Unit 1.1: Integration by Substitution

If you have forgotten how to find basic antiderivatives or need a little "polishing up", you might want to review section 4.9 in the text before trying to master more complex integration techniques. See the end of this unit summary for the list of antiderivative rules you must know.

While finding an antiderivative can generally be viewed as reversing differentiation, the method of *U-substitution* found in this section can be viewed as reversing the *Chain-Rule*. When applying the chain-rule (to take derivatives), we make use of the following identity

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

If we view f as a function of x, we simply have $\int \frac{df}{dx} dx = f(x) + C$. However, if the function $\frac{df}{dx}$ is not a simple one, it might be easier to view f as a function of something more complicated which we can name u, and then we use the right-side of the above identity to obtain

$$\int \frac{df}{du} \cdot \underbrace{\frac{du}{dx} dx}_{du} = \int \frac{df}{du} du = f(u) + C$$

The above allows us to simplify our integrand by integrating with respect to a different variable. The goal is to have $\frac{df}{du}$ take the form of a basic function for which we have memorized its antiderivative. An initial list of such functions is provided at the end of this document. Each time we apply a substitution we will be using the identity $du = \frac{du}{dx} dx$.

Before looking at a few examples, I would like to point out there are a variety of ways that you might encounter for applying a u-substitution. Each of these methods is similar, but differs in the way that constant factors are dealt with. I will take the approach of factoring constants out of the integral; the author makes use of a slightly different procedure. Be sure only to factor out constant factors, not variable factors and not constants that are not factors of the integrand.

Example 1 Find the following integral. $\int 5x^2 \cos x^3 \, dx$

Solution: When applying a u-substitution, you typically want to let u equal the "inside" of a function. The hope is that the derivative of u can be found as a factor of the integrand, although it may be off by a constant factor. In this case we let $u = x^3$ and note that its derivative is $3x^2$. While our integrand contains $5x^2$ we need only be concerned about the variable factors. We proceed as follows.

Let $u = x^3$, therefore $du = 3x^2 dx$. Typically any constant generated in the du statement will be removed by multiplying both side of the identity by its reciprocal. In this case we multiply by $\frac{1}{3}$ to obtain $\frac{1}{3}du = x^2dx$. The factor of $\frac{1}{3}$ as well as factor the 5 in the original integral will both be factored out of the integral. The integral can be deconstructed as follows

$$\int 5\,x^2 \cos \overbrace{x^3}^{u}\, dx = 5 \cdot \frac{1}{3} \int \cos u\, du = \frac{5}{3} \sin u + C = \frac{5}{3} \sin x^3 + C$$

$$\frac{1}{3}du$$

Final Answer: $\frac{5}{3}\sin x^3 + C$ Don't forget to write the final answer in terms of the original variable and don't forget the $+C$.

Definite Integrals. Recall the *Fundamental Theorem of Calculus*, which says that if F is <u>any</u> antiderivative of f, then $\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$, when f is continuous on $[a, b]$.

Because we can use <u>any</u> antiderivative we will always choose the antiderivative where the $+C$ is $+0$. It is also important to note that the a and b limits of integration determine the interval for x. When we apply a u-substitution, we must convert the x-interval to a u-interval.

<u>Example 2</u> Evaluate $\int_0^2 x\sqrt[3]{4 + x^2}\, dx$

Solution: In this case, we let u equal the inside of the cube root function (a power function). Let $u = 4 + x^2$, therefore $du = 2xdx$, and thus $\frac{1}{2}du = xdx$. Now we must change the limits of integration. When $x = 0$ (the lower limit), $u = 4 + x^2 = 4 + 0^2 = 4$ and when $x = 2$ (the upper limit), $u = 4 + 2^2 = 8$. Thus the x-interval from 0 to 2, corresponds to the u-interval from 4 to 8. With these substitutions we obtain

$$\int_0^2 x\sqrt[3]{4 + x^2}\, dx = \frac{1}{2}\int_4^8 \sqrt[3]{u}\, du = \frac{1}{2}\int_4^8 u^{1/3}\, du = \frac{1}{2} \cdot \frac{3}{4} u^{4/3}\Big]_4^8 = \boxed{\frac{3}{8}\left[8^{4/3} - 4^{4/3}\right]}$$

At this point, we will not concern ourselves with the presentation of the answer—only the procedures. I will stress such simplifications in the narrated examples found within the section sub-modules. Unless directed otherwise NEVER approximate your answers by writing them in decimal form and rounding.

