Unit 4.2 Power Series

Definition: A **power series centered at $x = a$** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

where each a_i is some real number.

Example 1 Consider the following function defined by a power series.

$$f(x) = \sum_{n=0}^{\infty} (2x)^n = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

Let us first evaluate $f\left(\frac{1}{4}\right) = \sum_{n=0}^{\infty}\left(2 \cdot \frac{1}{4}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$

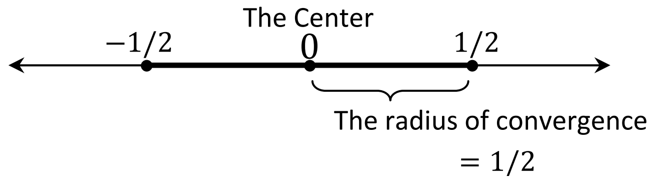
$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ (a geometric series with $a = 1$ and $r = \frac{1}{2}$)

$= \frac{1}{1-1/2} = 2$ (using the formula for the sum of a geometric series)

So $f\left(\frac{1}{4}\right) = 2$. Now look at $f(1) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + \cdots$

Once again evaluating the function/power series yields a geometric series, but in this second case the series diverges as $r = 2$. Thus we would say that $f(1)$ is undefined. This just means that 1 is not in the domain of the function, leading to the natural question: "what is the domain of this function?" To answer this question, we note that for any value of x, the function $f(x)$ is just a geometric series with $r = 2x$. The geometric series will converge so long as $|r| = |2x| < 1$. This is equivalent to $-1 < 2x < 1$ or even better $-\frac{1}{2} < x < \frac{1}{2}$. Thus the function $f(x)$ is defined on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$, which is its domain. This interval is also called the **interval of convergence** of the power series.

The terms of this power series were given by the expression $(2x)^n = 2^n(x-0)^n$. From this we can see that this series had a center $x = 0$, which is also the center of the interval of convergence given above. This is no coincidence. Let us take a look at this interval of convergence and its center on a number-line.



Half the width of the interval of convergence is called the **radius of convergence**. This terminology makes sense as it is the distance from the center to the boundary. It turns out that for any power series, the interval of convergence will always extend the same distance to the left and right of its center. However, convergence/divergence at the endpoints of the interval may differ as we will see.

So in summary we see that the function/power series

$$f(x) = \sum_{n=0}^{\infty} (2x)^n = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

has the interval of convergence $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and radius of convergence, $\frac{1}{2}$. Because it is a geometric series, we know what it converges to for values in its domain. We find that

$$f(x) = \sum_{n=0}^{\infty} (2x)^n = \frac{a}{1-r} = \frac{1}{1-2x} \quad \text{for} \quad -\frac{1}{2} < x < \frac{1}{2}$$

In other words, we have found an alternative way of expressing the ordinary function $\frac{1}{1-2x}$ for values of x between $-1/2$ and $1/2$. To see this in its most explicit form, we have shown that

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \cdots \quad \text{when} \quad -\frac{1}{2} < x < \frac{1}{2}$$

While this may not seem like the most useful thing at the moment, you can be certain that representing functions as power series can be a very useful and interesting concept. We will just get a small taste of their usefulness and "power" in this course.

Derivatives and Integrals of Power Series. Because we are in calculus, it might be advantageous to know how to differentiate and integrate functions when they are expressed in terms of their power series. This turns out to be a very simple task and begins to demonstrate the value of working with power series.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has a non − zero radius of convergence given by R.

Then f is differentiable (and thus continuous)on the interval $(a-R, a+R)$ and:

1. $f'(x) = \sum_{n=1}^{\infty} n \cdot a_n(x-a)^{n-1}$ (differentiate term – by – term)

2. $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{a_n(x-a)^{n+1}}{n+1}$ (integrate term – by – term)

There are a few notes to make regarding the above. First of all, notice that when taking the derivative, the starting value of n changed to $n = 1$. This is due to the fact that we get zero when we differentiate the constant term in the power series and do not have to include the zero when writing the derivative (when $n = 0$, we would get 0).

