Ficha nº 1 2,3,4

1.

$\langle h, k \rangle \cdot f = \langle h \cdot f, k \cdot f \rangle$

$\Leftrightarrow \begin{cases} \pi_1 \cdot (\langle h, k \rangle \cdot f) = h \cdot f \\ \pi_2 \cdot (\langle h, k \rangle \cdot f) = k \cdot f \end{cases}$ {6}

$\Leftrightarrow \begin{cases} h \cdot f = h \cdot f \\ k \cdot f = k \cdot f \end{cases}$ { Associatividade composição, 7 } (x2)

$\Leftrightarrow$ True

2. Como queremos fazer a prova, recorrendo à lei da fusão-x, é necessário conseguir definir dup sem recorrer a variáveis:

$dup\ x = (x, x)$

$\Leftrightarrow dup\ x = (id\ x, id\ x)$ {74 (x2)}

$\Leftrightarrow dup\ x = \langle id, id \rangle\ x$ {77}

$\Leftrightarrow dup = \langle id, id \rangle$ {72}

$dup \cdot f = \langle f, f \rangle$

$\Leftrightarrow \langle id, id \rangle \cdot f = \langle f, f \rangle$ {def dup}

$\Leftrightarrow \langle f, f \rangle = \langle f, f \rangle$ {(9), (1) (x2)}

$\Leftrightarrow$ True

3.

$assocl \cdot assocr = id$

~~$\Leftrightarrow assocl \cdot \langle id \times \pi_1, \pi_2 \cdot \pi_2 \rangle = id$ {def}~~

$\Leftrightarrow \langle id \times \pi_1, \pi_1 \cdot \pi_2 \rangle \cdot assocr = id$ {Def assocl}

$\Leftrightarrow \langle (id \times \pi_1) \cdot assocr, \pi_2 \cdot \pi_2 \cdot assocl \rangle = id$ {9}

$\Leftrightarrow \begin{cases} (id \times \pi_1) \cdot assocr = \pi_1 \\ \pi_2 \cdot \pi_2 \cdot assocr = \pi_2 \end{cases}$ {6, 1 (x2)}

$\Leftrightarrow \begin{cases} \langle \pi_1, \pi_1 \cdot \pi_2 \rangle \cdot assocr = \pi_2 \\ \underline{\qquad\qquad} \end{cases}$ {10, 1}

$\Leftrightarrow \begin{cases} \begin{cases} \pi_1 \cdot assocr = \pi_1 \cdot \pi_2 \\ \pi_1 \cdot \pi_2 \cdot assocr = \pi_2 \cdot \pi_2 \end{cases} \\ \underline{\qquad\qquad} \end{cases}$ {9, 6}

$$\Leftrightarrow \begin{cases} \pi_1 \cdot assocr = \pi_1 \cdot \pi_1 \\ \begin{cases} \pi_1 \cdot \pi_2 \cdot assocr = \pi_2 \cdot \pi_1 \\ \pi_2 \cdot \pi_2 \cdot assocr = \pi_2 \end{cases} \end{cases}$$ { Propriedade associativa da conjunção }

$$\Leftrightarrow \begin{cases} \pi_1 \cdot assocr = \pi_1 \cdot \pi_1 \\ \pi_2 \cdot assocr = \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \end{cases}$$ { 6 }

$\Leftrightarrow assocr = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, \pi_2 \rangle \rangle$ { 6 }

$\Leftrightarrow assocr = \langle \pi_1 \cdot \pi_1, \langle \pi_2 \cdot \pi_1, id \cdot \pi_2 \rangle \rangle$ { 1 }

$\Leftrightarrow assocr = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle$ { 10 }

$\boxed{\text{Definição de assocr com variáveis}}$

$assocr\ ((a, b), c) = \langle \pi_1 \cdot \pi_1, \pi_2 \times id \rangle\ ((a, b), c)$

$= (\pi_1 \cdot \pi_1\ ((a,b), c), (\pi_2 \times id)\ ((a,b), c))$ { 77 }

$= (a, (b, c))$ { 73, 79 (x3), 78, 74 }

4.

$xr \cdot \underbrace{\langle \langle f, g \rangle, h \rangle}_{id} = \langle \langle f, h \rangle, g \rangle$

Vamos tentar descobrir quais as funções f, g e h para que:

$\langle \langle f, g \rangle, h \rangle = id$

$$\Leftrightarrow \begin{cases} \langle f, g \rangle = \pi_1 \\ h = \pi_2 \end{cases}$$ { 6, 1 (x2) }

$$\Leftrightarrow \begin{cases} f = \pi_1 \cdot \pi_1 \\ g = \pi_2 \cdot \pi_1 \\ h = \pi_2 \end{cases}$$ { 6 }

Logo, $xr \cdot id = \langle \langle f, h \rangle, g \rangle$

$\Leftrightarrow xr = \langle \langle \pi_1 \cdot \pi_1, \pi_2 \rangle, \pi_2 \cdot \pi_1 \rangle$ { 1 }

$\Leftrightarrow \boxed{xr = \langle \pi_1 \times id, \pi_2 \cdot \pi_1 \rangle}$ { 1, 10 }

(5.)

$$\underline{(b,a)} = \langle \underline{b}, \underline{a} \rangle$$

$$\Leftrightarrow \begin{cases} \pi_1 \cdot \underline{(b,a)} = \underline{b} \\ \pi_2 \cdot \underline{(b,a)} = \underline{a} \end{cases} \qquad \{6\}$$

$$\Leftrightarrow \begin{cases} \underline{\pi_1 (b,a)} = \underline{b} \\ \underline{\pi_2 (b,a)} = \underline{a} \end{cases} \qquad \{4 \ (\times 2)\}$$

$$\Leftrightarrow \begin{cases} \underline{b} = \underline{b} \\ \underline{a} = \underline{a} \end{cases} \qquad \{79 \ (\times 2)\}$$

$$\Leftrightarrow \underline{\underline{True}}$$

(6.)

a)

$$f((a,b),c) = (a \wedge b) \oplus c \qquad \{\text{def } f\}$$
$$= \hat{\oplus}((a \wedge b), c) \qquad \{86\}$$
$$= \hat{\oplus}(\hat{\wedge}(a,b), c) \qquad \{86\}$$
$$= \hat{\oplus}(\hat{\wedge}(a,b), id\ c) \qquad \{74\}$$
$$= \hat{\oplus}(\hat{\wedge} \times id\ ((a,b),c)) \qquad \{78\}$$
$$= \hat{\oplus} \cdot (\hat{\wedge} \times id)\ ((a,b),c) \qquad \{73\}$$
$$= \hat{\oplus} \cdot (\hat{\wedge} \times id) \qquad \{72\}$$

$$\boxed{f = \hat{\oplus} \cdot (\hat{\wedge} \times id)}$$

Em Haskell $\quad$ `f = (uncurry (/=)) . ((uncurry (&&) >< id)`

b)

$$\begin{tikzcd} \mathbb{B} \times \mathbb{B} & (\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & \mathbb{B} \\ & (\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \arrow[ul] \arrow[u, "g"] \arrow[ur, "f"'] & \end{tikzcd}$$

$$(\mathbb{B} \times \mathbb{B}) \times \mathbb{B} \xrightarrow{g} (\mathbb{B} \times \mathbb{B}) \times \mathbb{B}$$