Homework 5

Chapter 9.14

p. 146 - 148: ex. 2, 5, 7ad(ii), 9.

2. Let $X_1, \dots, X_n \sim Uniform(a, b)$ where a and b are unknown parameters and $a < b$.

a) Find the method of moments estimators for a and b.

For Uniform distribution $\sim Uniform(a,b)$ the first moment will be

$$E(X) = \frac{a+b}{2}$$

$$E(X) = \int x \cdot f(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx =$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right] = \frac{\cancel{(b-a)}(b+a)}{\cancel{(b-a)} \cdot 2} =$$

$$= \frac{b+a}{2} = \frac{a+b}{2}.$$

The Second moment:

$$E(X^2) = \int x^2 \cdot f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx =$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{1}{b-a} \cdot \frac{1}{3}\left[b^3 - a^3\right] =$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{1}{3} \cdot \cancel{(b-a)}(b^2 + ba + a^2) = \frac{b^2+ba+a^2}{3}$$

b) Find the MLE $\hat{a}$ and $\hat{b}$

PDF is

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{if otherwise} \end{cases}$$

Likelihood function is

$$L(\theta) = f(X_1) \cdot f(X_2) \cdot \ldots \cdot f(X_n) = \begin{cases} \left(\frac{1}{b-a}\right)^n & \text{if } X_1 \ldots X_n \in [a, b] \\ 0, & \text{otherwise.} \end{cases}$$

The likelihood is maximazed if

$$\hat{a} = \min\{X_1, X_2, \ldots, X_n\}$$

$$\hat{b} = \max\{X_1, X_2, \ldots, X_n\}$$

c) Let $\tau = \int x \, dF(x)$. Find the MLE of τ.

$\tau = \int x \, dF(x) = \int x \cdot f(x) \cdot dx = \frac{a+b}{2}$, I found it earlier.

Let $\hat{a}$ be MLE of a, and $\hat{b}$ be MLE of b.

Then $\hat{\tau} = \frac{\hat{a}+\hat{b}}{2}$ is the MLE of τ.

d) Let $\hat{\tau}$ be the MLE of τ. Let $\tilde{\tau}$ be the nonparametric plug-in estimator of $\tau = \int x \, dF(x)$. Suppose that $a=1$, $b=3$, and $n=10$. Find the MSE of $\hat{\tau}$ by simulation. Find the MSE of $\tilde{\tau}$ analytically. Compare.

MSE (Mean Squared Error) for an estimator will be

$$MSE(\tilde{\tau}) = E[(\tilde{\tau} - \tau)^2] =$$
$$= Var(\tilde{\tau} - \tau) + (E(\tilde{\tau} - \tau))^2 =$$
$$= Var(\tilde{\tau}) + Bias^2(\tilde{\tau}) =$$
$$= \frac{(b-a)^2}{12 \cdot n} + (E(\tilde{\tau}) - \tau)^2 =$$
$$= \frac{(b-a)^2}{12 \cdot n} + 0 = \frac{(b-a)^2}{12 \cdot n}$$

If $a=1$; $b=3$; $n=10$

$$MSE = \frac{(3-1)^2}{12 \cdot 10} = 0.0333$$

The MSE of $\hat{\tau}$ by the simulation is about 0.015. Which is less than the $\tilde{\tau}$, the nonparametric estimator's MSE.

$0.015 < 0.033$

2EX_Zhetessov.R*

Source on Save | Run | Source

```r
a <- 1
b <- 3
n <- 10

rs <- runif(n, min=a, max=b)

m <- c()

for(i in 1:10000){
  s <- sample(rs, n, replace = TRUE)
  m <- append(m, (max(s)+min(s))/2 )
}

MSE <- var(m)

MSE

#Zhetessov Nur
```

