Quiz #111 Noor Mustafa diff eq 1 Question 1

1. $y' - y = 1 + te^t \quad y(0) = 0$

$L\{y' - y\} = L\{1 + te^t\}$

$L\{y' - y\} = 0$

$= L\{y'\} - L\{y\}$

$L\{y'\} = sL\{y\} - y(0)$

$= sL\{y\} - y(0)$

$= sL\{y\} - y(0) - L\{y\}$

$L\{1 + e^t t\} = 0$

$= L\{1\} + L\{te^t\}$

$L\{1\} = \frac{1}{x}$

$L\{f(t)\} - F_s$

$L\{t^k f(t)\} - (-1)^k \frac{d}{ds^k}(F(s))$

$L\{e^t t\} \rightarrow f(t) = e^t, \; k = 1$

$L\{e^t\} = \frac{1}{s-1}$

$$\frac{\partial}{\partial s}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

$$= \frac{\partial}{\partial s}\left((s-1)^{-1}\right)$$

$f = u^{-1}, \quad u = (s-1)$

$$= \frac{\partial}{\partial u}\left(u^{-1}\right) \frac{\partial}{\partial s}\left((s-1)\right)$$

$$\frac{\partial}{\partial u}\left(u^{-1}\right) = -\frac{1}{u^2}$$

$$= -\frac{1}{u^2} \frac{\partial}{\partial s}\left((s-1)\right)$$

$$= -\frac{1}{(s-1)^2} \frac{\partial}{\partial s}\left((s-1)\right)$$

$$\frac{\partial}{\partial s}\left((s-1)\right) = 1$$

$$\frac{\partial}{\partial s}(s) = 1$$

$$\frac{\partial}{\partial s}(1) = 0$$

$$= -\frac{1}{(s-1)^2} \cdot 1$$

$$= -\frac{1}{(s-1)^2}$$

$$= (-1)^1 \left(-\frac{1}{(s-1)^2}\right) \rightarrow \frac{1}{(s-1)^2}$$

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$sL\{y\} - y(0) - L\{y\} = \frac{1}{s} + \frac{1}{(s-1)^2}$

$y(0)=0$

$sL\{y\} - 0 - L\{y\} = \frac{1}{s} + \frac{1}{(s-1)^2}$

$sL\{y\} - L\{y\} = \frac{s^2-s+1}{s(s-1)^2}$ ← expand it

$sL\{y\} - L\{y\} = \frac{s^2-s+1}{s^3-2s^2+s}$

$\frac{L\{y\}\cancel{(s-1)}}{\cancel{s-1}} = \frac{\frac{s^2-s+1}{s^3-2s^2+s}}{s-1}$

$L\{y\} = \frac{\frac{s^2-s+1}{s^3-2s^2+s}}{s-1}$ ← fraction rule

$L\{y\} = \frac{s^2-s+1}{(s^3-2s^2+s)(s-1)}$

$y = L^{-1}\left\{\frac{s^2-s+1}{(s^3-2s^2+s)(s-1)}\right\}$ ← partial fraction now

$(s-1)(s^3-2s^2+1)$

s^3-2s^2+s

$= s(s^2-2s+1)$

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s^2-2s+1

$a=1,\ b=-2,\ c=1$

$u\cdot v=1,\ u+v=-2$

$u=-1,\ v=-1$

$=(s^2-s)+(-s+1)$

s^2-s

$=ss-s$

$=s(s-1)$

$-(s-1)$

$=s(s-1)(s-1)$

$s(s-1)^3$

$$=\frac{s^2-s+1}{s(s-1)^3}$$

$$\frac{s^2-s+1}{s(s-1)^3}=\frac{a_0}{s}+\frac{a_1}{s-1}+\frac{a_2}{(s-1)^2}+\frac{a_3}{(s-1)^3}$$

$$s^2-s+1=\frac{a_0s(s-1)^3}{s}+\frac{a_1s(s-1)^3}{s-1}+\frac{a_2s(s-1)^3}{(s-1)^2}+\frac{a_3s(s-1)^3}{(s-1)^3}$$

