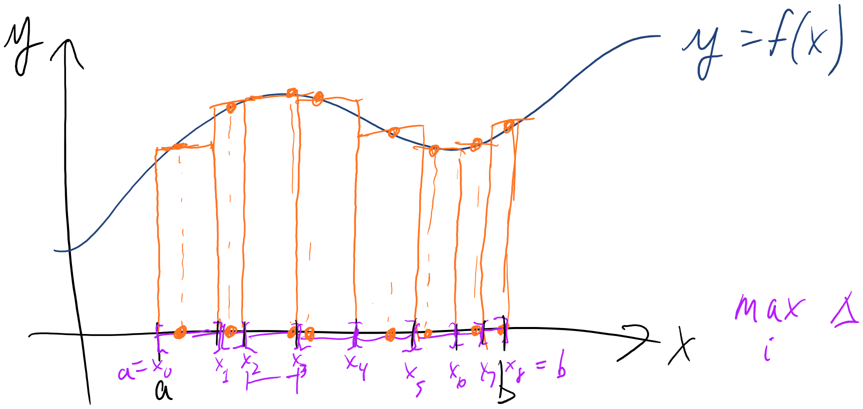
12.1: Double Integrals Over Rectangles

Wednesday, September 23, 2020 8:55 AM

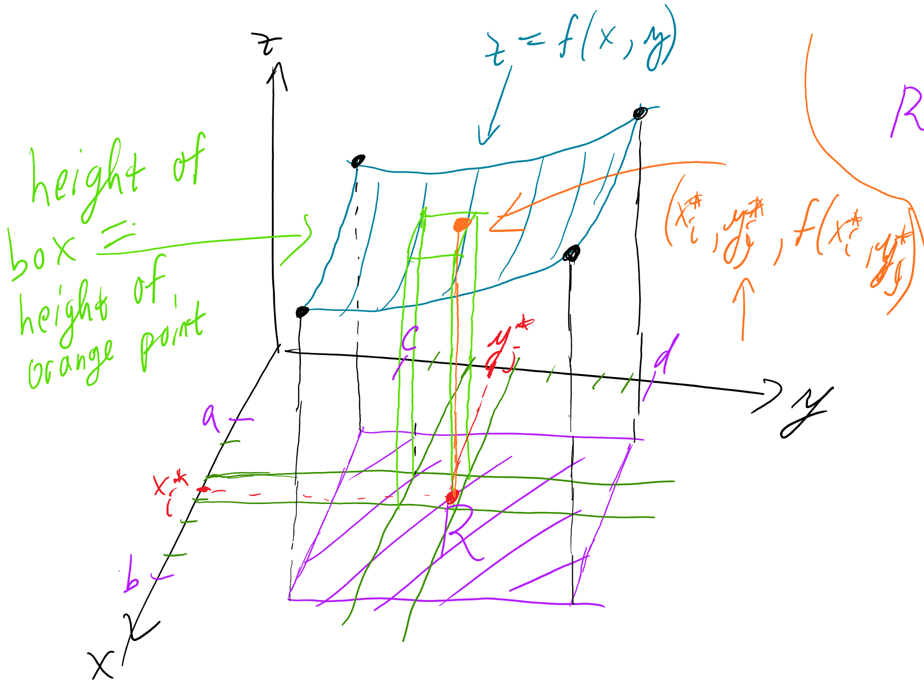


$\max_i \Delta x_i = x_1 - x_0 = \Delta x_1$

Riemann Sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i \approx$ area between curve & x-axis, if $f(x) \geq 0$

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Volumes and Double Integrals



