Unit 2.1: Trigonometric Substitution

In this section we will make use of a powerful technique called *Trigonometric Substitution*. We have already encountered u substitution where we typically took a relatively complicated function and made it appear simpler by renaming it u. The goal was to be able to simplify the apparent form of the integral. With trigonometric substitution (trig sub) we tend to replace a relatively simple function with a more complicated trig function, but still have the goal of simplifying the apparent form of the integral. In this class, we consider, sine, tangent, and secant substitutions. The following trig identities will be useful and should be memorized.

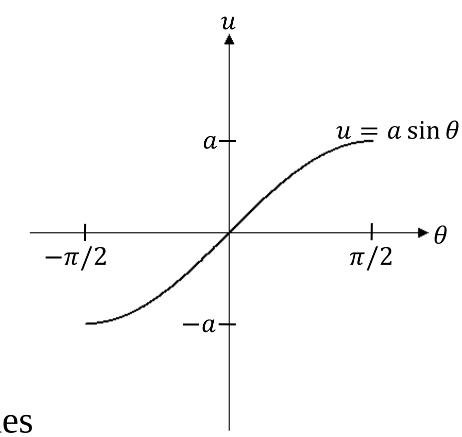
- Pythagorean Identities: $\sin^2 u + \cos^2 u = 1$ $\quad\quad$ $\tan^2 u + 1 = \sec^2 u$
- Power Reduction: $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$ $\quad\quad$ $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$
- Double Angle: $\sin 2u = 2 \sin u \cos u$ $\quad\quad$ $\cos 2u = \cos^2 u - \sin^2 u$

Sine Substitutions. Sine substitutions might be considered when we encounter integrands that contain $a^2 - u^2$ and most commonly $\sqrt{a^2 - u^2}$, where a is a positive real number and u is a variable expression. In this case substitution will be motivated by the Pythagorean Identity: $\sin^2 \theta + \cos^2 \theta = 1$ or equivalently $\cos^2 \theta = 1 - \sin^2 \theta$.

Let's just focus on the expression $\sqrt{a^2 - u^2}$, and apply the substitution $u = a \sin \theta$ where $a > 0$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. This yields

$$\sqrt{a^2 - u^2} = \sqrt{a^2 - (a \sin \theta)^2} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2}\sqrt{\cos^2 \theta} = a \cos \theta$$

The important thing to notice at this point is that we were able to rewrite our square root function as a simple cosine function (in terms of a different variable). A technical point to make is that in the original square root function requires that $-a \leq u \leq a$, otherwise one would obtain a negative number under the square root, yielding a non-real number. It is important that our substitution for u, namely $a \sin \theta$, has the same property, i.e. that it takes on values between $-a$ and a. The graph to the right shows that this is in fact the case. In the final step of simplifying above, we did not require absolute values on a and $\cos \theta$ as they will be positive for this interval of θ.



Let us now look at an example to see how this kind of substitution can be beneficial when finding antiderivatives.

Example 1 Evaluate the integral $\int \frac{10}{x^2\sqrt{25-x^2}}\,dx.$

Solution: We begin by first recognizing the $\sqrt{25-x^2} = \sqrt{(5)^2-(x)^2}$ in the denominator. This makes the sine substitution $x = 5\sin\theta$, reasonable to consider. Do not forget to also consider other possibilities such as u-sub or integration by parts, neither of which work here. We proceed as follows.
Let $x = 5\sin\theta$, so that $dx = 5\cos\theta\, d\theta$. While we could make all of our substitutions into the integral now, let's simplify the radical that motivated the substitution before doing so. We have

$$\sqrt{25-x^2} = \sqrt{(5)^2-(x)^2} = \sqrt{(5)^2-(5\sin\theta)^2} = \sqrt{25(1-\sin^2\theta)} = \sqrt{25}\sqrt{\cos^2\theta} = 5\cos\theta$$

Once you do this simplification a few times, you will likely be able to skip many or all of the steps above. Now when we substitute into our integral we obtain

