12.5: Triple Integrals

Friday, October 2, 2020 11:12 AM

- Instead of integrating $f(x,y)$ over rectangle R in 2-D, need to integrate $f(x,y,z)$ " box in 3-D (solid box)

$B = [a,b] \times [c,d] \times [r,s]$ is a 3-D solid box

$= \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$

- volume of a sub-box $= \Delta V_{ijk} = \Delta x_i \, \Delta y_j \, \Delta z_k$

$$\rightsquigarrow \sum_i \sum_j \sum_k f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk}$$

(triple Riemann sum)

<u>Def.</u> The triple integral of f over the box $B = [a,b] \times [c,d] \times [r,s]$ is

$$\iiint_B f(x,y,z)\, dV = \lim_{\max \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_i \sum_j \sum_k f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V_{ijk},$$

<u>if</u> the limit exists.

<u>Fubini's Theorem</u> If f continuous on B, $= \overset{x}{[a,b]} \times \overset{y}{[c,d]} \times \overset{z}{[r,s]}$ then

$$\iiint_B f\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz,$$ for any order of integration (6 possible):

order of integration

$$\int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz = \int_r^s \int_a^b \int_c^d f(x,y,z)\, dy\, dx\, dz$$

$$= \int_c^d \int_r^s \int_a^b f(x,y,z)\, dx\, dz\, dy$$

$$= \int_a^b \int_r^s \int_c^d f \; dy\, dz\, dx$$

$$= \int_c^d \int_a^b \int_r^s f \; dz\, dx\, dy$$

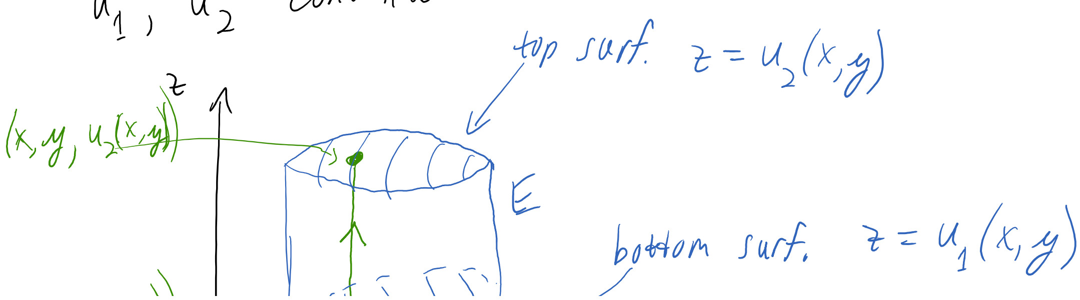
$$= \int_a^b \int_c^d \int_r^s f \; dz\, dy\, dx$$

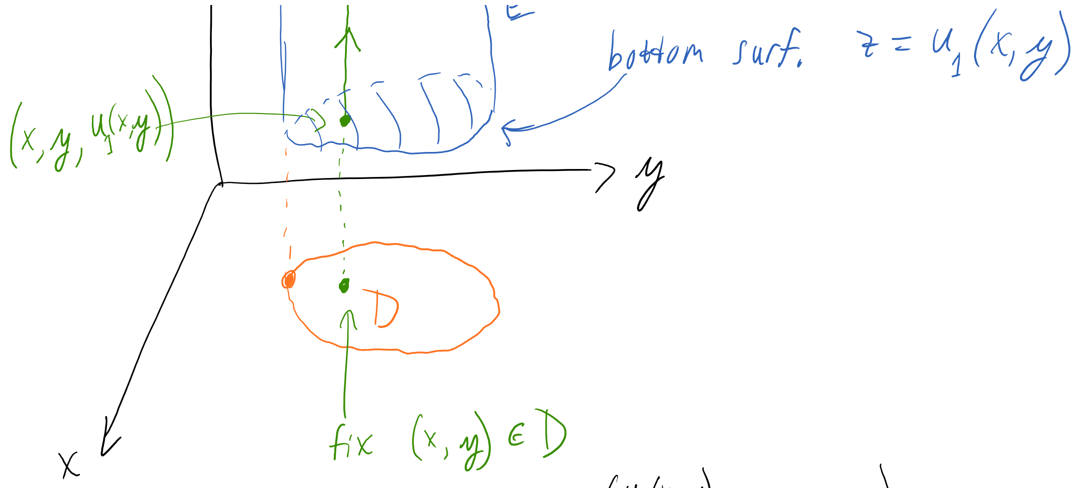
Regions more general than boxes?

