4.2 Inverse Transforms

- *Definition:* If $F(s)$ represents the Laplace transform of a function $f(t)$, we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

**Note: it is not hard to show that $\mathcal{L}^{-1}$ is also a linear operator.*

When evaluating inverse transforms, we do not have an explicit formula to follow, like we do for Laplace transforms. Instead, we usually try to "line up" our inverse transform from the table of transforms. It also often occurs that a function of s does not exactly match the form of a Laplace transform given by our table. In those cases it may be necessary to "fix up" the function of s before applying the inverse transform.

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$$

Solution: From the table of transforms, we can see that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

Since we are looking for $\mathcal{L}^{-1}\{1/s^4\}$, we need to "adjust" our function to make it line up. Using the property of linearity, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \boldsymbol{\frac{1}{6}t^3}.$$

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\}$$

Solution: From the table of transforms, we can see that

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \qquad \text{or} \qquad \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt.$$

Once again we need to "adjust" our function to make it line up. Using the property of linearity, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+5}\right\} = \frac{1}{\sqrt{5}}\mathcal{L}^{-1}\left\{\frac{\sqrt{5}}{s^2+5}\right\} = \boldsymbol{\frac{1}{\sqrt{5}}\sin\sqrt{5}t}.$$

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{-2s+3}{s^2+4}\right\}$$

Solution: From the table of transforms, we can see that

$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}=\sin kt \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}=\cos kt.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{-2s+3}{s^2+4}\right\}=-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}=\mathbf{-2\cos 2\boldsymbol{t}+\frac{3}{2}\sin 2\boldsymbol{t}}.$$

- *Example:* Evaluate

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$

Solution: To begin this problem, we need to break apart the fractions. By partial fraction decomposition we get

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}=\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+4} \quad \to \quad A=-\frac{16}{5},\ B=\frac{25}{6},\ C=\frac{1}{30}.$$

Therefore, we have

$$-\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}=-\boldsymbol{\frac{16}{5}e^{t}+\frac{25}{6}e^{2t}+\frac{1}{30}e^{-4t}}.$$

**If you do not remember the process of partial fraction decomposition, please consult your Calculus II text/notes.*

Theorem: **Transform of a Derivative**

If $f, f', f'', \dots, f^{(n-1)}$ are continuous on $[0,\infty)$ and of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$\mathcal{L}\{f^{(n)}(t)\}=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0),$$

where $F(s)=\mathcal{L}\{f(t)\}$.

Proof (by induction): For $n=1$,

$$\mathcal{L}\{f'(t)\}=\int_0^\infty e^{-st}f'(t)dt=e^{-st}f(t)|_0^\infty+s\int_0^\infty e^{-st}f(t)dt=-f(0)+s\mathcal{L}\{f(t)\}.$$

Thus,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Assume that the theorem holds for $n = k$, then

$$\mathcal{L}\left\{f^{(k+1)}(t)\right\} = \int_0^\infty e^{-st} f^{(k+1)}(t)dt = e^{-st} f^{(k)}(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f^{(k)}(t)dt$$
$$= -f^{(k)}(0) + s\mathcal{L}\left\{f^{(k)}(t)\right\}.$$

But since the theorem is true for $n = k$, we get

$$\mathcal{L}\left\{f^{(k+1)}(t)\right\} = s\left[s^k F(s) - s^{k-1} f(0) - s^{k-2} f'(0) - \cdots - f^{(k-1)}(0)\right] - f^{(k)}(0)$$

or

$$\mathcal{L}\left\{f^{(k+1)}(t)\right\} = s^{k+1} F(s) - s^k f(0) - s^{k-1} f'(0) - \cdots - s f^{(k-1)}(0) - f^{(k)}(0).$$

Therefore, by mathematical induction, the theorem holds for all values of n and

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \blacksquare$$

By using the above theorem, and applying the linearity of the Laplace transform, we can use the Laplace transform to solve linear initial-value problems in which the differential equation has constant coefficients.

**Note: if we do not have constant coefficients,* $\mathcal{L}\left\{a_n(t) f^{(n)}(t)\right\}$ *can be very complicated to work with.*

Solving a Linear IVP with the Laplace Transform

Apply the Laplace transform $\mathcal{L}$ to the linear differential equation to get an equation in $Y(s)$.

Solve the transformed equation for $Y(s)$.

Applying the inverse transform $\mathcal{L}^{-1}$ to the equation solved for $Y(s)$.

- *Example:* Solve

$$\frac{dy}{dt} + 3y = 13\sin 2t, \qquad y(0) = 6.$$

Solution: First we apply $\mathcal{L}$ to the differential equation, getting

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 13\mathcal{L}\{\sin 2t\} \quad \to \quad sY(s) - y(0) + 3Y(s) = 13 \cdot \frac{2}{s^2+4}.$$

Solving for $Y(s)$, we get

$$(s+3)Y(s)=6+\frac{26}{s^2+4} \quad \rightarrow \quad Y(s)=\frac{6}{s+3}+\frac{26}{(s+3)(s^2+4)}=\frac{6s^2+50}{(s+3)(S^2+4)}.$$

Using partial fraction decomposition, we get

$$Y(s)=\frac{8}{s+3}-\frac{2s}{s^2+4}+\frac{6}{s^2+4}.$$

Last, we need to apply $\mathcal{L}^{-1}$ to the equation, getting

$$\mathcal{L}^{-1}\{Y(s)\}=8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}-2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\},$$

or

$$\boldsymbol{y(t)=8e^{-3t}-2\cos 2t+3\sin 2t.}$$

- *Example:* Solve

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=e^{-4t}, \qquad y(0)=1, \qquad y'(0)=2$$

Solution: First we apply $\mathcal{L}$ to the left-hand side of the differential equation, getting

$$\begin{aligned}\mathcal{L}\{y''\}-3\mathcal{L}\{y'\}+2\mathcal{L}\{y\}&=s^2Y(s)-sy(0)-y'(0)-3Y(s)+3y(0)+2Y\\&=(s^2-3s+2)Y-s+1=(s-2)(s-1)Y-(s-1).\end{aligned}$$

Applying $\mathcal{L}$ to the right-hand side of the differential equation we get

$$\mathcal{L}\{e^{-4t}\}=\frac{1}{s+4}.$$

Solving for $Y(s)$, we get

$$Y(s)=\frac{s-1}{(s-1)(s-2)}+\frac{1}{(s+4)(s-1)(s-2)}=\frac{s^2+3s-3}{(s+4)(s-1)(s-2)}.$$

Using partial fraction decomposition, we get

$$Y(s)=\frac{1/30}{s+4}-\frac{1/5}{s-1}+\frac{7/6}{s-2}.$$

Last, we need to apply $\mathcal{L}^{-1}$ to the equation, getting

$$\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}-\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{7}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\},$$

or

$$\boldsymbol{y(t)=\frac{1}{30}e^{-4t}-\frac{1}{5}e^{t}+\frac{7}{6}e^{2t}.}$$

While both examples could have been solved using techniques from previous sections, such as undetermined coefficients, we will see problems where the Laplace transform is the better choice. Also, when solving an IVP, it was necessary to take derivatives and evaluate our solution in order to solve for the constants involved in y_c. Applying the Laplace transform to solve an IVP solves for these constants directly, so we do not need to go through the process of taking derivatives and evaluating. This can be very useful for higher-order initial-value problems, where we may normally need to take 4 or 5 derivatives to solve for the constants.