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1. find $L\{te^{3t}\sin 5t\}$

$L\{t^k f(t)\} = (-1)^k \frac{d^k}{ds^k} L\{f(t)\}$

$(-1)^1 \frac{d}{ds} L\{\sin(5t)\}$

$L\{\sin(5t)\} = \frac{5}{s^2+25}$

$\frac{d}{ds}\left(\frac{5}{s^2+25}\right) = \frac{0(s^2+25) - 5(2s)}{(s^2+25)^2} = \frac{-10s}{(s^2+25)}$

$(-1)^1\left(\frac{-10s}{(s^2+25)^2}\right) \rightarrow \frac{10s}{(s^2+25)^2}$

$L\{t\sin 5t\} = \frac{10s}{(s^2+25)^2}$

$L\{te^{3t}\sin 5t\} = \frac{10(s-3)}{((s-3)^2+25)^2}$

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2.

$$f(t) = \begin{cases} t & 0 < t \le 1 \\ 1 & 1 \le t \le 4 \\ 0 & t \ge 4 \end{cases}$$

$$f(t) = t + (1-t)u(t-1) + (0-1)u(t-4)$$

$$L\{t\} = \frac{1}{s^2}$$

$$L\{(1-t)u(t-1)\} = e^{-s} L\{1-(t+1)\}$$

$$L\{1-(t+1)\} = -\frac{1}{s^2}$$

$$L\{(1-t)u(t-1)\} = -\frac{e^{-s}}{s^2}$$

$$L\{(0-1)u(t-4)\} = -\frac{e^{-4s}}{s}$$

$$L\{f(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-4s}}{s}$$

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$y'' - 4y = f(t) \quad y(0) = 1 \quad y'(0) = -2$

$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

periodic function
period of 2π
$T = 2\pi$

$$\frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$f(t) = \sin(t)u(t) + (0 - \sin(t))\, u(t-\pi)$

$f(t) = \sin(t)u(t) - \sin(t)\, u(t-\pi)$

$$e^{-0s} \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

$$e^{-\pi s} \mathcal{L}\{-\sin(t+\pi)\} = \frac{-(\cancel{s\sin(\pi)} + \cos(\pi))}{s^2+1}$$

$0 \quad -1$

$$= \frac{e^{-\pi s}}{s^2+1}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \right]$$

$$= \frac{1 + e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}$$

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$L\{-4y\} = -4Y(s)$

$L\{y''\} = s^2 Y(s) - sy(0) - y'(0)$

$= s^2 Y(s) - s + 2$

$s^2 Y(s) - s + 2 - 4Y(s) = \dfrac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}$

$+s-2$

$Y(s)(s^2-4) = \dfrac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})} + (s-2)$

$Y(s) = \dfrac{1+e^{-\pi s}}{(s^2+1)(s^2-4)(1-e^{-2\pi s})} + \dfrac{(s-2)}{(s^2-4)}$

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4.

$$L^{-1}\left\{\frac{s^2+5s}{s^3-2s^2+3s-6}\right\}$$

$$s^3-2s^2+3s-6$$

$$\downarrow$$

$$(s-2)(s^2+3)$$

$$\frac{s^2+5s}{(s-2)(s^2+3)} = \frac{A\cancel{(s-2)}(s^2+3)}{\cancel{(s-2)}} + \frac{(bs+c)\cancel{(s^2+3)}(s-2)}{\cancel{(s^2+3)}}$$

$$s^2+5s = A(s^2+3) + (bs+c)(s-2)$$

$$s^2+5s = A(s^2+3) + bs(s-2) + c(s-2)$$

s^2: $1 = A+B$

s^1: $5 = -2B + c$

s^0: $0 = 3A - 2C$

$$B = 1-A$$

$$5 = -2(1-A) + (\tfrac{3}{2}A)$$

$$3A = 2c$$

$$\tfrac{3}{2}A = c$$

$$5 = -2 + 2A + \tfrac{3}{2}A$$

$$7 = 2A + \tfrac{3}{2}A$$

$$7 = \tfrac{7}{2}A$$

$A = 2$ $\quad B = -1$ $\quad C = 3$

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$$L^{-1}\left\{ \frac{2}{s-2} + \frac{-s+3}{s^2+3} \right\}$$

