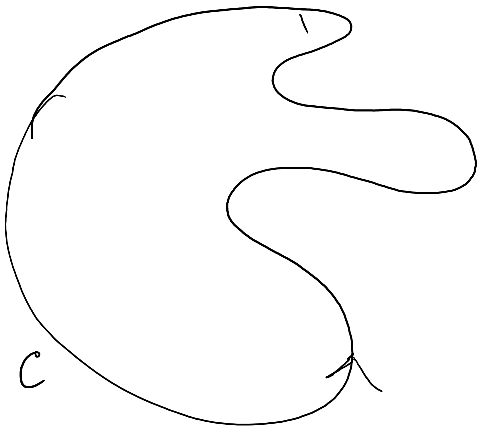
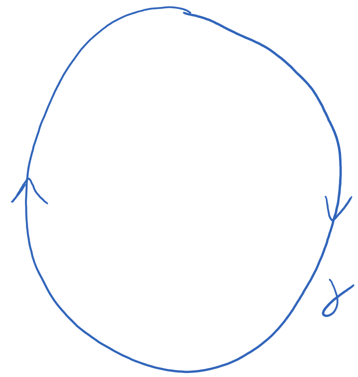
13.4: Green's Theorem

Thursday, November 5, 2020 9:52 AM



positive orientation (default)
(counterclockwise)

$$\int_C = \oint_C$$



negative orientation
(clockwise)

$$\oint$$

<u>Green's Th'm:</u> Let C be a positively oriented, p-w smooth, simple closed curve in the plane, and let D be the region bounded by C. If P and Q are C^1 on an open set containing D, <u>then</u>

$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

<u>Remark</u> Green's Th'm is analogous to FTC b/c

we integrate derivatives "inside" of D, and the function itself on ∂D.

Ex. $\int_C \left(3y - e^{\sin(x)}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$, where C is unit circle.

Sol'n Assume C has positive orientation and is traversed only once.

$$P(x,y) = 3y - e^{\sin(x)}, \qquad Q(x,y) = 7x + \sqrt{y^4+1}$$

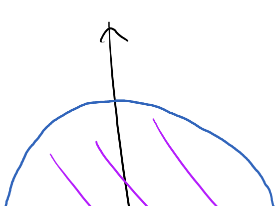
Check P_x, P_y, Q_x, Q_y all continuous on unit disk.

$$Q_y = \left(\left(y^4+1\right)^{1/2}\right)' = \frac{1}{2}\left(y^4+1\right)^{-1/2}\left(4y^3\right)$$

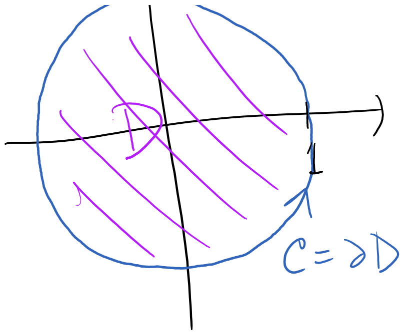
$$= \frac{2y^3}{\sqrt{y^4+1}} \quad \text{cont. all } \mathbb{R}^2 \text{ b/c } y^4+1 \geq 1 > 0$$

Green $\Rightarrow \int_C P\,dx + Q\,dy = \iint_D \left(Q_x - P_y\right) dA$

$$= \iint_D 7-3 \; dA$$



$= \iint 4 \; dA$



$$= \iint_D 4 \, dA$$

$$= \boxed{4\pi}$$

* Did <u>not</u> use a parametrization of C

<u>Note</u> $\iint_D dA = \text{area}(D)$; if P & Q are s.t.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$, there are 3 possibilities:

1) $P = 0$, $Q = x$

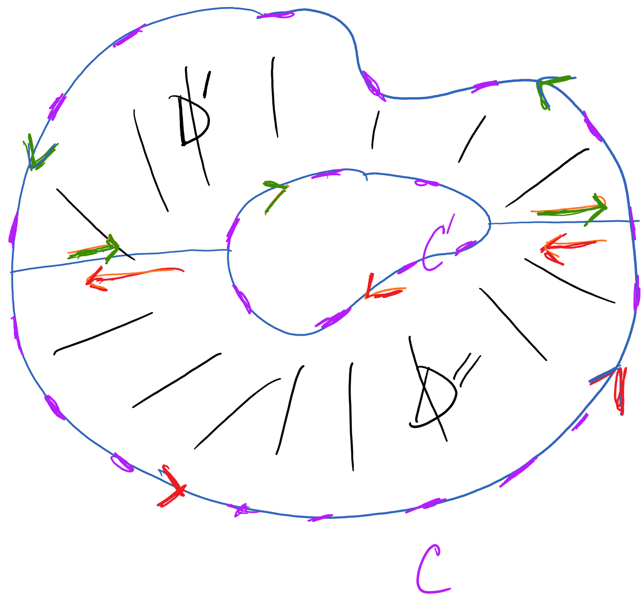
2) $P = -y$, $Q = 0$

3) $P = -\frac{1}{2} y$, $Q = \frac{1}{2} x$

Green $\Rightarrow$ $\text{area}(D) = \int_{\partial D} x \, dy = -\int_{\partial D} y \, dx$

$$= \frac{1}{2} \int_{\partial D} x\,dy - y\,dx.$$

<u>Green's Th'm for non-simply connected</u>



$D = D' \cup D''$ not SC, but is a union of SC D', D''

