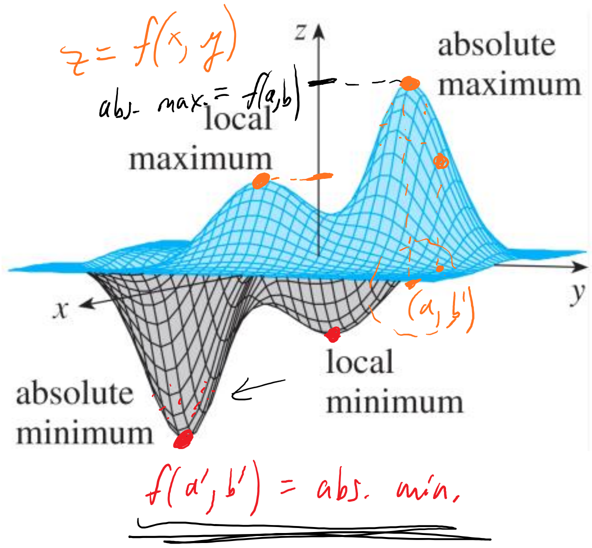
11.7: Maximum and Minimum Values

Friday, September 11, 2020 10:49 AM



Def. $z = f(x, y)$

1) f has a local max. at a point $(a, b) \in \mathbb{R}^2$ if $f(x, y) \le f(a, b)$ for all $(x, y) \in D$, some disk D centered at (a, b).

2) f has a local min. at (a, b) if $f(x, y) \ge f(a, b)$ for all $(x, y) \in D$, where D some disk cent. at (a, b).

3) If $f(x, y) \le f(a, b)$ for all (x, y) in domain of f, then $f(a, b)$ is an absolute max. of f.

4) If $f(x, y) \ge f(a, b)$ $\forall (x, y)$ in domain of f, then $f(a, b)$ is an absolute min. of f.

Th'm. If $f(x, y)$ has a local max/min at (a, b), and if $f_x(a, b)$ and $f_y(a, b)$ both exist, then

$$f_x(a, b) = f_y(a, b) = 0.$$

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<u>Def.</u> (a,b) is a <u>critical point</u> of f if $f_x(a,b)=0$ and $f_y(a,b)=0$, <u>or</u> if one of $f_x(a,b)$, $f_y(a,b)$ DNE.

<u>Ex.</u> Find extreme values of

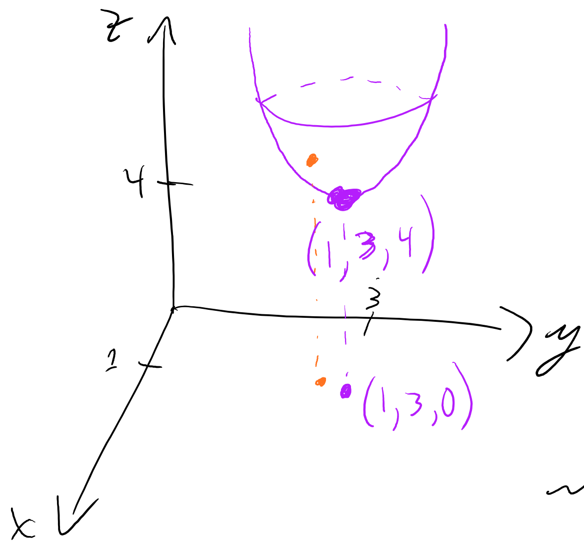
(a) $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

(b) $f(x,y) = y^2 - x^2$

<u>Sol'n:</u> (a) $z = x^2 - 2x + y^2 - 6y + 14$

$= (x^2 - 2x + 1) + (y^2 - 6y + 9) + 14 - 1 - 9$

$z = (x-1)^2 + (y-3)^2 + 4$



$f_x = 2x - 2 = 0$

$\Rightarrow x = 1$

$f_y = 2y - 6 = 0$

$\Rightarrow y = 3$

$\rightsquigarrow (a,b) = (1,3)$ is only crit. point b/c f_x & f_y exist all $\mathbb{R}^2$.

$f(1,3) = 4$ ... min, max, neither?

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$$\boxed{f(x,y) \geq f(1,3)}$$

$(x-1)^2+(y-3)^2+4 \geq 4 \iff$

$(x-1)^2+(y-3)^2 \geq 0$ ✓ $\implies f(1,3)=4$ is an abs. (and local) min. of f.

