10.3: The Dot Product

Thursday, August 13, 2020 10:00 PM

Def. Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

The dot product of $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

In V_2, $\langle a_1, a_2 \rangle \bullet \langle b_1, b_2 \rangle = a_1 a_2 + b_1 b_2$.

Note $\vec{a} \bullet \vec{b}$ is a scalar.

Properties $\vec{a}, \vec{b}, \vec{c} \in V_3$, $\alpha \in \mathbb{R}$

1) $\vec{a} \bullet \vec{a} = |\vec{a}|^2$

2) $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$

3) $\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$

4) $(\alpha\vec{a})\bullet\vec{b} = \alpha(\vec{a}\bullet b) = \vec{a}\bullet(\alpha\vec{b})$.

5) $\vec{0}\bullet\vec{a} = \vec{0}$

Th'm. If θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$, then

$$\vec{a}\bullet\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos(\theta).$$

Note If $\vec{a} \parallel \vec{b}$, then $\theta = 0$ or $\theta = \pi$.

Pf. (Law of cosines. ..)

Ex. 1 $\langle 2,4\rangle\bullet\langle 3,-1\rangle = 2(3)+4(-1)$
$= 2$

$\langle -1,7,4\rangle\bullet\langle 6,2,-\frac{1}{2}\rangle = (-1)6+7(2)$
$+4(-\frac{1}{2})$

$+4\left(-\frac{1}{2}\right)$

$= 6$

$(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2j - \vec{k})$

$= 1(0) + 2(2) + (-3)(-1)$

$= 7$

Ex. 2 If $\vec{a}, \vec{b}$ have lengths 4, 6 resp., and the angle between them is $\frac{\pi}{3}$, find $\vec{a} \cdot \vec{b}$.

$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$

$= 4(6) \cos\left(\frac{\pi}{3}\right)$

$$= \boxed{12}$$

Corollary If θ is the angle between $\vec{a}, \vec{b}$, where $\vec{a} \neq 0$ and $\vec{b} \neq \vec{0}$, then

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

Ex. 3 Find angle between $\vec{a} = \langle 2, 2, -1 \rangle$ and $\vec{b} = \langle 5, -3, 2 \rangle$.

Sol'n: $|\vec{a}| = 3$, $|\vec{b}| = \sqrt{38}$,

$\vec{a} \bullet \vec{b} = 2$

$$\cos(\theta) = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{2}{3\sqrt{38}}$$

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right) \approx \boxed{1.46}$$

<u>Def.</u> $\vec{a}$ and $\vec{b}$ are <u>perpendicular</u> (<u>orthogonal</u>) if $\vec{a} \bullet \vec{b} = \vec{0}$. (i.e. the angle between $\vec{a}$ & $\vec{b}$ is $\frac{\pi}{2}$, or one of $\vec{a}$, $\vec{b}$ is $\vec{0}$).

<u>Note</u> $$\vec{a} \bullet \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\left(\frac{\pi}{2}\right)$$
$$= 0$$

θ

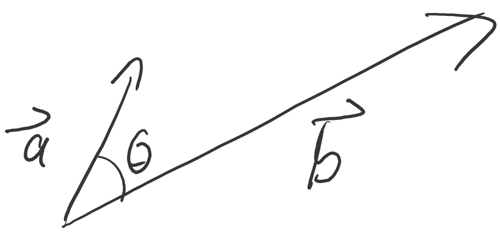
Ex. 4 Show $2\vec{i} + 2\vec{j} - \vec{k}$ perp. to $5\vec{i} - 4\vec{j} + 2\vec{k}$.

Sol'n: $(2\vec{i} + 2\vec{j} - \vec{k}) \cdot (5\vec{i} - 4\vec{j} + 2\vec{k})$

$= 10 - 8 - 2$

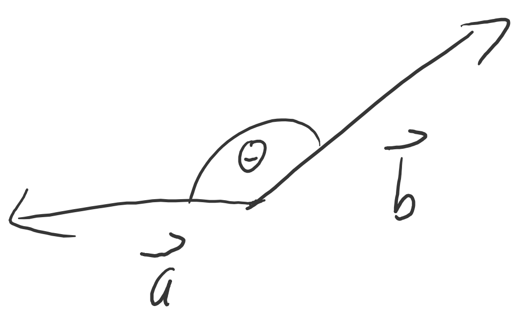
$= 0$ ✓

Rmk.