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$s^2 - s + 1 = a_0 (s-1)^3 + a_1 s(s-1)^2 + a_2 s(s-1) + a_3 s$

$0^2 - 0 + 1 = a_0 (0-1)^3 + a_1 \cdot 0 \cdot (0-1)^2 + a_2 \cdot 0 \cdot (0-1) + a_3 \cdot 0$

$1 = -a_0$

$1 + a_0 = -a_0 + a_0$

$1 + a_0 = 0$

$a_0 = -1$

$1^2 - 1 + 1 = a_0 (1-1)^3 + a_1 \cdot 1 \cdot (1-1)^2 + a_2 \cdot 1 \cdot (1-1) + a_3 \cdot 1$

$a_3 = 1$

$a_0 = -1, \quad a_3 = 1$

$s^2 - s + 1 = a_1 s^3 - s^3 - 2a_1 s^2 + a_2 s^2 + 3s^2$
$+ a_1 s - a_2 s - 2s + 1$

$$\begin{cases} a_1 - a_2 - 2 = -1 \\ -2a_1 + a_2 + 3 = 1 \\ a_1 - 1 = 0 \end{cases}$$

$a_1 = 1$ $\qquad$ $a_2 = 0$

$1 - a_2 - 2 = -1$
$-1 \qquad +2$

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$$\frac{-1}{s} + \frac{1}{s-1} + \frac{0}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}$$

$$= L^{-1}\left\{-\frac{1}{s} + \frac{1}{s-1} + \frac{1}{(s-1)^3}\right\}$$

$$= -L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$L^{-1}\left\{\frac{1}{s}\right\} = 1$ ← inverse leplace transform table

$$L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}$$

$$L^{-1}\left\{\frac{1}{(s-1)^3}\right\} = \frac{e^{t}t^{2}}{2}$$

$$L^{-1}\{F(s)\} = f(t) \rightarrow L^{-1}\{F(s-a)\} = e^{at} f(t)$$

$$\rightarrow L^{-1}\left\{\frac{1}{(s-1)^3}\right\} = e^{t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$e^{t} L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= e^{t} L^{-1}\left\{\frac{1}{2} \cdot \frac{2}{s^3}\right\}$$

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$= e^t \frac{1}{2} L^{-1}\left\{\frac{2}{s^3}\right\}$

$e^t L^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

$= e^t \frac{1}{2} t^2$

$= e^t \frac{t^2}{2}$

$= \frac{e^t t^2}{2}$

$= -1 + e^t + \frac{e^t t^2}{2}$

$y = -1 + e^t + \frac{e^t t^2}{2}$

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Question 2

$y(0)=0,\ y'(0)=4$

$y''-2y'+2y=1+t$

$L\{y''-2y'+2y\}=L\{1+t\}$

$L\{y''\}-2L\{y'\}+2L\{y\}$

$L\{y''\}=s^2L\{y\}-sy(0)-y'(0)$

$=s^2L\{y\}-sy(0)-y'(0)$

$L\{y'\}=sL\{y\}-y(0)$

$=s^2L\{y\}-sy(0)-y'(0)-2(sL\{y\}-y(0))$

$+2L\{y\}$

$=L\{1+t\}$

$=L\{1\}+L\{t\}$

$L\{1\}=\frac{1}{s}$

$L\{t\}=\frac{1}{s^2}$

$=\frac{1}{s}+\frac{1}{s^2}$

$s^2L\{y\}-sy(0)-y'(0)-2(sL\{y\}-0)+2L\{y\}=\frac{1}{s}+\frac{1}{s^2}$

$y(0)=0\quad y'(0)=4$

$s^2L\{y\}-s\cdot 0-4-2(sL\{y\}-0)+2L\{y\}=\frac{1}{s}+\frac{1}{s^2}$

$s^2L\{y\}+2L\{y\}-2sL\{y\}-4=\frac{1}{s}+\frac{1}{s^2}$

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isolate $L\{y\}$

$\downarrow$

$$s^2 L\{y\} + 2L\{y\} - 2sL\{y\} = \frac{1}{s} + \frac{1}{s^2} + 4$$

$$Lcm = s^2$$

$$s^2 L\{y\} s^2 + 2L\{y\} s^2 - 2sL\{y\} s^2 = \frac{1}{s} s^2 + \frac{1}{s^2} s^2 + 4s^2$$

