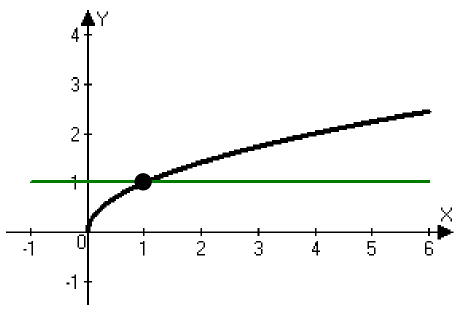
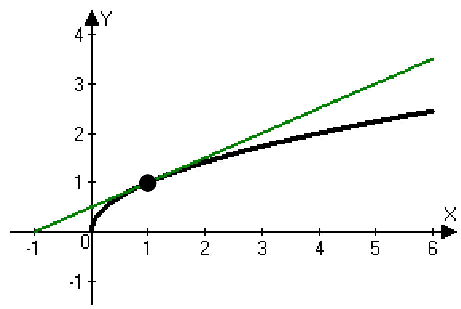
Unit 4.1 Taylor Polynomials and Approximations

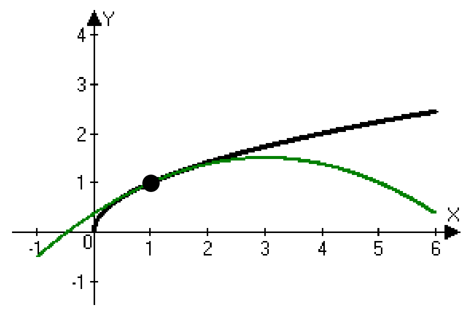
Now that we have completed our study of series tests, we turn our attention back to functions and will soon see the main reasons why we were studying series to begin with. Recall from Calculus I, that if a function is differentiable at a given point in its domain, then it can be approximated near that point by using a tangent line. The key feature of the tangent line is that at the point of tangency it shares the same output and slope (and thus direction) as the curve. However, what most curves have and what a tangent line lacks is concavity. It turns out that it is possible to use a parabola that not only shares the same output and slope as a given curve at some point, but also the same concavity. Consider the figure below showing how the function $y = \sqrt{x}$ can be approximated near $x = 1$ by a constant, linear-, and quadratic-function. The constant function only shares the same output as the square root function at $x = 1$. The linear function shares the same output and same slope (first derivative) at $x = 1$. The quadratic function shares the same output, same slope, and same concavity (second derivative) at $x = 1$.



Constant Approximation
Shares output with $\sqrt{x}$
when $x = 1$



Linear Approximation
Shares output and slope
with $\sqrt{x}$ when $x = 1$



Quadratic approximation
Shares output, slope, and
concavity with $\sqrt{x}$ when $x = 1$

But we don't have to stop at quadratic; we can use higher degree polynomials to locally approximate a function with even greater accuracy. Such polynomials are called Taylor polynomials and are defined below.

Definition: If f is function that is differentiable up to order at least n at $x = a$, then the polynomial

$$T_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called <u>the n^{th} degree Taylor polynomial for f centered at $x = a$.</u>

*In the case that $a = 0$, we sometimes call it a <u>Maclaurin polynomial</u>.

One can easily verify that the Taylor series defined above satisfies the following

$$T_n(a) = f(a), \qquad T_n'(a) = f'(a), \qquad T_n''(a) = f''(a), \dots, \qquad \text{and} \qquad T_n^{(n)}(a) = f^{(n)}(a)$$

So at $x = a$ (the center of the approximation), the Taylor polynomial shares the same output, first derivative (slope), second derivative (concavity), third derivative, and so on up to the n^{th} derivative, as the given function.

Example 1 Find the 3rd degree Taylor polynomial for $f(x) = e^x$ centered at $x = 0$.

Solution: First note that because we are centering the approximation at $x = 0$, we could have simply asked for the 3rd degree Maclaurin polynomial. In order to obtain a 3rd degree Taylor polynomial, we must compute all derivatives up to the 3rd derivative of e^x and evaluate each at the center $x = 0$.

$$f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \Rightarrow \quad f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$$

$$f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = e^0 = 1$$

We therefore obtain the following Taylor polynomial

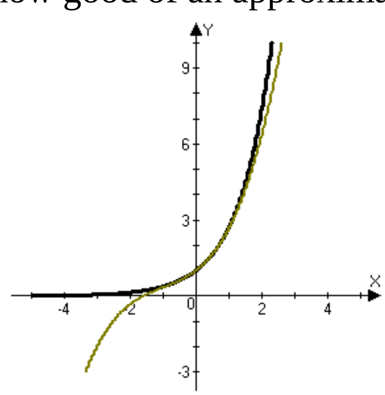
$$T_3(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$$= \frac{1}{1} + \frac{1}{1}x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

Thus we have $e^x \approx T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ near $x = 0$.

