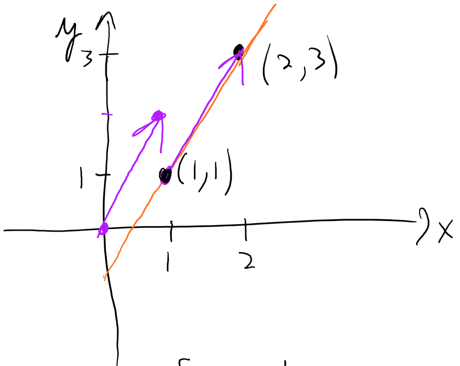
10.5: Equations of Lines and Planes

Thursday, August 13, 2020 10:00 PM

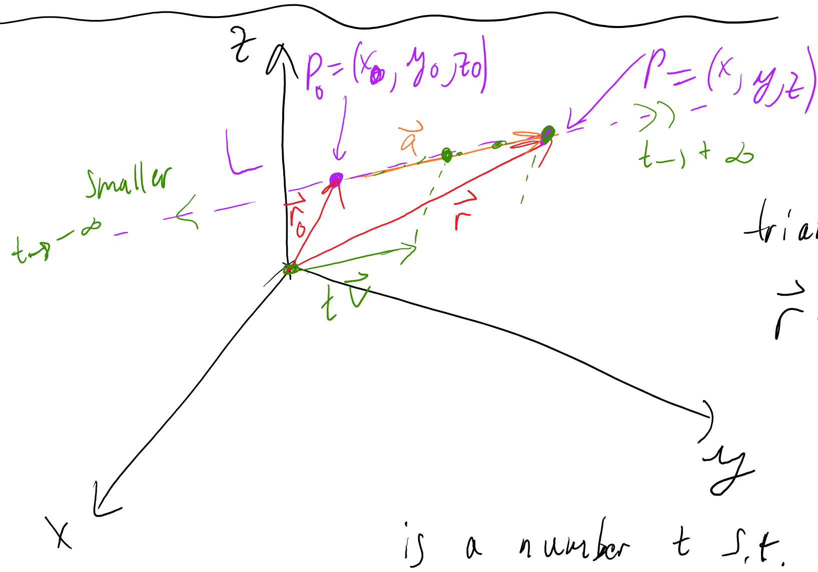
- In 2-D (i.e. in $\mathbb{R}^2$), which line connects $(1,1)$ to $(2,3)$? Parametric equation?
  - Could do $y-1=2(x-1)$, or start at $(1,1)$ and draw a line in direction of $(2,3)$:



$\langle 2-1, 3-1\rangle = \langle 1, 2\rangle$ has correct direction.

So starting at term. pt of $\langle 1,1\rangle$ going toward $\langle 2,3\rangle$, can use $r(t)=(1-t)\langle 1,1\rangle + t\langle 2,3\rangle$,

$-\infty < t < \infty$.



triangle law: $\vec{r}=\vec{r_0}+\vec{a}$

$\vec{a}\parallel\vec{v}$, so there is a number t s.t. $\vec{a}=t\vec{v}$.

$\therefore \underbrace{\vec{r}} = \underbrace{\vec{r_0}+t\vec{v}}$ varies with t

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Def. 1) The vector eq'n of the line L through $P_0=(x_0,y_0,z_0)$ in the direction of $\vec{v}=\langle a,b,c\rangle$ is $\vec{r}=\vec{r_0}+t\vec{v}$.

in the direction of $\vec{v} = \langle a, b, c \rangle$ is $\vec{r} = \vec{r}_0 + t\vec{v}$,

where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$.

(or $\vec{r} = \vec{r}_0 + t\vec{v} = \langle x_0, y_0, z_0 \rangle + \cancel{t}\langle ta, tb, tc \rangle$

$$\langle x, y, z \rangle = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$

2) The <u>parametric eq'ns</u> of L are

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

3) If two of a, b, c are $\neq 0$, can eliminate t and get the <u>symmetric eq'ns</u> of L:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

4) a, b, c are the <u>direction numbers of L</u>.

Any parallel vector to $\vec{v} = \langle a, b, c \rangle$ is OK, so any $k\langle a, b, c \rangle$, $k \neq 0$.

