11.1: Functions of Several Variables

Thursday, September 3, 2020 8:57 AM

ex. volume of a circular cylinder (finite height) is

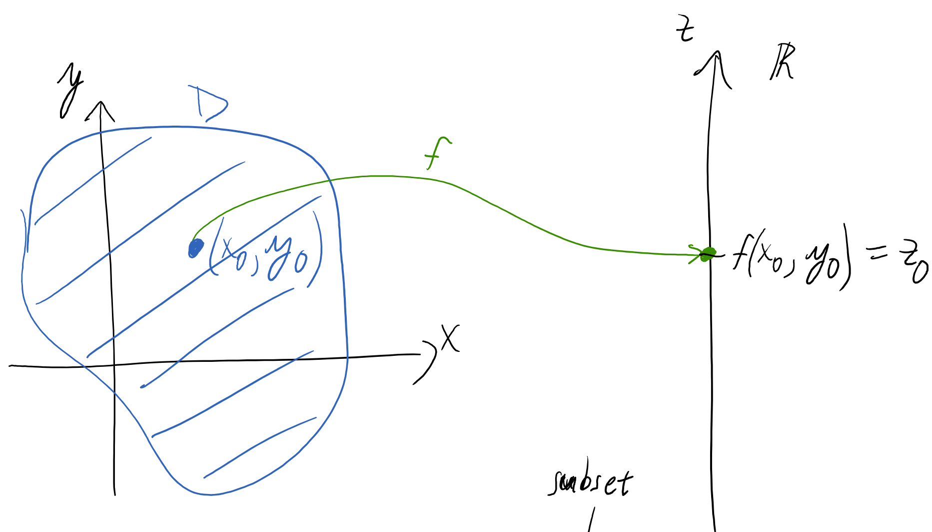
$V = \pi r^2 h$, r = radius, h = height

$V(r,h) = \pi r^2 h$ is a func. of r and h

Def. A function f of two variables is a rule that assigns to each ordered pair of real numbers (x,y) in a set D a unique real number $f(x,y)$. The set D is the domain of f, and its range is the set of values that f takes on, that is range$(f) = f(D)$

$$= \{ f(x,y) \mid (x,y) \in D \}$$

Note Write $z = f(x,y)$, so z is the dependent variable, and it's a func. of the two ind. var. x, y.



Def. If $f: D \to \mathbb{R}$, $D \overset{\text{subset}}{\subset} \mathbb{R}^2$, the set $f(D)$ is the range of f, and $\mathbb{R}$ is the codomain of f.

Note If no domain of $f(x,y)$ is specified, assume it's as large as possible:

$$D = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) \text{ is defined}\}.$$

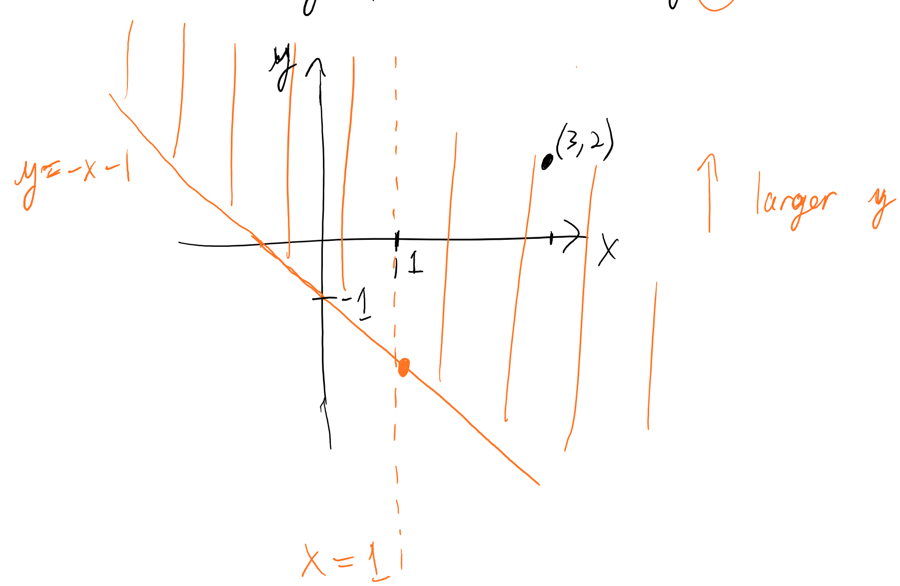
Ex. 1 Find the domains of the following func's and evaluate $f(3,2)$.