18:15 (Top Level) R Script

Console | Terminal | Jobs

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```
>
> for(i in 1:10000){
+   s <- sample(rs, n, replace = TRUE)
+   m <- append(m, (max(s)+min(s))/2 )
+ }
>
> MSE <- var(m)
>
> MSE
[1] 0.01576614
>
> #Zhetessov Nur
>
```

5. Let $X_1, \ldots, X_n \sim Poisson(\lambda)$. Find the method of moments estimator, the maximum likelihood estimator and Fisher information $I(\lambda)$.

For Poisson distribution we have $E(X) = \lambda$, then the method of moments estimator will be

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} X_i}{n}$$

The maximum likelihood estimator:

$$L(\lambda) = \prod_{i=1}^{n} f(X_i) = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!} = \frac{\lambda^{X_1+\ldots+X_n} e^{-\lambda \cdot n}}{X_1! \cdot X_2! \ldots X_n!}$$

$$\log L(\lambda) = (X_1+\ldots+X_n)\log\lambda - \lambda \cdot n \cdot \log e - \sum_{i=1}^{n} \log X_i!$$

$$\frac{d \log L(\lambda)}{\lambda} = \frac{X_1+\ldots+X_n}{\lambda} - n - 0$$

$$\frac{X_1+\ldots+X_n}{\lambda} - n = 0$$

$$\hat{\lambda} = \frac{X_1+\ldots+X_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n} \Rightarrow MLE$$

Fisher Information ($I(\lambda)$)

$$\frac{d^2 \log L(\lambda)}{d\lambda^2} = \frac{0\cdot\lambda - (X_1+\dots+X_n)\cdot 1}{\lambda^2} = -\frac{\sum_{i=1}^{n} X_i}{\lambda^2}$$

$$I(\lambda) = -E\left(-\frac{\sum_{i=1}^{n} X_i}{\lambda^2}\right) = \frac{1}{\lambda}$$

7. (Comparing 2 treatments)

n_1 - number of people given treatment 1

n_2 - # of people given treat. 2

X_1 - # of people favorably on treat. 1

X_2 - # of people favorably on treat. 2

$X_1 \sim Binomial(n_1, p_1)$

$X_2 \sim Binomial(n_2, p_2)$

Let $\psi = p_1 - p_2$

a) Find MLE $\hat{\psi}$ for ψ

$f_1(X_1) = p_1^{X_1}(1-p_1)^{n_1-X_1} \cdot C_{n_1}^{X_1}$

$f_2(X_2) = p_2^{X_2}(1-p_2)^{n_2-X_2} \cdot C_{n_2}^{X_2}$

$$L(p_1) = C_{n_1}^{X_1}\, p_1^{X_1} (1-p_1)^{n_1 - X_1}$$
$$L(p_2) = C_{n_2}^{X_2}\, p_2^{X_2} (1-p_2)^{n_2 - X_2}$$

$$\log L(p_1) = \log C_{n_1}^{X_1} + X_1 \log(p_1) + (n_1 - X_1)\log(1-p_1)$$
$$\log L(p_2) = \log C_{n_2}^{X_2} + X_2 \log(p_2) + (n_2 - X_2)\log(1-p_2)$$

$$\frac{d \log L(p_1)}{d p_1} = \frac{X_1}{p_1} + \frac{n_1 - X_1}{1-p_1} \cdot (-1)$$

$$\frac{d \log L(p_2)}{d p_2} = \frac{X_2}{p_2} + \frac{n_2 - X_2}{1-p_2} \cdot (-1)$$

$$\frac{X_1}{p_1} = \frac{n_1 - X_1}{1-p_1} \Rightarrow X_1 - \cancel{X_1 p_1} = p_1 \cdot n_1 - \cancel{p_1 X_1} \Rightarrow$$

$$\Rightarrow \hat{p}_1 = \frac{X_1}{n_1}$$

$$\frac{X_2}{p_2} = \frac{n_2 - X_2}{1-p_2} \Rightarrow \hat{p}_2 = \frac{X_2}{n_2}$$

So, MLE $\hat{\psi} = \hat{p}_1 - \hat{p}_2$ for $\psi = p_1 - p_2$ is

$$\hat{\psi} = \hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

d) Suppose that $n_1 = n_2 = 200$

$$X_1 = 160$$

$$X_2 = 148$$

Find $\hat{\psi}$. Find 90% CI for ψ.

II) Using the parametric bootstrap.

$$\hat{p}_1 = \frac{160}{200} = \frac{X_1}{n_1} = 0.8$$

$$\hat{p}_2 = \frac{148}{200} = \frac{X_2}{n_2} = 0.74$$

$$\hat{\psi} = \hat{p}_1 - \hat{p}_2 = 0.8 - 0.74 = 0.06$$

The parametric bootstrap method gave me $\hat{\psi}_2 = 0.06062$, which is very close to manually computed value of $\hat{\psi}$.

The 90% confidence interval using parametric bootstrap method computed

$$(\hat{\psi} - z \cdot SE; \hat{\psi} + z \cdot SE)$$

$$(0.06 - 1.645 \cdot 0.042; 0.06 + 1.645 \cdot 0.042)$$

$$(-0.0094; 0.1294)$$

7dii_Ex_Zhetessov.R

