13.3: Fundamental Theorem of Line Integrals

Thursday, November 5, 2020 9:52 AM

Recall: (FTC) If $F' = f$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

FTC for Line Integrals

Let C be a smooth curve given by $\vec{r}(t)$, $a \le t \le b$.
Let f be a differentiable func. of two (or three) variables, where ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

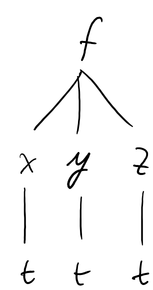
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Proof.

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t} \right\rangle dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} \right) dt$$

$$= \int_a^b \frac{\partial f}{\partial t}\,dt \quad \text{, by Chain Rule}$$



$$= \int_a^b \frac{\partial}{\partial t} f(\vec{r}(t))\,dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{, by FTC.}$$

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Independence of Path

Independence of Path

Note The FT for line integrals shows that for a conservative vector field $\vec{F} = \nabla f$, have $\int_C F \cdot d\vec{r}$ depends only on $\vec{r}(a)$ and $\vec{r}(b)$. (indep. of path)

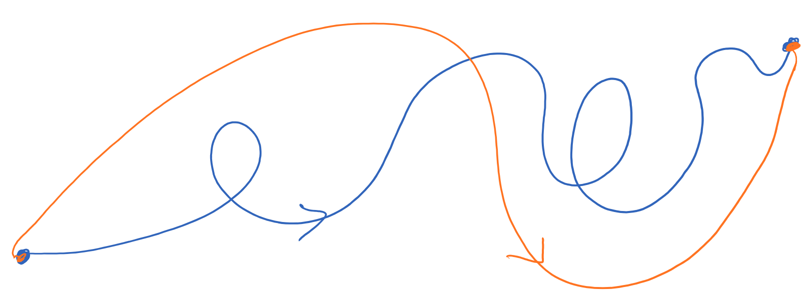
Def. 1) A path in $\mathbb{R}^n$ is a p-w smooth curve.

2) An open set $D \subset \mathbb{R}^n$ is connected if any two points in D can be joined by a path that lies entirely in D.

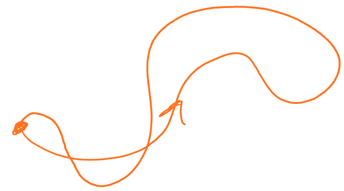
3) If $\vec{F}$ is a continuous vector field on a domain D, the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if

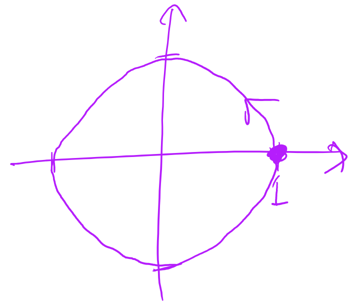
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \quad \text{for any two paths}$$

C_1, C_2 w/ same initial & terminal points.



4) A closed path is a path w/same initial point as terminal point.





Note In general, $\int_C \vec{F} \cdot d\vec{r}$ not IOP

(only IOP under special circumstances)

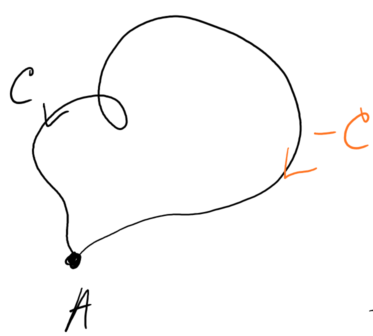
Fact $\vec{F}$ conservative on $D \Rightarrow \int_C \vec{F} \cdot d\vec{r}$ is IOP in D.
($\vec{F} = \nabla f$ some f)

Theorem $\int_C \vec{F} \cdot d\vec{r}$ IOP in D

$\Longleftrightarrow \int_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C in D.

Proof: ($\Rightarrow$) Assume $\int_C \vec{F} \cdot d\vec{r}$ IOP in D.

