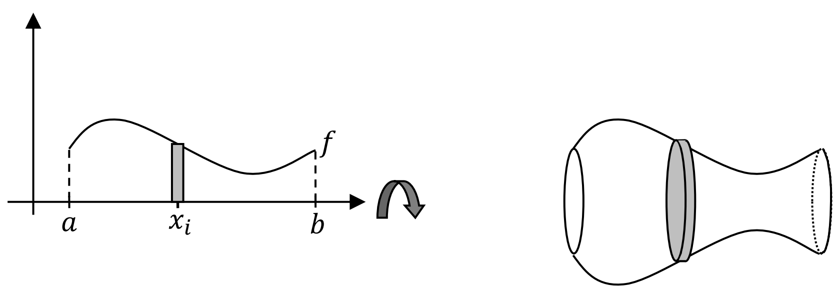
Unit 5.3 Volume—The Disk Method

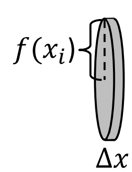
In this section, instead of finding the area of a given region, we will be interested in finding the volume of the solid formed by revolving the region about a particular axis. Depending on the situation, the integrals that we derive to find these volumes will take a different form. In this section we will look at the Disk and Washer methods.

Disks. Consider the following region bounded above by a function f and below by the x-axis on the interval from a to b. We would like to imagine revolving this region about the x-axis and then find the volume of the solid that we generate. To assist us, we will look at an arbitrary slice of the region approximated by one of our representative rectangles.



As you can see from the figure above, the representative rectangle sweeps out what we will refer to as a representative disk. Just as the representative rectangle was an approximation of a slice of the region, the representative disk will be an approximation of a slice of the generated solid. In order to find the volume of the solid, we first sum the volumes of each of the approximating disks to create a Riemann Sum that estimates the volume. Then we simply take the limit as the number of disks approaches infinity and we obtain the definite integral giving the exact value for the volume of the solid. Let us begin by writing an expression for the volume of the representative disk; we call this an increment of volume and denote it ΔV_i. First observe that the disk is simply a cylinder. To find its volume we need the radius and height (thickness) of this cylinder. The height (thickness) is what we denote Δx and the radius is the same as the height of the corresponding representative rectangle which is $f(x_i)$. Therefore,

$$\Delta V_i = \pi[f(x_i)]^2 \Delta x$$



The approximating Riemann Sum is thus given by

$$V \approx \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

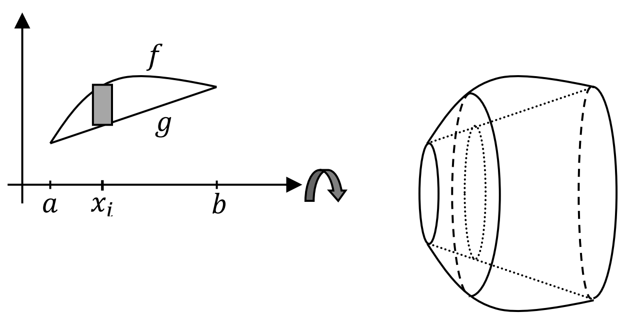
Now we let n go to infinity to obtain the definite integral giving the exact volume of this solid.

