10.2: Vectors

Thursday, August 13, 2020 9:59 PM

Def. 1) A position vector is a description of a point in space represented as a directed line segment emanating from the origin.

2) A representation (free) vector is the result of translating a position vector to a different location in space (perhaps not emanating from origin).

3) A vector is defined by its length and direction, or by its initial and terminal points.

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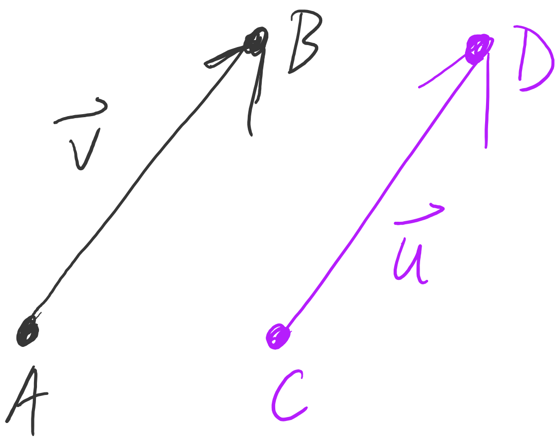
ex. A particle moves along a line segment from A to B.



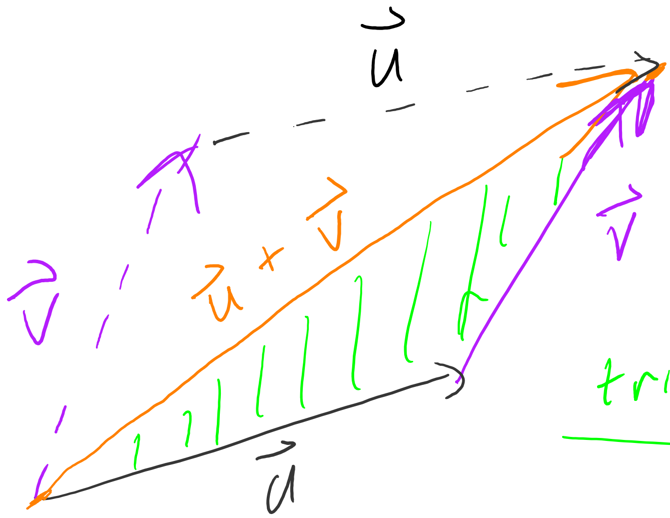
$\vec{v} = \overrightarrow{AB}$ is the displacement vector.

$\vec{u} = \overrightarrow{CD}$ has same length & direction as $\vec{v}$, so $\vec{u}$ is equivalent to $\vec{v}$; write $\vec{u} = \vec{v}$.

Def. $\vec{0}$ is the zero vector. It has length 0 and no direction.

Def. (vector addition)

If $\vec{u}$, $\vec{v}$ are such that the initial point of $\vec{v}$ is the terminal pt of $\vec{u}$, the sum $\vec{u}+\vec{v}$ is the vector from the initial pt of $\vec{u}$ to the terminal p't of $\vec{v}$.

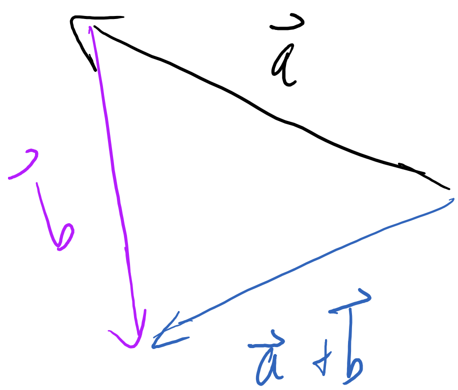


triangle law

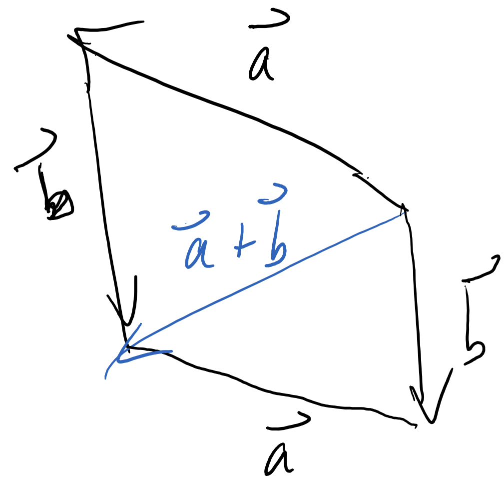
parallelogram law

Ex. 1 Draw the sum of $\vec{a}$ & $\vec{b}$:

Sol'n:



(triangle law)



(parallelogram law)

Def. (scalar multiplication)

Let c be a real number (i.e. scalar).

