Engineering Mechanics: Statics in SI Units, 12e

4 Force System Resultants

Chapter Objectives

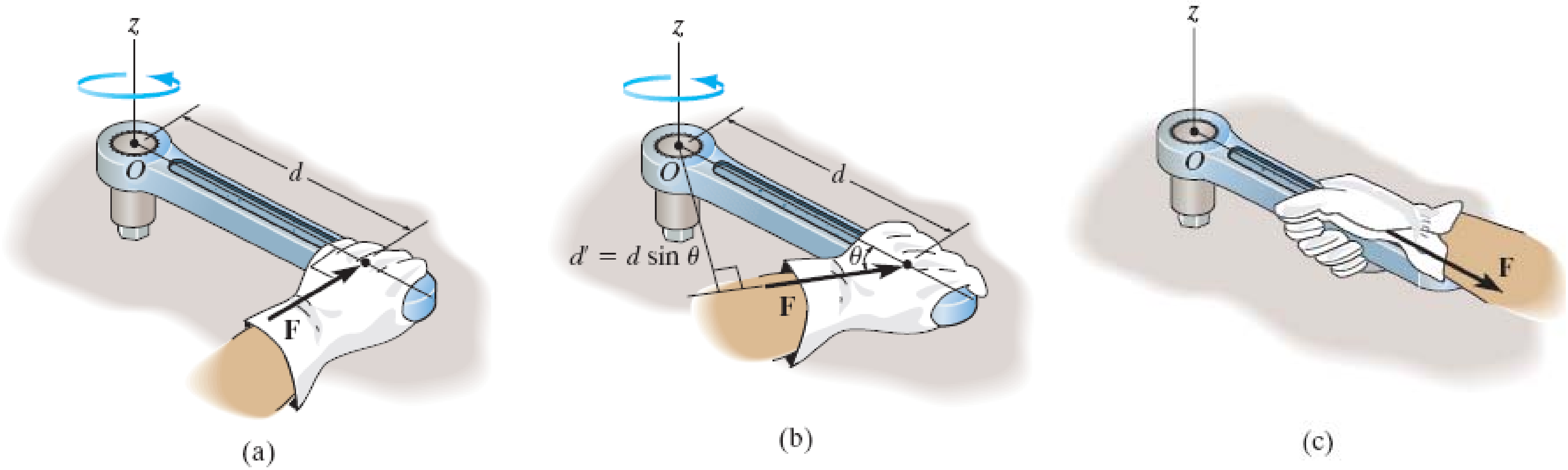
- Concept of moment of a force in two and three dimensions
- Method for finding the moment of a force about a specified axis.
- Define the moment of a couple.
- Determine the resultants of non-concurrent force systems
- Reduce a simple distributed loading to a resultant force having a specified location

Chapter Outline

1. Moment of a Force – Scalar Formation
2. Cross Product
3. Moment of Force – Vector Formulation
4. Principle of Moments
5. Moment of a Force about a Specified Axis
6. Moment of a Couple
7. Simplification of a Force and Couple System
8. Further Simplification of a Force and Couple System
9. Reduction of a Simple Distributed Loading

4.1 Moment of a Force – Scalar Formation

- *Moment* of a force about a point or axis – a measure of the tendency of the force to cause a body to rotate about the point or axis
- Torque – tendency of rotation caused by **F**$_x$ or simple moment (**M**$_o$) $_z$



4.1 Moment of a Force – Scalar Formation

Magnitude

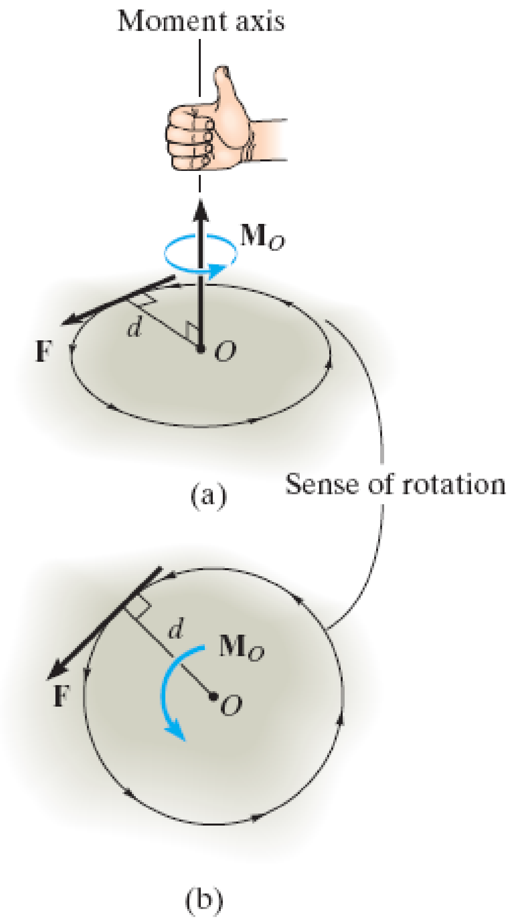
- For magnitude of $\mathbf{M}_O$,

$$\mathbf{M}_O = Fd \ (Nm)$$

where d = perpendicular distance from O to its line of action of force

Direction

- Direction using "right hand rule"

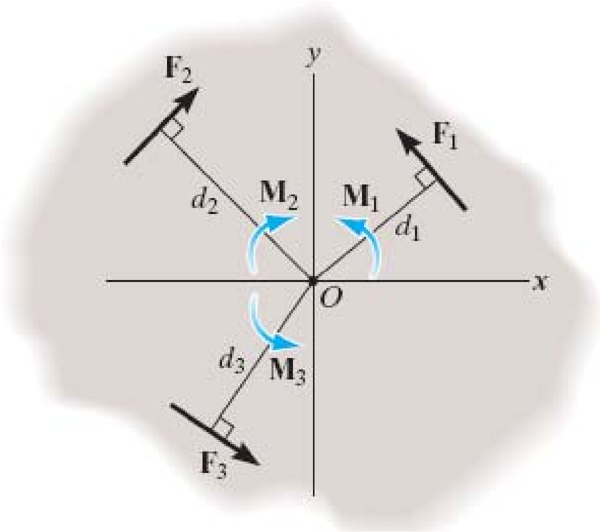


4.1 Moment of a Force – Scalar Formation

Resultant Moment

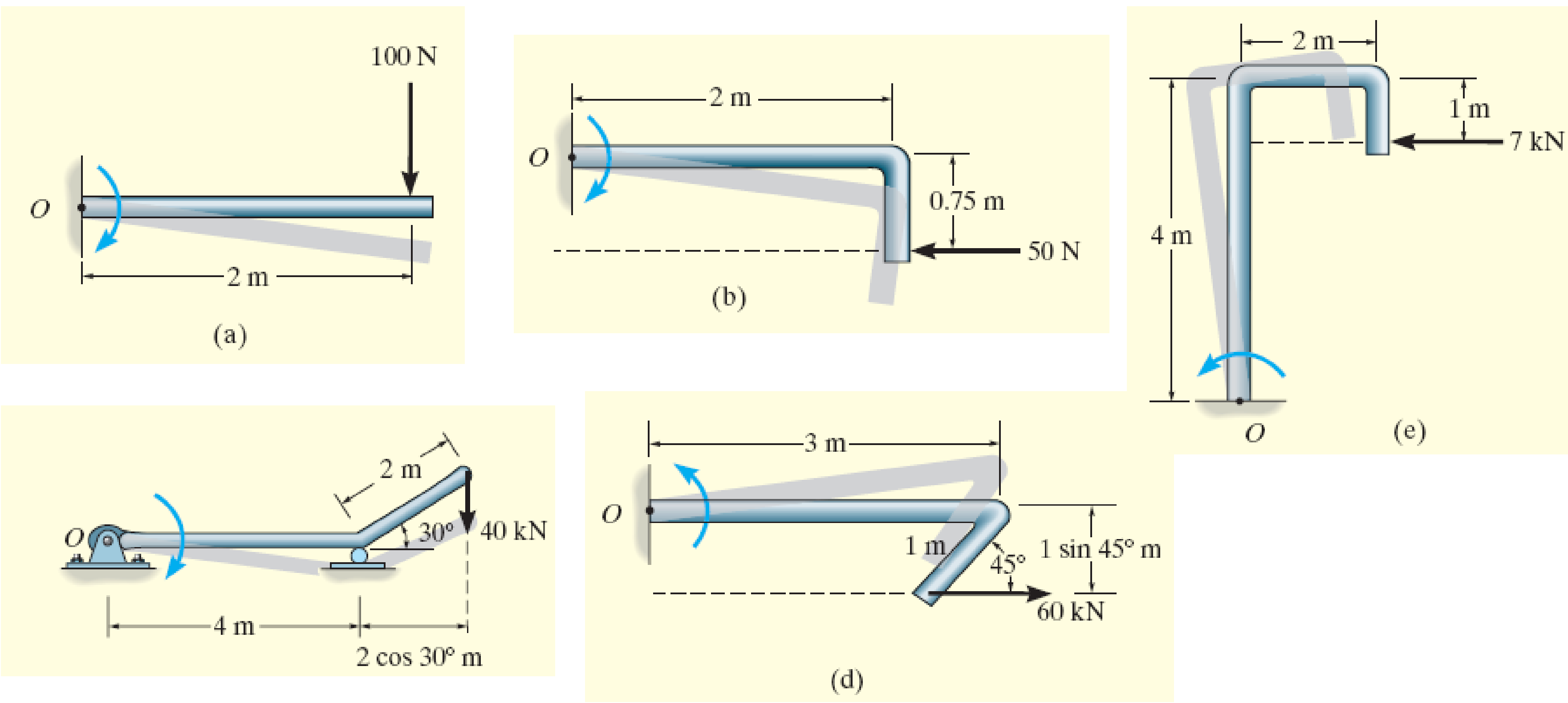
- Resultant moment, $\mathbf{M}_{Ro}$ = moments of all the forces

$$\mathbf{M}_{Ro} = \sum Fd$$



Example 4.1

For each case, determine the moment of the force about point **O**.

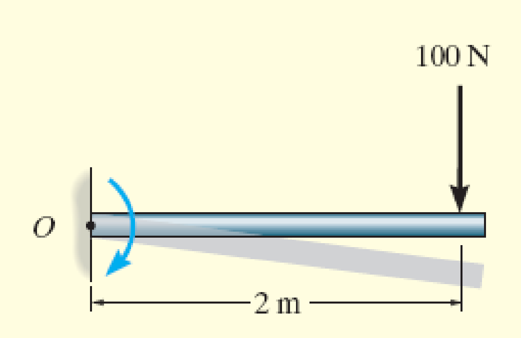


Solution

Line of action is extended as a dashed line to establish moment arm **d**.

Tendency to rotate is indicated and the orbit is shown as a colored curl.