It is also true that the derivative and antiderivative have the same center and radius of convergence. However, they do not necessarily have the same exact interval of convergence as things may differ at the endpoints of the interval. In particular, when you differentiate a power series you <u>may</u> lose convergence at the endpoints and when you integrate you <u>may</u> gain convergence at the endpoints. This will be demonstrated in the narrated examples.

To make sure you understood 1 & 2 above, let us consider a simple example to illustrate the basic idea.

<u>Example 2</u> Given $f(x) = \sum_{n=0}^{\infty} x^n$, find $f'(x)$ and $\int f(x)dx$.

Solution:

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \underbrace{0 \ +}_{\substack{\text{if } n=0 \\ \text{were} \\ \text{included}}} 1 + 2x + 3x^2 + \cdots$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

It turns out that the interval of convergence for f and f' is $(-1,1)$, while the interval of convergence for $\int f(x)\,dx$ is $[-1,1)$ and so the antiderivatives of f are actually defined at $x = -1$ even though f and f' are both undefined at $x = -1$. An explanation of this fact and additional examples can be found in the narrated examples.

Using One Power Series to Derive Others. Recall the following fact for a geometric series.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r} \qquad \text{when } -1 < r < 1$$

In the case where $a = 1$ and $r = x$ for $-1 < x < 1$, this allowed us to represent the function $\frac{1}{1-x}$ as a power series in the following way.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad \text{when } -1 < x < 1$$

Let us now expand upon this idea using algebraic manipulations, substitutions, derivatives, and antiderivatives. Each of the aforementioned will allow us to produce power series for other functions. In each case, we must always keep in mind what the associated interval of convergence is.

Example 3 Obtain a power series for $\frac{4}{5-x}$ centered at $x = 0$.

Solution: The function $\frac{4}{5-x}$ closely resembles the form $\frac{a}{1-r}$. All we need is a basic step of algebra to put it in that form exactly. In fact

$$\frac{4}{5-x} = \frac{\frac{4}{5}}{\frac{5}{5} - \frac{x}{5}} = \frac{\frac{4}{5}}{1 - \frac{x}{5}}$$

Expressed in that way, the given function takes the form $\frac{a}{1-r}$, where $a = \frac{4}{5}$ and $r = \frac{x}{5}$. With this identification, we can express our function as a geometric power series as follows.

$$\underbrace{\sum_{n=0}^{\infty} \frac{4}{5}\left(\frac{x}{5}\right)^n}_{\sum_{n=0}^{\infty} ar^n} = \underbrace{\frac{\frac{4}{5}}{1 - \frac{x}{5}}}_{\frac{a}{1-r}} \qquad \text{when } \underbrace{\left|\frac{x}{5}\right| < 1}_{|r|<1}$$

In this case, convergence requires that $\left|\frac{x}{5}\right| < 1 \;\Rightarrow\; -1 < \frac{x}{5} < 1 \;\Rightarrow\; -5 < x < 5$. If we rewrite our power series in expanded form and simplify the expression for the sum, we obtain

$$\frac{4}{5-x} = \frac{4}{5} + \frac{4}{25}x + \frac{4}{125}x^2 + \cdots \qquad \text{when } -5 < x < 5$$

Thus we have obtained a power series for the given function. The interval of convergence for this power series is $(-5,5)$, the center is 0, and the radius of convergence is 5.

One way to think about what a power series does, is to relate it to representing a number like π in decimal form. The symbol π is a nice compact way of writing the number, but by writing $\pi = 3.14159\cdots$ we are actually viewing π as an infinite sum (each decimal place corresponds to another term added on). We all know how convenient decimal notation can be—expressing a function as a power series can also be convenient as will be seen in this and the next section.

<u>Example 4</u> Obtain a power series for $\frac{1}{x}$ centered at $x = 1$.