Below we compare outputs of the exponential function and the Taylor polynomial for a few values of x near 0 in order to get an idea of how good of an approximation we have obtained.

$e^{-1} \approx .368$	$T_3(-1) \approx .333$
$e^0 = 1$	$T_3(0) = 1$
$e^{.5} \approx 1.649$	$T_3(.5) \approx 1.646$
$e^1 = e \approx 2.718$	$T_3(1) \approx 2.667$
$e^2 \approx 7.389$	$T_3(2) \approx 6.333$



The graph of $y = e^x$ and its 3rd degree Taylor approximation centered at $x = 0$.

Example 2 Find the 4th degree Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$.

Solution: First note that because the natural logarithm function is undefined at $x = 0$, we could not have centered a Taylor polynomial there. One might choose to center the polynomial at a value of x where you need the approximation to be accurate. Centering at $x = 0$ has the advantage of producing simpler looking polynomials as $(x-a)^n$ just becomes x^n. Let us now compute and evaluate up through the 4th derivative of $\ln x$.

$$f(x) = \ln x \quad \Rightarrow \quad f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow \quad f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow \quad f''(1) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow \quad f'''(1) = \frac{2}{1^3} = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad \Rightarrow \quad f^{(4)}(1) = -\frac{6}{1^4} = -6$$

We therefore obtain the following Taylor polynomial

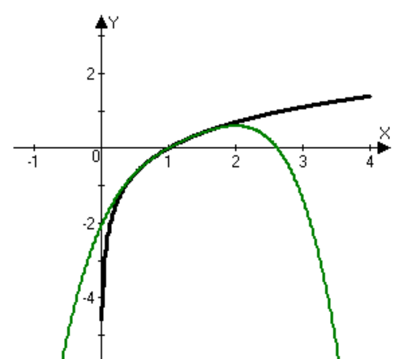
$$T_4(x) = \frac{f(1)}{0!} + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(0)}{4!}(x-1)^4$$

$$= \frac{0}{1} + \frac{1}{1}(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{6}(x-1)^3 + \frac{-6}{24}(x-1)^4$$

Thus we have $\ln x \approx T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$ near $x = 1$.

If we let $x = 2$ in the above, we find that

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .583$$



The graph of $y = \ln x$ and $y = T_4(x)$

The above calculation might remind you of approximating an infinite series (the alternating harmonic series) using a partial sum. Increasing the degree of the Taylor polynomial would give a better estimate, while the true value would be given by the infinite series. This leads us naturally to Power/Taylor Series which we will explore later in this unit.

Taylor's Remainder Theorem. We have seen that we can approximate a function f (that is differentiable up to degree at least n at $x = a$) using a Taylor Polynomial of the form

$$f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If we denote the error of the approximation by $R_n(x)$, which will depend on the value of x, then we can write $f(x) = T_n(x) + R_n(x)$. It can be shown using integration by parts that the error $R_n(x)$, can be written as

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$

As a result we get Taylor's Theorem which says:

If a function f is differentiable up to order $n+1$ on an interval I containing a, then for each x in I, there exists a number z between x and a where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

If we knew the value of z, then we could find our error exactly, but this is not a realistic expectation. Thus we look for an upper bound for this error term which is provided below.

$$|R_n(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!} \cdot \max\left\{\left|f^{(n+1)}\right|\right\}$$

Where $\max\left\{\left|f^{(n+1)}\right|\right\}$ is the maximum value of $\left|f^{(n+1)}\right|$ between x and a, and even this value can be approximated by something bigger and we will still obtain an upper bound for our error.

<u>Example 3</u> Recall that $T_6(x)$ for $f(x) = \cos x$, centered at 0 is $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ and that $f^{(6)}(x) = -\cos x$. Find an error bound for $R_6(x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Solution: Using the result of Taylor's Theorem as expressed above we have

$$\boxed{\underbrace{|R_6(x)|}_{|\text{Error}|} \leq \frac{|x-0|^7}{7!} \cdot \max \underbrace{\{|\sin x|\}}_{f^{(7)}(x)} \leq \frac{\left(\frac{\pi}{2}\right)^7}{7!} \cdot 1 \approx 0.005}$$

Because we are only interested in using values of x from the interval $\left[0, \frac{\pi}{2}\right]$, we replaced the x in the numerator above with $\frac{\pi}{2}$ to provide an upper bound for the error. We also used the fact that the sine function has a maximum value of 1. Our result means for any value of x between 0 and $\frac{\pi}{2}$, the cosine function and the 6th degree Taylor polynomial given above will differ by at most 0.005.