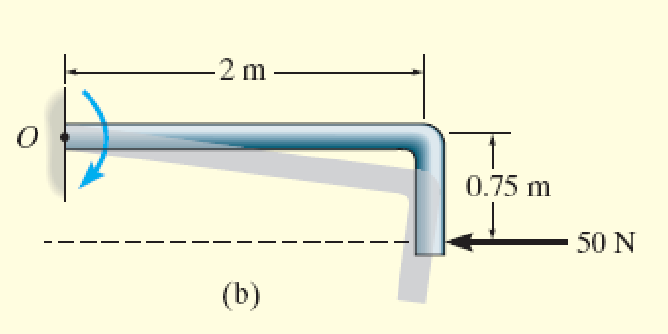
$$(a) M_o = (100N)(2m) = 200N.m(CW)$$



(a)

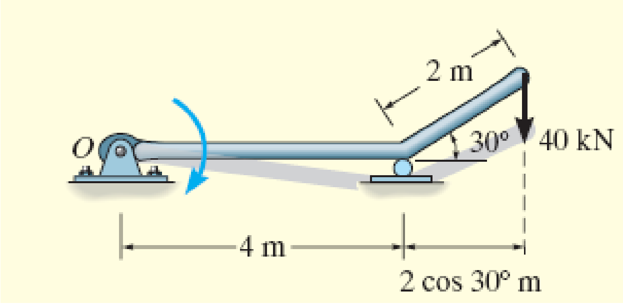
Solution

$$(b) M_o = (50N)(0.75m) = 37.5N.m(CW)$$



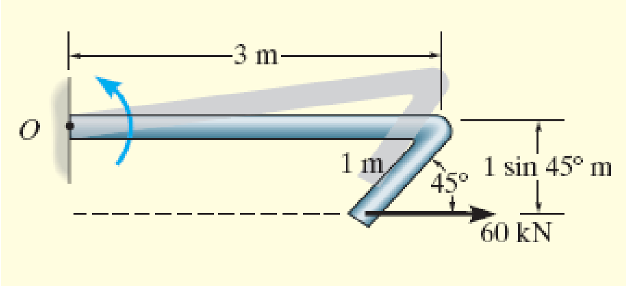
(b)

$$(c) M_o = (40N)(4m + 2\cos 30^\circ m) = 229N.m(CW)$$



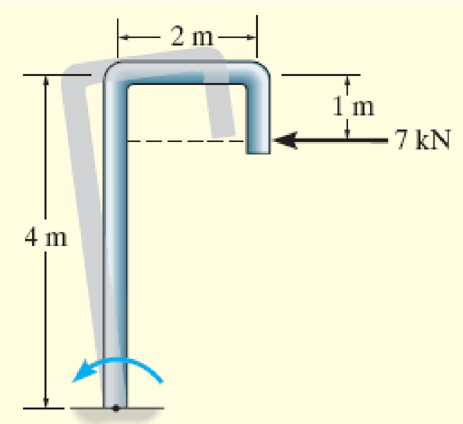
Solution

$$(d)M_o = (60N)(1\sin 45^\circ m) = 42.4N.m(CCW)$$



(d)

$$(e)M_o = (7kN)(4m - 1m) = 21.0kN.m(CCW)$$



(e)

4.2 Cross Product

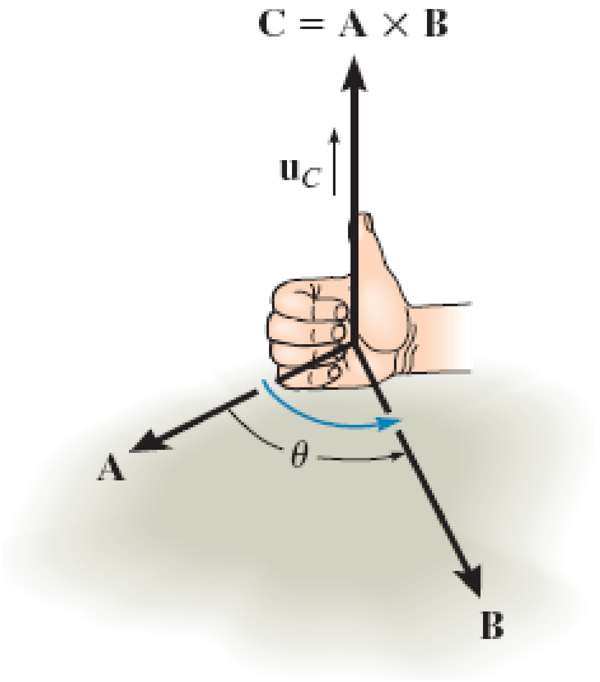
- Cross product of two vectors **A** and **B** yields **C**, which is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

Magnitude

- Magnitude of **C** is the product of the magnitudes of **A** and **B**
- For angle θ, $0° \leqslant \theta \leq 180°$

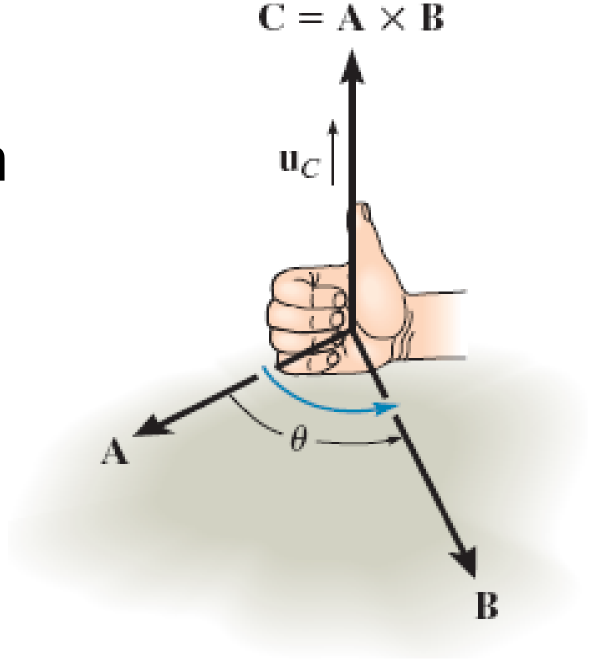
$$C = AB \sin\theta$$



4.2 Cross Product

Direction

- Vector **C** has a direction that is perpendicular to the plane containing **A** and **B** such that **C** is specified by the right hand rule
- Expressing vector **C** when magnitude and direction are known



$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (AB \sin\theta)\mathbf{u}_C$$

4.2 Cross Product

Laws of Operations

1. Commutative law is not valid

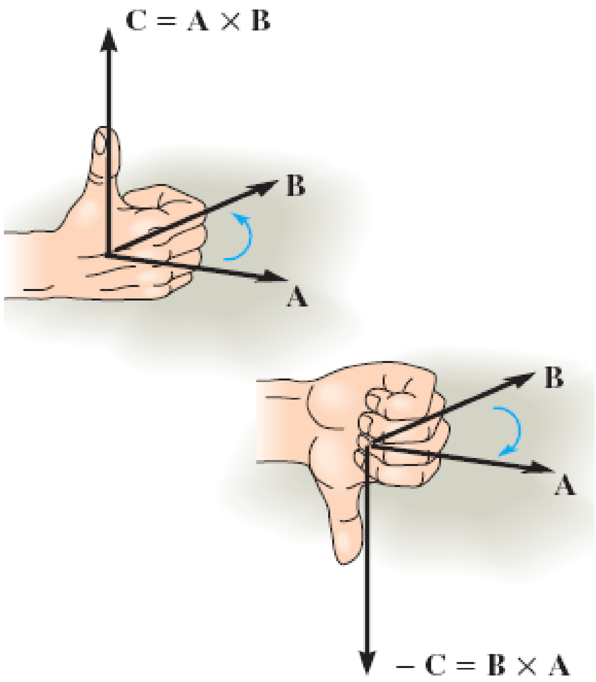
$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

- Cross product **A** X **B** yields a vector opposite in direction to **C**

$$\mathbf{B} \times \mathbf{A} = -\mathbf{C}$$



4.2 Cross Product

Laws of Operations

2. Multiplication by a Scalar

$a(\mathbf{A} \times \mathbf{B}) = (a\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\mathbf{B}) = (\mathbf{A} \times \mathbf{B})a$

3. Distributive Law

$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{D})$

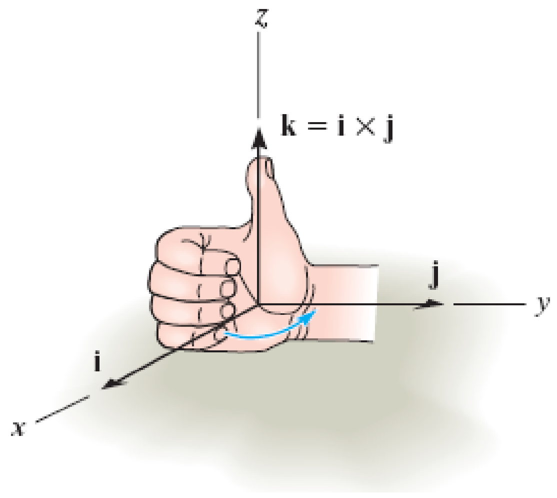
- Proper order of the cross product must be maintained since they are not commutative

4.2 Cross Product

Cartesian Vector Formulation

- Use $C = AB \sin\theta$ on pair of Cartesian unit vectors
- A more compact determinant in the form as

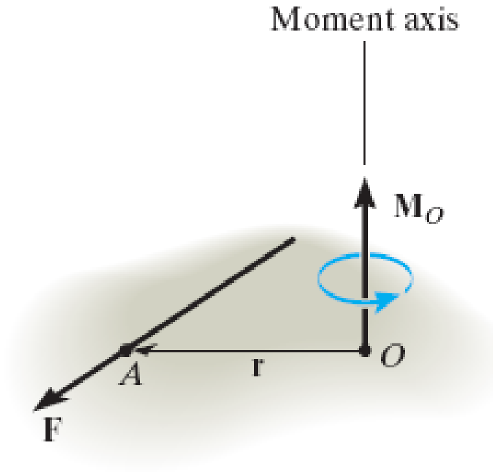
$$\vec{A} X \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



4.3 Moment of Force - Vector Formulation

- Moment of force **F** about point O can be expressed using cross product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$



Magnitude

- For magnitude of cross product,

$$M_O = rF \sin\theta$$

- Treat **r** as a sliding vector. Since $d = r \sin\theta$,

$$M_O = rF \sin\theta = F\,(r\sin\theta) = Fd$$

4.3 Moment of Force - Vector Formulation

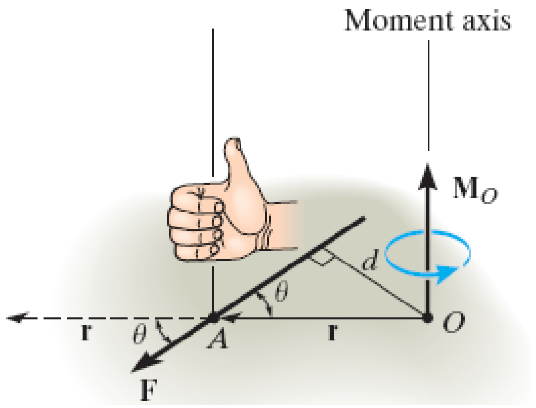
Direction

- Direction and sense of $\mathbf{M}_O$ are determined by right-hand rule

*Note:

- "curl" of the fingers indicates the sense of rotation

- Maintain proper order of **r** and **F** since cross product is not commutative

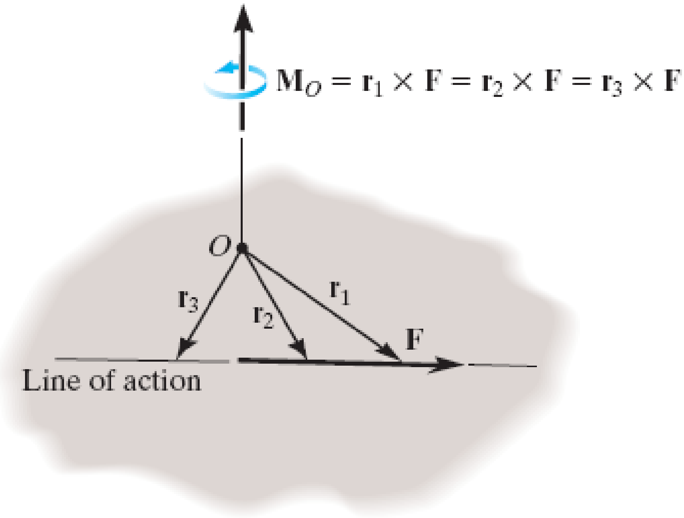


4.3 Moment of Force - Vector Formulation

Principle of Transmissibility

- For force **F** applied at any point A, moment created about O is $\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F}$
- **F** has the properties of a sliding vector, thus

$$\mathbf{M}_O = \mathbf{r}_1 \times \mathbf{F} = \mathbf{r}_2 \times \mathbf{F} = \mathbf{r}_3 \times \mathbf{F}$$

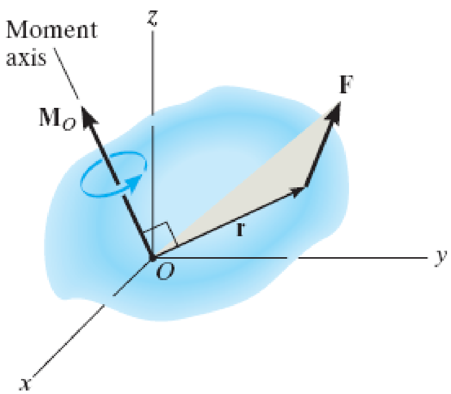


4.3 Moment of Force - Vector Formulation

Cartesian Vector Formulation

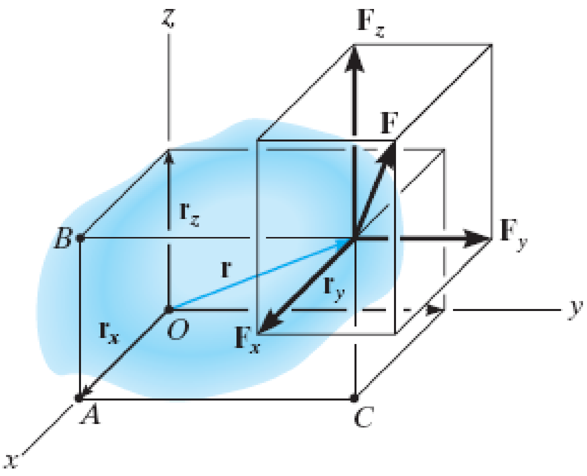
- For force expressed in Cartesian form,

$$\vec{M}_O = \vec{r} X \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



- With the determinant expended,

$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i}$
$\quad - (r_x F_z - r_z F_x)\mathbf{j} + (r_x F_y - {}_y F_x)\mathbf{k}$

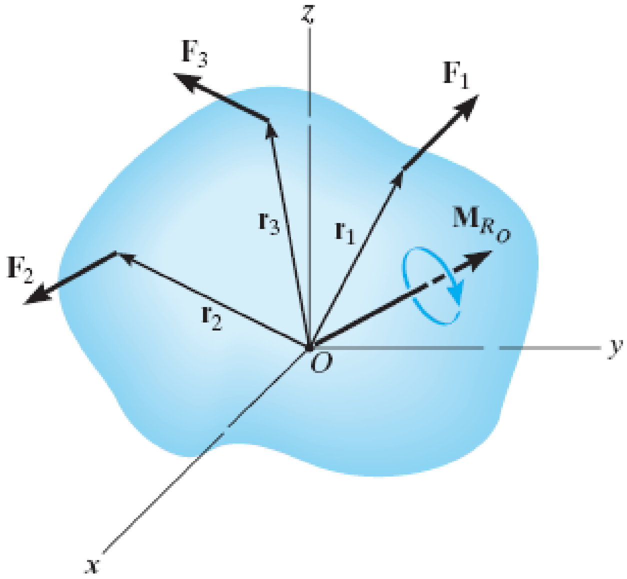


4.3 Moment of Force - Vector Formulation

Resultant Moment of a System of Forces

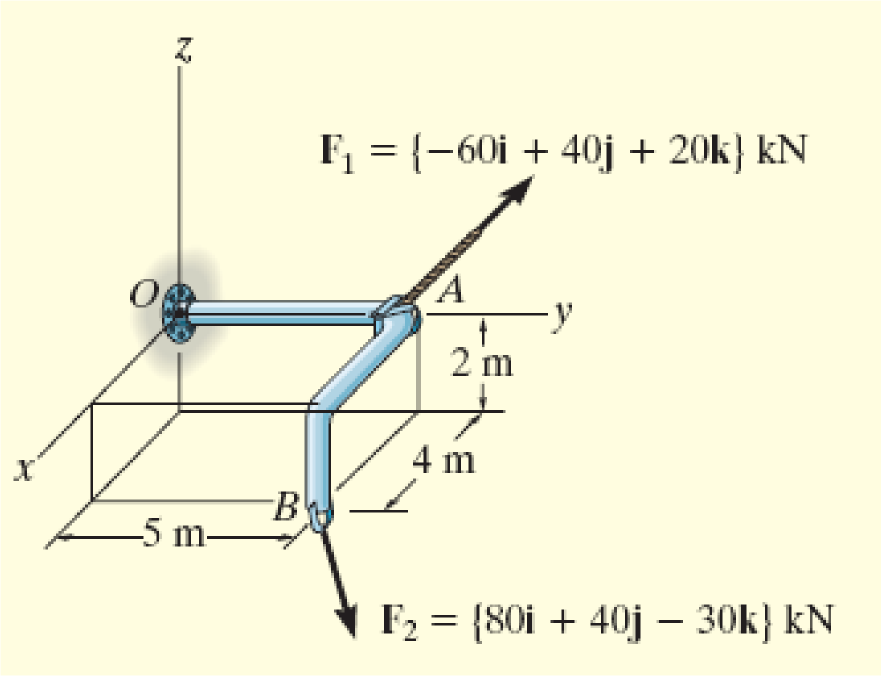
- Resultant moment of forces about point O can be determined by vector addition

$$\mathbf{M}_{Ro} = \sum(\mathbf{r} \times \mathbf{F})$$



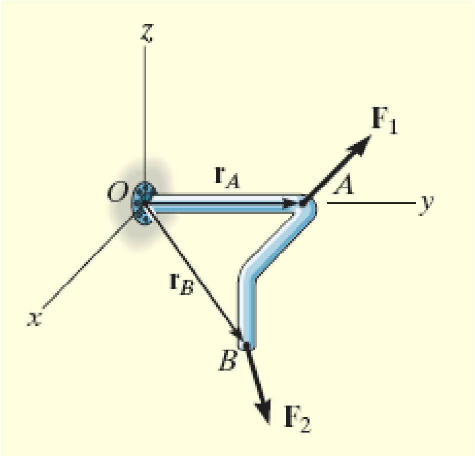
Example 4.4

Two forces act on the rod. Determine the resultant moment they create about the flange at *O*. Express the result as a Cartesian vector.



Solution

Position vectors are directed from point O to each force as shown.



These vectors are

$$r_A = \{5j\}\text{m}$$

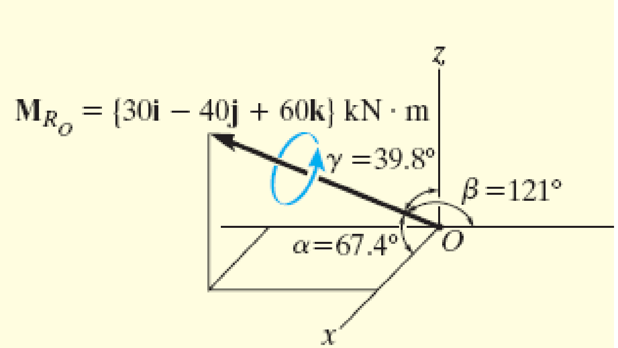
$$r_B = \{4i + 5j - 2k\}\text{m}$$

The resultant moment about O is

$$\vec{M}_O = \sum (r \times F) = r_A \times F + r_B \times F$$

$$= \begin{vmatrix} i & j & k \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= \{30i - 40j + 60k\}\text{kN} \cdot \text{m}$$

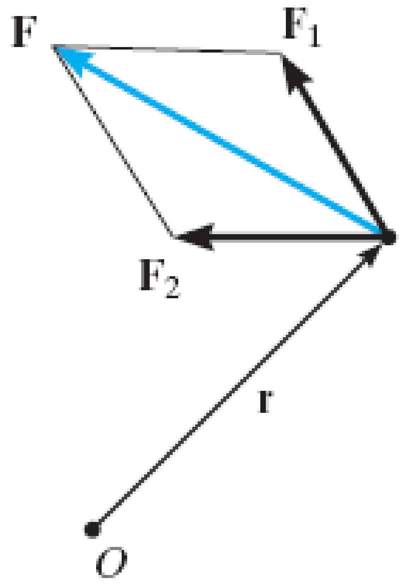


4.4 Principles of Moments

- Also known as Varignon's Theorem

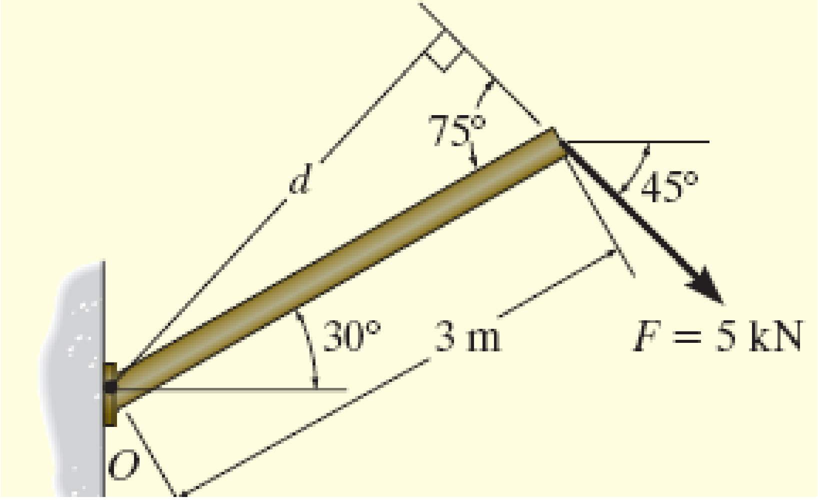
  "Moment of a force about a point is equal to the sum of the moments of the forces' components about the point"
- Since $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$,

