3.6 Cauchy-Euler Equations

While we can find explicit solutions to higher-order linear differential equations with constant coefficients with relative ease, this does not, in general, carry over to linear equations with variable coefficients. In fact, the best that we can usually do when presented with variable coefficients is a solution in the form of an infinite series. However, there are specific cases where we can still find a solution with relative ease.

- *Definition:* A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

  where the coefficients $a_0, a_1, \dots, a_n$ are constants, is called a **Cauchy-Euler equation**.

  **Note: the DE degenerates to $a_0 y = g(0)$ when $x = 0$, so we will only consider solutions on the interval $(0, \infty)$.*

First let us consider the homogeneous second-order case $ax^2y'' + bxy' + cy = 0$. If we can find a fundamental set of solutions to this equation, then we can use variation of parameters to find a particular solution y_p, and a general solution $y = y_c + y_p$.

Similar to what we did for constant coefficients, we will try to find a solution of the form $\boldsymbol{y = x^m}$. If we substitute y and its derivatives into the second-order equation, we get

$$ax^2[m(m-1)x^{m-2}] + bx[mx^{m-1}] + c[x^m] = x^m[am(m-1) + bm + c] = 0.$$

Since $x > 0$, we get a solution of $y = x^m$ whenever m is a solution to the auxiliary equation

$$\boldsymbol{am(m-1) + bm + c = 0 \quad \rightarrow \quad am^2 + (b-a)m + c = 0}.$$

Once again, we have a quadratic equation in m with three possible cases to consider.

Case I: Distinct Real Roots: In this case, we get two linearly independent solutions $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$. Since this produces a fundamental set of solutions, or general solution is

$$\boldsymbol{y = c_1 x^{m_1} + c_2 x^{m_2}}.$$

Case II: Repeated Real Roots: When $m_1 = m_2$, we only get one solution $y_1 = x^{m_1}$. We can find a second solution of the form $y_2(x) = u(x)x^{m_1}$ by using reduction of order. After going through the work (which is left as an exercise), we find that $y_2 = x^{m_1} \ln x$, giving us a general solution of

$$\boldsymbol{y = c_1 x^{m_1} + c_2 x^{m_1} ln\, x}.$$

*For higher-order equations, if m_1 is a root of multiplicity k, then it can be shown that our k linearly independent solutions are

$$x^{m_1}, \;\; x^{m_1}\ln x, \;\; x^{m_1}(\ln x)^2, \dots, x^{m_1}(\ln x)^{k-1}.$$

Case III: Complex Conjugate Roots: If m_1 and m_2 are complex conjugates, then we can write them as $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where $\beta > 0$. Formally, like before, we get the general solution

$$\boldsymbol{y = c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta}}.$$

However, once again, we prefer to write the solution in terms of real functions only. By the use of Euler's formula, and the identity

$$x^{i\beta} = \left(e^{\ln x}\right)^{i\beta} = e^{i\beta \ln x},$$

We get

$$x^{i\beta} = \cos(\beta \ln x) + i\sin(\beta \ln x)\,, \qquad x^{-i\beta} = \cos(\beta \ln x) - i\sin(\beta \ln x).$$

Using a process similar to before, we end up with

$$y_1 = x^{\alpha}\cos(\beta \ln x)\,, \qquad y_2 = x^{\alpha}\sin(\beta \ln x),$$

or

$$\boldsymbol{y_p = c_1 x^{\alpha}\cos(\beta \ln x) + c_2 x^{\alpha}\sin(\beta \ln x) = x^{\alpha}[c_1\cos(\beta \ln x) + c_2 \sin(\beta \ln x)]}.$$

- *Example:* Solve the following equations

$$\textbf{(a)}\; x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0.$$

Solution: We assume a solution of the form $y = x^m$. Taking derivatives, we get

$$y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}.$$

This gives us an auxiliary equation of

$$m(m-1) - 2m - 4 = 0 \;\;\rightarrow\;\; m^2 - 3m - 4 = 0 \;\;\rightarrow\;\; m_1 = -1, \;\; m_2 = 4.$$

Therefore our fundamental set of solutions would be $\{x^{-1}, x^4\}$, giving a general solution

$$\boldsymbol{y = c_1 x^{-1} + c_2 x^4}.$$

(b) $4x^3y'' + 8x^2y' + xy = 0.$

Solution: While this is not in the standard Cauchy-Euler form, we can divide by x to get

$$4x^2y'' + 8xy' + y = 0,$$

which is now standard Cauchy-Euler form. Taking derivatives and substituting in, we get an auxiliary equation of

$$4m(m-1) + 8m + 1 = 0 \quad \rightarrow \quad 4m^2 + 4m + 1 = 0 \quad \rightarrow \quad m_1 = m_2 = -\frac{1}{2}.$$

Therefore our fundamental set of solutions would be $\{x^{-1/2}, x^{-1/2}\ln x\}$, giving a general solution

$$\boldsymbol{y = c_1x^{-1/2} + c_2x^{-1/2}\ln x}.$$

(c) $x^3y''' + 5x^2y'' + 7xy' + 8y = 0.$

Solution: We assume a solution of the form $y = x^m$. Taking derivatives, we get

$$y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}, \qquad y''' = m(m-1)(m-2)x^{m-3}.$$

This gives us an auxiliary equation of

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0 \quad \rightarrow \quad m^3 + 2m^2 + 4m + 9 = 0$$

$$m_1 = -2, \qquad m_2 = 2i, \qquad m_3 = -2i.$$

Therefore our fundamental set of solutions would be $\{x^{-2}, \cos(2\ln x), \sin(2\ln x)\}$, giving a general solution

$$\boldsymbol{y = c_1x^{-2} + c_2\cos(2\ln x) + c_3\sin(2\ln x)}.$$

Note: for* *(b)****, we did not need to first divide by* x*; this was done to show that it was Cauchy-Euler. If we had substituted* $y = x^m$ *into the original form, we would have still gotten the same auxiliary equation, and same solution.*

- *Example:* Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

  Solution: First we consider the associated homogeneous case and assume a solution of the form $y = x^m$. Taking derivatives, we get

$$y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}.$$

This gives us an auxiliary equation of

$$m(m-1)-3m+3=0 \quad \to \quad m^2-4m+3=0 \quad \to \quad m_1=1,\ \ m_2=3.$$

Therefore our fundamental set of solutions would be $\{x, x^3\}$. Using variation of parameters, we want a particular solution of the form $y_p = u_1 x + u_2 x^3$. Before we proceed, we must get the differential equation in standard form, so we divide by x^2, giving

$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + \frac{3}{x^2}y = 2x^2e^x.$$

Next we find the Wronskian

$$W(x, x^3) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3,$$

and using the formulas from before, we get

$$u_1' = -\frac{x^3 \cdot 2x^2e^x}{2x^3} = -x^2e^x \quad \to \quad u_1 = -x^2e^x + 2xe^x - 2e^x$$

$$u_2' = \frac{x \cdot 2x^2e^x}{2x^3} = e^x \quad \to \quad u_2 = e^x.$$

Therefore,

$$y_p = -x^3e^x + 2x^2e^x - 2xe^x + x^3e^x = 2x^2e^x - 2xe^x,$$

and we get a general solution of

$$\boldsymbol{y = c_1x + c_2x^3 + 2x^2e^x - 2xe^x}.$$