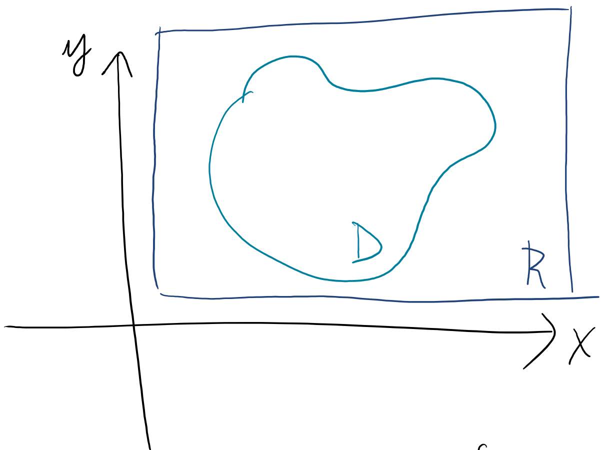
12.2: Double Integrals Over General Regions

Wednesday, September 30, 2020 1:46 PM



$$f : D \to \mathbb{R}$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D \end{cases}$$

- $\displaystyle\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$

- If $f(x,y) \geq 0$ on D, $\iint_D f(x,y)\, dA$ = vol. of solid lying below surface $z = f(x,y)$, and above D.

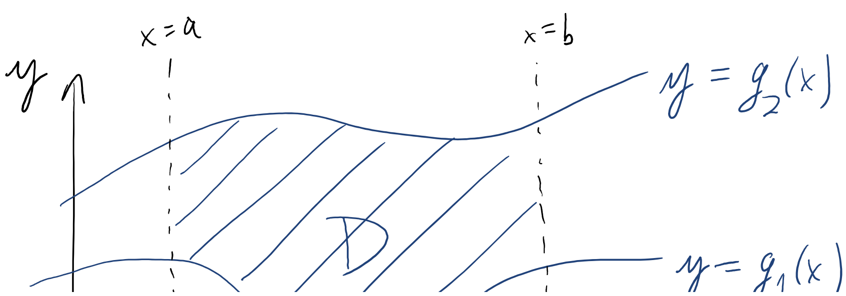
<u>Type I region</u> $D \subset \mathbb{R}^2$, $D = \{(x,y) : a \leq x \leq b,$

$g_1(x) \leq y \leq g_2(x)\}$

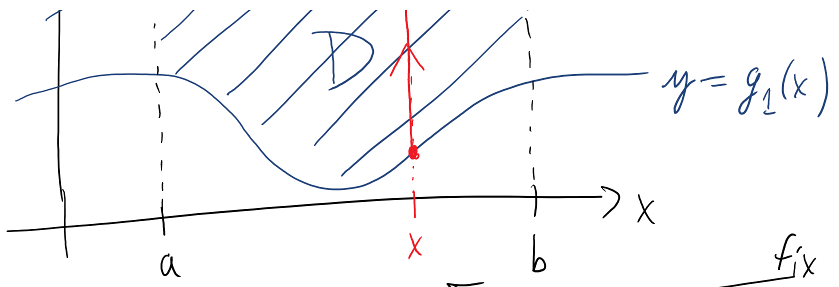
for some continuous func. $g_1, g_2 : [a,b] \to \mathbb{R}$.

If f continuous on D, then

$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy \right) dx$$



"bottom & top curves"



fix x, let y vary in inside integral

* Integrate over vertical segments, bottom to top.

In outter integral, let x vary.

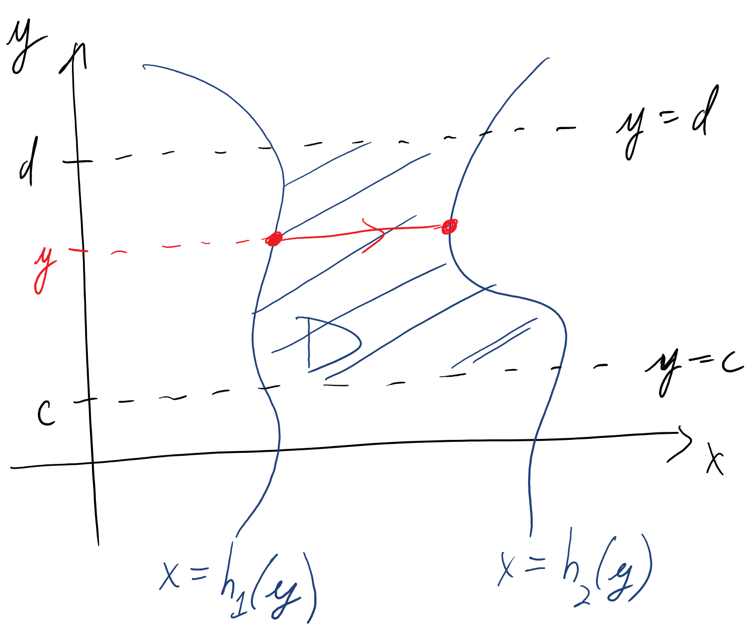
Type II region

$$D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

for some cont. $h_1, h_2 : [c,d] \longrightarrow \mathbb{R}$.