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$
$$= \mathbf{r} \times (\mathbf{F}_1 + \mathbf{F}_2)$$
$$= \mathbf{r} \times \mathbf{F}_1 + \mathbf{r} \times \mathbf{F}_2$$



Example 4.5

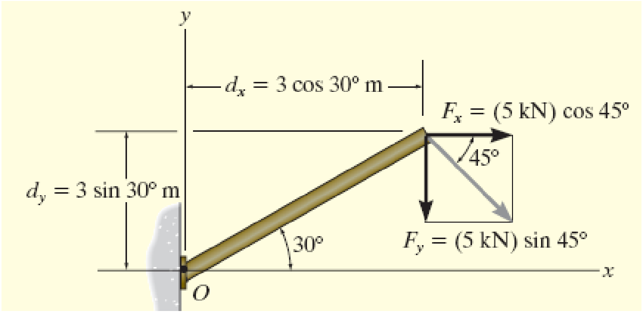
Determine the moment of the force about point *O*.



Solution

The moment arm *d* can be found from trigonometry,

$$d = (3)\sin 75° = 2.898\,\text{m}$$



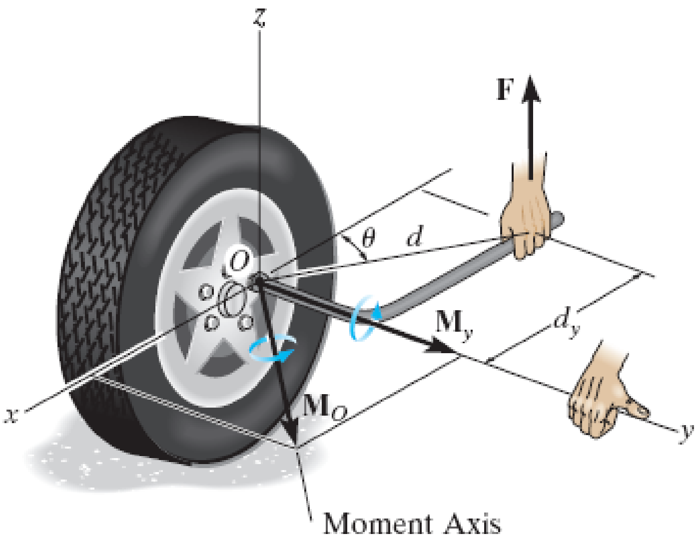
Thus,

$$M_O = Fd = (5)(2.898) = 14.5\,\text{kN}\cdot\text{m}$$

Since the force tends to rotate or orbit clockwise about point *O*, the moment is directed into the page.

4.5 Moment of a Force about a Specified Axis

- For moment of a force about a point, the moment and its axis is always perpendicular to the plane
- A scalar or vector analysis is used to find the component of the moment along a specified axis that passes through the point



4.5 Moment of a Force about a Specified Axis

Scalar Analysis

- According to the right-hand rule, M_y is directed along the positive *y* axis
- For any axis, the moment is

$$M_a = F d_a$$

- Force will not contribute a moment if force line of action is parallel or passes through the axis

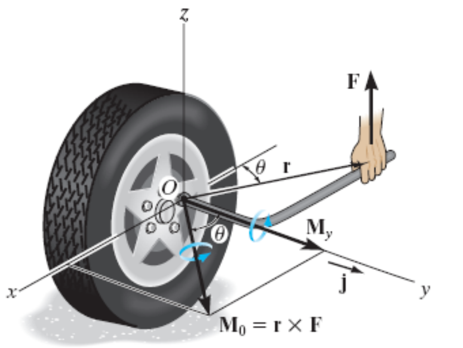
4.5 Moment of a Force about a Specified Axis

Vector Analysis

- For magnitude of $\mathbf{M}_A$,

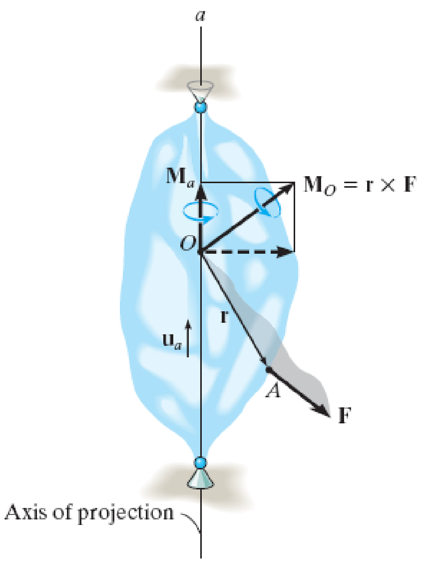
$$M_A = M_O \cos\theta = \mathbf{M}_O \cdot \mathbf{u}_a$$

where $\mathbf{u}_a$ = unit vector



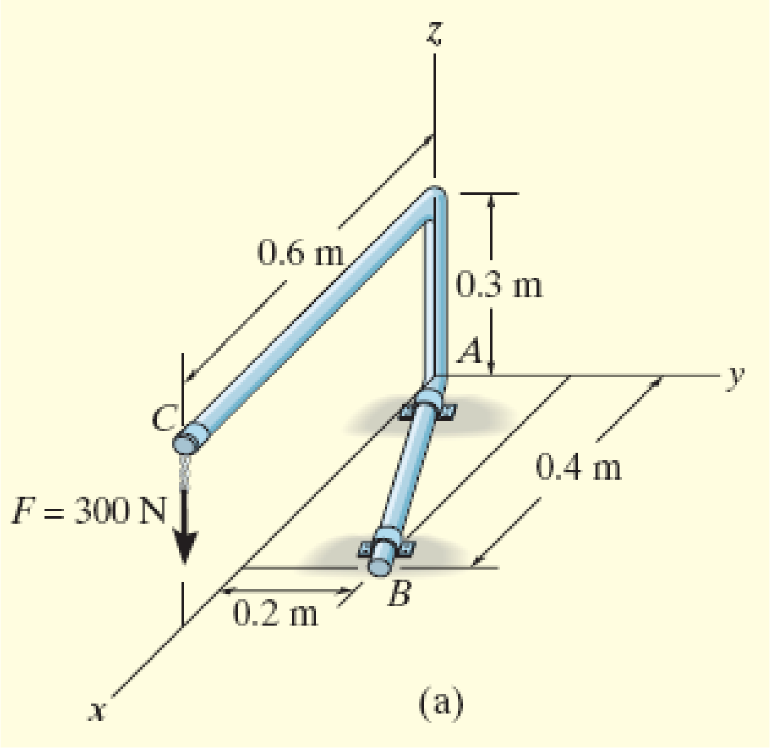
- In determinant form,

$$\left|\vec{M}_a\right| = \vec{u}_{ax} \cdot (\vec{r} X \vec{F}) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$



Example 4.8

Determine the moment produced by the force **F** which tends to rotate the rod about the *AB* axis.



Solution

Unit vector defines the direction of the *AB* axis of the rod, where

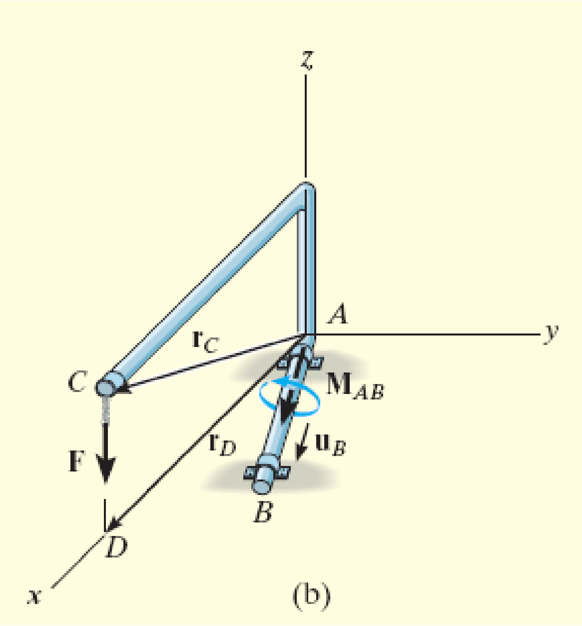
$$u_B = \frac{\vec{r}_B}{r_B} = \frac{\{0.4i + 0.2j\}}{\sqrt{0.4^2 + 0.2^2}} = 0.8944i + 0.4472j$$

For simplicity, choose $\mathbf{r}_D$

$$r_D = \{0.6i\}\text{m}$$

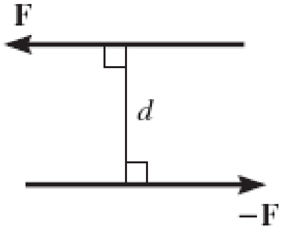
The force is

$$F = \{-300k\}\text{ N}$$



4.6 Moment of a Couple

- Couple
  - two parallel forces
  - same magnitude but opposite direction
  - separated by perpendicular distance d
- Resultant force = 0
- Tendency to rotate in specified direction
- Couple moment = sum of moments of both couple forces about any arbitrary point



Page148

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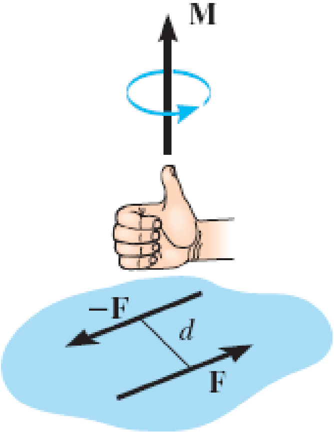
4.6 Moment of a Couple

Scalar Formulation

- Magnitude of couple moment

$$M = Fd$$

- Direction and sense are determined by right hand rule
- **M** acts perpendicular to plane containing the forces