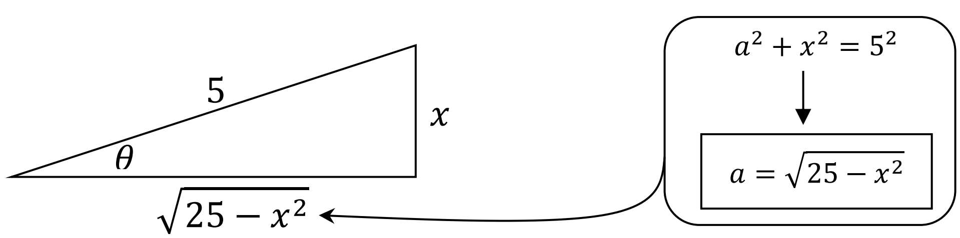
$$\int \frac{10}{x^2\sqrt{25-x^2}}\,dx = 10\int \frac{1}{\underbrace{(5\sin\theta)^2}_{x^2}\underbrace{\mathbf{5\cos\theta}}_{\sqrt{25-x^2}}}\overbrace{\mathbf{5\cos\theta}\,d\theta}^{dx} = 10\int \frac{1}{25\sin^2\theta}\,d\theta = \frac{2}{5}\int \csc^2\theta\, d\theta$$

$$= -\frac{2}{5}\cot\theta + C$$

You can now see that the substitution allowed us to express the integral in a form where we knew the antiderivative based on a basic integration rule. The only problem is that our antiderivative is now expressed in terms of θ and we would like to express it in terms of x. I will show you two ways to accomplish this. The first way is to recognize that $\cot\theta = \frac{\cos\theta}{\sin\theta}$. When we first made our substitutions, we had $x = 5\sin\theta$ and thus $\sqrt{25-x^2} = 5\cos\theta$. Thus we have

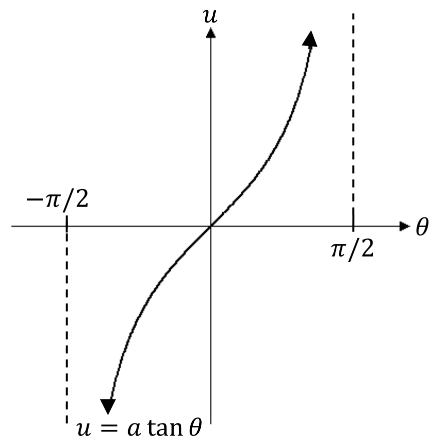
$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{5\cos\theta}{5\sin\theta} = \frac{\sqrt{25-x^2}}{x},$$

leaving us with a final answer of $\boxed{-\frac{2\sqrt{25-x^2}}{5x} + C}$. A second way to obtain an expression for $\cot\theta$ in terms of x is to draw a right triangle that exhibits the fact that $x = 5\sin\theta$ or equivalently that $\sin\theta = \frac{x}{5}$. Since the sine of an angle can be viewed as the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$, we obtain the triangle below, where the adjacent side is obtained by the Pythagorean Theorem.



Based on the above, we obtain $\cot\theta$ using the ratio $\frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{25-x^2}}{x}$, just as we obtained above.

Tangent Substitutions. We will now look at tangent substitutions, which we will consider when we encounter expressions of the form $a^2 + u^2$, and most commonly $\sqrt{a^2+u^2}$, where again, a is a positive real number and u is a variable expression. The substitution $u = a\tan\theta$, will be motivated by the identity $1 + \tan^2\theta = \sec^2\theta$. Again we let $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so that $a\tan\theta$ takes on all real number values, just like u is allowed to in this case (the $a^2 + u^2$ under the radical will never be negative, so there are no restrictions on u). Let us now take a look at the simplification that occurs when we apply the substitution $u = a\tan\theta$ to the expression $\sqrt{a^2+u^2}$.



$$\sqrt{a^2+u^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2}\sqrt{\sec^2\theta} = a\sec\theta$$

When applying this substitution in an integral, it might be beneficial to remember that the radical will simplify to $a\sec\theta$, rather than go through the simplification each time. Let us now take a look at an example.

<u>Example 2</u> Evaluate the integral $\int \frac{9x^3}{\sqrt{1+4x^2}}\,dx$.