```r
n <- 200
x1 <- 160
x2 <- 148

p1.hat = x1 / n
p2.hat = x2 / n

#Parametric bootstrap method

B <- 10000
tau.boot <- c()

for(i in 1:B){
  x1 <- rbinom(1,n,p1.hat)
  x2 <- rbinom(1,n,p2.hat)
  tau.boot <- append(tau.boot, (x1/n)-(x2/n) )
}

#Phi and phi.hat has been found

phi.hat = p1.hat - p2.hat

phi.hat          # 0.06000
mean(tau.boot)   # 0.06062

#--------------------------------------------------------------#
#90% CI

sprintf("Lower bound is %s", phi.hat - 1.645 * sd(tau.boot))
sprintf("Upper bound is %s", phi.hat + 1.645 * sd(tau.boot))

#Zhetessov Nur
```

Console

R 4.1.2 · ~/

```
>
> mean(tau.boot)
[1] 0.06062
> phi.hat
[1] 0.06
>
> #--------------------------------------------------------------#
>
> phi.hat = p1.hat - p2.hat
>
> sprintf("Lower bound is %s", phi.hat - 1.645 * sd(tau.boot))
[1] "Lower bound is -0.00942121594665671"
> sprintf("Upper bound is %s", phi.hat + 1.645 * sd(tau.boot))
[1] "Upper bound is 0.129421215946657"
>
```

9 Let $X_1, \ldots, X_n \sim Normal(\mu, 1)$. Let $\theta = e^{\mu}$ and let $\hat{\theta} = e^{\bar{x}}$ be the MLE. Create a data set, $\mu = 5$, consisting of $n = 100$ observations.

$$\theta = e^{\mu} = g(\mu)$$

$$\frac{dg(\mu)}{d\mu} = e^{\mu}$$

$$\widehat{SE}(\hat{\theta}) = |g'(\mu)| \widehat{SE}(\hat{\mu}) = \frac{e^{\hat{\mu}}}{\sqrt{n}}$$

Parametric bootstrap

Lower bound is 105.0035

Upper bound is 156.5074

Standard error is 13.1387

Non-parametric bootstrap

Lower bound is 103.8134

Upper bound is 157.6974

Standard Error is 13.7459

Delta Method
lower bound is 105.1274
Upper bound is 156.3835
Standard Error is 13.0755

The true distribution gives me the mean $\mu = 149.159$

In conclusion, parametric bootstrap, non-parametric bootstrap and delta method estimate true sampling distribution very good, where the estimates are very close to the true value.

9Ex_Zhetessov.R*

Source on Save

Run Source

