Unit 3.2 The Integral Test, p-Series and the Test for Divergence

Consider the following series.

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots$$

Looking at the partial sums we see that:

$$S_1 = 1,\ \ S_2 = 3,\ \ \ S_3 = 6,\ \ S_4 = 10,\ \text{etc…}$$

It is apparent that the partial sums are growing (without bound) and there aren't approaching any given number. In fact, there a nice formula for the n^{th} partial sum of the particular series given by $S_n = \frac{n(n+1)}{2}$. And as we can see $S_n \to \infty$, as $n \to \infty$. The above series clearly diverges.

One problem with the series given above is that the terms of the series were growing in magnitude. A series has no hope of converging if the terms that you add on do not get closer and closer to zero as n gets larger. The idea is that for a series to converge, after a while you need the terms that you are adding on to have very little effect on the sum, thus allowing it to settle down to a given value. To make this idea more formal, we present the following series test.

The n^{th} Term Test for Divergence (or The Divergence Test):

$$\text{If } \lim_{n\to\infty} a_n \neq 0, \text{then } \sum_{n=k}^{\infty} a_n \quad \text{Diverges}$$

$*$ **Warning**: If $\lim_{n\to\infty} a_n = 0$, then based on that information alone,

no conclusion can be made about the convergence or divergence of $\sum_{n=k}^{\infty} a_n$

In a sense, what the above test says is that for a series to even have a chance to converge, its terms must approach zero. However, the terms approaching zero is not enough to guarantee that the series will converge; it depends on how quickly they approach zero.

Examples 1 – 3 Determine whether each series converges or diverges. If the series converges, give its sum.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

Solution: This series is clearly not geometric as its terms are not determined by an exponential function, i.e. a_n is not an exponential function. Let's see what value, if any, the terms approach. In doing so we will rewrite the terms by dividing the numerator and denominators by n which is equal to $\sqrt{n^2}$ when n is positive, which it is as $n \to \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\underbrace{1/n^2}_{\to 0}}} = 1$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\boxed{\text{diverges}}$ by the n^{th} Term Test.

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2. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

Solution: While we could observe that this series diverges by the Geometric Series Test, where $r = -1$, let us apply the n^{th} term test. Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \quad \text{which does not exist since } (-1)^n \text{ will alternate between } \pm 1.$$

Since $\lim\limits_{n\to\infty} a_n \neq 0$, the series $\boxed{\text{diverges}}$ by the n^{th} Term Test.

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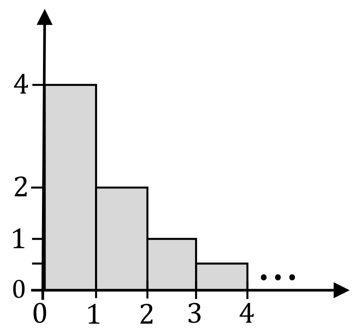
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Solution: Again, we first note that this is not a geometric series. Thus let us take a look at the limit of the terms.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Since the limit equals zero, the n^{th} Term Test, does not apply and at this time we cannot conclude whether or not this series converges or diverges. We will investigate this series in the next section and will find that it is a well known series called the *harmonic series*. Moreover, we will in fact find out that this series diverges. This will serve as a classic example of a series whose terms approach zero, yet the series still diverges.

The Integral Test. Let us now take a geometrical approach to determining the convergence or divergence of a given series. To do so, we will make use of improper integrals with an infinite upper limit of integration. Let us first consider the series $4 + 2 + 1 + \frac{1}{2} + \cdots$. Since the terms of this series are positive, we can view each as the area of a rectangle. In particular, we will view each as the area of a rectangle whose width is 1 and whose height is equal to the value of the term. We can then arrange these rectangles in the x, y-plane as follows.