Type I region in $\mathbb{R}^3$ (surface on bottom $z = u_1(x,y)$ and another surf. on top $z = u_2(x,y)$)

$$E = \{ (x,y,z) : (x,y) \in D,\; u_1(x,y) \le z \le u_2(x,y) \},$$

where D = projection of E onto xy-plane, and u_1, u_2 continuous on D.





$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

• method for $\iint_D$ depends on which type of region D is (type I, II, polar?)

(i) D type I (in $\mathbb{R}^2$)

$$E = \{(x,y,z) : a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$$

$$\iiint_E f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dy\,dx$$

(ii) D type II 2-D region

$$E = \{(x,y,z) : \boxed{c \le y \le d},\ h_1(y) \le x \le h_2(y),\ u_1(x,y) \le z \le u_2(x,y)\}$$

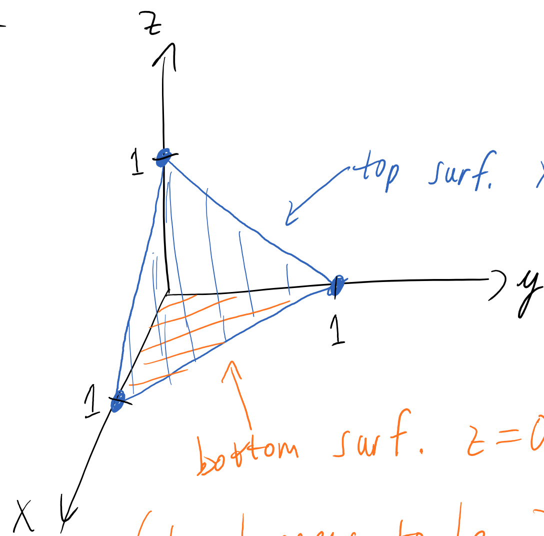
$$\iiint_E f\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right) dx\, dy$$

<u>Ex.</u> Evaluate $\iiint_E z\, dV$, E solid tetrahedron bdd by $x=0$, $y=0$, $z=0$, $x+y+z=1$.

$\vec{n} = \langle 1,1,1 \rangle$

<u>Sol'n:</u>

$x=0,\ y=0 \Rightarrow z=1$



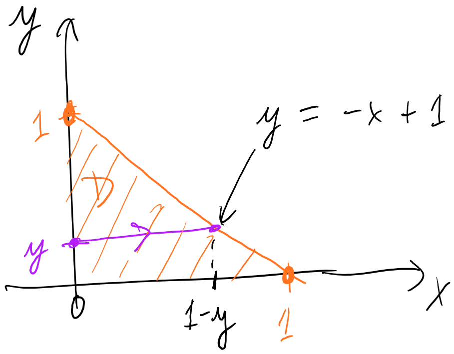
top surf. $x+y+z=1$

$z = 1-x-y$

$= u_2(x,y)$

bottom surf. $z=0=u_1(x,y)$

(also happens to be D here)



$y=-x+1 \Rightarrow x = 1-y$

$$\iiint_E z\, dV = \iint_D \left(\int_0^{1-x-y} z\, dz \right) dA$$

$$= \int_0^1 \int_0^{1-y} \left(\int_0^{1-x-y} z\, dz \right) dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \frac{z^2}{2}\Big|_0^{1-x-y} \, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-y} (1-x-y)^2 \, dx\, dy, \quad u = 1-x-y$$

