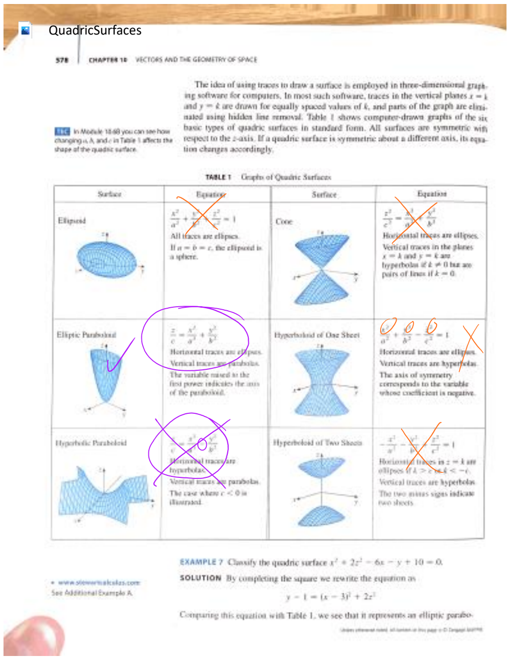
10.6: Cylinders and Quadric Surfaces

Thursday, August 13, 2020 10:00 PM



QuadricSurfaces

The idea of using traces to draw a surface is employed in three-dimensional graphing software for computers. In most such software, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k, and parts of the graph are eliminated using hidden line removal. Table 1 shows computer-drawn graphs of the six basic types of quadric surfaces in standard form. All surfaces are symmetric with respect to the z-axis. If a quadric surface is symmetric about a different axis, its equation changes accordingly.

TEC In Module 10.6B you can see how changing a, b, and c in Table 1 affects the shape of the quadric surface.

TABLE 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid	$\cancel{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone	$\cancel{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\cancel{\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets	$\cancel{-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

EXAMPLE 7 Classify the quadric surface $x^2 + 2z^2 - 6x - y + 10 = 0$.

SOLUTION By completing the square we rewrite the equation as

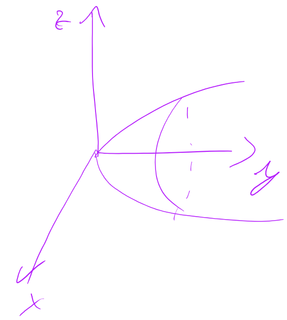
$$y - 1 = (x - 3)^2 + 2z^2$$

Comparing this equation with Table 1, we see that it represents an elliptic parabo-

• www.stewartcalculus.com See Additional Example A.

$x^2 + z^2 - y = 0$

$x^2 + z^2 = y$

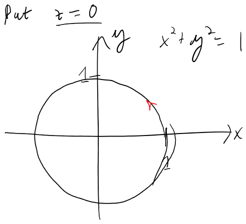


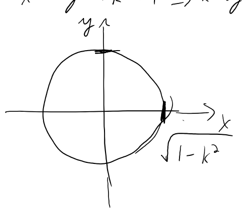
Def. 1) Traces are curves of intersection of surfaces with planes that are parallel to the coordinate planes.

ex. $x^2 + y^2 + z^2 = 1$ surface type?

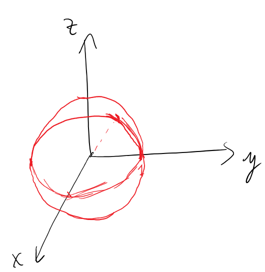
Put $z = 0$ $z = k$

$x^2 + y^2 = 1$ $x^2 + y^2 + k^2 = 1 \Rightarrow x^2 + y^2 = 1 - k^2$





3D



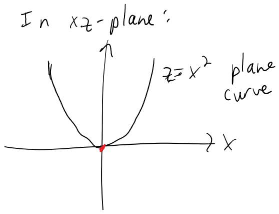
$x^2 + y^2 + z^2 = 1$

sphere cent. at $(0,0,0)$, rad. $= 1$

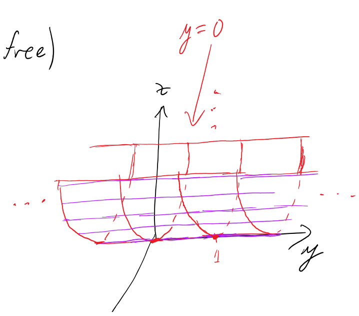
2) A cylinder is a surface that consists of all lines (called rulings) parallel to a given line and pass through a given plane curve.

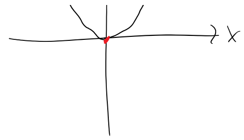
ex. $z = x^2$ in 3D (y free)

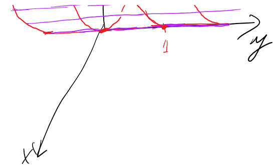
In xz-plane:



$y = 0$

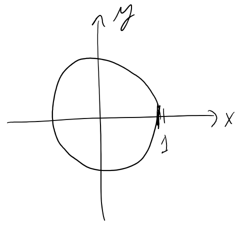


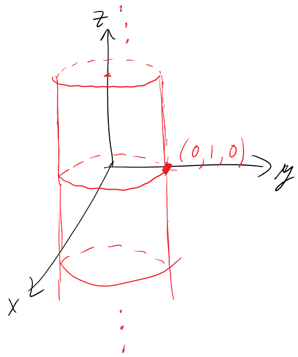




ex. $x^2+y^2=1$ (z free)

xy-plane





Def. A quadric surface is the graph of a degree 2 eq'n in x, y, z:

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0,$$

where $A, B, \dots, J$ real constants, and at least one of A, B, C is $\neq 0$.