$$s^4 L\{y\} + 2s^2 L\{y\} - 2s^3 L\{y\} = s + 1 + 4s^2$$

$$s^4 L\{y\} + 2s^2 L\{y\} - 2s^3 L\{y\}$$

$$s^4 = s^2 s^2, \quad s^3 = s^2 s$$

$$= s^2 s^2 L\{y\} + 2s^2 L\{y\} - 2s^2 s L\{y\}$$

$$= L\{y\} s^2 (s^2 + 2 - 2s)$$

$$\frac{s^2 L\{y\}(s^2 + 2 - 2s)}{s^2(s^2 + 2 - 2s)} = \frac{s + 1 + 4s^2}{s^2(s^2 + 2 - 2s)}$$

$$L\{y\} = \frac{s + 1 + 4s^2}{s^2(s^2 + 2 - 2s)}$$

$$y = L^{-1}\left\{\frac{s + 1 + 4s^2}{s^2(s^2 + 2 - 2s)}\right\}$$

$$\frac{s + 1 + 4s^2}{s^2(s^2 + 2 - 2s)} = \frac{a}{s} + \frac{a_1}{s^2} + \frac{a_3 s + a_2}{s^2 - 2s + 2}$$

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$$\frac{s^2(4s^2+s+1)(s^2-2s+2)}{s^2(s^2-2s+2)} = \frac{as^2(s^2-2s+2)}{s} +$$

$$\frac{a_1 s^2(s^2-2s+2)}{s^2} + \frac{s^2(a_3 s+a_2)(s^2-2s+2)}{s^2-2s+2}$$

$s+1+4s^2 = as(s^2-2s+2)+a_1(s^2-2s+2)+$
$s^2(a_3 s+a_2)$

$0+1+4\cdot 0^2 = a\cdot 0\cdot(0^2-2\cdot 0+2)+a_1(0^2-2\cdot 0+2)$
$+(a_3\cdot 0+a_2)\cdot 0^2$

$1 = 2a_1$

$a_1 = \frac{1}{2}$

$s+1+4s^2 = a_s(s^2-2s+2)+\frac{1}{2}(s^2-2s+2)+$
$s^2(a_3 s+a_2)$

$s+1+4s^2 = as^3-2as^2+2as+\frac{s^2}{2}-s+1$
$+a_3 s^3+a_2 s^2$

$s+1+4s^2 = as^3-2as^2+2as+\frac{1}{2}s^2-s$
$+1+a_3 s^3+a_2 s^2$

$4s^2+1\cdot s+1 = s^3(a+a_3)+s^2(a_2+\frac{1}{2}-2a)+s(2a-1)+1$

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$$\left[\begin{array}{l} 2a_1 - 1 = 1 \\ -2a + \frac{1}{2} + a_2 = 4 \\ a_1 + a_3 = 0 \end{array}\right]$$

$2a - 1 = 1$

$a_1 = 1$

$$\left[\begin{array}{l} -2 \cdot 1 + \frac{1}{2} + a_2 = 4 \\ 1 + a_3 = 0 \end{array}\right]$$