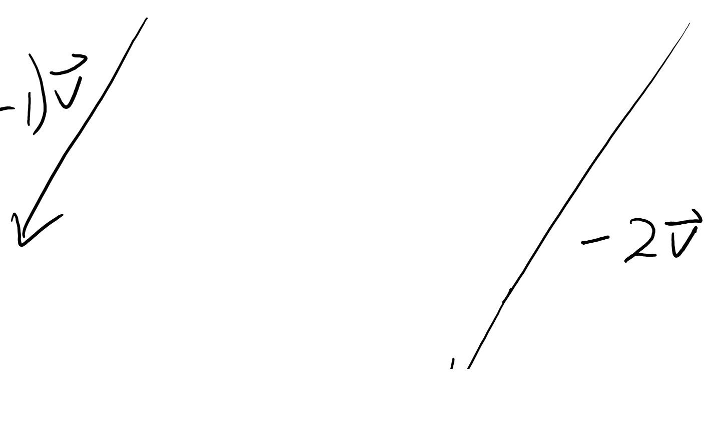
the <u>scalar multiple</u> $c\vec{v}$ is the vector with length $|c| \cdot (\text{length of } \vec{v})$ and direction

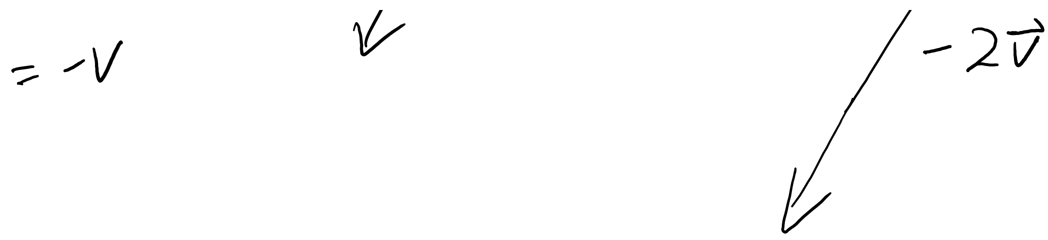
- same as direction of $\vec{v}$ if $c > 0$
- opposite $\vec{v}$ if $c < 0$
- none if $c = 0$

<u>ex.</u>

$(-1)\vec{v} = (-1)\vec{v}$

$= -v$



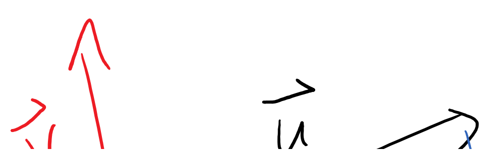


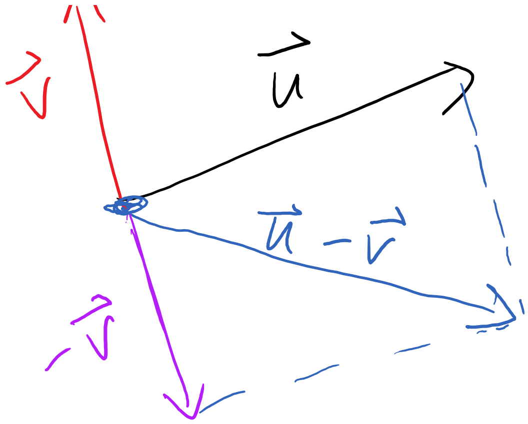
<u>Def.</u> 1) $\vec{u} \neq 0$ and $\vec{v} \neq 0$ are <u>parallel</u> if $\vec{u} = c\vec{v}$ for some scalar c.

<u>ex.</u> $\vec{u} \parallel (-1)\vec{u}$ for any $\vec{u} \neq \vec{0}$.

2) $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$,

$= \vec{u} + (-\vec{v})$.





<u>Components</u> need to treat vectors algebraically to do calculus.

