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1. $6y'' + y' - y = 0 \quad y_1 = e^{x/3}$

$ax^2y'' + xy' - xy = 0$

$y_2(x) = v(x)y_1(x)$

$y_2 = ve^{x/3}$

$y_2' = v'e^{x/3} + \frac{v}{3}e^{x/3}$

$y_2'' = v''e^{x/3} + \frac{v'}{3}e^{x/3} + \frac{v'}{3}e^{x/3} + \frac{v}{9}e^{x/3}$

$y_2'' = e^{x/3}[v'' + 2/3\,v' + 1/9\,v]$

$6e^{x/3}[v'' + 2/3\,v' + \frac{1}{9}v] + e^{x/3}[v' + \frac{1}{3}v]$

$- e^{x/3}[v] = 0$

$6v'' + v'[4+1] + v[\frac{2}{3} + \frac{1}{3} - 1] = 0$

$6v'' + 5v' = 0$

$\boxed{v' = w \quad v'' = w'}$

$6w' + 5w = 0$

$\int \frac{w'}{w} = \int -\frac{5}{6}$

$\ln(w) = -\frac{5}{6}x + C$

$w = ke^{-5/6x}$

continued →

1. $\frac{dv}{dx}(v') = ke^{-5/6x}$

$dv(v') = ke^{-5/6x}\,dx$

$v = -\frac{6}{5}ke^{-5/6x} + c$

$k = -5/6 \qquad c = 0$

general solution constants

$v = e^{-5/6x}$

$y_2 = e^{-5/6x}e^{1/3x}$

$y_2 = e^{(1/3-5/6)x}$

$= e^{-1/2x}$

$y_2 = e^{-x/2}$

$y(x) = c_1y_1 + c_2y_2 = c_1e^{x/3} + c_2e^{-x/2}$

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2. $xy'' + y' = 0 \quad y_1 = \ln x$

$y_2(x) = v(x)\, y_1(x) = v(x) \ln(x)$

$y_2' = v' \ln(x) + \frac{v}{x}$

$y_2'' = v'' \ln(x) + \frac{v'}{x} + \frac{v'}{x} + v\left(\frac{-1}{x^2}\right)$

$y_2'' = v'' \ln(x) + \frac{2v'}{x} - \frac{v}{x^2}$

$x\left[v'' \ln(x) + \frac{2v'}{x} - \frac{v}{x^2}\right] + v' \ln(x) + \frac{v}{x} = 0$

$v'' x \ln(x) + v'(2 + \ln(x)) + v\left[\underbrace{-\frac{1}{x} + \frac{1}{x}}_{0}\right] = 0$

$v'' x \ln(x) + v'(2 + \ln(x)) = 0$

$v' = w \quad v'' = w'$

$w' x \ln(x) + w(2 + \ln(x)) = 0$

$\frac{w'}{w} = \frac{2 + \ln(x)}{-x \ln x}$

$\int \frac{dw}{w} = \int \left(\frac{-2}{x \ln x} - \frac{1}{x}\right) dy$

$\ln(w) = -\ln(x) - 2 \int \frac{1}{x \ln(x)} dx$

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$$= \int \frac{1}{x\ln x} \quad \boxed{\begin{matrix} u = \ln x \\ du = \frac{1}{x}\,dx \end{matrix}} \text{ — u-sub}$$

$$= \int \frac{1}{u}\,du$$

$$= \ln(|u|) + \ln(c)$$

$$= \ln[\ln(x)] + \ln(c)$$

$$\ln(w) = -\ln(x) - 2(\ln(\ln(x)) + \ln(c))$$

$$\ln(w) = \ln\left(\frac{1}{x}\right) + \ln\left(\frac{1}{(\ln x)^2}\right) + \ln\left(\frac{1}{c^2}\right)$$

$$\ln(w) = \ln\left[\frac{1}{c^2 x(\ln x)^2}\right]$$

$$w = \frac{1}{c^2 x(\ln x)^2}$$

$$\boxed{v' = \frac{1}{c^2 x(\ln x)^2}}$$

$$w = v'$$

$$\int dv = \int \frac{1}{c^2 x(\ln x)^2}\,dx \quad \boxed{\begin{matrix} u = \ln(x) \\ dv = \frac{1}{x}\,dx \end{matrix}}$$

$$v = \frac{1}{c^2}\int \frac{1}{x(\ln x)^2}\,dx$$

$$\int u^{-2}\,du = -u^{-1} + K = -\frac{1}{u} + K$$

$$= -\frac{1}{\ln(x)} + K$$

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$$u = \frac{-1}{c^2 \ln(x)} + \frac{k}{c^2}$$

$$y_2 = \ln(x)\left[\frac{-1}{c^2 \ln(x)} + \frac{k}{c^2}\right]$$

$$y_2 = -\frac{1}{c^2} + \frac{k}{c^2} \ln(x)$$

$$y_2 = -1 + \ln(x)$$

$$y = c_1 + c_2 \ln(x)$$

Substitute k and c

$k = 1$

$c = 1$

$-\frac{1}{c^2} = c_1$

$c_1 \neq 0$

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3. $x^2y'' - 3xy' + 5y = 0$; $y_1 = x^2\cos(\ln x)$

$ax^2y'' + bxy' + cy = 0$ $\qquad y = x^r$

$x^2((x^r))'' - 3x((x^r))' + 5x^r = 0$

$(x^r)'' = rx^{r-2}(r-1)$

$x^2rx^{r-2}(r-1) - 3x(x^r)' + 5x^r = 0$

$(x^r)' = rx^{r-1}$

$x^2rx^{r-2}(r-1) - 3xrx^{r-1} + 5x^r = 0$

$x^2rx^{r-2}(r-1) - 3xrx^{r-1} + 5x^r = 0$

$r^2x^r - 4rx^r + 5x^r = 0$

*factor x^r

$x^r(r^2 - 4r + 5) = 0$

$x^r \neq 0$ $\quad$ x

$r^2 - 4r + 5 = 0$

$a = 1$ $\qquad r_{1,2} = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$

$b = -4$

$c = 5$

(continued)

3. $r = \dfrac{-(-4) + \sqrt{(-4)^2 - 4(1)(5)}}{2}$; $2+i$

$r = \dfrac{-(+4) - \sqrt{(-4)^2 - 4(1)(5)}}{2}$; $2-i$

$r = 2+i$, $r = 2-i$

$r_1 \neq r_2$ - $r_1 = a + i\beta$ $r_2 = a - i\beta$

<u>general solution</u>

$y = x^a(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$

$y = x^2(c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)))$