2. 5 Modeling with First-Order Differential Equations

Population Growth/Decay: *"The rate at which the population grows at a certain time is proportional to the total population at that time."*

In mathematical terms, with $P(t)$ representing the population at a time t, this translates to

$$\frac{dP}{dt} \propto P \quad \rightarrow \quad \frac{dP}{dt} = kP$$

where k is the constant of proportionality. This is a very basic (separable) model which is easy to solve for, and results in the familiar solution

$$P(t) = P_0 e^{kt}.$$

Newton's Law of Cooling/Warming: *"The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium."*

In mathematical terms, with $T(t)$ representing the temperature of the body at a time t and T_m representing the temperature of the medium, this translates to

$$\frac{dT}{dt} \propto T - T_m \quad \rightarrow \quad \frac{dT}{dt} = k(T - T_m).$$

This is once again an easy to solve separable DE. The solution is left as an exercise.

**You could also set up the model with $(T_m - T)$ instead, however this will lead to an extra negative sign in the solution. With initial conditions, both setups will give the same solution.*

- *Example:* When a cake is removed from an oven, its temperature is 300°F. After sitting in a 70°F room for three minutes, the cake has cooled to 200°F. What temperature will the cake be after 30 minutes?

  Solution: Unless told otherwise, assume that the temperature of the medium remains constant. This gives us a model of

  $$\frac{dT}{dt} = k(T - 70), \qquad T(0) = 300, \qquad T(3) = 200.$$

  Solving the differential equation we get

  $$\int \frac{dT}{T - 70} = \int k\,dt \quad \rightarrow \quad \ln|T - 70| = kt + c_1 \quad \rightarrow \quad T = 70 + ce^{kt}.$$

Using our initial condition $T(0) = 300$, we can find that $c = 230$. Then using the secondary condition $T(3) = 200$, we can get that $k = \frac{1}{3}\ln\left|\frac{13}{23}\right| \approx -0.19018$. Thus

$$T(t) = 70 + 230e^{-0.19018t}.$$

If we want the temperature of the cake after 30 minutes, we simply evaluate $T(30)$ to get a temperature of **71°F**.

**It is always acceptable to leave k in terms of logarithms, for a more exact answer.*

**When rounding, round according to what the question specifies.*

Spread of a Disease: *"The rate at which a disease spreads is proportional to the number of interactions between people who have the disease (x) and people who do not have the disease (y)."*

If we assume that the number of interactions is jointly proportional to $x(t)$ and $y(t)$, a fair assumption, then we get the model

$$\frac{dx}{dt} \propto xy \quad \rightarrow \quad \frac{dx}{dt} = kxy.$$

If the community involved has a fixed population of n, we can even eliminate y by using the substitution $y = n - x$ to get

$$\frac{dx}{dt} = kx(n - x).$$

- *Example:* A group of 30 contestants are sealed off in a house. Initially one contestant learns about a rumor from the outside world. The rumor then begins to spread around the house (like an infection). After 1 hour, 2 people know the rumor. How long until the whole house knows the rumor?

  Solution: The spread of the rumor can be modeled by the spread of a disease. If we let $x(t)$ represent the number of people who know the rumor after t hours, then we get

  $$\frac{dx}{dt} = kx(30 - x), \qquad x(0) = 1, \qquad x(1) = 2.$$

  Solving the differential equation we get

  $$\int \frac{dx}{x(30 - x)} = \int k dt \quad \rightarrow \frac{1}{30}\ln x - \frac{1}{30}\ln(30 - x) = kt + \ln c_1.$$

Solving for x, we get

$$x(t) = \frac{30e^{kt}}{c + e^{kt}}.$$

Using $x(0) = 1$, we can find that $c = 29$. And using $x(1) = 2$, we can find that $k = \ln\left(\frac{29}{14}\right) \approx 0.7282$. This gives us a final solution of

$$x(t) = \frac{30e^{0.7282t}}{29 + e^{0.7282t}}.$$

According to our model, $x(t) \to 30$ as $t \to \infty$, but $x(t)$ will never actually reach 30. However, by evaluating $x(t)$ at various times, we find that after 12 hours, 29.8612 people would know the rumor. So essentially **all 30 will know the rumor within 12 hours**.

**The integral with respect to x was evaluated using partial fraction decomposition. On an exam you would be expected to show the work in producing that answer.*

Keep in mind, this model does not take into account people recovering from a disease as it progresses. The way it is set up, the model assumes that anyone infected stays infected and continues to spread the disease. We will consider what happens when we include "recoveries" in the model in a later unit.

