## 2.1 Separable Variables

- *Definition:* A first-order DE is called **separable** if it can be written in the form

$$\frac{dy}{dx} = g(x)h(y)$$

We can solve a separable DE by using separation of variables (i.e. re-writing the equation and integrating both sides).

$$\frac{dy}{dx} = g(x)h(y) \quad \rightarrow \quad p(y)\frac{dy}{dx} = g(x),$$

where $p(y) = 1/h(y)$. If $y = \phi(x)$ is a solution, then $p\big(\phi(x)\big)\phi'(x) = g(x)$ and

$$\int p\big(\phi(x)\big)\phi'(x)\,dx = \int g(x)\,dx \quad \rightarrow \quad \int p(y)\,dy = \int g(x)\,dx \quad \rightarrow \quad P(y) = G(x) + c,$$

where $P(y)$ and $G(x)$ are the antiderivatives of $p(y)$ and $g(x)$, respectively.

**There is no need for two constants in the integration since they can be combined together.*

- *Example:* Solve the following differential equation. Give an explicit solution for $y(x)$.

$$\frac{dy}{dx} = \frac{y}{1+x}$$

Solution: By separation of variables we get

$$\int \frac{dy}{y} = \int \frac{dx}{1+x} \quad \rightarrow \quad \ln|y| = \ln|1+x| + c_1$$

This gives an implicit solution for the differential equation. If we want an explicit solution, we need to solve for y.

$$|y| = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} = |1+x|e^{c_1} \quad \rightarrow \quad y = \pm e^{c_1}(1+x)$$

Relabeling $\pm e^{c_1}$ as $c$ then gives the explicit solution $\boldsymbol{y = c(1+x)}$.

- *Example:* Solve the following IVP.

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(3) = -4$$

Solution: By separation of variables we get

$$\int y\,dy = -\int x\,dx \quad \rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can rewrite this expression as (the more familiar) $x^2 + y^2 = c$ by replacing the constant $2c_1$ by $c$. Now we consider the initial condition and get $(3)^2 + (-4)^2 = c$, giving us $c = 25$, and an implicit solution of $\boldsymbol{x^2 + y^2 = 25}$. We could also find an explicit solution (using the initial condition) of $\boldsymbol{y = -\sqrt{25 - x^2}}$ on the interval $-5 < x < 5$.

**Note: in both examples above, we re-labeled the constant of integration. When doing so, be sure to re-name the constant as well.*

Always be careful when separating variables. If $r$ is a zero of $h(y)$, then $y = r$ is a constant solution to the DE. However, $dy/h(y)$ is undefined at $r$, and you may lose this solution in the process.

Some problems may lead to nonelementary integrals. In those cases, your final solution may contain an integral, as with the example below.

- *Example:* Solve the following IVP.

$$\frac{dy}{dx} = \sin x^2, \qquad y(0) = -3$$

Solution: By separation of variables we get

$$\int dy = \int \sin x^2\,dx.$$

However the antiderivative of $\sin x^2$ is not an elementary function. However, using a dummy variable of integration, we can write the problem as

$$\int_0^x \frac{dy}{dt}\,dt = \int_0^x \sin t^2\,dt \quad \rightarrow \quad y(x) - y(0) = \int_0^x \sin t^2\,dt \quad \rightarrow \quad \boldsymbol{y(x) = 3 + \int_0^x \sin t^2\,dt.}$$

We must always be cautious about losing a solution, especially when solving a separable differential equation. This happens when the divisors are zero at a point. Specifically, if $k$ is a zero of the function $h(y)$, then $y = k$ is a constant solution to $dy/dx = g(x)h(y)$. However, when going through separation of variables, the left-hand side will be undefined at $y = k$, and the solution may be lost. Such a solution is referred to as a **singular solution**. While we will not spend a lot of time exploring singular solutions in this course, we must always be aware of them when "modifying" equations in the process of solving.

- *Example:* Solve the following differential equation. Give an explicit solution for $y(x)$.

$$x\frac{dy}{dx} = y^2 + y$$

Solution: By separation of variables we get

$$\int \frac{dy}{y^2 + y} = \int \frac{1}{x}\,dx.$$

Using partial fraction decomposition, we get

$$\int \frac{dy}{y^2 + y} = \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy.$$

Integrating and solving for $y$ we get

$$\ln|y| - \ln|y+1| = \ln|x| + \ln|c| \ \rightarrow \ln\left|\frac{y}{y+1}\right| = \ln|cx| \ \rightarrow \frac{y}{y+1} = cx \ \rightarrow \ \boldsymbol{y = \frac{cx}{1 - cx}}.$$

However, by examination of the original problem, we can see that $y = 0$ and $y = -1$ are both constant solutions to the differential equation. We can obtain $y = 0$ by setting $c = 0$, but we have lost the solution $y = -1$ in the process of solving the problem.