appendix F

The Fourier Transform

Fourier transforms (FTs) play an important role in tomographic reconstruction (see Chapter 16) and in computer implementation of convolutions (see Appendix G). As well, they are the basis for the modulation transfer function (MTF), one of the methods for evaluating spatial resolution of imaging systems (see Chapter 15) and many other methods for image analysis and image processing. This appendix is intended to provide an intuitive description of what the FT is, how it is calculated, and some of its properties. For simplicity, the analysis is presented for one-dimensional examples. The extension to higher dimensions is relatively straightforward. A detailed treatment is beyond the scope of this text. The interested reader is referred to the excellent texts by Bracewell for further information.[1,2]

A. THE FOURIER TRANSFORM: WHAT IT REPRESENTS

The FT is an alternative manner for representing a mathematical function or mathematical data. For example, suppose that the function $f(x)$ represents an image intensity profile. It can be shown that, so long as $f(x)$ has "reasonable properties," the profile can be represented as a sum of sine and cosine functions of different frequencies extending along the x-axis. The FT of $f(x)$, denoted as $F(k)$, represents the amplitudes of the sine and cosine functions for different *spatial frequencies, k*. Spatial frequency reflects how rapidly a sine or cosine function oscillates along the x-axis and has units of "cycles per distance," such as cycles per cm,* or cm^{-1}.

*As noted in Chapter 15, the notation k is used in physics to denote "spatial radians per distance" and the notation $\bar{k}$, or "k-bar," is used to denote "cycles per distance." Mathematically, $\bar{k} = k/2\pi$, because there are 2π radians per cycle. However, for notational simplicity, we use k for "cycles per distance" in this text.

The concept of spatial frequency is illustrated in Figure F-1. Slow oscillations represent low spatial frequencies and rapid oscillations represent high frequencies. If $f(x)$ represents an image profile, the former would represent primarily the coarse structures, whereas the latter would represent fine details.

Thus $F(k)$ is a representation of the image profile in *k-space,* or *spatial-frequency space,* whereas $f(x)$ is a representation of the profile in *object space* (sometimes also called *distance space*). Either $f(x)$ or $F(k)$ is a valid representation of the image intensity profile and, as shown subsequently, either one can be derived from the other. Another notation for the FT is

$$F(k) = \mathscr{F}[f(x)] \tag{F-1}$$

where the symbol $\mathscr{F}$ denotes the *operation* of computing an FT.

FTs can be computed for functions in other coordinate spaces as well. For example, in audio technology, signal intensity varies as a function of time, *t,* and its FT describes the function in terms of temporal frequencies, v, expressed as cycles per second, or Hz (sec^{-1}). Coordinate pairs that are represented by a function and its FT, such as x and k, or t and v, are referred to as *conjugate variables.*

B. CALCULATING FOURIER TRANSFORMS

FTs can be calculated for continuous functions, which have values for all values of x, or for discrete functions, which have values only at discrete *sampling intervals* Δx. A discrete function may be a sampled version of a continuous function. Continuous FTs often are employed for theoretical modeling or analysis. However, because the signals of interest in nuclear medicine imaging always are sampled (or "digitized") in projection bins, image pixels,

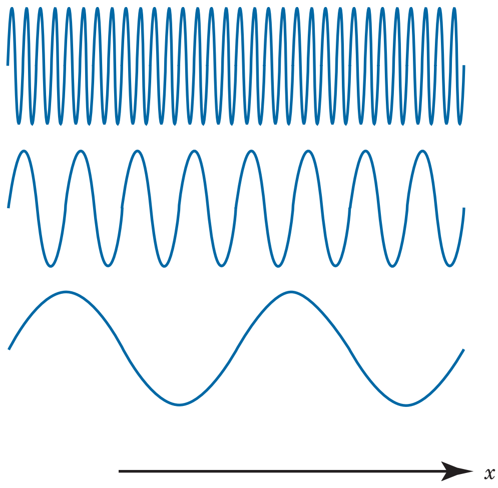

FIGURE F-1 Sinusoidal functions representing low, medium, and high spatial frequencies.

and so on, we will restrict our analysis to the *discrete Fourier transform* (DFT).