$$V = \int_a^b \pi[f(x)]^2 dx$$

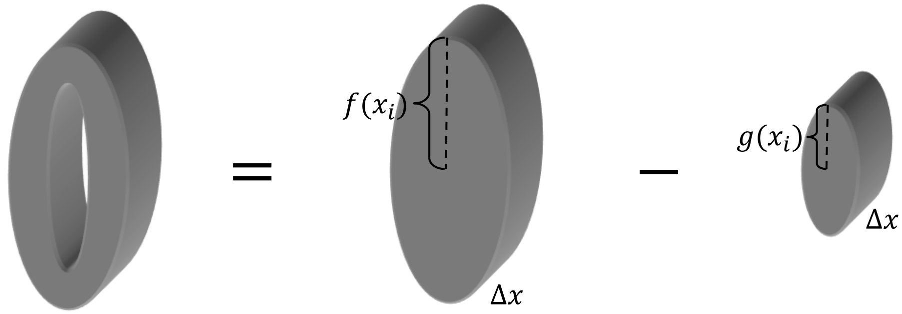
Again, I warn you not to just focus on memorizing this integral and plugging in whichever function is given in a particular problem. Our goal in this chapter is to learn how integrals are derived to perform various calculations. We all have a background in basic geometry, that's why we are looking at volumes. On exams you may be asked to write the formula for the increment of volume and/or the Riemann Sum in addition to writing and evaluating the definite integral so be sure to know this terminology and understand the notation.

Before moving on, let's recognize why it was that our representative rectangle generated a disk. Aside from the revolving part generating the circular dimensions of the disk, there are two main reasons why we obtained a disk (as opposed to a washer or shell which we will be looking at shortly). The first is that the bottom of the region and thus the bottom of the rectangle, bordered the axis of revolution. The second was that our vertical rectangle was perpendicular to the horizontal axis of revolution. As we will see in our examples, we do not have to revolve our regions about the x-axis; we will also revolve our regions about the y-axis and other horizontal and vertical lines. Regardless of what line we revolve the region about, if the axis of revolution borders one end of the region and we are using rectangles that are perpendicular to the axis of revolution, then we will get a disk. Now let's move on to Washers. Note: all examples for the next few sections will be in the narrated form only.

Washers. Now suppose that our representative rectangles are still perpendicular to the axis of revolution, but instead that the axis of revolution does not border the region. Consider the figure below where our region is bounded by two functions and then revolved about the x-axis.



While it requires a little thought to picture the entire solid, it is more important that we can identify the three dimensional shape generated by the rectangle. We call this shape a (representative) washer and an example of such a shape is shown on the next page. Again we must come up with an expression for the volume of the representative washer in terms of the variables involved. To do so, we simply observe that the washer is just a disk with an inner disk removed as depicted in the figure. Thus to get the volume of the washer, we simply take the volume of the big disk and subtract the volume of the inner little disk.



Of course the width of each disk is Δx as it was determined by the width of the rectangle. The radius of the big disk is determined by the distance of the top of the rectangle from the axis of revolution (the x-axis) and is therefore equal to $f(x_i)$. The radius of the little disk is determined by the distance of the bottom of the rectangle from the axis of revolution and is therefore equal to $g(x_i)$. Thus we get the following formula for ΔV_i, the increment of volume (i.e. the volume of the representative washer).

$$\Delta V_i = \pi[f(x_i)]^2\Delta x - \pi[g(x_i)]^2\Delta x = \pi([f(x_i)]^2 - [g(x_i)]^2)\Delta x$$

It is the final expression on the right-hand side that we focus on. When dealing with washers, it is helpful to remember that the increment of volume takes the basic form of: π times, the outer radius squared minus the inner radius squared, times the thickness. The outer radius is determined by the distance of the furthest end of the rectangle from the axis of revolution, while the inner radius is the distance of the nearest end of the rectangle from the axis of revolution. The terms furthest and nearest are used here as they will also apply when our rectangles are horizontal (see the narrated examples). Now it's time to add up the volumes of each of the washers to form our Riemann Sum approximating the volume of the entire solid. We obtain

$$V \approx \sum_{i=1}^{n} \pi([f(x_i)]^2 - [g(x_i)]^2)\Delta x$$

Again, we let n go to infinity to obtain the definite integral yielding the exact volume of the solid.

$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)dx$$

You should note in each of these derivations when subscripts are used and when they are not—when Δx is used and when dx is used. I will also point out that we could have factored the π out in front of the Riemann Sums as well as the definite integral, but I chose not to.

In conclusion, we note that disks and washers are formed when the representative rectangles are perpendicular to the axis of revolution. In the next section we will see what happens when they are parallel to the axis of revolution.