appendix G

Convolution

Convolution is a mathematical technique that is used in imaging to compute the effects of two (or more) simultaneously operating "blurring" functions. For example, the image projected by a collimator onto an Anger camera detector is blurred by the collimator holes (see Chapter 14, Sections C and D). Suppose that the collimator point-spread function is described by a one-dimensional function $f(x)$. That profile is projected onto the camera detector where it is blurred again by the intrinsic resolution of the detector (see Chapter 14, Section A.1). Suppose that the detector point-spread function is described by a function $g(x)$. Then the combined effects or the collimator and detector blurring are described by the convolution, $h(x)$, symbolized as

$$h(x) = f(x) * g(x) \tag{G-1}$$

An alternative notation is

$$h(x) = f(x) \circledast g(x) \tag{G-2}$$

The function $h(x)$ is thus the *system point-spread function* that accounts for both collimator and detector blurring. Mathematical convolutions find use in other fields as well, including statistics, electrical engineering, and general signal processing.

Mathematically, the convolution of two continuous functions is described by[1]

$$h(x) \int_{-\infty}^{\infty} f(u) g(x-u) du \tag{G-3}$$

Equation G-3 is useful for theoretical development and analysis (e.g., for system design). However, most practical uses of convolution in the nuclear medicine laboratory involve discrete, rather than continuous functions (e.g., a projection profile for emission computed tomography is a discrete series of numbers). One representation for discrete functions represented by a series of n values is

$$f = [a_0, a_1, \ldots, a_{n-1}] \tag{G-4}$$

$$g = [b_0, b_1, \ldots, b_{n-1}] \tag{G-5}$$

A more compact notation is $f(x_i)$. Here it is understood that $x_i = (i \times \Delta x)$, in which Δx is the sampling interval along the x-axis and $i = 0, 1, 2, \ldots, (n-1)$.

Using the notation of Equations G-4 and G-5, the convolution of two discrete functions is described by

$$h_j = \sum_{i=0}^{j} (a_i \times b_{j-i}) \tag{G-6}$$

The subscript j runs from 0 to $(2n-1)$; in other words, the convolution yields $(2n-1)$ values for h.†

Analogous expressions exist for two-dimensional convolutions (i.e., functions of two variables, x and y). Convolutions frequently are performed for two-dimensional image processing (e.g., image smoothing—another form of "blurring"); however, the concepts embodied in Equations G-3 and G-6 are most easily demonstrated using one-dimensional graphical examples.

Consider first the convolution of two continuous functions. In Figure G-1A, the function $f(u)$ is plotted as a function of u. (Note that this is only a change of notation for mathematical representation. It is not a

†Note that some of the values for a and b appearing in Equation G–6 do not exist. Specifically, there are no values of a or b when their subscripts exceed $(n-1)$, both of which occur in the equation as written. For computer implementation, the practical solution is to increase the length of the arrays for f and g by adding zeros to the list of values in Equations G–4 and G–5. This is sometimes called "zero padding." Additional values for h that are generated by this step are discarded.

