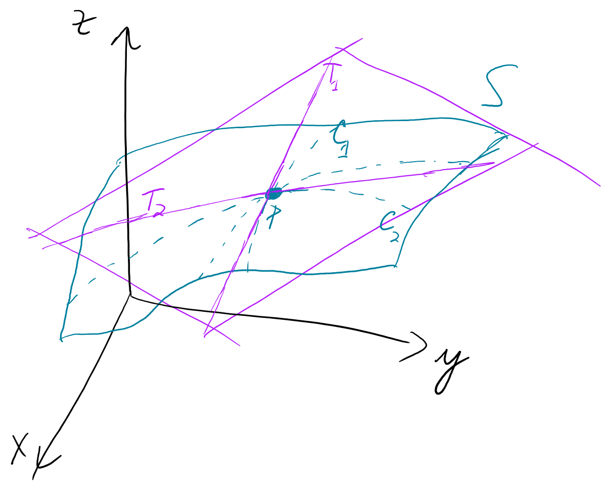
11.4: Tangent Planes and Linear Approximations

Friday, September 4, 2020 8:57 AM

& Polynomial Approx Degree 2



ex. $z = f(x,y) \rightsquigarrow$ surface S

$P = (x_0, y_0, f(x_0,y_0))$ on surface

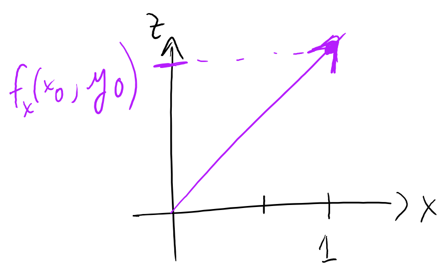
(plane $y = y_0) \cap S = C_1$

(plane $x = x_0) \cap S = C_2$

plane is tangent to S at point P

- Know the plane contains tan. lines in all directions, in particular, the x & y directions.

- Recall (i) tan. line to surface $z = f(x,y)$ in x direction has slope $f_x(x_0, y_0)$. Direction vector?



In 3D $\rightsquigarrow \vec{u} = \langle 1, 0, f_x(x_0,y_0)\rangle$ as dir. vector

(ii) tan. line in y-direction has direction vector

$$\langle 0, 1, \underbrace{f_y(x_0,y_0)}_{\parallel \; v_3} \rangle = \vec{v}$$

- So have two dir. vectors $\vec{u}, \vec{v}$ for the tangent plane, so $\vec{n} = \vec{u} \times \vec{v}$ is normal vect. for tan. plane:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & u_3 \end{vmatrix} = \vec{i}(0 - u_3) - \vec{j}(v_3 - 0) + \vec{k}(1-0)$$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & u_3 \\ 0 & 1 & v_3 \end{vmatrix} = \vec{i}(0-u_3) - j(v_3 - 0) + k(1-0)$$

$$= \langle -u_3, -v_3, 1 \rangle$$

$$\vec{n} = \langle -f_x(x_0,y_0), -f_y(x_0,y_0), 1 \rangle$$

• Know $P = (x_0, y_0, z_0)$ on plane, where $z_0 = f(x_0, y_0)$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0, \quad \vec{n} = \langle a,b,c \rangle$$

$$\Rightarrow -f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0) + z - f(x_0,y_0) = 0$$

$$\boxed{z - f(x_0,y_0) = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Def. ↑ is the scalar eq'n of the tan. plane to the surface $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$, if f_x & f_y both continuous at (x_0, y_0).

Linear Approximations

Close to (x_0, y_0, z_0) on surface $z = f(x,y)$, the tan. plane $z = f_x(x_0,y_0) + f_y(x_0,y_0) + z_0$

$$\approx f(x_0, y_0)$$

b/c plane is "close" to surface near (x_0, y_0, z_0)

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ex. $z = x^2 + y^2$, $(1,0,1)$ on surf.

$f_x = 2x \Rightarrow f_x(1,0) = 2$

$f_y = 2y \Rightarrow f_y(1,0) = 0$

tan. plane approx. near $(1,0,1)$:

$$z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$$
$$= 2(x-1) + 0(y-0) + 1$$
$$z = 2x - 1$$

Use $z = 2x-1$ to approx. $f(1,0)$

- $z = 2(1) - 1 = 1$ ✓
- $f(1,0) = 1^2 + 0^2 = 1$

same b/c used $(1,0,1)$ to build tan. plane

Near $(1,0)$, $f(x,y) \approx ?$

Def. The <u>linearization</u> of $f(x,y)$ at (a,b), assuming f_x & f_y cont., is

$$z = L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(solve for z)

<u>Recall</u> $y = f(x)$, if x changes from a to $a + \overset{h}{\Delta x}$, the <u>increment</u> of y is $\Delta y = f(a+\Delta x) - f(a)$, and $f'(a)$ exists if $\lim\limits_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a)$ exists.