In practice, a function representing a signal, such as an image intensity profile $f(x)$, is sampled at a finite number of points, N, at equally spaced intervals Δx. The range of x over which the function is sampled, $N\Delta x$, is the field-of-view along x, denoted by FOV_x. For detectors with discrete detector elements, the sampling interval generally is the width of a detector element and the number of samples is the number of elements across the detector array. For gamma cameras, N and Δx (or FOV_x) can be chosen somewhat arbitrarily under computer control. In this case, FOV_x may or may not equal the physical field of view of the detector. It simply refers to the distance along the x-axis over which data are sampled, which may be smaller (or even larger) than the physical dimensions of the detector. Although N can be any integer, it usually is a power of 2 (64, 128, 256, etc.). Using a power of 2 allows one to use a very fast computational algorithm called the *fast Fourier transform* (FFT). FFTs are provided in many mathematical software packages. The basic algorithm is described in reference 3.

Consider the case in which $f(x)$ is sampled at N points over a range of x values from x = 0 to $(N - 1)\Delta x$. The sampled data are represented as

$$f_j = f(j\Delta x) \quad j = 0, 1, 2, \ldots N-1 \tag{F-2}$$

Given N sampled points, the discrete FT of f is calculated from the following equation

$$F(m\Delta k) = F_m = (1/N)\sum_{j=0}^{N-1} f_j e^{-i2\pi mj/N} \quad m = 0, 1, 2, \ldots N-1 \tag{F-3}$$

where $i = \sqrt{-1}$ is a complex ("imaginary") number. Thus values for F are computed at N points at intervals Δk along the k-axis, ranging from $k = 0$ to $k_{max} = (N - 1)\Delta k$. The "zero-frequency" term actually represents a constant equal to the average value of $f(x)$ within the digitized field of view, FOV_x. We will designate the range of k-space over which the FT is computed, $N\Delta k$, as FOV_k (i.e., the "field of view" in k-space).

The complex exponential term in Equation F-3 represents *Euler's equation,* which states that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{F-4}$$

Thus in general, the values of F_m are complex numbers with "real" and "imaginary" parts. This does not imply that the FT is some sort of "imaginary" entity. Rather, the use of complex numbers in Equation F-3 provides a mathematically convenient way for computing the amplitudes of "cosine" functions versus "sine" functions in the representation of $f(x)$.

Conversely, if one is provided with a set of N values for F, they can be used to compute N values for f using the *inverse* discrete FT, described by

$$f(j\Delta x) = f_j = \sum_{\kappa=0}^{N-1} F_\kappa e^{i2\pi j\kappa/N} \tag{F-5}$$

This equation differs from Equation F-3 only in the sign of the exponent and the appearance of the $(1/N)$ term in the former. (Some alternative formulations for Equations F-3 and F-5 show different placements for N.) To distinguish between the two, Equation F-3 sometimes is referred to as the *forward* FT.

Finally, if the original data are properly sampled (e.g., if they meet the requirements of the sampling theorem discussed later), the inverse FT of the forward FT returns an exact replica of the original data. In other words,

$$\mathscr{F}^{-1}[\mathscr{F}[f(x)]] = f(x) \qquad \text{(F-6)}$$

C. SOME PROPERTIES OF FOURIER TRANSFORMS

Equations F-3 and F-5 are mathematically powerful, but not exactly transparent regarding how they operate. Therefore we will not attempt to analyze them in any detail. Rather, we will focus only on some properties of the discrete FT.

As noted previously, the FT of a function that is sampled at N points also has N values. It can be shown that the following relationships apply regarding data intervals and range in k-space:

$$\Delta k = 1/\text{FOV}_\text{x} \qquad \text{(F-7)}$$

$$\text{FOV}_\text{k} = 1/\Delta x \qquad \text{(F-8)}$$

Thus the *data interval in k-space* is inversely proportional to the *data range in object space,* and conversely, the *data range in k-space* is inversely proportional to the *data interval in object space*. The relationships between data ranges and intervals are completely symmetrical between object space and k-space. Figure F-2 illustrates these relationships.

Equations F-7 and F-8 have important practical ramifications. Specifically, a fundamental assumption underlying the discrete FT algorithm is that the sampled function $f(x)$ actually extends indefinitely along the x-axis, repeating itself at intervals $N\Delta x = \text{FOV}_\text{x}$. In other words, it assumes that the sampled data simply repeat themselves to infinity in both directions along the x-axis. Conversely, it assumes as well that $F(k)$ repeats itself at intervals $N\Delta k$, again to infinity, in both directions along the k-axis. This means, for example, that because of cyclic repetition, $F(N\Delta k) = F(0)$. This result can be understood by observing that $N\Delta k = \text{FOV}_\text{k}$ and, according to Equation F-8, $\text{FOV}_\text{k} = 1/\Delta x$.