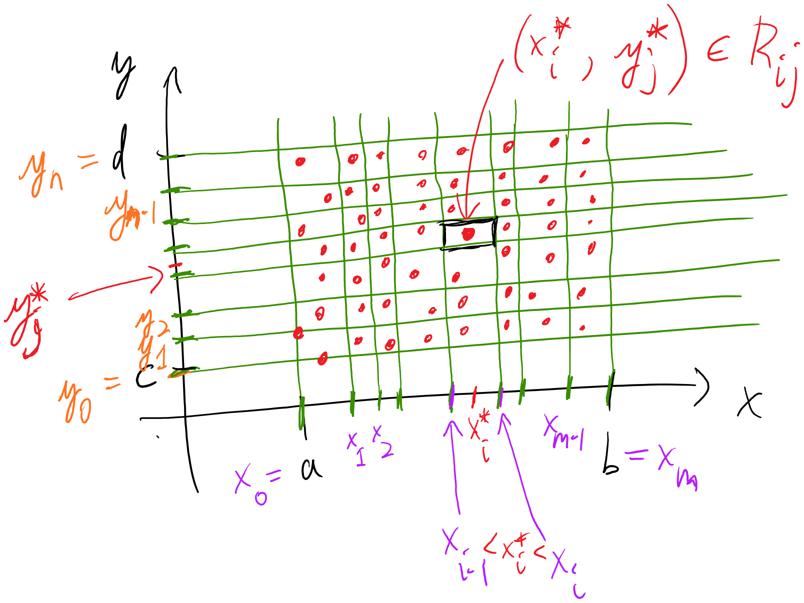
$R = [a,b] \times [c,d]$

$= \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$

$z = f(x,y) \geq 0$ for all $(x,y) \in R$

S = solid between surface $z = f(x,y)$ and rect. R.

$= \{(x,y,z) : 0 \le z \le f(x,y),\ (x,y) \in R\}$



$a = x_0 < x_1 \cdots < x_{m-1} < x_m = b$

$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$

(x_i^*, y_j^*) sample point in R_{ij} gives height of box $= f(x_i^*, y_j^*)$

$l_i = \Delta x_i = x_i - x_{i-1}$

$w_j = \Delta y_j = y_j - y_{j-1}$

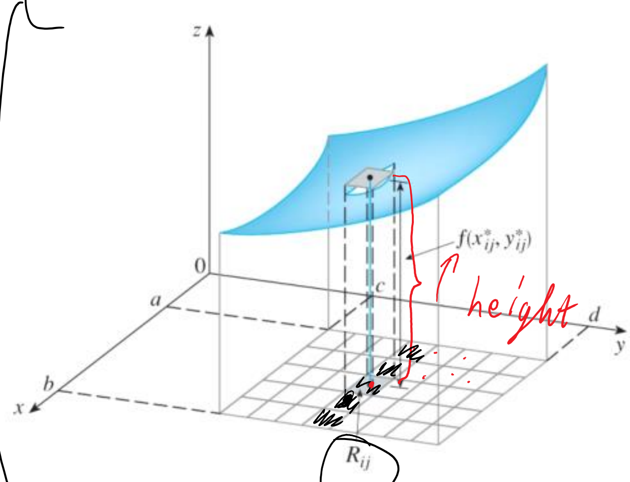


FIGURE 4

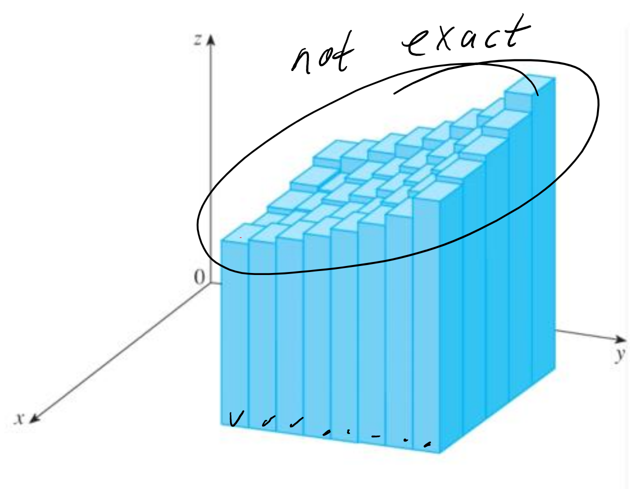


FIGURE 5

area of $R_{ij} = \Delta x_i \Delta y_j = \Delta A_{ij}$

V_{ij} = vol. of box above R_{ij}

$$= \Delta A_{ij}\, f(x_i^*, y_j^*)$$

$$= f(x_i^*, y_j^*)\, \Delta A_{ij}$$

V = total volume of S

$\approx$ sum of vol. of boxes

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}$$

$$V = \lim_{\max \Delta x_i, \Delta y_i \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}\,, \quad \text{where}$$

$\max \Delta x_i, \Delta y_j$ = largest of lengths of all subintervals

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<u>Def.</u> 1) The <u>double integral</u> of f over the rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{\max \Delta x_i, \Delta y_l} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}\,,$$

if the limit exists.

2) $f(x,y)$ is <u>integrable</u> on R if the double integrable exists.

3) $\sum^{m} \sum^{n} f(x_i^*, y_j^*)\, \Delta A_{ij}$ is a double Riemann

3) $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij}$ is a <u>double Riemann sum</u>.

<u>Facts</u> 1) f continuous $\Rightarrow$ f integrable

2) f bounded, and f continuous except on finitely many smooth curves, then f integrable.

