5.2 Homogeneous Linear Systems

When looking for solutions for a homogeneous linear system with constant coefficients, we use a similar technique as we did for general homogeneous linear equations with constant coefficients. Namely, we try to find solution vectors of the form

$$\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K}e^{\lambda t}.$$

If $\boldsymbol{X}$, as above, is a solution vector of $\mathbf{X}' = \mathbf{AX}$**,** then $\mathbf{X}' = \mathbf{K}\lambda e^{\lambda t}$, giving us

$$\mathbf{K}\lambda e^{\lambda t} = \mathbf{AK}e^{\lambda t} \quad \rightarrow \quad \lambda\mathbf{K} = \mathbf{AK} \quad \rightarrow \quad \mathbf{AK} - \lambda\mathbf{K} = \mathbf{0}.$$

Since $\mathbf{K} = \mathbf{IK}$, this last equation is the same as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = \mathbf{0}.$$

In order for this system to have a non-trivial solution (i.e. $\mathbf{K} \neq \mathbf{0}$), we must have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

This polynomial equation in λ is called the **characteristic equation** of the matrix $\mathbf{A}$; its solutions are the **eigenvalues** of $\mathbf{A}$; a solution $\mathbf{K} \neq \mathbf{0}$ corresponding to an eigenvalue λ is called an **eigenvector** of $\mathbf{A}$. A solution to the homogeneous system is then $\mathbf{X} = \mathbf{K}e^{\lambda t}$. Much like in previous work, there are three cases to discuss: real and distinct eigenvalues, repeated eigenvalues, and complex eigenvalues.

- **Distinct Real Eigenvalues:**

<u>Theorem:</u> **General Solution - Homogeneous System (Distinct Eigenvalues)**

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous linear system, and let $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_n$ be the corresponding eigenvectors. Then the general solution of the homogeneous system on the interval $(-\infty, \infty)$ is

$$\mathbf{X} = c_1\mathbf{K_1}e^{\lambda_1 t} + c_2\,\mathbf{K_2}e^{\lambda_2 t} + \cdots + c_n\mathbf{K_n}e^{\lambda_n t}.$$

- *Example:* Solve

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = 2x + y \ .$$

Solution: From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) = 0$$

we get eigenvalues of $\lambda_1 = -1$ and $\lambda_2 = 4$.

For $\lambda_1 = -1$, we get

$$(2+1)k_1 + 3k_2 = 0$$
$$2k_1 + (1+1)k_2 = 0.$$

Solving the system, we get that $k_1 = -k_2$. If we let $k_2 = -1$, we get the eigenvector

$$\mathbf{K_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 4$, we get

$$(2-4)k_1 + 3k_2 = 0$$
$$2k_1 + (1-4)k_2 = 0.$$

Solving the system, we get that $k_1 = \frac{3}{2}k_2$. If we let $k_2 = 2$, we get the eigenvector

$$\mathbf{K_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

While this solution is equivalent to the solution

$$x(t) = c_1 e^{-t} + 3c_2 e^{4t}, \qquad y(t) = -c_1 e^{-t} + 2c_2 e^{4t},$$

we tend to leave our solutions in matrix form, which gives us a more condensed form.

**When solving for the eigenvectors, we will always get a free variable, which we can let be any non-zero number. This means that the eigenvector for a specific eigenvalue is not unique. However, c_1 and c_2 in the general solution will "absorb" any difference that may come from our choice of eigenvector.*

When dealing with a system of more than 2 equations, it is typical to use Gauss-Jordan elimination to find eigenvectors.

- *Example:* Solve

$$\begin{aligned} x' &= -4x + y + z \\ y' &= x + 5y - z \; . \\ z' &= y - 3z \end{aligned}$$

Solution: From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = -(\lambda+3)(\lambda+4)(\lambda-5) = 0$$

we get eigenvalues of $\lambda_1 = -3, \lambda_2 = -4$, and $\lambda_3 = 5$.

For $\lambda_1 = -3$, we get

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - k_3 = 0$ and $k_2 = 0$. If we let $k_3 = 1$, then we get the eigenvector

$$\mathbf{K_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda_2 = -4$, we get

$$(\mathbf{A} + 4\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -10 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - 10k_3 = 0$ and $k_2 + k_3 = 0$. If we let $k_3 = 1$, then we get the eigenvector

$$\mathbf{K_2} = \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix}.$$

Last, for $\lambda_3 = 5$, we get

$$(\mathbf{A} - 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -9 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - k_3 = 0$ and $k_2 - 8k_3 = 0$. If we let $k_3 = 1$, then we get the eigenvector

$$\mathbf{K_3} = \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 10 \\ -1 \\ 1 \end{pmatrix} e^{-4t} + c_3 \begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix} e^{5t}.$$

- **Repeated Eigenvalues:**

If m is a positive integer and $(\lambda - \lambda_1)^m$ is a factor of the characteristic equation, but $(\lambda - \lambda_1)^{m+1}$ is not, then λ_1 is said to be an eigenvalue of multiplicity m. When this happens, we have two cases.

