New chapter

LTI Cont-Time Systems

* LTI systems can be be Modeled by

1- Differential Equations

2- Impulse Response

3- State Space Model

*Differential equation:

An LIT system can be modeled using an N'th order constant coefficient Differential equation (CCDE) as

$$b_N \frac{d^N y(t)}{dt^N} + b_{N-1} \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + b_0 y(t) = a_M \frac{d^M x(t)}{dt^M} + \cdots + a_0 x(t).$$

with initial conditions $y(0), y'(0), \cdots y^{N-1}(0)$, The solution of the CCDE is $y(t) = y_h(t) + y_p(t)$. Where $y_h(t)$ is the homogenous (natural, transiant) solution with zero input i,e:

$$b_N \frac{d^N y_h(t)}{dt^N} + \cdots + b_0 y_h(t) = 0.$$

And $y_p(t)$ is the particular (steady state, forced) solution with an input i,e:

$$b_N \frac{d^N y_p(t)}{dt^N} + \cdots + b_0 y_p(t) = a_M \frac{d^M x(t)}{dt^M} + \cdots + a_0 x(t).$$

* $y_h(t)$: let $y_h(t) = e^{rt}$, then $b_N r^N e^{rt} + b_{N-1} r^{N-1} e^{rt} + \cdots + b_0 e^{rt} = 0$

$$\underbrace{e^{rt}\left(b_N r^N + b_{N-1} r^{N-1} + \cdots + b_0\right)}_{\Delta r \text{ characteristic equation}} = 0$$

$$\Delta r = 0 = (r - r_1)(r - r_2) \cdots = 0$$

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots$$

* cases:

1. All roots are real and distinct;

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \dots + c_N e^{r_N t}$$

2. If a root r_i has multiplicity P, $(r - r_i)^P$;

$$c_{i1} e^{r_i t} + c_{i2} t e^{r_i t} + \dots + c_{ip} t^{P-1} e^{r_i t}$$

3. If we have a complex conjugate roots $(a+bj)$, $(a+bj)^*$; $a \pm bj$

solution → $\bar{c}_1 e^{(a+bj)t} + \bar{c}_2 e^{(a-bj)t} = e^{at}(\bar{c}_1 e^{bjt} + \bar{c}_2 e^{-bjt})$

solution

solution → $\boxed{e^{at}(c_1 \cos(bt) + c_2 \sin(bt))}$

* $y_p(t)$:

$y_p(t)$ has the same form as the input.

$x(t)$	$y_p(t)$
$e^{\pm at}$	$c_1 e^{\pm at}$
$\cos(w_0 t)$, $\sin(w_0 t)$	$c_1 \cos(w_0 t) + c_2 \sin(w_0 t)$
$e^{\pm at} \cos(w_0 t)$, $e^{\pm at} \sin(w_0 t)$	$e^{\pm at}(c_1 \cos(w_0 t) + c_2 \sin(w_0 t))$
t^3	$at^3 + bt^2 + ct + d$
$t^2 e^{\pm at}$	$e^{\pm at}(at^2 + bt + c)$
P	C
$u(t)$	C

* If the particular solution and the homogenous soln shares a common term, then that of the particular is multiplied by t.

25

Ex $\ddot{y}(t) + y(t) = \sin(t)$, $y(0) = 1$
$\dot{y}(0) = 0$

Solution:-

$y(t) = y_h(t) + y_p(t)$

$\rightarrow y_h(t)$: $D r = r^2 + 1 = 0 \rightarrow r = \pm j$

$\therefore y_h(t) = e^{0t}(c_1 \cos(t) + c_2 \sin(t))$

$= c_1 \cos(t) + c_2 \sin(t)$

$\rightarrow y_p(t) = t(a\cos(t) + b\sin(t))$ -- 1

$\ddot{y}_p(t) + y_p(t) = \sin(t)$ -- 2

substitute 1 in 2 and simplify:

$-2a\sin(t) + 2b\cos(t) = \sin(t)$

$\therefore -2a = 1 \longrightarrow a = -\frac{1}{2}$

$2b = 0 \longrightarrow b = 0$

$\rightarrow y_p(t) = -\frac{1}{2}t\cos(t)$

$y(t) = c_1\cos(t) + c_2\sin(t) - \frac{1}{2}t\cos(t)$

$y(0) = 1$ $c_1 = 1 \longrightarrow c_1 = 1$

$\dot{y}(0) = 0 \longrightarrow c_2 = \frac{1}{2}$

$\therefore y(t) = \cos(t) + \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t)$

- Stability:

Stability of LTI is governed by the roots of the natural solution $y_h(t)$.