$\frac{du}{dx} = -1$

$du = -dx$

$\underline{x=0} \Rightarrow u = 1-y$

$x = 1-y \Rightarrow u = 1-(1-y)-y$

$= 1-1+y-y$

$= 0$

$$= \frac{1}{2}\int_0^1 \int_{1-y}^{0} u^2 (-du)\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-y} u^2 \, du\, dy$$

$$= \frac{1}{2}\int_0^1 \frac{u^3}{3}\Big|_0^{1-y} dy$$

$$= \frac{1}{6}\int_0^1 (1-y)^3\, dy, \quad t = 1-y$$

$dt = -dy$

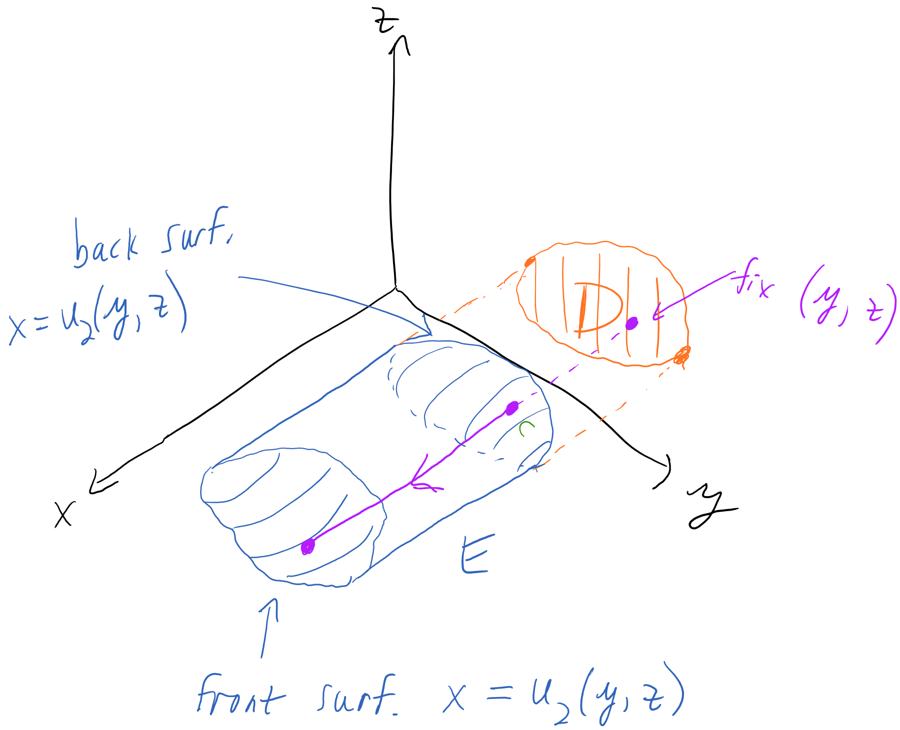
$$= \frac{1}{6}\int_0^1 t^3\, dt$$

$$= \boxed{\frac{1}{24}}$$

Type II region in 3D (surface on back $x = u_1(y,z)$ and on front $x = u_2(y,z)$)

