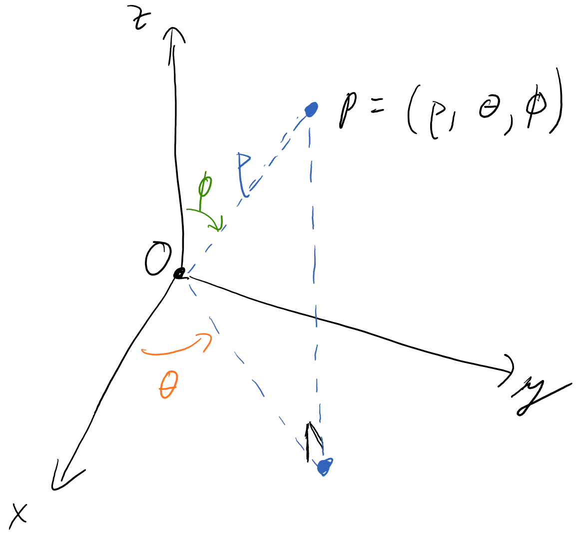
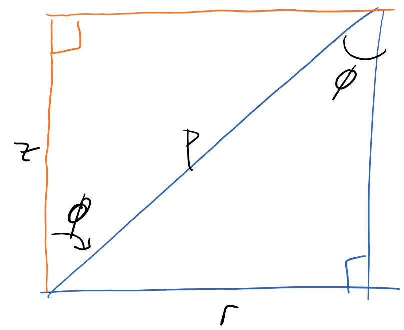
Thursday, October 8, 2020 11:24 AM

The <u>spherical coordinates</u> (ρ, θ, ϕ) of a point in space are:



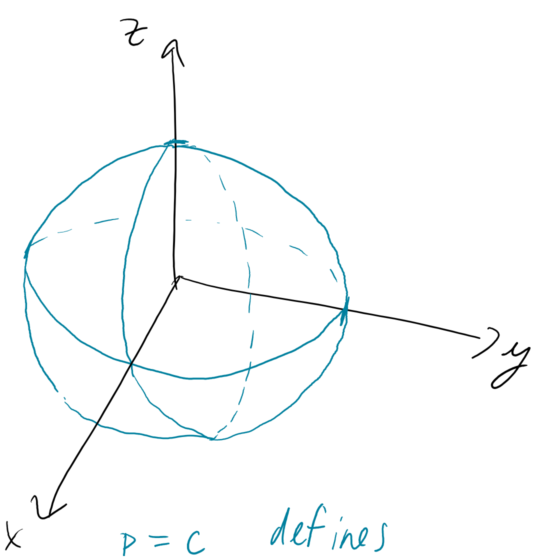


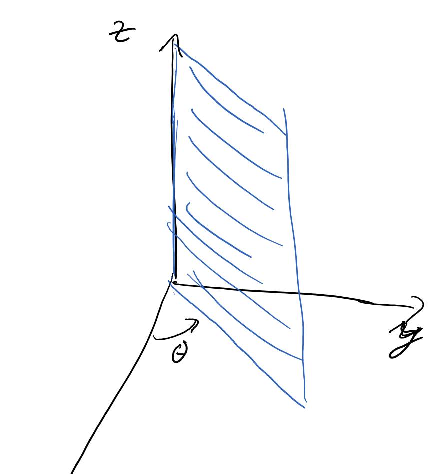
$\rho = |OP|$ = dist. from P to origin

θ = same angle as in cyl. coord.

ϕ = angle between pos. z-axis and line segment OP.

