5.3 Non-Homogeneous Linear Systems

When looking for a general solution to a non-homogeneous system, we found that it is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$, where $\mathbf{X}_c$ is the complimentary function, and $\mathbf{X}_p$ is any particular solution. We have already discussed how to find the complementary function. Now we will consider two methods for obtaining and $\mathbf{X}_p$.

Undetermined Coefficients:

The method of undetermined coefficients consists of making an educated guess about the form of $\mathbf{X}_p$. Just like in previous examples, this method is only applicable when the entries of $\mathbf{F}(t)$ are polynomials, exponential functions, sines and cosines, or finite sums and products of these. The process for "guessing" at $\mathbf{X}_p$ follows the same general guidelines as before, with some minor differences. In general, it is best to break down $\mathbf{F}(t)$ into different components and tailor your guess for $\mathbf{X}_p$ to each of these components.

- *Example:* Solve the system

$$\mathbf{X}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 6t \\ -10t + 4 \end{pmatrix}.$$

Solution: It is left as an exercise to show that associated homogeneous system produces a complementary function

$$\mathbf{X}_c = c_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{7t}.$$

When attempting to find $\mathbf{X}_p$, we note that

$$\mathbf{F}(t) = \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

Thus, we will attempt to find a particular solution of the form

$$\mathbf{X}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}.$$

Substituting this into the non-homogeneous system gives us

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix}\left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix}t + \begin{pmatrix} 0 \\ 4 \end{pmatrix},$$

or

$$\begin{pmatrix} (6a_2 + b_2 + 6)t + (6a_1 + b_1 - a_2) \\ (4a_2 + 3b_2 - 10)t + (4a_1 + 3b_1 - b_2 + 4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the 4-variable system

$$\begin{aligned} 6a_2 + b_2 + 6 &= 0 \\ 6a_1 + b_1 - a_2 &= 0 \\ 4a_2 + 3b_2 - 10 &= 0 \\ 4a_1 + 3b_1 - b_2 + 4 &= 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} a_1 &= -\frac{4}{7} \\ a_2 &= -2 \\ b_1 &= \frac{10}{7} \\ b_2 &= 6. \end{aligned}$$

Therefore we get

$$\mathbf{X}_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \frac{1}{7}\begin{pmatrix} -4 \\ 10 \end{pmatrix} \quad \rightarrow \quad \mathbf{X} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \frac{1}{7}\begin{pmatrix} -4 \\ 10 \end{pmatrix}.$$

- Example: Determine the form of a particular solution for the system

$$\mathbf{X}' = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \mathbf{X} + \begin{pmatrix} -2e^{5t} + 1 \\ e^{5t} - \cos 3t + 4 \end{pmatrix}.$$

Solution: In this case, we have

$$\mathbf{F}(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{5t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos 3t + \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Without first solving for $\mathbf{X}_c$, a natural guess for a particular solution would be

$$\mathbf{X}_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} e^{5t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \cos 3t + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \sin 3t + \begin{pmatrix} a_4 \\ b_4 \end{pmatrix}.$$

However, our associated homogeneous system produces eigenvalues of $\lambda_1 = 0$ and $\lambda_2 = 5$, giving us solutions of

$$X_1 = K_1, \qquad X_2 = K_2 e^{5t},$$

both of which show up in $\mathbf{F}(t)$. Therefore we would need to adjust our guess to create a linearly independent solution. Previously this was accomplished by multiplying by powers of t so that our guess no longer contained any terms common to our fundamental solution set. However, due to the complications of having a system of equations, we actually need to include two new terms, giving

$$\mathbf{X}_p = \left[\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} te^{5t} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} e^{5t}\right] + \left[\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \cos 3t + \begin{pmatrix} a_4 \\ b_4 \end{pmatrix} \sin 3t\right] + \left[\begin{pmatrix} a_5 \\ b_5 \end{pmatrix} t + \begin{pmatrix} a_6 \\ b_6 \end{pmatrix}\right].$$

Variation of Parameters:

If $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n$ is a fundamental set of solutions of the homogeneous system $\mathbf{X}' = \boldsymbol{AX}$ on the interval I, then its general solution on the interval is $\mathbf{X} = c_1\mathbf{X}_1 + c_2\,\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ or

$$\mathbf{X} = c_1 \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} + c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} + \cdots + c_n \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} c_1x_{11} + c_2x_{12} + \cdots + c_nx_{1n} \\ c_1x_{21} + c_2x_{22} + \cdots + c_nx_{2n} \\ \vdots \\ c_1x_{n1} + c_2x_{n2} + \cdots + c_nx_{nn} \end{pmatrix}.$$

We can therefore write $\mathbf{X} = \mathbf{\Phi}(t)\mathbf{C}$, where $\mathbf{C}$ is a column vector of arbitrary constants and

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

which is called the **fundamental matrix** of the system on the interval.

