3.5 Variation of Parameters

Variation of parameters works for second-order (and higher) differential equations in much the same way that it did for first-order differential equations.

First, we begin by putting our linear second-order differential equation into the standard form

$$y'' + P(X)y' + Q(x)y = f(x).$$

For a first-order differential equation, we used the assumption that $y_p = u(x)y_1(x)$. For a second-order differential equation, we use

$$y_p = u_1(x)y_1 + u_2(x)y_2,$$

where y_1 and y_2 form a fundamental set of solutions of the associated homogeneous equation. While we still have the drawback of needed to have solutions to the homogeneous case, we have seen that this is not a problem when we have constant coefficients (or for specific other cases we will see later). Therefore, we will assume that we are either given or can easily find y_1 and y_2.

Using the product rule to differentiate y_p, we get

$$y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$$

$$y_p'' = u_1y_1'' + 2u_1'y_1' + u_1''y_1 + u_2y_2'' + 2u_2'y_2' + u_2''y_2.$$

Substituting into the standard form, and grouping terms together, we get

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1[y_1'' + Py_1' + Qy_1] + u_2[y_2'' + Py_2' + Qy_2] + u_1''y_1 + u_1'y_1' + u_2''y_2 \\ &\quad + u_2'y_2' + P[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2' \\ &= 0 + 0 + \frac{d}{dx}[u_1'y_1] + \frac{d}{dx}[u_2'y_2] + P[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2' \\ &= \frac{d}{dx}[u_1'y_1 + u_2'y_2] + P[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2' = f(x). \end{aligned}$$

Now, to solve for the two unknowns, we need two equations. If we make the assumption that $u_1'y_1 + u_2'y_2 = 0$, then we are left with $u_1'y_1' + u_2'y_2' = f(x)$, giving us two equations. By Cramer's Rule, the solution to our system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(x) \end{aligned}$$

can be expressed in terms of determinants:

$$\boldsymbol{u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W}}, \qquad \boldsymbol{u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}},$$

where

$$\boldsymbol{W} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad \boldsymbol{W_1} = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad \boldsymbol{W_2} = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

The functions u_1 and u_2 are then found by integrating the results.

**Note: W is the Wronskian of y_1 and y_2, which are linearly independent solutions, so we do not run the risk of dividing by zero.*

While in the first-order case it was suggested to remember the process for variation of parameters instead of just the final formula, the process for second-order cases is too long and complicated to go through each time. For that reason, for second-order variation of parameter problems it is acceptable (and encouraged) to just use the resulting formulas for u_1' and u_2'.

- *Example:* Solve the following differential equation using variation by parameters.

$$y'' - 4y' + 4y = e^{2x}$$

Solution: We start by solving the associated homogeneous equation, and get the auxiliary equation

$$m^2 - 4m + 4 = 0 \quad \rightarrow \quad m_1 = m_2 = 2,$$

giving us a fundamental set of solutions $\{e^{2x},\ xe^{2x}\}$. Using variation of parameters, our particular solution will be of the form $y_p = u_1 e^{2x} + u_2 x e^{2x}$. Next we compute the Wronskian

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

By our formulas,

$$u_1' = -\frac{xe^{2x} \cdot e^{2x}}{e^{4x}} = -x \quad \rightarrow \quad u_1 = -\frac{1}{2}x^2$$

$$u_2' = \frac{e^{2x} \cdot e^{2x}}{e^{4x}} = 1 \quad \rightarrow \quad u_2 = x.$$

Hence,

$$y_p = -\frac{1}{2}x^2 e^{2x} + x^2 e^{2x} = \frac{1}{2}x^2 e^{2x}$$

and

$$\boldsymbol{y = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2}x^2 e^{2x}}.$$

**Note: we neglect the constants of integration when finding u_1 and u_2 (or set them equal to zero) since they would be absorbed into the complementary solution anyway.*

While variation of parameters will work for problems we looked at using undetermined coefficients, as shown above, it is usually better saved as an alternative when undetermined coefficients will not work. The process of finding W and integrating u_1' and u_2' can quite often lead to long and complicated calculations.

However, there are cases where variation of parameters does work well, and undetermined coefficients does not work at all.

- *Example:* Solve the following differential equation.

$$y'' + 9y = \frac{1}{4}\csc 3x$$

Solution: We start by solving the associated homogeneous equation, and get the auxiliary equation

$$m^2 + 9 = 0 \quad \to \quad m_1 = 3i, \;\; m_2 = -3i,$$

giving us a fundamental set of solutions $\{\cos 3x, \sin 3x\}$. Since undetermined coefficients will not work when $g(x) = \frac{1}{4}\csc 3x$, we must use variation of parameters in this case. Our particular solution will be of the form $y_p = u_1 \cos 3x + u_2 \sin 3x$. Next we compute the Wronskian

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3.$$

By our formulas,

$$u_1' = -\frac{\sin 3x \cdot \frac{1}{4}\csc 3x}{3} = -\frac{1}{12} \quad \to \quad u_1 = -\frac{1}{12}x$$

$$u_2' = \frac{\cos 3x \cdot \frac{1}{4}\csc 3x}{3} = \frac{1}{12}\frac{\cos 3x}{\sin 3x} \quad \to \quad u_2 = \frac{1}{36}\ln|\sin 3x|.$$

Hence,

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|,$$

and

$$\boldsymbol{y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln |\sin 3x|.}$$

- *Example:* Solve the following differential equation.

$$y'' - y = \frac{1}{x}$$

Solution: We start by solving the associated homogeneous equation, and get the auxiliary equation

$$m^2 - 1 = 0 \quad \rightarrow \quad m_1 = 1, \;\; m_2 = -1,$$

giving us a fundamental set of solutions $\{e^x, e^{-x}\}$. Since undetermined coefficients will not work when $g(x) = \frac{1}{x}$, we must use variation of parameters in this case. Our particular solution will be of the form $y_p = u_1 e^x + u_2 e^{-x}$. Next we compute the Wronskian

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

By our formulas,

$$u_1' = -\frac{e^{-x} \cdot \frac{1}{x}}{-2} = \frac{e^{-x}}{2x} \quad \rightarrow \quad u_1 = \frac{1}{2} \int \frac{e^{-x}}{x} dx$$

$$u_2' = \frac{e^x \cdot \frac{1}{x}}{-2} = -\frac{e^x}{2x} \quad \rightarrow \quad u_2 = -\frac{1}{2} \int \frac{e^x}{x} dx.$$

Since both integrals are non-elementary, we are forced to leave our answer in terms of definite integrals

$$y_p = \frac{1}{2} e^x \int_{x_0}^{x} \frac{e^{-t}}{t} dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t} dt,$$

and

$$\boldsymbol{y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} e^x \int_{x_0}^{x} \frac{e^{-t}}{t} dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t} dt.}$$

**Note: in our example, we can only integrate on an interval $[x_0, x]$ that does not contain the origin.*

*If you encounter a non-elementary integral, do not leave indefinite integrals in your final answer. Instead re-write the integrals as definite integrals as in the example above.

While variation of parameters will also work in third-order (or higher) cases, the work becomes very long and complicated, and it is not suggested that you tackle such problems without computer aided tools.