Unit 4.3 Taylor Series

In this final section, we combine the various concepts that we explored in the last three sections. Let us begin by reexamining the n^{th} degree Taylor polynomial for a function centered at $x = a$.

$$T_n(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

As the degree of the polynomial increases, the polynomial does a better and better job of approximating the function f (over some interval). It is natural to extend this idea by letting the terms go on indefinitely, thus creating an infinite power series called the Taylor series.

Definition: If f is function that is differentiable of all orders at $x = a$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series for f centered at $x = a$.

*In the case that $a = 0$, we sometimes call it a Maclaurin series.

In Unit 4.1, we found Taylor polynomials for e^x, $\sin x$, and $\cos x$ as well as for other functions. You might recall that upon calculating derivatives of these functions and evaluating them derivatives at 0, simple patterns arose. For example, every derivative (the first, second, etc…) of e^x is just e^x and so you will get 1, every time you evaluate these derivatives at $x = 0$. When you take repeated derivatives of the sine and cosine functions, you just get a repeating pattern of sines, cosines, and their opposites and when you evaluate them at $x = 0$you get repeating patterns of 0's, 1's, and -1's. This should give you an idea of where the power series for these functions come from that are given below (if not, derive them using definition above).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

You are expected to memorize these power series in both the Σ- and expanded- forms

It should be emphasized that in the above, our use of equality is intended. That is, each of these functions, is equal to the corresponding power series and each can be approximated by any partial sum for its series (these partial sums are the Taylor polynomials).

Example 1 Determine the interval of convergence for the Taylor series for e^x centered at 0.

Solution: From the above, we see that e^x has a Taylor series centered at 0 whose terms are given by $\frac{x^n}{n!}$. Let us use the Ratio Test to determine the condition for convergence. Observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \to 0 \quad \text{as } n \to \infty \quad \text{for all } x$$

Since the above limit is less than 1 for all x, the series will converge (to e^x) for all x. We express the interval of convergence as $(-\infty, \infty)$ and say that the series has an infinite radius of convergence.

Example 2 Use the Taylor series for $\sin x^2$ centered at 0 to evaluate and approximate

$$\int_0^1 \sin x^2 \, dx.$$

Solution: Let us first acknowledge that we have no way of expressing the antiderivative of $\sin x^2$ in terms of a finite number of elementary functions. Thus we turn to power series. To obtain the power series for $\sin x^2$ we need only replace x with x^2 in the power series for $\sin x$.

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n+1)!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

This series converges so long as x^2 (which replaced x) is a real number, and therefore it converges for all x. Let us now carry out the integration.

$$\int_0^1 \sin x^2 \, dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n+1)!}\right] dx = \int_0^1 \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right) dx$$

$$= \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}\right]_0^1 = \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \cdots\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \overbrace{(1)^{4n+3}}^{=1}}{(4n+3)(2n+1)!} - \sum_{n=0}^{\infty} 0 = \left(\frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \cdots\right) - 0$$

$$= \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} - \frac{1}{75600} + \cdots$$

We have expressed our definite integral as an alternating series. Using the first three terms as an estimate we find that

$$\boxed{\int_0^1 \sin x^2 \, dx \approx \frac{1}{3} - \frac{1}{42} + \frac{1}{1320} \approx .3103}$$

Since in the third term we added, this is an overestimate and our error is less than or equal to the magnitude of the fourth term, $\frac{1}{75600} \approx .00001323$ (this is our error bound). Thus we have a very good approximation to the value of the given integral.

<u>Example 3</u> Use a series to approximate the following integral with an error < 0.1.

$$\int_{.01}^{1} \frac{\ln(\sqrt{y}+1)}{y} \, dy$$

Solution: Let us first note that our techniques of integration do not allow us to find an antiderivative of the integrand above in the ordinary way. Thus we will express the integrand as a series and use it to evaluate the integral. In Unit 5.2 we obtained the following power series for $\ln x$.

