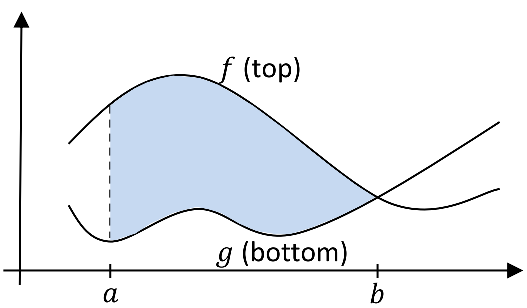
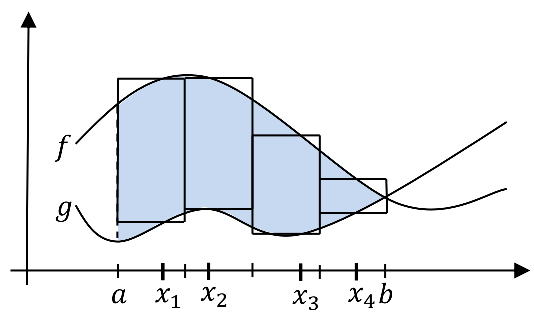
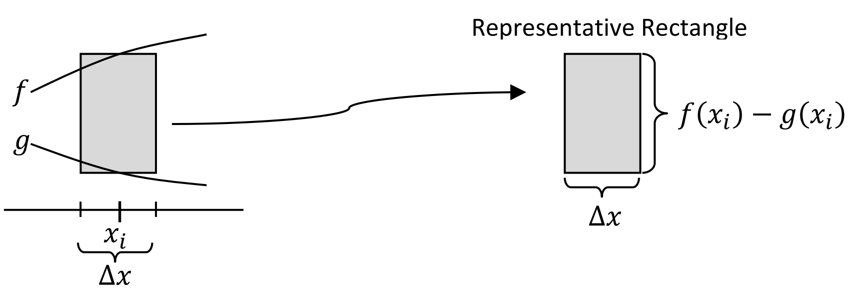
Unit 5.2: Area of a Region Between Two Curves

The concept of integration arises when the quantities that we are dealing with in certain calculations are changing (not constant). For example, if something moves at a constant speed, then we do not need calculus to determine how far it will go in a given amount of time. However, if the speed isn't constant, a definite integral can be used to calculate distance. The definite integral is derived by treating the speed as approximately constant over shorter and shorter time intervals and summing the approximated distances for each time interval. Such a sum is called a Riemann Sum and an appropriate limit (the shrinking of the time intervals towards zero) of the Riemann Sum gives us the definite integral. In this section we start with something simple—area. In calculus I, you would have looked at using integrals to find area *under a curve*, while here we turn our attention to finding area *between two curves* (generated by two functions). Typically these curves (functions) will not be constant, but over short intervals, they will be approximately constant. Thus to approximate the area between the curves, we use rectangles whose tops and bottoms are at a constant height (see the figure below).





Rather than looking at the entire region between two curves and each of the approximating rectangles, we will simply look at an arbitrary part of the region and a single representative rectangle, which is just a rectangle described very generally so that its properties pertain to each of the rectangles used in a given approximation. Such a rectangle is shown below.



Area of representative rectangle:

$\Delta A_i = [f(x_i) - g(x_i)]\Delta x$

Each rectangle is determined by a single x-value chosen from the corresponding interval. In general we denote this x-value by x_i. In the example above, the values of i ranged from 1 to 4 (integer values only) as we used four rectangles to approximate the region. The top and bottom of the rectangle is determined by the values of the two functions that enclose the region at the value x_i. Thus the height of each rectangle is given by $f(x_i) - g(x_i)$, while the width is denoted Δx. We will always choose the width of each rectangle (subinterval) to be the same for convenience sake. Therefore the area of each rectangle is $\Delta A_i = [f(x_i) - g(x_i)]\Delta x$. Any time you see a subscript, it is an indicator that the quantity with the subscript may depend on which particular representative element you are looking at. For example, ΔA_1 represents the area of the first rectangle, and this is likely to differ from ΔA_2—the area of the second rectangle. We call ΔA_i an increment of area. As we add up the areas of each of the rectangles, starting with the first, then the second and so on, we accumulate more and more of the area the forms the region. The sum of the areas of each rectangle is an example of a Riemann sum and is shown below. Let A be the exact area of the region.

$$A \approx \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x \qquad \text{(approximating Riemann Sum)}$$

The Riemann sum is an example of a *discrete* sum—there is a first term, then a second, then a third and so on until we get to the last term (the n^{th} term). Each of these terms is generated by looking at a finite number of x values as we increment along the interval from a to b. To create the definite integral giving the EXACT area of the region, we simply take the limit as n goes to infinity (thus $\Delta x \to 0$) as shown below.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \int_a^b [f(x) - g(x)]dx$$

There are two very important things that you should notice and develop an understanding of here. One is how the notation of the Riemann Sum translates over to the notation in the definite integral. The (discrete) summing symbol Σ is replaced with the (continuous) summing symbol $\int$, which is an elongated S for Sum. The x_i's turn into x's without subscripts and the Δx becomes dx. When you solve problems in this chapter, you will be expected to use appropriate notation. As indicated above, the definite integral can be thought of as a continuous sum. Instead of looking at separated x-values along the interval from a to b to form an approximating sum, the definite integral corresponds to having x take on all, infinitely many x-values from a to b. I will explain the notation more when we get to the narrated examples.

At this point, let's write down the final result. Please keep and mind, it isn't the final result that we are primarily interested in, otherwise, I wouldn't have explained everything as I did above. I refer you to the figures above to accompany this result.

$$\text{Area of Region} = \int_a^b [f(x) - g(x)]dx = \int_a^b (Top - Bottom)dx$$

Before we look at an example, I would like to point out that in the above explanations, we approximated the region by partitioning the interval from a to b along the x-axis and using what I will refer to as vertical rectangles (typically taller than they are wide). We could have instead partitioned the region along the y-axis and used horizontal rectangles, which then leads to integration along the y-axis. In this case, instead of integrating the Top function minus the Bottom function, we would instead integrate the Right minus the Left (always Bigger minus Smaller). I refer you to the textbook and the narrated examples for more on this.

Example Find the EXACT area of the region bound between the graphs of $y = 2x$ & $y = x^2$.

Solution: We must first determine the points of intersection for these to graphs. We do this by setting the two functions equal to each other and solving for x.

$$2x = x^2 \;\rightarrow\; x^2 - 2x = 0 \;\rightarrow\; x(x-2) = 0 \;\rightarrow\; x = 0, x = 2$$

Since the functions intersect only at 0 and 2, between 0 and 2, the functions will bound a region. Substituting a value between 0 and 2 into each of the functions tells us that $y = 2x$ is the top function and $y = x^2$ is the bottom function. Using a graphing utility also helps make the region clear (see the graph below). Therefore the exact area of the region bounded by these two functions is

$$\int_0^2 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \left(4 - \frac{8}{3}\right) - (0 - 0) = \boxed{\frac{4}{3}}$$