1.1 Definitions and Terminology

- *Definition:* An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is called a **differential equation** (**DE**).

  - *Ordinary Differential Equation (ODE)* - contains only ordinary derivatives with respect to a single independent variable.
  - *Partial Differential Equation (PDE)* - contains partial derivatives

The **order** of a differential equation is the order of the highest derivative in the equation.

- All n^{th}-order ordinary differential equations can be represented as a function

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

- If it is possible to solve for $y^{(n)}$, we often also represent the differential equation by the **normal form**

$$\frac{d^n y}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right)$$

A first-order differential equation can also be written in **differential form** using the conversion

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad \leftrightarrow \quad \boldsymbol{M(x,y)dx + N(x,y)dy = 0}.$$

**Note: while often thought of as "multiplying" both sides by dx, this is inaccurate. The exact process by which this works is highlighted in the section on separable differential equations.*

An n^{th}-order differential equation is **linear** if it can be written as a linear combination of the derivatives,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_o(x)y = g(x)$$

- *Example:* Classify the following differential equations.

$\frac{dy}{dx} + 5y = e^x$ -> 1st-order, linear, ODE

$\frac{d^4y}{dt^4} + \left(\frac{dx}{dt}\right)^5 = 2x + t$ -> 4th-order, non-linear, ODE

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2 y$ -> 1st-order, PDE

$\frac{dy}{dx} + 3xy^2 = x + 1$ -> 1st-order, non-linear, ODE

$4\frac{d^3y}{dx^3} - \frac{dy}{dx} + x^3 y = \sin x$ -> 3rd-order, linear, ODE

- <u>*Definition:*</u> Any function ϕ, defined on an interval I and having at least n continuous derivatives on I, which when substituted into $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$ reduces the equation to an identity, is called a **solution** of the equation on the interval I.

  A solution of a differential equation that is identically zero on an interval I is called a **trivial solution.*

- *Example:* Verify that the following are solutions on the interval $(-\infty, \infty)$

  (a) $\frac{dy}{dt} = y^2 \sin t$; $y = \sec t$

$$LHS: \quad \frac{dy}{dt} = \frac{d}{dt}(\sec t) = \sec t \tan t$$
$$RHS: \quad y^2 \sin t = \sec^2 t \sin t = \sec t \tan t$$

  Since the left-hand side and right-hand side are the same for all values of t, we have a solution on $(-\infty, \infty)$.

  (b) $y'' + 2y' - 3y = 0$; $y = e^x$

$$LHS: \quad y'' + 2y' - 3y = e^x + 2e^x - 3e^x = 0$$
$$RHS: \quad 0$$

  Since the left-hand side and right-hand side are the same for all values of x, we have a solution on$(-\infty, \infty)$.

- *Definition:* A solution of a differential equation that is represented in the form $y = \phi(x)$ is called an **explicit solution**.

- *Definition:* A relation $G(x, y) = 0$ is called an **implicit solution** of an ordinary differential equation on an interval I, if there exists at least one function $y = \phi(x)$ that satisfies the relation as well as the differential equation on I.

For this course, if we "arrive" at a solution of $G(x, y) = 0$, we will assume that there exists at least one function that satisfies the relation and the equation.

- *Example:* Show that on the interval $(-2, 2)$, $x^2 + y^2 = 4$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

By implicit differentiation, we get

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}4 \quad \rightarrow \quad 2x + 2y\frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Families of Solutions:

We experienced problems like the following in Calculus, and solved them using integration.

$$y' = \cos x \quad \rightarrow \quad y = \sin x + c$$

$$y'' = 6x \quad \rightarrow \quad y' = 3x^2 + c_1 \quad \rightarrow \quad y = x^3 + c_1 x + c_2$$

In both cases, we ended up with arbitrary constants of integration. Both also represent (very) basic differential equations, so we would expect similar things to happen when solving a DE.

When solving a first-order differential equation $F(x, y, y') = 0$, we (*usually*) obtain a solution containing an arbitrary constant c, giving a **one-parameter family of solutions** $G(x, y, c) = 0$.

When solving an n^{th}-order differential equation, we seek an **n-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$.

If every solution of an n^{th}-order ODE on an interval I can be obtained from $G(x, y, c_1, c_2, \ldots, c_n) = 0$, we say that the family is the **general solution** of the DE.

A solution that is free of arbitrary parameters is called a **particular solution**.

Our overall goal for a differential equation is to answer the following questions:

1. *Does a solution exist?*
2. *If a solution exists, is it unique?*
3. *If a solution exists, how do we find it?*

The next two sections give us some tools to start answering those questions.