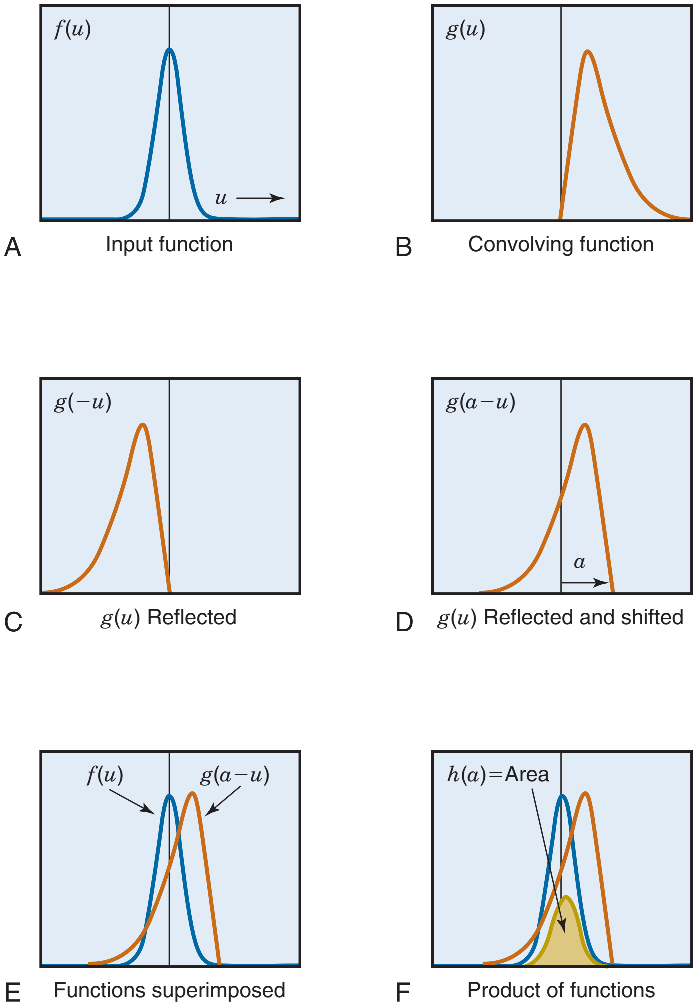

FIGURE G-1 Illustration of the steps involved in determining the value of the convolution of two functions, $f(x)$ and $g(x)$, at $x = a$. *A* and *B,* the variable x is replaced by u. *C,* The convolving function, g, is reflected through the origin, $u = 0$. *D,* The reflected function is shifted by a distance, $x = a$, to the right. *E,* The function f and the shifted function, g, are overlapped. *F,* The product of f and the shifted function g is taken at each value of u. The area under the curve representing this product is $h(a)$, the value of the convolution at the value of $x = a$.

change of coordinate systems.) The convolving function, $g(u)$ is shown in Figure G-1B. Figure G-1C shows $g(–u)$, which is $g(u)$ reflected through the origin. In Figure G-1D, g is shifted by a distance a along the u-axis. This is $g(a – u)$, or $g(x – u)$ for $x = a$ in Equation G-3. According to Equation G-3, the integral of the product of f and the reflected and shifted function g is the value of the convolution function, h at $x = a$; in other words, $h(a)$. This is proportional to the shaded area in Figure G-1F. These steps (reflect, shift, take the product, and integrate) are repeated for all values of x (i.e., a in Figure G-1D) to obtain the full functional representation of $h(x)$.

Figure G-2 illustrates the convolution of two discrete functions, f = [1,2,1] and g = [0,2,1]. As in Figure G-1, the convolving function, g, first is reflected through the origin (middle row, left). The product of the reflected but unshifted (a = 0) version of g and f is formed and summed across both functions. This is $h(0)$, which happens to have a value of zero in this example. The reflected and shifted version of g then is shifted along the horizontal axis and the process is repeated at unit increments of a. In each successive illustration for different values of a, the product is proportional to the shaded area. The process is repeated until g has passed completely over the function f and only values of zero are obtained (bottom row, right). As indicated earlier, this requires $(2n – 1)$ steps, yielding $(2n – 1)$ values for h.

Some important properties of convolutions, either continuous or discrete, are that they are commutative, distributive over addition, and associative. These properties are respectively described by the following equations.