$a_3 = -1$

$a_2 - \frac{3}{2} = 4$

$+\frac{3}{2} \quad +\frac{3}{2}$

$a_2 = \frac{11}{2}$

$$\frac{1}{s} + \frac{\frac{1}{2}}{s^2} + \frac{(-1)s + \frac{11}{2}}{s^2 - 2s + 2}$$

$$\frac{1}{s} + \frac{1}{2s^2} + \frac{-s + \frac{11}{2}}{s^2 - 2s + 2}$$

$-s + \frac{11}{2}$

$$= \frac{-s \cdot 2}{2} + \frac{11}{2} \rightarrow \frac{-s \cdot 2 + 11}{2} \rightarrow = \frac{\frac{-2s + 11}{2}}{s^2 - 2s + 2}$$

$$= \frac{-s \cdot 2 + 11}{2(s^2 - 2s + 2)}$$

$$= \frac{1}{s} + \frac{1}{2s^2} + \frac{-2s + 11}{2(s^2 - 2s + 2)}$$

$$= L^{-1}\left\{ \frac{1}{s} + \frac{1}{2s^2} + \frac{-2s+11}{2(s^2-2s+2)} \right\}$$

$$\frac{-2s+11}{2(s^2-2s+2)} \rightarrow = \frac{1}{2} \cdot \frac{-2s+11}{s^2-2s+2}$$

$$2a = -2$$

$$a = -1$$

$$s^2 - 2s + 2 + (-1)^2 - (-1)^2$$

$$s^2 - 2s + (-1)^2 = (s-1)^2$$

$$(s-1)^2 + 1$$

$$= \frac{1}{2} \cdot \frac{-2(s-1) + 9}{(s-1)^2 + 1}$$

$$= -\frac{s-1}{(s-1)^2+1} + \frac{9}{2} \cdot \frac{1}{(s-1)^2+1}$$

$$= L^{-1}\left\{ \frac{1}{s} + \frac{1}{2s^2} - \frac{s-1}{(s-1)^2+1} + \frac{9}{2} \cdot \frac{1}{(s-1)^2+1} \right\}$$

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$= L^{-1}\{\frac{1}{s}\} + L^{-1}\{\frac{1}{2s^2}\} - L^{-1}\{\frac{s-1}{(s-1)^2+1}\}$
$+ \frac{9}{2} L^{-1}\{\frac{1}{(s-1)^2+1}\}$

$L^{-1}\{\frac{1}{s}\} = 1$

$L^{-1}\{\frac{1}{2s^2}\} \rightarrow L^{-1}\{\frac{1}{2} \cdot \frac{1}{s^2}\} \rightarrow \frac{1}{2} L^{-1}\{\frac{1}{s^2}\}$

$L^{-1}\{\frac{1}{s^2}\} = t$

$= \frac{1}{2}t \rightarrow \frac{t}{2}$

$L^{-1}\{\frac{s-1}{(s-1)^2+1}\}$

$L^{-1}\{\frac{s}{s^2+1}\} = \cos(t)$

$L^{-1}\{\frac{s-1}{(s-1)^2+1}\} = e^t \cos t$

$L^{-1}\{\frac{1}{s^2+1}\} = \sin(t)$

$L^{-1}\{\frac{1}{(s-1)^2+1}\} = e^t \sin t$

$= 1 + \frac{t}{2} - e^t \cos(t) + \frac{9}{2} e^t \sin(t)$

$y = 1 + \frac{t}{2} - e^t \cos(t) + \frac{9}{2} e^t \sin(t)$

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3.

$\rightarrow y'' + 4y' + 3y = 1 - 2u(t-3) + u(t-6)$

$L\{y''\} = s^2 L\{y\} - sy(0) - y'(0)$

$L\{4y'\} = 4sL\{y\} - y(0)$

$L\{3y\} = 3L\{y\}$

$L\{1\} = \frac{1}{s}$

$L\{-2u(t-3)\} = \frac{-2e^{-3s}}{s}$

$L\{u(t-6)\} = \frac{e^{-6s}}{s}$

laplace transform table

$L\{y'' + 4y' + 3y\} + L\{1 - 2u(t-3) + u(t-6)\}$

$s^2 L\{y\} + 4sL\{y\} + 3L\{y\} = \frac{1}{s} - \frac{2e^{-3s}}{s} + \frac{e^{-6t}}{s}$