$$E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$$

D = proj. of E onto yz-plane

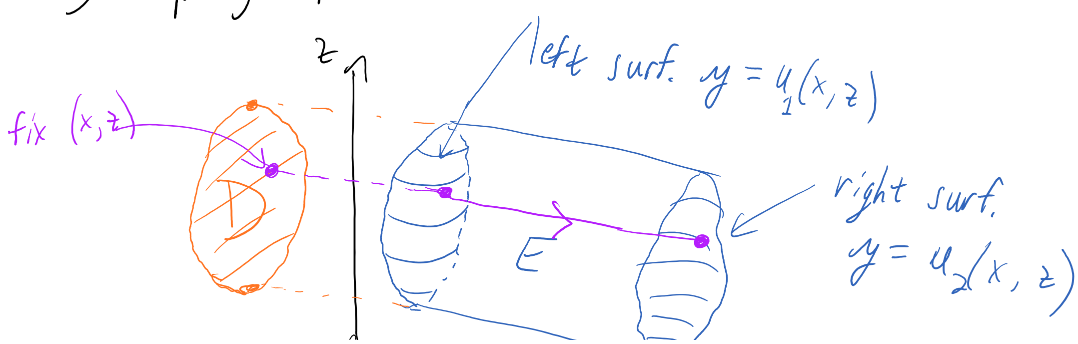


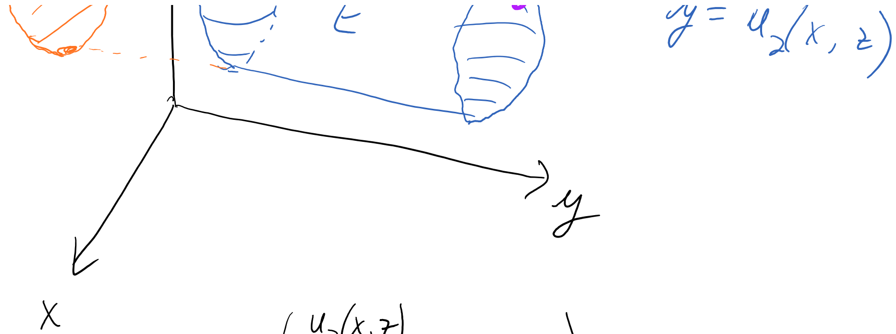
$$\iiint_E f \, dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) \, dx \right) dA$$

<u>Type III region in $\mathbb{R}^3$</u> (surf. on left $y = u_1(x, z)$ and on right $y = u_2(x, z)$)

$$E = \{ (x, y, z) : (x, z) \in D, \; u_1(x, z) \le y \le u_2(x, z) \}$$

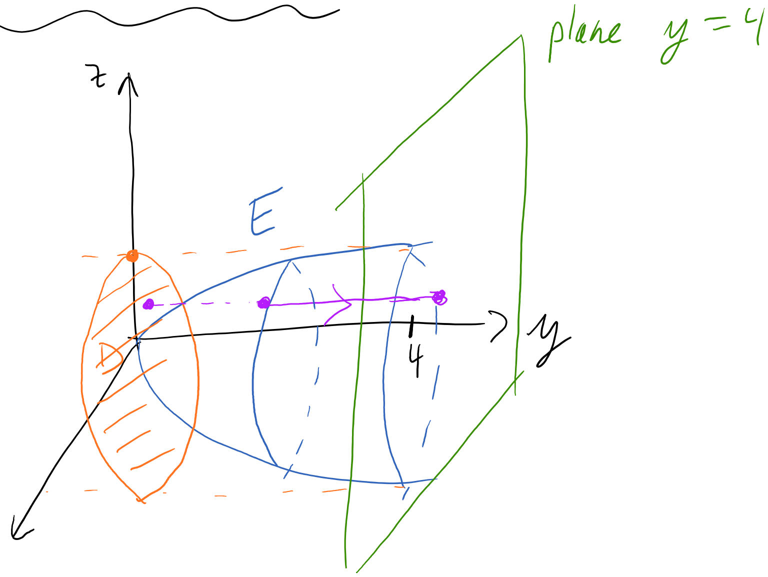
D = proj. of E onto xz-plane



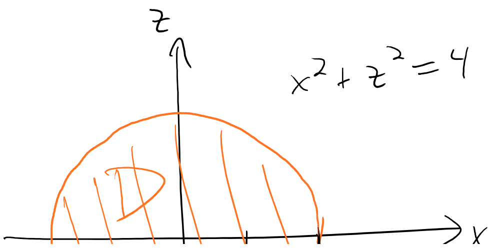


$$\iiint_E f \, dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right) dA$$

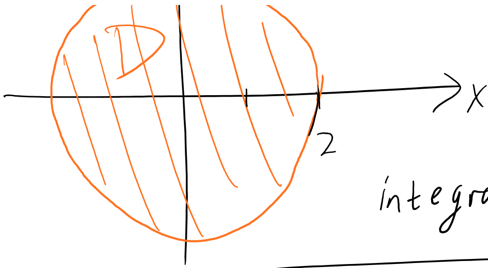
Ex. $\iiint_E \sqrt{x^2+z^2} \, dV$, E bdd by paraboloid $y = x^2 + z^2$ and plane $y = 4$.