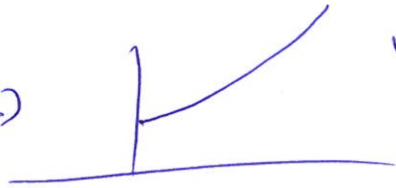
note!

- roots are real: $\rightarrow ce^{rt}$

If r is < 0 $\longrightarrow$

system is stable
exponentially decaying

If $r > 0$ $\longrightarrow$



unstable

- complex roots $\pm bj$
real part $a = 0$

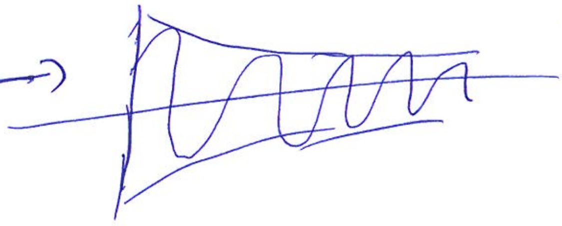
solution $\rightarrow c_1\cos(bt) + c_2\sin(bt)$

marginally (critically stable)

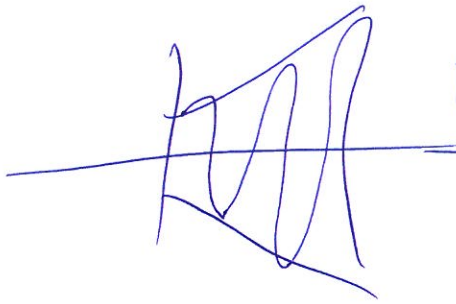
- complex roots $a \pm bj$ solution $\rightarrow e^{at}(c_1\cos bt + c_2\sin bt)$

stable

$a < 0 \longrightarrow$



$a > 0 \longrightarrow$



unstable

<u>conclusion</u>: an LTI causal system is stable if all real parts of all roots of the CCDE are < 0.

* The Impulse response:

Let a system be LTI, then we define the impulse response $h(t)$ of the system as

$$x(t) = \delta(t) \longrightarrow \boxed{T(\;)} \longrightarrow y(t) = h(t)$$

$$\underset{\text{input}}{\delta(t-t_0)} \longrightarrow \boxed{} \longrightarrow \underset{\text{output}}{h(t-t_0)}$$

→ **Conclusions:** if a system is LTI then

1) The system is comletely described by its impulse response $h(t)$.

2) The output of the system $y(t)$ can be calculated using the convolution between $x(t)$ and $h(t)$ as

حفظ $$\left(y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \right)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

→ **Properties of the convolution (LTI system):**

1) $$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$ → let $t - \lambda = L$, $\lambda = t - L$

$$= \int h(L)\, x(t-L)\, dL$$

$$= h(t) * x(t)$$

→ The convolution integral is symmetrical w.r.t $x(t)$ and impulse response $h(t)$.

② $x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$ these are equivalent

$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$

cascade system

$x(t) \longrightarrow \boxed{h_1(t) * h_2(t)} \longrightarrow y(t) \equiv x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$h(t) = h_1(t) * h_2(t)$

Total impulse response of system

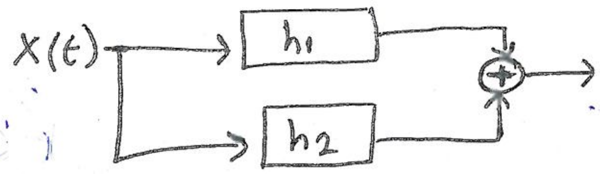
→ Equivelent or Total Impulse Response:

let $x(t) = \delta(t)$

$\delta(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{h_1} \boxed{h_2(t)} \longrightarrow y(t) = h_1 * h_2 = h$