$\lim \left[\dfrac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right] = 0$, say

$$\Rightarrow \lim_{\Delta x \to 0} \left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right] = 0, \quad \text{say}$$

$$\varepsilon = \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)$$

$$\Rightarrow \varepsilon \Delta x = f(a+\Delta x) - f(a) - f'(a)\Delta x$$
$$= \Delta y - f'(a)\Delta x$$

$$\Rightarrow \Delta y = f'(a)\Delta x + \varepsilon \Delta x, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0.$$

Def. 1) The increment of z is

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

2) If $z = f(x,y)$, say f is differentiable at (a,b) if Δz can be written

hard to use

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$, and $\varepsilon_2 \to 0$ as $\Delta y \to 0$.

Note The above def. says f differentiable at (a,b) if the tan. plane approximates the graph of f well near the point of tangency $(a, b, f(a,b))$.

Th'm If f_x & f_y continuous in a disk cent. at (a,b), then f differentiable at (a,b).

(i.e. f is C^1 near (a,b))

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Differentials

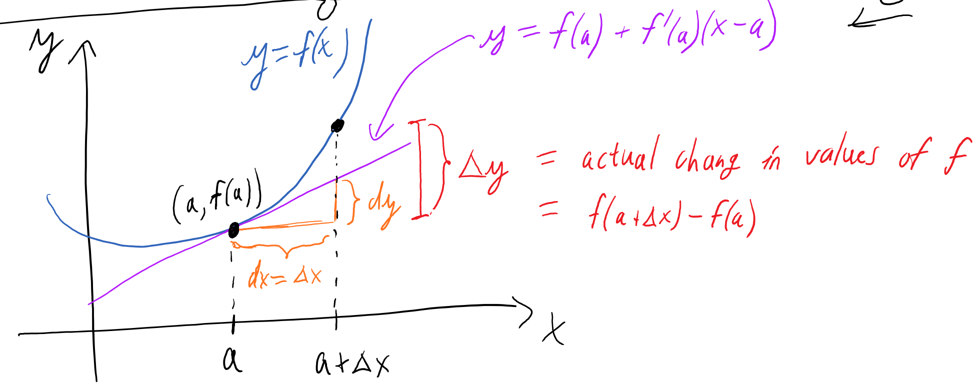
- For $y=f(x)$, let dx be an ind. var.
  Then $dy = f'(x)\,dx$
- For differentiable $z=f(x,y)$, let dx & dy ind. var.,
  and $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

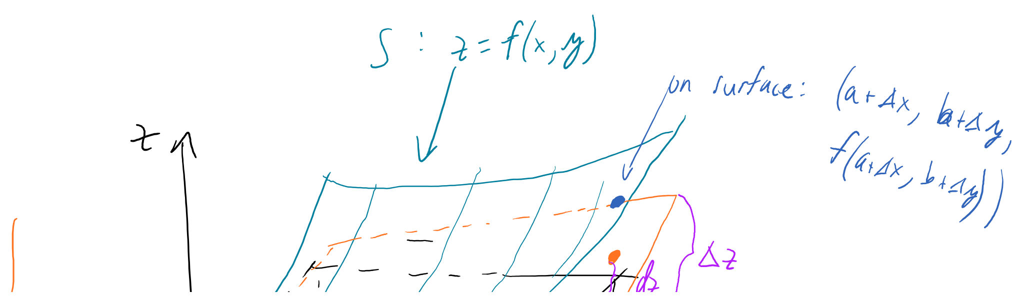
Def. dz is the (total) differential.