(a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

(b) $f(x,y) = x \ln(y^2 - x)$

Sol'n: • Can't have denom. $= 0$, so $x - 1 \neq 0 \iff x \neq 1$

• $x + y + 1 \geq 0 \implies y \geq -x - 1$ $\quad (y = -x-1)$



$$\boxed{D = \{(x,y) \mid x + y + 1 \geq 0 \text{ and } x \neq 1\}}$$

$$f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \boxed{\frac{\sqrt{6}}{2}}$$

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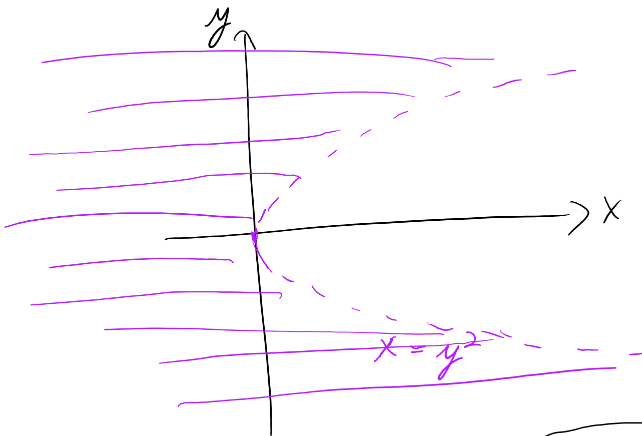
(b) $f(x,y) = x \ln(y^2 - x)$

- Can't have $y^2 - x \le 0$, so $y^2 - x > 0$

$\iff y^2 > x$ $\quad (y^2 = x)$



$$\boxed{D = \{(x,y) \mid y^2 > x\}}$$

$x = 3, \quad y = 2$

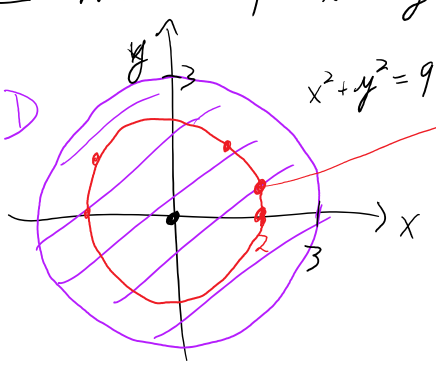
$y^2 > x$

$4 > 3$ ✓

$$f(3,2) = 3\ln(2^2 - 3)$$
$$= 3\ln(1)$$
$$= \boxed{0}$$

Ex. 2 Find domain & range of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

Sol'n: Need $9 - x^2 - y^2 \ge 0 \iff x^2 + y^2 \le 9$



$g \to \sqrt{9-(x^2+y^2)} = \sqrt{9-4} = \sqrt{5} \in [0,3]$

$$\boxed{D = \{(x,y) : 9 - x^2 - y^2 \ge 0\}}$$

$z = g(x,y)$, so $z = \sqrt{9-(x^2+y^2)} \geq 0$

* Any point (x,y) on $x^2+y^2=9$ has $z=0$

$z = \sqrt{9-(x^2+y^2)}$ is largest when x^2+y^2 smallest,

i.e. when $(x,y)=(0,0)$

$\rightsquigarrow z = \sqrt{9-0} = 3$

$\therefore \text{range}(g) = g(D) = \{z \mid 0 \leq z \leq 3\}$

$= \boxed{[0,3]}$

Def. The level curves/sets of $f(x,y)$ are the curves in $\mathbb{R}^2$ with equations $f(x,y)=k$, k constant, for each $k \in \text{range}(f)$.

ex. Level sets in Ex. 2 are concentric circles in D.

Graphs

Def. If a func. $f(x,y)$ has domain D, the graph of f is the set of all points $(x,y,z) \in \mathbb{R}^3$ s.t. $z=f(x,y)$ and $(x,y) \in D$.

$$\Gamma_f = \{(x,y,z) : (x,y) \in D \text{ and } z = f(x,y)\}.$$

Note (i) Graph of $f(x)$ is a curve C w/eq'n $y=f(x)$.