Note Translations & rotations can bring any quadric surface to one of these two standard forms:

$$Ax^2+By^2+Cz^2+J=0, \quad \text{or} \quad Ax^2+By^2+Iz=0$$

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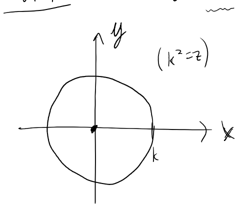
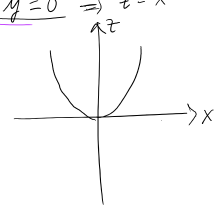
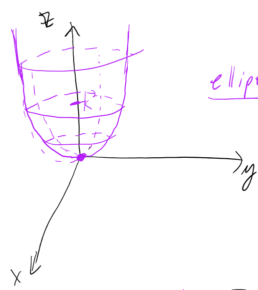
ex. Sketch $z=x^2+y^2$.

Sol'n: Fix $z=k^2$. Then $x^2+y^2=k^2$

$y=0 \Rightarrow z=x^2$







elliptic paraboloid

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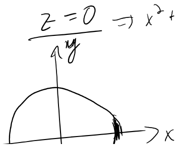
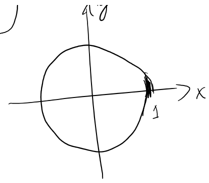
ex. $x^2+y^2-z^2=1$

$x^2+y^2=z^2+1$

$z=0 \Rightarrow x^2+y^2=1$

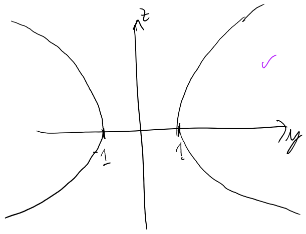


$x^2+y^2=z^2+1$



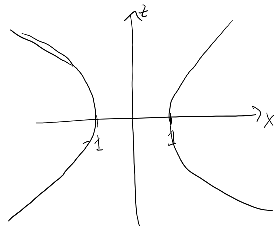
$x=0$

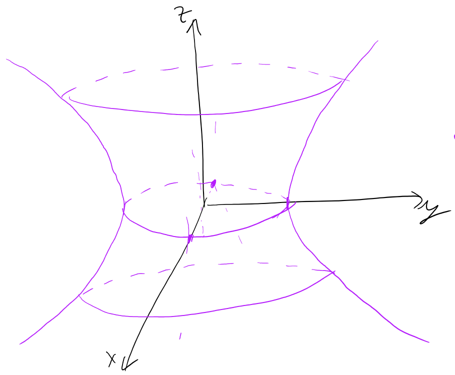
$y^2=z^2+1$



$y=0$

$x^2-z^2=1$





hyperboloid of one sheet

• To help classify & draw quadric surfaces, use chart/table.

ex. Classify the quadric surface w/eq'n $x^2+2z^2-6x-y+10=0$.

Sol'n: Complete the square

$$x^2-6x+9-9+2z^2-y+10=0$$

$$(x-3)^2+2z^2-y=-1$$

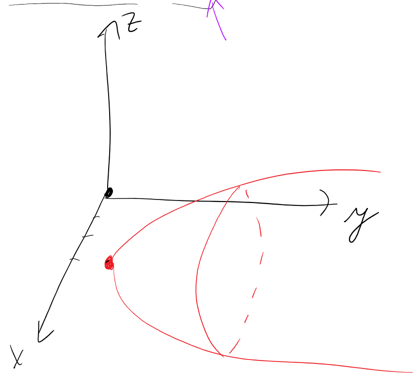
→ table? Basically

$$x^2+z^2-y=-1$$

$$x^2+z^2-(y-1)=0$$

$$x^2+z^2-y=0$$

$$\boxed{(x-3)^2+2z^2=y-1}$$



elliptic paraboloid

Identify Quadric Surfaces

$(x-3)^2+2z^2=y-1$

2nd deg. polyn' eq'n in x, y, z

in x, y, z

- Only two of x, y, z are squared
  - terms with x^2, y^2, z^2 have same sign → elliptic paraboloid
  - terms with x^2, y^2, z^2 have diff. signs → hyperbolic paraboloid (saddle)
- all three of x, y, z squared
  - variable terms sum to 0 → cone
  - variable terms sum to positive constant
    - all 3 terms with x^2, y^2, z^2 are positive → ellipsoid
    - 2 terms with x^2, y^2, z^2 pos., 1 negative → hyperboloid one sheet
    - 1 term with x^2, y^2, z^2 pos., 2 negative → hyperboloid two sheets