Case I: We can find m linearly independent eigenvectors $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_m$ corresponding to our eigenvalue. In this case our general solution will contain

$$c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_1 t} + \cdots + c_m\mathbf{K}_m e^{\lambda_1 t}.$$

Case II: We can only find one eigenvector corresponding to our eigenvalue. Then we can find m linearly independent solutions of the form

$$\begin{aligned} \mathbf{X_1} &= \mathbf{K}_{11} e^{\lambda_1 t} \\ \mathbf{X}_2 &= \mathbf{K}_{21} t e^{\lambda_1 t} + \mathbf{K_{22}} e^{\lambda_1 t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + \mathbf{K}_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + \mathbf{K}_{mm} e^{\lambda_1 t}. \end{aligned}$$

Looking at a basic case, suppose that λ_1 is an eigenvector of multiplicity two and that there is only one eigenvector associated. Then a second solution of the form

$$\mathbf{X}_2 = \mathbf{K} t e^{\lambda_1 t} + \mathbf{P} e^{\lambda_1 t}.$$

To find $\mathbf{K}$ and $\mathbf{P}$, we substitute this solution into our system and simplify to get

$$(\mathbf{AK} - \lambda_1 \mathbf{I}) t e^{\lambda_1 t} + (\mathbf{AP} - \lambda_1 \mathbf{P} - \mathbf{K}) e^{\lambda_1 t} = \mathbf{0}.$$

Since this holds for all values of t, we must have that

$$(\mathbf{AK} - \lambda_1 \mathbf{I}) = \mathbf{0}, \qquad (\mathbf{AP} - \lambda_1 \mathbf{P}) = \mathbf{K}.$$

**The first equation states that $\mathbf{K}$ must be an eigenvector associated with λ_1, so we only need to solve the second equation for $\mathbf{P}$.*

- *Example:* Solve

$$\mathbf{X}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\mathbf{X}.$$

Solution: From the characteristic equation

$$\det(\,\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = -(\lambda+1)^2(\lambda-5) = 0$$

we get eigenvalues of $\lambda_1 = \lambda_2 = -1$, and $\lambda_3 = 5$.

For $\lambda_1 = \lambda_2 = -1$, we get

$$(\mathbf{A} + \mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -2 & 2 & -2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - k_2 + k_3 = 0$. Since we only have one equation, and three known variables, we have two free variables. If we let $k_2 = 1, k_3 = 0$, then we get $k_1 = 1$. If we let $k_2 = 0, k_3 = 1$, then we get $k_1 = -1$. This gives us two eigenvectors

$$\mathbf{K_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{K}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Since neither eigenvector is a constant multiple of the other, we have two linearly independent eigenvectors.

For $\lambda_3 = 5$, we go through the normal process and get

$$(\mathbf{A} + 5\mathbf{I}|\mathbf{0}) = \left(\begin{array}{ccc|c} -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - k_3 = 0$ and $k_2 + k_3 = 0$. If we let $k_3 = 1$, then we get the eigenvector

$$\mathbf{K_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{5t}.$$

- *Example:* Solve

$$\mathbf{X}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{X}.$$

Solution: From the characteristic equation

$$\det(\, \mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix} = (\lambda+3)^2 = 0$$

we get eigenvalues of $\lambda_1 = \lambda_2 = -3$.

For $\lambda_1 = \lambda_2 = -3$, we get

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{0}) = \left(\begin{array}{cc|c} 6 & -18 & 0 \\ 2 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $k_1 - 3k_2 = 0$. If we let $k_2 = 1$, then we get the eigenvector

$$\mathbf{K_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Since there was only one free variable, we cannot find another linearly independent eigenvector. Therefore, we need to find a second solution of the form

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t}.$$

By earlier work, we showed $\mathbf{K} = \mathbf{K_1}$, so we only need to find $\mathbf{P}$. Using $(\mathbf{AP} - \lambda_1 \mathbf{P}) = \mathbf{K}$**,** we get

$$(\mathbf{A} + 3\mathbf{I}|\mathbf{K}) = \left(\begin{array}{cc|c} 6 & -18 & 3 \\ 2 & -6 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -6 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore, $2p_1 - 6p_2 = 1$. If we let $p_1 = 1/2$, then we get that $p_2 = 0$, giving us

$$\mathbf{P} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

Hence, our second solution is

$$\mathbf{X}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{-3t},$$

and the general solution is

$$\mathbf{X} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right].$$

For a case where we have an eigenvalue of multiplicity 3, where we can only find one eigenvector, we follow a similar pattern and get a third solution of

