# 10.4: The Cross Product

Thursday, August 13, 2020 10:00 PM

<u>Def.</u>: the <u>cross product</u> (<u>vetor product</u>) of

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ with $\vec{b} = \langle b_1, b_2, b_3 \rangle$

is $\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

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<u>Def.</u> 1) The <u>determinant</u> of a $2\times 2$ matrix of form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

need

2) The det. of a $3\times 3$ matrix is

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{array} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

aside

$$= a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$

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<u>Fact</u>

$$\vec{a} \times \vec{b} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ \ldots \end{vmatrix} \end{array} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ \ldots \end{vmatrix} - \vec{i} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k}$$

$$= \vec{a} \times \vec{b}. \checkmark$$

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<u>Ex. 1</u> $\vec{a} = \langle 1, 3, 4 \rangle$, $\vec{b} = \langle 2, 7, -5 \rangle$

$$\begin{array}{c} \quad + \quad\;\; - \quad\;\; + \\ \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \end{array} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \vec{k}$$

$\begin{vmatrix} 2 & 7 \end{vmatrix}$ $k$

$$= -43\vec{i} + 13\vec{j} + \vec{k}.$$

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**Fact** $\vec{a} \times \vec{a} = \vec{0}$ for all $\vec{a} \in V_3$.

**Pf.** Obvious

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**Thm.*** $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

**Pf. (sketch)**

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \ldots = 0 \quad \checkmark$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0. \quad \checkmark$$

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Rk the direction of $\vec{a} \times \vec{b}$ given by

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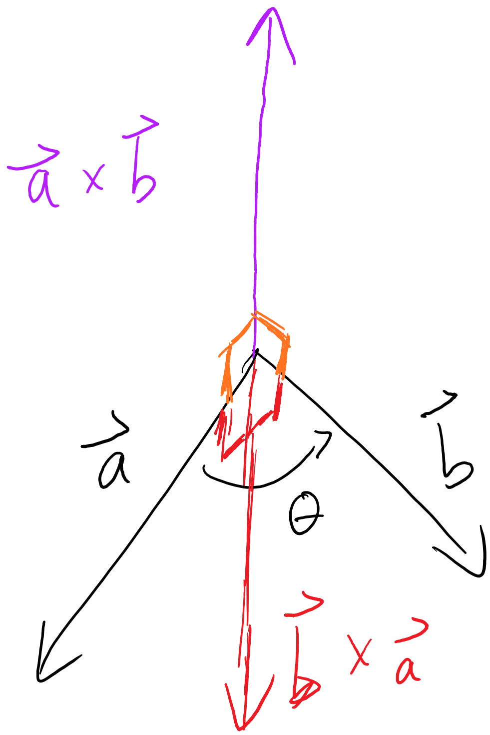
**Rmk.** the direction of $\vec{a} \times \vec{b}$ given by "right hand rule"




- fingers curl inward from $\vec{a}$ to $\vec{b}$ direction
- youtube video right hand rule?

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**Th'm.** If $\theta$ is angle between $\vec{a}$ and $\vec{b}$ (so $0 \le \theta \le \pi$), then

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin(\theta)$$

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Alternate def. of $\vec{a} \times \vec{b}$: the vector w/ magnitude $|\vec{a}| \cdot |\vec{b}| \sin(\theta)$ and direction determined by RHR.

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Corollary Two nonzero vect. $\vec{a}, \vec{b}$ are parallel $\Longleftrightarrow$ $\vec{a} \times \vec{b} = \vec{0}$.

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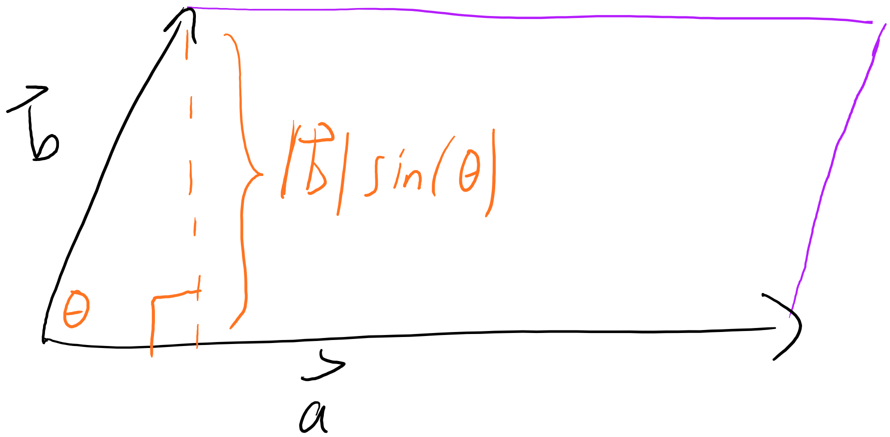
Fact. $|\vec{a} \times \vec{b}|$ = area of parallelogram det. by $\vec{a}$ & $\vec{b}$.




