1.3 Autonomous First-Order DEs

- *Definition:* An ordinary differential equation in which the independent variable does not explicitly appear is called **autonomous.**

An autonomous first-order differential equation can be written as

$$f(y, y') = 0 \quad or \quad \frac{dy}{dx} = f(y)$$

The zeros of $f(y)$ are called **critical points** or **equilibrium points**.

**If c is a critical point of f, then $y(x) = c$ is a constant solution to the differential equation.*

We can create a one-dimensional **phase portrait** of an autonomous differential equation to help visualize solution curves, which can aid in finding solutions to the differential equation. This is done by finding the critical points and checking the sign of $f(y)$ in the intervals around the critical points.

- *Example:* Create a phase portrait for the following autonomous DE.

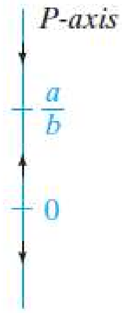
$$\frac{dP}{dt} = P(a - bP)$$

$$P(a - bP) = 0 \quad \rightarrow \quad P = 0, \qquad P = \frac{a}{b}$$

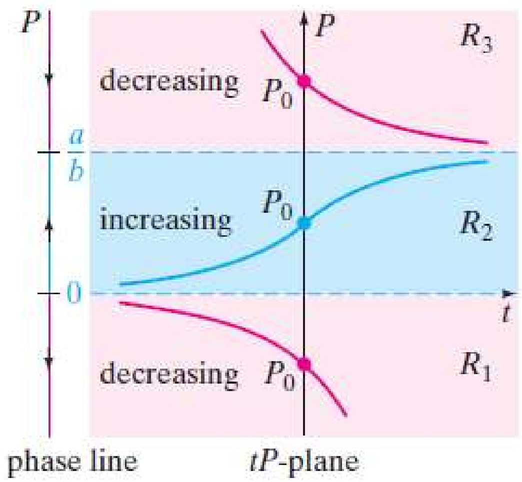
Next we check the sign of $f(P)$.

Interval	**Sign of $\boldsymbol{f(P)}$**	$\boldsymbol{P(t)}$
$(-\infty, \mathbf{0})$	negative	decreasing
$(\mathbf{0}, \boldsymbol{a/b})$	positive	increasing
$(\boldsymbol{a/b}, \infty)$	negative	decreasing

Finally we use this information to draw the (vertical) P-axis with labels and arrows.



We can then use the phase portrait to give a (rough) sketch of the solution curves.



Critical points can be further categorized based on the behavior that $y(x)$ exhibits near c. There are three categories.

1. **Asymptotically Stable** (or an **attractor**) - both arrows point towards c
   Solutions that are near, but not at c, is attracted towards c
2. **Unstable** (or a **repeller**) - both arrows point away from c
   Solutions that are near, but not at c, are repelled away from c
3. **Semi-stable** - one arrow points towards c and the other points away from c
   Solutions are one side are attracted; solutions on the other are repelled