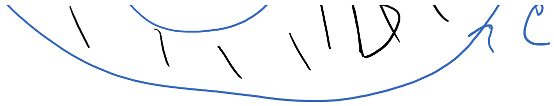
- Use Green on each portion
  $\Rightarrow$ inside curve has opposite orientation

- ∂D comes in two pieces:
  C, C'
  outter boundary

Green $\Rightarrow$ $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy$$

$$= \int_C P\,dx + Q\,dy - \int_{C'} P\,dx + Q\,dy$$

Similarly, for n inside curves, subtract n integrals:

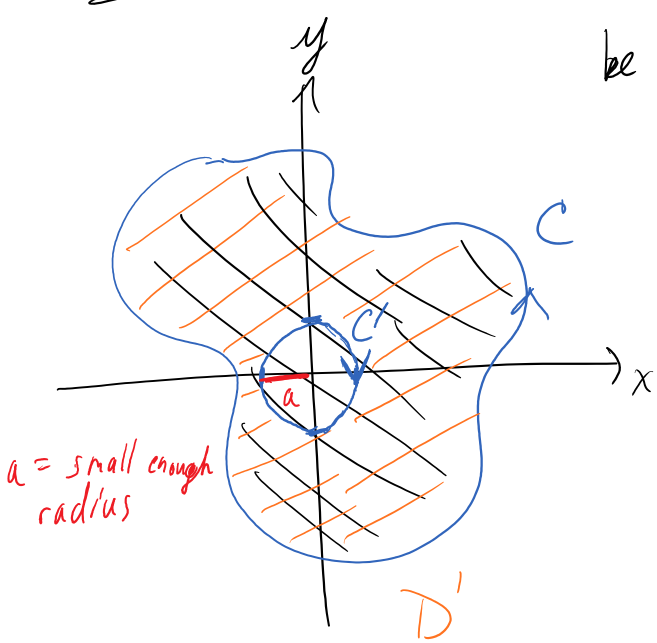


$$\int_{\partial D} = \int_C - \int_{C'} - \int_{C''}$$

Ex. $\vec{F}(x,y) = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$. Show $\displaystyle\int_C \vec{F}\cdot d\vec{r} = 2\pi$

for every positively oriented simple closed path C that encloses the origin.

Sol'n: Let C be any such path, and let C' be a circle in D cent. at $(0,0)$. (D = region enclosed by C)



Let D' be the region inside C, and outside of C'.

$$Q(x,y) = \frac{-y}{x^2+y^2}, \quad P(x,y) = \frac{x}{x^2+y^2}$$

$$\text{Green} \Rightarrow \int_{\partial D'} P\,dx + Q\,dy = \iint_{D'} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

gives C' negative

gives C negative orientation

$$\int_C P\,dx + Q\,dy - \int_{C'} P\,dx + Q\,dy = \iint_{D'} 0 \; dA = 0$$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

easy to compute:

Use $\vec{r}(t) = \langle a\cos(t),\ a\sin(t)\rangle$, $0 \le t \le 2\pi$

$$\vec{r}\,'(t) = \langle -a\sin(t),\ a\cos(t)\rangle$$

$$\int_{C'} P\,dx + Q\,dy = \int_{C'} \vec{F} \cdot d\vec{r}$$

, $\vec{F}(x,y) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2}$

$$\vec{F}(r(t)) = \frac{-a\sin(t)}{a^2}\vec{i} + \frac{a\cos(t)}{a^2}\vec{j}$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin(t)}{a}, \frac{\cos(t)}{a} \right\rangle \cdot \langle -a\sin(t), a\cos(t)\rangle \, dt$$

$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t) \; dt$$

$$\int_0 \sin^2(t) + \cos^2(t) \, dt$$

$$= 2\pi.$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 2\pi \text{ for all such } C. \qquad \square$$

Rmk $\vec{F} = \langle P, Q \rangle$ has property $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(on last example)

(i.e. $\vec{F} \cdot d\vec{r}$ is a closed differential), yet

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0 \text{ for many closed contours } C.$$

Note 1) domain D' not simply connected

2) $\vec{F}$ is conservative on any simply-conn. domain not containing origin $(0,0)$.

Ex. $\vec{F} = \langle x-y, x-2 \rangle$.

$$\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = 1 \Rightarrow \vec{F} \text{ not conservative}$$

∂y $\qquad$ ∂r

on <u>any</u> domain D.

(never equal, regardless of x, y.)

$\int_C \vec{F} \cdot d\vec{r}$ IOP? No.

$\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed C? No.

$\vec{F}$ circulation free? No.

<u>Note</u> C = unit circle $\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \pi \neq 0$.