## 2.4 Solving by Substitution

We can transform differential equations by the use of a substitution. A typical option is to transform the first-order DE $y' = f(x, y)$ by the substitution $y = g(x, u)$, where $u$ is a function of $x$. By using the chain rule, we get

$$\frac{dy}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial u}\frac{du}{dx} = g_x + g_u\frac{du}{dx}.$$

Substituting this into the original equation then gives

$$g_x + g_u\frac{du}{dx} = f\big(x, g(u, x)\big) \quad \rightarrow \quad \frac{du}{dx} = \frac{f(x, g) - g_x}{g_u} = F(x, u).$$

If we can find a solution $u = \phi(x)$ for this new equation, then $y = g(x, \phi(x))$ is a solution to the original equation. We will explore three types of problems that can be solved by substitution.

**1. Homogeneous Differential Equations**

- *Definition:* If a function $f$ possesses the property that $f(tx, ty) = t^{\alpha} f(x, y)$ for some real $\alpha$, then $f$ is called a **homogeneous function** of degree $\alpha$. A first-order DE in differential form is said to be **homogeneous** if both $M$ and $N$ are homogeneous functions of the *same* degree.

A first-order DE in normal form

$$\frac{dy}{dx} = f(x, y)$$

is homogeneous if $f(tx, ty) = f(x, y)$. Proof of this is left as an exercise.

**Note: homogeneous has a different meaning here than it did in linear equations. Pay close attention to the context of the problem.*

**Solving:** For a homogeneous equation, using either the substitution $\boldsymbol{y = ux}$ or $\boldsymbol{x = vy}$ will reduce the problem to a separable DE.

Using the substitution $y = ux$, we get that the differential $dy = u\,dx + x\,du$. Substituting this into the differential form, and using the fact that $M$ and $N$ are homogeneous functions, we get

$$M(x, ux)dx + N(x, ux)[u\,dx + x\,du] = x^{\alpha}M(1, u)dx + x^{\alpha}N(1, u)[u\,dx + x\,du] = 0$$

Dividing by $x^{\alpha}$ and distributing terms then gives us

$$[M(1,u) + u\,N(1,u)]dx + x\,N(1,u)du = 0 \quad \rightarrow \quad \frac{N(1,u)}{M(1,u) + N(1,u)}du + \frac{dx}{x} = 0.$$

**Once again, it is not important to memorize this final solution. You should instead remember the appropriate substitution and work through the procedure each time.*

**The process works similarly for the substitution $x = vy$.*

- *Example:* Solve $(x^2 + y^2)dx + (x^2 - xy)dy = 0$.

  Solution: First note by inspection that both $M$ and $N$ are homogeneous of degree 2. So using the substitution$y = ux$ gives us

$$(x^2 + u^2x^2)dx + (x^2 - ux^2)[u\,dx + x\,du] = 0$$

$$x^2(1 + u^2)dx + x^2(1 - u)[u\,dx + x\,du] = 0$$

$$[1 + u^2 + u - u^2]dx + (1 - u)xdu = 0$$

$$\frac{1-u}{1+u}du + \frac{dx}{x} = 0 \quad \rightarrow \quad \frac{dx}{x} = \frac{u-1}{1+u}du.$$

  By polynomial division and integration, we get

$$\int \frac{dx}{x} = \int 1 - \frac{2}{1+u}du \quad \rightarrow \quad \ln|x| + \ln|\mathrm{c}| = \mathrm{u} - 2\ln|1 + \mathrm{u}| \quad \rightarrow \quad \mathrm{u} = \ln|\mathrm{cx}(1 + \mathrm{u})^2|.$$

  By substituting back for $u$ we get the implicit solution

$$\frac{y}{x} = \ln\left|cx\left(1 + \frac{y}{x}\right)^2\right| = \ln\left|\frac{c(x+y)^2}{x}\right| \quad \rightarrow \quad \boldsymbol{c(x+y)^2 = xe^{\frac{y}{x}}}.$$

  **Note: the use of $\ln|c|$ for our constant of integration is a common practice for simplification when dealing with integrals that produce logarithms.*

  **While it is common practice to "clean" up the logarithms using the properties of logarithms, anything on the final line would be considered an acceptable simplified solution.*

## 2. Bernoulli's Equation

- *Definition:* A first-order DE of the form

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where $n$ is any real number, is called **Bernoulli's equation**.

**For $n = 0$ or $n = 1$, this is an linear equation.*

**Solving:** For Bernoulli's equation, using the substitution $\boldsymbol{u = y^{1-n}}$ will reduce the problem to a linear DE.

- *Example:* Solve $x\frac{dy}{dx} + y = x^2y^2$.

Solution: First we re-write the equation as

$$\frac{dy}{dx} + \frac{1}{x}y = xy^2.$$

With $n = 2$, we use $u = y^{1-2} = y^{-1}$ or $y = u^{-1}$. This gives us

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -u^{-2}\frac{du}{dx}$$

which we then substituted into the original equation, resulting in

$$-u^{-2}\frac{du}{dx} + \frac{1}{x}u^{-1} = xu^{-2} \quad \rightarrow \quad \frac{du}{dx} - \frac{1}{x}u = -x.$$

The integrating factor, on the interval $(0, \infty)$, is

$$e^{-\int dx/x} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying and integrating, we get

$$\frac{d}{dx}[x^{-1}u] = -1 \quad \rightarrow \quad x^{-1}u = -x + c \quad \rightarrow \quad u = -x^2 + cx.$$

Since $u = y^{-1}$, our final solution is then

$$y^{-1} = cx - x^2 \quad \rightarrow \quad \boldsymbol{y = (cx - x^2)^{-1}}.$$

**3. Reduction to Separable**

A differential equation of the form $y' = f(Ax + By + C)$ can always be reduced to separable by using the substitution $\boldsymbol{u = Ax + By + C}$.

- Example: Solve $y' = (3x - y)^2 + 4, \quad y(0) = 0.$

  Solution: If we let $u = 3x - y$, then

  $$\frac{du}{dx} = 3 - \frac{dy}{dx} \quad \rightarrow \quad \frac{dy}{dx} = 3 - \frac{du}{dx}$$

  Substituting this into the original equation, we get

  $$3 - \frac{du}{dx} = u^2 + 4 \quad \rightarrow \quad \frac{du}{dx} = -(1 + u^2) \quad \rightarrow \quad \frac{du}{1 + u^2} = -dx.$$

  By integration we have

  $$\tan^{-1} u = -x + c.$$

  By using the initial condition that $y(0) = 0$, we know that $u(0,0) = 0$, giving

  $$\tan^{-1} 0 = 0 + c \quad \rightarrow \quad c = 0.$$

  Substituting back in for $u$, we get the solution

  $$\boldsymbol{\tan^{-1}(3x - y) = -x \quad \rightarrow \quad y = 3x + \tan x}.$$