5.1 Preliminary Theory - Linear Systems

- *Definition:* A system of linear first-order differential equations that have the normal form

$$\frac{dx_1}{dt} = g_1(t, x_1, x_2, \dots, x_n)$$
$$\frac{dx_2}{dt} = g_2(t, x_1, x_2, \dots, x_n),$$
$$\vdots$$
$$\frac{dx_n}{dt} = g_n(t, x_1, x_2, \dots, x_n)$$

  is called a **first-order system**.

When each of the functions $g_1, g_2, \dots, g_n$ are linear in the dependent variables $x_1, x_2, \dots, x_n$, then we get the normal form

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t)$$
$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t).$$
$$\vdots$$
$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)$$

We will refer to this system simply as a **linear system**. When $f_i(t) = 0, i = 1,2, \dots, n$, the linear system is said to be **homogeneous**.

Matrix Form: If $\mathbf{X}, \mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \qquad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

then the linear system can be written as

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F}.$$

- *Definition:* A **solution vector** on an interval I is any column matrix

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

  whose entries are differential functions satisfying the linear system on the interval.

- *Example:* Write the following system in matrix form and verify that the solution vector.

$$\begin{aligned} x' &= x + 3y \\ y' &= 5x + 3y \end{aligned} \qquad x(t) = e^{-2t}, \qquad y(t) = -e^{-2t}.$$

Solution: In matrix form, we get

$$\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\mathbf{X}, \qquad \mathbf{X_1}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

To verify the solution, we first need to find the derivative of $\mathbf{X_1}(t)$

$$\mathbf{X}_\mathbf{1}'(t) = \begin{pmatrix} -2 \\ 2 \end{pmatrix} e^{-2t}.$$

Next, we substitute in $\mathbf{X_1}(t)$ and simplify to get

$$\mathbf{AX_1} = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-2t} - 3e^{-2t} \\ 5e^{-2t} - 3e^{-2t} \end{pmatrix} = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \mathbf{X}_\mathbf{1}'(t).$$

**When dealing with only two independent variables, it is common practice to use x and y as opposed to x_1 and x_2.*

Much of the theory of systems of n linear-first order differential is similar to that of linear n^{th}-order differential equations. For that reason, we will leave the proofs of most of the theorems as exercises.

<u>Theorem:</u> **Existence of a Unique Solution**

Let the entries of the matrices $\mathbf{A}(t)$ and $\mathbf{F}(t)$ be continuous functions on a common interval I that contains the point t_0. Then there exists a unique solution to the initial-value problem

$$\mathbf{X}' = \mathbf{AX} + \mathbf{F}, \qquad \mathbf{X}(t_0) = \mathbf{X_0}$$

on the interval I.

<u>Theorem:</u> **Superposition Principle**

Let $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_k}$ be a solution set of vectors of the homogeneous linear system on an interval I. Then the linear combination

$$c_1\mathbf{X_1} + c_2\,\mathbf{X_2} + \cdots + c_k\mathbf{X_k},$$

where $c_i, i = 1,2, \dots, k$ are arbitrary constants, is also a solution on the interval.

- *<u>Definition:</u>* Let $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_k}$ be a set of solution vectors of the homogeneous system on an interval I. We say that the set is **dependent** on the interval if there exist constants $c_1, c_2, \dots, c_k$, not all zero, such that

$$c_1\mathbf{X_1} + c_2\,\mathbf{X_2} + \cdots + c_k\mathbf{X_k} = \mathbf{0}$$

for every t in the interval. If the set is not linearly dependent, it is said to be **linearly independent**.

Theorem: **Criterion for Linearly Independent Solutions**

Let $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_n}$ be solution vectors of the homogeneous linear system on an interval I. Then the set of solution vectors is linearly independent on I if and only if the **Wronskian**

$$W(\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_k}) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \dots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \dots & x_{2n}(t) \\ \vdots & & & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix} \neq 0$$

for every t in the interval.

**Once again it can be shown that for every t in I, $W = 0$, or for every t in I, $W \neq 0$. Therefore, if we can show that $W \neq 0$ for some t_0, then our set is linearly independent on I.*

- *Definition:* Any set $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_n}$ of n linearly independent solution vectors of the linear homogeneous system of n equations on an interval I is said to be a **fundamental set of solutions** on I.

Theorem: **Existence of a Fundamental Set**

There exists a fundamental set of solutions for the linear homogeneous system of differential equations on I.

Theorem: **General Solution - Homogeneous Systems**

Let $\mathbf{X_1}, \mathbf{X_2}, \dots, \mathbf{X_n}$ be a fundamental set of solutions of the homogeneous system on I. Then the **general solution** of the system on the interval is

$$\mathbf{X} = c_1\mathbf{X_1} + c_2\,\mathbf{X_2} + \cdots + c_n\mathbf{X_n},$$

where $c_i, i = 1,2, \dots n$ are arbitrary constants.

Theorem: **General Solution - Non-Homogeneous Systems**

Let $\mathbf{X_p}$ be a given solution of the non-homogeneous system on an interval I and let $\mathbf{X_c}$ be the general solution of the associated homogeneous system. The **general solution** of the non-homogeneous system on the interval is

$$\mathbf{X} = \mathbf{X_c} + \mathbf{X_p}.$$

The general solution $\mathbf{X_c}$ of the associated homogeneous system is called the complementary function of the non-homogeneous system.