Th'm (Second Derivatives Test)

Suppose 2nd partials of f are continuous on a disk cent. at (a,b), and $f_x(a,b) = f_y(a,b) = 0$.

(In particular, (a,b) is a crit pt of f).

Let $D = \begin{vmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$.

(a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local min.

(b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local max.

(c) If $D(a,b) < 0$, then $f(a,b)$ is neither a local max nor local min.

Def. In (c), for $D(a,b) < 0$, the point (a,b) is a <u>saddle point</u> of f.

Note If $D(a,b)=0$, the theorem is useless.

Ex. $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

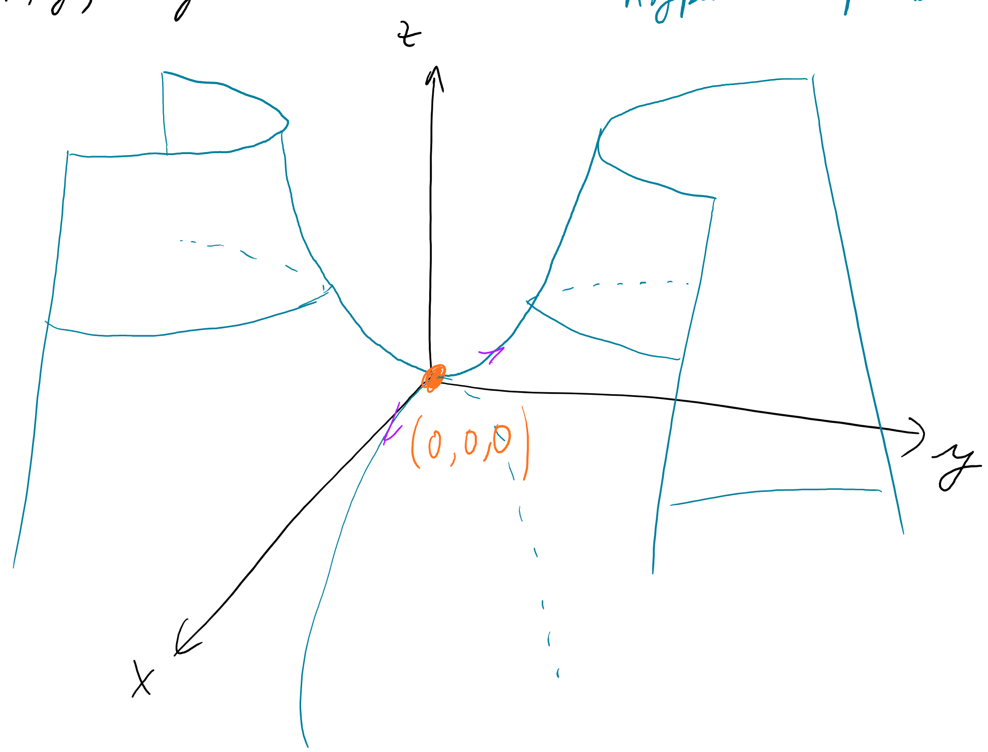
(a) $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0 = f_{yx}$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 4 - 0 = 4 \quad \text{constant}$$

$\Rightarrow D(1,3) = 4 > 0$. Also $f_{xx}(1,3) = 2 > 0$

2nd Der. Test (a) $\Rightarrow$ $f(1,3) = \boxed{4 \text{ local min.}}$

(b) $f(x,y) = y^2 - x^2$ hyperbolic paraboloid



$$\left.\begin{array}{l} f_x = -2x = 0 \Rightarrow x = 0 \\ f_y = 2y = 0 \Rightarrow y = 0 \end{array}\right\} \Rightarrow (0,0) \text{ crit pt}$$

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4$$

$$\Rightarrow D(0,0) = -4 < 0$$

2nd Der Test $\Rightarrow$ $\boxed{f(0,0) = 0 \text{ neither local min nor local max, } (0,0) \text{ saddle point}}$

