2.2 Linear Equations

- *Definition:* A first-order DE is called a **linear equation** in y if it can be written in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

When $\mathrm{g(x)} = 0$, the equation is also called **homogeneous**; otherwise it is **non-homogeneous**.

The **standard form** of a linear equation is obtained by dividing by $a_1(x)$, giving

$$\frac{dy}{dx} + P(x)y = f(x).$$

Claim: If y_c is a solution to the homogeneous DE $y' + P(x)y = 0$, and y_p is a solution to the non-homogeneous $y' + P(x)y = f(x)$, then $y = y_c + y_p$ is a solution to the equation $y' + P(x)y = f(x)$.

Proof: By substitution in the left-hand side of the linear equation

$$\frac{d}{dx}\left[y_c + y_p\right] + P(x)\left[y_c + y_p\right] = \left(\frac{dy_c}{dx} + P(x)y_c\right) + \left(\frac{dy_p}{dx} + P(x)y_p\right) = 0 + f(x)$$

since y_c is a solution to the homogeneous equation and y_p is a solution to the non-homogeneous case. Therefore, $y = y_c + y_p$ is a solution to the non-homogeneous case as well.

So in order to find a general solution to a linear equation, we need to find a y_c and a y_p.

The equation $y' + P(x)y = 0$ is a separable DE. By separation of variables, we get

$$\int \frac{dy}{y} = -\int P(x)dx \quad \rightarrow \quad \boldsymbol{y_c = ce^{-\int P(x)dx}}.$$

For convenience in future work, we let $y_c = cy_1$, where $y_1 = e^{-\int P(x)dx}$.

We can now use a technique called **variation of parameters** to find y_p. For this we want to find a function $u(x)$ so that $y_p = u(x)y_1(x)$ is a solution to the non-homogeneous case. By substitution we get

$$\frac{d}{dx}[uy_1] + P(x)uy_1 = uy_1' + y_1u' + P(x)uy_1 = u\left[\frac{dy_1}{dx} + P(x)y_1\right] + y_1\frac{du}{dx} = f(x).$$

Since y_1 is a solution to the homogeneous case, we are left with

$$y_1 \frac{du}{dx} = f(x) \quad \rightarrow \quad u = \int \frac{f(x)}{y_1(x)} dx \quad \rightarrow \quad \boldsymbol{y_p = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\, dx}.$$

Thus we have a solution of $\boldsymbol{y = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\, dx.}$ It is not hard to show that y represents a **general solution** to the problem.

In general, when solving a linear equation we do not go through all of that work. The key part of the formula is $e^{\int P(x)dx}$, which is referred to as the **integrating factor**.

Solving a Linear First-Order Equation

i. Put the linear equation into standard form.
ii. Identify $P(x)$ and find the integrating factor $e^{\int P(x)dx}$.
iii. Multiply the standard form by the integrating factor. The left-hand side can then be re-written as a derivative, resulting in:
$$\frac{d}{dx}\left[e^{\int P(x)dx} y\right] = e^{\int P(x)dx} f(x)$$
iv. Integrate both sides of the equation and solve for y.

**The constant of integration resulting from $\int P(x)dx$ can be dropped (or assumed to be 0) when find the integrating factor. The proof of this is left as an exercise.*

- *Example:* Solve the following DE.

$$2x\frac{dy}{dx} - 6xy = 12x$$

Solution: First get the equation in standard form by dividing by $2x$.

$$\frac{dy}{dx} - 3y = 6$$

Since $P(x) = -3$, the integrating factor would be $e^{\int -3dx} = e^{-3x}$. Multiplying both sides by the integrating factor we get

$$\frac{d}{dx}[e^{-3x}y] = 6e^{-3x} \quad \rightarrow \quad e^{-3x}y = \int 6e^{-3x}\, dx = -2e^{-3x} + c.$$

Dividing by e^{-3x}, we get a solution of $\boldsymbol{y = ce^{3x} - 2}$.

**If we wished to break down the solution, we would get that $y_c = e^{3x}$ and $y_p = -2$.*

- *Example:* Solve the following DE.

$$x\frac{dy}{dx} - 5y = x^7 e^x$$

Solution: First get the equation in standard form by dividing by x.

$$\frac{dy}{dx} - \frac{5}{x}y = x^6 e^x$$

We identify that $P(x) = -5/x$, which is continuous on the interval $(0, \infty)$. The integrating factor would be $e^{\int -5/x dx} = e^{-5\ln|x|} = |x|^{-5} = x^{-5}$, since $x > 0$. Multiplying both sides by the integrating factor we get

$$\frac{d}{dx}[x^{-5}y] = xe^x \quad \rightarrow \quad x^{-5}y = \int xe^x\,dx = xe^x - e^x + c.$$

Dividing by x^{-5}, we get a solution of $\boldsymbol{y = x^6 e^x - x^5 e^x + c x^5}$.