Special Integrals. It will be very beneficial (especially in later chapters) if you learn to avoid making a substitution when u would be *linear*, i.e. of the form $ax + b$. In this case we can make use of the following fact:

$$\text{If } \int f(x)dx = F(x) + C, \text{ then } \int f(ax + b)dx = \frac{1}{a}F(ax + b) + C$$

Example 3 Find the integrals $\int e^{3x} dx$, $\int \sin(\pi\theta) d\theta$, and $\int (5t-9)^6\, dt$

Solution: $\int e^{3x} dx = \frac{1}{3} e^{3x} + C$

$\int \sin(\pi\theta) d\theta = -\frac{1}{\pi}\cos(\pi\theta) + C$

$\int (5t-9)^6\, dt = \frac{1}{5} \cdot \frac{1}{7}(5t-9)^7 + C$

You are encouraged to solve each of the above using substitutions to see why this works. As soon as you feel comfortable, start taking advantage of this special case.

Integrals Involving the Natural Logarithm. From Calculus I, you should recall the rule $\frac{d}{dx}\ln x = \frac{1}{x}$. It is also true that $\frac{d}{dx}\ln|x| = \frac{1}{x}$. The more general version of this rule based on the chain rule says that

$$\frac{d}{dx}\ln|u| = \frac{u'}{u} \quad \Rightarrow \quad \int \frac{u'}{u}\, dx = \boxed{\int \frac{1}{u}\, du = \ln|u| + C}$$

When applying a substitution to fit the above form, we typically let u equal the denominator or a factor of the denominator. Before carrying out the substitution, make sure that the derivative of u can be found as a factor of the integrand (often in the numerator).

Example 1 Find the integral $\int \frac{2x^2+4x}{x^3+3x^2-4} dx$.

Solution: Let $u = x^3 + 3x^2 - 4$, therefore $du = (3x^2 + 6x)dx = 3(x^2 + 2x)dx$ and thus $\frac{1}{3} du = (x^2 + 2x)dx$. From the integrand, we can factor out a 2 (from the numerator) and thus we obtain

$$\int \frac{2x^2+4x}{x^3+3x^2-4} dx = \int \frac{2(x^2+2x)}{x^3+3x^2-4} dx = 2 \cdot \frac{1}{3} \int \frac{1}{u} du = \frac{2}{3}\ln|u| + C$$

Final Answer: $\frac{2}{3}\ln|x^3 + 3x^2 - 4| + C$

Example 2 Evaluate $\int_e^{e^3} \frac{2}{x \ln x} dx$.

Solution: In this case, we first note that the inside of the logarithm function is just x, so it doesn't make sense to let u be the inside of the log function. Letting u be the entire denominator will require the product rule which would generate variable factors not contained in the integrand. But if we simply let $u = \ln x$, we then obtain $du = \frac{1}{x} dx$. It is important to realize that the dx factor is always a numerator factor as it can be thought of as $\frac{dx}{1}$. Thus the $\frac{1}{x}\,dx$ is perfect for our situation. We must convert our limits of integration to u-values as follows: when $x = e,\ u = \ln e = 1$ and when $x = e^3,\ u = \ln e^3 = 3$. Factoring out a 2 from the integrand and applying our substitution yields

$$\int_e^{e^3} \frac{2}{x \ln x}\,dx = 2\int_1^3 \frac{1}{u}\,du = 2\ln|u|\Big]_1^3 = 2[\ln 3 - \underbrace{\ln 1}_{0}] = 2\ln 3 = \boxed{\ln 9}$$

At this point in the course you should have the following integration rules memorized. These functions may be encountered on any of the exams (there will not be any formula sheets allowed).

Below n represents a real number, a represents a positive real number different from 1, and C is an arbitrary constant (or parameter) sometimes called the constant of integration.

1) $\int u^n du = \frac{u^{n+1}}{n+1} + C$ (when $n \neq -1$)

2) $\int u^{-1} du = \int \frac{1}{u}\,du = \ln|u| + C$

3) $\int e^u du = e^u + C$

4) $\int a^u du = \frac{1}{\ln a} a^u + C$

5) $\int \cos u\,du = \sin u + C$

6) $\int \sin u\,du = -\cos u + C$

7) $\int \sec u \tan u\,du = \sec u + C$

8) $\int \sec^2 u\,du = \tan u + C$

9) $\int \tan u\,du = -\ln|\cos u| + C$

10) $\int \cot u\,du = \ln|\sin u| + C$

11) $\int \sec u\,du = \ln|\sec u + \tan u| + C$

12) $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

As we proceed in the course, additional rules will be added and I will update the list.