# Unit 2.4: Improper Integrals

If we go back to the definition of an improper integral (based on a limit of a Riemann Sum), we would find that this definition does not allow our function to have discontinuities on the interval we are integrating over and it also doesn't allow for unbounded intervals such as the interval from 2 to infinity. In these cases, we have to modify (in a natural way) the definition of our definite integrals, thus these will not be "ordinary" integrals, but what we refer to as "improper" integrals. We will split our definitions into two types: those with intervals containing positive and/or negative infinity, and those with discontinuous integrands.

***Type 1 – Intervals Involving $\pm\infty$.***

**Definition:** Suppose that $f$ is continuous on the intervals of integration below.

1. $$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$$

2. $$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

3. $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx \quad \text{, where } a \text{ is any real number}$$

*If the limits above exist, we say that the integral **<u>converges</u>** to the limit and if the limits do not exist, we say that the integral **<u>diverges</u>**.

The idea behind these definitions is that for us to deal with an unbounded (infinite) interval, the natural thing to do is to imagine approaching infinity; thus the limit. In the above definitions, we replaced the infinity with the variable $t$. We didn't have to use $t$, we could have used another variable name, as long as we don't use the variable that defines our function. Let us consider the first rule where we integrate from $a$ to infinity. By replacing the infinity with $t$ and letting $t$ approach infinity, the idea is that for any value $t$ takes on as it approaches infinity, the integral would be an ordinary one. For example, on its way to infinity, $t$ might have the value of 1,000,000. Integrating from $a$ to 1,000,000, would correspond to an ordinary definite integral. As we continue to increase the upper limit of integration, whatever value the integrals are approaching, if there is one, that's the value we will assign to the integral. A quick note about the third definition above before we look at an example: In this definition we did not introduce the limit notation as the two integrals that we generated are already predefined (in definitions 1 and 2) in terms of limits. Also, the value that we use to "split" up the interval from negative to positive infinity is completely arbitrary.

**Example 1** Evaluate the integral $\int_1^\infty xe^{-x^2}\,dx$ or indicate that it diverges.

Solution: We begin by rewriting our improper integral as a limit of an ordinary integral.

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx$$

Before we apply the limit, we must evaluate the integral, which will come out as a function of $t$, since it depends on the value of $t$ chosen as the upper limit. In this case our integration involves a $u$-substitution. If we let $u = -x^2$, then $du = -2x\,dx$, so that $-\frac{1}{2}du = x\,dx$. We will also change our limits of integration as follows: $x = t$ corresponds to $u = -t^2$ and $x = 1$ corresponds to $u = -1^2 = -1$. Thus we obtain

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\int_{-1}^{-t^2} e^u du\right] = \lim_{t\to\infty}\left[-\frac{1}{2}e^u\right]_{-1}^{-t^2}$$

$$= \lim_{t\to\infty} -\frac{1}{2}\left(\underbrace{e^{-t^2}}_{\to\, 0} - e^{-1}\right) = -\frac{1}{2}\left(-\frac{1}{e}\right) = \boxed{\frac{1}{2e}}$$

When evaluating the limit, we used the fact that $e^{-t^2}$ approaches 0 as $t$ approaches infinity. While $\frac{1}{2e}$ is the answer, one might also say that the integral converges to $\frac{1}{2e}$.

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**Example 2** Evaluate the integral $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$ or indicate that it diverges.

Solution: Because both our upper and lower limits of integration are infinite, we first split up the integral at any value we wish, and 0 is probably a natural choice. Once we split the integral up, we then express each of those integrals as limits.

$$\int_{-\infty}^\infty \frac{x}{x^2+1}dx = \int_{-\infty}^0 \frac{x}{x^2+1}dx + \int_0^\infty \frac{x}{x^2+1}dx = \lim_{t\to-\infty}\int_t^0 \frac{x}{x^2+1}dx + \lim_{t\to\infty}\int_0^t \frac{x}{x^2+1}dx$$

The two limits obtained above are separate calculations. If either limit does not exist, then the original integral diverges. If both limits exist, then the original integral will converge to the sum of the two limits. Let us consider one of the integrals at a time. In this case we could again use $u$-sub and change our limits of integration like we did in the last problem. However, some of us might be able to come up with the antiderivative without $u$-sub and thus I will show it done that way in this example. We can see that since the derivative of the denominator is a multiple of the numerator, we are going to get a log function. We proceed as follows

$$\lim_{t\to-\infty}\int_t^0 \frac{x}{x^2+1}dx = \lim_{t\to-\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_t^0 = \lim_{t\to-\infty}\frac{1}{2}[\ln 1 - \underbrace{\ln(t^2+1)}_{\infty}] \quad (DNE)$$

Since the limit above does not exist (DNE), the original integral diverges. There is no point in checking the second limit, as regardless of the outcome, the integral we started with diverges.