Choose an arbitrary closed path C in D.



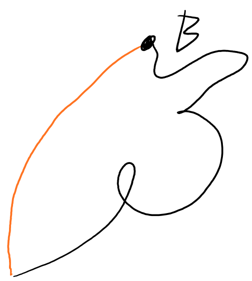
$$\int_C \vec{F} \cdot d\vec{r} = \int_{-C} \vec{F} \cdot dr = -\int_C \vec{F} \cdot d\vec{r}$$

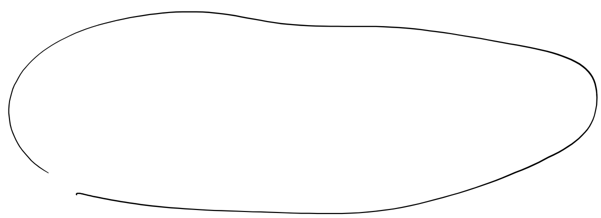
$$\Rightarrow \int_C \vec{F} \cdot dr = -\int_C \vec{F} \cdot d\vec{r}$$

$\int_C$

Choose any two points $A, B \in D$.

Let $C = C_1 \cup (-C_2)$





Rmk If $\vec{F}$ "flows" in

a) same direction as C, then $\int_C \vec{F} \cdot d\vec{r} > 0$

b) opposite " " ", " $\int_C \vec{F} \cdot d\vec{r} < 0$

c) perpendicular " " ", " $\int_C \vec{F} \cdot d\vec{r} = 0$

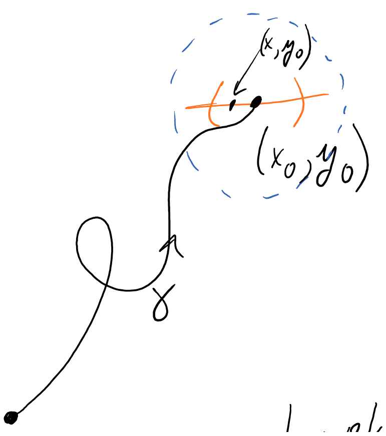
Th'm Let $\vec{F} = \langle P, Q \rangle$ be continuous on D, where D is open & connected. If $\int_C \vec{F} \cdot d\vec{r}$ is IOP in D, then $\vec{F}$ is conservative on D.

Proof. Let A be any point in D. Define the func.

$$h(B) = \int_A^B \vec{F} \cdot d\vec{r}.$$

Fix a point $(x_0, y_0) \in D$.

Because D open, there is a ball cent. at (x_0, y_0), lying in D. Let γ be any path in D from A to (x_0, y_0).

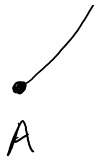


Near (x_0, y_0), consider the line parametrized by

$x = x(t)$, $y = y_0$.

$\vec{r}(t) = \langle x(t), y_0 \rangle$

Look at $h(x, y_0)$ near x_0:



Look at $h(x, y_0)$ near x_0:

[Integrate over γ, then follow line from (x_0, y_0) to (x, y_0).]

$$h(x, y_0) = \int_A^{(x, y_0)} \vec{F} \cdot d\vec{r} = \int_\gamma \vec{F} \cdot d\vec{r} + \int_{(x_0, y_0)}^{(x, y_0)} \vec{F} \cdot d\vec{r}$$

$$= \int_\gamma \vec{F} \cdot d\vec{r} + \int_{(x_0, y_0)}^{(x, y_0)} P\,dx + Q\,dy$$

$= 0$ b/c horizontal segment: $y(t) = y_0$, $y'(t)dt = 0dt = 0$

$$= \int_\gamma \vec{F} \cdot d\vec{r} + \int_{x_0}^{x} P(x(t), y_0)\,dt$$

$$\Rightarrow \frac{\partial h}{\partial x} = 0 + P(x, y_0) \quad \text{by FTC}$$

$$\Rightarrow \frac{\partial h}{\partial x}(x_0, y_0) = P(x_0, y_0).$$

Because $(x_0, y_0) \in D$ arbitrary,

$$\frac{\partial h}{\partial x}(x, y) = P(x, y) \quad \text{for all } (x, y) \in D.$$

$$\Rightarrow \frac{\partial h}{\partial x} = P \text{ on } D.$$

Similarly $\frac{\partial h}{\partial y} = Q$ on D.

