 2020 11:52 AM

ex. A rectangular box w/o lid is to be made from 12 cm^2 of cardboard. Find the max. volume of such a box.

Sol'n: $V = xyz$, maximize V given constraint

$12 = xy + 2zy + 2xy$ (no lid)

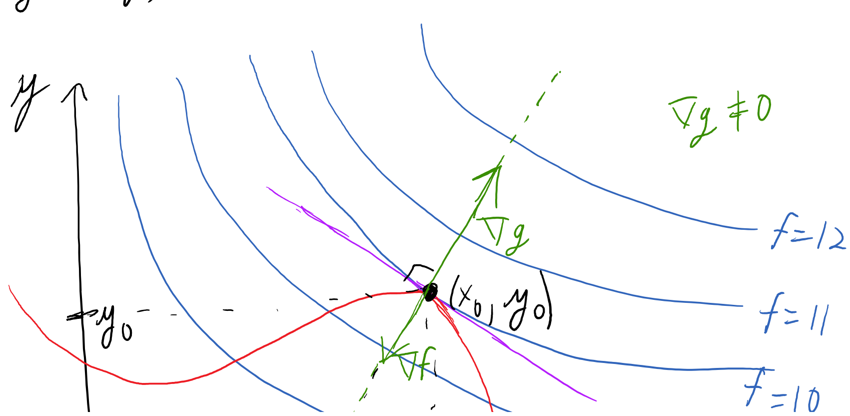


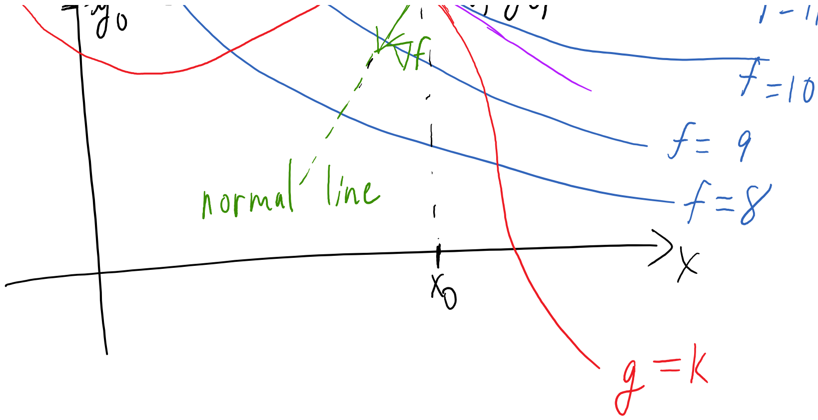
. . . ?

Lagrange Multipliers in 2-D

Find extreme values of $f(x, y)$ subject to a constraint $g(x, y) = k$ (which defines a level curve of g).

- So only consider (x, y) in domain of f where $g(x, y) = k$.





- To maximize $f(x,y)$ means find the largest value c s.t. $f(x,y)=c$ <u>and</u> $g(x,y)=k$.

- Happens when curves just touch each other, i.e. curves share a tan. line.

- If $\nabla g(x_0,y_0) \neq \vec{0}$, then $\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0)$ for some real λ.

<u>Def.</u> λ as above is a <u>Lagranga multiplier.</u>

<u>In 3D:</u>

Find extreme values of $f(x,y,z)$ subject to constraint $\underbrace{g(x,y,z)=k.}$ $\leftarrow$ level <u>surface</u> of g in 3D

- If the max. value of f is $f(x_0,y_0,z_0)=c$, then the level surface $f(x,y,z)=c$ is tangent to the

  " " $g(x,y,z)=k$, so the corresponding

gradient vectors are parallel:

Method of Lagrange Multipliers

To find max. & min. values of a func. $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming these extreme values exist & $\nabla g \neq \vec{0}$ on the surface $g(x, y, z)$):

(a) Find all (x, y, z) s.t. $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

and $g(x, y, z) = k$.

(b) Evaluate f at each point found in (a)

- largest f-value is max.
- smallest " " min.

Ex. $V = xyz$ subject to $12 = xy + 2xz + 2yz$.

