4.4 Operational Properties

Theorem: **Derivatives of Transforms**

If $F(s) = \mathcal{L}\{f(t)\}$ and $n = 1,2, \ldots$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

- *Example:* Evaluate $\mathcal{L}\{t^2 \sin 2t\}$.

  Solution: By the above theorem,

  $$\mathcal{L}\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\{\sin 2t\} = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = \boldsymbol{\frac{16 - 12s^2}{(s^2+4)^3}}.$$

- *Definition:* If f and g are piece-wise continuous on the interval $[0, \infty)$, then the **convolution** of f and g, denoted by $f * g$, is defined as

  $$f * g = \int_0^t f(\tau) g(t - \tau) d\tau.$$

**It is left as an exercise to show that $f * g = g * f$.*

Theorem: **The Convolution Theorem**

If $f(t)$ and $g(t)$ are piece-wise continuous on $[0, \infty)$ and of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s).$$

Inverse Form: $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$.

- *Example:* Evaluate

  $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+5)^2}\right\}.$$

  Solution: Let

  $$F(s) = G(s) = \frac{1}{s^2+5}.$$

Then,

$$f(t) = g(t) = \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\} = \frac{1}{\sqrt{5}}\sin\sqrt{5}t.$$

Therefore, by the inverse form of the theorem,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+5)^2}\right\} = \left(\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right) * \left(\frac{1}{\sqrt{5}}\sin\sqrt{5}t\right) = \frac{1}{5}\int_0^t \sin\sqrt{5}\tau \sin\sqrt{5}(t-\tau)\,d\tau.$$

By using the trig identity that $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$, with $A = \sqrt{5}\tau$ and $B = \sqrt{5}(t-\tau)$, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+5)^2}\right\} = \frac{1}{10}\int_0^t [\cos\sqrt{5}(2\tau - t) - \cos\sqrt{5}t]d\tau = \boldsymbol{\frac{\sin\sqrt{5}t - \sqrt{5}t\cos\sqrt{5}t}{10}}.$$

**Trig functions lend themselves to many identities, so we were able to evaluate the convolution integral. While we should always try to evaluate the convolution integral, there will sometimes arise cases where we cannot.*

When $g(t) = 1$, the convolution theorem gives us the Laplace transform of the integral of f:

$$\boldsymbol{\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s} \quad \leftrightarrow \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau}$$

*The inverse form of the rule above can be used in place of partial fractions when s^n is a factor of the denominator and $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is easy to integrate.

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2(s^2+4)}\right\}.$$

Solution: Let

$$F(s) = \frac{2}{s^2+4}.$$

Then,

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t.$$

Therefore, by the inverse form,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t \sin 2\tau\, d\tau = \frac{1}{2} - \frac{1}{2}\cos 2t,$$

and

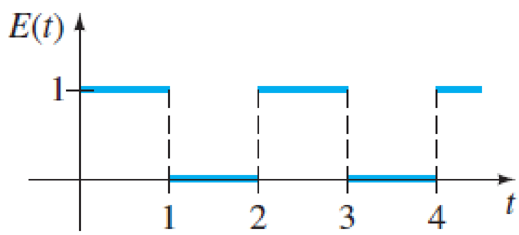
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \frac{1}{2}\int_0^t (1-\cos 2\tau)d\tau = \boldsymbol{\frac{1}{2}t - \frac{1}{4}\sin 2t}.$$

Theorem: **Transform of a Periodic Function**

If $f(t)$ is piece-wise continuous on $[0,\infty)$, of exponential order, and periodic with period T, then

$$\boldsymbol{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt.}$$

- *Example:* Find the Laplace transform of the square wave $E(t)$.



Solution: The function $E(t)$ has a period of $T = 2$. For $0 \le t < 2$, $E(t)$ can be defined by

$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2' \end{cases}$$

and outside the interval it can be defined by $E(t) = E(t+2)$. From the transformation of a periodic function theorem,

$$\mathcal{L}\{E(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}E(t)dt = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot 1dt + \int_1^2 e^{-st}\cdot 0dt\right]$$
$$= \frac{1}{1-e^{-2s}}\frac{1-e^{-s}}{s} = \boldsymbol{\frac{1}{s(1+e^{-s})}}.$$