6.2 Solutions About Singular Points

In the previous section we looked at finding series solutions about an ordinary point. If $a_2(x_0) \neq 0$ in the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then $x = x_0$ was an ordinary point. However, if $a_2(x_0) = 0$, then $x = x_0$ was called a **singular point.** Dealing with singular points is much more complicated. While we can always find two linearly independent power series solutions about an ordinary point, that is not always the case when dealing with a singular point.

Before exploring how to (try) to find solutions about a singular point, we want to look at two different classifications of singular points.

- *Definition:* Given the linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \Leftrightarrow \quad y'' + P(x)y' + Q(x)y = 0,$$

a singular point x_0 is called a **regular singular point** if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is called an **irregular singular point**.

- *Example:* Classify the singular points of the equation

$$(x^2 - 9)^2 y'' + 2(x + 3)y' + 6y = 0.$$

Solution: By examination, we have singular points of $x = 3$ and $x = -3$. Writing the equation in standard form we get

$$y'' + \frac{2}{(x+3)(x-3)^2} y' + \frac{6}{(x+3)^2(x-3)^2} = 0,$$

or

$$P(x) = \frac{2}{(x+3)(x-3)^2}, \qquad Q(x) = \frac{6}{(x+3)^2(x-3)^2}.$$

Taking $x = -3$, we get

$$p(x) = \frac{2}{(x-3)^2}, \qquad q(x) = \frac{6}{(x-3)^2}.$$

Therefore, since both $p(x)$ and $q(x)$ are analytic at $x = -3$, it is a **regular singular point**.

Taking $x = 3$, we get

$$p(x) = \frac{2}{(x+3)(x-3)}, \qquad q(x) = \frac{6}{(x+3)^2}.$$

Therefore, since $p(x)$ is not analytic at $x = 3$, it is an **irregular singular point**.

Theorem: **Frobenius' Theorem**

If $x = x_0$ is a regular singular point of the equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r},$$

where r is a constant. The series will converge at least on some interval $0 < x - x_0 < R$.

**Note: Frobenius' Theorem gives no assurance that we will be able to find two solutions of the given form.*

**Note: if $x = x_0$ is an irregular singular point, we are not even guaranteed to be able to find one solution of the given form.*

Using the method of Frobenius, we can attempt to find solutions about a regular singular point. The method itself is very similar to the method of finding series solutions about ordinary points. However, before we try to solve for the constants, we need to find the appropriate values for r. For simplicities sake, we will focus on cases where the singular point is $x = 0$.

- *Example:* Find series solutions to the equation about the singular point $x = 0$.

$$3xy'' + y' - y = 0.$$

Solution: First we take derivatives of our "guess" to get

$$y' = \sum_{n=0}^{\infty} c_n (n + r) x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} c_n (n + r)(n + r - 1) x^{n+r-2}.$$

By substitution, we get

$$3\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= \sum_{n=0}^{\infty} c_n(n+r)(3n+3r-2)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= x^r\left[r(3r-2)c_0x^{-1} + \sum_{n=1}^{\infty} c_n(n+r)(3n+3r-2)x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right]$$

$$= x^r\left[r(3r-2)c_0x^{-1} + \sum_{k=0}^{\infty} c_{k+1}(k+1+r)(3k+3r+1)x^k - \sum_{k=0}^{\infty} c_k x^k\right]$$

$$= x^r\left[r(3r-2)c_0x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(3k+3r+1)c_{k+1} - c_k]x^k\right] = 0.$$

Therefore, $r(3r-2)c_0 = 0$ and $(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0, k = 0,1,\ldots$ Since we gain nothing by letting $c_0 = 0$, we are left with

$$r(3r-2) = 0 \;\;\rightarrow\;\; r_1 = 0, \qquad r_2 = \frac{2}{3}.$$

When $r_1 = 0$, we get the relation

$$c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \qquad k = 0,1,2,\ldots$$

This gives us the sequence

$$c_1 = \frac{c_0}{1\cdot 1},\;\; c_2 = \frac{c_1}{2\cdot 4} = \frac{c_0}{2!\cdot 1\cdot 4},\;\; c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{3!\cdot 1\cdot 4\cdot 7}, \ldots ,$$
$$c_n = \frac{c_0}{n!\cdot 1\cdot 4\cdot \ldots\cdot (3n-2)}.$$

When $r_2 = \frac{2}{3}$, we get the relation

$$c_{k+1} = \frac{c_k}{(k+\frac{5}{3})(3k+3)} = \frac{c_k}{(3k+5)(k+1)}, \qquad k = 0,1,2,\ldots$$

