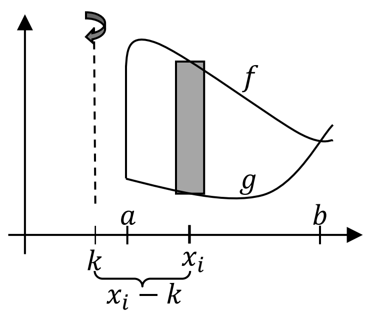
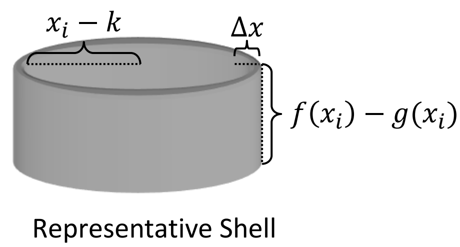
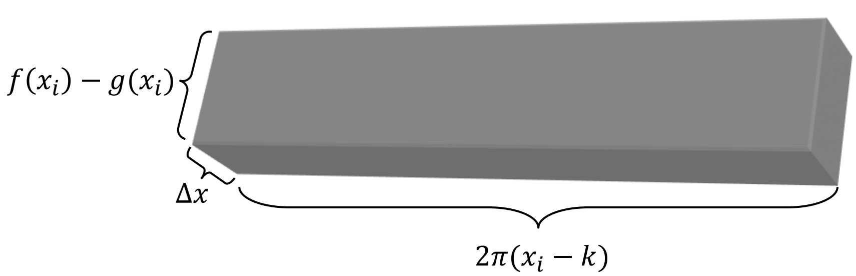
Unit 5.4 Volume—The Shell Method

As was mentioned at the end of the Unit 2.3 summary, we get disks or washers when our representative rectangles are perpendicular to the axis of revolution. In this section we consider the case when our rectangles are parallel to the axis of revolution. When this is the case, the representative rectangle from our region generates a *representative shell* in the generated solid. In the figure below we are using a vertical rectangle and I have chosen to revolve the region about a vertical line of the form $x = k$, where k is some number. In this figure we do not show the entire solid that would be generated, but rather one of the approximating shells generated by the representative rectangle.





One way of picturing a shell is to take a piece of paper and join two of the opposite ends (without folding the paper). The very small thickness of paper would correspond to a very small Δx. To obtain an expression for ΔV_i, the volume of the shell, imagine cutting the shell vertically and laying it out flat (like the piece of paper before we rolled it up). We obtain what I will refer to as a rectangular slab or just slab. A slab is just a rectangle with a little thickness. It's volume is determined by $L \times W \times H$. In the figure below we have labeled the slab with all of its dimensions. The two dimensions that are relatively easy to identify have length Δx and $f(x_i) - g(x_i)$. The other dimension can be thought of as the circumference of the shell. The circumference is 2π multiplied by the radius. The radius of the shell is determined by the distance of the representative rectangle from the axis of revolution. Thus the radius is $x_i - k$ or $k - x_i$ depending on which is bigger, x_i or k. Always subtract, bigger minus smaller.



Based on the above we find the volume of the representative shell is

$$\Delta V_i = 2\pi(x_i - k)[f(x_i) - g(x_i)]\Delta x$$

As a result the Riemann Sum approximating the volume of the entire solid is

$$V \approx \sum_{i=1}^{n} 2\pi(x_i - k)[f(x_i) - g(x_i)]\Delta x$$

Letting n go to infinity gives us the definite integral for the EXACT volume of the solid.

$$V = \int_a^b 2\pi(x - k)[f(x) - g(x)]dx$$

Once again, do not bother memorizing this final form. Different variations of the problem might lead to a slightly different looking integral. If you understand the construction of the integral, you should be able to handle the different variations with little or no difficulty.

As a final note, I would like to point out that when we wish to find the volume of a solid of revolution, we sometimes have a choice whether or not we will use disks/washers or shells. Regardless of the axis of revolution, we sometimes have a choice whether or not we want to use horizontal or vertical rectangles. Which one we use will determine whether we get disks/washers or shells. It might be that using one versus the other makes the problem easier.

*A helpful rhyming mnemonic for remembering which kind of representative volume element you are going to get is:

“If it’s (the representative rectangle) ***parallel*** (to the axis of revolution)***, then it yields a shell”***

$\Downarrow$

“If it’s parallel, then it yields a shell”