Thus if a sinusoidal oscillation with spatial frequency $N\Delta k$ is present in the underlying sampled function, it would be sampled precisely once per cycle. Furthermore, with precise timing, it would be sampled always at the same point during the cycle. Thus the sampled data for a spatial frequency $k = N\Delta k$ would be indistinguishable from a “constant” signal level (i.e., $k = 0$). Similarly, the next higher frequency, $(N + 1)\Delta k$, would be sampled less than once per cycle and would be indistinguishable from a lower frequency, which turns out to be k_1. This effect, in which a higher

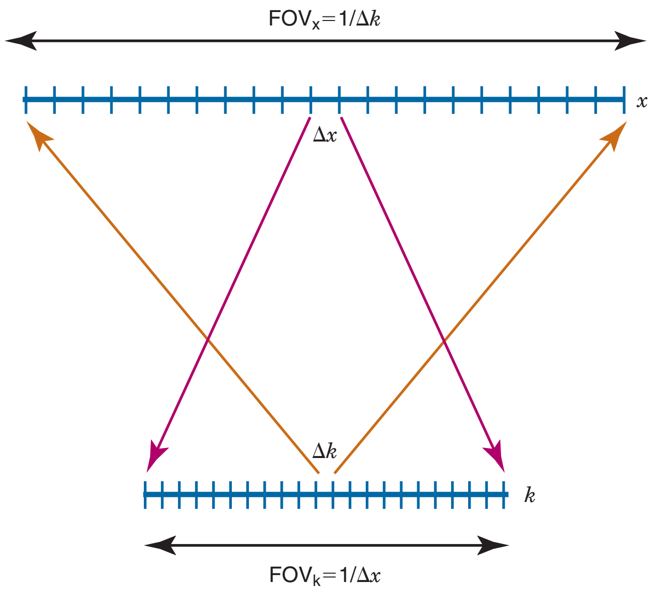

FIGURE F-2 Illustration of the relationships between sampling ranges and sampling intervals in object space and k-space.

frequency is falsely identified as a lower one, is called *aliasing*.*

From this illustration, it is apparent that the highest spatial frequency that can be represented accurately in a discrete FT is determined by the sampling interval along the x-axis. However, there is yet another complication. Specifically, an image (or any object-space profile) can contain both positive and negative frequencies. For a cosine function, there is no difference, because $\cos(-x) = \cos(x)$. However, for a sine function, $\sin(-x) = -\sin(x)$, so the positive and negative oscillations are reversed. In terms of the level of "detail" contained in an image or profile, the difference between positive and negative frequencies is unimportant. The practical importance is that the use of "negative frequencies" allows one to obtain consistent representations for the FT of a function $f(x)$ that is shifted along the x-axis, but otherwise unchanged. This is the basis of the "shift theorem," which is discussed in advanced texts.[4]

The presence of negative frequencies means that the highest spatial frequency that can be computed by a discrete FT is not $N\Delta k$, but only half this value. This frequency, called the *Nyquist frequency,* can be derived from Equations F-6 and F-7 and usually is expressed as

$$k_{\text{Nyquist}} = 1/(2 \times \Delta x) \tag{F-9}$$

This equation is the basis of the *sampling theorem,*[5] which states that, to accurately compute (or "recover") a specified spatial frequency requires a sampling rate of at least two samples per cycle at that frequency. Frequencies higher than the Nyquist frequency cannot be distinguished from lower frequencies. An example is illustrated in Figure F-3. Profiles (or signals) containing frequencies that are higher than the Nyquist frequency specified in Equation F-9 are said to be *undersampled.*

Undersampling and aliasing can lead to image distortions, errors in data analysis, and so forth. Figure F-4 illustrates their effects in k-space. In this hypothetical example, a sampling rate was used corresponding to k_{Nyquist} = 8 cm^{-1}; however, the sampled function actually contained frequencies out to ±10 cm^{-1}. Because of undersampling, frequencies between 8 and 10 cm^{-1} are "wrapped around" and overlap the lower end of the negative frequency portion of the spectrum. A similar wraparound of negative frequencies occurs at the high end of the spectrum. When the FT is calculated, the overlapped portions are simply added together in the *aliased regions* of the spectrum.