Solution: Again we will manipulate this function to express it in the form $\frac{a}{1-r}$. In this case, the problem is that we don't have the subtraction in our denominator. However, we can rewrite x as $1 - 1 + x = 1 - (1 - x)$. Therefore,

$$\frac{1}{x} = \frac{1}{1-(1-x)}$$

In this way, we have re-expressed $\frac{1}{x}$ in the form $\frac{a}{1-r}$ where $a = 1$ and $r = 1 - x$. Therefore, we can express $\frac{1}{x}$ as a geometric power series in the following way.

$$\boxed{\frac{1}{x} = \sum_{n=0}^{\infty} 1(1-x)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots}$$

The above equality holds true if and only if the series converges. In this case convergence requires that $|r| = |1 - x| < 1$. Let us solve this inequality for x.

$$|1-x| < 1 \quad \Rightarrow \quad -1 < 1 - x < 1 \quad \Rightarrow \quad -2 < -x < 0 \quad \Rightarrow \quad 0 < x < 2$$

We have therefore expressed $\frac{1}{x}$ as a power series with interval of convergence $(0,2)$. The center is $x = 1$ and the radius is also 1 (half the width of the interval).

Note: in the above we made use of the following simplification

$$(1-x)^n = [(-1)(x-1)]^n = (-1)^n (x-1)^n$$

Example 5 Obtain a power series for $\ln x$ centered at $x = 1$.

Solution: The last problem set us up for this one. The idea is to realize that $\ln x$ is simply an antiderivative of $\frac{1}{x}$. Thus a power series for $\ln x$ can be obtained by integrating a power series for $\frac{1}{x}$. We proceed below

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots \qquad \text{for } 0 < x < 2$$

Integrating both sides with respect to x and collecting arbitrary constants on the right we obtain

$$\ln x = C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = C + (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

By letting $x = 1$, we find that $C = 0$. Thus we have generated a power series for $\ln x$, centered at 1. While the center and radius of convergence is the same as for the original series, we may have gained convergence at the endpoints 0 and -2. Let us check them separately.

When $x = 0$, the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$$

This series is just the opposite of the harmonic series and therefore diverges.

When $x = 2$, the series becomes

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This series is the alternating harmonic series and therefore it converges.

Thus we have gained convergence at the endpoint $x = 2$ and we can now say that

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad \text{for } 0 < x \le 2$$

From this we can also see that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ as we noted in Unit 5.1.

Example 6 Obtain a power series for $x^2\ln(x^3+1)$ centered at $x=0$.

Solution: Once again, the last problem has set us up for this one. We already know that

$$\ln x = \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{n+1}}{n+1} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad \text{for } \ 0 < x \leq 2$$

In order to obtain a power series for $x^2\ln(x^3+1)$, we can simply replace x with x^3+1 and then multiply by x^2. While this can all be done at once, we will show this in two separate steps.

Step 1: Replace x with x^3+1.

$$\ln(x^3+1) = \sum_{n=0}^{\infty}(-1)^n\frac{[(x^3+1)-1]^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+3}}{n+1}$$

Step 2: Multipy by x^2.

$$x^2\ln(x^3+1) = x^2\sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+3}}{n+1} \overset{\text{distribute}}{\triangleq} \sum_{n=0}^{\infty}(-1)^n\frac{x^2\cdot x^{3n+3}}{n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+5}}{n+1}$$

Since the power series that we started with converged for $0 < x \leq 2$, and since we replaced x with x^3+1, this new power series will converge when $0 < x^3+1 \leq 2$, which upon solving for x, yields $-1 < x \leq 1$. Multiplying by x^2 has no effect on the interval of convergence. In summary we have obtained the following

$$x^2\ln(x^3+1) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{3n+5}}{n+1} = x^5 - \frac{x^8}{2} + \frac{x^{11}}{3} - \frac{x^{14}}{4} + \cdots \qquad \text{for } -1 < x \leq 1$$