Chemical Reactions: *"The rate at which a chemical reaction proceeds is proportional to the product of the amounts of the reactants left. "*

For this situation, we are assuming that we start with a units of substance A and b units of substance B. We also assume that M parts of A and N parts of B used to form a substance C. If $X(t)$ is used to denote the amount of C at time t, then the number of units of A and B at time t are, respectively,

$$a - \frac{M}{M+N}X \quad \text{and} \quad b - \frac{N}{M+N}X.$$

Therefore,

$$\frac{dX}{dt} \propto \left(a - \frac{M}{M+N}X\right)\left(b - \frac{N}{M+N}X\right) \quad \to \frac{dX}{dt} = k(\alpha - X)(\beta - X),$$

where $\alpha = a(M+N)/M$ and $\beta = b(M+N)/N$.

- *Example:* Cadburium is formed when one gram of Agarium interacts with 4 grams of Butterium. Initially there are 50 grams of Agarium and 32 grams of Butterium. After 10 minutes it is found that 30 grams of Cadburium has formed. What is the limiting amount of Cadburium in the long run? How much of Agarium and Butterium will remain at that point?

  Solution: If we let $X(t)$ be the amount of Cadburium at time t, we first find α and β:

$$\alpha = \frac{50(1+4)}{1} = 250, \qquad \beta = \frac{32(1+4)}{4} = 40$$

  Thus,

$$\frac{dX}{dt} = k(250-X)(40-X), \qquad X(0)=0, \qquad X(10)=30.$$

  Solving gives us the solution

$$\int \frac{dX}{(250-X)(40-X)} = \int k dt \;\rightarrow\; \frac{1}{210}\ln\left(\frac{250-X}{40-X}\right) = kt + c_1 \;\rightarrow\; \frac{250-X}{40-X} = ce^{210t}.$$

  Using $X(0)=0$, we can find that $c = 25/4$. And using $X(10)=30$, we can find that $210k = \frac{1}{10}\ln\left(\frac{88}{25}\right) \approx 0.1258$. With some cleanup, this gives us a final solution of

$$X(t) = \frac{1000 - 1000e^{-.1258t}}{25 - 4e^{-.1258t}}.$$

  According to our model, $X(t) \to 40$ as $t \to \infty$, so in the long run, the amount of Cadburium will approach **40 grams**. If we ever got to 40 grams of Cadburium, there would be

$$50 - \frac{1}{5}(40) = \mathbf{42\ g\ of\ } \boldsymbol{A} \qquad \text{and} \qquad 32 - \frac{4}{5}(40) = \mathbf{0\ g\ of\ } \boldsymbol{B} \mathbf{\ left.}$$

  **Much of the cleanup work was omitted to save space. On an exam you would be expected to show all of the work in producing that answer.*

<u>Mixtures:</u>

The mixing of two solutions of differing concentrations gives rise to a first-order differential equation. If we let $A(t)$ represent the amount of solute in a tank at time t, then the rate of change of A is equal to the difference of the rate at which the solute enters the tank and the rate at which the solute leaves the tank:

$$\frac{dA}{dt} = \begin{pmatrix} input\ rate \\ of\ solute \end{pmatrix} - \begin{pmatrix} output\ rate \\ of\ solute \end{pmatrix} = R_{in} - R_{out}.$$

In the above equation, R_{out} is usually a function of A, while R_{in} tends to be a constant.

- *Example:* A large tank holds 300 gallons of a brine (salt) solution. A brine solution is being poured into the tank at a rate of 3 gal/min; it mixes with the solution in the tank, and then is pumped out at a rate of 3 gal/min. If the concentration of salt in the brine solution coming in is 2 lb/gal, and the tank initially had 50 lbs of salt in it, how much salt is in the tank after 2 hours?

  Solution: The rate of salt coming into the tank is a constant (2 lb/gal)*(3 gal/min), giving us $R_{in} = 6$ lb/min. The rate of salt leaving the tank based on the concentration of salt in the 300 gal tank at a given time and the flow rate out of 3 gal/min. This gives us that $R_{out} =$ (A/300 lb/gal)*(3 gal/min) $= A/100$ lb/min. Thus

  $$\frac{dA}{dt} = 6 - \frac{A}{100}, \qquad A(0) = 50.$$

  While this is a separable equation, it is also a linear equation

  $$\frac{dA}{dt} + \frac{1}{100}A = 6.$$

  Using an integrating factor of $e^{t/100}$, we get that

  $$A(t) = 600 + ce^{-t/100} \quad \rightarrow \quad A(t) = 600 - 550e^{-t/100}.$$

  If we want the amount of salt after 2 hours, we now simply calculate

  $$A(120) = 600 - 550e^{-1.2} \approx \boldsymbol{434\ lb}.$$

  **Our model was in terms of minutes, so make sure to convert 2 hours into 120 minutes before evaluating.*