$$f(x) * g(x) = g(x) * f(x) \qquad \text{(G-7)}$$

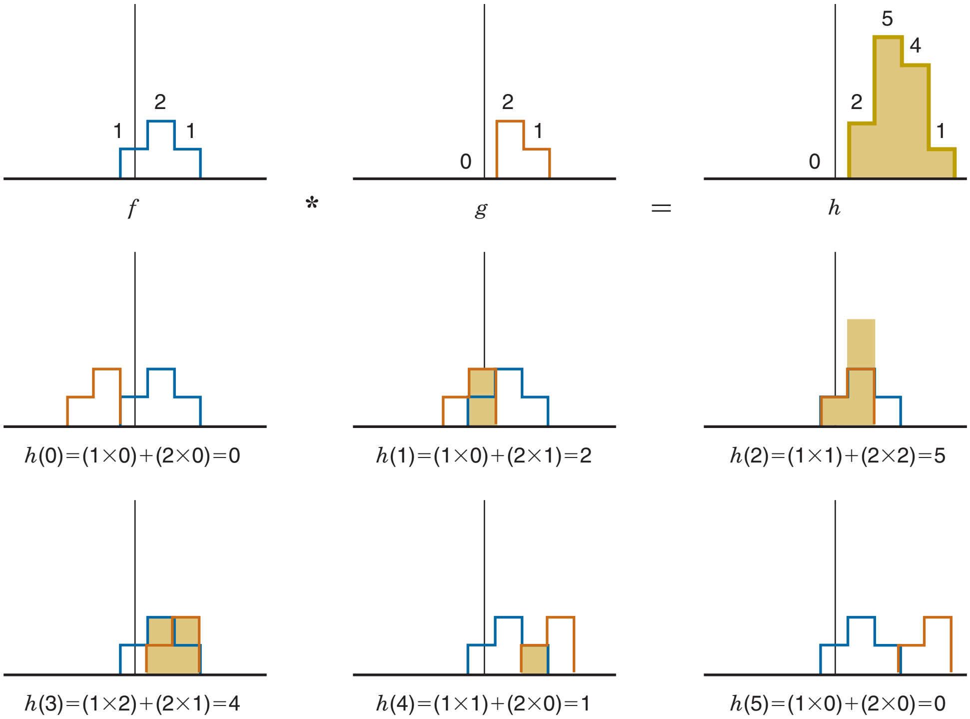

FIGURE G-2 *Top row,* The convolution of two discrete functions, $f(x)$ = [1,2,1] and $g(x)$ = [0,2,1], is given by $h(x)$ = [0,2,5,4,1]. *Bottom rows,* The convolving function, $g(x)$, is reflected and then progressively shifted (*orange curves*) in unit increments of x across the stationary function, $f(x)$ (*blue curve*). The product of the overlapping functions is formed at each shift increment (*shaded areas*). The summation of the product (*shaded yellow areas*) is the value of the convolution, h, for the value of x = to the shift distance. Where the reflected and shifted version of g does not overlap with f (e.g., x = 0 and x = 5), the convolution h is zero.

$$f(x) * [g(x) + h(x)] = [f(x) * g(x)] + [f(x) * h(x)] \tag{G-8}$$

$$f(x) * [g(x) * h(x)] = [f(x) * g(x)] * h(x) \tag{G-9}$$

Equation G-7 (commutative property) says that the order of the convolution can be reversed. Thus the result would be the same for a gamma camera image if the intrinsic blurring occurred before or after collimator blurring.

Equation G-8 (distributive property) says that the convolution of the sum is the same as the sum of the convolutions. Thus if two images are projected simultaneously onto the detector and then blurred, the same result is obtained as if the two images were projected separately, blurred, and then added.

Finally, Equation G-9 (associative property) says that the order of convolution does not affect the outcome. Thus if multiple blurring effects are present, the order in which they occur does not affect the outcome. This property is useful for image processing operations that employ convolutions. (Note that $h(x)$ in Equation G-9 refers to a third blurring function, not the convolution of f and g as used in Equations G-1 to G-3.)

Convolutions have other useful properties. For example, Figure G-3 illustrates the convolution of two Gaussian functions. Gaussian functions often are used to approximate the blurring caused by collimators, intrinsic resolution of the detector, and so on. These functions are characterized by a mean value, μ, and variance, σ^2, and are of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \tag{G-10}$$

It can be shown that the convolution of two Gaussian functions, with means μ_1 and μ_2 and variances σ_1 and σ_2, is a third Gaussian function with mean and variance given by

$$\mu_3 = \mu_1 + \mu_2 \tag{G-11}$$

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2 \tag{G-12}$$

Equation G-12 applies as well to the full width at half-maximum (FWHM) for the Gaussian functions. Thus, if two Gaussian

