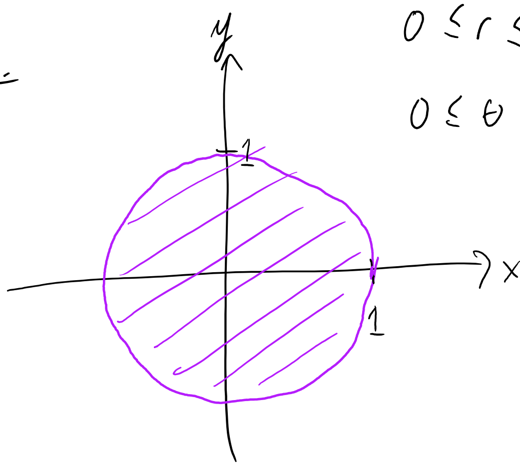
12.3: Double Integrals in Polar Coordinates

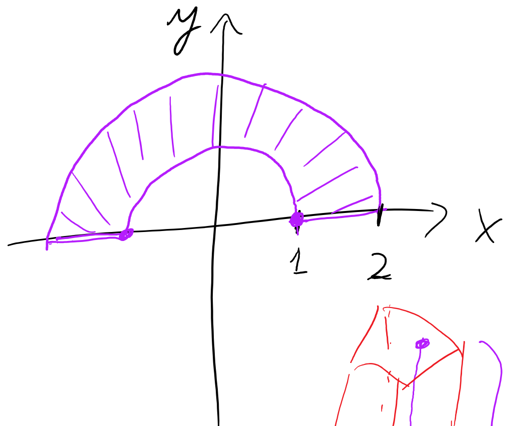
Thursday, October 1, 2020 7:04 AM

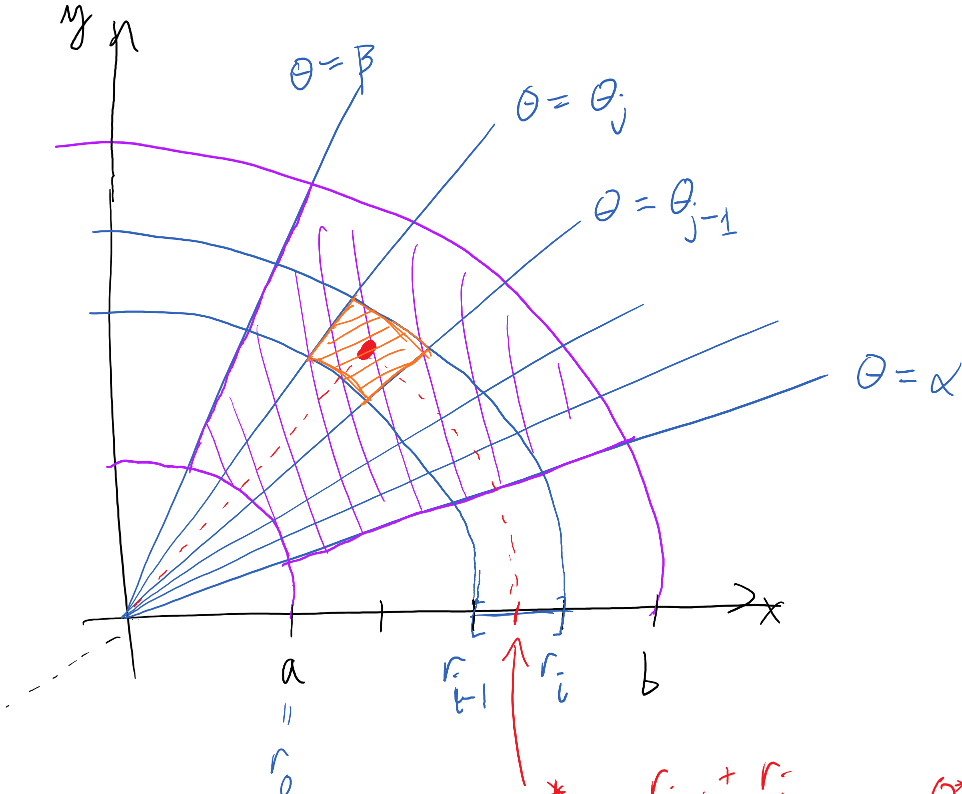
Def. $R = \{(r,\theta) : a \le r \le b,\ \alpha \le \theta \le \beta\}$ is a polar rectangle.