$h(t) = h_1 * h_2$

parallel system

③ 

$\equiv x(t) \rightarrow \boxed{h_1 + h_2} \rightarrow y(t)$

$x(t) \rightarrow \boxed{h} \rightarrow y(t)$

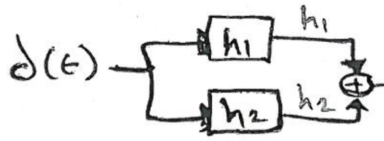
Total impulse Respons

$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$
$= x(t) * (h_1(t) + h_2(t))$

note → in cascade we convolve, in parallel we sum.

→ Equivelent : or Total impulse Response $h(t)$

let $x(t) = \delta(t)$



$y(t) = h(t) = h_1 + h_2$

④ $y(t) = \int h(\lambda)\, x(t-\lambda)\, d\lambda$

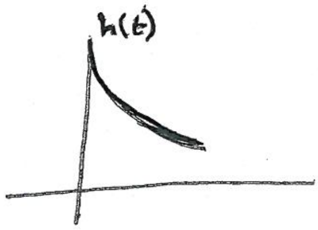
$$\approx \Delta\lambda \sum_{k} h(k\Delta\lambda)\, x(t-k\Delta\lambda)$$

$$\approx \Delta\lambda \left(\cdots + h(-2\Delta\lambda)\, x(t+2\Delta\lambda) + h(-\Delta\lambda)\, x(t+\Delta\lambda) + h(0)\, x(t) + \cdots \right)$$

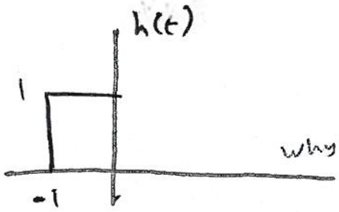
The system is causal when $\boxed{h(t) = 0,\ t<0}$

$\boxed{h(t) = 0;\ t<0}$

Ex: $h(t) = u(t)e^{-t}$

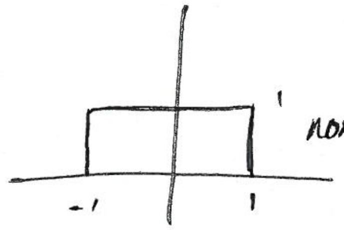


Causal ; $h(t)$ at $t<0$ $=0$



non-causal

why? at $t<0$ $h=1$



non-Causal

at $t<0$ $h=1$

⑤ let $x(t)$ be bounded, $|x(t)| \leq \alpha < \infty$

$$|y(t)| = \left| \int h(\lambda)\, x(t-\lambda)\, d\lambda \right| < \infty$$

$$\leq \int |h(\lambda)\, x(t-\lambda)|\, d\lambda$$

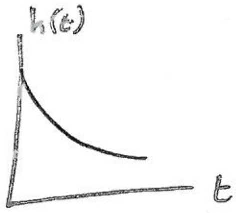
$$\leq \int |h(\lambda)|\, |x(t-\lambda)|\, d\lambda \leq \alpha \int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda$$

Then $|y(t)|$ is bounded when

$\int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda < \infty$ $\therefore$ system is stable when

$$\boxed{\int_{-\infty}^{\infty} |h(t)|\, d(t) < \infty}$$

EX: $h(t) = u(t)\,e^{-t}$



$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} e^{-t}\,dt = e^{-t}\Big]_{\infty}^{0} = 1 < \infty \quad \text{stable.}$$

EX: $x(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow y(t)$, Find first $h(t)$?

$$\delta(t) \longrightarrow \boxed{\int_{-\infty}^{t}} \longrightarrow h(t)\ ; \quad h(t) = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda = u(t)$$

check:

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} d(t) = t\Big]_{0}^{\infty} = \infty \quad \text{unstable.}$$