***Type 2—Discontinuous Integrands.*** We now consider the situations where our integrands have a discontinuity on the interval we are integrating over. The natural way to handle such a situation is to approach the discontinuity by way of a limit. Thus we obtain the following definitions.

**Definition:**

1. Suppose $f$ is continuous for $a \le x < b$ and discontinuous at $x = b$. In this case

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

2. Suppose $f$ is continuous for $a < x \le b$ and discontinuous at $x = a$. In this case

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

3. Suppose $f$ is continuous for $a \le x \le b$ except at $x = c$, where $a < c < b$. In this case

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

*If any of the above limits do not exist, we say the integral diverges, otherwise it converges to the limit.

Make sure you notice that in this case, we have one-sided limits (i.e. left-hand and right-hand limits). For example, if we look at the first definition where our discontinuity is at the right endpoint of the interval, we would have to approach the right endpoint from the left side if we are to stay in the interval. Let's see a couple of examples.

**Example 3** Evaluate the integral $\int_0^1 \frac{1}{x}dx$

Solution: Here we should note that we have a discontinuity at $x = 0$. Thus we must express this integral in terms of a limit and evaluate as follows.

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}[\ln x]_t^1 = \lim_{t\to 0^+}[\ln 1 - \underbrace{\ln t}_{-\infty}] \quad (DNE)$$

Since the limit does not exist, the integral diverges.

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**<u>Example 4</u>** Evaluate the integral $\int_1^{10} \frac{1}{(x-2)^{2/3}}\,dx$

Solution: In this problem, we have a discontinuity at $x = 2$ which is strictly between the endpoints of our interval. Thus we must split the integral at $x = 2$ to obtain two integrals with endpoint discontinuities requiring limits.

$$\int_1^{10} \frac{1}{(x-2)^{2/3}}\,dx = \int_1^{2} \frac{1}{(x-2)^{2/3}}\,dx + \int_2^{10} \frac{1}{(x-2)^{2/3}}\,dx = \lim_{t\to 2^-}\int_1^{t} \frac{1}{(x-2)^{2/3}}\,dx + \lim_{t\to 2^+}\int_t^{10} \frac{1}{(x-2)^{2/3}}\,dx$$

Again we must evaluate each of the limits separately, if either doesn't exist, the original integral will diverge. In both cases, the integrand is the same and can be viewed as $(x-2)^{-2/3}$. Thus the antiderivative follows by the power rule for integration. We obtain

$$\lim_{t\to 2^-}\int_1^{t} \frac{1}{(x-2)^{2/3}}\,dx = \lim_{t\to 2^-}\left[3(x-2)^{1/3}\right]_1^t = \lim_{t\to 2^-} 3[\underbrace{(t-2)^{1/3}}_{0} - (1-2)^{1/3}] = 3[0-(-1)] = 3$$

and

$$\lim_{t\to 2^+}\int_t^{10} \frac{1}{(x-2)^{2/3}}\,dx = \lim_{t\to 2^+}\left[3(x-2)^{1/3}\right]_t^{10} = \lim_{t\to 2^-} 3\left[\underbrace{(10-2)^{1/3}}_{\sqrt[3]{8}\,=\,2} - \underbrace{(t-2)^{1/3}}_{0}\right] = 3[2-0] = 6$$

Since both limits exist, the original integral converges to their sum. That is,

$$\int_1^{10} \frac{1}{(x-2)^{2/3}}\,dx = \int_1^{2} \frac{1}{(x-2)^{2/3}}\,dx + \int_2^{10} \frac{1}{(x-2)^{2/3}}\,dx = 3 + 6 = \boxed{9}$$

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One final note: if you attempted to solve the integral in example 4 without ever considering the discontinuity; i.e. you just found the antiderivative and subbed in the 10 and 1 and subtracted, you would have gotten the same answer that we did. However, that will not always happen and finding the value that way would be <u>completely invalid</u> (and you would earn little to no credit if you did it on an exam). A simple example like $\int_{-1}^{1} \frac{1}{x}\,dx$ would show you that without applying the limits, you wouldn't get a correct answer. The integral just mentioned diverges, however a student who just found the antiderivative, subbed in the 1 and -1 and then subtracted would get a value of 0, which is the wrong answer. So the bottom line is: you must use limits to deal with discontinuities, even if you <u>sometimes</u> might get the correct answer without them.