Unit 2.3 Indeterminate Forms and L'Hopital's Rule

We now return to the world of limits. From Calculus I, you should recall the concept of an *indeterminate form* that might arise when attempting to evaluate a limit using substitution methods. For example, let us take a look at the limit: $\lim_{x\to 2}\frac{x-2}{x^2-x-2}$. The approach that you most likely learned to take is to substitute in 2 for x to obtain

$$\lim_{x\to 2}\frac{x-2}{x^2-x-2} \overset{\text{sub}}{\cong} \frac{(2)-2}{(2)^2-(2)-2} = \frac{0}{0} = ???$$

What do we make of the result $0/0$? First of all 0 divided by 0 is undefined, but in our usage above, that is not what we really meant when we wrote 0/0. In fact, our use of the = above is not technically appropriate, but as long as we have an understanding of what we mean by the above, it can be justified. So what do we mean? We are interested in the value that the entire function approaches as x approaches 2. Our substitution approach only told us that the numerator and denominator were both approaching zero, although they never become zero. Thus our form 0/0 is only an indicator that the numerator and denominator both approach zero, however, from that information alone, we cannot predict the behavior of the function as a whole; thus we say that 0/0 is an indeterminate form. The technique learned in Calc I, to deal with this particular kind of example is demonstrated below.

$$\lim_{x\to 2}\frac{x-2}{x^2-x-2} = \lim_{x\to 2}\frac{x-2}{(x-2)(x+1)} = \lim_{x\to 2}\frac{1}{x+1} \overset{\text{sub}}{\cong} \frac{1}{(2)+1} = \boxed{\frac{1}{3}}$$

The value that we assign to the limit is therefore 1/3. The idea behind the method above is based on the fact that the functions $\frac{x-2}{x^2-x-2}$ and $\frac{1}{x+1}$ are equal except when $x = 2$. Since in the limit, we are only interested in letting x approach 2 (and thus never equal 2), both functions will approach the same value. When we applied our substitution into $\frac{1}{x+1}$, it yielded a form 1/3. While that was our final answer, we have to keep in mind that technically at that point we were only indicating that the numerator was approaching 1, while the denominator was approaching 3. Since there is only ONE number that we can multiply by 3 to get 1 (which is what $\frac{1}{3} = 1 \div 3$ correspondes to), namely 1/3, the form one over three was determinate and gave us an answer of 1/3. If we compare that to the indeterminate form 0/0, we can multiply 0 by anything to get 0, so we have no idea what to make of this ratio.

Before we look at the Calc II version of limits and L'Hopital's Rule, let us briefly list a handful of limit forms that you should know at this point; each is based on the properties of some of our common functions. Each of the statements below will take the form $(Limit\ Form) \rightsquigarrow (value\ of\ limit)$. Here we use $\rightsquigarrow$ to mean "corresponds to". Keep in mind that this is a non-technical/informal list that is meant to communicate basic ideas. Be careful not to misinterpret these statements. Ask me for clarification if needed.

Common Determinate Forms: Let a and b be constants with $b \neq 0$.

1) $\frac{a}{b} \rightsquigarrow \frac{a}{b}$ 2) $\frac{b}{0} \rightsquigarrow \pm\infty$ ($\pm$ here might depend on a left- vs. right-hand limit)

3) $\frac{a}{\pm\infty} \rightsquigarrow 0$ 4) $\frac{\pm\infty}{a} \rightsquigarrow \pm\infty$ 5) $b \cdot \pm\infty \rightsquigarrow \pm\infty$

6) $e^{\infty} \rightsquigarrow \infty$ 7) $e^{-\infty} \rightsquigarrow 0$ 8) $\ln 0 \rightsquigarrow -\infty$ 9) $\ln \infty \rightsquigarrow \infty$

These would apply to any exponential function with base bigger than 1.	These would apply to any log function with base bigger than 1.

10) $\arctan(\infty) \rightsquigarrow \frac{\pi}{2}$ 11) $\arctan(-\infty) \rightsquigarrow -\frac{\pi}{2}$

To make sure it is clear what we mean by the above, let us consider #8, $\ln 0 \rightsquigarrow -\infty$. What is meant by this is that as the inside of the logarithm function approaches 0, the value of the logarithm approaches $-\infty$. By itself, $\ln 0$ is undefined.

The Indeterminate Quotients $\mathbf{\frac{0}{0}}$ **and** $\frac{\pm\infty}{\pm\infty}$**.** We will now turn our attention to dealing with several indeterminate forms. We begin with the cases where our numerator and denominator both approach 0 or both approach an infinity (positive or negative). In these cases, we often turn to L'Hopital's Rule, which is given below.

> **Theorem:** (L'Hopital's Rule) Suppose that f and g are differentiable on an open interval containing c, except possibly at c, and suppose that $g' \neq 0$ on that interval except possibly at c.
>
> If $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ yields an indeterminate form of $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then
>
> $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

This rule does not apply to any other limit forms. What it tells us is that we can replace our original function with a new function obtained by separately taking the derivative of the numerator and denominator and the resulting limit will be the same as for the original function. Note: we are not taking the derivative of the original function as that would require the Quotient Rule.