⑥ unit step response ($g(t)$): (It is easier to practically determine $g(t)$ than $h(t)$)

$$\underset{u(t)}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow g(t)$$

$$g(t) = \int_{-\infty}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda = \int_{0}^{\infty} h(t-\lambda)\,d\lambda \qquad \text{let } L = t-\lambda$$

$$g(t) = -\int_{t}^{-\infty} h(L)\,dL = \int_{-\infty}^{t} h(L)\,dL$$

$$\boxed{\frac{dg(t)}{dt} = h(t)} \qquad u(t) \longrightarrow \boxed{h(t)} \xrightarrow{g(t)} \boxed{\frac{d}{dt}} \longrightarrow h(t)$$

⟹ relation between $h(t)$ and $g(t)$

7. $x(t) \overset{\text{convolve}}{*} \delta(t) = x(t)$

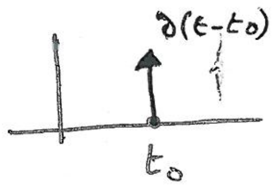
any function

of $\rightarrow \int \delta(\lambda)\, x(t-\lambda)\, d\lambda = \int \delta(\lambda)\, x(t-\lambda)\Big|_{\lambda=0} d\lambda = \int \delta(\lambda)\, x(t)\, d\lambda$

$\lambda = 0$

$= x(t) \underbrace{\int \delta(\lambda)\, d\lambda}_{1} = x(t)$

EX: $x(t) \overset{\text{convolve}}{*} \delta(t-t_0) = x(t-t_0)$



use property 1

$= \int \delta(\lambda - t_0)\, x(t-\lambda)\, d\lambda = \int \delta(\lambda - t_0)\, x(t-\lambda)\Big|_{\lambda = t_0} d\lambda$

$\lambda - t_0 = 0$
$\lambda = t_0$

$= x(t-t_0) \int \delta(\underset{\lambda - t_0}{\cdot})\, d\lambda = x(t-t_0)$

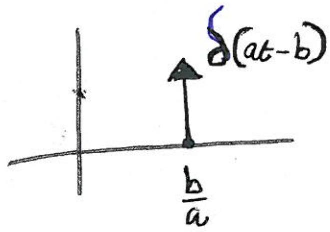
EX: $x(t) * \delta(at-b) = \frac{1}{|a|}\, x(t - \frac{b}{a})$

$= \int \delta(a\lambda - b)\, x(t-\lambda)\, d\lambda$

$= \int \delta(a\lambda - b)\, x(t-\lambda)\Big|_{\lambda = \frac{b}{a}} d\lambda = x(t - \frac{b}{a}) \int \delta(a\lambda - b)\, d\lambda$

$a\lambda = b$
$\lambda = \frac{b}{a}$

$= \frac{1}{|a|}\, x(t - \frac{b}{a})$



* How to perform convolution?

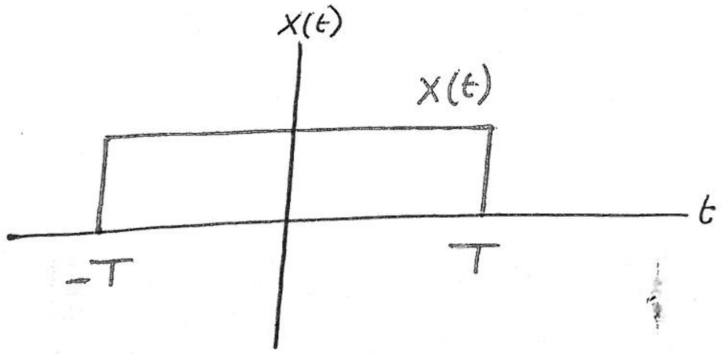
$$y(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

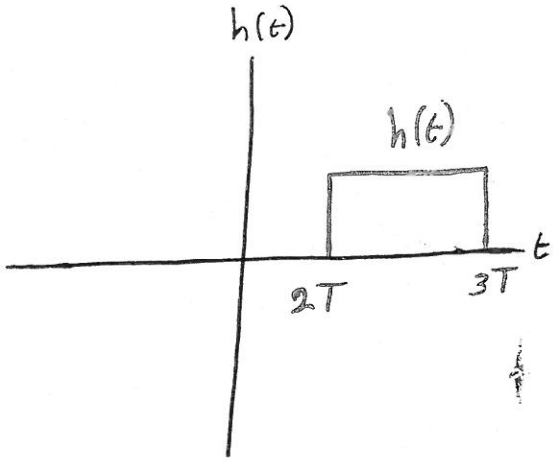
$\rightarrow y(0) = \int_{-\infty}^{\infty} x(\lambda)\, h(-\lambda)\, d\lambda \rightarrow$ multiplication and integration.