4.6 Moment of a Couple

Vector Formulation

- For couple moment,

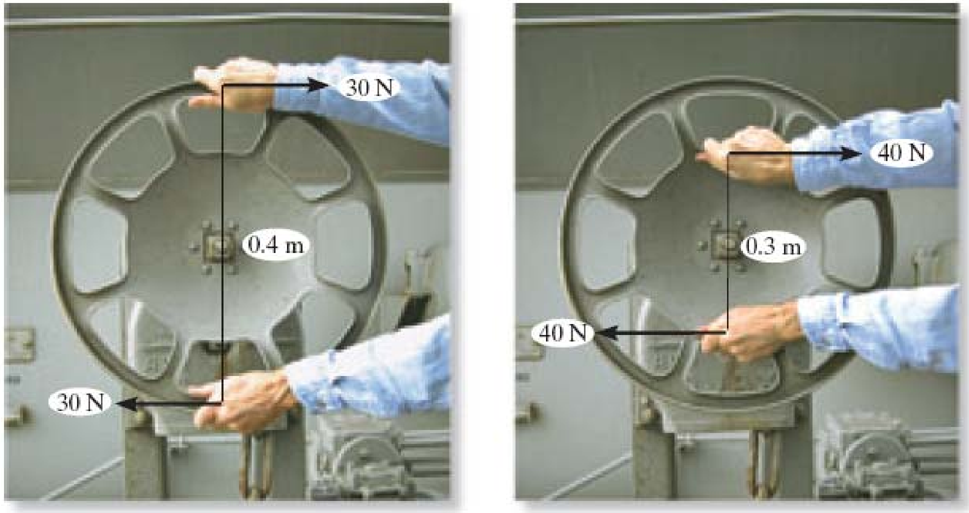
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

- If moments are taken about point A, moment of –**F** is zero about this point
- **r** is crossed with the force to which it is directed

4.6 Moment of a Couple

Equivalent Couples

- 2 couples are equivalent if they produce the same moment
- Forces of equal couples lie on the same plane or plane parallel to one another



4.6 Moment of a Couple

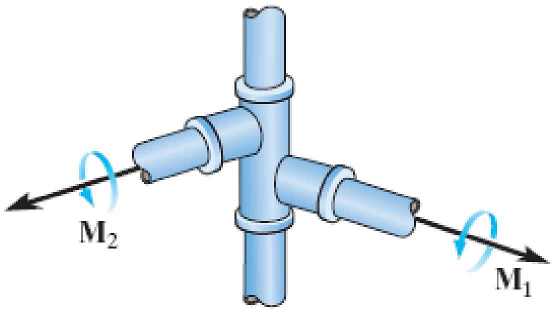
Resultant Couple Moment

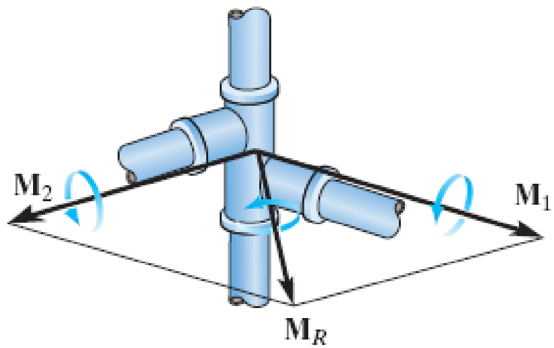
- Couple moments are free vectors and may be applied to any point P and added vectorially
- For resultant moment of two couples at point P,

$$\mathbf{M}_R = \mathbf{M}_1 + \mathbf{M}_2$$

- For more than 2 moments,

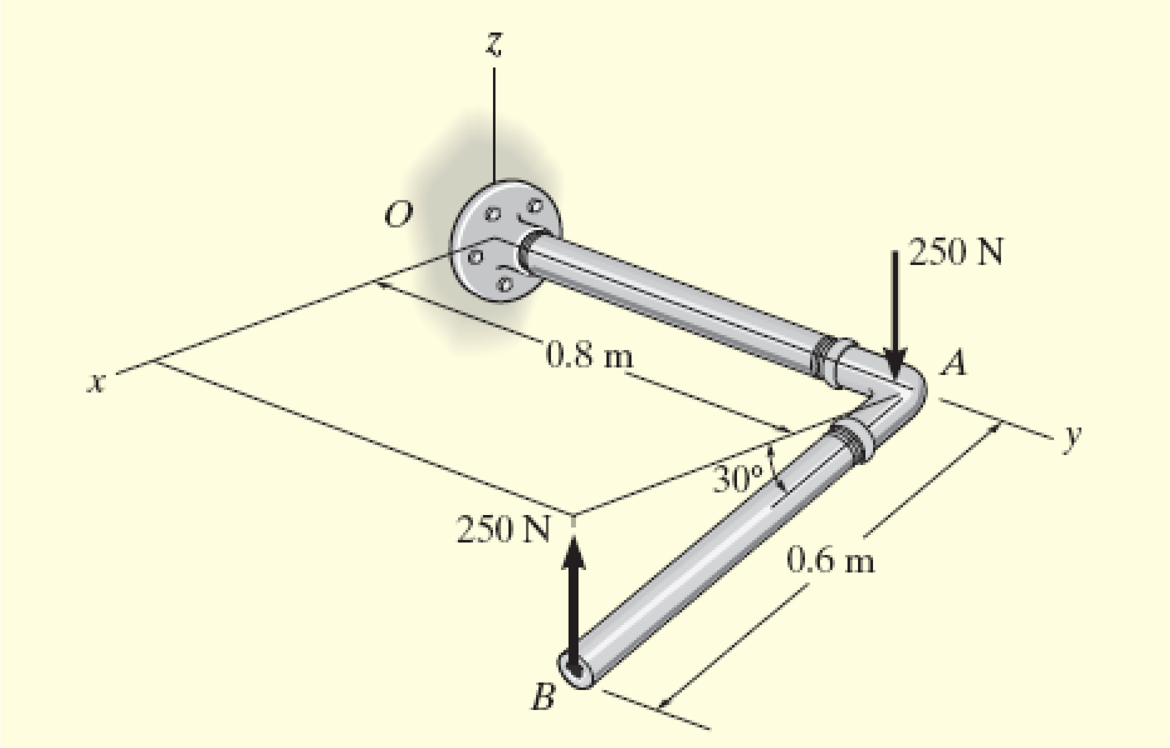
$$\mathbf{M}_R = \sum(\mathbf{r} \times \mathbf{F})$$





Example 4.12

Determine the couple moment acting on the pipe. Segment *AB* is directed 30° below the *x–y* plane.



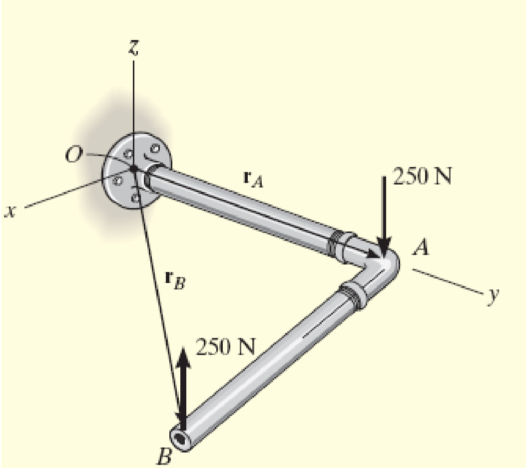
SOLUTION I (VECTOR ANALYSIS)

Take moment about point O,

$\mathbf{M} = \mathbf{r}_A \text{ X } (-250\mathbf{k}) + \mathbf{r}_B \text{ X } (250\mathbf{k})$

$= (0.8\mathbf{j}) \text{ X } (-250\mathbf{k}) + (0.66\cos 30^{o}\mathbf{i} + 0.8\mathbf{j} - 0.6\sin 30^{o}\mathbf{k}) \text{ X } (250\mathbf{k})$

$= \{-130\mathbf{j}\}\text{N.cm}$

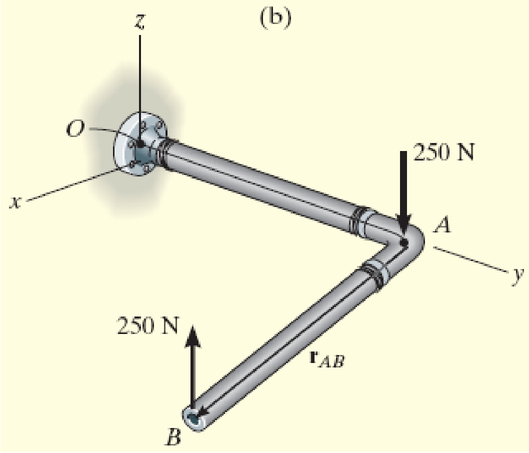


Take moment about point A

$\mathbf{M} = \mathbf{r}_{AB} \text{ X } (250\mathbf{k})$

$= (0.6\cos 30^\circ\ \mathbf{i} - 0.6\sin 30^\circ\ \mathbf{k}) \text{ X } (250\mathbf{k})$

$= \{-130\mathbf{j}\}\text{N.cm}$



SOLUTION II (SCALAR ANALYSIS)

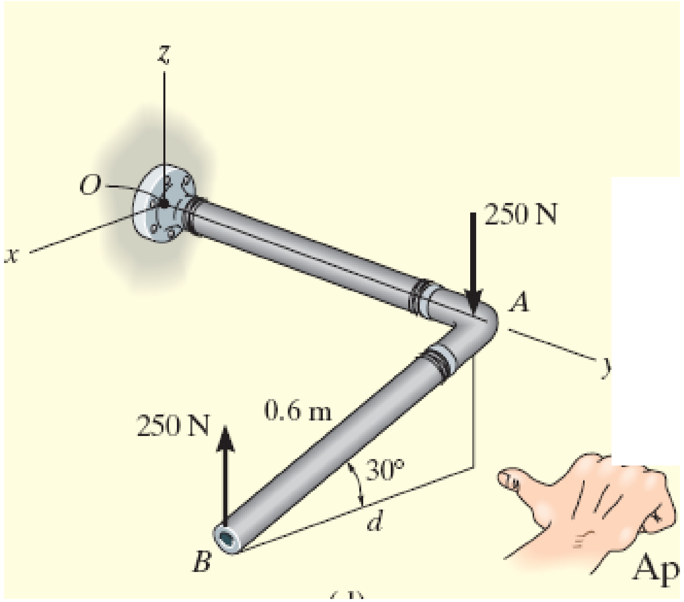
Take moment about point A or B,

$M = Fd = 250\text{N}(0.5196\text{m})$

$= 129.9\text{N.cm}$

Apply right hand rule, M acts in the –j direction

$\mathbf{M} = \{-130\mathbf{j}\}\text{N.cm}$



4.7 Simplification of a Force and Couple System

- An equivalent system is when the *external effects* are the same as those caused by the original force and couple moment system
- External effects of a system is the *translating and rotating motion* of the body
- Or refers to the *reactive forces* at the supports if the body is held fixed

