Unit 1.2: Inverse Trigonometric Functions—Integration

We continue our investigation of u-substitution by looking at functions whose antiderivatives yield inverse trig. functions; in particular the inverse sine, inverse tangent, and inverse secant functions. From Calculus I, you may recall the derivative rules

$$\frac{d}{dx}\sin^{-1}u = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\tan^{-1}u = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\sec^{-1}u = \frac{u'}{|u|\sqrt{u^2-1}}$$

As a result one obtains the following antiderivative rules (slightly generalized to include the a)

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+C \qquad \int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C \qquad \int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right|+C$$

There are a few important points to emphasize before looking at examples. In both of the above rules, a is a positive real number and u is a variable expression. In the cases where the rule generates an inverse sine or secant function, the order of subtraction under the radical is very important. If the order of subtraction were reversed in either case, we would have to make use of *trigonometric substitution* which we will learn later in the course. Also make special note of the absolute value associated with the inverse secant rule. It is also important to remember the du is replacing $u'dx$ in the original integral. We require that u' take care of any variable factors aside from the radical in the denominator and leave us with a constant numerator for which this rule applies. Let us now look at a few examples.

Example 1 Find the integral $\int \frac{5}{\sqrt{1-4x^2}}\,dx$.

Solution: First we note that letting $u = 1-4x^2$ (the inside of the radical function) will not be a useful substitution as it leads to $du = -8x dx$. The factor of x needed for du is not present in the integral, thus we look to another method. After seeing and doing several problems, you will likely identify this as an inverse sine function right away. However, one must always be cautious and make sure that all of the details work out (as you will see in examples 4 and 5). Let us first rewrite the integral as follows

$$\int \frac{5}{\sqrt{1-4x^2}}\,dx = 5\int \frac{1}{\sqrt{(1)^2-(2x)^2}}\,dx$$

We can now see that this integral gives us an inverse sine function and identify $a = 1$ and $u = 2x$. Because u is not simply x (or x plus a constant), we must carry out the substitution. We thus have $du = 2dx$ or equivalently $\frac{1}{2}du = dx$. With this substitution we obtain

$$5\int \frac{1}{\sqrt{(1)^2-(2x)^2}}\,dx = 5\cdot\frac{1}{2}\int \frac{1}{\sqrt{1^2-u^2}}\,du = \frac{5}{2}\sin^{-1}\left(\frac{u}{1}\right)+C = \boxed{\frac{5}{2}\sin^{-1}2x+C}$$

Example 2 Find the integral $\int \frac{1}{9+x^2}\,dx$.

Solution: Because the derivative of the denominator yields a factor of x, which is not found as a factor of the integrand, this integral does not give us a log function. Instead we identify it as an inverse tangent function. First we rewrite the integral to emphasize a and u.

$$\int \frac{1}{9+x^2}\,dx = \int \frac{1}{(3)^2+(x)^2}\,dx$$

We identify $a=3$ and $u=x$. Since $u=x$, we need not apply the substitution as the integral perfectly fits our memorized rule.

Final Answer: $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C$

__Example 3__ Find the integral $\int \frac{1}{7x\sqrt{10x^2-5}}\,dx$.

Solution: First we note that letting $u=10x^2-5$ (the inside of the radical function) will not be a useful substitution as it leads to $du=20x dx$. The factor of x needed for du is not present in the integral, thus we look to another method. Let us first rewrite the integral as follows

$$\int \frac{1}{7x\sqrt{10x^2-5}}\,dx = \frac{1}{7}\sqrt{10}\int \frac{1}{\sqrt{10}\cdot x\sqrt{(\sqrt{10}\cdot x)^2-(\sqrt{5})^2}}\,dx$$

We can now see that this integral gives us an inverse secant function and identify $a=\sqrt{5}$ and $u=\sqrt{10}\cdot x$. Because u is not simply x (or x plus a constant), we must carry out the substitution. We thus have $du=\sqrt{10}dx$ or equivalently $\frac{1}{\sqrt{10}}du = dx$. With this substitution we obtain

$$\frac{\sqrt{10}}{7}\int \frac{1}{\sqrt{10}\cdot x\sqrt{\left(\sqrt{10}\cdot x\right)^2-\left(\sqrt{5}\right)^2}}\,dx = \frac{\sqrt{10}}{7}\cdot\frac{1}{\sqrt{10}}\int \frac{1}{u\sqrt{u^2-\left(\sqrt{5}\right)^2}}\,du = \frac{1}{7}\cdot\frac{1}{\sqrt{5}}\sec^{-1}\left|\frac{u}{\sqrt{5}}\right|+C$$

$$=\frac{\sqrt{5}}{35}\sec^{-1}\left|\frac{\sqrt{10}\cdot x}{\sqrt{5}}\right|+C=\boxed{\frac{\sqrt{5}}{35}\sec^{-1}\left|\sqrt{2}\cdot x\right|+C}$$

Example 4 Find the integral $\int \frac{x}{5+9x^2} dx$.

Solution: Here the derivative of the denominator is $18x$. Because the integrand contains a factor of x (in the numerator), this integral will yield a logarithm, not an inverse tangent function. Had we first tried to view this as an inverse tangent function, we would have identified $u = 3x$, but then $du = 3dx$ and we wouldn't have been able to account for the x in the numerator. It is important that we know what the possibilities are and what options we might have for u anytime we encounter an integral. The order in which one investigates these options will vary from one person to the next. No one ever makes the correct substitution on the first try for every problem they ever encounter. As you practice, your intuition will strengthen and you will make better choices more often. Even an expert can try and fail several times before finding the correct solution. Let us proceed with the correct substitution.