$y(1) = \int_{-\infty}^{\infty} x(\lambda)\, h(1-\lambda)\, d\lambda \rightarrow$ find $h(t)$, shift and ....

EX: $x(t) = u(t+T) - u(t-T)$, $h(t) = u(t-2T) - u(t-3T)$

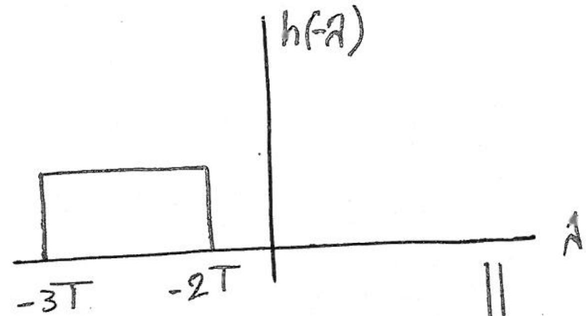
Find $y(t) = x(t) * h(t)$.

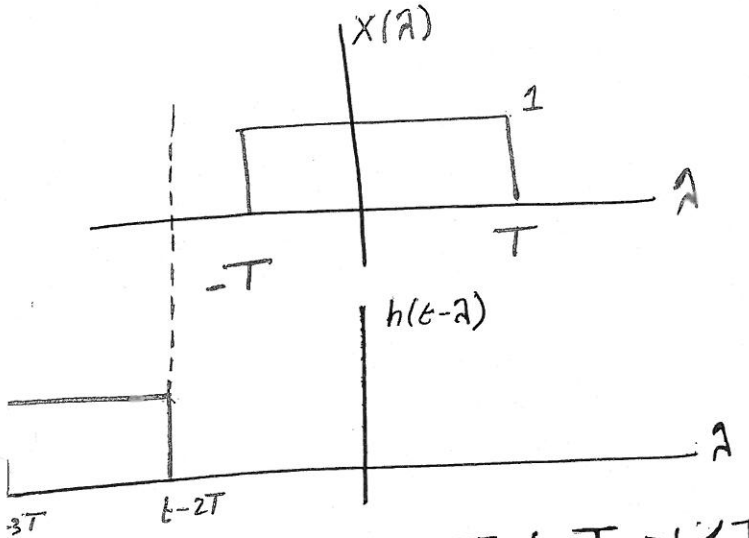


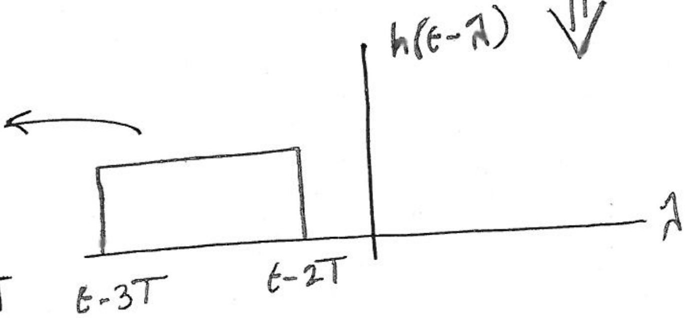


$$y(t) = \int x(\lambda)\, h(t-\lambda)\, d\lambda$$

I need $x(\lambda)$ I need $h(t-\lambda)$







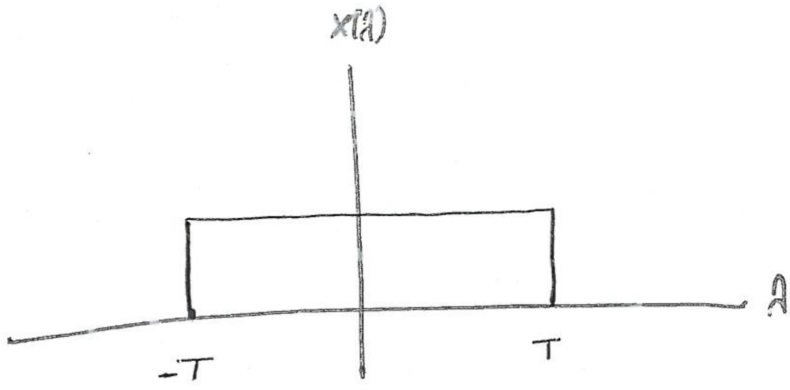
$\Longrightarrow$ As long As $t-2T < -T \longrightarrow t < T$

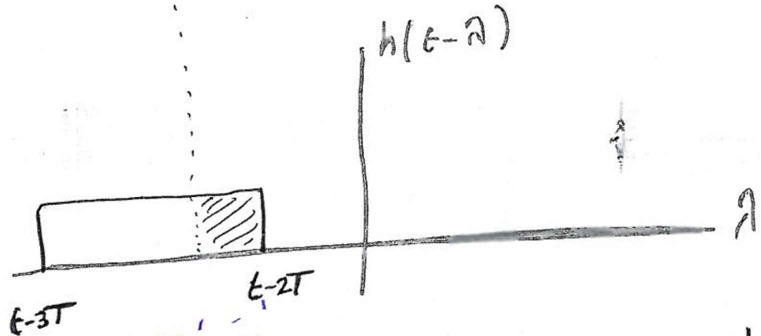
$$\boxed{y(t) = 0 \quad , \; -\infty < t < T}$$

$\Longrightarrow$ As long As $t-2T \geq -T \longrightarrow t \geq T$ until $t-3T = -T \longrightarrow t = 2T$

$$y(t) = \int_{-T}^{t-2T} (1)(1)\, d\lambda = t-2T+T = t-T$$

$$\boxed{y(t) = t-T \; , \; T \leq t \leq 2T}$$