Sol'n: Note $g(x, y, z) = xy + 2xz + 2yz$, $k = 12$,

$V(x, y, z) = xyz$.

(a) $\nabla V = \langle yz, xz, xy \rangle$

$\lambda \nabla g = \lambda \langle y + 2z, x + 2z, 2x + 2y \rangle$

$\nabla V = \lambda \nabla g \Rightarrow yz = \lambda(y + 2z)$

$$\nabla V = \underbrace{\lambda \nabla g} \Rightarrow \begin{cases} yz = \lambda(y+2z) \\ xz = \lambda(x+2z) \\ xy = \lambda(2x+2y) \\ * \; 12 = xy + 2xz + 2yz \quad (\text{constraint}) \end{cases}$$

- If $\lambda = 0 \Rightarrow \nabla V = \vec{0}$, which it's not. $\Rightarrow \lambda \neq 0$.
- If $x=0$ or $y=0$ or $z=0 \Rightarrow V=0 \Rightarrow V$ not maximized, so assume $x \neq 0$, $y \neq 0$, $z \neq 0$.

$$\begin{cases} xyz = x\lambda(y+2z) = \underline{\lambda(yx+2zx)} \\ xyz = y\lambda(x+2z) = \lambda(yx+2zy) \\ xyz = z\lambda(2x+2y) = \underline{\lambda(2xz+2yz)} \end{cases}$$

$$\lambda(yx+2zx) = \lambda(yx+2zy)$$

$$yx + 2zx = yx + 2zy \quad , \lambda \neq 0$$

$$\underline{x = y} \quad , z \neq 0$$

$$\lambda(x^2 + 2zx) = \lambda(2xz + 2xz)$$

$$x^2 + 2zx = 4xz \quad , \quad \lambda \neq 0$$

$$x^2 = 2xz \quad , \quad x \neq 0$$

$$x = 2z$$

$$z = \frac{1}{2}x$$

$$\Rightarrow * \quad 12 = x^2 + \cancel{2}x\left(\frac{x}{\cancel{2}}\right) + \cancel{2}x\left(\frac{x}{\cancel{2}}\right) = 3x^2$$

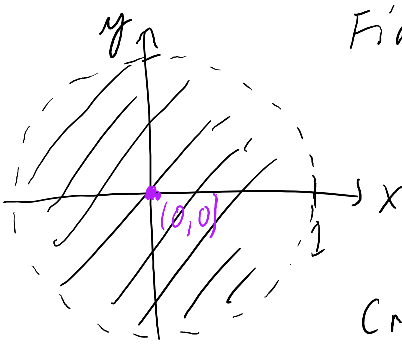
$$\Rightarrow x^2 = 4 \quad \Rightarrow x = \pm 2 \quad \Rightarrow x = 2$$

$$\Rightarrow y = 2, \quad z = 1$$

$\therefore$ Only sol'n is $(2, 2, 1)$.

(b) So $V(2, 2, 1) = 2(2)(1) = \boxed{4\text{ cm}^3 = \text{max. volume.}}$

Ex. Find extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \le 1$.



First, consider $x^2 + y^2 < 1$, deal w/ boundary $x^2 + y^2 = 1$ later.

