Unit 1.4: Trigonometric Integrals

This unit will make significant use of trig functions and several trig identities. Below is a list of trig identities that you will be expected to have memorized and use at any time throughout the remainder of the course. These identities along with others can be found on the inside of the cover of our textbook.

- Pythagorean Identities: $\sin^2 u + \cos^2 u = 1$ $\quad\quad$ $\tan^2 u + 1 = \sec^2 u$
- Power Reduction: $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$ $\quad\quad$ $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$

Integrals Involving Sine and/or Cosine. Let us first consider integrals that have an odd power of sine or cosine. In these cases we will use a u-substition. Whichever function has the odd power (the sine or the cosine), we will let u equal the other function. To make this more clear, let's look at a simple example.

Example 1 Determine the integral $\int \sin^6 x \cos^3 x\, dx$

Solution: We note here that $\cos x$ is being raised to an odd power. What we will do is take one factor of $\cos x$ and "put it with the dx", leaving an even power of cosine which will be ideal for using the first Pythagorean Identity above. The $\cos x$ being with the dx should motivate the choice of $u = \sin x$. Here's what it looks like.

$$\int \sin^6 x \cos^3 x\, dx = \int \sin^6 x \underbrace{\cos^2 x}_{1-\sin^2 x} \underbrace{\cos x\, dx}_{\text{for } du} = \int \sin^6 x\,(1 - \sin^2 x) \cos x\, dx$$

We now let $u = \sin x$, so that $du = \cos x\, dx$. Substituting into the last integral above yields

$$\int u^6(1 - u^2)du = \int (u^6 - u^8)\, du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \boxed{\frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C}$$

One could factor out $\frac{1}{63}\sin^7 x$ in the answer above, but I will refrain from doing that here. In examples like the one above, we look for an odd power of sine or cosine so that when we take one of the factors and put it with the dx, it leaves us with an even power, thus allowing us to use the Pythagorean Identity. Even if it left us with $\cos^6 x$, we can use the identity by observing that $\cos^6 x = (\cos^2 x)^3 = (1 - \sin^2 x)^3$. If we are going to let u equal a sine function, then we want to write the integral all in terms of the sine function and let the du take up the one factor of cosine that was put with dx. A similar approach would be taken if the sine function were raised to an even power.

Now we look at integrals that only have even powers of sines and/or cosines. In these situations we will use the power reduction identities to reduce all powers down to one.

Example 2 Determine the integral $\int \cos^2(\pi x)\, dx$.

Solution: Because we only have even powers of our trig function, we will use the power reduction identity: $\cos^2 u = \frac{1}{2}(1 + \cos 2u)$, which in our case gives $\cos^2(\pi x) = \frac{1}{2}(1 + \cos(2\pi x))$. Thus we simply do a rewrite on our integral to obtain

$$\int \cos^2(\pi x)\, dx = \frac{1}{2}\int [1 + \cos(2\pi x)] dx = \boxed{\frac{1}{2}\left[x + \frac{1}{2\pi}\sin(2\pi x)\right] + C}$$

It is important that you look at more examples than the one above to see how to deal with powers higher than 2 and examples where both the sine and cosine functions have even powers. You can find examples of such problems in the next and in the narrated examples.

Because of the various trig identities, you should be aware that sometimes you can obtain a significantly different looking answer to these problems and it still may be correct. Keep that in mind when you look at the answers in the back of the book and/or solutions manual. There are also cases where you might have two options for a u-sub, that both work.

Integrals Involving Secant and/or Tangent. There is a little bit more variety that can occur with secants and tangents, but the primary examples that we will consider will involve us making a substitution where u is a secant function or a tangent function and thus du will have to contain their derivatives. Let us recall the following derivative rules:

$$\frac{d}{dx}\tan x = \sec^2 x \qquad\qquad \frac{d}{dx}\sec x = \sec x \tan x$$

Based on the above, we will want to let u equal a tangent function if we can put two factors of the corresponding secant function with the dx and additionally leave only an even power of secant so that we can use a Pythagorean Identity to express the integral in terms of tangent alone. Similarly, we will let u equal a secant function if we can put one factor each of the corresponding secant and tangent functions with the dx and additionally leave only an even power of tangent again allowing us to use a Pythagorean Identity to express the integral in terms of only the secant function. If neither of these options work, one then might consider integration by parts, or rewriting the integral in terms of sines and/or cosines and manipulating the expressions into a convenient form. These kinds of examples are often difficult to solve without a decent amount of time allowed for trial-and-error, so they do not make good test questions. Nonetheless, they are important for you to encounter and you may see some in the homework.

