3.4 Undetermined Coefficients

In order to solve a non-homogeneous linear differential equation, we need to do two things:

- Find the complementary solution y_c.
- Find *any* particular solution of the non-homogeneous equation.

We saw how to find y_c when we had constant coefficients. Now we will explore how to find a particular solution for the non-homogeneous equation.

The Method of Undetermined Coefficients.

The main idea behind this approach is to take an educated guess as to the form of y_p based on $g(x)$ and work from there. However, this method is really limited to linear differential equations where

- the coefficients are constants
- $g(x)$ is a polynomial, exponential function, sine or cosine function, or a finite sum and product of these functions.

There are two different techniques to determine our guess for y_p: the annihilator approach and the superposition approach. We will start with the annihilator approach, and then build up the superposition approach from there.

Undetermined Coefficients - Annihilator Approach

If L is a linear differential operator with constant coefficients and f is a sufficiently differentiable function such that $L\big(f(x)\big) = 0$, then L is called an **annihilator** of the function f.

Using appropriate annihilators, we can transform certain non-homogeneous equations into homogeneous equations that we can solve. This is the main idea of the annihilator approach.

Below is a table of useful operators for common $g(x)$ functions.

Operator	**Annihilates...**
D^n	$1,\ x,\ x^2, \dots,\ x^{n-1}$
$(D-\alpha)^n$	$e^{\alpha x},\ xe^{\alpha x},\ x^2e^{\alpha x}, \dots,\ x^{n-1}e^{\alpha x}$
$[D^2 - 2\alpha D + (\alpha^2+\beta^2)]^n$	$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x, \dots, x^{n-1}\ e^{\alpha x}\cos\beta x$ $e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x, \dots, x^{n-1}\ e^{\alpha x}\sin\beta x$

Steps for Annihilator Approach

1. Find a fundamental set of solutions for the homogeneous equation $L(y) = 0$.
2. Determine the appropriate differential operator L_1 that annihilates $g(x)$.
3. Apply L_1 to both sides of the non-homogeneous equation $L(y) = g(x)$.
4. Find the general solution to the homogeneous differential equation $L_1\{L(y)\} = 0$.
5. Delete the "repeated" terms from the fundamental set of solutions from the first step.
6. Form a linear combination of the remaining terms to create y_p.
7. Substitute y_p into $L(y) = g(x)$ and solve for the unknown coefficients.
8. The general solution is $y = y_c + y_p$.

**When applying the annihilator, it is common practice to use D for our derivatives. This allows to easier simplification.*

- *Example:* Solve the following non-homogeneous equation using the annihilator approach.

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$$

Solution: First we must solve $y'' + 3y' + 2y = 0$. The auxiliary equation gives us

$$m^2 + 3m + 2 = 0 \quad \rightarrow \quad m_1 = -1,\ \ m_2 = -2 \quad \rightarrow \quad y_c = c_1e^{-x} + c_2e^{-2x}.$$

Next we find the appropriate annihilator for $g(x) = 4x^2$. In this case, we would use $L_1 = D^3$. Applying L_1 we get

$$D^3(D^2y + 3Dy + 2y) = D^3(4x^2) \quad \rightarrow \quad (D^5 + 3D^4 + 2D^3)y = 0.$$

The corresponding auxiliary equation would give

$$m^5 + 3m^4 + 2m^3 = 0 \quad \rightarrow \quad m_1 = m_2 = m_3 = 0,\ \ m_4 = -1,\ \ m_5 = -2.$$

This would give us a general solution of

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^{-2x}.$$

Eliminating the pieces that were in y_c we are left with

$$y_p = A + Bx + Cx^2.$$

By substitution

$$\begin{aligned} L(y_p) &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= 2Cx^2 + (2B + 6C)x + (2A + 3B + 2C). \end{aligned}$$

Since $L(y_p) = g(x)$, we get that

$$2Cx^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2,$$

or

$$2C = 4, \qquad 2B + 6C = 0, \qquad 2A + 3B + 2C = 0 \quad \rightarrow \quad A = 7,\ \ B = -6,\ \ C = 2.$$

Therefore, $y_p = 7 - 6x + 2x^2$, and we get a general solution of

$$\boldsymbol{y = c_1 e^{-x} + c_2 e^{-2x} + 7 - 6x + 2x^2}.$$

- *Example:* Solve the following non-homogeneous equation using the annihilator approach.