<u>Def.</u> For a position vector $\vec{u}$ with terminal point (a, b, c), write

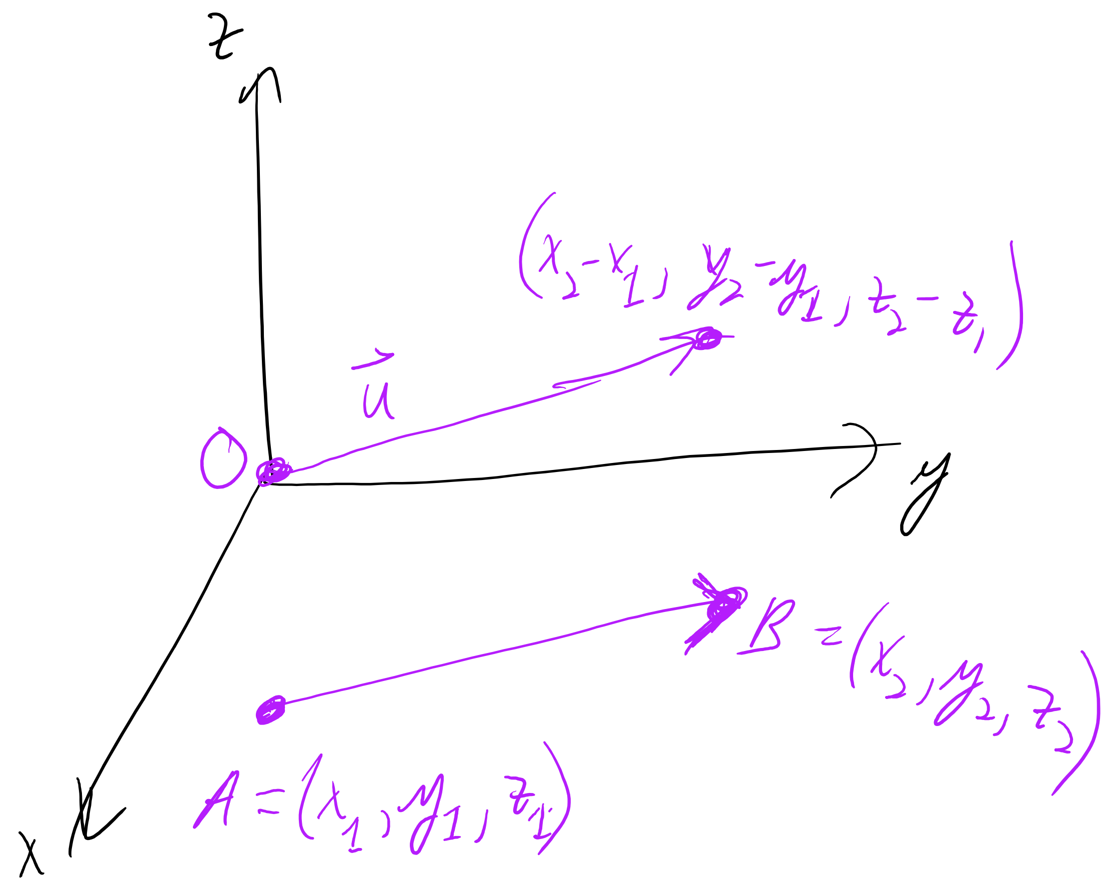
$\vec{u} = \langle a, b, c \rangle$ for 3-D ;

if term. pt of $\vec{v}$ is $(a, b) \in \mathbb{R}^2$,

$\vec{v} = \langle a, b \rangle$.

Note To find the vector $\vec{u}$ w/rep. starting at $(x_1, y_1, z_1) = A$ and terminating at $(x_2, y_2, z_2) = B$,

$$\vec{u} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



ex. Find the vector rep. by $\overrightarrow{AB}$ where $A = (2, -3, 4)$ and $B = (-2, 1, 1)$.

Sol'n. $\vec{u} = \langle -2-2, 1+3, 1-4 \rangle$

$= \boxed{\langle -4, 4, -3 \rangle}$

(is the ~~pos~~ corresp. pos. vector)

Def. The magnitude (i.e. length, modulus, norm) of $\vec{v}$ in 3D is

$$\|\vec{v}\| = |\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}, \text{ where}$$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$.

In 2-D, $\vec{v} = \langle v_1, v_2 \rangle$, so

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}.$$

<u>Arithmetic</u> $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$, $c \in \mathbb{R}$

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\vec{a} = \langle ca_1, ca_2, ca_3 \rangle$$

(same in 2-D and n-D, $n \geq 4$)

Ex. $\vec{a} = \langle 4, 0, 3 \rangle$, $\vec{b} = \langle -2, 1, 5 \rangle$.

Find $|\vec{a}|$, $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $3\vec{b}$, $2\vec{a}+5\vec{b}$.

(i) $|\vec{a}| = \sqrt{16+9} = \boxed{5}$

(ii) $\vec{a}+\vec{b} = \langle 4-2,\ 0+1,\ 3+5 \rangle$
$= \boxed{\langle 2, 1, 8 \rangle}$

(iii) $\vec{a}-\vec{b} = \langle 4+2,\ 0-1,\ 3-5 \rangle$
$= \boxed{\langle 6, -1, -2 \rangle}$