<u>Ex. 1:</u> (a) Find vector & parametric eq'ns for line through $(5, 1, 3)$ and is parallel to $\vec{i} + 4\vec{j} - 2\vec{k}$.

b) Find two other points on the line.

(b) Find two other points on the line.

~~(c) Below~~

(a) $\vec{r}_0 = \langle 5, 1, 3 \rangle = 5\vec{i} + \vec{j} + 3\vec{k}$, $\vec{v} = \vec{i} + 4\vec{j} - 2\vec{k}$

$$\boxed{\vec{r} = \underbrace{\vec{r}_0 + t\vec{v}}_{} = (5+t)\vec{i} + (1+4t)\vec{j} + (3-2t)\vec{k}}$$

vector eq'n

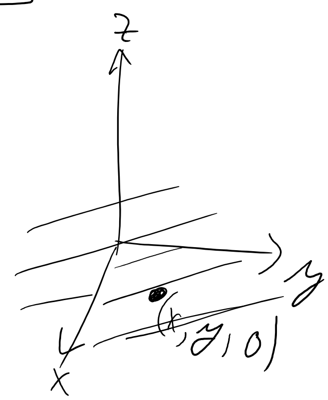
$$\boxed{x = 5+t, \quad y = 1+4t, \quad z = 3-2t \quad \text{param. eq'ns}}$$

(b) Choose any 2 convenient t:

$\underline{t=1}$ $\quad x = 6, \; y = 5, \; z = 1 \rightsquigarrow \boxed{(6, 5, 1)}$

$\underline{t=-1}$ $\quad x = 4, \; y = -3, \; z = 5 \rightsquigarrow \boxed{(4, -3, 5)}$

(c) At which point does L intersect xy-plane?



$\underline{z=0}$ $\quad 3 - 2t = 0 \Rightarrow 3 = 2t$

$t = \frac{3}{2}$

$\Rightarrow x = 5 + \frac{3}{2}, \; y = 1 + 4\left(\frac{3}{2}\right), \; z = 0$

$\Rightarrow x = \frac{13}{2}, \; y = 7, \; z = 0$

$\rightsquigarrow \boxed{\left(\frac{13}{2}, 7, 0\right)}$

Fact Let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\vec{r}_1 = \langle x_1, y_1, z_1 \rangle$.

The line segment from (x_0, y_0, z_0) to (x_1, y_1, z_1)

has vector eq'n $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$.

(Also works in n-D for all $n \in \mathbb{N}$).

<u>Def.</u> Lines L_1 & L_2 are <u>skew</u> if they are neither parallel, nor have intersection.

<u>ex.</u> x-axis: $\vec{r} = \vec{r}_0 + t\vec{v}$

$= \vec{0} + t\vec{i}$

$\vec{r} = t\vec{i}$

y-axis, translated up one unit: $\vec{r} = \vec{r}_0 + t\vec{v}$

$= \vec{0} + t\vec{j} = \langle 0,0,0 \rangle + \langle 0,t,0 \rangle$

$= \langle 0, t, 0 \rangle$

move up $\rightsquigarrow$ $\langle 0, t, 1 \rangle = t\vec{j} + \vec{k}$

skew

$\vec{r} = t\vec{j} + \vec{k}$.

<u>Ex. 3</u> Show L_1, L_2 are skew, where

L_1: $x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$

L_2: $x = 2s$, $y = 3 + s$, $z = -3 + 4s$.

<u>Sol'n:</u> The lines aren't parallel $\rightsquigarrow$ why?

If $L_1 \parallel L_2$, then their direction vectors would be scalar multiples.

scalar multi...

coeff. of t in parametric eq'ns:

L_1 has dir. vector $\langle 1, 3, -1 \rangle$

L_2 " " " $\langle 2, 1, 4 \rangle$

There is $c \neq 0$ s.t. $\langle 1, 3, -1 \rangle = c \langle 2, 1, 4 \rangle$

$\Rightarrow 1 = 2c \Rightarrow c = \frac{1}{2} \Rightarrow 3 = \frac{1}{2}$ (contradiction)

$\therefore L_1, L_2$ not parallel.