$$\text{Area} = |\vec{a}|\left(|\vec{b}|\sin(\theta)\right)$$
$$= |\vec{a} \times \vec{b}|.$$

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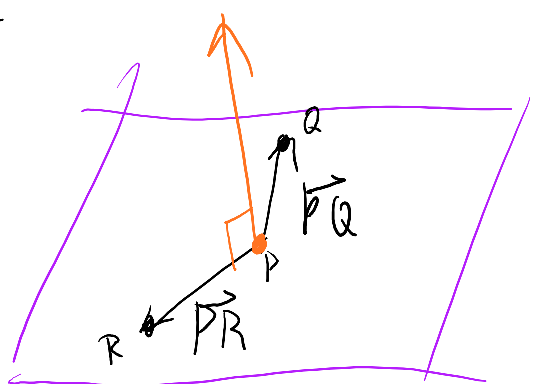
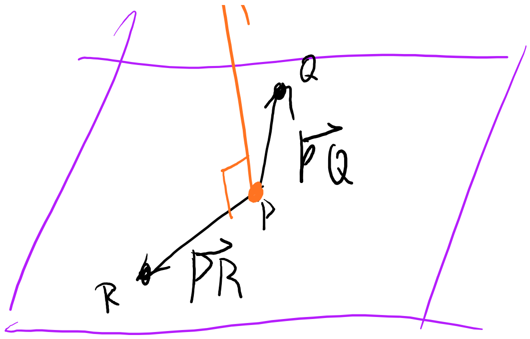
<u>Ex. 3</u> Find a vect. perp. to the plane that passes through $P=(1,4,6)$, $Q=(-2,5,-1)$, $R=(1,-1,1)$.

<u>Sol'n:</u>






$$(\vec{PQ} \times \vec{PR}) \perp \vec{PQ} \quad \text{and} \quad (\vec{PQ} \times \vec{PR}) \perp \vec{PR}.$$

$$\vec{PQ} = -3\vec{i} + \vec{j} - 7\vec{k}$$

$$\vec{PR} = \quad -5\vec{j} - 5\vec{k}$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \boxed{-40\vec{i} - 15\vec{j} + 15\vec{k}}$$

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Ex. 4 Find area of triangle w/vertices

$P = (1, 4, 6)$, $Q = (-2, 5, -1)$, $R = (1, -1, 1)$.

Sol'n: Know area of parallelogram w/ adjacent sides $\vec{PQ}$ and PR is

$$\left|\vec{PQ} \times \vec{PR}\right| = 5\sqrt{82}$$

$$\Rightarrow \text{area of triangle} = \frac{1}{2}\left(5\sqrt{82}\right)$$

$$= \boxed{\frac{5}{2}\sqrt{82}}.$$

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Rmk 1) cross product not commutative

ex. $\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{i} = -\vec{k}$

$\Rightarrow \vec{i} \times \vec{j} \neq \vec{j} \times i$

2) cross prod. not associative:

$$\vec{i} \times (\vec{i} \times \vec{j}) \neq (\vec{i} \times \vec{i}) \times \vec{j}$$

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Th'm $\vec{a}, \vec{b}, \vec{c} \in V_3$, $k \in \mathbb{R}$

1) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2) $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

4) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

6) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

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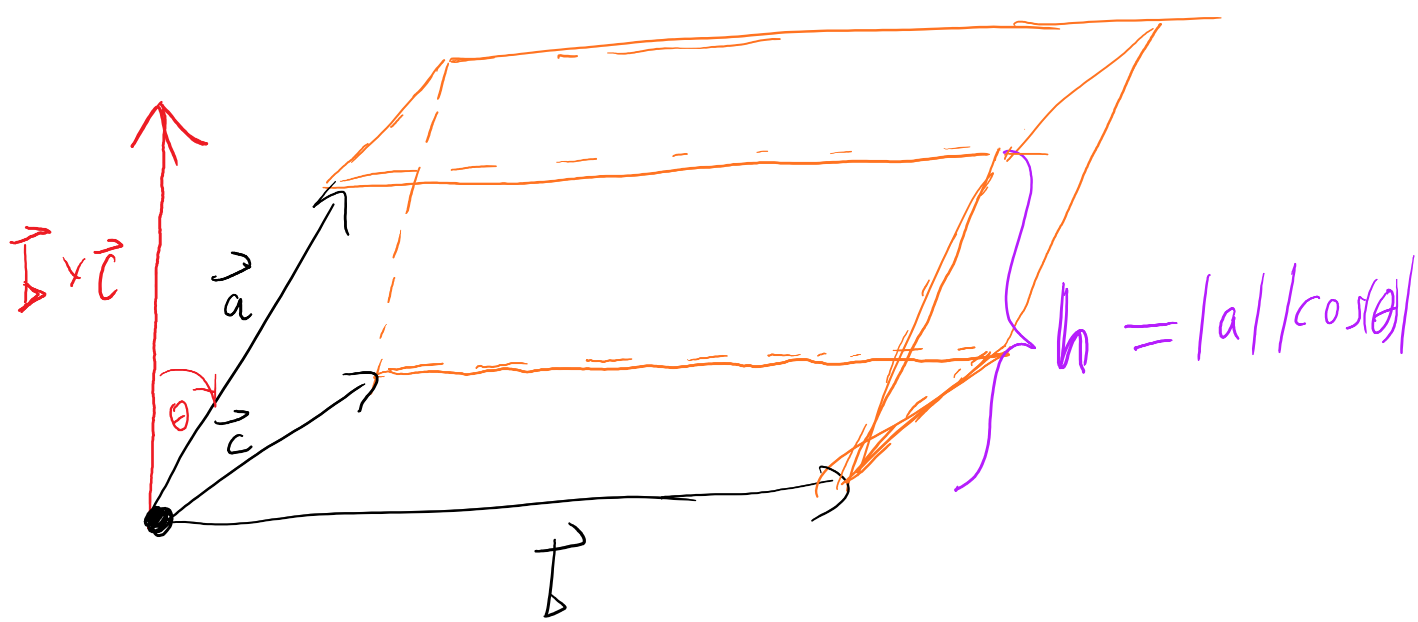
**Def.** $\vec{a} \cdot (\vec{b} \times \vec{c})$ is the <u>scalar triple product</u> of $\vec{a}, \vec{b}, \vec{c}$.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \pm$ <u>volume of parallelepiped</u> determined by $\vec{a}, \vec{b}, \vec{c}$.




Volume = (area of base)(height)

$$= Ah$$

$$A = |\vec{b} \times \vec{c}|, \quad h = |\vec{a}| \cdot |\cos(\theta)|$$

$$Vol = |\vec{a}| \cdot |\vec{b} \times \vec{c}| \cdot |\cos(\theta)|$$

$$\boxed{V = |\vec{a} \cdot (\vec{b} \times \vec{c})|.}$$

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**Torque** Force $\vec{F}$ acts on a rigid body at ~~the~~ a point given by pos. vect. $\vec{r}$.

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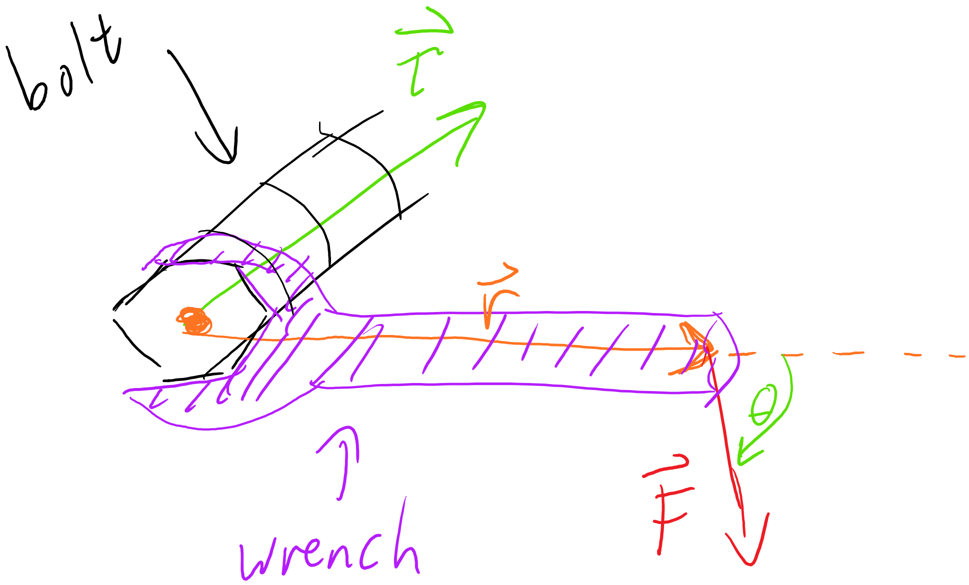
**Def.** the **torque** relative to the origin is $\vec{\tau} = \vec{r} \times \vec{F}$.
(measures tendency of the body to rotate about origin)




$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| \cdot |\vec{F}| \cdot \sin(\theta)$$

$\theta$ = angle between $\vec{r}$ and $\vec{F}$.

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Ex. 6 A bolt tightened by applying a 40 N force to a 0.25-m wrench. Find magnitude of torque about the center of the bolt. Assume $\theta = 75°$.

Sol'n $|\vec{\tau}| = |\vec{r}| \cdot |\vec{F}| \sin(\theta)$

$= (0.25)(40)\sin(75^\circ)$

$\approx 9.66 \text{ J}.$