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t},$$

where

$$(\mathbf{AK} - \lambda_1 \mathbf{I}) = \mathbf{0}, \qquad (\mathbf{AP} - \lambda_1 \mathbf{P}) = \mathbf{K}, \qquad (\mathbf{AQ} - \lambda_1 \mathbf{Q}) = \mathbf{P}.$$

**For cases where the multiplicity of λ_1 is $m \geq 3$, it may be possible to find $k < m$ linearly independent eigenvectors. In this case, we would have to produce $m - k$ other linearly independent solutions like above.*

- **Complex Eigenvalues:**

Theorem: **Solutions Corresponding to a Complex Eigenvalue**

Let $\mathbf{A}$ be the coefficient matrix with real entries of a homogeneous systems of equations, and let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$. Then

$$\mathbf{K}_1 e^{\lambda_1 t} \qquad \text{and} \qquad \overline{\mathbf{K}_1} e^{\overline{\lambda_1} t}$$

are solutions.

Much like in other cases, it is desirable to rewrite the solution in terms of real functions. To do this, we once again employ Euler's formula to write

$$\mathbf{K}_1 e^{(\alpha+\beta i)t} = \mathbf{K}_1 e^{\alpha t}(\cos\beta t + i\sin\beta t)$$
$$\overline{\mathbf{K}_1} e^{(\alpha-\beta i)t} = \overline{\mathbf{K}_1} e^{\alpha t}(\cos\beta t - i\sin\beta t).$$

By combining these two solutions together, in a similar fashion as before, we end up with a simplified set of solutions.

Theorem: **Real Solutions Corresponding to a Complex Eigenvalue**

Let $\lambda_1 = \alpha + \beta i$ be a complex eigenvalue of the coefficient matrix $\mathbf{A}$ in the homogeneous systems of equations. Let $\mathbf{K}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$ and let

$$\mathbf{B}_1 = \frac{1}{2}(\mathbf{K}_1 + \overline{\mathbf{K}}_1), \qquad \mathbf{B}_2 = \frac{i}{2}(-\mathbf{K}_1 + \overline{\mathbf{K}}_1).$$

Then

$$\mathbf{X}_1 = [\mathbf{B}_1\cos\beta t - \mathbf{B}_2\sin\beta t]e^{\alpha t} \qquad \text{and} \qquad \mathbf{X}_2 = [\mathbf{B}_2\cos\beta t + \mathbf{B}_1\sin\beta t]e^{\alpha t}$$

are linearly independent solutions on $(-\infty, \infty)$.

**Since $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ represent the real and imaginary parts of $\boldsymbol{K}_1$, respectively, we often denote them by $\boldsymbol{B}_1 = Re(\boldsymbol{K}_1)$ and $\boldsymbol{B}_2 = Im(\boldsymbol{K}_1)$.*

- *Example:* Solve

$$\mathbf{X}' = \begin{pmatrix} 2 & 8 \\ -1 & -2 \end{pmatrix}\mathbf{X}.$$

Solution: From the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 8 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

we get eigenvalues of $\lambda_1 = 2i$, $\lambda_2 = -2i$. When $\lambda = 2i$, we get the system

$$(2 - 2i)k_1 + 8k_2 = 0$$

$$-k_1 + (-2 - 2i)k_2 = 0.$$

This means that $k_1 = (-2 - 2i)k_2$. If we let $k_2 = 1$, we get

$$\mathbf{K}_1 = \begin{pmatrix} -2-2i \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + i\begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Therefore the general solution is

$$\mathbf{X} = c_1\left[\begin{pmatrix} -2 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} -2 \\ 0 \end{pmatrix}\sin 2t\right] + c_2\left[\begin{pmatrix} -2 \\ 0 \end{pmatrix}\cos 2t + \begin{pmatrix} -2 \\ 1 \end{pmatrix}\sin 2t\right].$$

**We did not have to choose λ_1 when looking for our eigenvector. The process would work just as well starting with λ_2. However, it is common practice to choose the eigenvalue with $\beta > 0$ so that we do not have negative arguments inside the trig functions.*

**Note: while it can be done, it is generally not advised to use row-reduction to solve a system involving complex values.*

While we have explored solving homogeneous first-order systems of linear equations, we can adapt our techniques to solve second-order systems of the form $\mathbf{X}'' = \mathbf{AX}$. For the base case where we have two equations in terms of x_1 and x_2, we can introduce two new variables $x_3 = x_1'$ and $x_4 = {x_2}'$. This give us the larger system

$$\mathbf{X}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \mathbf{A} & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}\mathbf{X}.$$

However, since this method doubles the number of variables, we can very quickly produce very large systems.