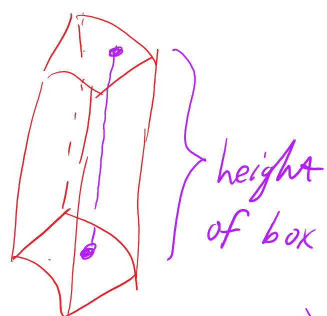
ex. $(a=0,\ b=1)$ $0 \le r \le 1$, $0 \le \theta \le 2\pi$



ex. $1 \le r \le 2$, $0 \le \theta \le \pi$







$\text{vol} = \begin{pmatrix}\text{area of} \\ \text{base}\end{pmatrix} \cdot \text{ht}$

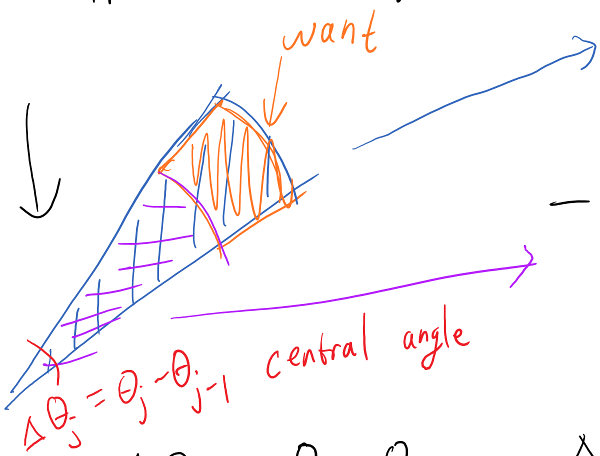
$= \Delta A_{ij}\, f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)$

$r_i^* = \dfrac{r_{i-1} + r_i}{2}$, $\theta_j^* = \dfrac{\theta_{j-1} + \theta_j}{2}$

Sample point (r_i^*, θ_j^*) in middle of subrectangle $[r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$

(orange piece = (area of ... of circle from)

· Area of orange piece $= \begin{pmatrix} \text{area of sector of circle from} \\ \theta_{j-1} \text{ to } \theta_j \text{, with radius } r_i \end{pmatrix}$



$- \begin{pmatrix} \text{area of sector from } \theta_{j-1} \text{ to } \theta_j \\ \text{w/ radius } r_{i-1} \end{pmatrix}$

· $\Delta\theta_j = \theta_j - \theta_{j-1}$, $\Delta r_i = r_i - r_{i-1}$

Recall area of a sector of a circle w/radius r, central angle θ, is $\frac{1}{2} r^2 \theta$.

$\therefore$ area of orange piece $= \Delta A_{ij}$

$$= \frac{1}{2} r_i^2 \Delta\theta_j - \frac{1}{2} r_{i-1}^2 \Delta\theta_j$$

$$= \frac{1}{2} \Delta\theta_j \left(r_i^2 - r_{i-1}^2 \right)$$

$$= \frac{1}{2} \left(r_i - r_{i-1}\right)\left(r_i + r_{i-1}\right) \Delta\theta_j$$

$$= \left(\frac{r_i + r_{i-1}}{2}\right)\left(r_i - r_{i-1}\right) \Delta\theta_j$$

$$\Delta A_{ij} = r_i^* \Delta r_i \Delta\theta_j \approx dA$$

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $f(x, y)$

$$\rightsquigarrow \sum_i \sum_j f\left(r_i^* \cos(\theta_j^*),\ r_i^* \sin(\theta_j^*)\right) \cancel{\Delta A_{ij}}\ r_i^* \Delta r_i \Delta \theta_j$$

$$\rightsquigarrow \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r \, dr \, d\theta$$