As type III: • left surf. $y = x^2 + z^2 = u_1(x, z)$
• right " $y = 4 = u_2(x, z)$
• D is disk cent. at $(0,0)$, radius = 2



$$\iiint_E \sqrt{x^2+z^2} \, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2} \, dy \right)$$



$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{x^2+z^2} \sqrt{x^2+z^2}\, dy \right)$$

integrate over D in polar:

$$= \int_0^{2\pi}\int_0^{2} \left(\int_{r^2}^{4} r^2 \quad dy \right) \cancel{r}\, dr\, d\theta$$

$x = r\cos(\theta)$

$z = r\sin(\theta)$

$\sqrt{x^2+z^2} = \sqrt{r^2} = r$

$(r \geq 0)$

$$= \int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} r^2 \, dy\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} r^2 \left(\int_{r^2}^{4} dy \right) dr\, d\theta$$

$4r^2 - r^4$

$$= \int_0^{2\pi} \int_0^{2} r^2 (4 - r^2)\, dr\, d\theta$$

$$= 2\pi \left[\frac{4}{3} r^3 - \frac{r^5}{5} \right]_0^2$$

$$= 2\pi \left(\frac{4}{3}(2^3) - \frac{2^5}{5} \right)$$

$$= 2^6 \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

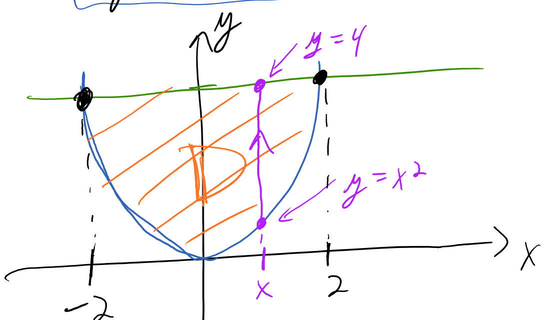
$$= 2^6 \pi \left(\frac{2}{15} \right)$$

$$= \boxed{\frac{128\pi}{15}}$$

As type I:

$y = x^2 + z^2$, $y = 4$ $z = 0 \Rightarrow y = x^2$

$y = x^2 + z^2$, $\quad y = 4$ $\quad\longrightarrow\quad z = 0 \Rightarrow y = x^2$



$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} \sqrt{x^2+z^2}\, dz \right) dA$$

$$= \iint_D \left(\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz \right)$$

$y = x^2 + z^2$

$z^2 = y - x^2$

$z = \pm\sqrt{y-x^2}$

$$= \int_{-2}^{2} \int_{x^2}^{4} \left(\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz \right) dy\, dx \qquad \left(\text{type I in 3-D}\right)$$

As type II (project E onto yz-plane)

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{-\sqrt{y-z^2}}^{\sqrt{y-z^2}} \sqrt{x^2+z^2}\, dx \right),$$

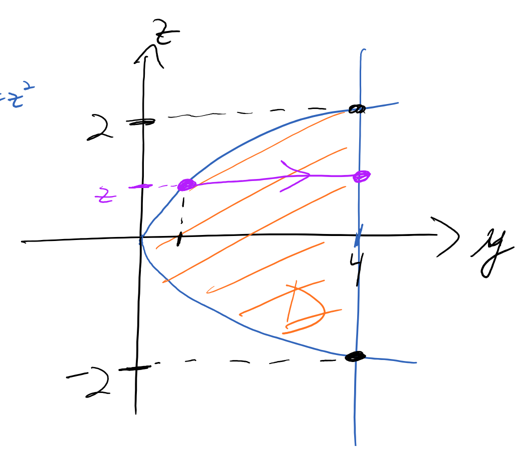
$y = x^2 + z^2$

$x = \pm\sqrt{y - z^2}$

$$= \int_{-2}^{2} \int_{z^2}^{4} \int_{-\sqrt{y-z^2}}^{\sqrt{y-z^2}} \sqrt{x^2+z^2}\, dx$$