If f cont. on D,

$$\iint_D f(x,y)\, dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$$

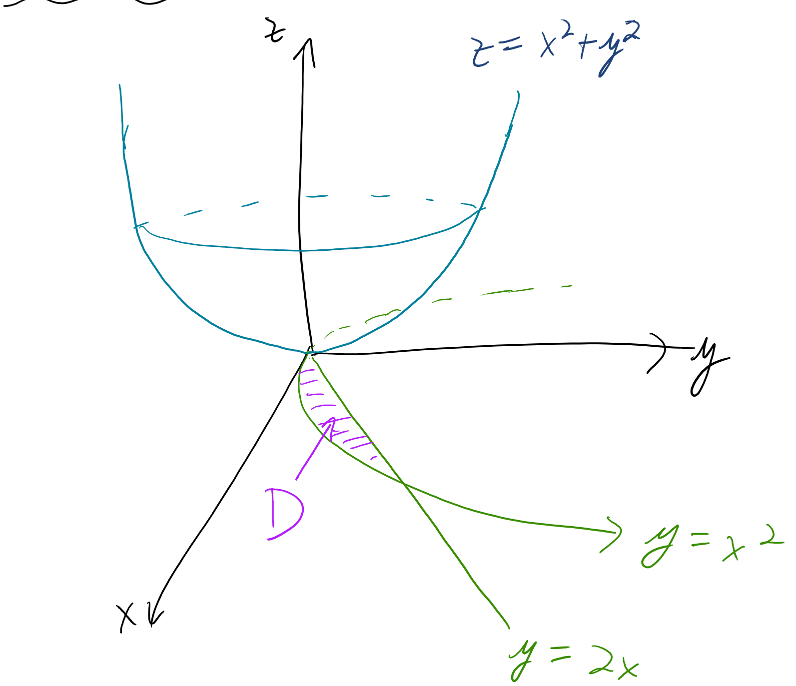


"left & right curves"

$x = h_1(y)$ $\qquad$ $x = h_2(y)$

* Fix y, integrate over horizontal segments, left to right.

Ex. Find the volume of solid under $z = x^2 + y^2$, above D, where D is the region in xy-plane bdd by $y = 2x$ and $y = x^2$.



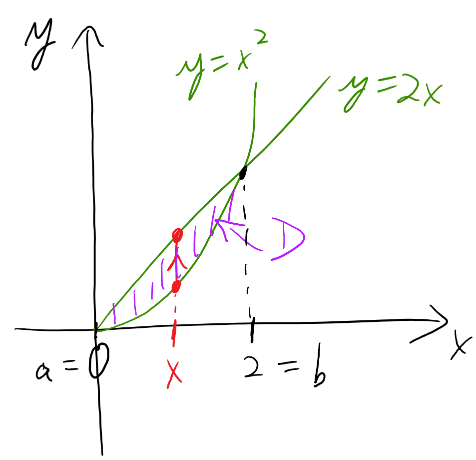
Have $f(x,y) = x^2 + y^2 \geq 0$ on D.

$$\Rightarrow V = \iint_D x^2 + y^2 \, dA$$

$$= ?$$

$$x^2 = 2x$$

$$x = 2$$



$$V = \iint_D x^2 + y^2 \, dA$$

$$= \int_0^2 \int_{x^2}^{2x} x^2 + y^2 \, dy \, dx \qquad \text{(type I)}$$

$$= \int_0 \int_{x^2} x^2 + y^2 \, dy \, dx \qquad (\text{type I})$$

$$= \int_0^2 \int_{x^2}^{2x} x^2 \, dy \, dx + \int_0^2 \int_{x^2}^{2x} y^2 \, dy \, dx$$

$$= \int_0^2 x^2 \left(\int_{x^2}^{2x} 1 \, dy \right) dx + \int_0^2 \int_{x^2}^{2x} y^2 \, dy \, dx$$

$$= \int_0^2 x^2 \left(2x - x^2 \right) dx + \int_0^2 \left. \frac{y^3}{3} \right|_{x^2}^{2x} dx$$

$$= \int_0^2 2x^3 - x^4 \, dx + \frac{1}{3} \int_0^2 8x^3 - x^6 \, dx$$

$$= \left[\frac{x^4}{2} - \frac{x^5}{5} \right]_0^2 + \frac{1}{3} \left[2x^4 - \frac{x^7}{7} \right]_0^2$$

