Def. 1) $D \subset \mathbb{R}^2$. A <u>vector field on $\mathbb{R}^2$</u> is a func. $\vec{F}$ that assigns, to each point $(x, y) \in D$, a 2-D vector $\vec{F}(x, y)$.

2) $E \subset \mathbb{R}^3$. A vector field on $\mathbb{R}^3$ is a func. $\vec{F}$ that assigns, to each point $(x, y, z) \in E$, a 3-D vector $\vec{F}(x, y, z)$.

3) Writing
$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = P(x, y)\vec{i} + Q(x, y)\vec{j},$$
P and Q are the <u>component func's of $\vec{F}$</u>. Similarly for $\vec{F} = \langle P, Q, R \rangle$.

Example 1 $\vec{F}(x, y) = \underbrace{-y\vec{i} + x\vec{j}}_{= \langle -y, x \rangle}$. Describe $\vec{F}$ and sketch some of the vectors.

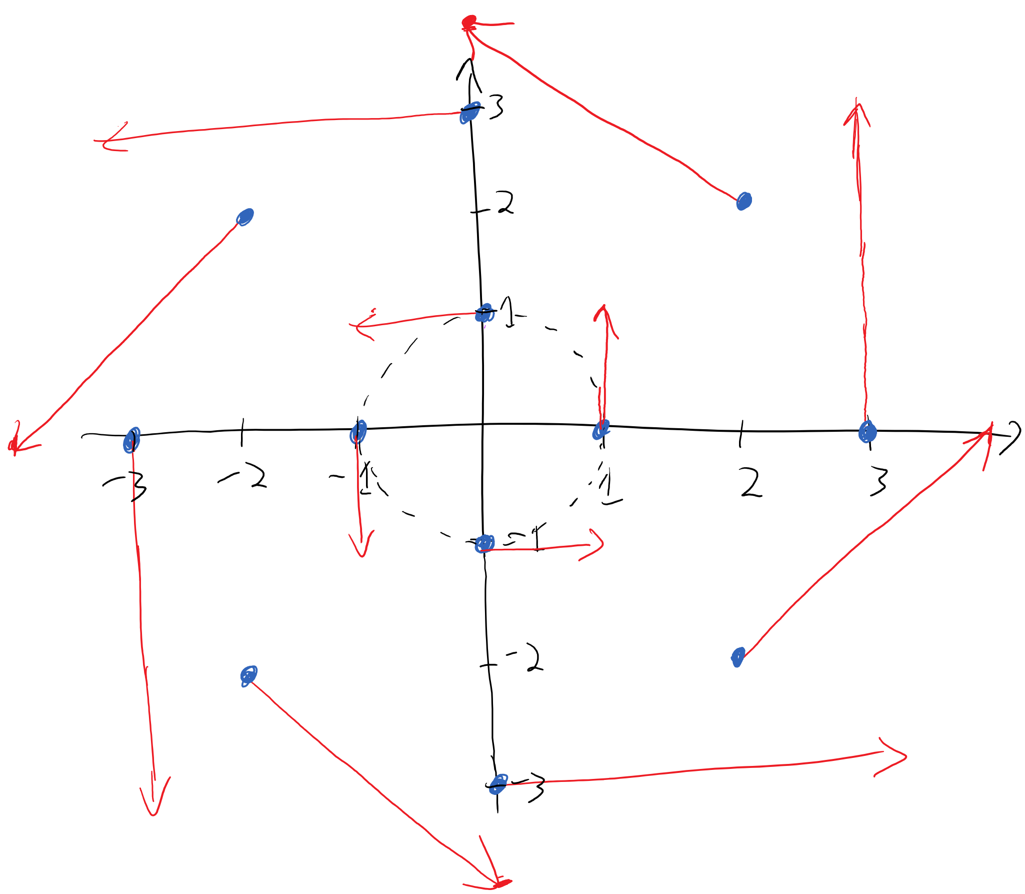
Sol'n:

(x, y)	$\vec{F}(x, y)$
✓ $(1, 0)$	$\langle 0, 1 \rangle$
✓ $(-1, 0)$	$\langle 0, -1 \rangle$
✓ $(2, 2)$	$\langle -2, 2 \rangle$

(x, y)	$\vec{F}(x, y)$
✓ $(-2, 2)$	$\langle -2, -2 \rangle$
✓ $(2, -2)$	$\langle 2, 2 \rangle$
✓ $(0, 3)$	$\langle -3, 0 \rangle$

✓ $(2,2)$	$\langle -2, 2\rangle$
✓ $(-2,-2)$	$\langle 2, -2\rangle$
✓ $(3,0)$	$\langle 0, 3\rangle$
✓ $(-3,0)$	$\langle 0, -3\rangle$
✓ $(0,1)$	$\langle -1, 0\rangle$
✓ $(0,-1)$	$\langle 1, 0\rangle$

✓ $(0,3)$	$\langle -3, 0\rangle$
✓ $(0,-3)$	$\langle 3, 0\rangle$



- Each arrow tangent to a circle cent. at $(0,0)$...

