4.3 Translation Theorems

Theorem: **First Translation Theorem**

If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Proof: By definition,

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a). \quad \blacksquare$$

If we consider that s is a real variable, then the graph of $F(s-a)$ is the graph of $F(s)$ shifted on the s-axis by a units. For emphasis of this point, we sometimes use the notation

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}|_{s\to s-a}.$$

- *Example:* Evaluate $\mathcal{L}\{e^{3t}t^5\}$.
  Solution: By applying the translation theorem,
  $$\mathcal{L}\{e^{3t}t^5\} = \mathcal{L}\{f(t)\}|_{s\to s-3} = \frac{5!}{s^6}\bigg|_{s\to s-3} = \frac{\mathbf{120}}{(\boldsymbol{s}-\mathbf{3})^{\mathbf{6}}}.$$

**Note: as we progress, it is common to leave off the middle steps in the process above.*

- *Example:* Evaluate $\mathcal{L}\{e^{-2t}\sin 4\mathrm{t}\}$.
  Solution: By applying the translation theorem,
  $$\mathcal{L}\{e^{-2t}\sin 4t\} = \frac{\mathbf{4}}{(\boldsymbol{s}+\mathbf{2})^{\mathbf{2}}+\mathbf{16}}.$$

**Note: when dealing with the translation of the transform of sine and cosine, it is common practice to not multiply out the $(s-a)^2$ term.*

Inverse Form: If we can recognize that we are dealing with $F(s-a)$, then we can apply the inverse transform to get

$$\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t),$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

- *Example:* Evaluate
  $$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$$
  Solution: First we must use partial fraction decomposition to re-write $F(s)$, which gives us

$$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2} \quad \rightarrow \quad A = 2, \qquad B = 11.$$

Therefore, we have

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = 2e^{3t} + 11\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\},$$

or

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \boldsymbol{2e^{3t} + 11e^{3t}t}.$$

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+6}\right\}$$

Solution: Since the denominator is not in a form we can find on our table, we use completing the square to get

$$s^2 + 4s + 6 = (s+2)^2 + 2.$$

Therefore we have,

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} = \mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+2}\right|_{s\to s+2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\},$$

or

$$\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+6}\right\} = \boldsymbol{e^{-2t}\cos\sqrt{2}t}.$$

- <u>*Definition:*</u> The **unit step function** $\mathcal{U}(t-a)$ is defined to be

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}.$$

**We have restricted $\mathcal{U}$ to the non-negative t −axis since that is all that is needed for the Laplace transform. We can easily extend $\mathcal{U}$ to include negative values for t.*

By multiplying a function by the unit step function, we can "turn off" the portion of the function where $t < a$. The unit step function also gives us a way of writing piece-wise defined functions in compact form.

- *Example:* Express the following piece-wise function in terms of unit step functions.

$$f(t) = \begin{cases} 3t, & 0 \le t < 2 \\ 0, & 2 \le t < 5. \\ \sin t, & t \ge 5 \end{cases}$$

Solution: Since $3t$ is active until $t = 2$, and then inactive after, we would include the expression

$$3t - 3t\,\mathcal{U}(t-2).$$

Then, at $t = 5, \sin t$ becomes active, so we would include the expression

$$\sin t\ \mathcal{U}(t-5).$$

Putting this together, we get that

$$\boldsymbol{f(t) = 3t - 3t\,\mathcal{U}(t-2) + \sin t\ \mathcal{U}(t-5).}$$

**In general, if we "turn off" $a(t)$ and "turn on" $b(t)$ at $t = \alpha$, we include the expression $[b(t) - a(t)]\mathcal{U}(t-\alpha)$ in our formula for $f(t)$.*

<u>Theorem:</u> **Second Translation Theorem**

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

Proof: By the additive property of integrals, and the definition of $\mathcal{U}$,

$$\int_0^\infty e^{-st}\, f(t-a)\mathcal{U}(t-a)dt = \int_a^\infty e^{-st}\, f(t-a)dt.$$

If we let $v = t - a$, then we get

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = \int_0^\infty e^{-s(v+a)}\, f(v)dv = e^{-as}\int_0^\infty e^{-sv}\, f(v)dv = e^{-as}F(s). \quad \blacksquare$$

- *Example:* Evaluate $\mathcal{L}\{\sin(t-\pi)\,\mathcal{U}(t-\pi)\}$.

  Solution: By noticing that $a = \pi$, and $f(t) = \sin t$, we can apply the second translation theorem to get

$$\mathcal{L}\{\sin(t-\pi)\,\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin t\} = \boldsymbol{\frac{e^{-\pi s}}{s^2+1}}.$$

Inverse Form: If $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then for $a > 0$,

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+4}\right\}.$$

  Solution: First we note that $a = 2$ and $F(s) = 1/(s+4)$. By the inverse of the first translation theorem,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}.$$

Therefore, by the inverse of the second translation theorem, we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+4}\right\} = \boldsymbol{e^{-4(t-2)}\,\mathcal{U}(t-2)}.$$

Alternative Form of the Second Translation Theorem: While we will occasionally need to find the Laplace transform of a function of the form $f(t-a)\mathcal{U}(t-a)$, more often we will want to find the Laplace transform of a function of the form $g(t)\mathcal{U}(t-a)$. By applying the Laplace transform, and using the substitution $u = t - a$, we get

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = \int_a^\infty e^{-st} g(t)dt = \int_0^\infty e^{-s(u+a)} g(u+a)du,$$

or

$$\boldsymbol{\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}}.$$

- *Example:* Evaluate $\mathcal{L}\left\{\cos 2t\,\mathcal{U}(t-\frac{\pi}{2})\right\}$.

  Solution: First we note that $a = \pi/2$ and $g(t) = \cos 2t$. Then we can apply the alternate version of the second translation theorem to get

  $$\mathcal{L}\left\{\cos 2t\,\mathcal{U}(t-\frac{\pi}{2})\right\} = e^{-\pi s/2}\mathcal{L}\left\{\cos 2(t+\frac{\pi}{2})\right\} = -e^{-\pi s/2}\mathcal{L}\{\cos 2t\} = \boldsymbol{\frac{-se^{-\pi s/2}}{s^2+2}}.$$