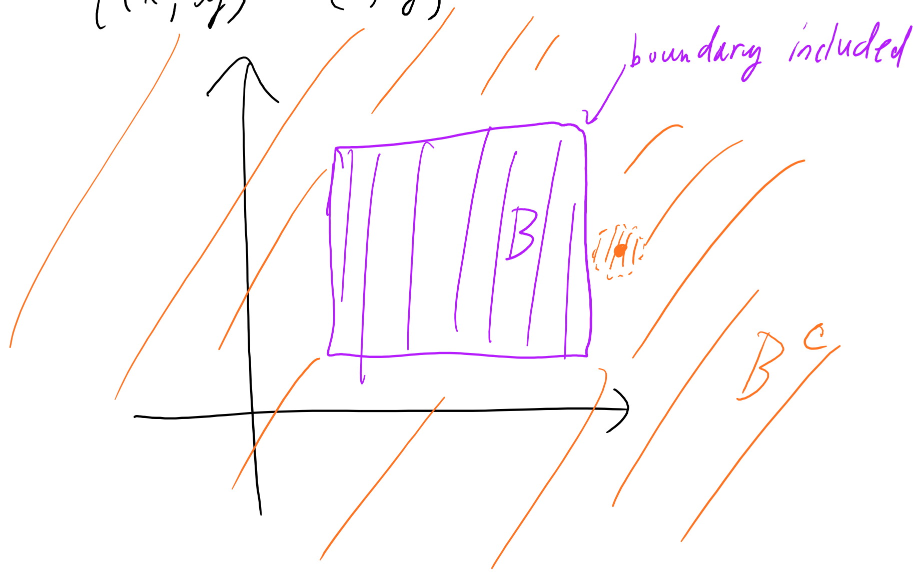
Absolute max & min values

Recall $f : [a,b] \longrightarrow \mathbb{R}$ continuous $\Rightarrow$ f attains absolute max. & min. values in $[a,b]$.

Def. 1) $A \subset \mathbb{R}^2$ is open if for any $(x_0, y_0) \in A$, there is a disk $D_\varepsilon = \{(x,y) : \sqrt{(x-x_0)^2 - (y-y_0)^2} < \varepsilon\}$, $\varepsilon > 0$, where $D \subset A$.

2) $B \subset \mathbb{R}^2$ is <u>closed</u> if B^c is open, where

$$B^c = \{(x,y) : (x,y) \text{ not in } B\}$$

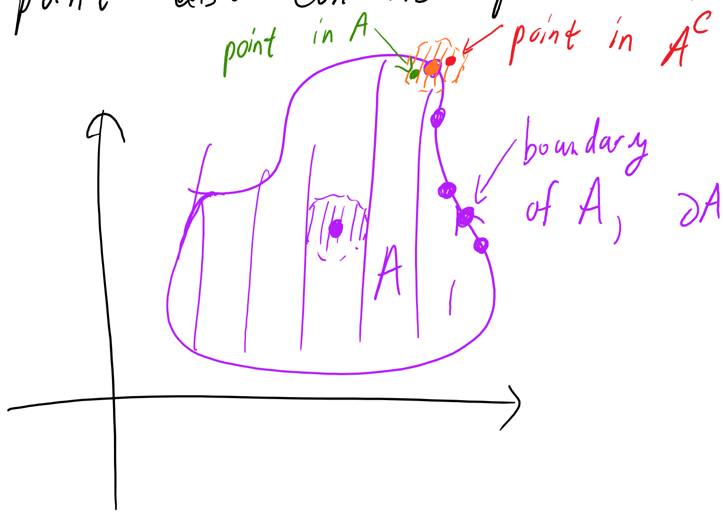


3) $C \subset \mathbb{R}^2$ is <u>bounded</u> if $C \subset D$ some disk D of finite radius.

(B^c not bounded in def. 2 above, but
B <u>is</u> " " " " ")

4) The <u>boundary</u> of $A \subset \mathbb{R}^2$ is the set of

points in $\mathbb{R}^2$ s.t. every disk containging such a point also contains points of A and A^c.



Extreme Value Th'm $D \subset \mathbb{R}^2$ closed and bounded.

If $f : D \longrightarrow \mathbb{R}$ is continuous on D, then f attains an abs. max. and an abs. min. on D.

$$f(x_1, y_1) = \text{abs. min.}, \quad f(x_2, y_2) = \text{abs. max},$$

$$(x_1, y_1), \ (x_2, y_2) \in D.$$

Note Let $f : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^2$ closed & bdd. (bounded)

If f_x and f_y both exist at $(a,b) \in D$, and if f has an extreme value at (a,b), then either (a,b) is a critical p't of f, or $(a,b) \in \partial D$.

boundary of D

Method for finding abs. max. & min. values on closed & bdd D

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1) Find f values at every crit. pt.