```
63
64 #Zhetessov Nur
65
```

64:15 (Top Level) R Script

Console Terminal Jobs

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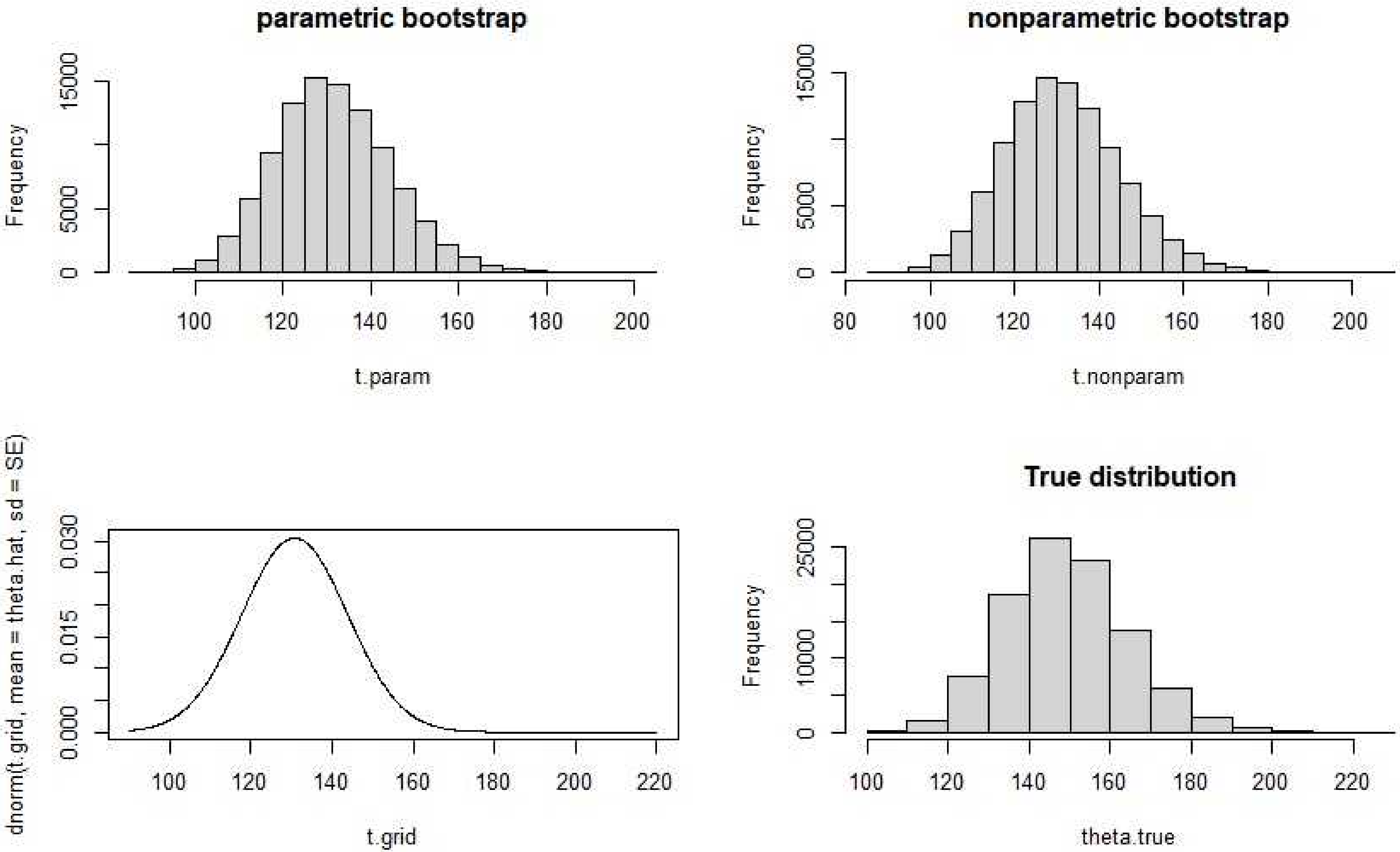
```
> #Delta method
>
> sprintf("Delta Method")
[1] "Delta Method"
> sprintf("Lower bound is %s", theta.hat - 1.96 * SE)
[1] "Lower bound is 105.127368702197"
> sprintf("Upper bound is %s", theta.hat + 1.96 * SE)
[1] "Upper bound is 156.383498716203"
>
> #Bootstraps
>
> B <- 100000
>
> # Non-parametric bootstrap
> t.nonparam <- c()
>
> for(i in 1:B){
+   x <- sample(X, n, replace = TRUE)
+   t.nonparam <- append(t.nonparam, exp(mean(x)))
+ }
>
>
> sprintf("Non-parametric bootstrap")
[1] "Non-parametric bootstrap"
> sprintf("Lower bound is %s", theta.hat - 1.96 * sd(t.nonparam))
[1] "Lower bound is 103.813433992396"
> sprintf("Upper bound is %s", theta.hat + 1.96 * sd(t.nonparam))
[1] "Upper bound is 157.697433426004"
> sprintf("Standard Error is %s", sd(t.nonparam))
[1] "Standard Error is 13.7459182228591"
>
>
> #Parametric bootstrap
>
> t.param <- c()
>
> for(i in 1:B){
+   x <- rnorm(n, mean = mean(X), sd=1)
+   t.param <- append(t.param, exp(mean(x)))
+ }
> sprintf("Parametric bootstrap")
[1] "Parametric bootstrap"
> sprintf("Lower bound is %s", theta.hat - 1.96 * sd(t.param))
[1] "Lower bound is 105.003488572056"
> sprintf("Upper bound is %s", theta.hat + 1.96 * sd(t.param))
[1] "Upper bound is 156.507378846344"
> sprintf("Standard Error is %s", sd(t.param))
[1] "Standard Error is 13.1387475189511"
>
> #Histograms and plots
>
> hist(t.param, main="parametric bootstrap")
```

```
>
> #True distribution
> theta.true <- c()
> for(i in 1:B){
+   x <- rnorm(n, mean = 5, sd=1)
+   theta.true <- append(theta.true, exp(mean(x)))
+ }
> hist(theta.true, main = "True distribution")
> mean(theta.true)
[1] 149.159
>
```



parametric bootstrap
Frequency
0
5000
15000
100
120
140
160
180
200
t.param
nonparametric bootstrap
Frequency
0
5000
15000
80
100
120
140
160
180
200
t.nonparam
dnorm(t.grid, mean = theta.hat, sd = SE)
0.000
0.015
0.030
100
120
140
160
180
200
220
t.grid
True distribution
Frequency
0
10000
25000
100
120
140
160
180
200
220
theta.true