Example 3 Determine the integral $\int \tan^5 x \sec^7 x \, dx$.

Solution: We have plenty of factors of both $\tan x$ and $\sec x$ to allow us an attempt of letting $u = \tan x$ or $u = \sec x$. Splitting off the appropriate factors for du would present us with the following options

$$\int \tan^5 x \sec^5 x \underbrace{\sec^2 x \, dx}_{\text{for } du??} \qquad or \qquad \int \tan^4 x \sec^6 x \underbrace{\sec x \tan x \, dx}_{\text{for } du??}$$

The first option is not promising as having $\sec^2 x$ with the dx should motivate us to have $u = \tan x$, requiring us to eliminate the other secant functions from the integral (namely $\sec^5 x$) by way of our identities. But the remaining secant function is to an odd power, thus preventing us from our goal. Luckily the second option will work. In the second option, having the $\sec x \, tanx$ with the dx motivates us to have $u = \sec x$. With $u = \sec x$, we wish to eliminate all other tangent functions from the integral by expressing them in terms of secants using the identities. To do this we need an even power of tangent left over, which fortunately we do in this case as we have our tangent function to the 4th power. The identity allows us to write $\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2$. We proceed as follows:

$$\int \tan^5 x \sec^7 x \, dx = \int \tan^4 x \sec^6 x \underbrace{\sec x \tan x \, dx}_{\text{for } du} = \int \underbrace{(\sec^2 x - 1)^2}_{\text{identity}} \sec^6 x \underbrace{\sec x \tan x \, dx}_{\text{for } du}$$

Let $u = \sec x$, so that $du = \sec x \tan x \, dx$. Substituting into the last integral above yields

$$\int \underbrace{(u^2 - 1)^2}_{\text{expand}} u^6 du = \int \underbrace{(u^4 - 2u^2 + 1)u^6}_{\text{distribute}} du = \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{1}{11}u^{11} - \frac{2}{9}u^9 + \frac{1}{7}u^7 + C = \boxed{\frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C}$$

A Special Integral to Know. In this section we will complete our list of integrals that must be memorized for the entire course. The final integral of a basic function that we will examine is that of the secant function. How we obtain the result will not be based on the most intuitive step, but we can be thankful for the creativity and insight provided by those who came before us. Let us consider the integral $\int \sec x \, dx$. To find the antiderivative, we multiply our integrand by 1 in a creative way as follows

$$\int \sec x \, dx = \int \sec x \cdot \underbrace{\frac{\sec x + \tan x}{\sec x + \tan x}}_{=1} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

We should now recognize that the numerator is the derivative of the denominator. That means we are going to get a log function out of this. We proceed with the substitution $u = \sec x + \tan x \;\; \rightarrow \;\; du = (\sec x \tan x + \sec^2 x)dx$, which yields

$$\int \frac{1}{u} du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

We conclude by taking a look at our now complete list of basic integration rules for this course.

Below n represents a real number, a represents a positive real number, and C is an arbitrary constant (or parameter) sometimes called the constant of integration.

1) $\int u^n du = \frac{u^{n+1}}{n+1} + C$ (when $n \neq -1$)

2) $\int u^{-1} du = \int \frac{1}{u}\, du = \ln|u| + C$

3) $\int e^u du = e^u + C$

4) $\int a^u du = \frac{1}{\ln a} a^u + C \quad (a \neq 1)$

5) $\int \cos u \, du = \sin u + C$

6) $\int \sin u \, du = -\cos u + C$

7) $\int \sec u \tan u \, du = \sec u + C$

8) $\int \sec^2 u \, du = \tan u + C$

9) $\int \csc u \cot u \, du = -\csc u + C$

10) $\int \csc^2 u \, du = -\cot u + C$

11) $\int \tan u \, du = -\ln|\cos u| + C$

12) $\int \cot u \, du = \ln|\sin u| + C$

13) $\int \sec u \, du = \ln|\sec u + \tan u| + C$

14) $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

15) $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C$

16) $\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$