2) Find extreme values of f on ∂D

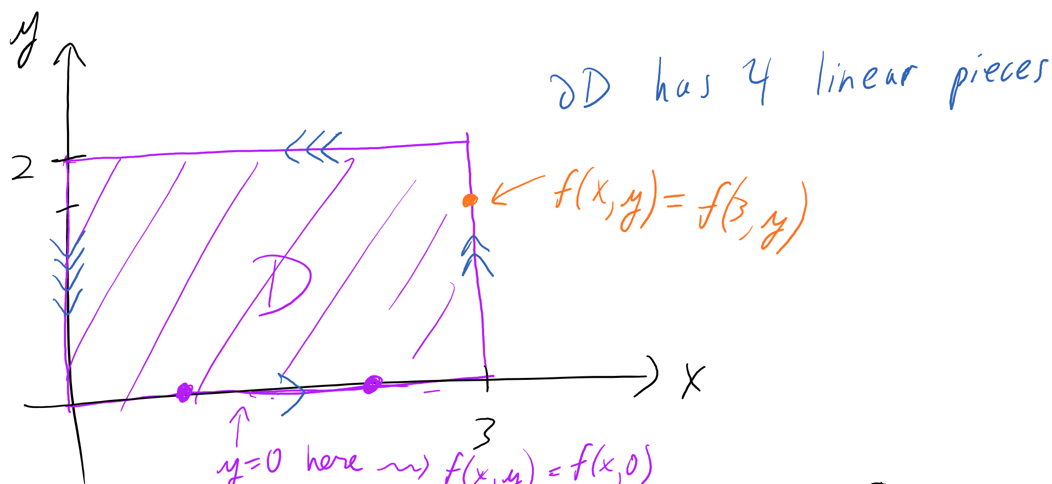
3) · Largest f value from 1) & 2) is the abs. max.
 · Smallest " " " " " " " " " min

Ex. Find absolute extrema of $f(x,y) = x^2 - 2xy + 2y$ on $D = \{(x,y) : 0 \le x \le 3,\ 0 \le y \le 2\}$.

Sol'n: Note f polyn. $\Rightarrow$ f continuous on all $\mathbb{R}^2$

$\Rightarrow f$ cont. on D.

Also D closed & bdd.



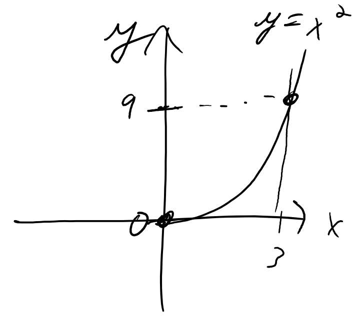
1) Deal w/ ∂D later. Find crit. pts inside D

$$\left.\begin{array}{l} f_x = 2x - 2y = 0 \Rightarrow x = y \\ f_y = -2x + 2 = 0 \Rightarrow x = 1 = y \end{array}\right\} \rightsquigarrow (a,b) = (1,1)$$

$f_x = 2x - 2y = 0$

$f_y = -2x + 2 = 0 \Rightarrow x = 1 = y \Big] \rightsquigarrow (a,b) = (1,1)$

$\therefore$ Only crit. p't inside D is $(1,1) \rightsquigarrow f(1,1) = 1$



2) (i) $f(x,y) = f(x,0) = x^2, \quad 0 \le x \le 3$

x^2 has min. & max. values

$f(0,0) = 0$ and $f(3,0) = 9$ resp.

(ii) $f(x,y) = f(3,y) = 9 - 6y + 2y = 9 - 4y, \quad 0 \le y \le 2$

is a linear func. of y, dec.

$\Rightarrow 9 - 4y$ has max. $f(3,0) = 9$, min. $= f(3,2) = 1$

(iii) $f(x,2) = x^2 - 4x + 4, \quad 0 \le x \le 3$

$g(x) = (x-2)^2, \quad g'(x) = 2(x-2) = 0 \Rightarrow x = 2$

$\Rightarrow x^2 - 4x + 4$ has min. $f(2,2) = 0$,

$g(0) = 4, \quad g(3) = 1$

(iv) $f(0,y) = 2y, \quad 0 \le y \le 2$

has min $f(0,0) = 0$, max $= f(0,2) = 4$

3) $\boxed{\begin{array}{l} \text{abs. max} = 9 = f(3,0) \\ \text{abs. min} = 0 = f(0,0). \end{array}}$