Check intersection:

$1 + t = 2s$, $-2 + 3t = 3 + s$, $4 - t = -3 + 4s$

$\Rightarrow t = 2s - 1$, $\Rightarrow -2 + 3(2s-1) = 3 + s$

$\Rightarrow -2 + 6s - 3 = 3 + s$

$5s = 8$

$s = \frac{8}{5}$

$t = 2\left(\frac{8}{5}\right) - 1 = \frac{11}{5}$

$4 - \frac{11}{5} = -3 + 4\left(\frac{8}{5}\right)$

$7 = \frac{51}{5} > 10$

$\Rightarrow$ no solutions

$\Rightarrow L_1 \cap L_2 = \phi$

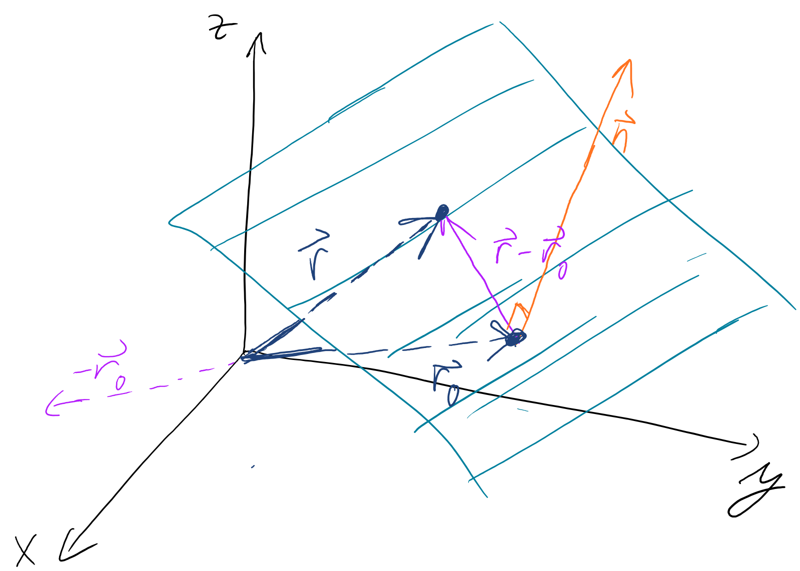
$\therefore L_1, L_2$ skew ✓

Planes Need 3 points to determine a plane, or a point and a vector orthogonal to the plane will do.

Def. 1) A vector $\vec{n} \neq \vec{0}$ is orthogonal to a plane if $\vec{n}$ is orthogonal to every vector in the plane.

2) A vector $\vec{n}$ orthogonal to a plane is a normal vector of that plane.

Fact. Let $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{r} = \langle x, y, z \rangle$ lie on a plane. Then $\vec{r} - \vec{r}_0$ also lies in that plane.



So if $\vec{n}$ is a normal vector, then $\vec{n} \perp (\vec{r} - \vec{r}_0)$, i.e.

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad (5), \quad \text{or} \quad \vec{n} \cdot \vec{r} - \vec{n} \cdot \vec{r}_0 = 0$$

$$\text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot r_0 \quad (6)$$

<u>Def.</u> 1) Equations (5), (6) are the <u>vector equations of the plane</u>.

2) $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$$\overset{\vec{n}}{\langle a, b, c \rangle} \cdot \overset{(\vec{r}-\vec{r}_0)}{\langle x - x_0, y - y_0, z - z_0 \rangle} = 0$$

$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \qquad (7)$$

3) Eq'n (7) is the scalar eq'n of the plane through (x_0, y_0, z_0), with normal vector $\langle a, b, c \rangle$.

(*Can read scalar eq'n to identify $\vec{n} = \langle a, b, c \rangle$)

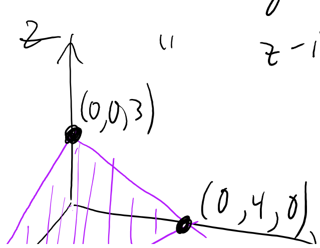
<u>Note</u> $ax + by + cz + d = 0$, where $d = -(ax_0 + by_0 + cz_0)$.
(<u>linear equation in x, y, z</u>)

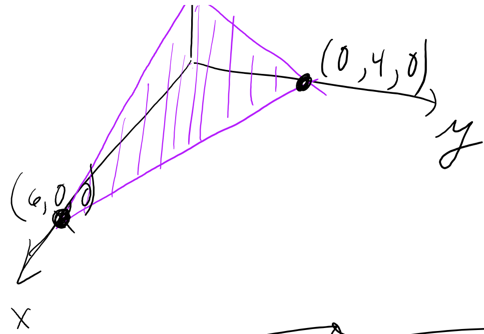
<u>Ex. 4</u> Find an eq'n of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 2, 3, 4 \rangle$. Find the intercepts & sketch the plane.