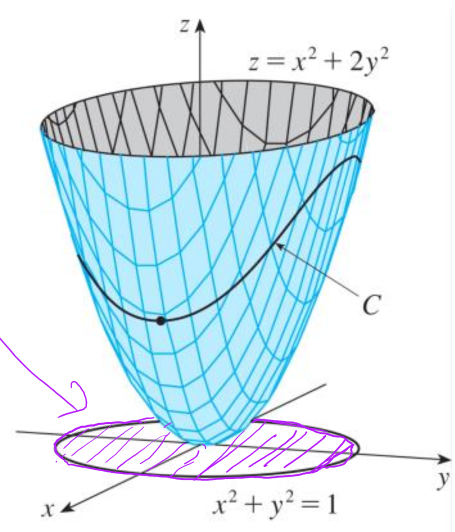


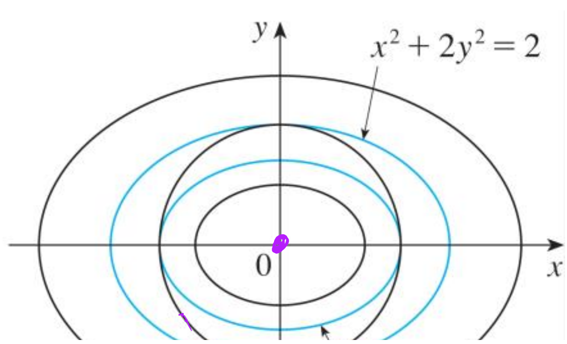
FIGURE 2

Crit. points:

$$f_x = 2x = 0 \Rightarrow x = 0$$

$$f_y = 4y = 0 \Rightarrow y = 0$$

$\therefore$ $(0, 0)$ is only crit. p't



$\therefore$ $(0,0)$ is only crit. pt

$\rightsquigarrow f(0,0) = 0$

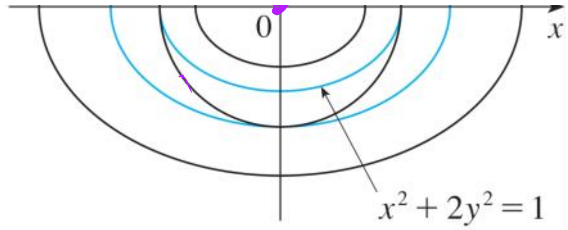


FIGURE 3

<u>Now look at ∂D</u>: $x^2 + y^2 = 1$, use

$1 = x^2 + y^2 = g(x,y)$ as constraint for Lagrange.

$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$

Clearly $\lambda \neq 0$.

$\cancel{2}x = \cancel{2}\lambda x \Rightarrow x = \lambda x.$

- If $x = 0$, then $1 = y^2$
  $\Rightarrow y = \pm 1$, so test
  $(0, \pm 1)$

- If $x \neq 0$, then $\lambda = 1$
  $\Rightarrow 4y = \lambda 2y = 2y$
  $\Rightarrow y = 0 \Rightarrow 1 = x^2$
  $\Rightarrow x = \pm 1$,
  so test $(\pm 1, 0)$

For Lag. mult, solutions are $\{(0, \pm 1), (\pm 1, 0)\}$

$f(x,y) = x^2 + 2y^2$, $f(0, \pm 1) = 2$

$f(\pm 1, 0) = 1$

$\therefore$ $\min(f) = 0$, $\max(f) = 2$

$\therefore$ $\boxed{\min(f) = 0, \quad \max(f) = 2}$

Note Did method from 11.7, using Lag. for 2D.

Ex. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are farthest from the point $(3, 1, -1)$.

Sol'n: maximize

$d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2 = f(x, y, z)$ subject to

constraint $x^2 + y^2 + z^2 = 4$, so $g(x, y, z) = x^2 + y^2 + z^2$.

(a) $\nabla f = \lambda \nabla g \Rightarrow$

$$2(x-3) = 2x\lambda$$
$$2(y-1) = 2y\lambda$$
$$2(z+1) = 2z\lambda$$

$$* \quad x^2 + y^2 + z^2 = 4$$

Solve for x, y, z in terms of λ:

$x - 3 = x\lambda$

$x(1-\lambda) = 3$

$x = \dfrac{3}{1-\lambda}$

If $1 - \lambda = 0$

$\lambda = 1$

$\Rightarrow 2(x-3) = 2x$ ※

$\Rightarrow \lambda \neq 1$

$y = \dfrac{1}{1-\lambda}$, $\quad z = \dfrac{-1}{1-\lambda}$

$$* \Rightarrow \frac{3^2}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 4$$

quadr. formula $\Rightarrow$ $\lambda = 1 \pm \frac{\sqrt{11}}{2}$

$$\rightsquigarrow P_0 = \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right), \left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right) = P_1$$

$\therefore$ P_0 closest point on sphere to P, P_1 farthest point on sphere from P

Two Constraints

- Find extreme values of $f(x, y, z)$ subject to $g(x, y, z) = k$ and $h(x, y, z) = c$.
- (x, y, z) restricted to curve of intersection of level surfaces.
- If f has an extreme value at $(x_0, y_0, z_0) = P$, then ∇f orthogonal to curve C at P.