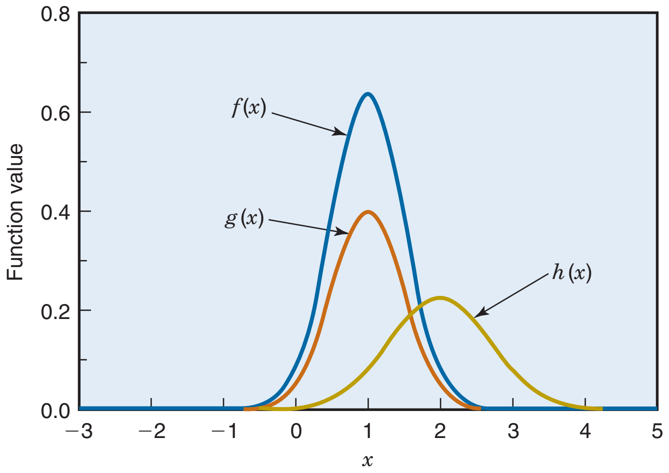

FIGURE G-3 The convolution of two Gaussian functions, $f(x)$ and $g(x)$, each having a mean value $\mu = 1.0$ and variance $\sigma^2 = 0.25$, is another Gaussian, $h(x)$, with $\mu = 2.0$ and variance $\sigma^2 = 0.5$.

functions, f and g, are characterized by FWHM(f) and FWHM(g), the full width at half-maximum of their convolution, h, is given by

$$\text{FWHM}(h) = \sqrt{\text{FWHM}(f)^2 + \text{FWHM}(g)^2} \tag{G-13}$$

This can be used to calculate the effect of combining the spatial resolutions of two components in an imaging system. For example, if a scintillation camera has an intrinsic resolution of 4 mm FWHM and a collimator resolution of 10 mm FWHM, and both can be reasonably approximated by a Gaussian shape, then their combined blurring effect is described by another Gaussian function with FWHM = $(10^2 + 4^2)^{1/2}$ = 10.8 mm. This would describe the system resolution for the detector/collimator combination (see Chapter 14, Section C.4). Note that blurring effects are not simply additive (i.e., 10 mm + 4 mm = 14 mm) and that the combined effect is dominated by the component that produces the greatest amount of blurring (the collimator, in this example).

One final useful property of convolution is given by the *convolution theorem*. If the Fourier transforms (FTs) of two functions $f(x)$ and $g(x)$ are $F(k)$ and $G(k)$, then the FT of their convolution has the properties that

$$\mathscr{F}[f(x) * g(x)] = F(k) \times G(k) \tag{G-14}$$

$$\mathscr{F}^{-1}[F(k) \times G(k)] = f(x) * g(x) \tag{G-15}$$

Here, $\mathscr{F}$ represents the operation of computing the FT and $\mathscr{F}^{-1}$ represents the inverse FT, as described in Appendix F. Thus convolution of two functions in the spatial domain is equivalent to point-by-point multiplication of their spectra in the spatial-frequency domain.

One practical application of the convolution theorem is that the modulation transfer function (MTF) of a system (see Chapter 15, Section B.2) is equal to the product of the MTFs of its individual components. This can be very helpful when the shapes of their point-spread functions are varied and non-Gaussian. Another practical application is that it provides a convenient alternative to Equation G-6 for computing the convolution of two functions. That equation requires many multiplications and data shifts, which can be very time consuming and tedious to implement in a computer. The convolution approach requires only the calculation of two FTs, a single point-by-point multiplication of these FTs, and the calculation of an inverse FT, all which can be performed much more rapidly by a computer than the tedious process described by Equation G-6. It should be noted that the convolution theorem can be extended to an arbitrary number of functions or dimensions.[2]

REFERENCES

1. Bracewell RN: *The Fourier Transform and Its Applications*, New York, 2000, McGraw-Hill, Chapter 3.
2. Bracewell RN: *Fourier Analysis and Imaging*. New York, 2004, Kluwer Academic Publishers, Chapter 5.