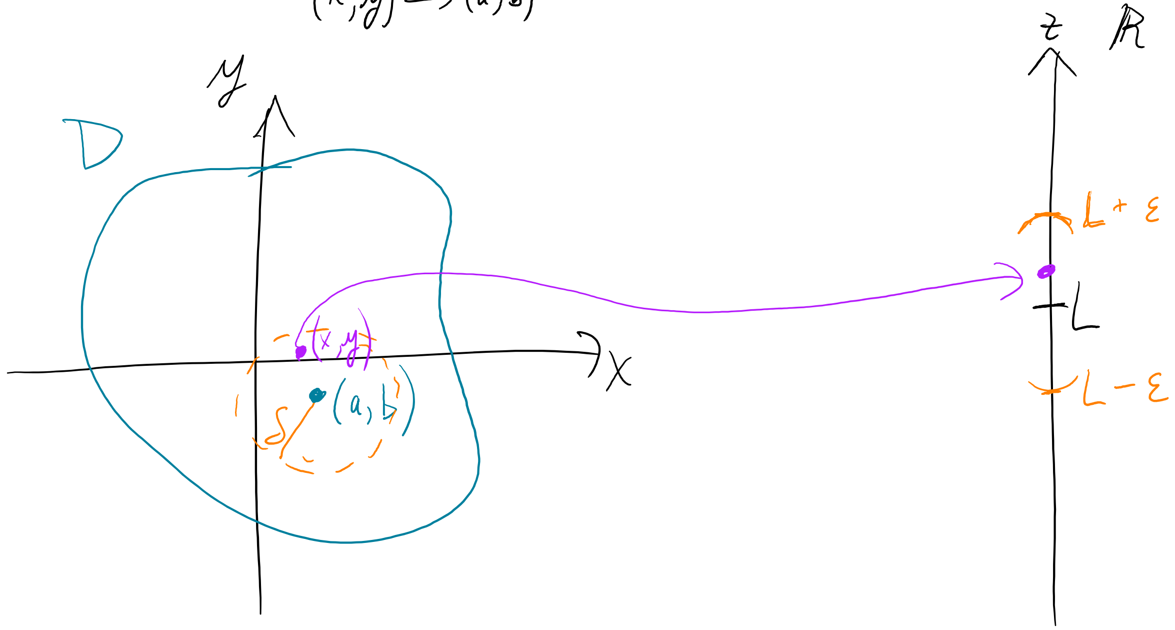
11.2: Limits and Continuity

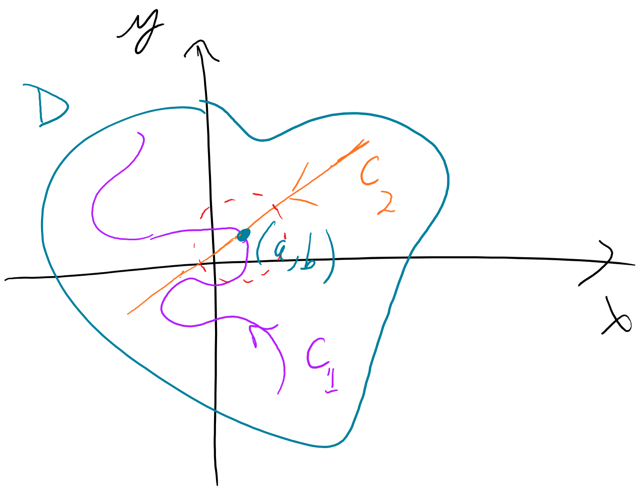
Thursday, September 3, 2020 11:37 AM

$f: D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^2$ (i.e. $f(x,y)$)

<u>Def.</u> If there is a real constant L s.t. for every $\varepsilon > 0$, there is $\delta > 0$ s.t. $\sqrt{(x-a)^2 + (x-b)^2} < \delta$ implies $|f(x,y) - L| < \varepsilon$, say L is the <u>limit of f</u> at (a,b), and $\lim\limits_{(x,y) \to (a,b)} f(x,y) = L$.



<u>Fact</u> If there are $L_1 \neq L_2$, and paths C_1, C_2 s.t. $f(x,y) \longrightarrow L_1$ along C_1, and $f(x,y) \longrightarrow L_2$ along C_2, where C_1 and C_2 both lead to (a,b), $(x,y) \to (a,b)$, then $\lim\limits_{(x,y) \to (a,b)} f(x,y)$ DNE.