$L\{y\}[s^2 + 4s + 3] = \frac{1 - 2e^{-3s} + e^{-6s}}{s}$

$L\{y\} = \frac{1 - 2e^{-3s} + e^{-6s}}{s(s^2 + 4s + 3)}$

$= \frac{1 - 2e^{-3s} + e^{-6s}}{s(s+3)(s+1)}$

$= \frac{1}{s(s+3)(s+1)} + \frac{-2e^{-3s}}{s(s+3)(s+1)} + \frac{e^{-6s}}{s(s+3)(s+1)}$

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$$= \left(1 - 2e^{-3s} + e^{-6s}\right) \frac{1}{s(s+3)(s+1)}$$

$$\frac{A_0}{s} + \frac{A_1}{s+3} + \frac{A_2}{s+1} = \frac{1}{s(s+3)(s+1)}$$

$$A_0(s+3)(s+1) + A_1(s)(s+1) + A_2(s)(s+3) = 1$$

$$A_0(s^2+4s+3) + A_1(s^2+s) + A_2(s^2+3s) = 1$$

$$A_0 + A_1 + A_2 = 0$$

$$4A_0 + A_1 + 3A_2 = 0$$

$$3A_0 = 1$$

$$A_0 = \frac{1}{3}$$

$eq_2 - eq_1$

$$4A_0 + A_1 + 3A_2 = 0$$
$$-\ A_0 + A_1 + A_2 = 0$$

$$3A_0 + 2A_2 = 0$$

$$A_2 = -\frac{1}{2}$$

$$\frac{1}{3} + A_1 + -\frac{1}{2} = 0$$

$$A_1 = \frac{1}{6}$$

$$\mathcal{L}\{y\} = \left(1 - 2e^{-3s} + e^{-6s}\right) \cdot \left[\frac{1}{3s} + \frac{1}{6(s+3)} - \frac{1}{2(s+1)}\right]$$

$$\mathcal{L}\{y\} = \frac{1}{3s} - \frac{2e^{-3s}}{3s} + \frac{e^{-6s}}{3s} + \frac{1}{6(s+3)} - \frac{2e^{-3s}}{6(s+3)}$$
$$+ \frac{e^{-6s}}{6(s+3)} - \frac{1}{2(s+1)} + \frac{e^{-3s}}{s+1} - \frac{e^{-6s}}{2(s+1)}$$

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$$L^{-1}\left\{\frac{1}{3s}\right\} = \frac{1}{3}$$

$$L^{-1}\left\{\frac{-2e^{-3s}}{3s}\right\} = -\frac{2}{3}u(t-3)$$

$$L^{-1}\left\{\frac{e^{-6s}}{3s}\right\} = \frac{1}{3}u(t-6)$$

$$L^{-1}\left\{\frac{1}{6(s+3)}\right\} = \frac{1}{6}e^{-3t}$$

$$L^{-1}\left\{-\frac{e^{-3s}}{3(s+3)}\right\} = -\frac{1}{3}e^{-3(t-3)}u(t-3)$$

$$L^{-1}\left\{\frac{e^{-6s}}{6(s+3)}\right\} = \frac{1}{6}e^{-3(t-6)}u(t-6)$$

$$L^{-1}\left\{-\frac{1}{2}\cdot\frac{1}{s+1}\right\} = -\frac{1}{2}e^{-t}$$

$$L^{-1}\left\{\frac{e^{-3s}}{s+1}\right\} = e^{-(t-3)}u(t-3)$$

$$L^{-1}\left\{\frac{-e^{-6s}}{2(s+1)}\right\} = -\frac{1}{2}e^{-(t-6)}u(t-6)$$

$$y = \frac{1}{3} - \frac{2}{3}u(t-3) + \frac{1}{3}u(t-6) + \frac{1}{6}e^{-3t}$$
$$- \frac{1}{3}e^{-3(t-3)}u(t-3) + \frac{1}{6}e^{-3(t-6)}u(t-6)$$
$$- \frac{1}{2}e^{-t} + e^{-(t-3)}u(t-3)$$
$$- \frac{1}{2}e^{-(t-6)}u(t-6)$$