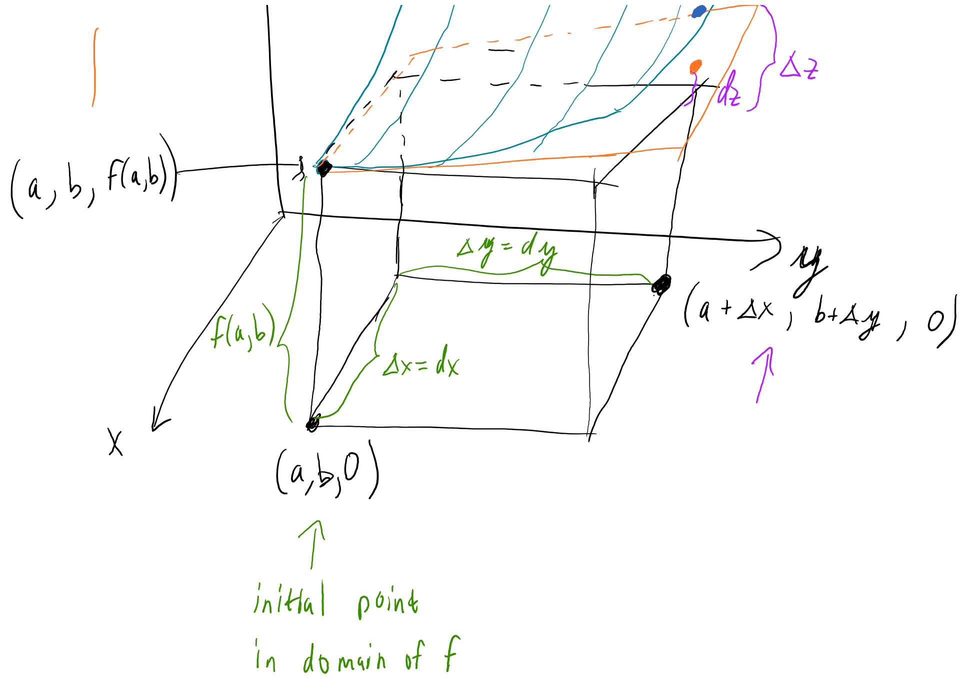
$$dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Aside If we take $dx = \Delta x = x-a$, and $dy = \Delta y = y-b$, have $dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$, so in differential notation, the linear approx. is

$$f(x,y) \approx f(a,b) + dz$$







Ex. $z = f(x, y) = x^2 + 3xy - y^2$

(a) Find dz

(b) If x changes from 2 to 2.05, and y changes from 3 to 2.96, compare Δz with dz.

Sol'n: (a) $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$

$$= (2x + 3y)dx + (3x - 2y)dy$$

(b) $x = 2, \quad \Delta x = 0.05 = dx$

$y = 3, \quad \Delta y = -0.04 = dy$

$$dz = [2(2) + 3(3)](0.05) + [3(2) - 2(3)](-0.04)$$

$$= \underline{\underline{0.65}}$$

$$\Delta z = f(a+\Delta x,\ b+\Delta y) - f(a,b)$$
$$= f(2.05,\ 2.96) - f(2,3)$$
$$= \left[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2\right] - \left[2^2 + 3(2)(3) - 3^2\right]$$
$$= \underline{0.6449}$$

$\therefore\ \Delta z \approx dz$, but dz easier to compute

ex. The radius & height of a right circular cone are measured as 10 cm and 25 cm resp., with a possible error in meas. of as much as 0.1 cm in each. Use differentials to estimate the max. error in the calculated volume of the cone.

Sol'n: $V = \frac{1}{3}\pi r^2 h$

$$dV = V_r\, dr + V_h\, dh$$
$$= \frac{\pi}{3} h(2r)\, dr + \frac{\pi}{3} r^2\, dh.$$

Note $|\Delta r| \leq 0.1 = dr$
$|\Delta h| \leq 0.1 = dh$

$$= \frac{\pi}{3}(25)(2\cdot 10)(0.1) + \frac{\pi}{3} 10^2(0.1)$$
$$= \boxed{20\pi\ cm^3}$$

Func. of ≥ 3 Var.

Linear approx. of $f(x,y,z)$ at (a,b,c) is

$$L(x,y,z) = f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) + f(a,b,c)$$

$\approx f(x, y, z)$

Def. The <u>increment</u> of $w = f(x,y,z)$ is

$$\Delta w = f(x+\Delta x,\ y+\Delta y,\ z+\Delta z) - f(x,y,z),$$ and

the <u>differential</u> of w is

$$dw = w_x\,dx + w_y\,dy + w_z\,dz.$$

Degree 2 Taylor Approximations

$f: D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^2$, f is $C^2(D)$, i.e.,

all 2nd partials of f are cont. on D, where $(a,b) \in D$.

For (x,y) near (a,b), $f(x,y) \approx P_2(x,y)$, where

$$P_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2 f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

ex. (mathinsight.org) Approximate f by P_2, where

$f(x,y) = e^{x+y}$ at $(0,0) = (a,b)$.

Sol'n:

$f_x = e^{x+y} = f_y$ $\qquad f(0,0) = 1$

$f_{xx} = e^{x+y} = f_{yy}$ $\qquad f_x(0,0) = 1 = f_y(0,0) = f_{xy}(0,0) = f_{yx}(0,0)$

$\qquad = f_{xx}(0,0) = f_{yy}(0,0)$

$f_{xy} = e^{x+y}$

$$\therefore P_2(x,y) = 1 + 1(x-0) + 1(y-0)$$

$$+ \frac{1}{2}\left[(x-0)^2 + 2(x-0)(y-0) + (y-0)^2\right]$$

$$\Rightarrow \boxed{e^{x+y} \approx 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 \quad \text{near } (0,0)}$$