**The fundamental matrix is nonsingular, and therefore $\mathbf{\Phi}^{-1}(t)$ exists.*

Following the technique of variation of parameters from before, our goal is to find a particular solution of the form

$$\mathbf{X}_p = \mathbf{\Phi}(t)\mathbf{U}(t),$$

where $\mathbf{U}(t)$ is a column vector of functions $u_1(t), u_2(t), \dots u_n(t)$. By applying the product rule, we get that

$$\mathbf{X}_p' = \mathbf{\Phi}(t)\mathbf{U}'(t) + \mathbf{\Phi}'(t)\mathbf{U}(t).$$

By substituting this into our non-homogeneous system, and using the fact that $\mathbf{\Phi}(t)\mathbf{U}'(t)$, we get

$$\mathbf{\Phi}(t)\mathbf{U}'(t) + \mathbf{\Phi}'(t)\mathbf{U}(t) = \mathbf{A}\mathbf{\Phi}(t)\mathbf{U}(t) + \mathbf{F}(t) \quad \to \quad \mathbf{\Phi}(t)\mathbf{U}'(t) = \mathbf{F}(t).$$

Since $\mathbf{\Phi}(t)$ is non-singular, we can multiply both sides of the equation by $\mathbf{\Phi}^{-1}(t)$ to get

$$\mathbf{U}'(t) = \mathbf{\Phi}^{-1}(t)\mathbf{F}(t) \quad \to \quad \mathbf{U}(t) = \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt.$$

Therefore, our particular solution is

$$\mathbf{X}_p = \mathbf{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt,$$

and our general solution would be

$$\mathbf{X} = \mathbf{\Phi}(t)\boldsymbol{C} + \boldsymbol{\Phi}(t) \int \mathbf{\Phi}^{-1}(t)\mathbf{F}(t)dt.$$

**As always, be very careful with the order of parts when dealing with matrices. The general rule that $a \cdot b = b \cdot a$ does not apply to matrix multiplication.*

While the overall process for find a particular solution does not look overly complicated, the process of finding $\mathbf{\Phi^{-1}}(\mathrm{t})$, $\mathbf{\Phi^{-1}}(\mathrm{t})\mathbf{F}(\mathrm{t})$, and $\mathbf{\Phi}(\mathrm{t})\int\mathbf{\Phi^{-1}}(\mathrm{t})\mathbf{F}(\mathrm{t})\mathrm{dt}$ can be very long and time consuming, especially as n gets larger. For that reason, it is perfectly acceptable (and encouraged) to use computer software to aid in these calculations.

- Example: Solve the system

$$\mathbf{X}' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

on the interval $(-\infty, \infty)$ by use of variation of parameters.

Solution: We must first solve the associated homogeneous system to get $\mathbf{\Phi}(t)$. By the characteristic equation and our techniques from before, we get eigenvalues of $\lambda_1 = -2$ and $\lambda_2 = -5$, with corresponding eigenvectors

$$\mathbf{K}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{K}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Therefore,

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \;\rightarrow\; \mathbf{\Phi^{-1}}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}.$$

By multiplication and integration, we get

$$\mathbf{\Phi^{-1}}(t)\mathbf{F}(t) = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} = \begin{pmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{pmatrix}$$

$$\int \mathbf{\Phi^{-1}}(t)\mathbf{F}(t)\,dt = \int \begin{pmatrix} 2te^{2t} + \frac{1}{3}e^{t} \\ te^{5t} - \frac{1}{3}e^{4t} \end{pmatrix} dt = \begin{pmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{pmatrix}$$

$$\mathbf{\Phi}(t)\int \mathbf{\Phi^{-1}}(t)\mathbf{F}(t)\,dt = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} te^{2t} - \frac{1}{2}e^{2t} + \frac{1}{3}e^{t} \\ \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} - \frac{1}{12}e^{4t} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}$$

$$\mathbf{X}_p = \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}.$$

Therefore we get a general solution of

$$\mathbf{X} = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 6/5 \\ 3/5 \end{pmatrix}t - \begin{pmatrix} 27/50 \\ 21/50 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix}e^{-t}.$$

**The general solution has been written in expanded form to highlight that the particular solution does have the form that would be expected from undetermined coefficients. In fact, for this problem, we could have used undetermined coefficients to find the same answer.*