from $\longrightarrow$ $t-3T > -T \Rightarrow t > 2T$

$$y(t) = \int_{t-3T}^{t-2T} (1)(1)\, d\lambda = T$$

until $\longrightarrow$ $t-2T \leqslant T \Rightarrow t = 3T$

$$\boxed{y(t) = T \quad , 2T < t \leqslant 3T}$$

from $\longrightarrow$ $t-2T > T \Rightarrow t > 3T$

$$y = \int_{t-3T}^{T} (1)(1)\, d\lambda = 4T - t$$

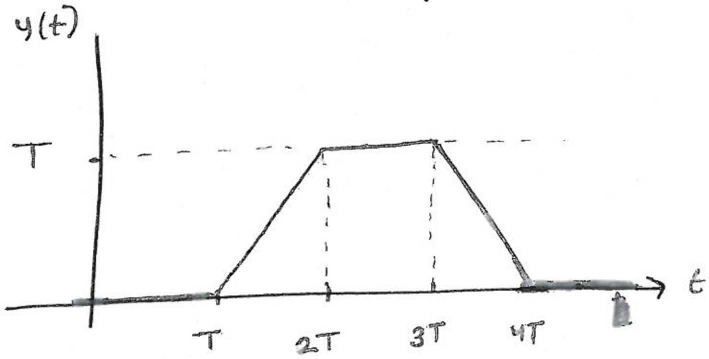
until $\longrightarrow$ $t-3T = T \Rightarrow t = 4T$

$$\boxed{y(t) = 4T - t \quad , 3T < t \leqslant 4T}$$

for $\longrightarrow$ $t-3T > T \Rightarrow t > 4T$

$$\boxed{y(t) = 0 \quad , t > 4T}$$

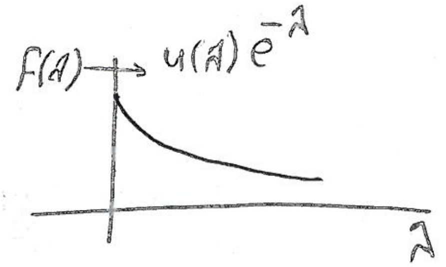
$$y(t) = \begin{cases} 0 & , -\infty < t \leqslant T \\ t - T & , T < t \leqslant 2T \\ T & , 2T < t \leqslant 3T \\ 4T - t & , 3T < t \leqslant 4T \\ 0 & , 4T < t < \infty \end{cases}$$

