Unit 3.3 The Comparison-Tests

The idea of the first comparison tests that we are going to be looking at in this section is very similar to the idea conveyed in the following example.

Suppose that we use the criteria that people over 6 feet will be considered tall and people under 5 feet will be considered short. Now suppose that we are given that Tim is Tall, and Sam is Short. In which of the following cases, could we conclude the status of the so mentioned Bob?

1) If Bob is taller than Tim, then Bob must be ________ ?
2) If Bob is shorter than Tim, then Bob must be ________?
3) If Bob is taller than Sam, then Bob must be ________?
4) If Bob is shorter than Sam, then Bob must be ________?

After a little thought, hopefully you realize that you can only draw a conclusion about Bob in the first and fourth cases above. If Bob is taller than Tim, and Tim is tall, then Bob must be tall. If Bob is shorter than Sam and Sam is short, then Bob must be short. However, as in case 2 above, knowing that Bob is shorter than Tim (who is tall) doesn't tell us anything about whether or not Bob is short or tall. The same idea applies to case 3. So what does this have to do with infinite series? If we think of tall as being like a divergent series and short as being like a convergent series, the following series test should make sense and can be considered analogous to the example above.

The Direct Comparison Test:

If $0 < a_n \leq b_n$, then if

$$1. \quad \underbrace{\sum b_n \text{ converges}}_{\substack{\text{the sum of the bigger} \\ \text{terms converges}}}, \text{then } \underbrace{\sum a_n \text{ converges}}_{\substack{\text{the sum of the smaller} \\ \text{terms converges}}}$$

and

$$2. \quad \underbrace{\sum a_n \text{ diverges}}_{\substack{\text{the sum of the smaller} \\ \text{terms diverges}}}, \text{then } \underbrace{\sum b_n \text{ diverges}}_{\substack{\text{the sum of the bigger} \\ \text{terms diverges}}}$$

The main idea is that if we have a convergent series like $\sum b_n$ and then another series that was term-by-term smaller, then the series with the smaller terms would also have to converge (smaller than small is small). If we had a divergent series like $\sum a_n$, then because these terms are positive, we can interpret the divergence as an indication that the sum is infinite. Thus if another series was term-by-term bigger, then this other series would also have to diverge (bigger than big is big).

There are a couple of points to make before looking at a few examples. First note that this series test only applies to series that contain positive terms due to the inequality $0 < a_n \leq b_n$. We did not bother to indicate the starting value of n and the ∞ on the sums, as the starting value of n makes no difference. A final very important note to make is that the test would be inconclusive if we knew that either

$$\sum a_n \text{ converges} \quad \text{or} \quad \sum b_n \text{ diverges}$$

These two situations where no conclusion can be made are analogous to the 2nd and 3rd cases in the tall versus small example. Make sure you take the time to make sense of which comparisons are useful and which are not.

One final note before moving on to examples: We will typically look to use the direct comparison test when our given series resembles a geometric or p-series.

Examples 1 & 2 Determine whether each series converges or diverges. Justify your response.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

Solution: This series closely resembles the geometric series, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ where $r = 1/2$. Since $|r| < 1$, the geometric series converges.

The only way that this comparison will prove to be useful based on the Direct Comparison Test is if the original series is term-by-term smaller. But it is clear that

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

since both fractions have the same numerator, but the fraction on the left has the bigger denominator. Thus the Direct Comparison Test applies and the convergence of the geometric series guarantees the $\boxed{\text{convergence of the original series}}$.

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2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Solution: In this case we compare the given series to the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ with $p = 1$, also known as the harmonic series. Since $p \leq 1$, the series diverges.

This comparison will only be useful if the given series is term-by-term bigger. But clearly

$$\frac{\ln n}{n} > \frac{1}{n}$$

as both fractions have the same denominator, while the fraction on the left has the bigger numerator (so long as $n \geq 3$). Thus by the Direct Comparison Test, the divergence of the p-series allows us to conclude the $\boxed{\text{divergence of the given series}}$.