$x=0$

$y = z^2$

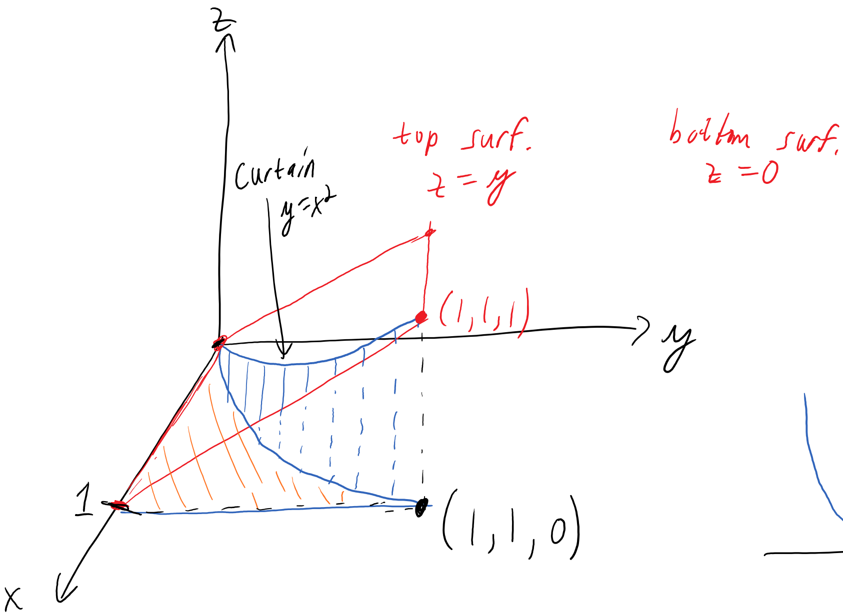


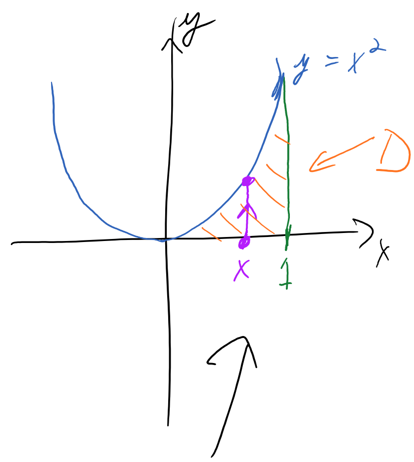
$*$ Easiest to do $\iint_D$ when D disk (i.e. type $\overline{\underline{III}}$)

Ex. Change $\displaystyle\int_{0=x}^{1=x}\int_{0=y}^{x^2=y}\int_{0=z}^{y=z} f(x,y,z)\,dz\,dy\,dx$ to integrate w.r.t. x, z, y, in that order (i.e. $dx\,dz\,dy$)

$$:= \iint_D \left(\int_0^y f\,dz\right) dA$$

Sol'n:





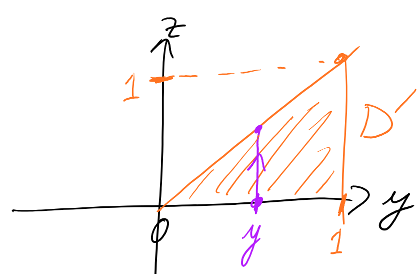
$$\iiint_E f\,dV = \iint_{D'}\left(\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right)$$

$$= \iint_{D'}\left(\int_{\sqrt{y}}^{1} f(x,y,z)\,dx\right)$$

$y = x^2$

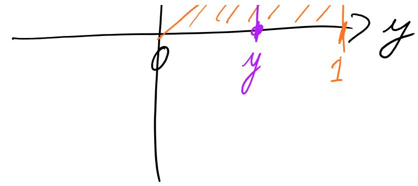
$x = \pm\sqrt{y}$

$$= \boxed{\int_0^1\int_0^y\int_{\sqrt{y}}^1 f(x,y,z)\,dx\;dz\;dy}$$



$_0 \qquad dy$

(type II)



Note

1) $E \subset \mathbb{R}^3$, $\quad vol(E) = \iiint_E 1 \, dV$

- $\int_a^b 1 \, dx = b - a$. If $a < b$, then $\int_a^b dx$ = length of $[a,b]$

$$= \int_{[a,b]} dx$$

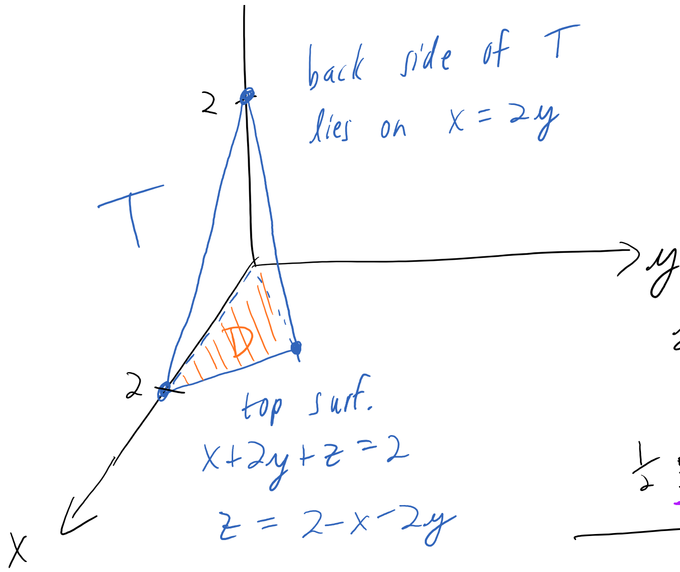
- $\iint_R 1 \, dA = \text{area}(R)$

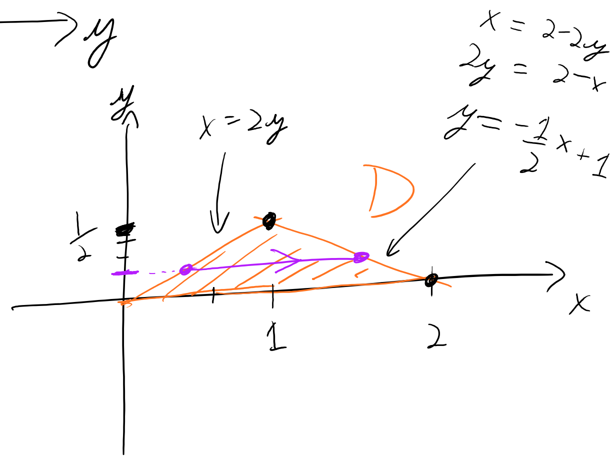
2) E type I in 3-D $\Rightarrow \iiint_E dV = vol(E)$

$$= \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} dz \right) dA$$

$$= \iint_D u_2(x,y) - u_1(x,y) \, dA$$

Ex. Use a triple int. to find the vol. of the tetrahedron T bdd by $x + 2y + z = 2$, $x = 2y$, $x = 0$, $y = 0$.





$$V = \iiint_T dV = \iint_D \left(\int_0^{2-x-2y} dz \right) dA$$

$$= \int_0^{1/2} \int_{2y}^{2-2y} \int_0^{2-x-2y} 1 \; dz \; dx \; dy$$

$$= \boxed{\frac{1}{3}}$$