$$= 2L^{-1}\left\{\frac{1}{s-2}\right\} - L^{-1}\left\{\frac{s}{s^2+3}\right\} + 3L^{-1}\left\{\frac{1}{s^2+3}\right\}$$

$$2L^{-1}\left\{\frac{1}{s-2}\right\} = 2e^{2t}$$

$$-L^{-1}\left\{\frac{s}{s^2+3}\right\} = -\cos(\sqrt{3}t)$$

$$3L^{-1}\left\{\frac{1}{s^2+3}\right\} = \frac{3}{\sqrt{3}}\sin(\sqrt{3}t)$$

$$= \boxed{2e^{2t} - \cos(\sqrt{3}t) + \sqrt{3}\sin\sqrt{3}t}$$

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5. $y'' + 6y' + 9y = e^{-3t}$ $y(0) = 0$
$y'(0) = 5$

$L\{y''\} = s^2 L\{y\} - 5$

$L\{6y'\} = 6sY(s)$

$L\{9y\} = 9Y(s)$

$L\{e^{-3t}\} = \frac{1}{s+3}$

$s^2 Y(s) - 5 + 6sY(s) + 9Y(s) = \frac{1}{s+3}$
$+5$

$\frac{Y(s)[s^2+6s+9]}{[s^2+6s+9]} = \frac{\frac{1}{s+3} + 5}{[s^2+6s+9]}$

$Y(s) = \frac{1}{(s+3)(s^2+6s+9)} + \frac{5}{(s^2+6s+9)}$

$Y(s) = \frac{1}{(s+3)^3} + \frac{5}{(s+3)^2}$

$L^{-1}\left\{\frac{5}{(s+3)^2}\right\} = 5te^{-3t}$

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$$L^{-1}\left\{\frac{1}{(s+3)^3}\right\} = \frac{t^2 e^{-3t}}{2}$$

$$y(t) = \frac{t^2 e^{-3t}}{2} + 5te^{-3t}$$

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(8) $\frac{dx}{dt} = -6x+2y \quad x(0)=5$

$\frac{dy}{dt} = -3x+1y \quad y(0)=0$

$A = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$

$\det(A-\lambda I)=0 \rightarrow \begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = 0$

$(-6-\lambda)(1-\lambda)+6=0$

$-6+6\lambda-\lambda+\lambda^2+6=0$

$\lambda^2+5\lambda=0$

$\lambda=-5, \lambda=0$

$\lambda=-5$

$(A-\lambda I)K \Rightarrow \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -k_1+2k_2 \\ -3k_1+6k_2 \end{bmatrix} \quad K_{\lambda=-5} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$k_1 = 2k_2$

$\lambda=0$

$\begin{pmatrix} -6 & 2 \\ -3 & 1 \end{pmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = 0$

$K_{\lambda=0} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -6k_1+2k_2 \\ -3k_1+k_2 \end{bmatrix} = 0$

$3k_2 = k_1$

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$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$x(0) = 5$
$y(0) = 0$

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$c_1 = 3$

$c_2 = -1$

Praticular solution is $\begin{bmatrix} x \\ y \end{bmatrix} = 3e^{-5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\boxed{\begin{aligned} x(t) &= 6e^{-5t} - 1 \\ y(t) &= 3e^{-5t} - 3 \end{aligned}}$$

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(7)

$$X' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} X$$

$$\det\left(\begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{vmatrix}\right) = \begin{pmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix}$$

$$\det \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4\lambda + 2$$

$$-(\lambda - 1)(\lambda^2 - 2\lambda + 2) = 0$$

quadratic formula

$\lambda - 1 = 0$

$\lambda_1 = 1$

$$\lambda_{2,3} = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$\lambda_{2,3} = \frac{2 \pm 2i}{2}$$