Solution: We begin by first recognizing the $\sqrt{1+4x^2} = \sqrt{(1)^2+(2x)^2}$ in our denominator. This fits the form for a tangent substitution with $a = 1$ and $u = 2x$. Before carrying out the tangent substitution, it is important to realize that a simple u-sub will not work. If we were going to attempt a u-sub, we would most likely consider $u = 1 + 4x^2$, which would then give us $du = 8x dx$. The fact that we have x^3 in our numerator as opposed to just x is an indicator that u-sub probably isn't the best approach. Let us now proceed with our tangent substitution. Let $2x = 1 \cdot \tan\theta = \tan\theta$, so that $2dx = \sec^2\theta\, d\theta$ or rather $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta\, d\theta$. While our general simplification that preceeded this example indicates that the radical in the denominator will simplify to $1 \cdot \sec\theta = \sec\theta$, let us see this simplification again.

$$\sqrt{1+4x^2} = \sqrt{(1)^2+(2x)^2} = \sqrt{(1)^2+(1\cdot\tan\theta)^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

Applying these substitutions we obtain

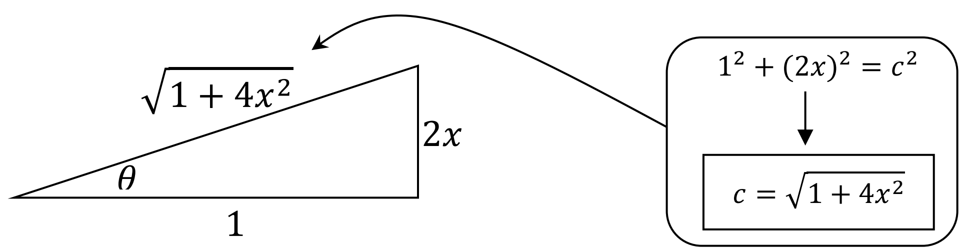
$$\int \frac{9x^3}{\sqrt{1+4x^2}}\,dx = 9\int \frac{\overbrace{\left(\frac{1}{2}\tan\theta\right)^3}^{x^3}}{\underbrace{\sec\theta}_{\sqrt{1+4x^2}}}\overbrace{\frac{1}{2}\sec^2\theta\,d\theta}^{dx} = \underbrace{9\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)}_{\frac{9}{16}}\int \tan^3\theta\sec\theta\,d\theta$$

This substitution has generated an integral involving powers of tangent and secant like we looked at in the previous section. Thus we consider a u-sub with u equal to $\tan\theta$ or $\sec\theta$. In this case we let $u=\sec\theta$, so that $du=\sec\theta\tan\theta\,d\theta$. This substitution yields

$$\frac{9}{16}\int \tan^3\theta\sec\theta\,d\theta = \frac{9}{16}\int \underbrace{(\sec^2\theta-1)}_{\tan^2\theta}\sec\theta\tan\theta\,d\theta = \frac{9}{16}\int (u^2-1)du$$

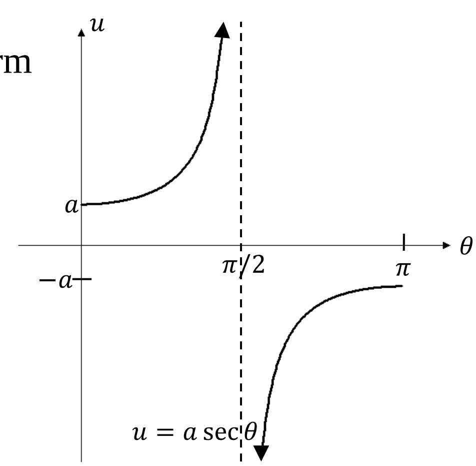
$$= \frac{9}{16}\left(\frac{u^3}{3}-u\right)+C = \frac{9}{16}\left(\frac{(\sec\theta)^3}{3}-\sec\theta\right)+C = \boxed{\frac{9}{16}\left(\frac{\left(\sqrt{1+4x^2}\right)^3}{3}-\sqrt{1+4x^2}\right)+C}$$

In the final step above we made use of the fact that $\sec\theta=\sqrt{1+4x^2}$. One could have obtained this from our initial substitutions and simplifications, or based on the following triangle based on our substitution, $2x=\tan\theta=\frac{\text{opposite}}{\text{adjacent}}$.



Using the ratio, $\sec\theta=\frac{\text{hypotenuse}}{\text{adjacent}}=\frac{\sqrt{1+4x^2}}{1}=\sqrt{1+4x^2}$, we obtain the same result.

Secant Substitutions. We will now look at secant substitutions, which we will consider when we encounter expressions of the form u^2-a^2, and most commonly $\sqrt{u^2-a^2}$, where again, a is a positive real number and u is a variable expression. The substitution $u=a\sec\theta$, will be motivated by the identity $\sec^2\theta-1=\tan^2\theta$. Here we have $0\le\theta<\frac{\pi}{2}$ when $u\ge a$ and $\frac{\pi}{2}<\theta\le\pi$ when $u\le -a$ so that $u=a\tan\theta$ takes on all real number values that result in $u^2-a^2\ge 0$.