(ii) " " $f(x,y)$ " " surface S with eq'n

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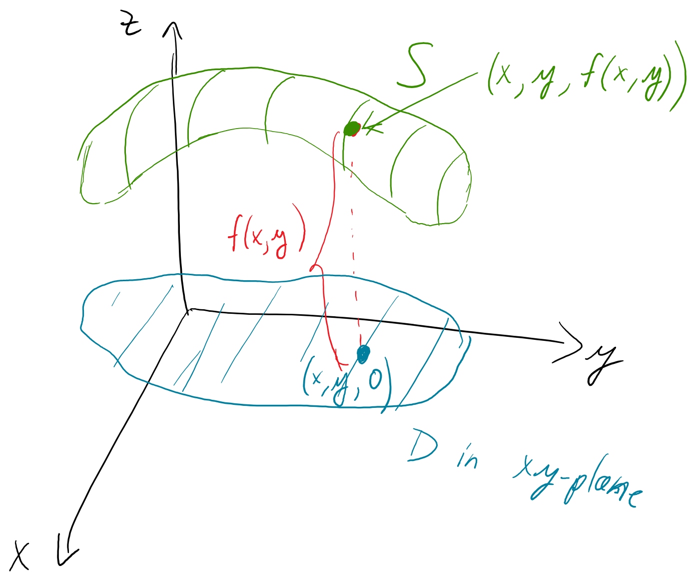
$z = f(x,y)$.

(Visualize S as above/below D)



Ex. 3 Sketch the graph of $f(x,y) = 6 - 3x - 2y$

Sol'n: $z = 6 - 3x - 2y \iff 3x + 2y + z = 6$ eq'n of plane with $\vec{n} = \langle 3, 2, 1 \rangle$.

$\Rightarrow$ plane intersects coord. axes

z-axis $x = y = 0$

$\Rightarrow z = 6$

$(0, 0, 6)$

y-axis $x = z = 0$

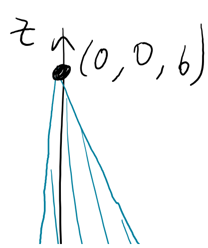
$\Rightarrow y = 3$

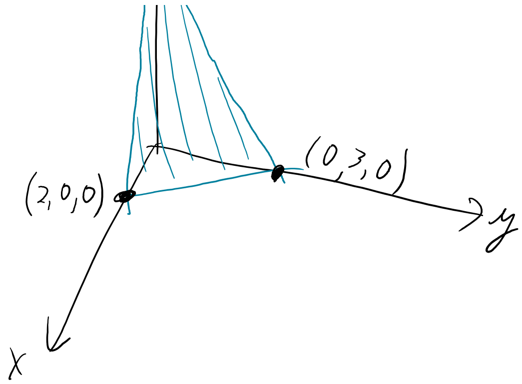
$(0, 3, 0)$

x-axis $y = z = 0$

$\Rightarrow x = 2$

$(2, 0, 0)$





Def. For constants a, b, c; $f(x,y) = ax + by + c$ is a linear func. (graph is a plane)

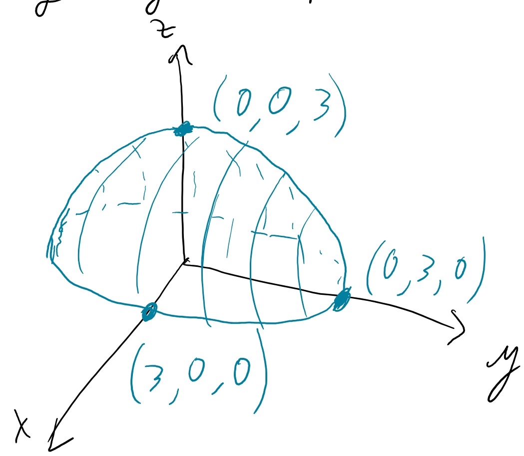
Ex. 4 Graph $g(x,y) = \sqrt{9 - x^2 - y^2}$

graph of g lies on sphere

Sol'n: $z = \sqrt{9 - x^2 - y^2} \Rightarrow z^2 = 9 - x^2 - y^2$

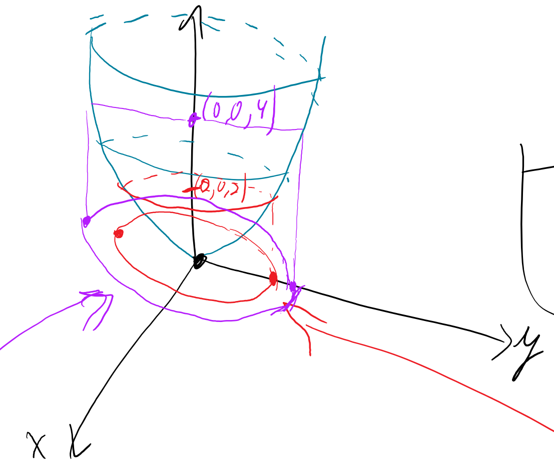
$\Leftrightarrow x^2 + y^2 + z^2 = 9$, which rep. a sphere, cent. at origin, radius 3.