<u>Sol'n:</u> $a = 2$, $b = 3$, $c = 4$, $x_0 = 2$, $y_0 = 4$, $z_0 = -1$

$$\Rightarrow \boxed{2(x-2) + 3(y-4) + 4(z+1) = 0}$$

<u>intercepts:</u> For x-int., put $y = z = 0 \Rightarrow x = 6. \rightsquigarrow (6, 0, 0)$
" y-int., " $x = z = 0 \Rightarrow y = 4 \rightsquigarrow (0, 4, 0)$
" z-int., " $x = y = 0 \Rightarrow z = 3 \rightsquigarrow (0, 0, 3)$



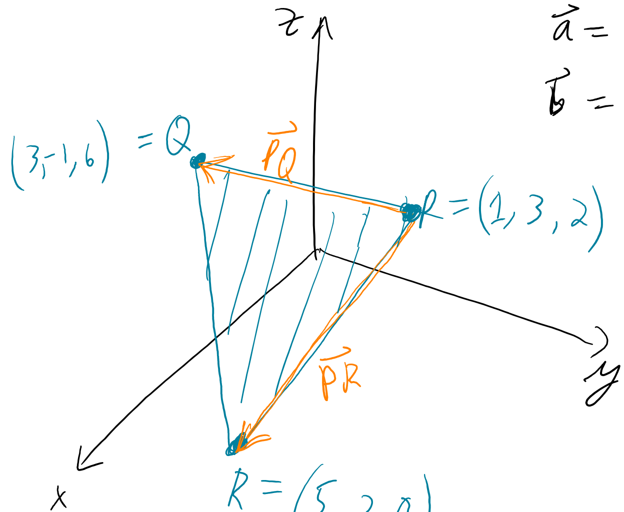


Note Given a linear eq'n in x, y, z like $ax+by+cz+d=0$, if one of a, b, c is $\neq 0$, then this is the eqn of a plane w/normal vector $\langle a, b, c\rangle$.

Ex. 5 Find an eq'n of plane through $P=(1,2,3)$, $Q=(3,-1,6)$, $R=(5,2,0)$.

Find $\vec{n}$ by using two vectors in plane $\vec{a}, \vec{b}$, then $\vec{n} = \vec{a} \times \vec{b}$.

$$\vec{a} = \overrightarrow{PQ} = \langle 2, -4, 4\rangle$$
$$\vec{b} = \overrightarrow{PR} = \langle 4, -1, -2\rangle$$



$\vec{a}, \vec{b}$ both lie in plane, so $\vec{a} \times \vec{b}$ is orthogonal to plane, and can take $\vec{n} = \vec{a} \times \vec{b}$ normal vect.

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\vec{i} + 20\vec{j} + 14\vec{k}$$

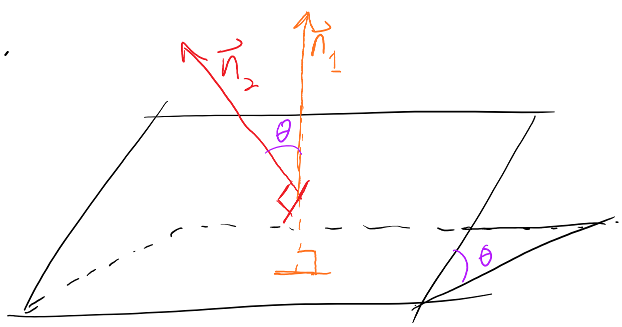
Use $P=(1,3,2)$ and $\vec{n}=\langle 12, 20, 14\rangle$ for eqn:

$$12(x-1)+20(y-3)+14(z-2)=0 \quad \underline{\text{or}}$$

$$6x+10y+7z=50.$$

(another normal vect. is $\frac{1}{2}\vec{n}=\langle 6,10,7\rangle$)

<u>Def.</u> Two planes are <u>parallel</u> if their normal vectors are parallel.



