3.7 Modeling with Higher-Order Differential Equations

Spring/Mass Systems: Suppose that a flexible spring is suspended vertically from a rigid support and then a mass m is attached to the free end. The mass stretches the spring and, by Hooke's law, the spring will exert a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s.

After the mass stretches the spring, it attains an equilibrium at which its weight $W = mg$ is balanced by the restoring force of the spring $F = ks$. This equilibrium implies that $mg = ks$ or $mg - ks = 0$.

**The mass is measured in slugs, kilograms, or grams, giving us a $g = 32\ ft/s^2$, $9.8\ m/s^2$, or $980\ cm/s^2$, respectively.*

If the mass is displaced by an amount x from the equilibrium position, the restoring force of the spring is $k(x+s)$. Assuming that there are no external forces acting on the spring/mass system, then by Newton's second law, we have

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx.$$

**We will adopt the convention that displacements below the equilibrium are positive.*

**Despite the fact that the setup for the problem is in the vertical direction, we could do the same for a horizontal situation. Therefore it is common practice to use $x(t)$ instead of $y(t)$.*

Case I: Free Undamped Motion

By dividing the previous equation by m, we get the second-order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega^2 = k/m$. This equation describes **simple harmonic motion** or **free undamped motion**. Since we have constant coefficients, we can find an auxiliary equation of

$$m^2 + \omega^2 = 0 \quad \to \quad m_1 = \omega i,\ \ m_2 = -\omega i,$$

giving a general solution of

$$\boldsymbol{x(t) = c_1 \cos \omega t + c_2 \sin \omega t}.$$

- *Example:* A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of motion.

Solution: First we must convert our measurements from inches to feet and from pounds to slugs:

$$6\ in. = \frac{1}{2}ft, \qquad 8\ in. = \frac{2}{3}ft, \qquad 2\ lbs = \frac{1}{16}\ slugs$$

From Hooke's law,

$$F = ks \quad \to \quad 2 = k\left(\frac{1}{2}\right) \quad \to k = 4\ lb/ft.$$

Putting everything together, we get

$$\frac{d^2x}{dt^2} + 64x = 0, \qquad x(0) = \frac{2}{3}, \qquad x'(0) = -\frac{4}{3}.$$

Solving the equation, we get a general solution of $\boldsymbol{x(t) = c_1 \cos 8t + c_2 \sin 8t}$. Using the first condition $x(0) = 2/3$, we get that $c_1 = 2/3$. Using the condition $x'(0) = -4/3$, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t \quad \to \quad 8c_2 = -\frac{4}{3} \quad \to \quad c_2 = -\frac{1}{6}.$$

Therefore the equation of motion is

$$\boldsymbol{x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t.}$$

**While units will only be required in final answers, it is a good idea to keep track of units within the problem to make sure things line up properly.*

In an ideal situation, the characteristics of the spring would not change, and k would remain constant. This is not usually the case and over time, the spring would age, giving rise to a variable spring "constant" function $K(t) = ke^{-at}$. However, solving the resulting system is beyond the scope of our techniques so far.

Case II: Free Damped Motion

Free harmonic motion is unrealistic unless we are working in a vacuum. Taking the resisting forces to the motion of the spring/mass system into account, we need to include an extra term into our model. The extra force is proportional to the velocity of the mass. Adding in this damping term, we arrive at **free damped motion**

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt},$$

where $\beta > 0$ is our damping constant of the system. By moving terms around and dividing by m, we get

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0,$$

where

$$2\lambda = \frac{\beta}{m}, \qquad \omega^2 = \frac{k}{m}.$$

**In the formula we use 2λ and ω^2 in our equation for simplicity of the solution. In practice we usually do not explicitly find λ and ω.*

Once again we get a homogeneous linear second-order differential equation with constant coefficients. Solving for m in the resulting auxiliary equation, we get

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \qquad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}.$$

Depending on the value of $\lambda^2 - \omega^2$, we get one of three cases.

- ***Case I:* $\boldsymbol{\lambda^2 - \omega^2} > 0$**

  In this situation we get two real distinct solutions for m, giving us a general solution of

  $$\boldsymbol{x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)}.$$

  This case is said to be **overdamped** and results in a smooth, non-oscillatory motion.

- ***Case II:* $\boldsymbol{\lambda^2 - \omega^2 = 0}$**

  In this situation we get one repeated solution for m, giving us a general solution of

  $$\boldsymbol{x(t) = e^{-\lambda t}(c_1 + c_2 t)}.$$

  This case is said to be **critically damped** since the slightest adjustment in β would push the system into either case I or case III.

- ***Case III:* $\boldsymbol{\lambda^2 - \omega^2} < 0$**

  In this situation we get two complex solutions for m, giving us a general solution of

$$\boldsymbol{x(t) = e^{-\lambda t}\left(c_1 \cos\sqrt{\lambda^2-\omega^2}\,t + c_2 \sin\sqrt{\lambda^2-\omega^2}\,t\right)}.$$

This case is said to be **underdamped** and results in oscillatory motion. However, due to presence of $e^{-\lambda t}$ in the solution, the amplitude of the vibration goes to zero as time goes to ∞.

