4.1 The Laplace Transform

- *Definition:* Let f be a function defined for $t > 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

is called the **Laplace transform** of f, provided the integral converges.

When the integral does converge, the result is a function of s. In this unit we will use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote the transformed function. For example,

$$\mathcal{L}\{f(t)\} = F(s), \qquad \mathcal{L}\{g(t)\} = G(s), \qquad \mathcal{L}\{y(t)\} = Y(s).$$

- *Example:* Evaluate $\mathcal{L}\{1\}$.

Solution: From the definition,

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \lim_{b\to\infty}\int_0^b e^{-st}dt = \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty}\frac{1-e^{-bs}}{s} = \frac{1}{s},$$

provided $s > 0$. If $s \le 0$, then the integral diverges.

The use of limit notation can become tedious when dealing with the Laplace transform, so it is acceptable to use the shorthand notation of $\left.\right|_0^\infty$ in place of $\lim\limits_{b\to\infty} \left.\right|_0^b$. However, it is **never acceptable to "plug in" ∞ when evaluating the functions.*

- *Example:* Evaluate $\mathcal{L}\{e^{-3t}\}$.

Solution: From the definition,

$$\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}e^{-3t}dt = \int_0^\infty e^{-(s+3)t}dt = \left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty = \frac{1}{s+3},$$

provided $s + 3 > 0$, or $s > -3$.

**While we will quickly start to rely on a table of Laplace transforms, it is still good to remember the definition and be able to use it.*

- *Example:* Evaluate $\mathcal{L}\{\alpha f(t) + \beta g(t)\}$.

  Solution: From the definition,
  $$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t)dt$$
  Therefore, $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$, and $\mathcal{L}$ is a linear operator.

- <u>*Definition:*</u> A function f is said to be of **exponential order *c*** if there exist constants $c, M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.
  **This definition essentially says that f is of exponential order if it does not grow faster than the exponential function e^{ct}.*

<u>***Theorem:***</u> **Sufficient Conditions for Existence**

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

Proof: By the additive property of definite integrals,
$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt = I_1 + I_2.$$
The integral I_1 exists because it can be written as a sum of integrals over intervals on which $e^{-st}f(t)$ is continuous. Since f is of exponential order, there exist constants $c, M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$. We can then write
$$|I_2| \leq \int_T^\infty |e^{-st} f(t)|dt \leq M \int_T^\infty e^{-st} e^{ct} dt = M\frac{e^{-(s+c)T}}{s-c}$$
for $s > c$. Since
$$M \int_T^\infty e^{-st} e^{ct} dt$$
converges, the integral I_2 exists for $s > c$. Since I_1 and I_2 both exist, then $\mathcal{L}\{f(t)\}$ exists for $s > c$. ■

**These conditions are sufficient, but not necessary for the existence of a Laplace transform.*

Theorem: **Behavior of $F(s)$ as $s \to \infty$**

If f is piecewise continuous on $(0, \infty)$ and of exponential order, and $F(s) = \mathcal{L}\{f(t)\}$, then

$$\lim_{s \to \infty} F(s) = 0.$$

Proof: Since f is of exponential order, there exist constants $\gamma, M_1 > 0$, and $T > 0$ such that $|f(t)| \leq M_1 e^{\gamma t}$ for all $t > T$. Also, since f is piecewise continuous for $0 \leq t \leq T$, it is necessarily bounded on the interval, which means $|f(t)| \leq M_2 = M_2 e^{0t}$. Let $M = \max\{M_1, M_2\}$ and let $c = \max\{\gamma, 0\}$, then

$$|F(s)| \leq \int_0^\infty |e^{-st} f(t)| dt \leq M \int_0^\infty e^{-st} e^{ct} dt = \frac{M}{s - c}$$

for $s > c$. Therefore, as $s \to \infty$, $|F(s)| \to 0$, and $F(s) = \mathcal{L}\{f(t)\} \to 0$. ■

**As a consequence of this theorem, we can say that functions such as $F(s) = 1$ or $F(S) = s/(s+1)$ are not the Laplace transform of piecewise continuous functions of exponential order. However, we cannot conclude that they are not Laplace transforms of other kinds of functions.*