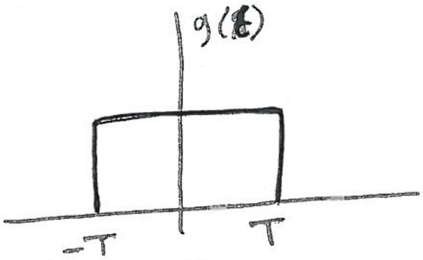


X:

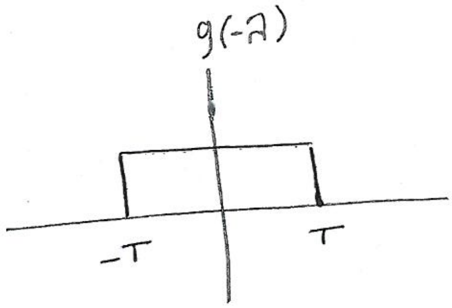
$f(t) = u(t)\, e^{-t}$, $g(t) = u(t+T) - u(t-T)$ Find $y(t) = f * g$?

$$y(t) = \int f(\lambda)\, g(t-\lambda)\, d\lambda$$

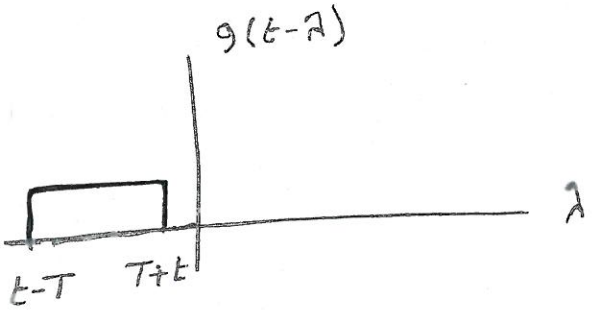




$\Downarrow$



$\Leftarrow$



$\Longrightarrow$ when $t+T < 0 \Longrightarrow t < -T$

$$\boxed{y(t) = 0 \;,\; -\infty < t < -T}$$

$\Longrightarrow$ when $t+T \geq 0 \Longrightarrow t \geq -T$

$$y(t) = \int_0^{t+T} e^{-\lambda}(1)\, d\lambda = e^{-\lambda}\Big]_{t+T}^{0} = 1 - e^{-t-T}$$

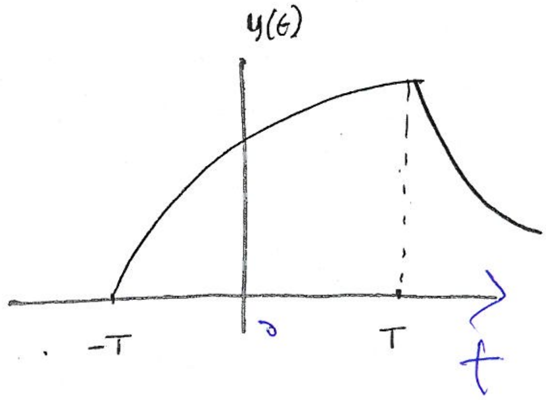
$\Longrightarrow$ until $t-T=0 \Longrightarrow t=T$