- Graph of g only has points (x,y,z) where $z \geq 0$



Ex. 5 Find domain, range, & sketch $h(x,y) = 4x^2 + y^2$

Sol'n: Recognize elliptic paraboloid: $z = 4x^2 + y^2$



$$\boxed{D = \mathbb{R}^2}$$

$$\boxed{\text{range}(h) = [0, \infty)}$$

$\underline{z=2}$
ellipse $2 = 4x^2 + y^2$
in xy-plane is a level
curve of h

$\underline{z=4}$ ellipse $4 = 4x^2 + y^2$
in xy plane is level curve
for $z = 4 = h(x,y)$

Ex. The level curves of $h(x,y) = 4x^2 + y^2$ are concentric ellipses. (See above)

Ex. 7 The plane $f(x,y) = 6 - 3x - 2y$ has level curves

$k = 6 - 3x - 2y$ for constants k.

$\underline{k=0}$ $0 = 6 - 3x - 2y \iff 2y = -3x + 6$

$y = \frac{-3}{2}x + 3$ (line in xy-plane)

$\underline{k=6}$ $6 = 6 - 3x - 2y$

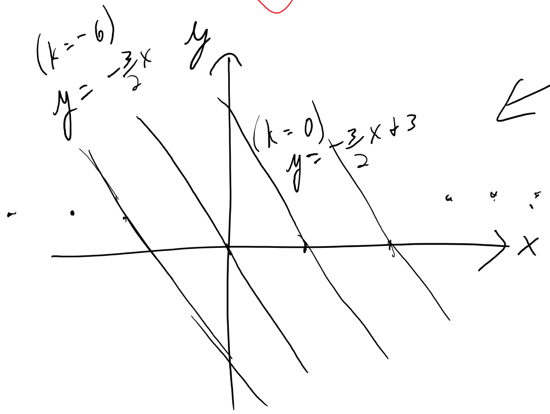
same slope $\rightsquigarrow$ level curves ...

$\underline{k=6}$ $\quad 6 = 6 - 3x - 2y$

$\iff 2y = -3x$

$\iff y = -\frac{3}{2}x$

same slope ⟶ level curves are parallel lines in plane



contour plot of $f(x,y)$

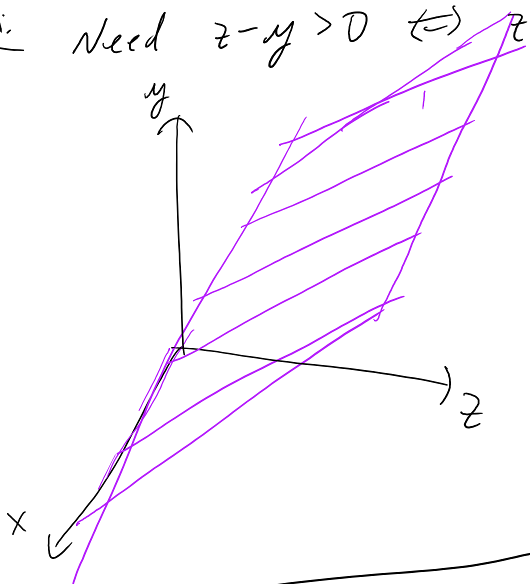
Functions of 3 or more variables

$f : D \to \mathbb{R}$, $\quad D \subset \mathbb{R}^3$ $\quad$ (domain in 3D)

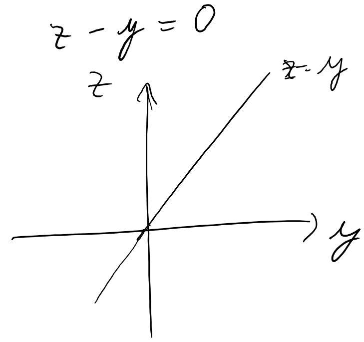
assigns a real number to each $(x,y,z) \in D$.

Ex. 10 $\quad$ Find domain of $f(x,y,z) = \ln(z-y) + xy\sin(z)$

Sol'n. $\quad$ Need $z - y > 0 \iff z > y$ $\quad (z = y)$



$z - y = 0$



$$\boxed{D = \{(x,y,z) \mid z > y\}}$$ (upper half space)

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<u>Rmk</u> $f(x,y,z)$ has <u>level surfaces</u>, which are given by $k = f(x,y,z)$, k constant, but are usually hard to draw.

<u>Ex. 11</u> Find level surfaces of $f(x,y,z) = x^2+y^2+z^2$

<u>Sol'n:</u> $k = x^2+y^2+z^2$

<u>$k=1$</u>

level surf. is sphere of rad. 1, cent. at $(0,0,0)$.

<u>k</u>

sphere rad. $+\sqrt{k}$ cent. at $(0,0,0)$

$\therefore$ level surf. are concentric spheres cent. at $(0,0,0)$.