## 1.2 Initial-Value Problems & Boundary Value Problems

- *Definition:* On some interval $I$ containing $x_0$, the problem

$$\text{Solve:} \qquad \frac{d^n y}{dx^n} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$
$$\text{Subject to:} \quad y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

  where $y_0, y_1, \dots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem (IVP)**, and $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$ are called the **initial conditions.**

- *Example:* Given that $x = c_1 \cos t + c_2 \sin t$ is a general solution of $x'' + x = 0$, find a solution to the initial-value problem

$$x' + x = 0, \qquad x\left(\frac{\pi}{2}\right) = -1, \qquad x'\left(\frac{\pi}{2}\right) = 1$$

  Solution:

$$x\left(\frac{\pi}{2}\right) = -1 \quad \to \quad c_1 \cos\frac{\pi}{2} + c_2 \sin\frac{\pi}{2} = \boldsymbol{c_2 = -1}$$
$$x'\left(\frac{\pi}{2}\right) = 1 \quad \to \quad -c_1 \sin\frac{\pi}{2} + c_2 \cos\frac{\pi}{2} = -c_1 = 1 \quad \to \quad \boldsymbol{c_1 = -1}$$

  Hence, $\boldsymbol{x(t) = -\cos t - \sin t}$ is a solution of the IVP.

Note that in the example it is stated that $x(t) = -\sin t$ is ***a*** solution to the IVP, not ***the*** solution. It is not always the case that a differential equation will have a unique solution (or a solution at all). However in some cases we can be sure to have a unique solution.

***Theorem:*** **Existence of a Unique Solution**

Let $R$ be a rectangular region defined by $a \leq x \leq b, c \leq y \leq d$ that contains the point $(x_0, y_0)$ in its interior. If $f(x, y)$ and $\partial f / \partial y$ are continuous on $R$, then there exists some interval $I_0$ centered at $x_0$, contained in $[a, b]$, and a unique function $y(x)$, defined on $I_0$, that is a solution of the first-order initial value problem

$$\frac{dy}{dx} = f(x, y); \;\; y(x_0) = y_0$$

*These conditions are sufficient, but not necessary for a unique solution.

*This does not (yet) apply to higher-order IVP's.

- *Definition:* On some interval $I$, containing $a$ and $b$, the problem

$$\begin{aligned} &Solve: && \frac{d^2y}{dx^2} = f(x, y, y') \\ &Subject\ to: && y(a) = y_0,\ \ y(b) = y_1 \end{aligned}$$

  where $y_0, y_1$ are arbitrary specified real constants, is called a second-order **boundary-value problem (BVP)**, and $y(a) = y_0, y(b) = y_1$ are called the **boundary conditions.**
  *We could have other possible boundary conditions which include derivatives of $y$ evaluated at $a$ and $b$ as well.*
  *A BVP can be of higher order than 2.*

Unlike an IVP, we cannot guarantee a unique solution to a BVP. A boundary-value problem can have one, many, or no solutions.

- *Example*: Given that $x = c_1 \cos t + c_2 \sin t$ is a general solution of $x'' + x = 0$, find a solution to the following boundary-value problems.

  a) $x'' + x = 0,\ x(0) = 0,\ x(2\pi) = 0$

  Solution: Using $x(0) = 0$, we get that $c_1 = 0$. Applying $x(2\pi) = 0$, we get $0 = 0$, leaving us free to choose any value we wish for $c_2$. This gives us infinitely many solutions of the form $\boldsymbol{x(t) = c_2 \sin t}$.

  b) $x'' + x = 0,\ x(0) = 0,\ x\left(\frac{\pi}{2}\right) = 0$

  Solution: Using $x(0) = 0$, we once again get that $c_1 = 0$. Applying $x(\pi/2) = 0$, we get $c_2 = 0$, giving us the single (trivial) solution $\boldsymbol{x(t) = 0}$.

  c) $x'' + x = 0,\ x(0) = 0,\ x(2\pi) = 1$

  Solution: Using $x(0) = 0$, we once again get that $c_1 = 0$. Applying $x(2\pi) = 1$, we get $0 = 1$, a contradiction. This leaves us with **no solution** to the BVP.