appears b/c using polar rectangles

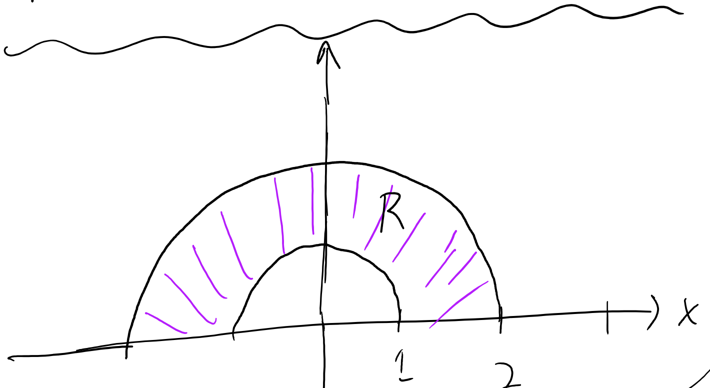
Change to Polar in Double Integrals

f continuous on polar rectangle R given by

$0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$,

then $$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Ex. Evaluate $\iint_R 3x + 4y^2 \, dA$, where R region in upper half plane bdd by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$.



$1 \le r \le 2$, $0 \le \theta \le \pi$

$$\iint_R 3x + 4y^2 \, dA = ?$$

$x = r\cos(\theta)$, $y = r\sin(\theta)$, $dA = r\,dr\,d\theta$

$x = r\cos(\theta)$, $y = \ldots$

$$= \int_0^{\pi}\int_1^2 \left[3r\cos(\theta) + 4(r\sin(\theta))^2\right] r \, dr \, d\theta$$

$$= \int_0^{\pi}\int_1^2 3r^2\cos(\theta) + 4r^3\sin^2(\theta) \, dr \, d\theta$$

$$= \int_0^{\pi}\int_1^2 \underline{3r^2}\,\underline{\cos(\theta)} \, dr \, d\theta + 4\int_0^{\pi}\int_1^2 r^3\sin^2(\theta) \, dr \, d\theta$$

$$= \left(\int_0^{\pi}\cos(\theta)\,d\theta\right)\left(\int_1^2 3r^2\,dr\right) + 4\left(\int_0^{\pi}\sin^2(\theta)\,d\theta\right)\left(\int_1^2 r^3\,dr\right)$$

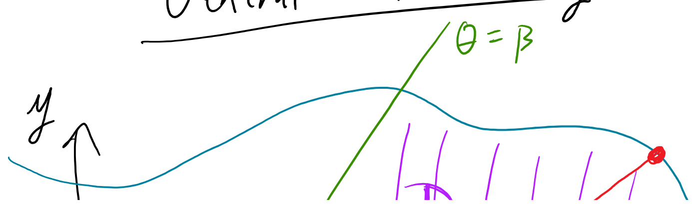
$$= 0 + 4\left(\int_0^{\pi}\frac{1-\cos(2\theta)}{2}\right)\left[\frac{r^4}{4}\right]_1^2$$

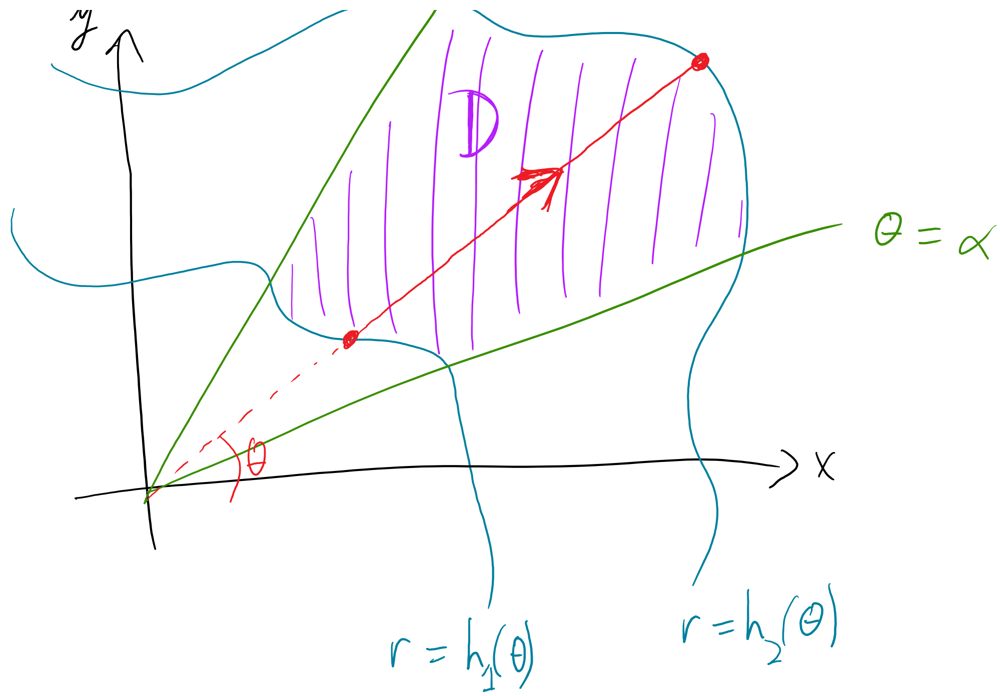
$$= 2\left(\pi - \frac{\sin(2\theta)}{2}\Big|_0^{\pi}\right)\frac{1}{4}\left(2^4 - 1\right)$$