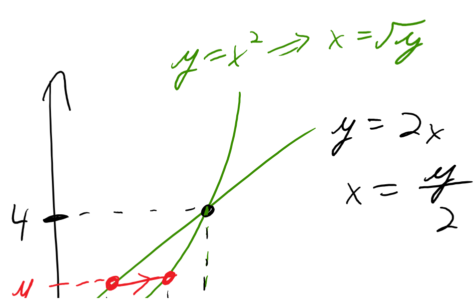
$$= \frac{2^4}{2} - \frac{2^5}{5} + \frac{1}{3} \left(2^5 - \frac{2^7}{7} \right)$$

$$= 2^4 \left(\frac{1}{2} - \frac{2}{5} + \frac{1}{3} \left(2 - \frac{2^3}{7} \right) \right)$$

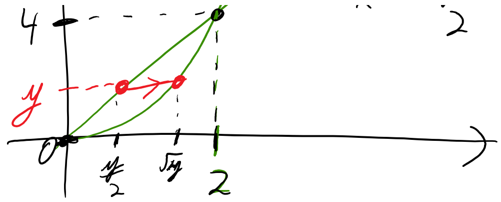
$$= 2^4 \left(\frac{1}{2} - \frac{2}{5} + \frac{2}{3} \left(1 - \frac{4}{7} \right) \right) \qquad \frac{2}{3} \; \frac{3}{7}$$

$$= 2^4 \left(\frac{1}{2} - \frac{2}{5} + \frac{2}{7} \right)$$

$$= \boxed{\frac{216}{35}}$$



Or as type II :



$$\iint_D x^2+y^2\,dA = \int_0^4\int_{y/2}^{\sqrt{y}} x^2+y^2\,dx\,dy$$

$$= \boxed{\frac{216}{35}}$$

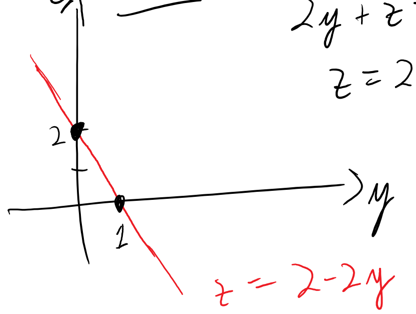
Ex. Find vol. of tetrahedron bdd by planes

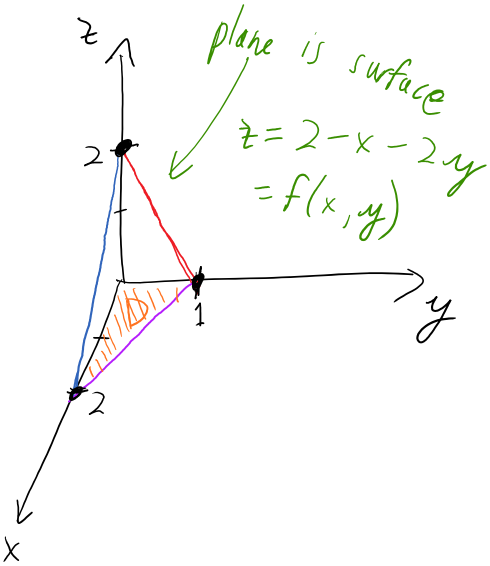
$x+2y+z=2$, $x=0$, $y=0$, $z=0$.

