6.1 Solutions About Ordinary Points

Recall: A power series about $x = a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Every power series has a radius of convergence R. If $R > 0$, then the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. The radius of convergence then gives us an interval of convergence $(a-R, a+R)$.

We can find the radius of convergence for a power series by the ratio test. Suppose $c_n \neq 0$ for all n, and that

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L.$$

When $0 < L < \infty$, then the radius of convergence $R = 1/L$. If $L = 0$, then $R = \infty$, and the series converges for all values of x.

A power series defines a function $f(x)$, whose domain is the interval of convergence. Moreover, $f'(x)$ and $\int f(x)dx$ can be found by term-by-term differentiation or integration.

A function f is analytic at a point a if it can be represented by a power series about $x = a$ with a positive, or infinite, radius of convergence.

An important skill for the application of power series to differential equations is the ability to reindex series in order to combine two or more summations into a single series.

- *Example:* Write the following summation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

as a single power series whose general term involves x^k.

Solution: In order to combine the two series, both summation indices must start with the same number and the powers of x in each series must be "in phase" (i.e. they must start with the same power of x). First we will look at getting our series in phase with one another.

For our example the first series begins with x^0 while the second series begins with x^1. To get the two in phase, we "pull" the first term of the first series outside the summation, giving us

$$2c_2x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}.$$

Now both series start with x^1. Next we will reindex the two series so that we can combine them together. In general, any reindexing will work, but for our problem we are asked to have a general term with x^k, so we must choose our reindexing accordingly. In order to get the proper indexing, we let $k = n - 2$ for the first series and $k = n + 1$ for the second series. This means that we need to substitute $n = k + 2$ and $n = k - 1$ into the first and second series, respectively. Putting this together, we get

$$2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k = 2c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-1}]x^k.$$

The lower bound in the summation for both now begins at $k = 1$, which lines up with the fact that they both begin with x^1.

- *Definition:* A point x_0 is said to be an **ordinary point** of the differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \to \quad y'' + P(x)y' + Q(x)y = 0$$

if both $P(x)$ and $Q(x)$ are analytic at x_0. A point that is not an ordinary point is called a **singular point** of the equation.

We will be mainly interested in the case where $P(x)$ and $Q(x)$ are rational functions, which are analytic whenever the denominator is not zero. Therefore, for our situation, the point $x = x_0$ is an **ordinary point** if $a_2(x_0) \neq 0$, and $x = x_0$ is a **singular point** if $a_2(x_0) = 0$.

Theorem: Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, we can find two linearly independent solutions in the form of a power series centered at x_0. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the nearest singular point.

The method of finding a series solution to a homogeneous linear second-order differential equation is similar to the process used in finding particular solutions by the method of undetermined coefficients. The general idea is to substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ into the DE, combine the series together, then solve for the coefficients.

*While any ordinary point x_0 will work, it is usually preferable to use $x_0 = 0$, when possible.

- *Example:* Solve $y'' + xy = 0$

Solution: First we note that there are no singular points for the equation, so we may choose $x = 0$ for our center. Taking derivatives of $y = \sum_{n=0}^{\infty} c_n x^{\,n}$, we get

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\,.$$

Substituting these series into the equation, we get

$$y'' + xy = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \; + x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \; + \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

By our earlier example, we showed that these series combine to give us

$$2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-1}]x^k = 0.$$

At this point, since the sum is equal to zero, then by the identity property for series, the coefficients for each power of x must be 0. Therefore, we get that $c_0 = 0$ and

$$(k+2)(k+1)c_{k+2} + c_{k-1} = 0, \qquad k = 1,2,\ldots$$

This expression is referred to as a **recurrence relation**, and it determines c_k in terms of previous terms. Solving for c_{k+2}, we get

$$c_{k+2} = -\frac{c_{k-1}}{(k+2)(k+1)}, \qquad k = 1,2,\ldots$$

Expanding this out, we find

$$c_3 = -\frac{c_0}{3*2} \qquad c_6 = -\frac{c_3}{6*5} = \frac{c_0}{180} \qquad c_9 = -\frac{c_6}{9*8} = \frac{-c_0}{12960}$$

$$c_4 = -\frac{c_1}{4*3} \qquad c_7 = -\frac{c_4}{7*6} = \frac{c_1}{504} \qquad c_{10} = -\frac{c_7}{10*9} = \frac{-c_1}{45360}$$

$$c_5 = -\frac{c_2}{5*4} = 0 \qquad c_8 = -\frac{c_5}{8*7} = 0 \qquad c_{11} = -\frac{c_8}{11*10} = 0$$

and so on. Following this pattern, we notice that all of the coefficients c_{3k} are multiples of c_0, all the coefficients c_{3k+1} are multiples of c_1, and all the coefficients of c_{3k+2} are multiples of c_2, and therefore zero. Therefore, since our recursive formula leaves c_0 and c_1 completely undetermined, we can allow them to be whatever we like. When $c_0 = 1$ and $c_1 = 0$, we get $y_1(x)$ below, and when $c_0 = 1$ and $c_1 = 0$, we get $y_2(x)$ below.

$$\boldsymbol{y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \frac{1}{12960}x^9 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k}}{2 \cdot 3 \cdots (3k-1)(3k)}}$$

$$\boldsymbol{y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \frac{1}{45360}x^{10} + \cdots = x + \sum_{k=1}^{\infty} \frac{(-1)^k x^{3k+1}}{3 \cdot 4 \cdots (3k)(3k+1)}}$$

We can then put these two solutions together to get a general solution of

$$\boldsymbol{y(x) = C_1 y_1(x) + C_2 y_2(x)},$$

where C_1 and C_2 are arbitrary constants.