If (x,y) on C_1 and $f(x,y) \to L_1$

and $f(x,y) \to L_2$ for (x,y) on C_2,

and $L_1 \neq L_2$.

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ex. Does $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ have a limit at $(0,0)$?

If so, find it. If not, show it DNE.

Note f not defined at $(0,0)$ $\rightsquigarrow$ can't plug it in.

$x=0$

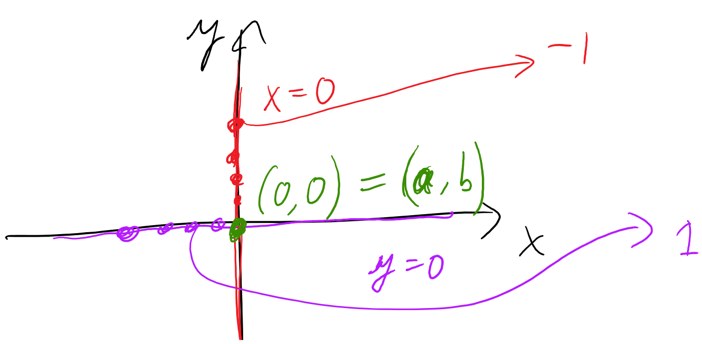
$$f(x,y) = \frac{-y^2}{y^2} = -1$$

$f(x,y) \longrightarrow -1$ along $x=0$

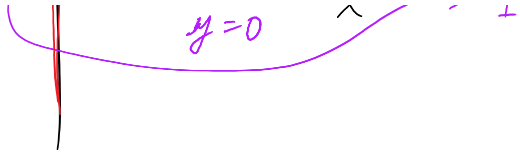
$y=0$

$$f(x,y) = \frac{x^2}{x^2} = 1$$

$f(x,y) \longrightarrow 1$ along $y=0$



$1 \neq -1$



$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DNE}.$$

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ex. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2-y^2}{x^2+y^2} = \boxed{-1}$ by plugging in $x=0,\ y=1.$

i.e. $f(a,b) = f(0,1) = -1.$ Also f cont. everywhere but $(0,0)$.

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ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$?

(makes denom. one term, so can cancel)

$x=0$	$y=x$
$f(x,y)=0$	$f(x,y) = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$
$f(x,y)\to 0$	$f(x,y) \to \dfrac{1}{2}$

$\frac{1}{2} \neq 0 \implies$ limit $\boxed{\text{DNE}}$

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ex. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$?

$x^2 \rightsquigarrow y^4$?

$x=0$	$x=y^2$
$f(x,y)=0$	$f = \dfrac{y^2y^2}{(y^2)^2+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$

$x = 0$

$f(x,y) = 0$

$x = y^2$

$$f = \frac{y^2 y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$$

$\frac{1}{2} \neq 0 \Rightarrow \lim \boxed{DNE}$

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ex. $\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2+y^2}$ $\qquad |f(x,y)| \to 0 \iff f(x,y) \to 0.$

$x=0$	$y=0$	$y=x$
$f=0$	$f=0$	$f \to 0$

... if lim. exists, must be $L = 0$.

$$0 \le \left| \frac{3x^2 y}{x^2+y^2} \right| = \frac{3x^2|y|}{x^2+y^2} \le \frac{3x^2|y|}{x^2}$$

$$= 3|y| \xrightarrow{(x,y)\to 0} 0$$

$\therefore \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0.}$

Continuity

Def 1) $f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, $(a,b) \in D$. We say f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

f is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

2) f continuous on D if f cont. at (a,b) for all $(a,b)\in D$.

Fact • Polynomial functions are cont. on $\mathbb{R}^2$

• Rational " " " on their domains.

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ex. Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ cont.?

Sol'n: f rational $\Rightarrow$ f cont. all $\mathbb{R}^2$ except $(0,0)$

$$\boxed{\mathbb{R}^2 \setminus \{(0,0)\}}$$

Func of ≥ 3 Variables

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$

Def. f has a limit at $(a_1, a_2, \dots, a_n) \in D$ if there is a # L s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$ where

$$\sqrt{(x_1-a_1)^2 + (x_2-a_2)^2 + \dots + (x_n-a_n)^2} < \delta \Rightarrow |f(x_1,\dots,x_n) - L| < \varepsilon.$$