This effect and its consequences for emission CT reconstruction are discussed in Chapter 16, Section C.1. Similar effects occur in other applications of FTs [e.g., computations of MTFs (see Chapter 15, Section B.2)]. The sampling theorem also has practical ramifications for data acquisition with arrays of discrete detector elements. It can be shown that a linear array of detector elements, each of width *d,* with negligibly small spacings between elements, can detect spatial frequencies out to $k = 1/d$ and even beyond. (See Figure F-5*C*, which is essentially the MTF for an aperture of width d = 1.) However, with only one sample per distance *d,* the Nyquist frequency is $k_{\text{Nyquist}} = 1/(2d) = (1/2) \times (1/d)$ (i.e., one half the frequency that would be desired for $k_{\max} = 1/d$). Thus spatial frequencies in the range $1/(2d) < k < 1/d$ are aliased back into lower frequency portions of the spectrum, where they can cause distortions of the recorded profile. One way to avoid this is to acquire a second set of data with the detector array "shifted" by one half the width of the individual detector elements and to insert these data points between those for the "unshifted" data. This reduces the sampling interval from d to $d/2$, thereby suppressing the effects of aliasing.

This analysis applies when the spatial resolution of the individual detector elements is essentially the width of the element. This would apply, for example, to collimated linear arrays operating in single-photon counting mode. The situation is yet more complicated in positron emission tomography (PET), for which the spatial resolution of a pair of detectors of width d can be as small as $d/2$ (see Chapter 18, Section A.3). In this case, interleaving is essential to achieve the required sampling frequency. Techniques for accomplishing this in PET imaging are described in Chapter 18, Section A.6.

It is important to realize that aliasing cannot be "undone" by postprocessing of undersampled data. If one starts with an undersampled data set, there is no way to

*Most readers will have seen examples of aliasing in motion pictures, in which undersampling in the frame rate leads to the appearance of false and even backward rotations of the spokes of wagon wheels, helicopter blades, and so forth.

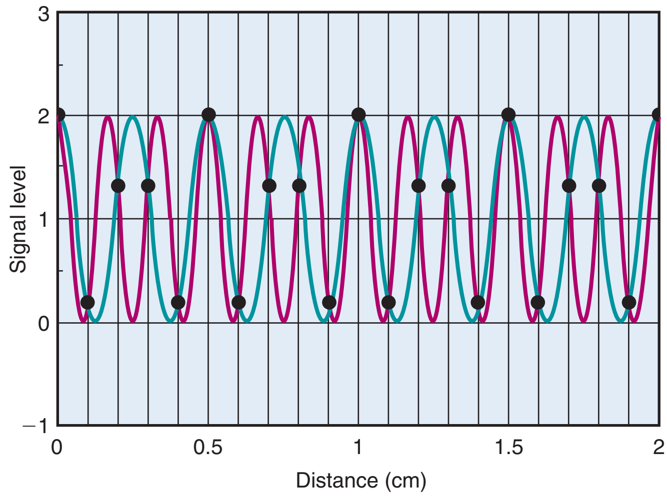

FIGURE F-3 Illustration of the effects of undersampling of a spatially varying source distribution. The *red curve* represents the actual signal, which has a spatial frequency of 6 cycles per cm (6 cm^{-1}). *Black dots* represent sampled points. The sampling distance is $\Delta x = 0.1$ cm, for which the Nyquist frequency is 5 cm^{-1}. As a result of undersampling, the sampled data for a spatial frequency of 6 cm^{-1} cannot be distinguished from a signal corresponding to a spatial frequency of 4 cm^{-1} (*green line*). The false representation of higher frequencies as lower frequencies is called *aliasing*.

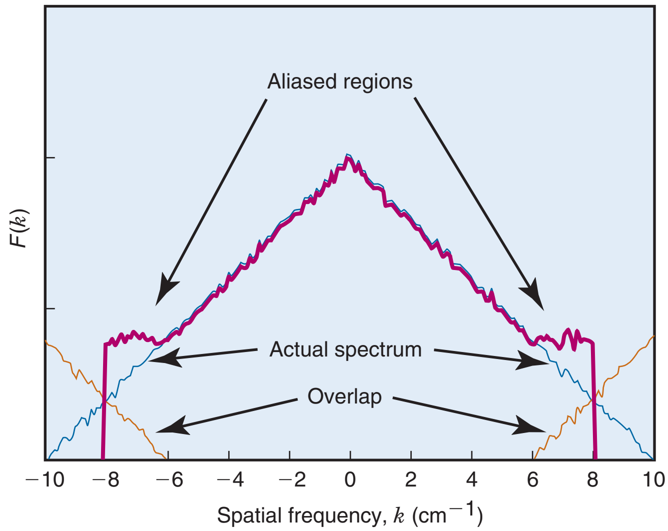

FIGURE F-4 Illustration of the effects of aliasing in k-space. The graphs represent hypothetical Fourier transforms (FTs). Data in object space were sampled at 16 points per cm (cm^{-1}) for computing the FT. The Nyquist frequency for this sampling rate is 8 cm^{-1}. However, the actual spatial-frequency spectrum (*blue line*) has significant frequency content extending out to 10 cm^{-1}. As a result of undersampling, the replicated ends of the actual spectrum (*orange line*) overlap and add to the calculated spectrum (*heavy red line*), creating an aliased region of the spectrum. The calculated spectrum does not accurately represent the true FT for the object.