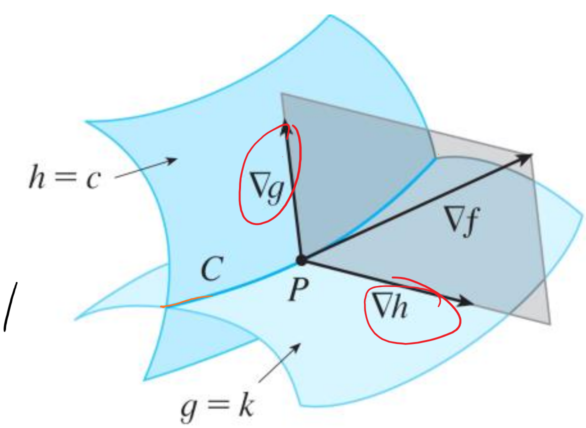


FIGURE 5

Also ∇g ortho to $g(x, y, z) = k$

FIGURE 5

Also ∇g orthog. to $g(x,y,z)=k$

and ∇h " " $h(x,y,z)=c$.

$\Rightarrow \nabla f(x_0, y_0, z_0)$ in plane determined by

$\nabla g(x_0, y_0, z_0)$ & $\nabla h(x_0, y_0, z_0)$

(assuming $\nabla g(P)$ & $\nabla h(P)$ non-zero, non-parallel)

$\Rightarrow \exists$ numbers λ, μ s.t.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0),$$

i.e.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x,y,z) = k$$

$$h(x,y,z) = c$$

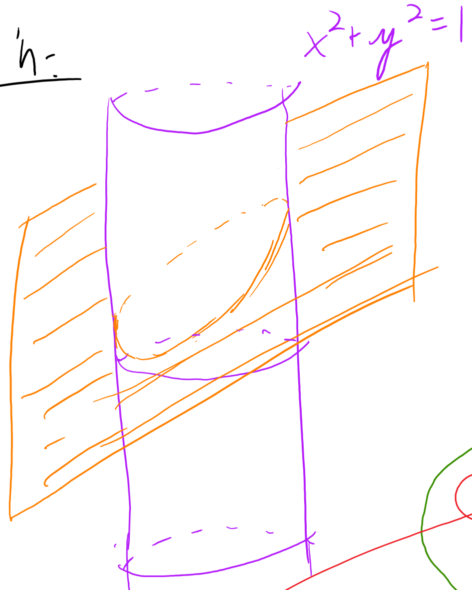
Ex. Find max. of $f(x,y,z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ & the cylinder $x^2 + y^2 = 1$.

$\nabla f = \langle 1, 2, 3 \rangle$

$x-y+z=1$ & the cylinder $x^2+y^2=1$.

$\nabla f = \langle 1, 2, 3 \rangle$

Sol'n:



$g = x-y+z = 1, \quad \nabla g = \langle 1, -1, 1 \rangle$

$h = x^2+y^2 = 1, \quad \nabla h = \langle 2x, 2y, 0 \rangle$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$1 = \lambda + 2x\mu$$

$$2 = -\lambda + 2y\mu$$

$$3 = \lambda$$

$$x - y + z = 1$$

$$x^2 + y^2 = 1$$

$$2x\mu = -2 \Rightarrow x = -1/\mu, \quad \mu \neq 0$$

$$y = \frac{5}{2\mu}$$

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1 \Rightarrow \mu = \pm\frac{\sqrt{29}}{2}$$

$$\Rightarrow x = \mp\frac{2}{\sqrt{29}}, \quad y = \pm\frac{5}{\sqrt{29}}$$

$$z = 1 - x + y = 1 \pm 7/\sqrt{29}$$

$\rightsquigarrow$ f-values are $3 \pm \sqrt{29}$

$\therefore \boxed{\max = 3 + \sqrt{29}}$