Unit 2.2 Partial Fractions

In this section our main focus is on a method of algebra, whereby we decompose a proper rational function into what is called its **P**artial **F**raction **D**ecomposition (PFD). To motivate the idea and how it might be useful for integration, note the following simplification

$$\underbrace{\frac{2}{x+1}+\frac{1}{x+2}}_{\substack{\text{partial fraction}\\ \text{decomposition of}}}=\frac{2(x+2)+1(x+1)}{(x+1)(x+2)}=\frac{3x+5}{x^2+3x+2}$$

Now suppose that we were given the integral $\int \frac{3x+5}{x^2+3x+2}\,dx$. Up to this point, we might consider a u-sub by letting u equal the denominator and getting log functions or completing the square in the denominator to get inverse tangent functions or possibly a combination of both logs and inverse tangent functions by splitting up the numerator. However, none of these considerations work out here. Instead, if we could somehow figure out a way of reversing the simplification above, we could express our integrand as a sum of its partial fractions and integrate each partial fraction. Such an approach is demonstrated below.

$$\int \frac{3x+5}{x^2+3x+2}\,dx=\int \frac{2}{x+1}\,dx+\int \frac{1}{x+2}\,dx=2\ln|x+1|+\ln|x+2|+C$$

Once we obtain our partial fractions, the integration should typically be "straightforward", thus our focus in this section summary will only be on getting the partial fraction decomposition (PFD). Full examples that include the integration will be demonstrated in the narrated examples.

Obtaining the PFD will first require us to have a "proper" rational function. Recall that a rational function is proper, when the numerator has a lower degree than the denominator. If we encounter an "improper" rational function, we will first have to use long division to express the function as a polynomial plus a proper rational function (analogous to expressing an improper fraction as a mixed number; i.e. an integer plus a proper fraction). The second thing that we have to do before obtaining the decomposition is factor our denominator as much as possible. From prior algebra courses, you might recall that every polynomial can be factored into linear factors (degree 1) and irreducible quadratic factors (degree 2). An irreducible quadratic factor is one that has only non-real zeros (it's graph would be a parabola with no x-intercepts). We will separate our study of PFD's into four cases.

Case 1: Distinct Linear Factors. Suppose we wished to integrate the function $\frac{x^2+2x-1}{2x^3+3x^2-2x}$. First note that we do have a proper rational function, so we begin by factoring the denominator as $x(2x^2+3x-2)=x(2x-1)(x+2)$. Here we have three distinct linear factors in our denominator and each of these factors will get its own partial fraction. For linear factors, the corresponding partial fraction will have constant numerators. This gives us the following form for the PFD:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

where $A, B,$ and C are constants that we must determine. To do so, we first multiply both sides of the equation above by the denominator $x(2x-1)(x+2)$ to clear fractions and obtain

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Notice that each factor of the denominator remains in all but one of the terms on the right. Because these factors are linear, there is a value of x for each that we can substitute to make them turn out to be zero. Therefore we proceed as follows:

- Let $x = 0$ to obtain $0^2 + 2(0) - 1 = A(-1)(2) + B(0)(2) + C(0)(-1)$ or simply $-2A = -1$. Thus we find that $\boxed{A = \frac{1}{2}}$
- Let $x = \frac{1}{2}$ to obtain $\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 1 = A(0)\left(\frac{5}{2}\right) + B\left(\frac{1}{2}\right)\left(\frac{5}{2}\right) + C\left(\frac{1}{2}\right)(0)$ or simply $\frac{1}{4} = \frac{5}{4}B$. We thus have $\boxed{B = \frac{1}{5}}$
- Let $x = -2$ to obtain $(-2)^2 + 2(-2) - 1 = A(-5)(0) + B(-2)(0) + C(-2)(-5)$ or simply $-1 = 10C$, giving us $\boxed{C = -\frac{1}{10}}$

This gives us the following PFD

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} == \frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}$$

While we are not looking at antiderivatives at this time, it should be noted that with linear factors and constant numerators, the antiderivatives would all be logarithms and each of the constants that we found, can be factored out of the corresponding integrals for each partial fraction.