$x=0$

$2y+z=2$

$z=2-2y$

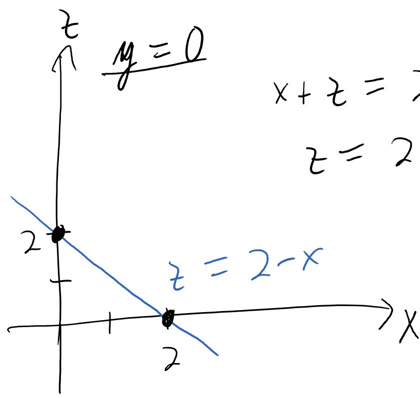




$y=0$

$x+z=2$

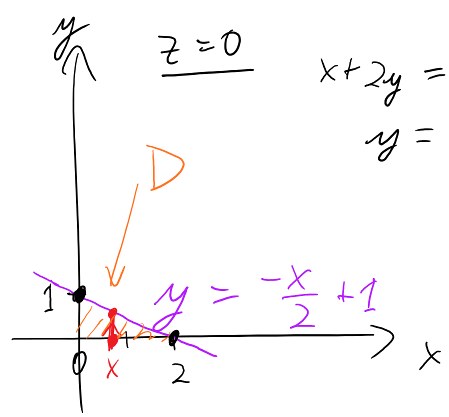
$z=2-x$



$z=0$

$x+2y=2$

$y=\frac{-x}{2}+1$



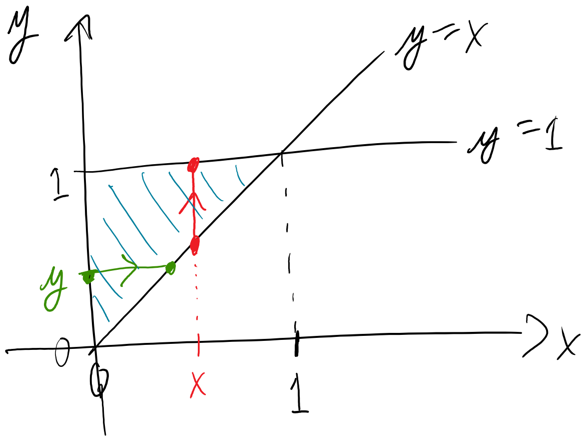
$V = \iint \quad dA = \int^{2}\int^{-\frac{x}{2}+1}$

$$V = \iint_D f(x,y)\, dA = \int_0^2 \int_0^{-\frac{x}{2}+1} 2 - x - 2y \; dy \; dx \qquad (\text{type I})$$

$$= \ldots \quad (\text{solve})$$

<u>Ex.</u> Evaluate $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \sin(y^2)\, dy\, dx.$ (this is currently type I)

Hard to do $\int_x^1 \sin(y^2)\, dy$ ..., so switch order of integration?



$$\iint_D f \, dA = \int_0^1 \int_0^y \sin(y^2)\, dx \; dy \qquad (\text{type II})$$

$$= \int_0^1 \sin(y^2) \int_0^y dx \; dy$$

$$= \int_0^1 \sin(y^2)\, y \; dy \,, \qquad u = y^2$$

$$\frac{du}{2} = \cancel{2} y \, dy$$

$$= \int_0^1 \sin(u)\, \frac{du}{2}$$

$0 \qquad 2$

$$= -\frac{1}{2}\cos(u)\Big|_0^1$$

$$= \boxed{-\frac{1}{2}\left(\cos(1) - 1\right)}$$

Properties Assume $\iint_D f$, $\iint_D g$ exist, $D \subset \mathbb{R}^2$.

(i) $\iint_D f + g = \iint_D f + \iint_D g$

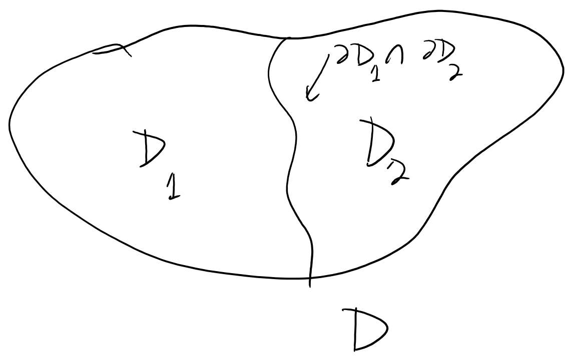
(ii) $\iint_D f - g = \iint_D f - \iint_D g$

(iii) $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$

(iv) If $f(x,y) \geq g(x,y)$ on D, then $\iint_D f \geq \iint_D g$

(v) $D = D_1 \cup D_2$, where $(D_1 \cap D_2) \subset (\partial D_1 \cap \partial D_2)$,

then $\iint_D f = \iint_{D_1} f + \iint_{D_2} f$.



(vi) $\displaystyle\iint_D 1 \, dA = \text{area}(D)$

(vii) $m \le f(x, y) \le M$ on D, then

$$m \cdot \text{area}(D) \le \iint_D f \, dA \le M \cdot \text{area}(D)$$

i.e. $$m \le \frac{1}{\text{area}(D)} \iint_D f \, dA \le M$$