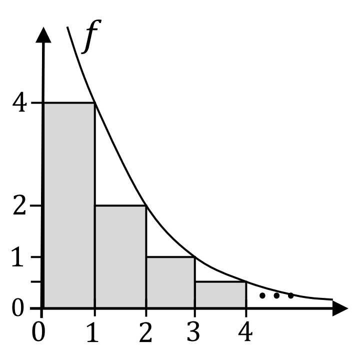


The area of the first rectangle is 4
The area of the second rectangle is 2
The area of the third rectangle is 1
The area of the fourth rectangle is 1/2
and so on…

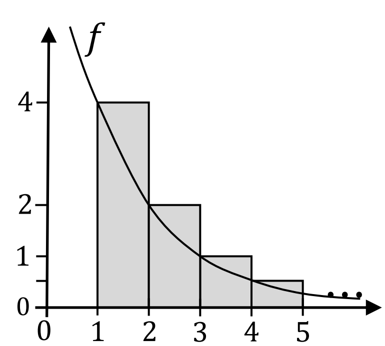
Thus the total area of all of the rectangles is

$$4 + 2 + 1 + \frac{1}{2} + \cdots.$$

We have therefore translated our problem of summing the series $4 + 2 + 1 + \frac{1}{2} + \cdots$, to the problem of finding the total area of all of the rectangles (if such a total exists). Why is that useful? Well… we have already developed tools for dealing with area—namely integrals. Moreover, we have previously used rectangles to approximate the value of definite integrals. In fact we can view the rectangles as being either an over- or under-estimate of the area under a curve as the curve extends to infinity. To see how this may be done, let's once again consider the figure above along with a slight modification of it obtained by shifting each rectangle 1-unit to the right. On each we have superimposed the graph of a function that may be approximated by these rectangles using left- and right-hand sums.



f as an overestimate of the rectangles



f as an underestimate of the rectangles

Based on the first figure, we can see that total area of the rectangles is less than the area under the graph of f from 0 to ∞. The second figure shows that the total area of the rectangles is greater than the area under the graph of f from 1 to ∞. It follows that if the area under the graph of f from 1 to ∞ is finite, then the total area of all of the rectangles must be finite and thus the corresponding series must converge. If the area under the graph of f is infinite, then the total area of all of the rectangles will be infinite and thus the corresponding series must diverge. This idea is formalized in the following series test.

The Integral Test:

Let $f(x)$ be a positive, continuous, decreasing function for $x \geq k$ and suppose that $f(n) = a_n$. In this case

the series $\displaystyle\sum_{n=k}^{\infty} a_n$ and the improper integral $\displaystyle\int_k^{\infty} f(x)\,dx$ both converge or both diverge.

Before looking at an example, note that in the above, we can start the sum and integral at any integer value we wish—thus the use of k. The main difference between $f(x)$ and a_n is the domain over which they are used. For example, if we have $f(x) = x^2$ and $a_n = n^2$, both are given by the same formula. The difference between the two is that we let x take on all <u>real</u> values over some interval, whereas we only let n take on <u>integer</u> values. The graph of $f(x) = x^2$ would consist of a continuous curve, while the graph of $a_n = n^2$ would consist of only the points on the graph of f corresponding to integer inputs.

<u>**Example 4**</u> Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{4}{9+n^2}$ converges or diverges.

Solution: First note that the terms of the series approach zero, thus the n^{th} term test does not apply. Consider the continuous function $f(x) = \frac{4}{9+x^2}$ which is also positive and decreasing for $x \geq 1$. Since $f(n) = \frac{4}{9+n^2} = a_n$, the integral test applies and the given series will have the same behavior (converge or diverge) as the improper integral $\int_1^{\infty} \frac{4}{9+x^2}\,dx$. Thus we evaluate the integral as follows

$$\int_1^{\infty} \frac{4}{9+x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{4}{3^2+x^2}\,dx = \lim_{t\to\infty} \left[\frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_1^t = \lim_{t\to\infty} \frac{4}{3}\left[\underbrace{\tan^{-1}\frac{t}{3}}_{\pi/2} - \tan^{-1}\frac{1}{3}\right]$$

The important thing to note is that the above limit exists and thus the improper integral converges. It does not matter to us what the integral above converges to. Since the improper integral converges the integral test tells us that the given series must also $\boxed{\text{converge}}$.

Example 5 Determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ converges or diverges.

