4.5 The Dirac Delta Function

- *Definition:* The function

$$\delta_a(t-t_0)=\begin{cases}0, & 0\le t<t_0-a\\ \dfrac{1}{2a}, & t_0-a\le t<t_0+a,\\ 0, & t\ge t_0+a\end{cases}$$

Is called the **unit impulse function**.

The unit impulse function gives us a short "burst" on the interval of width a and has the property that $\int_0^\infty \delta_a(t-t_0)dt=1$. In practice, we often work with another impulse "function" where the width of the interval goes to zero.

- *Definition:* If we let $a\to 0$ in the unit impulse function, we get the **Dirac delta function** $\delta(t-t_0)$, defined by the properties

$$(i)\ \ \delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\ne t_0\end{cases}\qquad \text{and}\qquad \int_0^\infty \delta(t-t_0)dt=1.$$

**The Dirac delta function produces an "infinite" pulse at one instant.*

The Dirac delta function does not behave like our normal real-valued functions and it is usually quite helpful to think of the function based on how it effects other functions. If f is a continuous function, then

$$\int_0^\infty f(t)\ \delta(t-t_0)dt=f(t_0),$$

which is sometimes used as the definition of the Dirac delta function.

While the Dirac delta function is not a standard real-valued piecewise function, we can still find the Laplace transform of it.

Theorem: **Transform of the Dirac Delta Function**

For $t_0>0$, $$\boldsymbol{\mathcal{L}\{\delta(t-t_0)\}=e^{-st_0}}.$$

Proof: We start by expressing $\delta_a(t-t_0)$ in terms of the unit step function:

$$\delta_a(t-t_0)=\frac{1}{2a}\big[\mathcal{U}\big(t-(t_0-a)\big)-\mathcal{U}\big(t-(t_0+a)\big)\big].$$

By linearity of the Laplace transform,

$$\mathcal{L}\{\delta_a(t-t_0)\} = \frac{1}{2a}\left[\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s}\right] = e^{-st_0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right).$$

By using L'Hopital's Rule, we get that

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{a\to 0}\mathcal{L}\{\delta_a(t-t_0)\} = e^{-st_0}\lim_{a\to 0}\left(\frac{e^{sa}-e^{-sa}}{2sa}\right) = e^{-st_0}. \quad \blacksquare$$

**When $t_0 = 0$, we get that $\mathcal{L}\{\delta(t)\} = 1$. By an earlier theorem, for piece-wise continuous functions of exponential order, $F(s) \to 0$ as $a \to \infty$. This emphasizes that the Dirac delta function is not our typical type of function.*

- Example: Solve $y'' + y = 4\,\delta(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$.

  **This problem represents a spring/mass system with no damping, where the mass is released from rest at a point 1 unit below equilibrium and is given a sharp blow at $t = 2\pi$.*

  Solution: While we can get a solution to the associated homogeneous equation, our technique form the previous unit do not help us in finding a particular solution to the non-homogeneous case. However, we can find a solution using the Laplace transform. By applying the Laplace transform and solving for $Y(s)$, we get

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 4\mathcal{L}\{\,\delta(t-2\pi)\} \quad \to \quad s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s}$$

$$(s^2+1)Y(s) = s + 4e^{-2\pi s} \quad \to \quad Y(s) = \frac{s}{s^2+1} + \frac{4e^{-2\pi s}}{s^2+1}.$$

  Using the inverse transform, we get

$$\mathcal{L}^{-1}\{\mathrm{Y(s)}\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} \quad \to \quad y(t) = \cos t + 4\sin(t-2\pi)\,\mathcal{U}(t-2\pi).$$

  Using the fact that $\sin(t-2\pi) = \sin t$, the solution then becomes

$$\boldsymbol{f(t)} = \begin{cases} \boldsymbol{\cos t}, & 0 \le t < 2\boldsymbol{\pi} \\ \boldsymbol{\cos t + 4\sin t}, & t \ge 2\boldsymbol{\pi} \end{cases}.$$

  **Essentially the mass was experiencing simple harmonic motion until it was struck at $t = 2\pi$, where the impulse increased the amplitude of the vibration.*