**Note: when finding y'', our lower bound changed from 0 to 2. This is because the first two terms in series, when starting at $n = 0$, would both be zero, so we drop them from the series.*

Two comments as about the series solutions $y_1(x)$ and $y_2(x)$ above. First, note that while there is a "pattern" to the terms in both series solutions, the sigma notation is not the easiest to create in all cases. For this reason, it is common to just list the first 4 terms of the series solution, with a "$+\cdots$" after. For many cases, four terms will give you an acceptable margin of error in your solutions.

The second thing to note is that our series solutions do not represent nice clean functions. This is the power of power series solutions. Even when we cannot find "nice" solutions, we can still find a series solution to our problems.

In the above example, we used C_1 and C_2 for our arbitrary constants in the linear combination. We could have just as easily saved a step and used the c_0 and c_1 as our arbitrary constants in our general solution. However, in some problems we will find that the recurrence relation gives us constants that are combinations of c_0 and c_1. In those cases, the process of "choosing" values for c_0 and c_1 before creating y_1 and y_2, and then creating "new" arbitrary constants for our general solution does come in handy.

- *Example:* Solve $y'' - (1 + x)y = 0$

Solution: First we note that there are no singular points for the equation, so we may choose $x = 0$ for our center. Taking derivatives of $y = \sum_{n=0}^{\infty} c_n x^{\,n}$, we get

$$y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\,.$$

Substituting these series into the equation, we get

$$y'' - (1+x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - x\sum_{n=0}^{\infty} c_n x^n$$
$$= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0.$$

Next we need to adjust our series to be able to combine them, getting

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = 2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} = 2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k,$$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + \sum_{n=1}^{\infty} c_n x^n, \quad \text{and} \quad \sum_{n=0}^{\infty} c_n x^{n+1} = \sum_{k=1}^{\infty} c_{k-1}x^k.$$

Combining the three together, we get

$$2c_2 - c_0 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} - c_k - c_{k+1}]x^k.$$

At this point, since the sum is equal to zero, then by the identity property for series, the coefficients for each power of x must be 0. Therefore, we get that $c_2 = \frac{1}{2}c_0$ and

$$(k+2)(k+1)c_{k+2} + c_{k-1} = 0, \qquad k = 1,2,\ldots$$

This expression is referred to as a **recurrence relation**, and it determines c_k in terms of previous terms. Solving for c_{k+2}, we get

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}, \qquad k = 1,2,\ldots$$

Therefore, all coefficients c_n, for $n \geq 3$, are expressed in terms of both c_0 and c_1. To simplify things, and find our two series, we can choose the following values for c_0 and c_1: giving us

$c_0 \neq 0, \qquad c_1 = 0$

$$c_2 = \frac{c_0}{2}$$

$$c_3 = \frac{c_1 + c_0}{3 * 2} = \frac{c_0}{3 * 2} = \frac{c_0}{6}$$

$$c_4 = \frac{c_2 + c_1}{4 * 3} = \frac{c_0}{4 * 3 * 2} = \frac{c_0}{24}$$

$$c_5 = \frac{c_3 + c_2}{5 * 4} = \frac{c_0}{5 * 4}\left[\frac{1}{6} + \frac{1}{2}\right] = \frac{c_0}{30}$$

$c_0 = 0, \qquad c_1 \neq 0$

$$c_2 = \frac{c_0}{2} = 0$$

$$c_3 = \frac{c_1 + c_0}{3 * 2} = \frac{c_1}{3 * 2} = \frac{c_1}{6}$$

$$c_4 = \frac{c_2 + c_1}{4 * 3} = \frac{c_1}{4 * 3} = \frac{c_1}{12}$$

$$c_5 = \frac{c_3 + c_2}{5 * 4} = \frac{c_1}{6 * 5 * 4} = \frac{c_1}{120}$$

and so on. Letting $c_0 = 1$ in the first case and $c_1 = 1$ in the second case, we get

$$\boldsymbol{y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots}$$

$$\boldsymbol{y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots.}$$

We can then put these two solutions together to get a general solution of

$$\boldsymbol{y(x) = C_1 y_1(x) + C_2 y_2(x)},$$

where C_1 and C_2 are arbitrary constants.

**Note: while $y_1(x)$ and $y_2(x)$ contain common powers of x, by examination, it is easy to see that they do represent two linearly independent solutions.*

The method we used above was applied when we had polynomial coefficients. However, we can apply the same method to other problems as long as we are able to find a power series expansion of the coefficients. That means we can use this process whenever our coefficients include functions such as e^x, $\sin x$, $\cos x$, and $\ln x$. However, the process does get more complicated due to the need to multiply two power series together.