11.3: Partial Derivatives

Thursday, September 3, 2020 12:43 PM

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$. Fix $y = b$ and let x vary.

$g(x) = f(x, b)$ only depends on x. If $g'(a)$ exists,

write $f_x(a,b) = g'(a)$.

Def. 1) The partial derivative of $f(x,y)$ w.r.t. x at a point $(a,b) \in D$ is

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}, \quad \text{if the}$$

limit exists.

2) The partial deriv. of $f(x,y)$ w.r.t. y at (a,b) is

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}, \quad \text{if the}$$

lim. exists.

3) Derivatives as functions: The partial derivatives of $f(x,y)$ as functions are

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}, \quad \text{for } x, y$$

s.t. the limit exists

both func. of two variables

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}, \quad \text{for } x, y$$

for two variables

- $f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$, for x, y

s.t. lim. exists.

Notation

$$f_x = D_x f = D_1 f = f_1 = \frac{df}{dx} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = \frac{\partial z}{\partial x} = z_x$$

Rule for Partial Differentiation

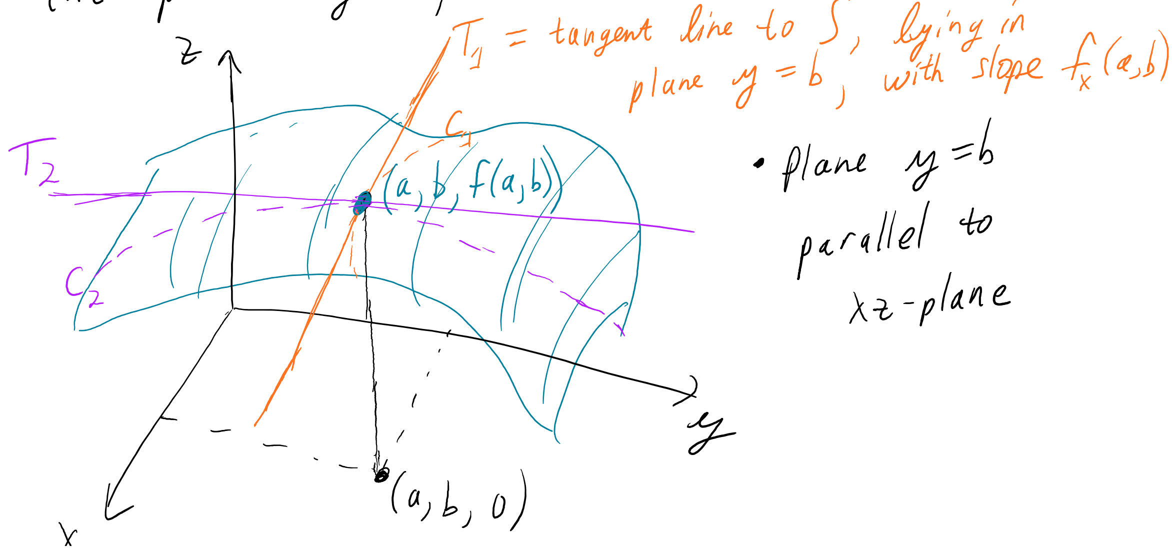
To find f_x, hold y constant, differentiate w.r.t. x.

" " f_y, " x " , " " y.

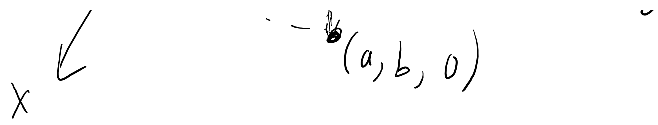
Interpretation

$f(x,y) = z$

$f_x(a,b)$ = slope of tangent line to surface $z = f(x,y)$, in the plane $y=b$, at the point $(a, b, f(a,b)) \in \mathbb{R}^3$.



- plane $y=b$ parallel to xz-plane

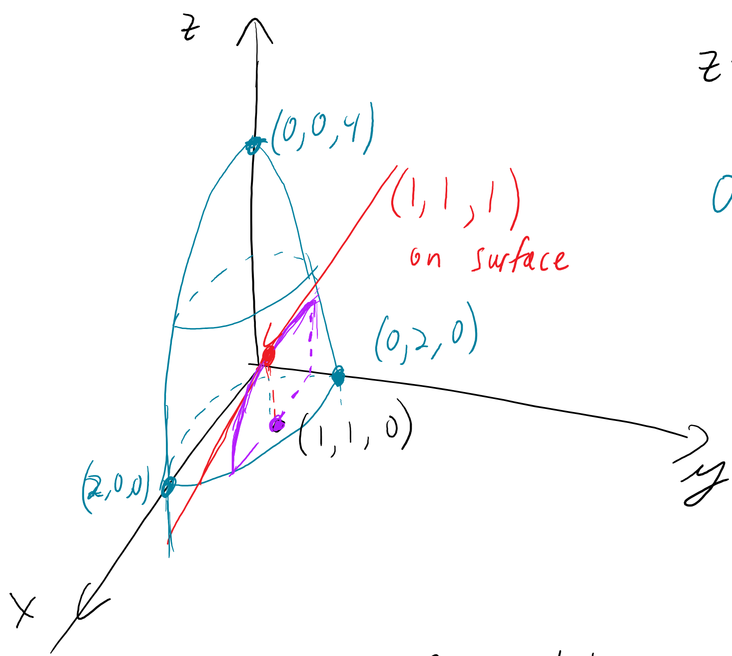


- plane $x = a$ parallel to yz-plane
- The inters. of plane $x=a$ with surface S is a curve C_2
- " " " " $y=b$ " " " " " " C_1

Ex. 2 If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$, and interpret as slopes.

