Unit 3.5 The Ratio-, Root-, and Comparison-Tests

The last two series tests that we will be considering are the *Ratio* and *Root Tests*. The *Ratio Test* is arguably one of, if not the most important and applicable series test covered in this course. It should also be noted that the Ratio and Root Tests are equivalent—whenever one of them applies, the other applies. However, for a given series, one might be easier to use than the other. Let's take a look at these tests as well as a few examples.

The Ratio Test:

1. If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges (absolutely; see Unit 4.4 for abs. convergence)

2. If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ (or $= \infty$), then $\sum a_n$ diverges

3. If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then no conclusion can be made based on the *Ratio Test*.

*The *Ratio Test* is particularly useful when dealing with series involving factorials.

The n^{th} Root Test:

1. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$, then $\sum a_n$ converges (absolutely; see Unit 4.4 for abs. convergence)

2. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} > 1$ (or $= \infty$), then $\sum a_n$ diverges

3. If $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = 1$, then no conclusion can be made based on the n^{th} *Root Test*.

Examples 1 - 3 Determine whether each series converges or diverges. Justify your response.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n}{n!}$

Solution: The presence of the factorial in this problem makes the *Ratio Test* a very good test to consider first. Observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!}\right| = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} \to 0 \text{ as } n \to \infty$$

We have shown that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ and thus by the $Ratio\ Test,$ the series $\boxed{\text{converges.}}$

Note: Because the ratio above was positive, the absolute values could be dropped. We also could have indicated that the series converged absolutely, but you will not be expected to make such an indication unless requested to do so.

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2. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{3n+1}{2n}\right)^n$

Solution: The fact that the terms of our series take the form U^n, makes the n^{th} *Root Test*, a good test to consider trying to use. Observe that

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{3n+1}{2n}\right)^n\right|} = \frac{3n+1}{2n} = \frac{3+1/n}{2} \to \frac{3}{2} \text{ as } n \to \infty$$

Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1,$ the series $\boxed{\text{diverges}}$ by the $n^{th} Root\ Test$.

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3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$

Solution: One can easily conclude that the following series $\boxed{\text{converges}}$ as it is a p-series with $p = 2 > 1$. However, let us suppose we wished to analyze the series using the *Ratio Test*. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)^2} \div \frac{1}{n^2}\right| = \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \left(\frac{n}{n+1}\right)^2 = \left(\frac{1}{1+1/n}\right)^2 \to 1 \text{ as } n \to \infty$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$ the $Ratio\ Test$ does not apply. The purpose of this example is to make it clear that you cannot always use the *Ratio Test*. In fact, the *Ratio Test* will never work for a p-series, so ideally one makes use of the p-series (or integral) test in such a case.
