Unit 3.4 Alternating Series

Consider the series $\quad -\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ whose terms alternate between positive and negative. Such a series is called an alternating series and this series in particular is known as the *alternating harmonic series* as it closely resembles the harmonic series that we encountered earlier. The series given above can be represented using sigma notation as $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. In general we will express an alternating series in the form

$$\sum_{n=1}^{\infty}(-1)^{n}\cdot a_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1}\cdot a_n \qquad \text{where } a_n>0.$$

Whether the -1 has a power of n or $n+1$ has to do with getting the signs of the terms to alternate in the correct order. You should also be reminded that our series need not begin at $n=1$. Also notice that in this form, a_n does not represent the general term, but instead it represents the magnitude of the general term as it does not include the ± 1 factor. The following series test can often be used to show that an alternating series converges.

The Alternating Series Test:

Given that $a_n>0$

If 1. $a_{n+1}\le a_n$ (i.e. the magnitudes of the terms decrease)

and 2. $\lim\limits_{n\to\infty} a_n$ (i.e. the magnitudes of the terms decay/limit to zero)

$$\text{then } \sum_{n=k}^{\infty}(-1)^{n}\cdot a_n \quad \text{and} \quad \sum_{n=k}^{\infty}(-1)^{n+1}\cdot a_n \qquad \text{converge}$$

Note: If either condition above fails, no conclusion can be made about the series using the Alternating Series Test. If condition 1 fails, the series may converge or may diverge and another series test must be used to determine which is the case. If condition 2 fails, then the series will diverge, but not by the alternating series test. One test that would immediately apply in that situation is the n^{th}-term test.

Examples 1 & 2 Determine whether each series converges or diverges.

. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\quad 1-\frac{}{2}+\frac{}{3}-\frac{}{4}+\cdots \qquad (\textit{the alternating harmonic series})$

Solution: We first note that this is in fact an alternating series and identify $a_n = 1/n$. We now observe that

$$\underbrace{\frac{1}{n+1} < \frac{1}{n}}_{a_{n+1} < a_n} \qquad \text{and} \qquad \underbrace{\lim_{n\to\infty} \frac{1}{n} = 0}_{\lim\limits_{n\to\infty} a_n = 0}$$

Thus by the alternating series test, the given series $\boxed{\text{converges}}$.

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2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^2}{n^2+5}$

Solution: While this is an alternating series with $a_n = \frac{n^2}{n^2+5}$, we observe that

$$\lim_{n\to\infty} \frac{n^2}{n^2+5} \overset{\div n^2}{=} \lim_{n\to\infty} \frac{1}{1+\frac{5}{n^2}} = 1 \neq 0.$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, the alternating series test does not apply. However, the n^{th} term test applies and we conclude that the given series $\boxed{\text{diverges}}$. Note: $a_n \nrightarrow 0 \implies (-1)^n a_n \nrightarrow 0$.

The Remainder of an Alternating Series. Once again, S_n denotes the n^{th} partial sum and R_n denotes the n^{th} remainder. Given an alternating series where $a_{n+1} \leq a_n$ and $\lim\limits_{n\to\infty} a_n = 0$,

$$|R_n| \leq a_{n+1} \quad \text{(the error is less than the magnitude of the first neglected term)}$$

In addition, if we let S denote the sum of the alternating series, then we will have

$$S_n \leq S \leq S_{n+1} \quad \text{or} \quad S_{n+1} \leq S \leq S_n$$

That means that the true sum of the series, will always lie between two consecutive partial sum estimates. You are referred to the book and/or narrated examples for an explanation of where the above inequalities come from. However, we provide a basic explanation of the idea here. An alternating series can be viewed as a series of alternating additions and subtractions of positive numbers. If those numbers decrease to a limiting value of zero, then every time we add, we will go above S (the sum of the series), and every time we subtract we go below S. If we are doing a partial sum where we ended with an addition, then we will have an overestimate; if we end with a subtraction, we will have an underestimate. To illustrate the idea, let us revisit the alternating harmonic series,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. The first partial sum estimate will consist of just the first term of the series 1. Because this term is positive, 1 will be an overestimate of the sum of the entire series. The second partial sum estimate is $1 - 1/2 = 1/2$, which is an underestimate since we ended with a subtraction. Thus we know the exact value of the series is between $1/2$ and 1. Below we show several additional partial sums

$$\underbrace{S_1 = 1}_{\text{over}}, \quad \underbrace{S_2 = \frac{1}{2}}_{\text{under}}, \quad \underbrace{S_3 = \frac{5}{6}}_{\text{over}}, \quad \underbrace{S_4 = \frac{7}{12}}_{\text{under}}, \quad \underbrace{S_5 = \frac{47}{60}}_{\text{over}}, \quad \underbrace{S_6 = \frac{37}{60}}_{\text{under}}, \quad \underbrace{S_7 = \frac{319}{420}}_{\text{over}}, \quad \ldots$$

