5.4 Applications Involving Systems

Some applications lead to more complex cases which result in systems of equations. Assuming the system produced is a linear first-order system with constant coefficients, we can use the previous techniques to solve.

Competition Models:

Suppose that two different species of animal occupy the same ecosystem and compete for the same resources. We can then model the growth/decline of each population using a competition model. The model itself depends on the exact interaction between the two species and can be modeled in various ways.

The simplest model would assume that the population of one species would decline based purely on the existence of the other species. This gives rise to

$$\begin{aligned} \frac{dx}{dt} &= ax - by \\ \frac{dy}{dt} &= cy - dx \end{aligned},$$

a linear system that we can solve, assuming we know or can find the constants $a, b, c,$ and d.

A slightly more complex model would assume that the growth rate would be reduced by the interactions between the two species, leading to

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= cy - dxy \end{aligned},$$

a non-linear, and much more difficult system to solve. This is very similar to a predator-prey model, where one species hunts the other.

Both of the first two models assume that either species on their own would grow exponentially. Since this is not generally accurate in the long run, we could adjust the model to have logistic growth, instead of exponential, resulting in

$$\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 x^2 - c_1 xy \\ \frac{dy}{dt} &= a_2 y - b_2 y^2 - c_2 xy \end{aligned},$$

yet another more complicated non-linear model.

While these are interesting to consider, and do show the ways that one species can influence another, we will not spend a lot of time solving such models. The next two examples tie back to earlier work and do lead to problems that we will attempt to solve.

Mixtures:

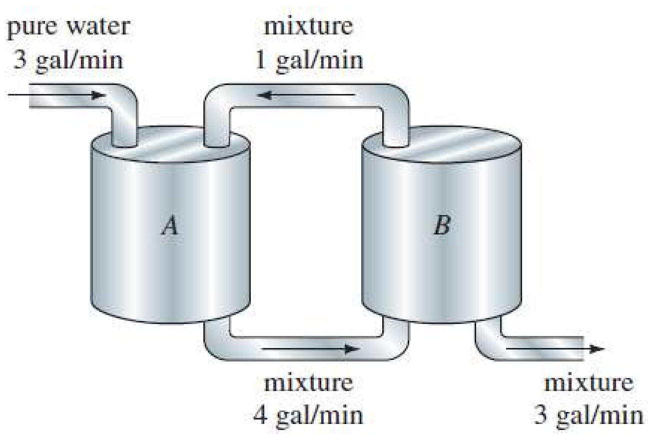
As we saw previously, the mixing of two solutions of differing concentrations gives rise to a first-order differential equation. When looking at a problem involving a single tank, we got

$$\frac{dA}{dt} = \begin{pmatrix} input\ rate \\ of\ solute \end{pmatrix} - \begin{pmatrix} output\ rate \\ of\ solute \end{pmatrix} = R_{in} - R_{out},$$

where $A(t)$ was the amount of solute in the tank at time t.

If instead we have multiple tanks, we must consider the amount of solute in each tank at a given time. This gives rise to a system of first-order differential equations, each of which is governed by the equation above.

- *Example:* Consider the system of two tanks shown. Tank A and tank B can each hold 50 gallons of liquid. Liquid is pumped into and out of the tanks at the rates shown on the diagram. If tank A starts with 25 pounds of salt, and tank B starts with pure water, and if pure water is pumped into tank A from an outside source, set up and solve a linear model to find the amount of salt in tank A, given by $x_1(t)$, and the amount of salt in tank B, given by $x_2(t)$ at a given time t.



Solution: Looking at tank A first, pure water is entering the tank at a rate of 3 gal/min, and liquid from tank B is entering at a rate of 1 gal/min. The concentration of salt coming from tank B is $x_2/50$ lb/gal, giving a rate in of

$$R_{inA} = 3 * 0 + 1 * \frac{x_2}{50} = \frac{1}{50}x_2.$$

The rate of salt leaving tank A would be $R_{outA} = (x_1/50$ lb/gal)*(4 gal/min) $= 2x_1/25$ lb/min. Thus

$$\frac{dx_1}{dt} = -\frac{2}{25}x_1 + \frac{1}{50}x_2.$$

Similarly, for tank B, we have salt coming in at a rate of $R_{inB} = 2x_1/25$ lb/min. The mixture in tank B would then flow out at a rate of $R_{outB} = (x_2/50$ lb/gal)*(4 gal/min) $= 2x_2/25$ lb/min, giving