- *Example:* A half-kilogram mass is attached to a 5-meter-long spring. At equilibrium the spring measures 5.98 meters. It is also known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity of the mass. If the mass is released from rest from a point 2 meters above the equilibrium position, find the displacement $x(t)$.

Solution: From Hooke's law,

$$F = ks \quad \rightarrow \quad \frac{1}{2}\cdot 9.8 = k(5.98-5) \quad \rightarrow k = 5\ N/m.$$

Putting everything together, we get

$$\frac{1}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 0, \qquad x(0) = -2, \qquad x'(0) = 0.$$

Solving the resulting auxiliary equation, we get $m_1 = -1+3i,\ m_2 = -1-3i$, giving a general solution of

$$\boldsymbol{x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)}.$$

Using the first condition $x(0) = -2$, we get that $c_1 = -2$. Using the condition $x'(0) = 0$, we get

$$x'(t) = -e^{-t}(c_1 \cos 3t + c_2 \sin 3t\,) + e^{-t}(-3c_1 \sin 3t + 3c_2 \cos 3t)$$

$$-c_1 + 3c_2 = 0 \quad \rightarrow \quad c_2 = -\frac{2}{3}.$$

Therefore the equation of motion is

$$\boldsymbol{x(t) = -e^{-t}\left(2\cos 3t + \frac{2}{3}\sin 3t\right)}.$$

Case III: Driven Motion

Suppose we now have an external force $f(t)$ acting on the mass. Including this force into our model produces **driven motion** or **forced motion** represented by

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t) \quad \to \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t),$$

where

$$2\lambda = \frac{\beta}{m}, \qquad \omega^2 = \frac{k}{m}, \qquad F(t) = \frac{f(t)}{m}.$$

Once again we get a linear second-order differential equation with constant coefficients, so we can solve the associated homogeneous equation to get y_c. We can then use either undetermined coefficients or variation of parameters to get a particular solution.

- *Example:* Interpret and solve the following IVP.

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t\,, \qquad x(0) = \frac{1}{2}, \qquad x'(t) = 0.$$

Solution: This problem represents a driven, damped spring/mass system consisting of a $\frac{1}{5}$ slug (or kilogram) mass attached to a spring with a constant $k = 2$ lb/ft (or N/m). The mass starts at rest $\frac{1}{2}$ ft (or m) below the equilibrium position. The medium provides a damping that is numerically equal to 1.2 times the velocity of the mass and there is an external periodic driving force of $f(t) = 5\cos 4t$.

Solving the equation, we get a fundamental set of solutions of $\{e^{-3t}\cos t\,,\ e^{-3t}\sin t\}$ to the associated homogeneous system. Using the method of undetermined coefficients, we assume a particular solution of the form $x_p = A\cos 4t + B\sin 4t$. Differentiating x_p and substituting into the DE gives

$$(-6A + 24B)\cos 4t = (-24A - 6B)\sin 4t = 25\cos 4t.$$

Solving the system of equations

$$\begin{aligned} -6A + 24B &= 25 \\ -24A - 6B &= 0\,' \end{aligned}$$

we get $A = -\frac{25}{102},\ B = \frac{50}{51}$. Therefore the general equation of motion is given by

$$\boldsymbol{x(t) = e^{-3t}(c_1\cos t + c_2\sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t}.$$

Using the initial conditions we can solve for c_1 and c_2, getting

$$x(0) = c_1 - \frac{25}{102} = \frac{1}{2} \quad \to \quad c_1 = \frac{38}{51}$$

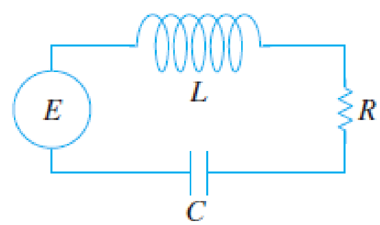
$$x'(0) = -3c_1 + c_2 - \frac{200}{51} = 0 \quad \rightarrow \quad c_2 = -\frac{86}{51}.$$

Therefore the equation of motion is

$$\boldsymbol{x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t.}$$

When $F(t)$ is a periodic function, as in the example above, the general solution is the sum of a non-periodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover, as $t \rightarrow \infty$, $x_c(t) \rightarrow 0$, leaving only $x_p(t)$. In this case, $x_c(t)$ is called the **transient term** or **transient solution**, while $x_p(t)$, the part that remains after an interval of time, is called the **steady-state term** or **steady-state solution**.