(iv) $3\vec{b} = 3\langle 2, 1, -5 \rangle = \boxed{\langle 6, 3, -15 \rangle}$

(v) $2\vec{a} + 5\vec{b} = 2\langle 4, 0, 3\rangle + 5\langle -2, 1, 5\rangle$

$= \langle 8, 0, 6\rangle + \langle -10, 5, 25\rangle$

$= \boxed{\langle -2, 5, 31\rangle}$

Properties $\vec{u}, \vec{v}, \vec{w}$ vectors; $a, b \in \mathbb{R}$

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

3) $\vec{u} + \vec{0} = \vec{u}$

4) $\vec{u} + (-\vec{u}) = \vec{0}$

5) $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$

6) $(a+b)\vec{u} = a\vec{u} + b\vec{u}$

7) $(ab)\vec{u} = a(b\vec{u})$

8) $1\vec{u} = \vec{u}$

Def. V_2 = set of all 2-D vectors

V_3 = " " " 3-D "

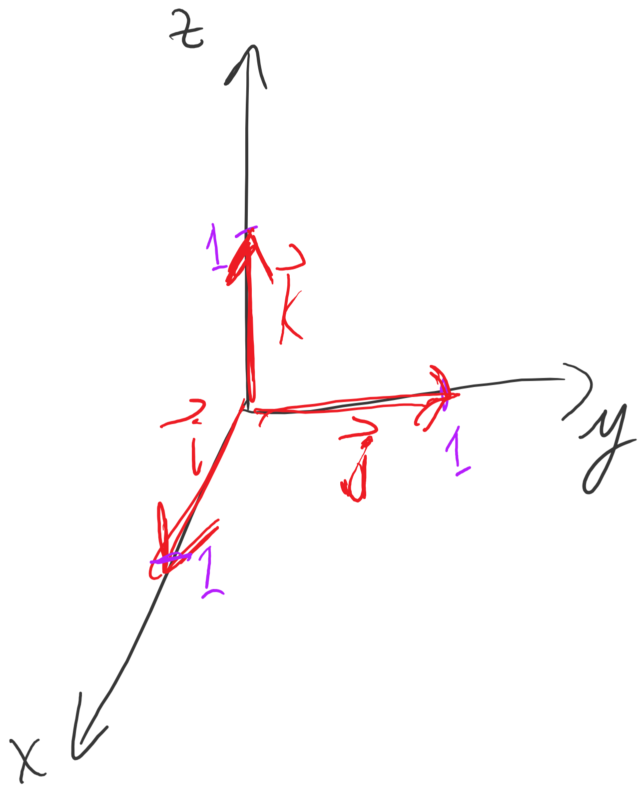
$\vdots$

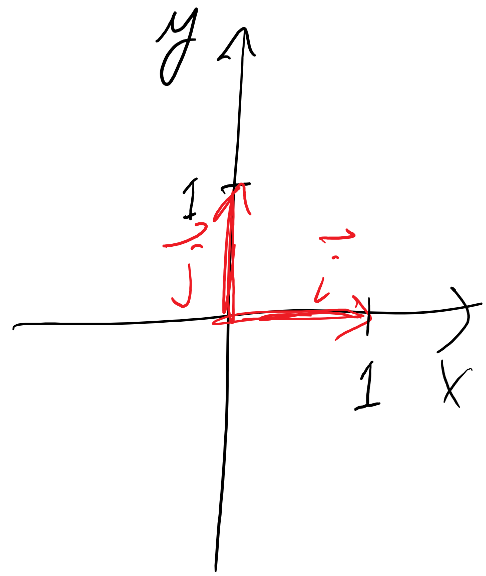
V_n = " " " n-D "

Def. $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$ are the standard basis vectors in V_3.

In V_2, the <u>st'd basis vect.</u> are

$\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.





<u>Fact</u> Any $\vec{u} \in V_n$ can be written in terms of the st'd basis vect. of V_n.

In 3-D, $\vec{u} = \langle a, b, c \rangle$

$$= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle$$

$$= a \langle 1, 0, 0 \rangle + b \langle 0, 1, 0 \rangle + c \langle 0, 0, 1 \rangle$$

$$\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}.$$

Def. A <u>unit vector</u> is a vector with length 1.

Note 1) $\vec{i}, \vec{j}, \vec{k}$ are unit vectors

2) For any $\vec{a} \neq \vec{0}$, the vector $\left(\frac{1}{|\vec{a}|}\right) \cdot \vec{a}$ has length 1.

I.e. $\left| \frac{\vec{a}}{\ldots} \right| = 1.$

I.e. $\left|\dfrac{\vec{a}}{|\vec{a}|}\right| = 1.$

So given $\vec{a}$, if need unit vect. in same direction as $\vec{a}$, use

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}.$$

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ex. Find a unit vect. in dir. of $2\vec{i} - \vec{j} - 2\vec{k}$.

Sol'n. $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k} = \langle 2, -1, -2 \rangle$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}(2\vec{i} - \vec{j} - 2\vec{k})$$

$$= \boxed{\frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}}.$$

Read "Applications" in Sect. 10.2.