Let us now take a look at the simplification that occurs when we apply the substitution $u = a\sec\theta$ to the expression $\sqrt{u^2 - a^2}$.

$$\sqrt{u^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2}\sqrt{\tan^2\theta} = a|\tan\theta|$$

When $0 \le \theta < \frac{\pi}{2}$ or equivalently $u \ge a$, we have $|\tan\theta| = \tan\theta$. When $\frac{\pi}{2} < \theta \le \pi$ or equivalently $u \le -a$, we have $|\tan\theta| = -\tan\theta$. For definite integrals the appropriate interval will be determine by the limits of integration. For indefinite integrals, you will typically be given the interval in the statement of the problem.

Example 3 Evaluate the integral $\int \frac{\sqrt{9x^2-4}}{x^2}\,dx$ where $x > 2/3$.

Solution: We begin by first recognizing the $\sqrt{9x^2 - 4} = \sqrt{(3x)^2 - (2)^2}$. This fits the form for a secant substitution with $a = 2$ and $u = 3x$. We therefore let $3x = 2\cdot\sec\theta$, so that $x = \frac{2}{3}\sec\theta$ and $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$. While our general simplification that preceeded this example indicates that the radical in the denominator will simplify to $2\cdot\tan\theta$, let us see this simplification again. Note: Since $x > 2/3$, we will have $\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$.

$$\sqrt{9x^2 - 4} = \sqrt{(3x)^2 - (2)^2} = \sqrt{(2\sec\theta)^2 - (2)^2} = \sqrt{4}\cdot\sqrt{\sec^2\theta - 1} = 2\sqrt{\tan^2\theta} = 2\tan\theta$$

Applying these substitutions we obtain

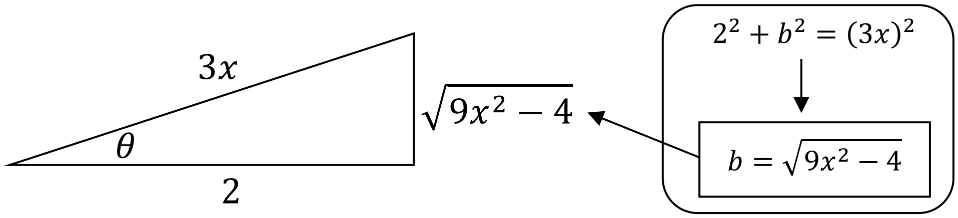
$$\int \frac{\sqrt{9x^2 - 4}}{x^2}\,dx = \int \frac{\overbrace{2\tan\theta}^{\sqrt{9x^2-4}}}{\underbrace{\frac{4}{9}\sec^2\theta}_{x^2}}\overbrace{\frac{2}{3}\sec\theta\tan\theta\,d\theta}^{dx} = \underbrace{2\left(\frac{2}{3}\right)\left(\frac{9}{4}\right)}_{3}\int \frac{\tan^2\theta}{\sec\theta}\,d\theta$$

There are a few ways to proceed from here, each of which will lead to the same way of rewriting the integrand. We will take the route using the identity $\tan^2\theta = \sec^2\theta - 1$.

$$3\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = 3\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = 3\int\left(\frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta}\right)d\theta = 3\int(\sec\theta - \cos\theta)\,d\theta$$

$$= 3(\ln|\sec\theta + \tan\theta| - \sin\theta + C = \boxed{3\left(\ln\left|\frac{3x}{2} + \frac{\sqrt{9x^2-4}}{2}\right| - \frac{\sqrt{9x^2-4}}{3x}\right) + C}$$

In the final step above we rewrote our resulting in terms of x using the following triangle based on our substitution, $\frac{3x}{2} = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$.



As you can see, trig substitution is a relatively involved process which can sometime require additional techniques of integration like u-sub. It is recommended that before applying a trig-sub, you make sure that a simpler approach wouldn't work. For example, consider the following two integrals, that both could be determined with trig sub.

$$\int \frac{1}{\sqrt{1 - 4x^2}}\,dx \qquad \text{and} \qquad \int \frac{x}{\sqrt{9 + x^2}}\,dx$$

While a sine-sub would work for the first integral and a tan-sub for the second one, each of these integrals can be evaluated using u-sub only. In the first case we let $u = 2x$ and get inverse sine functions, and in the second case we let $u = 9 + x^2$ and get power functions.