$$y'' - 3y' = 8e^{3x} + 4\sin x$$

Solution: First we must solve $y'' - 3y' = 0$. The auxiliary equation gives us

$$m^2 - 3m = 0 \quad \rightarrow \quad m_1 = 0,\ \ m_2 = 3 \quad \rightarrow \quad y_c = c_1 + c_2 e^{3x}.$$

Next we find the appropriate annihilator for $g(x) = 8e^{3x} + 4\sin x$. In this case, since $(D-3)$ annihilates e^{3x} and (D^2+1) annihilates $4\sin x$, we would use $L_1 = (D-3)(D^2+1)$. Applying L_1 we get

$$(D-3)(D^2+1)(D^2y - 3Dy) = (D-3)(D^2+1)(8e^{3x} + 4\sin x)$$

$$D(D-3)^2(D^2+1)y = 0.$$

The corresponding auxiliary equation would give

$$m(m-3)^2(m^2+1) = 0 \quad \rightarrow \quad m_1 = 0,\ \ m_2 = m_3 = 3,\ \ m_4 = i,\ \ m_5 = -i.$$

This would give us a general solution of

$$y = c_1 + c_2 e^{3x} + c_3 x e^{3x} + c_4 \cos x + c_5 \sin x.$$

Eliminating the pieces that were in y_c we are left with

$$y_p = Axe^{3x} + B\cos x + C\sin x.$$

Since m_3 was a repeated root, we do need xe^{3x} in y_p. Taking derivatives we get

$$y_p' = 3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x$$

$$y_p'' = 9Axe^{3x} + 6Ae^{3x} - B\cos x - C\sin x.$$

By substitution

$$(9Axe^{3x} + 6Ae^{3x} - B\cos x - C\sin x) - 3(3Axe^{3x} + Ae^{3x} - B\sin x + C\cos x)$$
$$= 3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x.$$

Since $L(y_p) = g(x)$, we get that

$$3Ae^{3x} + (-B - 3C)\cos x + (3B - C)\sin x = 8e^{3x} + 4\sin x,$$

or

$$3A = 8, \quad -B - 3C = 0, \qquad 3B - C = 4 \quad \rightarrow \quad A = \frac{8}{3}, \; B = \frac{6}{5}, \; C = -\frac{2}{5}.$$

Therefore,

$$y_p = \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x,$$

and we get a general solution of

$$\boldsymbol{y = c_1 + c_2e^{3x} + \frac{8}{3}xe^{3x} + \frac{6}{5}\cos x - \frac{2}{5}\sin x.}$$

Undetermined Coefficients - Superposition Approach

The annihilator approach will work for any $g(x)$ that is a polynomial, exponential function, sine or cosine function, or a finite sum and product of these functions. However, as we use the annihilator approach more and more, we beginning to notice some patterns in the form of y_p, which leads us to the superposition approach.

Rather than going through the process of the annihilator approach, with the superposition approach we jump straight into a "guess" for y_p. To obtain our guess, we first break down g into the sum of parts

$$g(x) = g_1(x) + g_2(x) + \cdots + g_k(x).$$

We then find an appropriate guesses $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ for each part. Then by the superposition principle, $y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$.

Below is a table of common choices for y_p.

$\boldsymbol{g(x)}$	**Form of $\boldsymbol{y_p}$**
$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
$\sin kx, \quad \cos kx$	$A\cos kx + B\sin kx$

e^{kx}	Ae^{kx}
$p(x)\sin kx\,,\ \ p(x)\cos kx$	$(A_n x^n + \cdots + A_0)(B\ \cos kx + C\sin kx)$
$p(x)e^{kx}$	$(A_n x^n + \cdots + A_0)e^{kx}$
$e^{kx}\sin\beta x\,,\ \ e^{kx}\cos\beta x$	$(A\cos\beta x + B\sin\beta x)e^{kx}$

Steps for Superposition Approach

1. Find a fundamental set of solutions for the homogeneous equation $L(y) = 0$.
2. Determine the appropriate guess for y_p.
3. Adjust y_p as needed.
4. Substitute y_p into $L(y) = g(x)$ and solve for the unknown coefficients.
5. The general solution is $y = y_c + y_p$.

*While the steps for the superposition approach may make it seem like an easier process, we need to be extra careful about overlap in y_c and y_p, as we will see.

- *Example:* Solve the following non-homogeneous equation using the superposition approach.