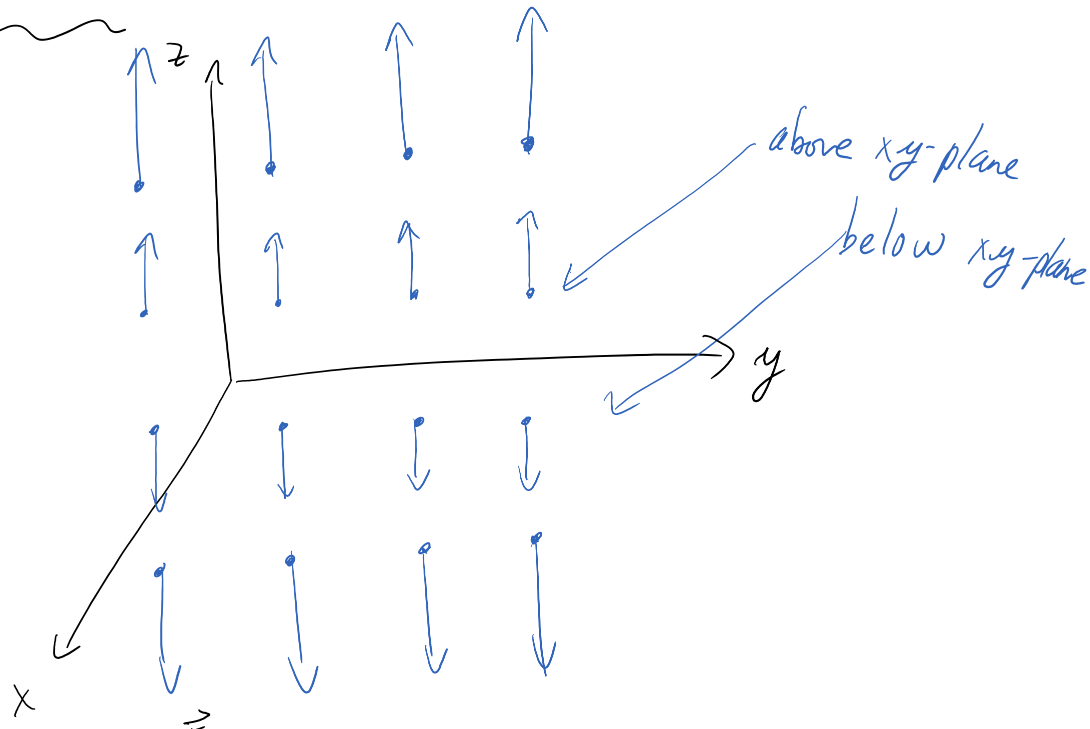
true if $\vec{x} \cdot \vec{F}(\vec{x}) = 0$ for all $\vec{x}$, b/c this means $\vec{x} \perp \vec{F}(\vec{x})$.

$$\vec{x} \cdot \vec{F}(\vec{x}) = \langle x, y \rangle \cdot \langle -y, x \rangle = -xy + yx = 0 \quad \checkmark$$

Also $\left|\vec{F}(x,y)\right| = \left|\langle -y, x\rangle\right| = \sqrt{x^2+y^2} = |\vec{x}|$,

so each vector has length equal to the radius of that same circle.

Ex. 2 Sketch $\vec{F}(x,y,z) = z\vec{k}$.

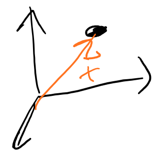


- All vectors $\vec{F}$ vertical, with length = dist. from (x,y,z) to xy-plane
- Point upward if $z>0$, down. if $z<0$
- Get longer as (x,y,z) gets farther from xy-plane.

Ex. 3 Newton's Law of Gravitation $\Rightarrow$ magnitude of of grav. force between two objects w/ masses m and M is $|\vec{F}| = \dfrac{mMG}{r^2}$, where

r = dist. between obj., G = grav. constant.

- Assume obj. w/ mass M at origin in $\mathbb{R}^3$.
- $\vec{x} = \langle x, y, z \rangle$ = position vector of obj. w/ mass m.



- Then $r = |\vec{x}| \Rightarrow r^2 = |\vec{x}|^2$.
- The grav. force exerted on the second obj. acts toward origin, and the unit vector is $\dfrac{-\vec{x}}{|\vec{x}|}$, so

$$\vec{F}(\vec{x}) = \frac{mMG}{|\vec{x}|^2}\left(\frac{-\vec{x}}{|\vec{x}|}\right)$$

$$= \frac{-mMG\,\vec{x}}{|\vec{x}|^3}$$

$\vec{x} = \langle x, y, z \rangle$

$$= \left\langle \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

Gradient Fields

2 variables: $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$

2 variables: $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$
$(x,y) \longmapsto z$,

(i) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ is a vector field on $\mathbb{R}^2$,

(ii) $\nabla g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z) \rangle$ is a v.f. on $\mathbb{R}^3$.

Ex. 4 Find gradient vector field of $f(x,y) = x^2y - y^3$.

Sol'n: $\nabla f(x,y) = \langle 2xy, x^2 - 3y^2 \rangle$.

Def. 1) A vector field $\vec{F}$ is conservative if it is the gradient of some scalar func.
(i.e. $\vec{F} = \nabla f$ some f)

2) If $\vec{F} = \nabla f$, f is a potential function for $\vec{F}$ (f like antideriv.).

Warning Not all vector fields are conservative.

$f(x,y,z) = \dfrac{mMG}{\sqrt{x^2+y^2+z^2}}$

$\left(\dfrac{1}{\sqrt{x^2+1}}\right)' = \left((x^2+1)^{-1/2}\right)'$
$= -\dfrac{1}{\cancel{2}}(x^2+1)^{-3/2}(\cancel{2}x)$

$$\Rightarrow \nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$$

$$= \frac{-mMGx}{(x^2+y^2+z^2)^{3/2}} \vec{i} + \frac{-mMGy}{(x^2+y^2+z^2)^{3/2}} \vec{j} + \frac{-mMGz}{(x^2+y^2+z^2)^{3/2}} \vec{k}$$

$= \vec{F}$ from Ex. 3

$\Rightarrow \vec{F}$ is a conservative vector field.