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$$\Rightarrow \nabla h = \left\langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right\rangle = \langle P, Q \rangle = \vec{F} \text{ on } D.$$

$\Rightarrow h$ is a primitive of $\vec{F}$ on $\vec{D}$

$\Rightarrow \vec{F}$ conservative on $\vec{D}$. $\square$

<u>Th'm (5):</u> If $\vec{F} = \langle P, Q \rangle$ is conservative and C^1 on D, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

<u>Proof</u> $\vec{F}$ cons. $\Rightarrow \exists f$ s.t. $\nabla f = \vec{F} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ $= \langle P, Q \rangle$

$$\Rightarrow P = \frac{\partial f}{\partial x} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = f_{xy}.$$

Because $\vec{F}$ is C^1, have f is C^2.

Clairaut $\Rightarrow f_{xy} = f_{yx}$. OTOH,

$$Q = \frac{\partial f}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} = f_{yx}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \quad \square$$

<u>Note</u> $\vec{F} \cdot d\vec{r} = \langle P, Q \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$

Note $\underbrace{\vec{F}}_{\text{vector field}} \cdot d\vec{r} = \langle P, Q \rangle \cdot \langle x'(t), y'(t) \rangle \, dt$

$= \underbrace{P\,dx + Q\,dy}$

differential is (i) exact if $\exists h$ s.t.

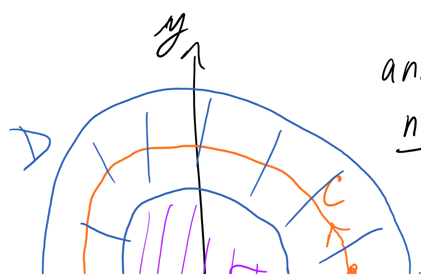
$$\frac{\partial h}{\partial x} = P \text{ and } \frac{\partial h}{\partial y} = Q$$

(ii) closed if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

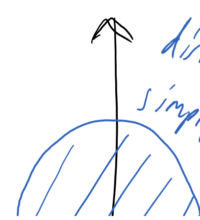
Def. 1) A simple curve does not intersect itself anywhere between its endpoints.

2) A domain in $\mathbb{R}^2$ is an open, connected set.

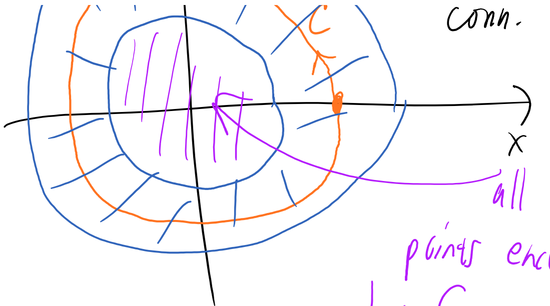
3) A connected $D \subset \mathbb{R}^2$ is simply connected if every simple closed curve in D encloses only points in D (i.e. D has no "holes.")



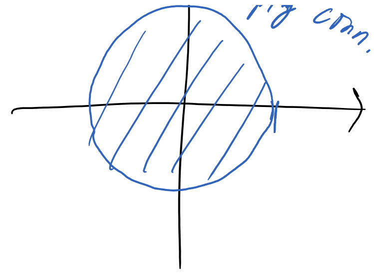
annulus is not simply conn. (SC)

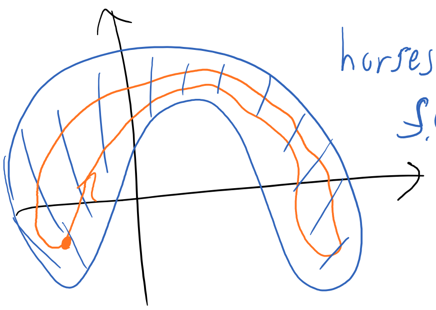


disk is simply conn.



all purple points enclosed by C are outside of D





horseshoe is S.C.

not S.C.