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad \text{for } \ 0 < x \le 2$$

To obtain a series for the integrand in the problem we need only replace x with $\sqrt{y}+1$ and then divide through by y. We do have to make sure that the series will converge on the interval we are integrating over, namely $.01$ to 1. For the resulting series to converge, the results of example 3 tell us that we would need $0 < \sqrt{y}+1 \le 2 \ \Rightarrow \ -1 < \sqrt{y} \le 1 \ \Rightarrow \ 0 \le y \le 1$. Because of the y in the denominator of the integrand, we cannot have $y = 0$, so the resulting series will converge for $0 < y \le 1$. Therefore we can use the series when integrating over the interval from $.01$ to 1. We now proceed and will only focus on the expanded form of the series. Given that $0 < y \le 1$,

$$\frac{\ln(\sqrt{y}+1)}{y} = \frac{\left((\sqrt{y}+1)-1\right)}{y} - \frac{\left((\sqrt{y}+1)-1\right)^2}{2y} + \frac{\left((\sqrt{y}+1)-1\right)^3}{3y} - \frac{\left((\sqrt{y}+1)-1\right)^4}{4y} + \cdots$$

$$= \frac{\sqrt{y}}{y} - \frac{\left(\sqrt{y}\right)^2}{2y} + \frac{\left(\sqrt{y}\right)^3}{3y} - \frac{\left(\sqrt{y}\right)^4}{4y} + \cdots$$

$$= y^{-1/2} - \frac{1}{2} + \frac{1}{3}y^{1/2} - \frac{1}{4}y + \frac{1}{5}y^{3/2} - \frac{1}{6}y^2 + \cdots$$

Therefore,

$$\int_{.01}^{1} \frac{\ln\left(\sqrt{y}+1\right)}{y}\,dy = \int_{.01}^{1} \left(y^{-1/2} - \frac{1}{2} + \frac{1}{3}y^{1/2} - \frac{1}{4}y + \frac{1}{5}y^{3/2} - \frac{1}{6}y^2 + \cdots\right) dy$$

$$= \left[2y^{1/2} - \frac{1}{2}y + \frac{2}{9}y^{3/2} - \frac{1}{8}y^2 + \frac{2}{25}y^{5/2} - \frac{1}{18}y^3 + \cdots\right]_{.01}^{1}$$

$$= \left(2\cdot 1^{1/2} - \frac{1}{2}\cdot 1 + \frac{2}{9}\cdot 1^{3/2} - \frac{1}{8}\cdot 1^2 + \cdots\right) - \left(2(.01)^{1/2} - \frac{1}{2}(.01) + \frac{2}{9}(.01)^{3/2} - \frac{1}{8}(.01)^2 + \cdots\right)$$

$$= \left(2 - \frac{1}{2} + \frac{2}{9} - \frac{1}{8} + \cdots\right) - \left(\frac{1}{5} - \frac{1}{200} + \frac{1}{4500} - \frac{1}{80000} + \cdots\right)$$

We have now expressed the value of the integral as the difference of two series. In fact each is an alternating series and we can approximate each using a partial sum estimate. Recall that the sum of an alternating series (whose terms decrease to zero in magnitude) will always be between two consecutive partial sum estimates. Using 4^{th} and 5^{th} partial sum estimates for each of the above series we find that

$$1.59 < \left(2 - \frac{1}{2} + \frac{2}{9} - \frac{1}{8} + \cdots\right) < 1.68 \quad \text{and} \quad \left(\frac{1}{5} - \frac{1}{200} + \frac{1}{4500} - \frac{1}{80000} + \cdots\right) \underbrace{\approx .195}_{\substack{\text{correct} \\ \text{to 3 dec.} \\ \text{places}}}$$

Thus we find the difference of the two is between 1.395 and 1.485. Therefore we conclude

$$\boxed{1.39 < \int_{.01}^{1} \frac{\ln(\sqrt{y}+1)}{y}\,dy < 1.49}$$

Note: using the TI-89 calculator, I computed the value of the integral to be approximately 1.4497 which is correct to the number of decimal places shown.