Because $S_6 \approx .61$ is an underestimate and $S_7 \approx .76$ is an overestimate we know that the true sum of the series is somewhere between .61 and .76. Later in the course we will see that the true sum of this series is $\ln 2 \approx .693$.

Example 3 Estimate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 4}{\ln(n+1)}$ using S_6. Given an error bound inequality for this estimate and an inequality satisfied by the entire sum S.

Solution: First note that this alternating series does converge and the magnitudes of the terms decrease to a limiting value of zero. We now compute the sixth partial sum as follows

$$S_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1} \cdot 4}{\ln(n+1)} = \frac{4}{\ln 2} - \frac{4}{\ln 3} + \frac{4}{\ln 4} - \frac{4}{\ln 5} + \frac{4}{\ln 6} - \frac{4}{\ln 7} \approx 2.7067$$

Because we ended with a subtraction, this approximation will be an underestimate. The size of our error will be less than the magnitude of the next term. In other words,

$$\boxed{\text{Error } = |R_6| \leq a_7 = \frac{4}{\ln 8} \approx 1.9236}$$

This error bound inequality is related to the fact that the true sum of the series will have to between S_6 and $S_7 = S_6 + a_7$. In this case $S_7 \approx 4.6303$ and thus

$$\boxed{\underbrace{2.7067}_{\approx S_6} \leq \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}}_{S} \leq \underbrace{4.6303}_{\approx S_7}}$$

Absolute and Conditional Convergence.

Definition:

$\sum a_n$ is **<u>absolutely convergent</u>** if and only if $\sum|a_n|$ converges.

$\sum a_n$ is **<u>conditionally convergent</u>** if and only if $\sum a_n$ converges but $\sum|a_n|$ diverges.

Note: Any series that converges absolutely or conditionally is a convergent series. Absolute and conditional are just two categories that we can split convergent series into. Any convergent series consisting of only positive terms, automatically converges absolutely since in that case $\sum a_n$ and $\sum|a_n|$ are the same series (if a_n is positive, then $|a_n| = a_n$). One can think of absolute convergence as a "stronger" type of convergence. For a series to converge, its terms must get small in a sufficiently fast manner. A series with both positive and negative terms has, in some sense, a much better chance of converging than a series with only positive terms. With positive and negative terms, you do a little adding and then a little subtracting, thus reducing the chances of the sum blowing up to infinity. So the idea is that if you take a convergent series and then switch all of its negative terms (subtractions) to positive ones (additions) and the series still converges, then the original series has a "strong" type of convergence that we call absolute convergence. One of the reasons that we care about such classifications as absolute and conditional convergence is that for example, with a conditionally convergent series, you cannot use associative and/or commutative laws (i.e. rearrangement of terms) in carrying out the sum, while in an absolutely convergent series you can. For the majority of the series problems in this course, you will not be required to indicate whether a convergent series converges absolutely or conditionally. However, you should be prepared to determine such classifications or provide their definitions if requested to do so in a given problem. Let us look at two simple examples to clarify how we determine these classifications.

<u>Example 4</u> Classify the convergence (abs. vs. cond.) of the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

Solution. First note that both series can be shown to be convergent using the alternating series test. In the first series we identify $a_n = \frac{(-1)^n}{n^2}$ and thus $|a_n| = \left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$. Thus we consider the associated series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges as a p-series with $p = 2 > 1$.

Because this associate series also converged, we conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ $\boxed{\text{converges absolutely}}$.

On the other hand, for the second series (alternating harmonic), we look at the associated series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ which we recognize as the harmonic series which diverges. Because of this divergence, we conclude that the <u>alternating</u> harmonic series $\boxed{\text{converges conditionally}}$.