Case 2: Linear Factors with Repetition. Let us consider the rational function $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$. In this example, our rational function is not proper as the degree of the numerator is not less than that of the denominator. Thus we first use long division to write the function as $x + 1 + \frac{4x}{x^3-x^2-x+1}$. The long division steps can be found in the narrated examples, but I will refrain from showing such steps at this time. We therefore direct our attention to finding the PFD of the rational expression $\frac{4x}{x^3-x^2-x+1}$. To do so we first factor the denominator as follows

$$x^3 - x^2 - x + 1 = x^2(\boldsymbol{x - 1}) - 1(\boldsymbol{x - 1}) = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

This time we again have all linear factors, however, the factor $(x-1)$ is repeated twice and will thus have two partial fractions associated with it in the PFD. One of the partial fractions associated with $(x-1)^2$ will contain the denominator $(x-1)^2$ and in addition we must include partial fractions with every lesser (positive) power of $(x-1)$. In this case, that just means we must also have a partial fraction with denominator $x-1$. However, if we had a factor of $(x-1)^5$, then we would have five corresponding partial fractions with denominators $(x-1)^5, (x-1)^4, (x-1)^3, (x-1)^2$, and $(x-1)$. We therefore have the following form for our PFD

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Again, the linear factors, whether repeated or not, give us partial fractions with constant numerators. We proceed by multiplying through by the denominator $(x-1)^2(x+1)$ to obtain

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Because there are only two linear factors, there are only two values of x that can be chosen to yield zero factors. That will take care of determining two of our constants. The third constant can be obtained by picking any other value for x; we will use $x = 0$ for simplicity.

- Let $x = 1$, to obtain $4 = 2B$ and thus $\boxed{B = 2}$.
- Let $x = -1$, to obtain $-4 = 4C$ and thus $\boxed{C = -1}$.
- Let $x = 0$, to obtain $0 = A(-1)(1) + \underbrace{2}_{B}(1) + \underbrace{(-1)}_{C}(1)$. This gives us $\boxed{= 1 .}$

We now express the original rational function as the polynomial obtained from the long division process plus the decomposition obtained above.

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}$$

If we were integrating this function, the polynomial part would be straight forward, while the terms with the first degree denominators would yield logs and the term with the second degree denominator would yield a power function. Again, you can find the integrals worked out in the narrated examples.

Cases 3 and 4: Distinct and Repeated Irreducible Quadratic Factors. Before looking at a decomposition, let us first give the following fact that might be helpful for determining when a quadratic polynomial is irreducible.

Fact: the polynomial $ax^2 + bx + c$ is irreducible if any of the following equivalent conditions are satisfied:

1) $ax^2 + bx + c$ has no real zeros (i.e. no real value for x yields a zero output)
2) $b^2 - 4ac < 0$ (this comes from the quadratic formula)
3) The graph of $y = ax^2 + bx + c$ (parabola) has no x-intercepts.

Each irreducible quadratic factor will get its own partial fraction with a linear numerator (as opposed to constant numerators for linear denominators). If the quadratic factor is repeated, we proceed just as we did for repeated linear factors by including partial fractions with denominators of each degree, starting at 1 and up to the factor corresponding to the repetition number. To make this more clear, let us consider finding the PFD for the rational function $\frac{7x^2-3x+2}{(x^2+x+5)(x^2+1)^3}$. First note that both $x^2 + x + 5$ and $x^2 + 1$ are irreducible. The form of the decomposition will therefore be

$$\frac{7x^2 - 3x + 2}{(x^2 + x + 5)(x^2 + 1)^3} = \frac{Ax + B}{x^2 + x + 5} + \frac{Cx + D}{x^2 + 1} + \frac{Ex + F}{(x^2 + 1)^2} + \frac{Gx + H}{(x^2 + 1)^3}$$

I will not attempt to find the constants in this case as it would be a long and tedious process, best suited for a computer. We will look at a simpler example in a moment. It should be noted integrating functions in this kind of decomposition could yield logarithms, inverse tangent functions, power functions and could in the worst case, require tangent substitutions. Most of the examples that we do by hand will not get too difficult.

Before concluding let us observe one PFD that involves an irreducible quadratic.

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Like before, we clear fractions by multiplying through by the denominator $x(x^2 + 4)$ to obtain

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

Because we have the linear factor x, we can let $x = 0$ to find one of the constants, but the fact that $x^2 + 4$ is an irreducible quadratic factor, there is no (real) value of x that will make it zero. While we could find the other constants by picking two other values of x (any we choose), we will show another method that can often be simpler in cases like this. Before doing so, let us let $x = 0$ to obtain $4 = 4A$ and thus $\boxed{A = 1}$. We now expand the right-hand side of the equation above, combine like terms and then "equate coefficients" as follows.

$$2x^2 - 1x + 4 = 1 \cdot x^2 + 4 + Bx^2 + Cx = (1 + B)x^2 + Cx + 4$$

For this to be true, the coefficient of each term must match in the first and last expression. This gives the following $2 = 1 + B \;\rightarrow\; \boxed{B = 1}$ and $\boxed{-1 = C}$.