<u>Remark</u> If f integrable, can choose <u>uniform</u> partitions, where $\Delta x_i \equiv \Delta x = \frac{b-a}{m}$ and $\Delta y_j \equiv \frac{d-c}{n} = \Delta y$, so that $\Delta A_{ij} \equiv \Delta A = \Delta x \Delta y$. Also can use (x_i, y_j) as sample points $\rightsquigarrow f(x_i, y_j)$, and

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i) \Delta A$$

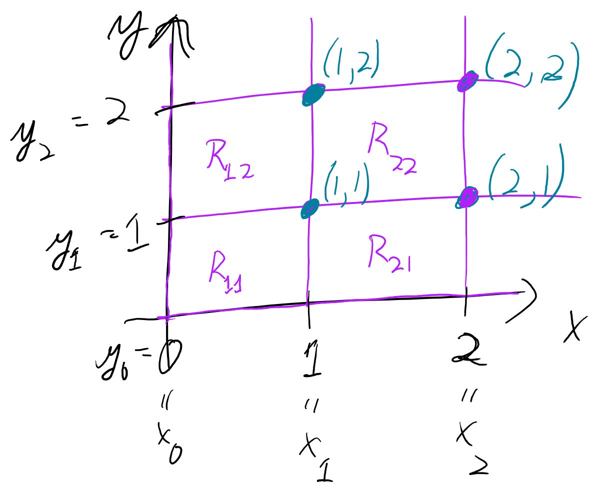
$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A$$

$$= \iint_R f(x, y) \, dA$$

Fact If $f(x, y) \geq 0$, then the volume V of the solid that lies between the rect. R and surf. $z = f(x, y)$

is $$V = \iint_R f(x,y)\, dA.$$

Ex. Estimate vol. of solid lying above $R = [0,2] \times [0,2]$ and below $z = \underbrace{16 - x^2 - 2y^2}_{}$ $= f(x,y)$ using four subrectangles with uniform (regular) partitions. Use upper right corner of each rect. as sample point.



Sol'n: $m = n = 2$

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= \underset{(i=1)}{\sum_{j=1}^{2} f(x_1, y_j) \Delta A} + \underset{(i=2)}{\sum_{j=1}^{2} f(x_2, y_j) \Delta A}$$

$$= \underset{(j=1)}{f(x_1, y_1) \Delta A} + \underset{(j=2)}{f(x_1, y_2) \Delta A} + \underset{(j=1)}{f(x_2, y_1) \Delta A} + \underset{(j=2)}{f(x_2, y_2) \Delta A}$$