$\vec{a} \cdot \vec{b} > 0$

θ acute



$\vec{a} \cdot \vec{b} < 0$

θ obtuse

$\vec{a} \cdot \vec{b}$ measures how close $\vec{a}, \vec{b}$ are

~~to~~ to pointing same direction

Projections

$\vec{p}$

$\cos(\theta) = \frac{|\vec{p}|}{|\vec{b}|}$

$|\vec{p}| = |\vec{b}| \cos(\theta)$

$= \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \text{comp}_{\vec{a}} \vec{b}$

$\text{proj}_a b$

Def. 1) The vector projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a}$$

$$\text{proj}_{\vec a}\vec b = \left(\frac{\vec a \cdot \vec b}{|\vec a|}\right)\frac{\vec a}{|\vec a|} = \frac{\vec a \cdot \vec b}{|\vec a|^2}\,\vec a$$

2) The <u>scalar projection</u> (<u>component</u>) of $\vec b$ along $\vec a$ is

$$\text{comp}_{\vec a}\vec b = \frac{\vec a \cdot \vec b}{|\vec a|}.$$

= signed magnitude of $\text{proj}_{\vec a}\vec b$:

- negative if opposite
- positive if same

<u>Ex. 5</u> Find the scalar & vector project. of $\vec b = \langle 1, 1, 2\rangle$ onto $\vec a = \langle -2, 3, 1\rangle$.

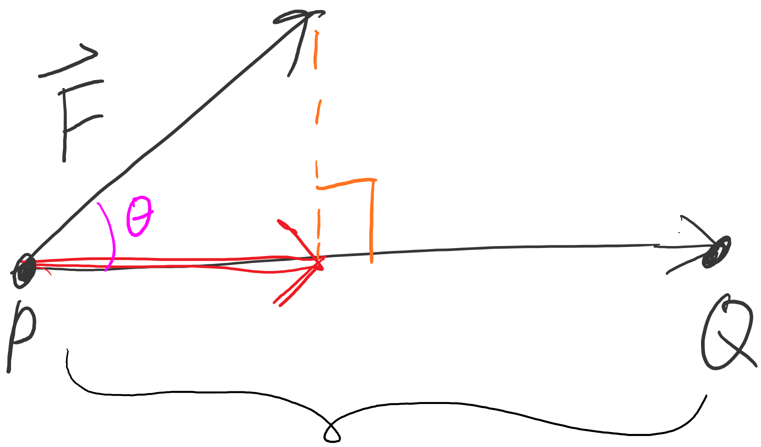
Sol'n: $|\vec{a}| = \sqrt{14}$

$$\operatorname{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{3}{\sqrt{14}}$$

$$\operatorname{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right) \frac{\vec{a}}{|\vec{a}|} = \frac{3}{\sqrt{14}} \frac{1}{\sqrt{14}} \langle -2, 3, 1 \rangle$$

$$= \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle$$

Rmk. Constant force $\vec{F}$ applied to move an object from point P to Q.



$\vec{D} = \vec{PQ}$ = displ. vector

$$\text{Work} = W = \left(|\vec{F}|\cos(\theta)\right)|\vec{D}|$$

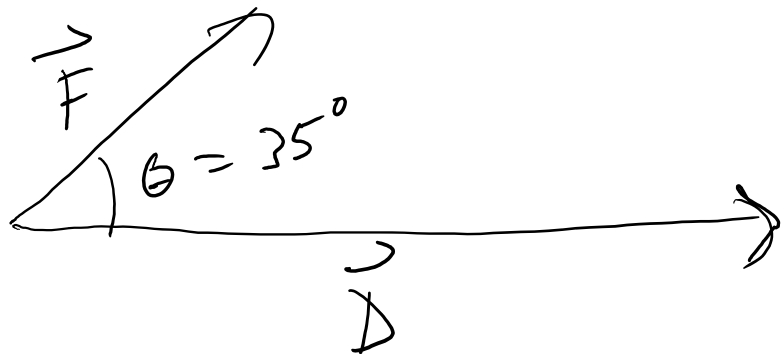
$$= |\vec{F}| \cdot |\vec{D}| \cos(\theta)$$

$$= \vec{F} \bullet \vec{D}$$

<u>Ex.</u> A wagon pulled 100m along a horizontal path by a constant force of 70 N. The handle of the wagon is held $35°$ above horizontal.

Find the work done.

Sol'n:



$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos(35^\circ)$$

$$= (70\ N)(100\ m)\cos(35^\circ)$$

$$\approx 5734\ Nm$$

$$= \boxed{5734\ J}$$

<u>Ex. 7</u> Force $\vec{F} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

moves particle from $P = (2, 1, 0)$

to $Q = (4, 6, 2)$. Find work.

<u>Sol'n:</u> $\vec{D} = \overrightarrow{PQ} = \langle 2, 5, 2 \rangle$

$$W = \vec{F} \cdot \vec{D} = \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle$$

$$= \boxed{36} \ldots \text{units?}$$