This gives us the sequence

$$c_1 = \frac{c_0}{5\cdot 1},\;\; c_2 = \frac{c_1}{8\cdot 2} = \frac{c_0}{2!\cdot 5\cdot 8},\;\; c_3 = \frac{c_2}{11\cdot 3} = \frac{c_0}{3!\cdot 5\cdot 8\cdot 11}, \ldots$$

or

$$c_n = \frac{c_0}{n!\cdot 5\cdot 8\cdot \ldots\cdot (3n+2)}.$$

Using these two cases, we get the series solutions

$$y_1(x) = x^0\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \cdot 1 \cdot 4 \cdot \ldots \cdot (3n-2)}\right]$$

$$y_2(x) = x^{2/3}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \cdot 5 \cdot 8 \cdot \ldots \cdot (3n+2)}\right].$$

**Note: when taking derivatives of our original "guess" we did not change the lower bound like we did for ordinary points. This is because in this case we do not know exact values of exponents without knowing the values of* r.

**While we found an explicit formula for the coefficients in the example above, we will generally concern ourselves with only find the first few non-zero terms for the series solutions.*

The equation $r(3r-2) = 0$ is referred to as the **indicial equation** of the problem, and the roots are called the **indicial roots** of the singularity. While it is possible to find the indicial equation without the above substitutions, we will generally be concerned finding the solution to the differential equation, so it is usually easiest to find the indicial roots in the process.

- *Example:* Find at least one series solution to the equation about the point $x = 0$.

$$xy'' + y = 0.$$

Solution: First we take derivatives of our "guess" to get

$$y' = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

By substitution, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} \\
&\qquad = x^r\left[r(r-1)c_0 + \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n\right] \\
&\qquad = x^r\left[r(r-1)c_0 + \sum_{k=0}^{\infty}(k+1+r)(k+r)c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k\right] \\
&\qquad = x^r\left[r(r-1)c_0 + \sum_{k=0}^{\infty}[(k+1+r)(k+r)c_{k+1} + c_k]x^k\right] = 0.
\end{aligned}$$

Therefore we have an indicial equation of $r(r-1) = 0$, giving us roots of $r_1 = 0,\ r_2 = 1$.

When $r_1 = 0$, we get the relation

$$k(k+1)c_{k+1} + c_k = 0, \qquad k = 0,1,2,\dots$$

When $k = 0$, we get that $c_0 = 0$, leaving us with

$$c_{k+1} = -\frac{c_k}{k(k+1)}, \qquad k = 1,2,\dots,$$

giving

$$c_2 = \frac{-c_1}{1\cdot 2}, \quad c_3 = \frac{-c_2}{2\cdot 3} = \frac{c_1}{2!\cdot 3!}, \quad c_4 = \frac{-c_3}{3\cdot 4} = \frac{-c_1}{3!\cdot 4!}, \dots, \quad c_{n+1} = \frac{(-1)^n c_1}{n!\cdot(n+1)!}.$$

When $r_2 = 1$, we get the relation

$$c_{k+1} = -\frac{c_k}{(k+2)(k+1)}, \qquad k = 0,1,2,\dots,$$

giving

$$c_1 = \frac{-c_0}{1\cdot 2}, \quad c_2 = \frac{-c_1}{2\cdot 3} = \frac{c_0}{2!\cdot 3!}, \quad c_3 = \frac{-c_2}{3\cdot 4} = \frac{-c_0}{3!\cdot 4!}, \dots, \quad c_n = \frac{(-1)^n c_0}{n!\cdot(n+1)!}.$$

Using these two cases, we get the series solutions

$$y_1(x) = x^0\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n!\cdot(n+1)!}\right]$$

$$y_2(x) = x^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{n}}{n!\cdot(n+1)!}\right].$$

However, y_1 and y_2 both represent the same series, so we are left with only one series solution.

When dealing with the method of Frobenius, we will encounter one of three cases. For the sake of convenience, we will assume that we are working with a singular point $x = 0$ and r_1 and r_2 are our indicial roots, with $r_1 > r_2$.

Case I: If r_1 and r_2 are distinct and $r_1 - r_2$ is not an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad c_0 \neq 0, \ \ b_0 \neq 0.$$

Case II: If r_1 and r_2 are distinct and $r_1 - r_2$ is an integer, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad c_0 \neq 0, \;\; b_0 \neq 0,$$

where C is a constant that could be zero.

Case III: If $r_1 = r_2$, then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1}, \qquad c_0 \neq 0, \;\; b_0 \neq 0.$$

In Case III (or in Case II when $C \neq 0$), a second solution can be found using variation of parameters, giving

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx.$$

However, due to the fact that we have the square of a infinite series in the denominator, it is generally not recommended to evaluate the given integral without the aid of a computer algebra system.

**Note: since our indicial equation is a quadratic equation, there is a chance that our indicial roots could be complex. We will not explore those cases in this class.*