3.2 Reduction of Order

Suppose that y_1 denotes a non-trivial solution of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ on the interval I. If we want y_2, a linearly independent solution to the equation, then $u(x) = y_2(x)/y_1(x)$ is non-constant on I. Therefore, $y_2(x) = u(x)y_1(x)$ would give a second linearly independent solution. We can solve for $u(x)$ by substituting y_2 into the differential equation. This results in solving a first-order deferential equation, and is known as **reduction of order.**

- *Example:* Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, find a second solution.

  Solution: By reduction of order, we can find a second solution of the form $y_2 = ue^x$. By the product rule, we get that

  $$y_2' = ue^x + u'e^x, \qquad y_2'' = ue^x + 2u'e^x + u''e^x,$$

  giving

  $$y'' - y = ue^x + 2u'e^x + u''e^x - ue^x = e^x(u'' + 2u') = 0.$$

  Since $e^x \neq 0$, we are left with $u'' + 2u' = 0$. If we let $w = u'$, this becomes $w' + 2w = 0$, which is a first-order linear DE in w. By use of the integrating factor e^{2x}, we get the solution

  $$w = c_a e^{-2x} \quad \rightarrow \quad u' = c_a e^{-2x} \quad \rightarrow \quad u = -\frac{1}{2}c_a e^{-2x} + c_b.$$

  Thus,

  $$y_2 = ue^x = -\frac{1}{2}c_a e^{-x} + c_b e^x.$$

  Since we don't want arbitrary constants in our answer, we can let $c_a = -2,\ c_b = 0$, resulting in a second solution of $\boldsymbol{y_2 = e^{-x}}$. Clearly y_1 and y_2 are linearly independent solutions, so we have created a fundamental set of $\{e^x, e^{-x}\}$, or a general solution of $\boldsymbol{y = c_1 e^x + c_2 e^{-x}}$.

  **Our choices for c_a and c_b were arbitrary, and any choice where $c_a \neq 0$ would have worked in producing a linearly independent solution. However, any other choice of c_a and c_b would be "absorbed" by c_1 and c_2 in the general solution. Therefore, we tend to choose c_a and c_b so as to give the "cleanest" y_2.*

While reduction of order can be used on a generic case to obtain a formula for y_2, it is generally better to remember the process above as opposed to the formula.