$\rho \geq 0$, $0 \leq \phi \leq \pi$



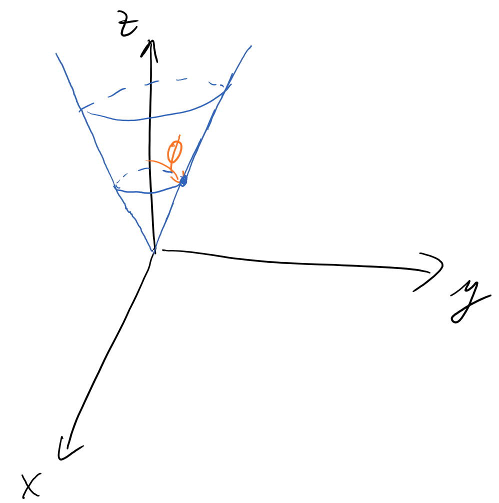


$\rho = c$ defines

$\theta = c$ def.

x $\rho = c$ defines sphere radius c, cent. at origin

x $\theta = c$ def. a <u>half</u> plane



$\phi = c, \quad 0 < c < \frac{\pi}{2}$

(half cone)

$z = \rho\cos(\phi), \qquad r = \rho\sin(\phi)$

$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta)$

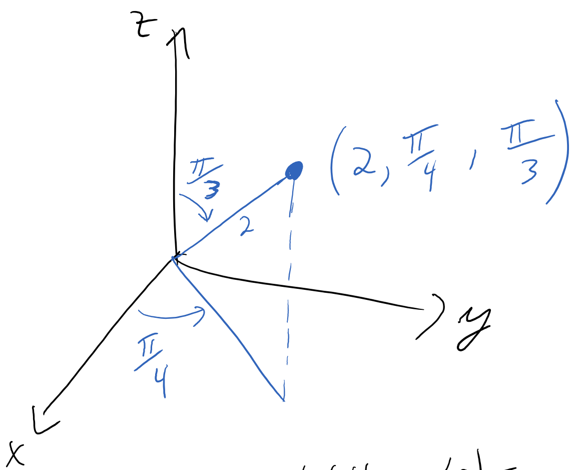
$$\boxed{x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)}$$

(spherical to rectangular)

$$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

<u>Ex. 1</u> The point $\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right)$ is in spherical coord.
Plot & convert to rect. coord.

Sol'n: $\rho = 2, \quad \theta = \pi/4, \quad \phi = \frac{\pi}{3}$



$$x = \rho \sin(\phi) \cos(\theta) = 2 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin(\phi) \sin(\theta) = 2 \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{4}\right) = \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos(\phi) = 2 \cos\left(\frac{\pi}{3}\right) = 1$$

$$\left(2, \frac{\pi}{4}, \frac{\pi}{3}\right) \longmapsto \boxed{\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)}$$

Ex. 2 $\quad (0, 2\sqrt{3}, -2)$ is in rectangular; convert to spherical.

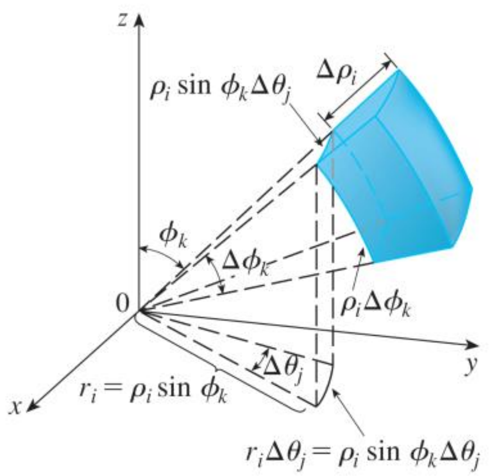
Sol'n: $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{12 + 4} = 4$

$$\cos(\phi) = \frac{z}{\rho} = \frac{-2}{4} = \frac{-1}{2} \Rightarrow \phi = \frac{2\pi}{3}$$

$$\cos(\theta) = \frac{x}{\rho \sin(\phi)} = \frac{0}{4\sin(\phi)} = 0 \Rightarrow \frac{\pi}{2} \quad (b/c \quad y > 0)$$

$\therefore \boxed{\left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right)}$

Evaluating Triple Integrals w/Spherical Coord.



spherical wedge E_{ijk}

$$E = \{(\rho, \theta, \phi) : a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