$$= f(1,1)(1) + f(1,2)(1) + f(2,1)(1) + f(2,2)(1)$$

$= 13 + 7 + 10 + 4$

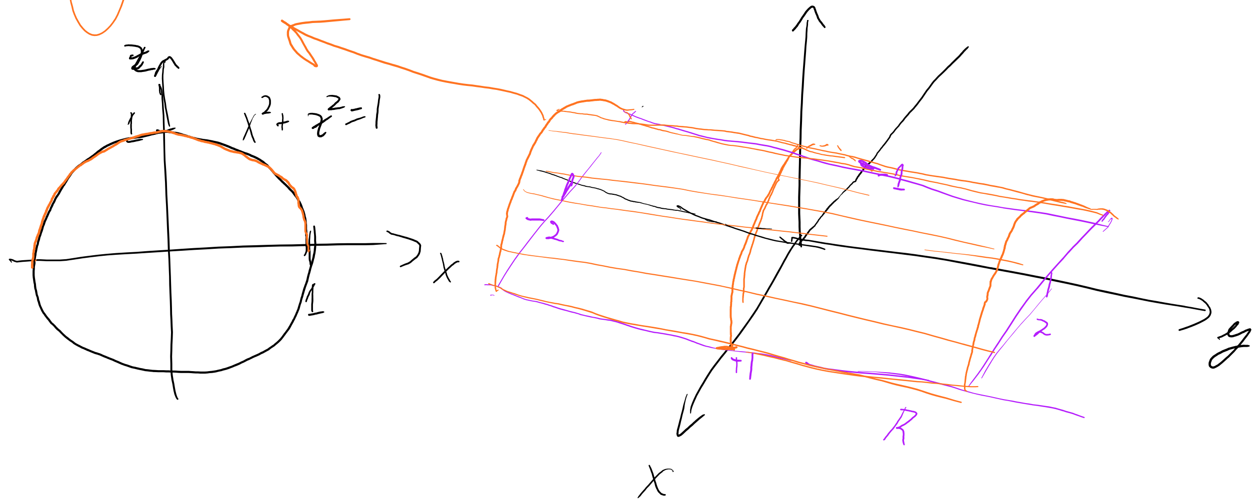
$= \boxed{34}$

Ex. If $R = \{(x,y) : -1 \le x \le 1,\ -2 \le y \le 2\}$, evaluate

$$\iint_R \sqrt{1-x^2}\, dA.$$

Sol'n: $\sqrt{1-x^2} \ge 0$, so interpret as volume

$$z = \sqrt{1-x^2} \implies z^2 = 1 - x^2 \implies x^2 + z^2 = 1$$



$$\iint_R \sqrt{1-x^2}\, dA = \text{vol. of half cylinder w/ base radius 1, height 4}$$

$$= \frac{1}{2}\pi r^2 h$$

$$= \frac{1}{2}\pi(4) = \boxed{2\pi}$$

(too hard for brute force computation)

Midpoint Rule

uniform partitions

$R = [a,b] \times [c,d]$, $a = x_0 < x_1 < \dots < x_{m-1} < x_n = b$

$c = y_0 < y_1 < \dots < y_{n-1} < y_n = d$

$\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$, $\bar{y}_j = \dfrac{y_{j-1} + y_j}{2}$ are

midp'ts of $[x_{i-1}, x_i]$ & $[y_{j-1}, y_j]$ resp.

Def. The midp't rule for double integrals is

$$\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$$

Iterated Integrals

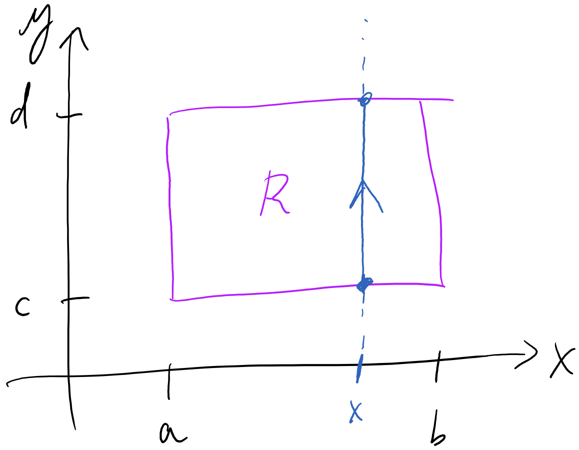
$R = [a,b] \times [c,d]$, $f: R \longrightarrow \mathbb{R}$ continuous.

Put $A(x) = \int_c^d f(x,y)\, dy$ (partial integration w.r.t. y)

$$\int_a^b A(x)\, dx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$



$$R = \bigcup_{x \in [a,b]} \{x\} \times [c,d]$$

$$= \bigcup_{x \in [a,b]} \{(x,y) : c \le y \le d\}$$

$$= \{(x,y) : c \le y \le d,\ a \le x \le b\}$$

__Ex.__ (a) $\displaystyle\int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx = ?$

__Note__ $R = [0,3] \times [1,2]$, $f(x,y) = x^2 y$ cont. on R b/c polynomial.

$\Rightarrow \displaystyle\iint_R f \, dA$ exists.

Fix x. Then $\displaystyle A(x) = \int_1^2 x^2 y \, dy = x^2 \int_1^2 y \, dy = x^2 \left. \frac{y^2}{2} \right|_1^2$

$$= \frac{x^2}{2}(4-1)$$

$$= \frac{3}{2} x^2$$

$$\Rightarrow \int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx = \int_0^3 \frac{3}{2} x^2 \, dx$$

$$= \frac{\cancel{3}}{2} \left. \frac{x^3}{\cancel{3}} \right|_0^3 = \frac{1}{2}(27) = \boxed{\frac{27}{2}}$$

Shorter

$$\int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx = \int_0^3 x^2 \left(\int_1^2 y \, dy \right) dx$$

x^2: indep. of y

$\left(\int_1^2 y \, dy \right)$: constant

$$= \left(\int_1^2 y \, dy \right) \left(\int_0^3 x^2 \, dx \right)$$

$$= \left(\frac{y^2}{2} \Big|_1^2 \right) \left(\frac{x^3}{3} \Big|_0^3 \right) = \boxed{\frac{27}{2}}$$

(b) $$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 y \left(\int_0^3 x^2 \, dx \right) dy$$

y: indep. of x

$$= \int_1^2 y \left(\frac{27}{3} \right) dy = \frac{27}{3} \frac{y^2}{2} \Big|_0^2$$

$$= \boxed{\frac{27}{2}}$$

(same answer w/ reversed order of integration)

Fubini's Theorem If f continuous on a rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx$$

* More generally true if f bounded on R, and discontinuous only on finite # of smooth

curves, and both iterated integrals exist.