$$\frac{dx_2}{dt} = \frac{2}{25}x_1 - \frac{2}{25}x_2.$$

Putting this together, we get the system

$$\begin{aligned}\frac{dx_1}{dt} &= -\frac{2}{25}x_1 + \frac{1}{50}x_2 \\ \frac{dx_2}{dt} &= \frac{2}{25}x_1 - \frac{2}{25}x_2\end{aligned} \quad \rightarrow \quad \mathbf{X}' = \begin{pmatrix} -\frac{2}{25} & \frac{1}{50} \\ \frac{2}{25} & -\frac{2}{25} \end{pmatrix}\mathbf{X}, \qquad \mathbf{X}(0) = \begin{pmatrix} 25 \\ 0 \end{pmatrix}.$$

Leaving the work as an exercise, it can be show that this system has eigenvalue/eigenvector pairs of

$$\lambda_1 = -\frac{1}{25}, \;\; K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \lambda_2 = -\frac{3}{25}, \;\; K_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix},$$

giving a general solution of

$$\mathbf{X}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t/25} + c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-3t/25}.$$

Using the initial condition, we get

$$\mathbf{X}(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 25 \\ 0 \end{pmatrix} \quad \rightarrow \quad c_1 = \frac{25}{2}, \;\; c_2 = -\frac{25}{2},$$

giving a solution of

$$\mathbf{X}(t) = \frac{25}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t/25} - \frac{25}{2}\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-3t/25} \quad \rightarrow \quad \begin{aligned}\boldsymbol{x_1(t)} &= \boldsymbol{\frac{25}{2}e^{-t/25} + \frac{25}{2}e^{-3t/25}} \\ \boldsymbol{x_2(t)} &= \boldsymbol{25e^{-t/25} - 25e^{-3t/25}}\end{aligned}.$$

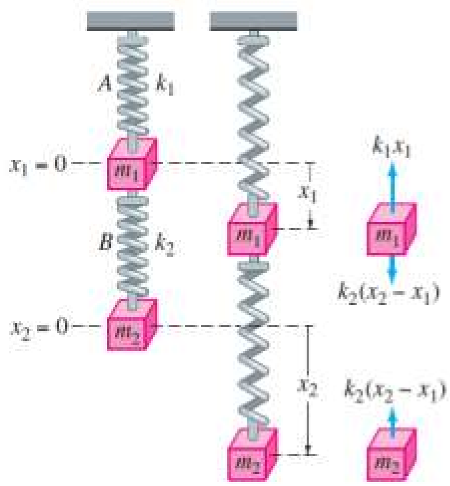
******Note: for our example, and most others, the flow rates of liquid into and out of each tank is a net zero. If this were not the case, then we would have one (or more) tanks either draining out or overflowing.*

Since pure water was flowing into the two tank system from the outside, we were left with a homogeneous system of equations. However, if a salt mixture had been flowing into the overall system, we would need to solve a non-homogeneous system.

The same process as above would work with three, or more, tanks as well, with each tank adding another variable and equation to the system.

Coupled Spring-Mass Systems:

If two masses m_1 and m_2 are connected to two springs A and B, or negligible mass, have spring constants k_1 and k_2, respectively, as shown below, the result is a coupled spring-mass system.



Let $x_1(t)$ and $x_2(t)$ represent the displacements of m_1 and m_2, respectively, from their equilibrium positions. When the system is in motion, spring B is subjected to both an elongation and a compression from the displacements of m_1 and m_2, resulting in a net elongation of $x_2 - x_1$. By applying Hooke's law, we get the linear second-order system

$$m_1 \frac{d^2 x_1}{dt^2} = -k_1 x_1 + k_2(x_2 - x_1)$$

$$m_2 \frac{d^2 x_2}{dt^2} = -k_2(x_2 - x_1).$$

While this is a second-order system, we have seen how to solve such a system by assigning extra variables $x_3 = x_1'$ and $x_4 = x_2'$ to create a 4-variable system. This model represents free undamped motion as well. But, adding in a damping term could easily be accomplished, and would not add too much extra work to finding our solution. However, adding driven motion to the system would create a non-homogeneous case, which can complicate the solution process.

We could also use the same techniques to create a system of equations for three, or more, masses coupled together. However, solving a system involving three masses would involve creating a 6-variable system (using our previous methods).