LRC Series Circuits: Consider the single loop LRC series circuit shown, consisting of a inductor, resistor, and capacitor. Let $i(t)$ denote the current in the circuit, and let $q(t)$ be the charge on the capacitor at time t. Assume that the inductance L, resistance R, and capacitance C are constants. According to Kirchhoff's second law, the impressed voltage $E(t)$ on the closed loop must be equal to the sum of the voltage drops in the loop. The voltage drops across the inductor, resistor, and capacitor are



$$L\frac{di}{dt}, \quad iR, \quad and \quad \frac{1}{C}q,$$

respectively. This gives us

$$L\frac{di}{dt} + iR + \frac{1}{C}q = E(t).$$

Using the fact that $i = dq/dt$, we can then get the linear second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

This is numerically equivalent to the spring/mass system problems, and can be solved in the same fashion (yielding similar results).

<u>Deflection of a Beam:</u> Assume that a beam of length L is homogeneous and has uniform cross sections along its length. A curve joining the centroids of all of its cross sections is a straight line called the **axis of symmetry**. If a load in applied to the beam (even just its own weight) in a vertical plane containing the axis of symmetry, the beam distorts. The curve connecting the centroids of all of the cross sections on the distorted beam is called the **deflection curve**. Let x correspond to the axis of symmetry, and let $y(x)$ be the deflection from the axis of symmetry at a position x along the beam.

**The downward direction is considered positive, much like with the spring/mass systems.*

Let E and I, which represent Young's modulus of elasticity of the material of the beam and the moment of inertia of a cross section of the beam, respectively, be constants. Their product EI is called the flexural rigidity of the beam. It can be shown that

$$\boldsymbol{EI\frac{d^4y}{dx^4} = w(x)},$$

where $w(x)$ is the load per unit length.

This is a very basic fourth-order differential equation, which can be solved through repeated integration. However, applications involving the deflection of a beam give us a boundary-value problem, as opposed to the initial-value problems we have seen.

The boundary conditions are based on how the two ends of the beam are supported. Each end of the beam may be **embedded** (or **clamped**), **simply supported** (or **hinged**), or **free**.

If the end of the beam is **embedded**, then that end of the beam is being firmly held in place. This means that there is no deflection and that the slope of the deflection curve must be zero at that end point.

If the end of the beam is **simply supported**, then that end of the beam is "resting" on the support. This means that there is no deflection at that end point and the bending moment (or second derivative of the deflection) of the beam must also be zero at that end point.

If the end of the beam is **free**, then there is nothing supporting the beam. This means that the bending moment (second derivative) and shear force (third derivative) must both be zero at that end.

To summarize, based on the how the end is supported, we get the following boundary conditions:

$$\begin{array}{ccc} \textbf{embedded} & \rightarrow & y = 0,\ y' = 0 \\ \textbf{simply supported} & \rightarrow & y = 0,\ y'' = 0 \\ \textbf{free} & \rightarrow & y'' = 0,\ y''' = 0 \end{array}$$

With the exception of both ends of the beam being free (in which case the beam would be falling through the air), any combination of supports is possible.

- *Example:* A 1 foot long beam is simply supported on the left and embedded on the right. Find the deflection of the beam if a load of $w(x) = 120EI \cdot x$ is distributed along its length.

Solution: The deflection of the beam $y(x)$ satisfies the equation

$$EI\frac{d^4y}{dx^4} = w(x) \quad \rightarrow \quad \frac{d^4y}{dx^4} = 120x.$$

Since the beam is simply supported on the left end and embedded on the right end, we would get the boundary conditions

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(1) = 0, \qquad y'(1) = 0.$$

Solving the differential equation, by repeated integration, we get

$$y(x) = x^5 + c_1x^3 + c_2x^2 + c_3x + c_4.$$

Using the left-end boundary conditions, we find that

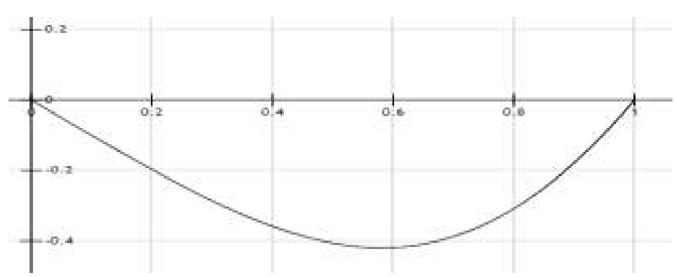
$$y(0) = c_4 = 0, \qquad y''(0) = 2c_2 = 0 \quad \rightarrow c_2 = c_4 = 0.$$

Using the right-end boundary conditions, we find that

$$y(1) = 1 + c_1 + c_3 = 0, \qquad y'(1) = 5 + 3c_1 + c_3 = 0.$$

Solving the resulting system of equations, we get that $c_1 = -2$ and $c_3 = 1$, giving a solution of

$$\boldsymbol{y(x) = x^5 - 2x^4 + x.}$$



*Note: we are not guaranteed of a unique solution (or any solution) when dealing with a BVP.