Sol'n $\quad f_x(x,y) = -2x \qquad f_y(x,y) = -4y$

$f_x(1,1) = \boxed{-2} \qquad f_y(1,1) = -4$



$z = 4 - x^2 - 2y^2 = 4 - (x^2 + 2y^2)$

$0 = 4 - 2y^2$

$y = 2$

- inters. of surf. w/plane $y=1 \rightsquigarrow z = 4 - x^2 - 2$
- $x = 1,\ y = 1 \Rightarrow z = 4 - 1 - 2 = 1$

• $f_x(1,1) = -2$ = slope of tan. line to curve of intersection of S with plane $y=1$.

• $f_y(1,1) = -4$ . . .

Ex. 4 Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ if z defined implicitly as a func. of x & y by $x^3 + y^3 + 6xyz + z^3 = 1$

Sol'n (Later)

Ex. 3 $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Sol'n:

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= \cos\left(\frac{x}{1+y}\right)\frac{1}{1+y} \\ \frac{\partial f}{\partial y} &= -\cos\left(\frac{x}{1+y}\right)\frac{x}{(1+y)^2} \end{aligned}}$$

$$\boxed{\frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = \frac{0-x}{(1+y)^2}}$$

Ex. 4 (Sol'n) Do implicit differentiation.

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Find $\frac{\partial z}{\partial x}$: $\frac{\partial}{\partial x}\left(x^3+y^3+z^3+6xyz\right)=\frac{\partial}{\partial x}(1)$

$$3x^2+0+3z^2\frac{\partial z}{\partial x}+6y\left(x\frac{\partial z}{\partial x}+z\right)=0$$

$$\boxed{\frac{\partial}{\partial x}(xz)=x\frac{\partial z}{\partial x}+z(1)}$$

$$3x^2+3z^2\frac{\partial z}{\partial x}+6yx\frac{\partial z}{\partial x}+6yz=0$$

$$\frac{\partial z}{\partial x}\left(3z^2\cancel{\frac{\partial z}{\partial x}}+6yx\cancel{\frac{\partial z}{\partial x}}\right)=-3x^2-6yz$$

$$\frac{\partial z}{\partial x}=\frac{-\cancel{3}x^2-\cancel{6}^{2}yz}{\cancel{3}z^2+\cancel{6}^{2}yx}=\boxed{-\frac{x^2+2yz}{z^2+2xy}}$$

Similarly

$$\frac{\partial}{\partial y}\left(x^3+y^3+z^3+6xyz\right)=\frac{\partial}{\partial y}(1)$$

$$0+3y^2+3z^2\frac{\partial z}{\partial y}+6x\left(y\frac{\partial z}{\partial y}+z\right)=0$$

$$\boxed{\frac{\partial}{\partial y}(yz)=y\frac{\partial z}{\partial y}+z(1)}$$

$$3y^2+3z^2\frac{\partial z}{\partial y}+6xy\frac{\partial z}{\partial y}+6xz=0$$

$$\frac{\partial z}{\partial y}\left(3z^2 + 6xy\right) = -3y^2 - 6xz$$

$$\boxed{\frac{\partial z}{\partial y} = -\frac{y^2 + 2xz}{z^2 + 2xy}}$$

Func. of ≥ 3 variables

$D \subset \mathbb{R}^n$, $f: D \to \mathbb{R}$

- $$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

  if lim. exists.

- $$f_x(x,y,z) = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x,y,z)}{h}$$

- $$f_{x_i} = \frac{\partial f}{\partial x_i} = D_i f = f_i$$

Ex. 5 Find f_x, f_y, f_z if $f(x,y,z) = e^{xy}\ln(z)$

Sol'n: $f_x = y\ln(z)e^{xy}$

$$f_y = x\ln(z)e^{xy}$$

$$f_z = e^{xy}/z$$

Higher Derivatives

Def. The 2nd partial derivatives of $f(x,y)$ are

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f_x = f_{xx}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial y}f_x = f_{xy}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}f_y = f_{yx}$$

mixed partial of f

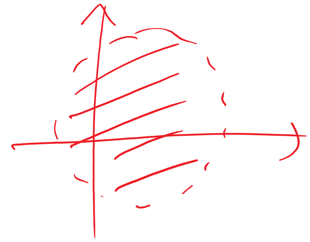
$$\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f_y = f_{yy}$$

Note $f_{xy} = (f_x)_y$ means diff. w.r.t. x, then y

Def. 1) A disk $D \subset \mathbb{R}^2$ is an open ball:

boundary not included

$$D = \{(x,y) : (x-h)^2 + (y-k)^2 < r^2\}$$



2) The open unit disk (unit ball) in $\mathbb{R}^2$ is

$$D = \{(x,y) : x^2 + y^2 < 1\}$$

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ex. $f(x,y) = x^3 + x^2y^3 - 2y^2$; find 2nd partials.

Sol'n: $f_x(x,y) = 3x^2 + 2xy^3$

$f_y(x,y) = 3x^2y^2 - 4y$

$f_{xx} = (f_x)_x = 6x + 2y^3$ | $f_{yx} = (f_y)_x = 6xy^2$

$f_{xy} = (f_x)_y = 6xy^2$ | $f_{yy} = (f_y)_y = 6x^2y - 4$

Note $f_{xy} = f_{yx}$ not coincidence

Clairaut's Th'm Let D be a disk in $\mathbb{R}^2$, and $(a,b) \in D$. If f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

<u>Rmk</u>

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

Clairaut $\Rightarrow$ $f_{xyy} = f_{yxy} = f_{yyx}$ if all continuous.