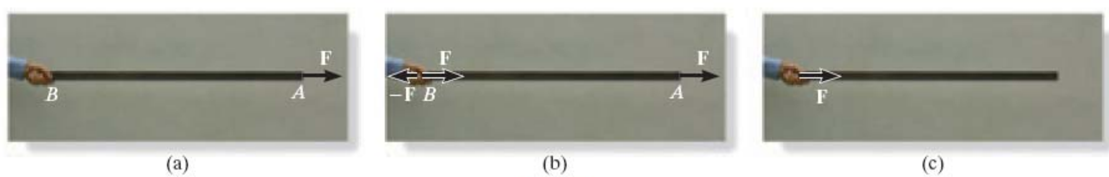
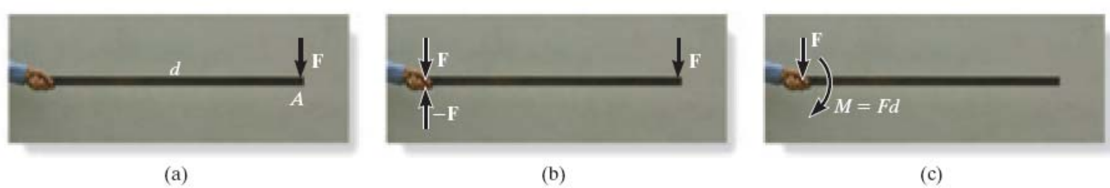


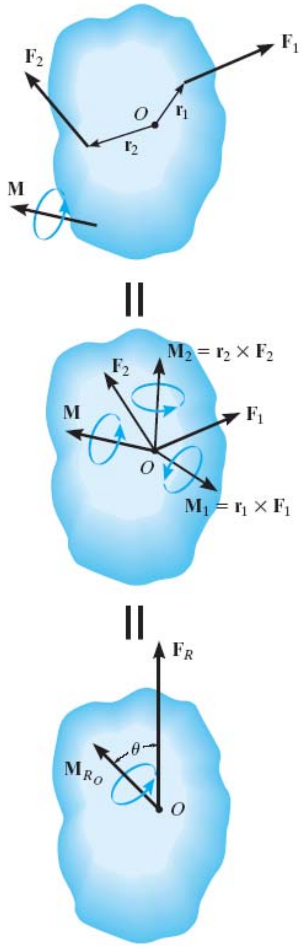
Fig. 4–34



4.7 Simplification of a Force and Couple System

- Equivalent resultant force acting at point *O* and a resultant couple moment is expressed as

$$F_R = \sum F$$
$$(M_R)_O = \sum M_O + \sum M$$



- If force system lies in the *x*–*y* plane and couple moments are perpendicular to this plane,

$$(F_R)_x = \sum F_x$$
$$(F_R)_y = \sum F_y$$
$$(M_R)_O = \sum M_O + \sum M$$

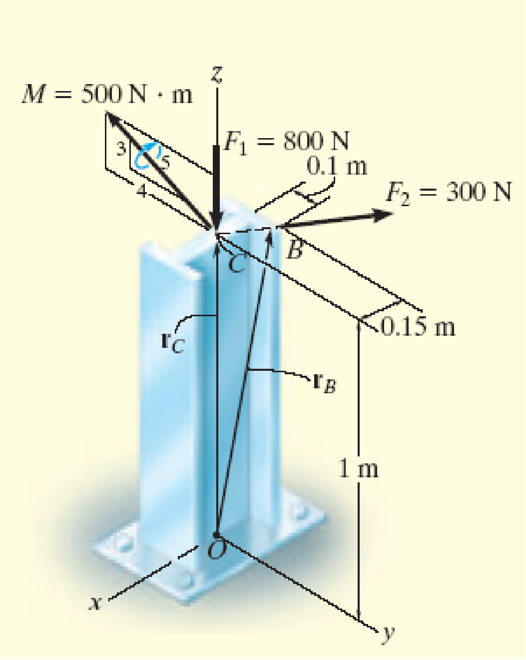
4.7 Simplification of a Force and Couple System

Procedure for Analysis

1. Establish the coordinate axes with the origin located at point *O* and the axes having a selected orientation
2. Force Summation
3. Moment Summation

Example 4.16

A structural member is subjected to a couple moment **M** and forces $\mathbf{F}_1$ and $\mathbf{F}_2$. Replace this system with an equivalent resultant force and couple moment acting at its base, point O.



Solution

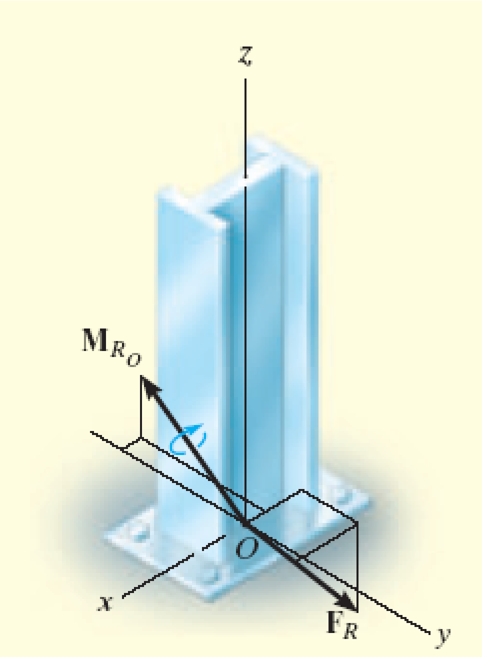
Express the forces and couple moments as Cartesian vectors.

$$\vec{F}_1 = \{-800\vec{k}\}N$$

$$\vec{F}_2 = (300N)\vec{u}_{CB} = (300N)\left(\frac{\vec{r}_{CB}}{|\vec{r}_{CB}|}\right)$$

$$= 300\left[\frac{-0.15\vec{i} + 0.1\vec{j}}{\sqrt{(0.15)^2 + (0.1)^2}}\right] = \{-249.6\vec{i} + 166.4\vec{j}\}N$$

$$M = -500\left(\frac{4}{5}\right)\vec{j} + 500\left(\frac{3}{5}\right)\vec{k} = \{-400\vec{j} + 300\vec{k}\}N.m$$



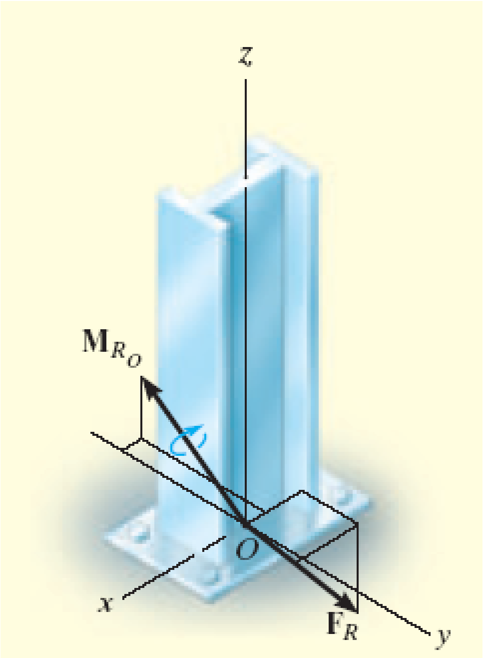
Solution

Force Summation.

$$\vec{F}_R = \Sigma \vec{F};$$

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 = -800\vec{k} - 249.6\vec{i} + 166.4\vec{j}$$

$$= \{-249.6\vec{i} + 166.4\vec{j} - 800\vec{k}\}N$$



$$\vec{M}_{Ro} = \Sigma \vec{M}_C + \Sigma \vec{M}_O = \vec{M} + \vec{r}_C X \vec{F}_1 + \vec{r}_B X \vec{F}_2$$

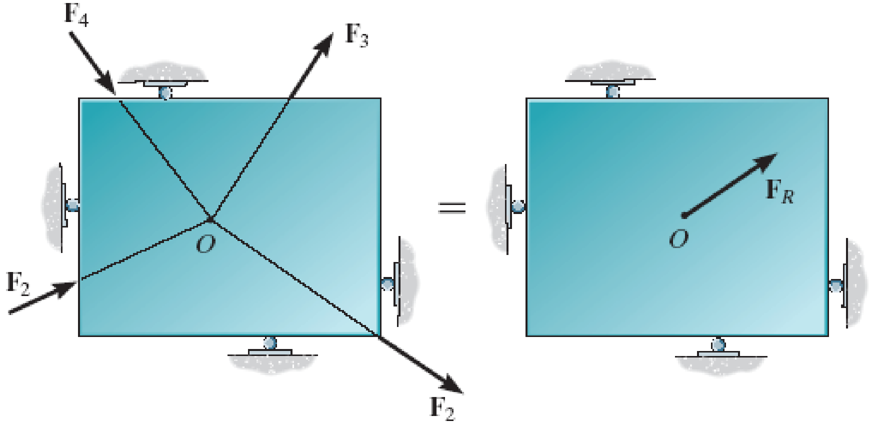
$$= (-400\vec{j} + 300\vec{k}) + (1\vec{k}) X (-800\vec{k}) + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -0.15 & 0.1 & 1 \\ -249.6 & 166.4 & 0 \end{vmatrix}$$

$$= \{-166\vec{i} - 650\vec{j} + 300\vec{k}\}N.m$$

4.8 Further Simplification of a Force and Couple System

Concurrent Force System

- A *concurrent force system* is where lines of action of all the forces intersect at a common point *O*