<u>Def.</u> $f(x,y)$ is <u>bounded</u> on D if there is constant M s.t. $|f(x,y)| \le M$ for all $(x,y) \in D$.

<u>Ex.</u> $\iint_R y \sin(xy)\, dA$, $R = [1,2] \times [0,\pi]$

<u>Sol'n:</u> $f(x,y) = y\sin(xy)$ continuous on R, so

Fubini $\Rightarrow$ can integrate either order.

$$\int_1^2 \left(\int_0^{\pi} y \sin(xy)\, dy \right) dx = \int_0^{\pi} \left(\int_1^2 y \sin(xy)\, dx \right) dy$$

IBP: annoying

$$= \int_0^{\pi} y \int_1^2 \sin(xy)\, dx\, dy$$

$$\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + C$$

$$= \int_0^{\pi} \cancel{y} \left[\frac{-\cos(xy)}{\cancel{y}} \right]_1^2 dy$$

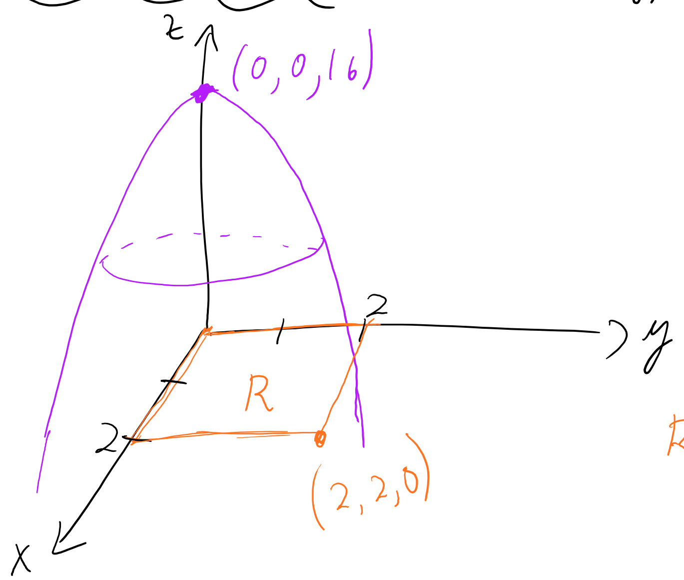
$$= -\int_0^{\pi} \cos(2y) - \cos(y)\, dy$$

$$= -\frac{\sin(2y)}{2}\Big|_0^{\pi} + \sin(y)\Big|_0^{\pi} = -\frac{1}{2}(0-0) + 0 = \boxed{0}$$

Ex. Find vol. of the solid S that is bounded by

Ex. Find vol. of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z=16$, the planes $x=2$ and $y=2$, and the three coord. planes.

$$f(x,y) = z = 16-(x^2+2y^2)$$



$$R = [0,2]\times[0,2]$$

$$V = \iint_R 16-x^2-2y^2 \, dA$$

$$= \int_0^2\int_0^2 16-x^2-2y^2 \, dx \, dy$$

$$= \int_0^2 \left[16x-\frac{x^3}{3}-2y^2x\right]_0^2 dy$$

$$= \int_0^2 \frac{88}{3}-4y^2 \, dy$$

$$= \boxed{48}$$

Fact Sometimes $f(x,y) = g(x)\,h(y)$ and hence

$$\iint_R f(x,y)\, dA = \int_c^d \left(\int_a^b g(x)\,h(y)\, dx \right) dy$$

$$= \int_c^d h(y) \left(\int_a^b g(x)\, dx \right) dy$$

$$= \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right), \quad \text{where}$$

$R = [a,b] \times [c,d]$.

Properties Assume f, g integrable on rect. R.

1) $\displaystyle\iint_R f + g \; dA = \iint_R f \, dA + \iint_R g \, dA$

2) $\displaystyle\iint_R f - g \; dA = \iint_R f \, dA - \iint_R g \, dA$

3) $\displaystyle\iint_R c f(x,y)\, dA = c \iint_R f \, dA$, c real constant

4) If $f(x,y) \geq g(x,y)$ on R, then

$$\iint_R f \, dA \geq \iint_R g \, dA.$$