Unit 3.1 Infinite Series and Convergence

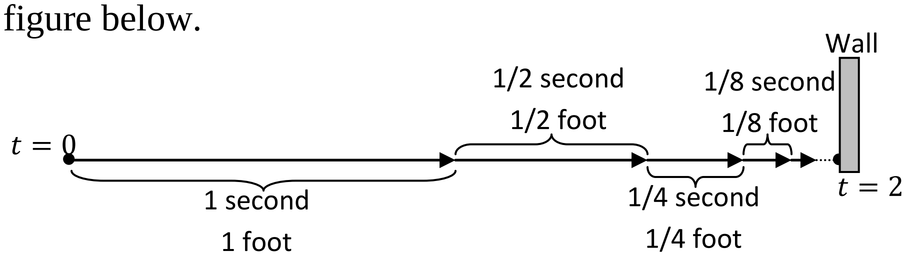
For the next several sections, we are going to be entertaining the concept of adding up infinitely many numbers. We certainly know what it means to add up a finite collection of numbers, but what would it mean to add up an infinite collection of numbers? If you add forever, do the numbers ever add up to anything? Well, that depends on how you define such a thing. Mathematically, we would like to define an infinite sum in a natural way that allows it to be useful in practical situations and consistent with other mathematics. Before we get into such a technical definition, let us consider two examples that may motivate a need to study infinite sums in the first place.

We are all familiar with the fact that $/3$ has a decimal representation consisting of repeating 3's. In fact observe that

$$\frac{}{3} = .333\cdots \quad = .3 + .03 + .003 + \cdots \quad = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots \quad \text{(an infinite sum)}$$

So here we are with a simple number like $1/3$, and when we view this number expressed in decimal notation, the repeating 3's actually correspond to an infinite sum. When we end up defining infinite sums, we better be sure that it yields $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots \ = \frac{1}{3}$.

The last example was purely mathematical and still might not convince you that infinite sums are something to take seriously (although it should). Let us consider an example involving motion. Suppose that an object is 2 feet from a wall and moves toward the wall at 1 ft/sec. Thus after 2 seconds the object will have moved 2 feet and hit the wall. Let us break down the motion over the 2 second time interval by dissecting it into pieces. We repeatedly look at the object as it moves half the remaining distance to the wall. Each move will also correspond to half of the remaining time left in the 2 second interval. After the first second, the object will have moved 1 foot; over the next half second, the object will move and additional ½ foot; over the next ¼ second the object will move an additional ¼ foot, and so on. This motion is illustrated in the figure below.



What we have done here is decomposed a 2 second time interval and a 2 foot distance into an infinite number of pieces. This leads to some kind of paradox. If we look at the problem from one angle, we think about the object moving half way to the wall, then half way again, half way again, over and over again. It appears that in this way, while the object will become arbitrarily close to the wall, it will never hit the wall since it just keeps going half way. However, it was moving at 1 ft/sec, so surely it must hit the wall after 2 seconds. To remedy this paradox, we must learn to analyze the infinite process in such a way that leads to the correct result. In this case, it requires us to obtain the following sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

Again, we encounter an infinite sum whose total we feel compelled to make sense of. Let us now turn to the formal definition of these infinite sums, or as we will call them *Series*.

Infinite Series. Consider a sum of the form $a_1 + a_2 + a_3 + \cdots + \underbrace{a_n}_{\text{the } n^{\text{th}} \text{ term}} + \cdots$. We can express this same sum in Sigma/Summation notation as follows

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

Since our ordinary notion of addition only involves a finite number of terms, the way that we handle the infinity, is to look at the "running total" as we add on more and more terms. We are interested in seeing whether or not this running total is approaching a definite value. If so, that value is what we will assign to the entire infinite sum. In other words, we have the following definition.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \underbrace{(a_1 + a_2 + a_3 + \cdots + a_n)}_{\text{the } n^{\text{th}} \text{ partial sum, } S_n}$$

As noted in the expression above, we call the sum of the first n terms, the n^{th} partial sum and denote it S_n. If the above limit exists and equals S, we say that the series *converges* to S, and consider S to be the sum of the series. If the limit does not exist, we say that the series *diverges*. To get a better handle on these concepts, let's revisit the sum in the example above given by

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \ = \ \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$$

For this series we obtain the following partial sums

$$S_1 = 1, \quad S_2 = 1 + \frac{1}{2} = 1.5, \quad S_3 = 1 + \frac{1}{2} + \frac{1}{4} = 1.75, \quad S_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1.875, \ldots$$

If we jump ahead to some later partial sums we see

$$S_{10} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^9} \approx 1.998 \quad \text{and} \quad S_{20} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{19}} \approx 1.999998$$

It appears that the more terms we add on, the closer we get to the number 2. That's a good thing, because in our above discussion, we wanted the sum to come out equal to 2, so that the motion problem made sense when we broke it down into an infinite process.