$a \geq 0$, $\beta - \alpha \le 2\pi$, $d - c \le 2\pi$

- Subdivide E into smaller spherical wedges E_{ijk} using spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$

- $E_{ijk} \approx$ rectanglular box w/dimensions

  $\Delta \rho_i$,

  $\rho_i \Delta \phi_k$ (arc of a circle w/ radius ρ_i, angle $\Delta \phi_k$),

  $\rho_i \sin(\phi_k) \Delta \theta_j$ (arc of circle w/ radius $\rho_i \sin(\phi_k)$, angle $\Delta \theta_j$)

- $\mathrm{Vol}(E_{ijk}) \approx \Delta V_{ijk} = (\Delta \rho_i)(\rho_i \Delta \phi_k)(\rho_i \sin(\phi_k) \Delta \theta_j)$

$$= \rho_i^2 \sin(\phi_k) \Delta \rho_i \Delta \theta_j \Delta \phi_k .$$

- Actually, MVT $\Rightarrow \exists \ (\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ in E_{ijk}

s.t. $\Delta V_{ijk} = \tilde{\rho}_i^{\,2} \sin(\tilde{\phi}_k) \Delta \rho_i \Delta \theta_j \Delta \phi_k$

- Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be in rect. coord.

- $$\iiint_E f(x,y,z)\, dV = \lim_{\max \Delta \rho_i, \Delta\theta_j, \Delta \phi_k \to 0} \sum_i \sum_j \sum_k f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}$$

$$= \lim_{\max \Delta \rho_i, \Delta\theta_j, \Delta \phi_k} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\tilde{\rho}_i \sin(\tilde{\phi}_k)\cos(\tilde{\theta}_j), \tilde{\rho}_i \sin(\tilde{\phi}_k)\sin(\tilde{\theta}_j), \tilde{\rho}_i \cos(\tilde{\phi}_k)\right) \tilde{\rho}_i^{\,2} \sin(\tilde{\phi}_k) \Delta \rho_i \Delta \theta_j \Delta \phi_k$$

$$\boxed{\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi}$$

where $E = \{(\rho, \theta, \phi) : a \le \rho \le b, \ \alpha \le \theta \le \beta, \ c \le \phi \le d\}$

- Can also integrate over

$$E = \{(\rho, \theta, \phi) : \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\}$$

Ex. 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is unit ball

$r^2 = x^2+y^2+z^2$

$$B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$$

Sol'n: ∂B = sphere $\Rightarrow$ use spher. coord.

$$B = \{(\rho,\theta,\phi) : 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

$$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 e^{(\rho^2)^{3/2}} \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$= \left(\int_0^{\pi} \sin(\phi)\, d\phi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 \rho^2 e^{\rho^3}\, d\rho\right)$$

$$= \left[-\cos(\phi)\Big|_0^{\pi}\right](2\pi)\left[\tfrac{1}{3}e^{\rho^3}\right]_0^1$$

$$= \boxed{\frac{4}{3}\pi(e-1)}$$

(harder, awkward in rect. coord.)

Ex. 4 Use sph. coord. to find vol. of the solid that lies above cone $z = \sqrt{x^2+y^2}$ and below sphere $x^2+y^2+z^2 = z$.

$$x^2+y^2+z^2-z+\tfrac{1}{4} = \tfrac{1}{4}$$
$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

Sol'n:

