## 3.1 Higher-Order Linear Equations

For first-order differential equations, we had techniques for solving a variety of different types of equations. For higher-order differential equations, we will mainly focus in on solving linear differential equations. Before we explore the specific techniques, we need to build up some of the framework that we will be using.

- *Definition:* A linear $n^{\text{th}}$-order DE is called **homogeneous** if it is of the form
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_o(x)y = 0.$$
A linear $n^{\text{th}}$-order DE is called **non-homogeneous** if it is of the form
$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_o(x)y = g(x).$$
**In this context, we are back to the original definition of homogeneous.*

  Our goal is to solve the non-homogeneous problem by first solving the homogenous case. In order to do that, we will first develop some extra tools.

In calculus, we can often denote differentiation by the **differential operator** $D$, where $Dy = dy/dx$. In general, $D^n y = d^n y/dx^n$. From this notation we can define a **($n^{\text{th}}$-order) polynomial operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x).$$

As a result of the properties of differentiation, the differential operator $L$ is a **linear operator,** meaning that for any constants $\alpha$ and $\beta$,

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L\{f(x)\} + \beta L\{g(x)\}.$$

Since any linear differential equation can be expressed in terms of $D$, any linear $n^{\text{th}}$-order DE can be written compactly as $L(y) = g(x)$.

***Theorem:*** **Superposition Principle - Homogeneous Equations**

Let $y_1, y_2, \ldots, y_k$ be solutions of $L(y) = 0$ on an interval $I$. Then the linear combination

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_k y_k,$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants, is also a solution on $I$.

***Proof:*** For simplicity we will prove the case where $k = 2$. The proof can easily be extended to any $k$. Let $y_1$ and $y_2$ be solutions of the $n^{\text{th}}$-order homogeneous equation $L(y) = 0$. If we define $y = c_1 y_1 + c_2 y_2$, by linearity of $L$ we get

$$L(y) = L\{c_1 y_1 + c_2 y_2\} = c_1 L(y_1) + c_2 L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \blacksquare$$

**Note: the trivial solution $y = 0$ is always a solution to a homogeneous linear equation.*

- <u>*Definition:*</u> A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval $I$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

  $$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

  for every $x$ in the interval. If the set is not linearly dependent, it is said to be **linearly independent.**

- *Example:* The set of functions $f_1(x) = \sqrt{x},\ f_2(x) = \sqrt{x} + 3x,\ f_3(x) = 2x$ is linearly dependent on the interval $(0, \infty)$. Let $c_1 = 2,\ c_2 = -2,\ c_3 = 3$, then we get

  $$2\sqrt{x} - 2\left(\sqrt{x} + 3x\right) + 3(2x) = 0.$$

  **Note: a set of functions is linearly dependent if you can express one function as a linear combination of the other functions.*

Since the superposition principle says that any linear combination of solutions is a solution, what we are interested in is finding a linearly independent set of solutions to a problem.

- <u>*Definition:*</u> Suppose that each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. Then the **Wronskian** of the functions is the determinant

  $$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

***<u>Theorem:</u>*** **Criterion for Linearly Independent Solutions**

Let $y_1, y_2, \ldots, y_n$ be solutions of $L(y) = 0$ on an interval $I$. Then the set of solutions is linearly independent on $I$ if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every $x$ in the interval.

**Note: if $y_1, y_2, \ldots, y_n$ are solutions of $L(y)$, then $W(y_1, y_2, \ldots, y_n)$ is either identically zero or never zero on $I$.*

- *Definition:* Any set $y_1, y_2, \ldots, y_n$ of $n$ linearly independent solutions the $n^{\text{th}}$-order DE $L(y) = 0$ on an interval $I$ is called a **fundamental set of solutions** on $I$.

***Theorem:*** **Existence of a Fundamental Set**
There exists a fundamental set of solutions to the homogeneous linear $n^{\text{th}}$-order differential equation $L(y) = 0$ on an interval $I$.

***Theorem:*** **General Solution - Homogeneous Equations**
Let $y_1, y_2, \ldots, y_n$ be *any* fundamental set of solutions to $L(y) = 0$ on an interval $I$. The general solution of the equation on the interval is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

- *Example:* By examination it is easy to check that $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions to the homogeneous equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By applying the Wronskian,

  $$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

  for every $x$. Therefore $y_1$ and $y_2$ are linearly independent, and $\{y_1, y_2\}$ is a fundamental set of solutions. Therefore, the general solution would be $\boldsymbol{y = c_1 e^{3x} + c_2 e^{-3x}}$**.**
  **Note: while you may be able to create a different fundamental set of solutions, this is the one and only general solution (up to relabeling).*
  **Note: for two functions it is usually to determine linear independence by inspection. However, for larger sets of functions the Wronskian is a very useful tool.*

- *Definition:* Any function $y_p$, free of arbitrary parameters, that satisfies $L(y) = g(x)$ is called a **particular solution** of the equation.

***Theorem:*** **General Solution - Non-homogeneous Equations**
Let $y_p$ be any particular solution of linear $n^{\text{th}}$-order differential equation $L(y) = g(x)$ on an interval $I$, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous equation on $I$. Then the **general solution** of $L(y) = g(x)$ on $I$ is

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p,$$
where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

***Proof:*** Let $Y(x)$ and $y_p(x)$ be two particular solutions of the non-homogeneous equation $L(y) = g(x)$. Define $u(x) = Y(x) - y_p(x)$. By linearity of $L$
$$L(u) = L\{Y(x) - y_p(x)\} = L(Y) - L(y_p) = g(x) - g(x) = 0.$$
Therefore, $u(x)$ is a solution to $L(y) = 0$. Hence, $u(x) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, and $Y(x) - y_p(x) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$, or $Y(x) = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p(x)$. ■

*$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ *is referred to as the complimentary function for* $L(y) = g(x)$.

***<u>Theorem:</u>*** **Superposition Principle - Non-homogeneous Equations**
Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be particular solutions of the linear $n^{\text{th}}$-order differential equations $L(y) = g_1(x), L(y) = g_2(x), \ldots, L(y) = g_k(x)$ on an interval $I$. Then
$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$
is a particular solution to $L(y) = g_1(x) + g_2(x) + \cdots + g_k(x)$.

***Proof:*** For simplicity we will prove the case where $k = 2$. The proof can easily be extended to any $k$. Let $y_{p_1}(x)$ and $y_{p_2}(x)$ be particular solutions of $L(y) = g_1(x)$ and $L(y) = g_2(x)$, respectively. Define $y_p(x) = y_{p_1}(x) + y_{p_2}(x)$. By linearity of $L$
$$L(y_p) = L\{y_{p_1} + y_{p_2}\} = L(y_{p_1}) + L(y_{p_2}) = g_1(x) + g_2(x). \qquad ■$$