Solution: Because the terms of this series approach zero, there is a chance that the series may converge. Let's consider the function $f(x) = 1/x$ which is positive, continuous, and decreasing for $x \geq 1$. In addition, $f(n) = 1/n = a_n$ and thus the integral test applies here. We therefore determine the corresponding improper integral as follows

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx = \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} [\ln t - \ln 1] = \infty \quad (DNE)$$

Since the limit does not exist, the integral diverges. By the Integral Test, the original series also $\boxed{\text{diverges}}$.

The series in example 5 is known as the *harmonic series*. It is a classic example of a series whose terms approach zero, yet the series still diverges. If you encounter this series again, you may immediately indicate that it diverges by identifying it as the harmonic series. The harmonic series is a member of a family of series called p-series. A p-series is a series whose terms are generated by certain **p**ower functions. More specifically a p-series takes the form

$$\sum_{n=k}^{\infty} \frac{1}{n^p} = \sum_{n=k}^{\infty} n^{-p} \quad \text{where } p \text{ is a positive real number.}$$

For these series, the exponent p will be a constant and the base n is the variable. Be careful not to confuse these with geometric series whose base is constant and exponent is the variable. For example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is a } p-\text{series with } p = 2, \text{whereas } \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is a geometric series with } r = \frac{1}{2}$$

The harmonic series is just the special case of a p-series where $p = 1$. Just as we used the integral test to show that the harmonic series diverged, one can use the integral test to show that p-series will diverge when $p \leq 1$ and converge when $p > 1$. Let us make this statement more formally as another series test. See the text for a proof of this theorem.

The p-Series Test:

The p-series $\sum_{n=k}^{\infty} \frac{1}{n^p}$, converges when $p > 1$ and diverges when $p \leq 1$.

Examples 6 & 7 Determine whether or not each series converges or diverges.

6. $\displaystyle\sum_{n=4}^{\infty} \frac{5}{\sqrt{n}}$

Solution: This series can be expressed as $\displaystyle 5\sum_{n=4}^{\infty} \frac{1}{n^{1/2}}$ and thus we can identify it as a p-series with $p = 1/2$. Since $p \le 1$, the series $\boxed{\text{diverges}}$ by the p-series test.

Note: in the above, we factored the 5 out of the summation (which was legitimate as 5 is a common factor of every term). This was not necessary, but like with integration, sometimes it makes it easier to recognize forms when constants are factored out.

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7. $\displaystyle\sum_{n=1}^{\infty} n^{-1.5}$

Solution: The series can be expressed as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ and thus we can identify it as a p-series with $p = 1.5$. Since $p > 1$, the series $\boxed{\text{converges}}$ by the p-series test.

Approximating Convergent Sums. Up to this point, our focus has mainly been on determining whether or not a series converges. Aside from geometric series, we have not concerned ourselves with what the series converge to. That is because for most series, obtaining a simple expression for the sum is impossible or very difficult at the least. But the fact that the series converges, means that it makes sense to approximate it. And what better way to approximate it than to use a partial sum. Rather than adding up all of the terms of the series, we will stop at some point. If a series diverges, it doesn't make sense to approximate it, since it won't add up to anything. Now, if we are going to approximate a series, it won't be very useful if we don't have any idea of how big our error might be. Thus we look to find ways of obtaining error bounds. Before doing so, let us introduce some notation.

Consider $\displaystyle S = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k = \underbrace{a_1 + a_2 + \cdots + a_n}_{S_n (\text{the } n^{\text{th}} \text{ partial sum})} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{R_n\ (the\ n^{\text{th}} \text{ remainder})}$

As before we have the n^{th} partial sum given by S_n, but now we call the sum of the remaining terms the n^{th} remainder and denote it R_n. If we use a partial sum to approximate a convergent series, the remainder is the sum of the terms that we have left out and therefore is the error of our approximation. Of course, we will not know what the remainder (error) is, but we will be looking for a bound on the error. For example, we might be able to say that our error is no more than one-tenth, but no less than one-hundredth of a unit. The following theorem gives us a nice error bound inequality for approximating certain series.