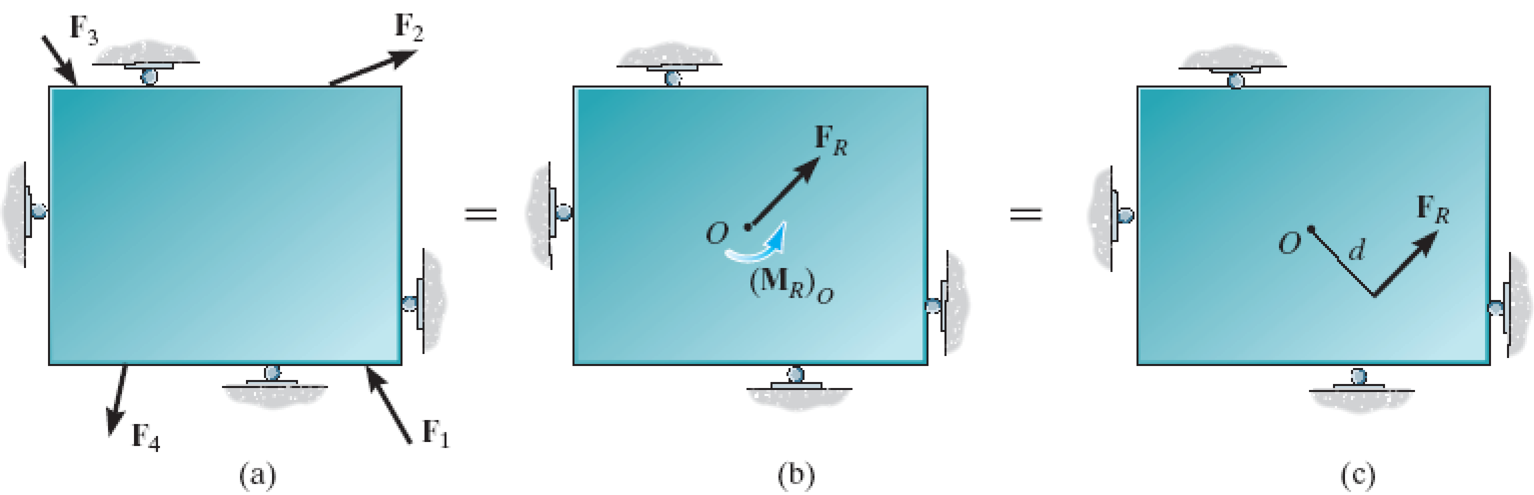
(a) (b)

$$F_R = \sum F$$

4.8 Further Simplification of a Force and Couple System

Coplanar Force System

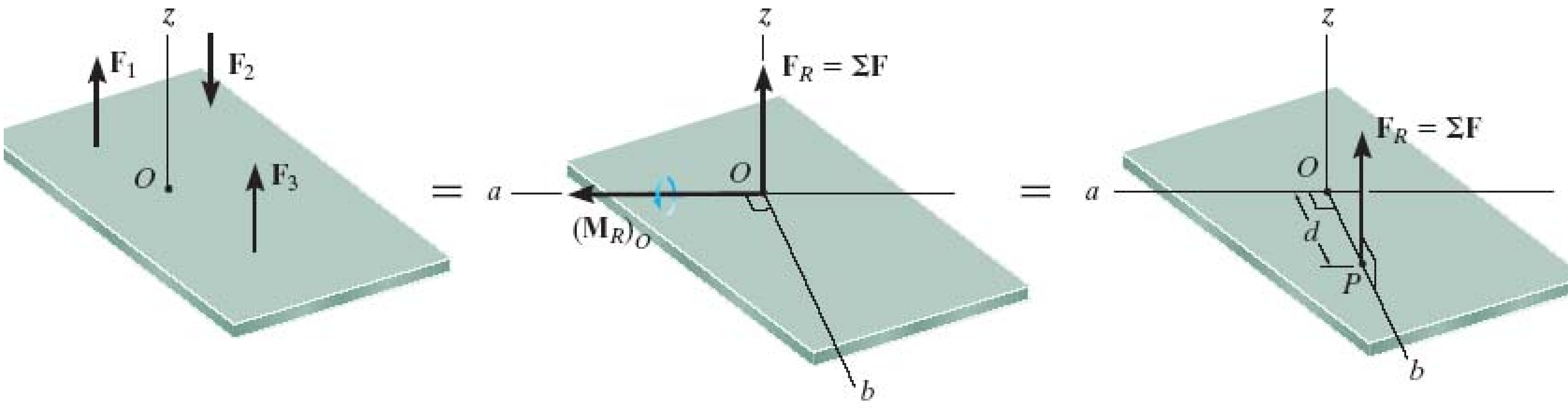
- Lines of action of all the forces lie in the same plane
- Resultant force of this system also lies in this plane



4.8 Further Simplification of a Force and Couple System

Parallel Force System

- Consists of forces that are all parallel to the *z* axis
- Resultant force at point *O* must also be parallel to this axis

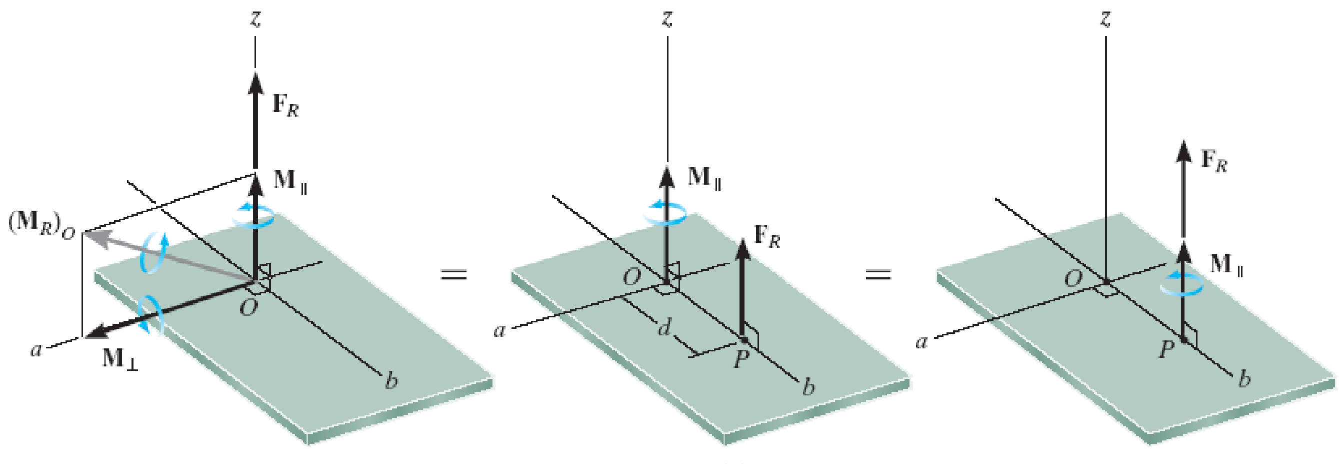


(a) (b) (c)

4.8 Further Simplification of a Force and Couple System

Reduction to a Wrench

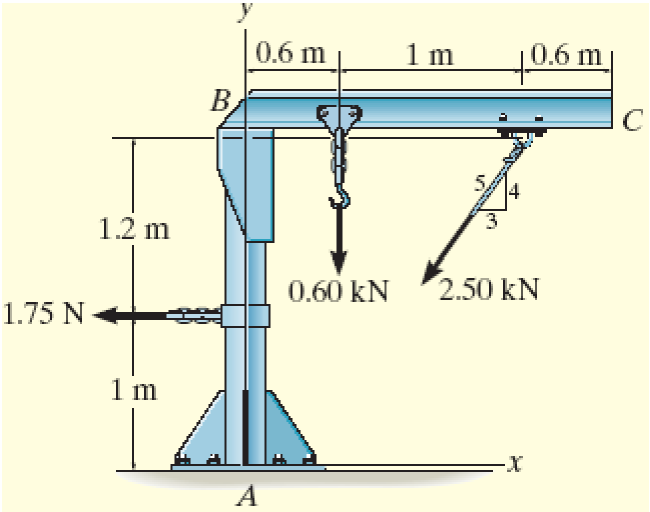
- 3-D force and couple moment system have an equivalent resultant force acting at point *O*
- Resultant couple moment *not perpendicular* to one another



(a) (b) (c)

Example 4.18

The jib crane is subjected to three coplanar forces. Replace this loading by an equivalent resultant force and specify where the resultant's line of action intersects the column AB and boom BC.



Solution

Force Summation

$$+\rightarrow F_{Rx} = \Sigma F_x;$$

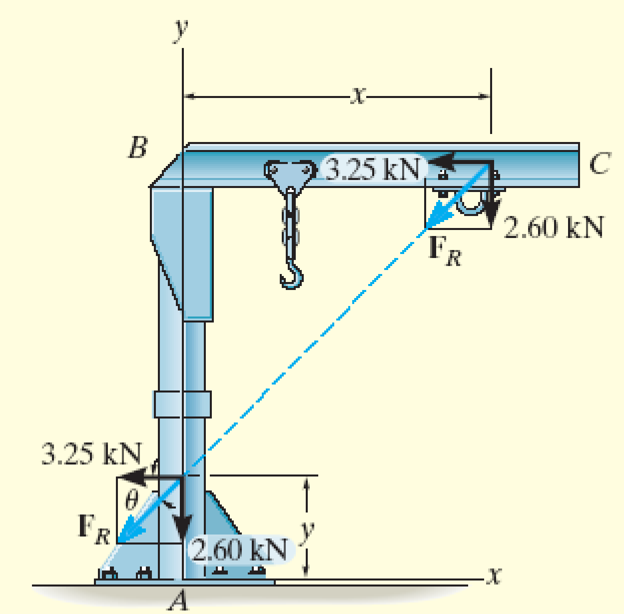
$$F_{Rx} = -2.5kN\left(\frac{3}{5}\right) - 1.75kN$$

$$= -3.25kN = 3.25kN \leftarrow$$

$$+\rightarrow F_{Ry} = \Sigma F_y;$$

$$F_{Ry} = -2.5N\left(\frac{4}{5}\right) - 0.6kN$$

$$= -2.60kN = 2.60N \downarrow$$



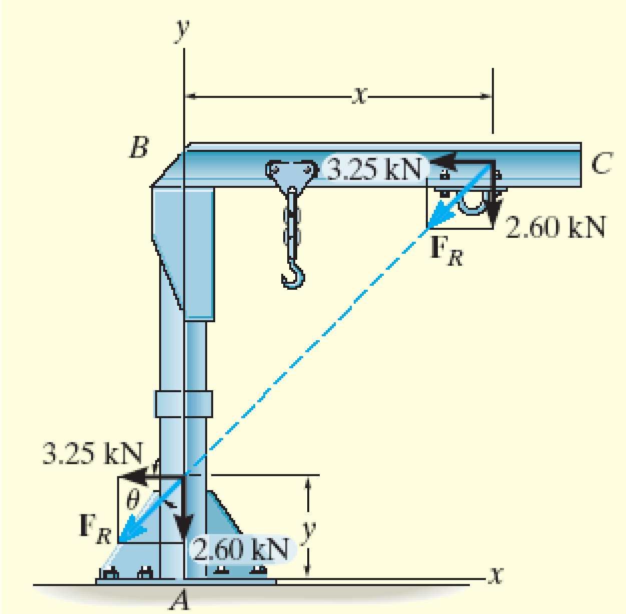
Solution

For magnitude of resultant force,

$$F_R = \sqrt{(F_{Rx})^2 + (F_{Ry})^2} = \sqrt{(3.25)^2 + (2.60)^2}$$
$$= 4.16kN$$

For direction of resultant force,

$$\theta = \tan^{-1}\left(\frac{F_{Ry}}{F_{Rx}}\right) = \tan^{-1}\left(\frac{2.60}{3.25}\right)$$
$$= 38.7^\circ$$