$$\boxed{y(t) = 1 - e^{-t} e^{-T} \;,\; -T \leq t \leq T}$$

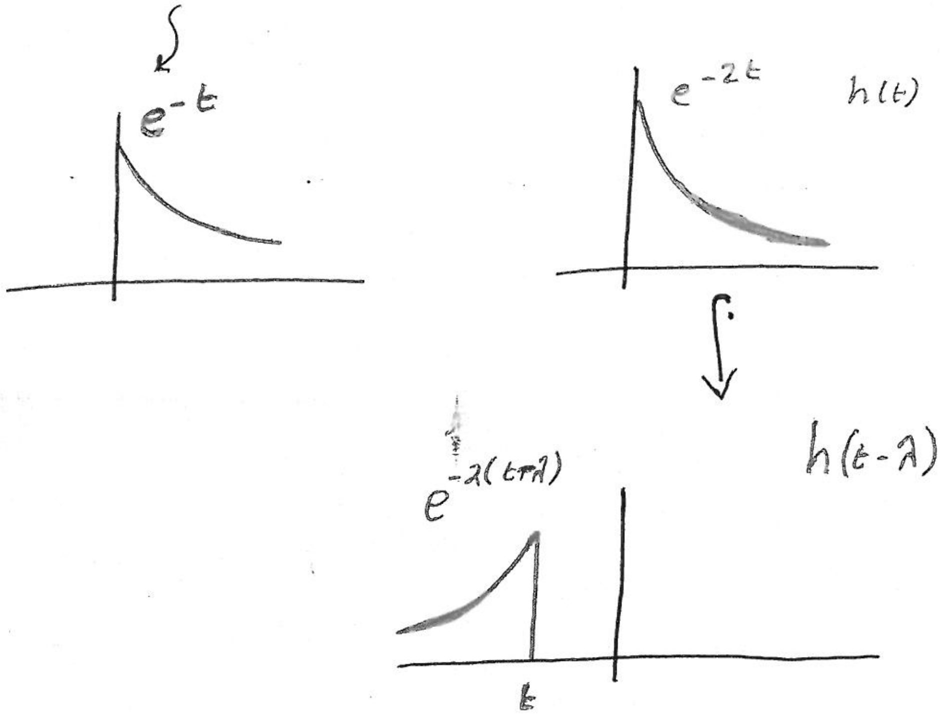
$\Longrightarrow$ when $t-T > 0 \Longrightarrow t > T$

$$y(t) = \int_{t-T}^{t+T} e^{-\lambda}(1)\, d\lambda = e^{-\lambda}\Big]_{t+T}^{t-T} =$$

$$\boxed{y(t) = e^{-t} e^{T} - e^{-t} e^{-T} \;,\; T < t < \infty}$$



H.W: $x(t) = e^{-t}u(t) \quad , h(t) = u(t)\, e^{-2t}$



F.A: $y(t) = u(t)\left(e^{-t} - e^{-2t}\right)$

$$\int_0^t e^{-\lambda}\, e^{-2(t-\lambda)}\, d\lambda = e^{-2t}\int_0^t e^{\lambda}\, d\lambda = e^{-t} - e^{-2t}$$

* Calculating the impulse response of a system described by the differential equation :

→ Solve the diff eq for :

$$b_N \frac{d^N h(t)}{dt^N} + b_{N-1} \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + b_0 h(t) = 0$$

with $h(0) = h'(0) = \dots = h^{N-2}(0) = 0 , h^{N-1}(0) = 1$, if $N=1$,

$h(0)=1$, $h(t) = (c_1 e^{r_1 t} + \dots) \underbrace{u(t)}_{\text{to be Causal}}$.

* Frequancy response to an LTI system :

– let $x(t) = e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$?

$$y(t) = \int h(\lambda)\, x(t-\lambda)\, d\lambda = \int h(\lambda)\, e^{j\omega(t-\lambda)}\, d\lambda$$

$$= \int_{-\infty}^{\infty} h(\lambda)\, e^{-\omega j \lambda}\, e^{j\omega t}\, d\lambda = e^{j\omega t} \int_{-\infty}^{\infty} h(\lambda)\, e^{-j\omega\lambda}\, d\lambda$$

$$= e^{j\omega t}\, \underbrace{H(j\omega)}_{\text{Frequency response of the system.}}$$

$$H(j\omega) = \int h(t)\, e^{-j\omega t}\, dt$$

End of chapter

Suggested Qs

3.2
3.6(a,d)
3.7
3.12
3.16
3.19
3.21(a)(f)
3.26
3.27(i)(u)