Th'm 6 Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vect. field on a simply conn. domain $D \subset \mathbb{R}^2$. If $\vec{F}$ is C^1 and if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D, then $\vec{F}$ is conservative.

Fact: Assume $\vec{F}$ is C^1 on an open, connected D. Then

$\int_C \vec{F} \cdot d\vec{r}$ IOP in D $\Leftrightarrow$ $\vec{F}$ conservative on D $\Rightarrow$ $\vec{F} \cdot d\vec{r}$ closed on D $\left(\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } D\right)$

$\Updownarrow$

$\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed C $\Leftarrow$ $\vec{F} \cdot d\vec{r}$ closed on D, and D ...

C all closed C in D $\Longleftarrow$ Four closed on D, and D simply conn.

Ex. Is $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$ conservative?

$= P\vec{i} + Q\vec{j}$

Sol'n. $\dfrac{\partial P}{\partial y} = -1$

$\dfrac{\partial Q}{\partial x} = 1 \quad \Rightarrow \quad \dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$

$\Rightarrow \vec{F}$ not conservative. $\boxed{\text{No}}$

Ex. Is $\vec{F}(x,y) = (3+2xy)\vec{i} + (x^2-3y^2)\vec{j}$ conservative?

Sol'n: $\dfrac{\partial P}{\partial y} = 2x$, $\dfrac{\partial Q}{\partial x} = 2x \Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(so far, know nothing)

Note $\vec{F}$ is C^1 on $\mathbb{R}^2$, and $\mathbb{R}^2$ open & SC

$\therefore$ Th'm 6 $\Rightarrow \vec{F}$ conservative on $\mathbb{R}^2$. $\boxed{\text{Yes}}$

How to find a potential func. of a conservative $\vec{F}$?

ex. (a) $\vec{F}(x,y) = (3x + 2xy)\,\vec{i} + (x^2 - 3y^2)\,\vec{j}$. Find a func. f s.t. $\nabla f = \vec{F}$.

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, C given by

$$\vec{r}(t) = e^t \sin(t)\,\vec{i} + e^t \cos(t)\,\vec{j}, \quad 0 \le t \le \pi.$$

Sol'n: (a) $\nabla f = \vec{F} = \langle P, Q \rangle = \langle 3 + 2xy,\ x^2 - 3y^2 \rangle$

$= \langle f_x, f_y \rangle$

$f_x = 3 + 2xy$

$\Rightarrow f(x,y) = \int 3 + 2xy \, dx$

$f(x,y) = 3x + x^2 y + g(y)$

const. w.r.t. x, could dep. on y

Can't stop here

$\Rightarrow f_y = x^2 + g'(y) = x^2 - 3y^2$

$\Rightarrow g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + C$

$\Rightarrow \boxed{f(x,y) = 3x + x^2 y - y^3.}$

Put $C = 0$

Check $f_x = 3 + 2xy = P$ ✓

Check $f_x = 3 + 2xy = P$ ✓

$f_y = x^2 - 3y^2 = Q$ ✓

(b) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot dr = f(\vec{r}(\pi)) - f(\vec{r}(0))$

$\begin{pmatrix} \vec{F} = \nabla f \\ \text{by (a)} \end{pmatrix}$

$= f(0, -e^{\pi}) - f(0, 1)$

$= -(-e^{\pi})^3 - (-1)^3$

$= \boxed{e^{3\pi} + 1}$

by FTLI using f, w/o hard computation