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

Solution: First we must solve $y'' - 2y' - 3y = 0$. The auxiliary equation gives us

$$m^2 - 2m - 3 = 0 \quad \rightarrow \quad m_1 = -1,\ \ m_2 = 3,$$

giving us a fundamental set of solutions of $\{e^{-x},\ e^{3x}\}$. Next we find the appropriate guess for y_p. In this case, for $4x - 5$ we would include $Ax + B$ in y_p and for $6xe^{2x}$ we would include $(Cx + E)e^{2x}$ in y_p. Since neither of these pieces overlaps with our fundamental set, $y_p = Ax + B + (Cx + E)e^{2x}$. Taking derivatives we get

$$y_p' = A + Ce^{2x} + 2(Cx + E)e^{2x} = A + 2Cxe^{2x} + (C + 2E)e^{2x}$$

$$y_p'' = 4xCe^{2x} + 2Ce^{2x} + 2(C + 2E)e^{2x} = 4xCe^{2x} + 4(C + E)e^{2x}.$$

By substitution

$$-3Ax + (-2A - 3B) - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x},$$

giving us the system

$$-3A = 4, \quad -2A - 3B = -5, \quad -3C = 6, \ \ 2C - 3E = 0$$

$$A = -\frac{4}{3}, \;\; B = \frac{23}{9}, \;\; C = -2, \;\; E = -\frac{4}{3}.$$

Therefore,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}$$

and we get the general solution

$$\boldsymbol{y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}}.$$

**Note: since D represents the differential operator, it is common practice to skip D when assigning undetermined coefficients.*

When using the superposition approach, we do still need to find y_c first so that we can make sure that our guess for y_p does not share any common terms with our fundamental set of solutions. If there are any repeated terms, we must multiply that (entire) portion of y_p by x^k, where k is a large enough exponent to get rid of any overlap. This process will be highlighted in the next example.

- *Example:* Solve the following non-homogeneous equation using the superposition approach.

$$y'' - 5y' + 4y = 8e^x$$

Solution: First we must solve $y'' - 5y' + 4y = 0$. The auxiliary equation gives us

$$m^2 - 5m + 4 = 0 \quad \rightarrow \quad m_1 = 1, \;\; m_2 = 4,$$

giving us a fundamental set of solutions of $\{e^x, \; e^{4x}\}$. Next we find the appropriate guess for y_p. In this case, for $8e^x$ we would use Ae^x. However, since e^x overlaps with our fundamental set, we need to adjust our guess by multiplying by powers of x to make it linearly independent. Therefore, we should use $y_p = Axe^x$ instead. Taking derivatives we get

$$y_p' = Axe^x + Ae^x, \qquad y_p'' = Axe^x + 2Ae^x.$$

By substitution

$$-3Ae^x = 8e^x \quad \rightarrow \quad A = -\frac{8}{3}.$$

Therefore,

$$y_p = -\frac{8}{3}xe^x$$

and we get the general solution

$$\boldsymbol{y = c_1e^x + c_2e^{4x} - \frac{8}{3}xe^x}.$$

**Note:* *if we had used $y_p = Ae^x$, after substituting in we would have been left with $0 = 8e^x$, a contradiction.*

This process of adjusting y_p sometimes does require multiplying by larger powers of x.

- *Example:* The equation $y'' - 2y' + y = 0$ gives us a fundamental set of $\{e^x, xe^x\}$. Therefore, if we were attempting to find a y_p for $y'' - 2y' + y = e^x$, our normal choice of $y_p = Ae^x$ would need to be multiplied by x^2 before we no longer had any overlap. Therefore, $\boldsymbol{y_p = Ax^2e^x}$.

- *Example:* The equation $y^{(4)} + y''' = 0$ gives us a fundamental set of $\{1, x, x^2, e^{-x}\}$. If we were attempting to find a y_p for $y^{(4)} + y''' = 1 - x^2e^{-x}$, our normal choice would be

  $$y_{p_1} = A, \qquad y_{p_2} = (Bx^2 + Cx + E)e^{-x}$$

  for

  $$g_1(x) = 1, \qquad g_2(x) = -x^2e^{-x},$$

  respectively. However, $1, x$, and x^2 are in the fundamental set above, y_{p_1} needs to be multiplied by x^3, giving $y_{p_1} = Ax^3$. Similarly, since e^{-x} is in the fundamental set above, y_{p_2} needs to be multiplied by x, giving $y_{p_2} = (Bx^3 + Cx^2 + Ex)e^{-x}$. Therefore,

  $$\boldsymbol{y_p = Ax^3 + (Bx^3 + Cx^2 + Ex)e^{-x}}.$$