<u>Ex. 6</u> (a) Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

(b) Find symmetric eq'ns for the line of intersection L of these 2 planes.

<u>Sol'n:</u> (a) $\vec{n}_1=\langle 1,1,1\rangle$, $\vec{n}_2=\langle 1,-2,3\rangle$.

Recall $\vec{n}_1\cdot\vec{n}_2=|\vec{n}_1|\cdot|\vec{n}_2|\cos(\theta) \Rightarrow \cos(\theta)=\dfrac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1|\cdot|\vec{n}_2|}$

$$\cos(\theta)=\frac{1-2+3}{\sqrt{1+1+1}\,\sqrt{1+4+9}}=\frac{2}{\sqrt{42}}$$

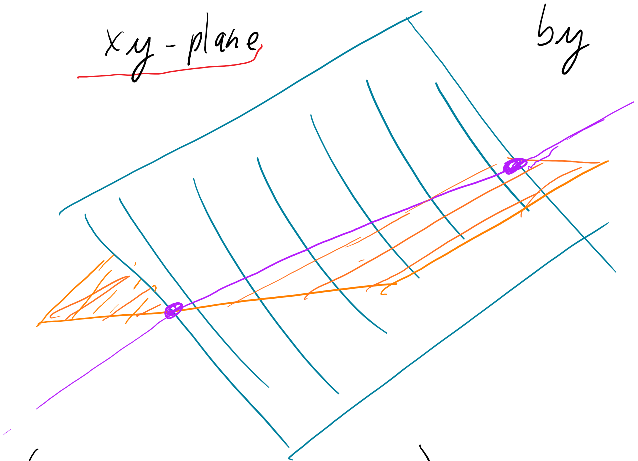
$$\Rightarrow \boxed{\theta=\cos^{-1}\left(\frac{2}{\sqrt{42}}\right)}$$

maybe try xz or yz too

$\theta = \cos^{-1}\left(\frac{\quad}{\sqrt{42}}\right)$

maybe try xz or yz too

(b) First, find a point on L. Use point where L intersects xy-plane by putting $z=0$ for both planes.



$\rightsquigarrow x+y=1$ and $x-2y=1$

$\Rightarrow x=1,\ y=0,\ z=0$

$\rightsquigarrow (1,0,0)$ lies on L

(Graph with Calcplot 3D)

Note L lies in both planes $\Rightarrow$ L perp. to both $\vec{n}_1$ & $\vec{n}_2$.

Can use dir. vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$ for L:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k}$$

$a=5$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

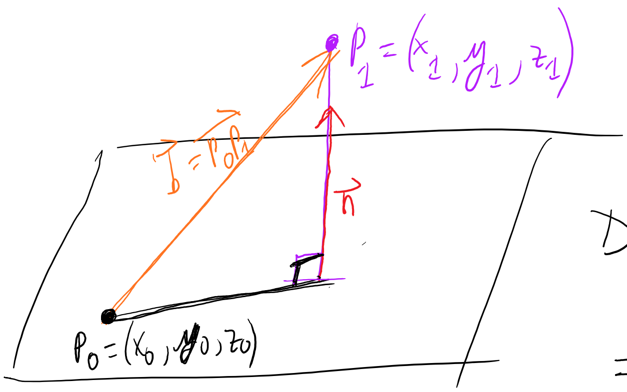
$$\boxed{\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}}$$

Fact: The distance from (x_1, y_1, z_1) to a plane $ax+by+cz+d=0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

$\vec{n} = \langle a, b, c \rangle$

$\vec{n} = \langle a, b, c \rangle$



$\vec{b} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$$D = \left| \operatorname{comp}_{\vec{n}} \vec{b} \right| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|}$$

$$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

B/c P_0 lies in plane, it satisfies eq'n of plane.

$$\Rightarrow ax_0 + by_0 + cz_0 + d = 0$$

$$\Rightarrow D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$