Example 1 Evaluate the limit $\lim_{x\to 0}\frac{2^x-1}{x}$.

Solution: Our initial attempt at a substitution yields the indeterminate form 0/0.

$$\lim_{x\to 0}\frac{2^x-1}{x} \overset{sub}{\cong} \frac{2^0-1}{0} = \frac{0}{0} \quad \text{(indetermintate)}$$

Let me remind us one last time that our use of = above is not a technical one, but more a matter of convenience. After all, the limit does not equal 0/0 since 0/0 has no value; it's undefined. Again, we are just noting that by way of a substitution, that both our numerator and denominator approach 0. Thus we can apply L'Hopital's Rule as follows.

$$\lim_{x\to 0}\frac{2^x-1}{x} \overset{LH}{\cong} \lim_{x\to 0}\frac{2^x\ln 2}{1} \overset{sub}{\cong} \frac{2^0\ln 2}{1} = \boxed{\ln 2}$$

I will use the LH and sub above the equal sign to make it clear when I am applying L'Hopital's Rule and when I am substituting.

Example 2 Evaluate the limit $\lim_{x\to\infty}\frac{6x-1}{e^{2x}}$.

Solution: We should first not that as $x\to\infty$, both the numerator and denominator approach infinity. Thus we have an indeterminate form of $\frac{\infty}{\infty}$ and we can use L'Hopital's Rule.

$$\lim_{x\to\infty}\frac{6x-1}{e^{2x}} \overset{LH}{\cong} \lim_{x\to\infty}\frac{6}{2e^{2x}} = \boxed{0}$$

We obtained our final answer based on the following two observations. As $x\to\infty$, the denominator will approach ∞, and since the numerator is 6, we get the form $\frac{6}{\infty} \rightsquigarrow 0$.

***The Indeterminate Products* $\mathbf{0(\pm\infty)}$.** We now consider examples where we view our function as a product of two factors, one of which approaches 0 and the other an infinity. The factor that approaches 0 is trying to make the entire function have a small value, while the factor approaching an infinity is trying to make the entire function have a value that is large in magnitude. So the question is, what happens when the two interact together? In these cases we use the fact that multiplication is equivalent to dividing by a reciprocal. Suppose that f is the factor approaching 0 and g is the factor approaching $\pm\infty$. Observe that

$$\underbrace{f}_{\substack{\downarrow\\0}}\cdot\underbrace{g}_{\substack{\downarrow\\\pm\infty}} = \frac{f}{1/g} \rightsquigarrow \frac{0}{1/\pm\infty} \rightsquigarrow \frac{0}{0} \qquad \text{and} \qquad \underbrace{f}_{\substack{\downarrow\\0}}\cdot\underbrace{g}_{\substack{\downarrow\\\pm\infty}} = \frac{g}{1/f} \rightsquigarrow \frac{\pm\infty}{1/0} \rightsquigarrow \frac{\pm\infty}{\pm\infty}$$

In either case, we can rewrite our function in a form where L'Hopitals Rule applies. Which is the best choice depends on the example. In making the decision, you will find a strong similarity to the decision you make when using integration by parts when you decide what to let u equal and what to let dv equal. Here the question is "what factor do we send to the denominator?"

Example 3 Evaluate the limit $\lim_{x\to 0^+} x\ln x$.

Solution: First note that we currently have a limit form of $0\cdot -\infty$, which is indeterminate. We have the following options for rewriting the limit

$$\lim_{x\to 0^+}\frac{x}{1/\ln x}\quad \text{(form } 0/0) \qquad \text{or} \qquad \lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{\ln x}{x^{-1}}\quad \text{(form } -\infty/\infty)$$

Only one of these is promising and you should attempt both to see why it is the second form above that will lead to a solution by way of L'Hopital's Rule. It basically comes down to what you would be left with after taking derivatives of your numerators and denominators. Let us proceed with the second rewrite above.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1}} \overset{LH}{\cong} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-x^{-2}}\cdot\frac{-x^2}{-x^2} = \lim_{x\to 0^+}(-x) \overset{sub}{\cong} 0$$

Example 4 Evaluate the limit $\lim_{x\to\infty} x^2e^{-5x}$.