Let $u = 5 + 9x^2$, therefore $du = 18xdx$ and thus $\frac{1}{18}du = xdx$. Substituting we obtain

$$\int \frac{x}{5+9x^2} dx = \frac{1}{18}\int \frac{1}{u} du = \frac{1}{18}\ln|u| + C = \boxed{\frac{1}{18}\ln(5+9x^2) + C}$$

Note: because $5 + 9x^2$ is never negative, we do not need absolute values in our final answer.

Example 5 Find the integral $\int \frac{x}{5+9x^4} dx$.

Solution: Based on example 3, one might jump to the conclusion that this integral does not yield an inverse tangent function because of the factor of x in the numerator. This is not a safe generalization to make. One must always consider the entire integrand. The derivative of the denominator yields a factor of x^3 which cannot be found as a factor of the integrand. So we can rule this out as a log function. Our only other option (at this point) is to consider whether or not it might be an inverse tangent. We first rewrite the integral as follows

$$\int \frac{x}{5+9x^4} dx = \int \frac{x}{\left(\sqrt{5}\right)^2 + (3x^2)^2} dx$$

We identify $a = \sqrt{5}$ and $u = 3x^2$, and therefore $du = 6xdx$, or rather $\frac{1}{6}du = xdx$.
Substituting, we obtain

$$\int \frac{x}{\left(\sqrt{5}\right)^2 + (3x^2)^2} dx = \frac{1}{6}\int \frac{1}{\left(\sqrt{5}\right)^2 + u^2} du = \frac{1}{6}\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + C = \boxed{\frac{1}{6\sqrt{5}}\tan^{-1}\left(\frac{3x^2}{\sqrt{5}}\right) + C}$$

Integrands requiring "completing the square". In this course we will occasionally have to complete the square to express a function in a desired format. Let us review the process briefly before seeing an example of an integral that requires it.

Recall the following formula for the square of a binomial.

$$(x+b)^2 = x^2 + \underbrace{2b}_{\text{halve it}}\,x + \underbrace{b^2}_{\text{square it}}$$

The important thing to note is the relationship between the coefficient of the x term and the constant term. We will make use of the "half it—square it" concept. It is also important to note that this relationship only exists when the leading coefficient is 1. The following examples should provide us with the basic procedure needed to complete the square as it will apply to some of the integrals in this section.

1) $x^2 + 4x + 12 = \overbrace{\boldsymbol{x^2} + \boldsymbol{4x} + 4}\,\underbrace{-4 + \mathbf{12}} = (x+2)^2 + 8$

2) $9 + 8x - x^2 = -[x^2 - 8x - 9] = -[\boldsymbol{x^2} - \boldsymbol{8x} + 16 - 16 - \mathbf{9}]$
$$= -[(x-4)^2 - 25]$$
$$= 25 - (x-4)^2$$

<u>Example 5</u> Find the integral $\int \frac{dx}{\sqrt{9+8x-x^2}}$.

Solution: Using the result of 2 above, we rewrite our integral as

$$\int \frac{dx}{\sqrt{9+8x-x^2}} = \int \frac{1}{\sqrt{25-(x-4)^2}}\,dx = \boxed{\sin^{-1}\left(\frac{x-4}{5}\right) + C}$$

Note: $25 = 5^2$ so $a = 5$. Since $u = x - 4, du = dx$, so we need not apply the substitution.

Special Note. The inverse sine, inverse tangent and inverse secant functions are also called the arcsine, arctangent, and arcsecant functions with the following notation:

$\arcsin x = \sin^{-1} x$ $\qquad$ $\arctan x = \tan^{-1} x$ $\qquad$ $\operatorname{arcsec} x = \sec^{-1} x$

Our textbook uses the -1 superscript notation, however, you are welcome to use either. I intend to use the -1 superscript notation for the inverse trig functions throughout the course, however, I may "slip" and use the arc notation, so be sure to be familiar with both.

At this point in the course you should have the following integration rules memorized. These functions may be encountered on any of the exams (there will not be any formula sheets allowed).

Below n represents a real number, a represents a positive real number, and C is an arbitrary constant (or parameter) sometimes called the constant of integration.

1) $\int u^n du = \frac{u^{n+1}}{n+1} + C$ (when $n \neq -1$)

2) $\int u^{-1} du = \int \frac{1}{u}\, du = \ln|u| + C$

3) $\int e^u du = e^u + C$

4) $\int a^u du = \frac{1}{\ln a} a^u + C \quad (a \neq 1)$

5) $\int \cos u\, du = \sin u + C$

6) $\int \sin u\, du = -\cos u + C$

7) $\int \sec u \tan u\, du = \sec u + C$

8) $\int \sec^2 u\, du = \tan u + C$

9) $\int \tan u\, du = -\ln|\cos u| + C$

10) $\int \cot u\, du = \ln|\sin u| + C$

11) $\int \sec u\, du = \ln|\sec u + \tan u| + C$

12) $\int \csc u\, du = -\ln|\csc u + \cot u| + C$

New Rules introduced this section:

13) $\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C$

14) $\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

15) $\int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

As we proceed in the course, additional rules may be added and I will update the list.