Unit 1.3: Integration by Parts

As we know, (indefinite) integration can be viewed as the "reverse" of differentiation. In particular, *Integration by Parts* can be viewed as the reverse of the product rule. Thus, let us first take a look at the product rule and then we'll "reverse" it.

Let u and v be differentiable functions of x. The product rule says that

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

If we integrate both side of this equation with respect to x we obtain

$$\int \frac{d}{dx}uv\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

The integral on the left hand side just cancels out the derivative, i.e. if you start with a function, take its derivative, and then the antiderivative, you end up with the function you started with (plus an arbitrary constant). On the right-hand side we can "cancel" the dx's. This yields

$$uv = \int u\,dv + \int v\,du \quad \text{or equivalently} \quad \boxed{\int u\,dv = uv - \int v\,du}$$

The equation on the right is our **I**ntegration-**B**y-**P**arts (IBP) formula. What it allows us to do, is take an integral and express it in terms of a different integral. The hope is that the new integral is in some way easier to deal with than the original. Let us look at an example of how this is used.

Example 1 Determine the integral $\int x \sin x\,dx$.

Solution: We must take the expression $x\sin x\,dx$ and identify it as a product of the form $u\,dv$. The factor dv must contain dx but may also contain a factor of the integrand. Here the most logical choices for u are x or $\sin x$. It turns out that in this case, it is best to view u as x and thus dv as $\sin x\,dx$. To use the IBP formula, we then have to determine du and v. To find du, we simply use the fact that $du = u'dx$. In this case, since $u = x$, we have $du = 1dx = dx$. To find v given that $dv = \sin x\,dx$, we need only determine an antiderivative of $\sin x$, giving us the most natural choice of $v = -\cos x$. We need not include the $+C$ here, as we can choose the antiderivative corresponding to $C = 0$. By the IBP formula we obtain

$$\int \underbrace{x}_{u}\underbrace{\sin x\,dx}_{dv} = \underbrace{x}_{u}\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\underbrace{dx}_{du}$$

$$= \boxed{-x\cos x + \sin x + C}$$

Example 2 Determine the integral $\int x^5 \ln x \, dx$.

Solution: In this particular problem, the most natural options to consider for u are x^5 or $\ln x$. Whichever we pick for u, it is important to realize that the other factor will go with the dx to form dv and we will have to know its antiderivative. At this point we don't know the antiderivative of $\ln x$, so it is best that we let:

$$u = \ln x \quad \rightarrow \quad du = \frac{1}{x}dx \qquad \text{and} \qquad dv = x^5 dx \quad \rightarrow \quad v = \frac{1}{6}x^6$$

By the IBP formula we obtain

$$\int x^5 \ln x \, dx = (\ln x)\left(\frac{1}{6}x^6\right) - \int \left(\frac{1}{6}x^6\right)\frac{1}{x}dx$$

The next step is crucial. We want the new integral on the right side of the equation to be simpler than what we started with. We have to recognize here, that the integrand can be simplified by canceling a factor of x. Once that is done, we see that this use of IBP, led to something that we can deal with. If you ever make a "bad" choice for u and dv, you typically don't realize it, until after you have formed the new integral and then see that it didn't get you anywhere. Continuing with what we had above, we have

$$\frac{1}{6}x^6 \ln x - \frac{1}{6}\int x^5 dx = \boxed{\frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C}$$

We could factor out $\frac{1}{36}x^6$, but I will refrain from such simplifications at this time.

Now that you've seen two examples, that natural questions you might have are: 1) how do you know when to use integration by parts, and 2) how do you know what to let u and dv equal. There are some very standard cases for which one should easily recognize that IBP is applicable and the choice for u and dv in those cases can be determined based on experience with similar problems (these cases can be found below). However, there will always be problems for which one would not initially suspect that IBP is the way to go. For these, it becomes one of those cases where you decide to try IBP after not having success with another approach.

1) For integrals of the form: $\int x^n e^{ax} dx$, $\int x^n \sin ax \, dx$, or $\int x^n \cos ax \, dx$,
   Let $u = x^n$ and let dv be the rest.
2) For integrals of the form: $\int x^n \ln ax \, dx$, $\int x^n \arcsin ax \, dx$, or $\int x^n \arccos ax \, dx$
   Let $dv = x^n dx$ and let u be the rest.
3) For integrals of the form: $\int e^{ax} \sin bx \, dx$ or $\int e^{ax} \cos bx \, dx$, you will have to use integration by parts twice. It doesn't matter whether you let u equal e^{ax} or the trig function, but be consistent in the two applications of IBP.