Theorem: Suppose that $f(x)$ is positive, continuous, and decreasing for $x \geq n$ and that $f(n) = a_n$.

If the series $\sum_{n=1}^{\infty} a_n$ converges, and is approximated by the n^{th} partial sum, S_n, then the remainder of this approximation, R_n, will satisfy the following inequality

$$\int_{n+1}^{\infty} f(x)\,dx < \underbrace{R_n}_{ERROR} < \int_{n}^{\infty} f(x)\,dx$$

Since the true sum of the series is $S = S_n + R_n$, we have the following inequality for the sum of our series. This inequality can be obtained by adding S_n to all parts of the above inequality.

$$S_n + \int_{n+1}^{\infty} f(x)\,dx < \underbrace{\sum_{n=1}^{\infty} a_n}_{S_n + R_n} < S_n + \int_{n}^{\infty} f(x)\,dx$$

An explanation of where these inequalities come from is provided in the narrated examples.

<u>**Example 8**</u> Estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ using the first four terms, i.e. S_4.

Provide an error bound for this approximation and then an inequality expressing an interval where the true sum of the series lies.

Solution: The given series is a convergent p-series with $p = 5 > 1$ and the corresponding function $f(x) = \frac{1}{x^5}$ satisfies the conditions of the above theorem.

$$\sum_{n=1}^{\infty} \frac{1}{n^5} \approx S_4 = \sum_{n=1}^{4} \frac{1}{n^5} = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} \approx 1.0363$$

By adding up the first four terms, we obtain an approximation of the entire sum. But how close are we? For all we know at this point, we might be off by thousands or millions; after all, we only used four terms. Let us now use the above theorem to find an error bound, i.e. an inequality satisfied by the remainder,

$$R_4 = \sum_{n=5}^{\infty} \frac{1}{n^5} = \frac{1}{3125} + \frac{1}{7776} + \cdots$$

The theorem indicates that

$$\int_5^{\infty} \frac{1}{x^5} dx < \underbrace{R_4}_{ERROR} < \int_4^{\infty} \frac{1}{x^5} dx$$

Let us now evaluate each of the above improper integrals

$$\int_5^{\infty} \frac{1}{x^5} dx = \lim_{t\to\infty} \left[-\frac{1}{4x^4}\right]_5^t = \lim_{t\to\infty} \left[-\frac{1}{4t^4} + \frac{1}{4(5)^4}\right] = .0004$$

and

$$\int_4^{\infty} \frac{1}{x^5} dx = \lim_{t\to\infty} \left[-\frac{1}{4x^4}\right]_4^t = \lim_{t\to\infty} \left[-\frac{1}{4t^4} + \frac{1}{4(4)^4}\right] = .0010$$

This means that our estimate is off by no more than .001 but it is off by at least .0004. That is, we have

$$.0004 < \underbrace{R_4}_{ERROR} < .0010$$

Adding our partial sum estimate yields the following inequality for the entire sum

$$1.0367 < \sum_{n=1}^{\infty} \frac{1}{n^5} < 1.0373$$

Partial Sums on the Calculator. For partial sums where n is small, it may not take too much effort to type each individual term into a calculator. However, when n is large, this may be very tedious. The TI-graphing calculator allows us to compute such sums with very little effort. You will be expected to be able to use the following commands. The general input for computing a partial sum, will take the following form

sum(seq(a_n formula, n, n start, n stop))

In place of n, you can use X as it is more easily accessible on the calculator. The sum and seq commands can both be found under the list menu which can be accessed by entering $\boxed{\text{2ND}}$, then $\boxed{\text{STAT}}$. The sum command will be found as item 5 under the MATH category, while the seq command will be found as item 5 under the OPS category.

For example to obtain the partial sum used in example 5, we would type

$$\text{sum(seq(x^(-5),x,1,4)} \approx 1.036341789$$

*For those that do not have a TI-graphing calculator, I have placed a video in the course showing how to compute partial sums using Microsoft Excel. You may also do an internet search to find free online programs that will compute partial sums for you.