$$= \frac{1}{2}\left(2^4-1\right)\left(\pi - \frac{1}{2}(0)\right)$$

$$= \boxed{\frac{15\pi}{2}}$$

General Polar Region





$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Ex. Find vol. of solid under $z = x^2 + y^2$, above xy-plane, inside $x^2 + y^2 = 2x$.

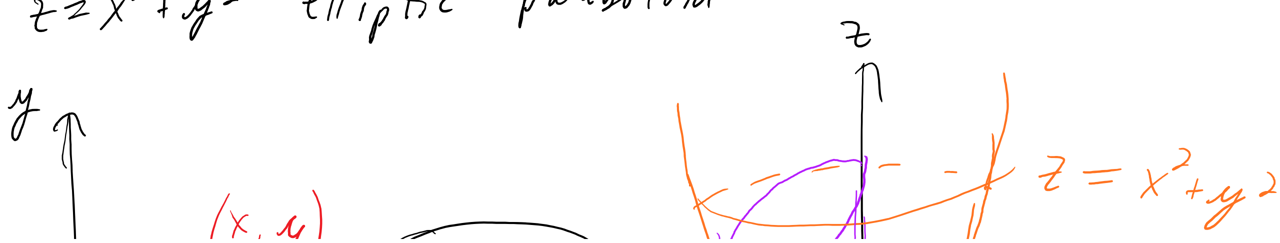
Sol'n:

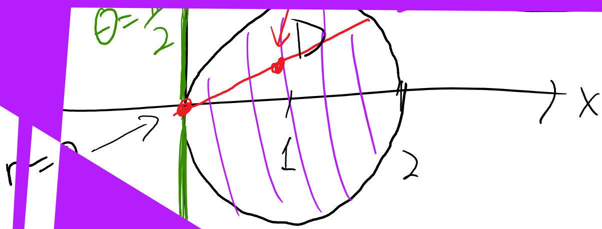
$$x^2 - 2x + y^2 = 0$$
$$x^2 - 2x + 1 + y^2 = 1$$
$$(x-1)^2 + y^2 = 1$$

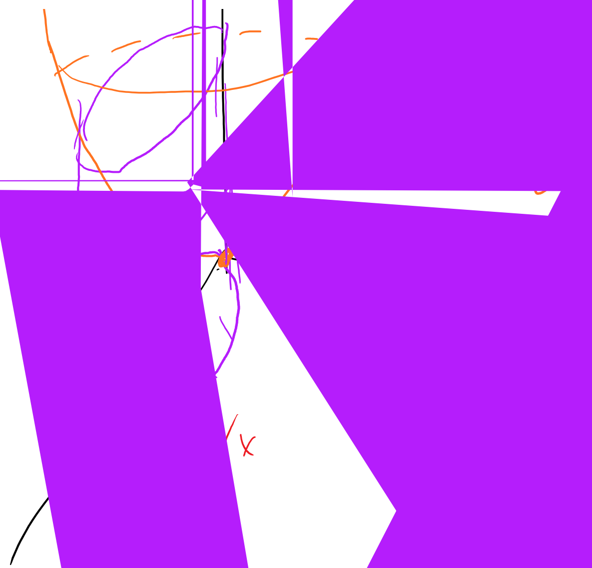
$$x^2 + y^2 = 2x = 2r\cos(\theta)$$
$$r^2 = 2r\cos(\theta)$$
$$r = 2\cos(\theta)$$

same circle, radius 1, cent. $(1,0)$

Note $z = x^2 + y^2$ elliptic paraboloid







$2\cos(\theta)$

$d\theta$

$= 8$

$$= \boxed{\frac{3\pi}{2}}$$