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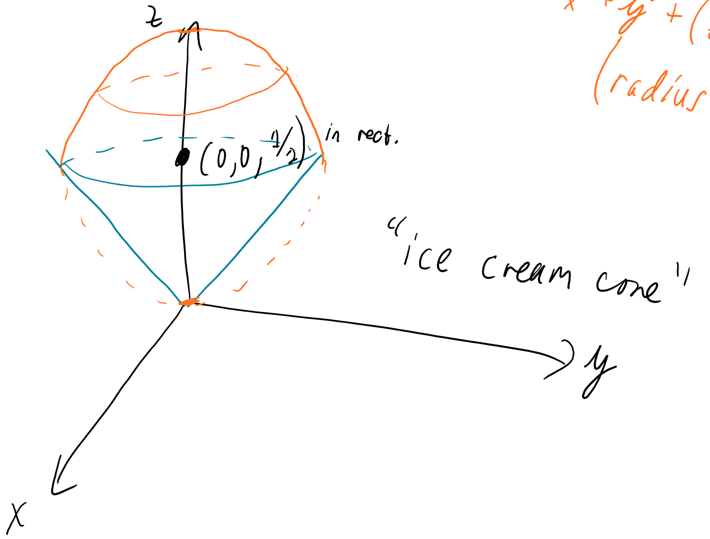

$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$
(radius $\frac{1}{2}$, cent. $(0,0,\frac{1}{2})$ in rect.)

sphere: $x^2+y^2+z^2 = z$

$$\rho^2 = z = \rho\cos(\phi) \iff \rho = \cos(\phi)$$

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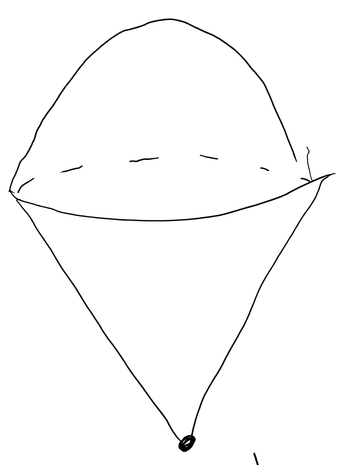
cone: $z = \sqrt{x^2+y^2}$

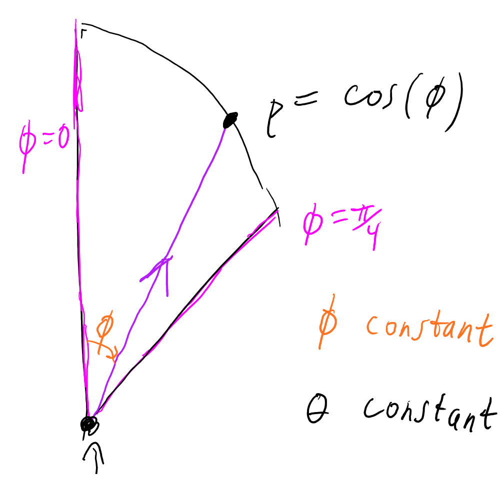
$$\rho\cos(\phi) = \sqrt{\rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)\sin^2(\theta)}$$

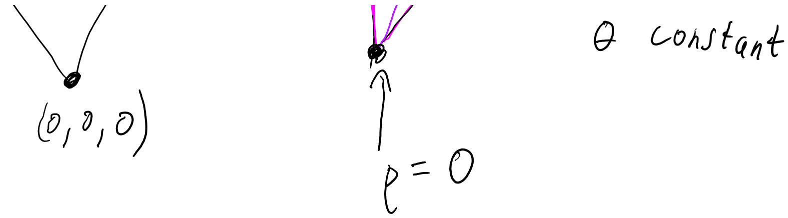
$$= \rho\sin(\phi)$$

$$\Rightarrow \cos(\phi) = \sin(\phi) \iff \phi = \frac{\pi}{4}$$

$$E = \left\{(\rho,\theta,\phi) : 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le \cos(\phi)\right\}.$$







$$V(E) = \iiint_E dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos(\phi)} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{\pi/4} \sin(\phi) \int_0^{\cos(\phi)} \rho^2\, d\rho\, d\phi\right) d\theta$$

$$= 2\pi \int_0^{\pi/4} \sin(\phi) \left[\frac{\rho^3}{3}\right]_0^{\cos(\phi)} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin(\phi) \cos^3(\phi)\, d\phi \, , \qquad u = \cos(\phi)$$

$$= \boxed{\frac{\pi}{8}}$$