Let's now consider the series $\sum_{n=2}^{\infty} \frac{1}{n^3 - 1}$, which if anything, is closest to the p-series $\sum_{n=2}^{\infty} \frac{1}{n^3}$ with $p = 3$. Since $p > 1$, this p-series converges. The problem here is that

$$\frac{1}{n^3 - 1} > \frac{1}{n^3}.$$

The two fractions have the same numerator, but the fraction on the left has a smaller denominator, making it bigger than the fraction on the right. This is like the situation where we knew that Bob was taller than Sam who was small. That didn't tell us anything about whether or not Bob was tall or small (or neither). Here we have a series that is term-by-term bigger than a series that converges. That doesn't tell us anything about its convergence. Are the terms so much bigger that the given series diverges or are they only "slightly bigger" leading to a bigger convergent sum? Well, the bottom line is that the Direct Comparison Test cannot be applied here. However, it should be hard to imagine that the two series above could have different behavior—they're almost the same series. In fact it will turn out that we can make such a comparison, it just can't be a <u>direct</u> comparison—it will be a <u>limit</u> comparison.

The Limit Comparison Test:

Given that a_n and b_n are positive,

$$\text{If } \lim_{n\to\infty} \frac{a_n}{b_n} > 0 \ (\neq \infty), \text{then } \sum a_n \text{ and } \sum b_n \text{ both converge or both diverge.}$$

The idea here is that as n gets larger and larger, the terms of one series are getting closer and closer to some positive multiple of the terms of the comparison series. For example, suppose that as n gets larger and larger, the terms of one series get closer and closer to twice as big as the terms of a comparison series. If the comparison series converged, then the series whose terms are roughly twice as big will still converge; we just might expect the sum of that series to be roughly (and I mean roughly) twice as big. Similarly, if the comparison series diverged, the series whose terms are roughly twice as big would also have to diverge.

The advantage of the Limit Comparison Test over the Direct Comparison Test is that it doesn't require the terms of one series to be bigger or smaller than the other. However, it does require to you to compare your given series to one very much like it and thus is less flexible in what you choose for a comparison series. We now revisit the series mentioned above in the following example.

Examples 3 & 4 Determine whether each series converges or diverges. Justify your response.

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - 1}$

Solution: As we mentioned above, a direct comparison with the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$ fails (see the above a reminder as to why). However, a limit comparison to this series will prove to be useful. As noted above, this comparison series converges as a p-series with $p = 3$, since $p > 1$.

We now look at the limit of the ratio of the corresponding terms of these series.

$$\lim_{n\to\infty} \left[\frac{1}{n^3 - 1} \div \frac{1}{n^3}\right] = \lim_{n\to\infty} \left[\frac{1}{n^3 - 1} \cdot \frac{n^3}{1}\right] = \lim_{n\to\infty} \frac{n^3}{n^3 - 1} \overset{\div n^3}{\overset{\text{🖩}}{=}} \lim_{n\to\infty} \frac{1}{1 - \frac{1}{n^3}} = 1$$

Since the above limit is positive (and not infinite), by the Limit Comparison Test, we conclude that both series must have the same behavior, that is, they both $\boxed{\text{converge}}$.

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4. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n + 2^n}{3^n + 1}$

Solution: In this example it might at first be less obvious as to what series to make a comparison to. The idea that we even consider a comparison is based on the complexity of the form of our terms. While the n^{th} term test could be applied here, let's take a look at an approach that might be more beneficial to handling a wider variety of problems. In a case like we have here, one might first ignore all but the most dominant (fastest growing) functions in both the numerator and denominators. In this case, the most dominant term in the numerator is 5^n, while the most dominant term in the denominator is 3^n. Thus we consider $\frac{5^n}{3^n} = \left(\frac{5}{3}\right)^n$ and attempt to apply the limit comparison test to the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n$ which is a geometric series with $r = 5/3$ and thus diverges as $|r| \geq 1$.

We now look at the limit of the ratio of corresponding terms.

$$\lim_{n\to\infty}\frac{5^n+2^n}{3^n+1}\div\left(\frac{5}{3}\right)^n=\lim_{n\to\infty}\frac{5^n+2^n}{3^n+1}\cdot\frac{3^n}{5^n}=\lim_{n\to\infty}\frac{15^n+6^n}{15^n+5^n}\overset{\div 15^n}{\cong}\lim_{n\to\infty}\frac{1+\left(\frac{6}{15}\right)^n}{1+\left(\frac{5}{15}\right)^n}=1$$

Since the limit is positive and not infinite, by the Limit Comparison Test, both series must have the same behavior, that is, both series $\boxed{\text{diverge}}$.