Unit 1.5: Differential Equations

In this course, we take only an introductory look at differential equations. After a brief explanation of the separation of variables technique, you will be directed to a handout for further coverage of this content.

Differential Equations (DE's). Without being technical, a differential equation in the variables x (the independent variable) and y (the dependent variable) is an equation expressing the relationship between x and y that involves derivatives of y. While the differential equations that we will be solving will only include the first derivative (we call them first order DE's), a differential equation can involve derivatives of any order. A few simple examples can be found below.

$$y' = 3x - \frac{y^2}{x} \qquad 4y'' - 3y' + y = \cos x \qquad 2\frac{d^2y}{dx} = 4\frac{dy}{dx} + x - 1$$

The goal in solving the differential equation is to find a relationship between x and y that is free of any derivatives. For example, suppose that we have an equation expressing an objects velocity (derivative of position) in terms of time. One may wish to instead express the objects position in terms of time. A simple example is shown below.

$$\underbrace{\frac{ds}{dt}}_{v(t)} = -9.8t \quad \Longrightarrow \quad s(t) = -4.9t^2 + s_0$$

The above is a simple example of a first order differential equation along with its general solution. The solution is <u>general</u> as it describes what all solutions look like by including the arbitrary constant s_0. A <u>particular solution</u> would be a solution where a specific value of s_0 would be included. Let us now look at a useful technique for solving a particular class of first order differential equations.

Separation of Variables. A <u>separable differential equation</u> is one that can be expressed as

$$\frac{dy}{dx} = f(y) \cdot g(x) \qquad \boxed{(\spadesuit)}$$

What this says is that when the derivative is isolated, the other side of the equation can be factored so that one of the factors is a function of only y and the other factor is a function of only x. For example, $\frac{dy}{dx} = \frac{y+1}{x}$ can be expressed as $\frac{dy}{dx} = (y+1) \cdot \frac{1}{x}$ and $xy + \frac{dy}{dx} = yx^2$ can be expressed as $\frac{dy}{dx} = y(x^2 - x)$ and thus both are separable. However, $\frac{dy}{dx} = y + x$ is not separable as we cannot express the right side as the necessary type of product.

Let us now look at a method of solution for such a DE. We begin by dividing both sides of the equation (♠) by $f(y)$ and then we integrate both sides of the equation with respect to x as follows.

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \quad \rightarrow \quad \int \frac{1}{f(y)}\underbrace{\frac{dy}{dx}dx}_{dy} = \int g(x)dx \quad \rightarrow \quad \int \frac{1}{f(y)}dy = \int g(x)dx$$

As long as we know the antiderivatives of $\frac{1}{f(y)}$ and $g(x)$, we will have solved the DE once we have removed the integral signs. In the separation of variables process, one simply treats $\frac{dy}{dx}$ as an ordinary fraction and manipulates the equation (♠) into the form $\frac{1}{f(y)}dy = g(x)dx$ using basic algebra and then simply attaches the integral symbol. Thus we will not see all of the steps as shown above when solving the DE. After we complete the solving process we will be left with a general solution of the DE. If initial conditions are included with the DE, we will then find a particular solution that satisfies the initial conditions.

<u>Example 1</u>. Find the general solution of the DE $(x^2+1)\frac{dy}{dx} = \frac{x}{y}$.

Solution: We begin by multiplying both sides of the DE by dx to separate the differentials. We then want all factor containing x to go with the dx and all factors containing y to go with the dy. To accomplish this we can simply divide both sides by (x^2+1) and multiply both sides by y. The last step is to attach the integral symbols and obtain the antiderivatives. These steps are shown below.

$$(x^2+1)\frac{dy}{dx} = \frac{x}{y} \quad \rightarrow \quad (x^2+1)dy = \frac{x}{y}dx \quad \rightarrow \quad ydy = \frac{x}{x^2+1}dx \quad \rightarrow \quad \int ydy = \int \frac{x}{x^2+1}dx$$

This leaves us with $\frac{1}{2}y^2 + C_1 = \frac{1}{2}\ln(x^2+1) + C_2$. A u-substition was used for the right-hand integral. Instead of including an arbitrary constant on both sides of the equation, one could simply subtract C_1 from both sides and obtain $C_2 - C_1$ which we then rename to be C. In future examples we will only introduce a single constant. The general solution expressed above is in an *implicit form* as it is not solved for y in terms of x. While this is ok, and sometimes our only option, it is nice to write the solution of a DE in an *explicit form* (solved for y) when possible. Both forms of the solution are shown below.

Implicit form of solution: $\frac{1}{2}y^2 = \frac{1}{2}\ln(x^2+1) + C$
Explicit form of solution: $y = \pm\sqrt{\ln(x^2+1) + C}$

More information regarding these forms and the renaming of arbitrary constants will be provided in the narrated examples in the DE handout.