The above series as well as the series for $1/3$ given at the beginning of this section summary, are examples of what we call *Geometric Series*. Geometric Series are going to be a rare type of series where we will have a simple formula giving their sum. For many other series we encounter, we will only be able to determine whether or not the series converges (i.e. whether or not we can assign it a sum), and if it does, we might then approximate the series using a partial sum.

Geometric Series. A geometric series, is a series of the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

It is more important that you view the sum expressed in the expanded form on the right-hand side of the equation above. There is no unique way to write a series in Σ-notation, so a series may be geometric yet when expressed in Σ-notation, it may not look like what you see on the left-side of the equation above. What makes the series geometric, is that the ratio of consecutive terms is constant. In particular we have $\frac{a_{n+1}}{a_n} = r$, which we call the geometric ratio. In addition, you might that **a geometric series has exponential terms.** The following result is proved in the book and it is strongly recommended that you look at the proof.

The Geometric Series Test:

The geometric series $a + ar + ar^2 + \cdots$

Converges when $|r| < 1$, i.e. when $-1 < r < 1$

and

Diverges when $|r| \geq 1$, i.e. when $r \leq -1$ or $r \geq 1$

In particular, when the series converges, it converges to the sum $\frac{a}{1-r}$ where a is the first term of the series and r is the geometric ratio which can be obtained by calculating $\frac{a_{n+1}}{a_n}$.

It is very important that you memorize the above result in its entirety. Before looking at an example, I would like to make it clear that there is more than one way to express a given series in summation notation. Each series given below yield the expanded form $\frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \cdots$

$$\sum_{n=0}^{\infty} \frac{8}{3}\left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{2^{n+2}}{3^n} = \sum_{n=3}^{\infty} \frac{2^n}{3^{n-2}}$$ (convince yourself the these are equal to the above)

Examples 1 – 3 Determine whether each series converges or diverges. If the series converges, give its sum, i.e. the value that it converges to.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

Solution: In expanded form, we see that we are looking at the sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, which is the same sum we generated in the motion problem earlier. This is a geometric series with ratio $r = 1/2$. Here the ratio is pretty obvious as each term is clearly $1/2$ times the previous term. Since $|r| < 1$, the series $\boxed{\text{converges}}$ by the Geometric Series Test. We therefore know that the series converges to $\frac{a}{1-r}$ where $r = 1/2$ as noted above and $a = 1$ (the first term).

Thus $\boxed{\text{this series converges to } \frac{1}{1-1/2} = 2}$.

Note: the sum of 2 is exactly what we would have expected in our motion problem where the object moved 2 feet in 2 seconds before it hit the wall.

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2. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$

Solution: In expanded form, we see that we are dealing with the sum introduced at the beginning of this section summary generated by the number $1/3$.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = .3 + .03 + .003 + \cdots \quad = \quad .33\overline{3}$$

We identify this series at geometric with a ratio of $r = 1/10$. Since $|r| < 1$, the series $\boxed{\text{converges}}$ by the Geometric Series Test. To obtain the sum of the series, we note that $a = 3/10$ (the first term). Therefore

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{a}{1-r} = \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = \boxed{\frac{1}{3}} \quad \text{(which is what we expected)}$$

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3. $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{5}{3}\right)^{n}$

Solution: In expanded form, we identify this series as $1-\frac{5}{3}+\frac{25}{9}-\frac{125}{27}+\cdots$. Here we have a geometric series with ratio $r=-5/3$ and thus $|r|=|-5/3|=5/3$. Since $|r|\nless 1$, i.e. $|r|\geq 1$, the series $\boxed{\text{diverges}}$ by the Geometric Series Test. Note: the formula for the sum of a convergent geometric series is of no use to us here since this series diverges and thus has no sum.

One other type of series is investigated in this section, namely Telescoping Series. For this, I direct you to the book and the narrated examples.