$2i$

$\lambda_2 = 1 + i$

$\lambda_3 = 1 - i$

For λ_1 =

$$(A - \lambda I) \begin{vmatrix} 1-1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

now reduce matrix

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ -1 & 0 & 0 \end{pmatrix}$$

$R_3 \leftarrow R_3 - 1 \cdot R_1$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_2 \leftarrow -1 \cdot R_2$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow R_1 \leftarrow -1 \cdot R_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A - 1I)\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{vmatrix} \begin{matrix} K_1 \\ K_2 \\ K_3 \end{matrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$K_1 = 0$

$K_2 - 2K_3 = 0$

$K_2 = 2K_3$

$k_1 = 0$

$k_2 = 2k_3$

$n = \begin{pmatrix} 0 \\ 2k_3 \\ k_3 \end{pmatrix} \quad k_3 \neq 0$

let $k_3 = 1$

$K_{\lambda=1} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$X = c_1 e^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

For $\lambda = 1 \pm i$

$\alpha = 1 \quad \beta = 1$

$n = \begin{pmatrix} \pm i \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$u = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$X = c_2 e^t \left(\cos(t) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \sin(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right) +$

$c_3 e^t \left(\cos(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$

so general solution is

$X = c_1 e^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 e^t \left(\cos(t) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \sin(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right)$

$+ c_3 e^t \left(\cos(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \sin(t) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right)$

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$$L\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - s$$

$$L\{4y'\} = 4s Y(s) - 4y(0) = 4s Y(s) - 4$$

$$L\{13y\} = 13 Y(s)$$

$$L\{\delta(t-\pi)\} = e^{-\pi s}$$

$$s^2 Y(s) - s + 4s Y(s) + 13 Y(s) - 4 = e^{-\pi s}$$

$$\frac{Y(s)[s^2+4s+13]}{(s^2+4s+13)} = \frac{e^{-\pi s} + s + 4}{s^2+4s+13}$$

$$Y(s) = \frac{e^{-\pi s} + s + 4}{s^2+4s+13}$$

$$Y(s) = \frac{e^{-\pi s}}{s^2+4s+13} + \frac{s+4}{s^2+4s+13}$$

$$\underbrace{s^2+4s}\,+13\,\underbrace{+4}\,-4$$

$$(s^2+4s+4)+13-4$$

$$(s+2)^2+9$$

$$Y(s) = \frac{e^{-\pi s}}{(s+2)^2+9} + \frac{s+4}{(s+2)^2+9}$$

$$= \frac{e^{-\pi s}}{(s+2)^2+9} + \frac{s+2}{(s+2)^2+9} + \frac{2}{(s+2)^2+9}$$

8

$$L^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} = e^{-2t} L^{-1}\left\{\frac{s}{s^2+9}\right\} = e^{-2t}\cos(3t)$$

$$L^{-1}\left\{\frac{2}{(s+2)^2+9}\right\} = e^{-2t} L^{-1}\left\{\frac{2}{s^2+9}\right\} = e^{-2t}\frac{2}{3}\sin(3t)$$

$$L^{-1}\left\{\frac{e^{-\pi s}}{(s+2)^2+9}\right\} = e^{-2t} L^{-1}\left\{\frac{e^{-\pi(s-2)}}{s^2+9}\right\}$$

$$L^{-1}\left\{\frac{e^{-\pi s + 2\pi}}{s^2+9}\right\} = e^{2\pi} L^{-1}\left\{\frac{e^{-\pi s}}{s^2+9}\right\}$$

$$= \frac{e^{2\pi}\, u(t-\pi)\sin 3(t-\pi)}{t}$$

$$= \frac{e^{-2t} e^{2\pi}\, u(t-\pi)\sin(3(t-\pi))}{3}$$

$$= \frac{1}{3} e^{-2(t-\pi)} u(t-\pi)\sin(3(t-\pi))$$

$$\boxed{y(t) = e^{-2t}\cos(3t) + e^{-2t}\frac{2}{3}\sin(3t) + \frac{1}{3}e^{-2(t-\pi)} u(t-\pi)\sin(3(t-\pi))}$$