Solution

Moment Summation

➔ Summation of moments about point A,

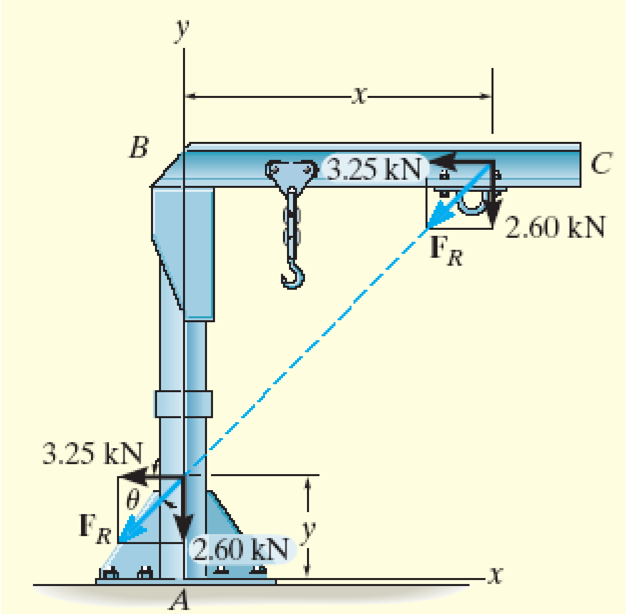
$$M_{RA} = \Sigma M_A;$$

$$3.25kN(y) + 2.60kN(0)$$

$$= 1.75kn(1m) - 0.6kN(0.6m)$$

$$+ 2.50kN\left(\frac{3}{5}\right)(2.2m) - 2.50kN\left(\frac{4}{5}\right)(1.6m)$$

$$y = 0.458m$$



Solution

Moment Summation

➔ Principle of Transmissibility

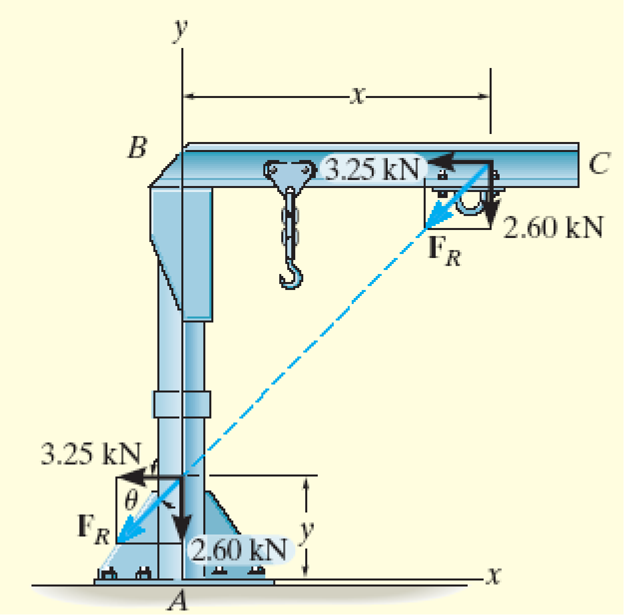
$$M_{RA} = \Sigma M_A;$$

$$3.25kN(2.2m) - 2.60kN(x)$$

$$= 1.75kn(1m) - 0.6kN(0.6m)$$

$$+ 2.50kN\left(\frac{3}{5}\right)(2.2m) - 2.50kN\left(\frac{4}{5}\right)(1.6m)$$

$$x = 2.177m$$

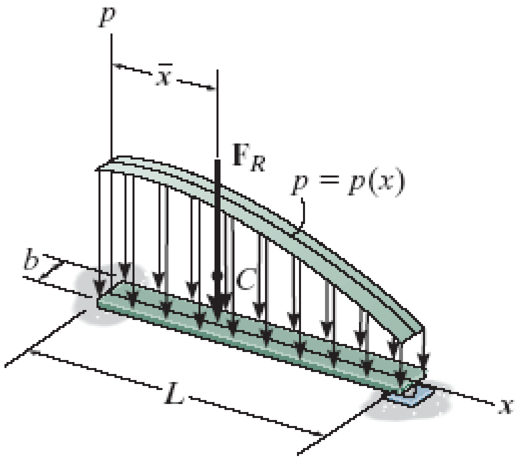


4.9 Reduction of a Simple Distributed Loading

- Large surface area of a body may be subjected to distributed loadings
- Loadings on the surface is defined as pressure
- Pressure is measured in Pascal (Pa): $\underline{1 \text{ Pa} = 1\text{N/m}^2}$

Uniform Loading Along a Single Axis

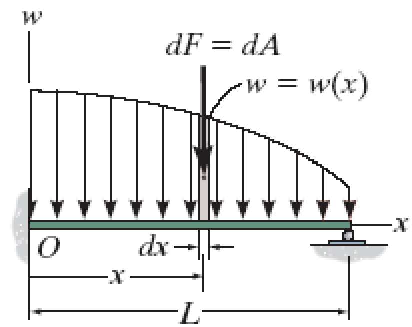
- Most common type of distributed loading is uniform along a single axis



4.9 Reduction of a Simple Distributed Loading

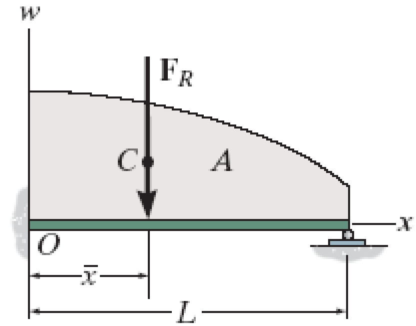
Magnitude of Resultant Force

- Magnitude of *d***F** is determined from differential *area dA* under the loading curve.
- For length *L*,



$$F_R = \int_L w(x)dx = \int_A dA = A$$

- *Magnitude of the resultant force is equal to the total area A under the loading diagram*.

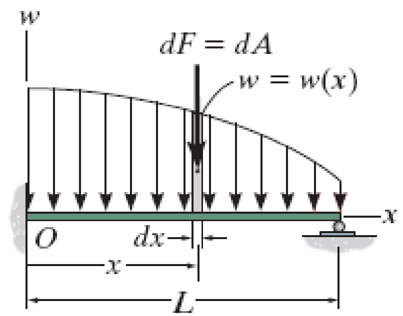


4.9 Reduction of a Simple Distributed Loading

Location of Resultant Force

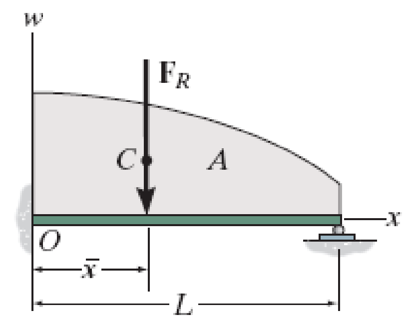
- $M_R = \Sigma M_O$
- d**F** produces a moment of $xdF = x\,w(x)\,dx$ about O
- For the entire plate,

$$M_{Ro} = \Sigma M_O \qquad \bar{x}F_R = \int_L x w(x) dx$$



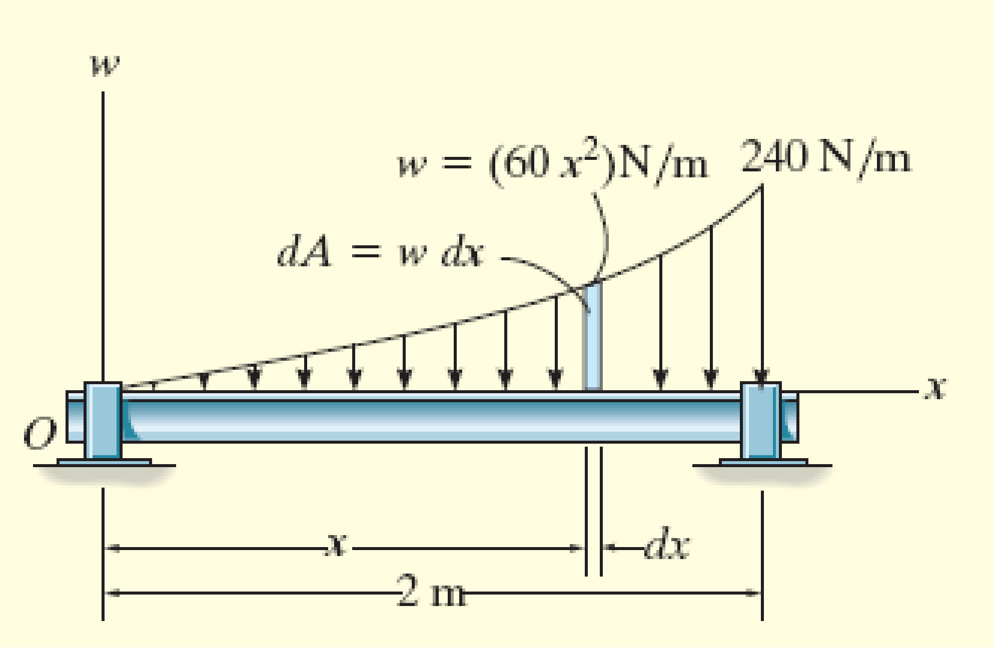
- Solving for $\bar{x}$

$$\bar{x} = \frac{\int_L x w(x) dx}{\int_L w(x) dx} = \frac{\int_A x dA}{\int_A dA}$$



Example 4.21

Determine the magnitude and location of the equivalent resultant force acting on the shaft.



Solution

For the colored differential area element,

$$dA = wdx = 60x^2dx$$

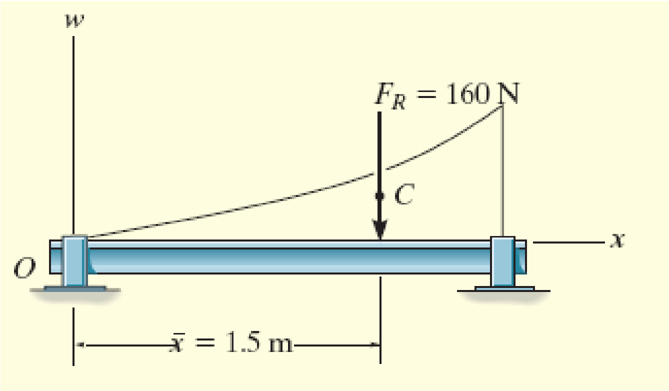
For resultant force

$$F_R = \Sigma F;$$

$$F_R = \int_A dA = \int_0^2 60x^2 dx$$

$$= 60\left[\frac{x^3}{3}\right]_0^2 = 60\left[\frac{2^3}{3} - \frac{0^3}{3}\right]$$

$$= 160N$$



Solution

For location of line of action,

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^2 x(60x^2)\,dx}{160} = \frac{60\left[\frac{x^4}{4}\right]_0^2}{160} = \frac{60\left[\frac{2^4}{4} - \frac{0^4}{4}\right]}{160}$$

$$= 1.5m$$

Checking,

$$A = \frac{ab}{3} = \frac{2m(240N/m)}{3} = 160$$

$$\bar{x} = \frac{3}{4}a = \frac{3}{4}(2m) = 1.5m$$