2.3 Exact Equations

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R, then its differential is given by

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

When $f(x, y) = c$, we are left with

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

- *Definition:* A differential expression $M(x, y)dx + N(x, y)dy$ is an **exact differential** in a region R if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is called an **exact equation** if the expression on the left-hand side is an exact differential.

Theorem: **Criterion for an Exact Differential**

Let $M(x, y)$ and $N(x, y)$ be continuous functions with continuous first partial derivatives in a rectangular region R. Then $M(x, y)dx + N(x, y)dy$ is an exact differential if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Proof: ***(→)*** Assume that $M(x, y)dx + N(x, y)dy$ is an exact differential. Then there exists some function $f(x, y)$ such that

$$M(x, y)dx + N(x, y)dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \rightarrow \quad M(x, y) = \frac{\partial f}{\partial x}, \qquad N(x, y) = \frac{\partial f}{\partial y}.$$

Therefore, by continuity of the first partials,

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}.$$

The proof of the other direction follows from the construction of $f(x, y)$.

We want to find an f such that $f_x = M(x, y)$ and $f_y = N(x, y)$.

If $f_x = M(x,y)$, then we can integrate both sides with respect to x to get

$$f(x,y) = \int M(x,y)\,dx + g(y),$$

where $g(y)$ is the "constant" of integration. If we then differentiate with respect to y we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y).$$

Thus, we get

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx.$$

Finally we integrate this with respect to y and substitute the result in for $g(y)$ in the equation

$$f(x,y) = \int M(x,y)\,dx + g(y).$$

The implicit solution is then $\boldsymbol{f(x,y) = c}$.

**Note: we could have just as easily started with $f_y = N(x,y)$ and integrated with respect to y.*

The exact formulas are not as important and remembering the overall process involved.

- *Example:* Solve $2xy\,dx + (x^2 - 1)dy = 0$.

  Solution: Check to see that we have an exact equation.

  $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 - 1) = 2x$$

  Since $M_y = N_x$, we do have an exact equation, and therefore a function $f(x,y)$ does exist. Starting with $f_x = M(x,y)$, we get

  $$f(x,y) = \int 2xy\,dx = x^2y + g(y).$$

  Taking the derivative of f with respect to y and setting equal to $N(x,y)$ gives

  $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2y + g(y)] = x^2 + g'(y) = x^2 - 1 \quad \rightarrow \quad g'(y) = -1 \quad \rightarrow \quad g(y) = y + c_1.$$

  Therefore our implicit solution is $\boldsymbol{x^2y + y + c_1 = 0}$ or $\boldsymbol{x^2y + y = c}$.

  The solution is **not $f(x,y) = x^2y + y$, it is $f(x,y) = c$.*

For exact equations it may not always be possible to find an explicit solution, so it is very common to give implicit solutions.

It may be possible to solve non-exact equations in a similar fashion using an integrating factor. The general method is to find a function $\mu(x,y)$ such that $\mu M dx + \mu N dy = 0$ is exact. By our earlier theorem, this means that

$$\frac{\partial}{\partial y}[\mu M] = \frac{\partial}{\partial x}[\mu N] \quad \rightarrow \quad \mu_x N - \mu_y M = (M_y - N_x)\mu.$$

However in order to find $\mu(x,y)$, we would need to solve a partial differential equation (not an easy task). If we assume μ is a function of a single variable, we would be able to solve for μ. This "assumption" leads to two cases:

i. If $(M_y - N_x)/N$ is a function of x alone, then $\mu = e^{\int \frac{M_y - N_x}{N} dx}$
ii. If $(N_x - M_y)/M$ is a function of y alone, then $\mu = e^{\int \frac{N_x - M_y}{M} dy}$

- *Example:* Solve $xy\, dx + (2x^2 + 3y^2 - 20)dy = 0$.

  Solution: Check to see that we have an exact equation.

  $$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(xy) = x, \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 3y^2 - 20) = 4x$$

  Since $M_y \neq N_x$, we do not have an exact equation. However, it is "close" to exact, so we may consider an integrating factor. Looking at our two ratios we get

  $$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = -\frac{3x}{2x^2 + 3y^2 - 20}, \qquad \frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3}{y},$$

  We notice that the second produces a function of just y. So we get an integrating factor of

  $$\mu = e^{\int \frac{3}{y} dy} = |y|^3 = y^3.$$

  Multiplying by μ then gives the exact equation

  $$xy^4\, dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0.$$

  It is left as an exercise to show that the above equation is in fact exact and that the general solution would be $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$.

**Note: when trying to determine if you have an exact equation, make sure that it is in the precise form $M(x,y)dx + N(x,y)dy = 0$, not the form $G(x,y)dx = H(x,y)dy$.*