Solution: As the problem is currently expressed, we have a limit form of $\infty\cdot 0$, which is indeterminate. We have the following options for rewriting the limit

$$\lim_{x\to\infty}\frac{x^2}{e^{5x}}\quad \text{(form } \infty/\infty) \qquad \text{or} \qquad \lim_{x\to\infty}\frac{e^{-5x}}{x^{-2}}\quad \text{(form } 0/0)$$

This situation is similar to the last problem, however in this case, the first option will lead to a solution by L'Hopital's Rule. In the last example, it was better to put the power of x in the denominator because the derivative of the log in the numerator allowed for a nice simplification to occur. That would not happen here and putting the power of x in the denominator here, would lead you to an endless repetition of L'Hopital's Rule, that would get you nowhere. I encourage you to try it in order to see why that is the case. Let us now proceed.

$$\lim_{x\to\infty} x^2e^{-5x} = \lim_{x\to\infty}\frac{x^2}{e^{5x}} \overset{LH}{\cong} \underbrace{\lim_{x\to\infty}\frac{2x}{5e^{5x}}}_{\substack{\text{still in the}\\ \text{form } \infty/\infty}} \overset{LH}{\cong} \underbrace{\lim_{x\to\infty}\frac{2}{25e^{5x}}}_{\substack{\text{now in the}\\ \text{form } 2/\infty}} = 0$$

As you can see, in the previous problem we used L'Hopital's Rule twice. Our first application of L'Hopital's Rule gave a new function that still had an indeterminate form which allowed for a second application of L'Hopital's Rule. Theoretically, you could have a function which requires any number of applications of L'Hopital's Rule. For example, if we started with the function $\frac{x^{30}}{e^{5x}}$, it would take 30 applications of L'Hopital's Rule to produce a determinate form. One however could anticipate the result without applying the rule (LH), as the exponential would never go away and eventually repeated differentiation of our numerator would eventually lead to a constant. Thus the limit would still be 0 as it would come from the form of a constant over infinity. I will never give you examples requiring more than three applications of L'Hopital's rule, but I do expect that you will show all steps that lead to a determinate form. You will not be allowed to give the answer based on what you know in your head. I will emphasize the point more in the narrated examples.

One last comment before we move to the next case. You might be wondering how you are supposed to know which factor to send to the denominator. While a complete list of situations is not reasonable to produce, I will mention a few common situations.

1) If you have a positive power of x multiplying an exponential, sine, or cosine function, keep the power of x in the numerator (like in example 4).

2) If you have a positive power of x multiplying a log or an inverse tangent function, send the power of x to the denominator (like in example 3).

3) In other cases, you might have to try out both approaches until you get something that works. Keep in mind that you may need more than one application of L'Hopital's Rule.

The Indeterminate Powers $\mathbf{0^0, \infty^0,}$ **and** $\mathbf{1^\infty}$**.** I first recommend that you try to convince yourself why these forms are indeterminate. Let me just speak to the first case of 0^0. First of all, remember we are talking about limit forms, not arithmetic. Zero to the power zero is undefined. However, by a limit form, we are only indicating that both the base and the exponent are approaching zero. Alone, the base approaching zero is trying to make the function small as a base of zero typically gives a value of 0 (e.g. $0^3 = 0$). Alone, the exponent approaching zero is trying to make the function get close to one as an exponent of zero typically gives a value of 1 (e.g. $2^0 = 1$). This conflict should give you an idea of why we have an indeterminate form. The way that we handle these indeterminate powers is to creatively transform them into a form involving a product and then rewrite the product as a quotient, like we did above. The method that I show here is slightly different than you might find in the book. Use whichever approach you prefer. I will base my approach on changing the base. We change the base by first applying a log to our function, followed by its inverse—an exponential function. Then using log properties, we create our desired product. This manipulation is shown below.

$$a^b = e^{\ln a^b} = e^{b \ln a}$$

In our examples, the a and b will be functions. Let us look at an example to see how it works.

<u>Example 5</u> Evaluate the limit $\lim\limits_{x \to 0^+} x^{\sqrt{x}}$.

Solution: First note that this limit currently yields the indeterminate form 0^0. We now change the base to e as shown above.

$$x^{\sqrt{x}} = e^{\ln x^{\sqrt{x}}} = e^{\sqrt{x} \ln x}$$

Thus

$$\lim_{x \to 0^+} x^{\sqrt{x}} = \lim_{x \to 0^+} e^{\sqrt{x} \ln x} = e^{\lim\limits_{x \to 0^+} \sqrt{x} \ln x}$$

The last step above is merely an indication that as we look at the limit in the middle, the constant base allows us to shift our attention to the exponent and see what the exponent approaches. Thus we will calculate $\lim\limits_{x \to 0^+} \sqrt{x} \ln x$. The answer to the original problem will therefore be e raised to the power given by this limit. Note that this limit is currently in the form $0 \cdot -\infty$ and can thus be expressed as a quotient that allows for L'Hopital's Rule. From our discussion above, it should be clear that we should send the radical factor (i.e. a power of x) to the denominator.

$$\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}} \overset{LH}{\cong} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-3/2}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-3/2}} \cdot \frac{-2x^{3/2}}{-2x^{3/2}} = \lim_{x \to 0^+} (-2x^{1/2}) \overset{sub}{\cong} 0$$

Thus our original limit gives us

$$\lim_{x \to 0^+} x^{\sqrt{x}} = \lim_{x \to 0^+} e^{\sqrt{x} \ln x} = e^{\lim\limits_{x \to 0^+} \sqrt{x} \ln x} = e^0 = \boxed{1}$$