"unwrap" the spectral overlap illustrated in Figure F-4.

Aliasing can be avoided in a number of ways, including the following:

1. The most direct way is to use a sampling interval along the x-axis (or along whatever object-space axis is being sampled) that is sufficiently small to meet the requirements of the sampling theorem for all frequencies present in the data. One way to achieve this is to acquire more data points along the sampled distance. However, for applications involving FFTs, this requires increasing the

number of samples (e.g., pixels) by a power of 2, which could be impractical for reasons of dataset size, computation time, and so on.

2. If increasing the number of samples is not practical, one could use the same number of samples but apply them over a smaller field-of-view, thereby decreasing the sampling interval Δx. This would increase the Nyquist frequency (Equation F-9); however, reducing FOV_x also has practical constraints. For example, the field-of-view must provide full coverage of the scanned portion of the body in computed tomography.
3. A third alternative is to deliberately "blur" the input profile (or image) before it is recorded by the detectors so as to suppress the potentially undersampled high-frequency components. In the context of image profiles, this means that the data projected onto the detector itself must be "blurred" (e.g., by using a coarser collimator). The obvious disadvantage of this approach is that the resulting image also is blurred.

If none of these options are practical, the effects of undersampling can be minimized (but not eliminated) by postprocessing steps. One approach for already undersampled data is to simply "lop off" or otherwise completely suppress portions of the frequency spectrum that might be affected by aliasing. However, as illustrated in Figure F-4, the potentially affected range is somewhat unpredictable and may be large. This also is a rather wasteful approach, because one ends up throwing away not only frequencies above the Nyquist frequency but any lower frequencies that they overlap after they are wrapped around in the spectrum.

D. SOME EXAMPLES OF FOURIER TRANSFORMS

Figure F-5 shows some one-dimensional examples of functions and their FTs. Note that the functions are represented as continuous functions. If DFTs were involved, the underlying functions would be the same, but the data would be represented by "points" on the curves, following the rules outlined in the preceding section.

In Figure F-5*A*, $f(x)$ is a constant, as would be all sampled values of it. Therefore the only spatial frequency that has nonzero amplitude is $k = 0$, and that amplitude is equal to the value of the constant.

Figure F-5*B* shows a cosine wave of spatial frequency 1 cycle per cm and amplitude 1.0 superimposed on a constant value of 1 unit. In this case, $f(x) = 0.5 \cos(2\pi x) + 1$. The FT has an amplitude of 1 at $k = 0$, and amplitudes of 0.5 each at $\pm 1\ \text{cm}^{-1}$. This ambiguity arises from the fact that the FT cannot distinguish between positive versus negative cosine functions at these frequencies. (They are in fact identical.) Thus it assigns half of the observed amplitude to each value. (The FT of a sine function is similar, except that the value for $k = -1\ \text{cm}^{-1}$ is -0.5. This is because the FT cannot distinguish a positive sine wave of negative frequency from a negative sine wave of positive frequency.)

Figure F-5*C* shows a "boxcar" function (sometimes also called a "rect" function) for which $f(x) = 1$ when x is within the range $-0.5 \leq x \leq +0.5$ and $f(x) = 0$ outside this range. This function crops up often in signal-processing techniques that employ "windowing" filters to eliminate portions of the spectrum outside a windowed range in k-space. In this case, the FT is a more complicated function (specifically, a "sinc function").

Figure F-5*C* illustrates that FTs can exist even for functions with unusual properties, such as discontinuities. Also, although $f(x)$ itself does not have apparent "high-frequency oscillations," its FT has high-frequency components. These components are necessary to represent the sharp edges of the boxcar. Thus the MTF of an imaging system must have a good high-frequency response to faithfully reproduce sharp edges and fine details. In addition, the figure indicates that windowing filters with sharp edges in k-space can create "ringing artifacts" (sometimes called *Gibbs phenomenon*) in object space. These features, as well as illustrations of FTs of many other mathematical functions, are presented and discussed in reference 1.

Finally, Figure F-6 shows a brain image, an intensity profile for a strip across the image, and the FT of that profile. Note that most of the "information" is contained in the low-frequency portion of the spectrum. This does not mean that the high-frequency response (MTF) of the imaging system is unimportant. Indeed, because the high-frequency components are relatively weak to begin with, it is important that the imaging system be able to faithfully preserve them.

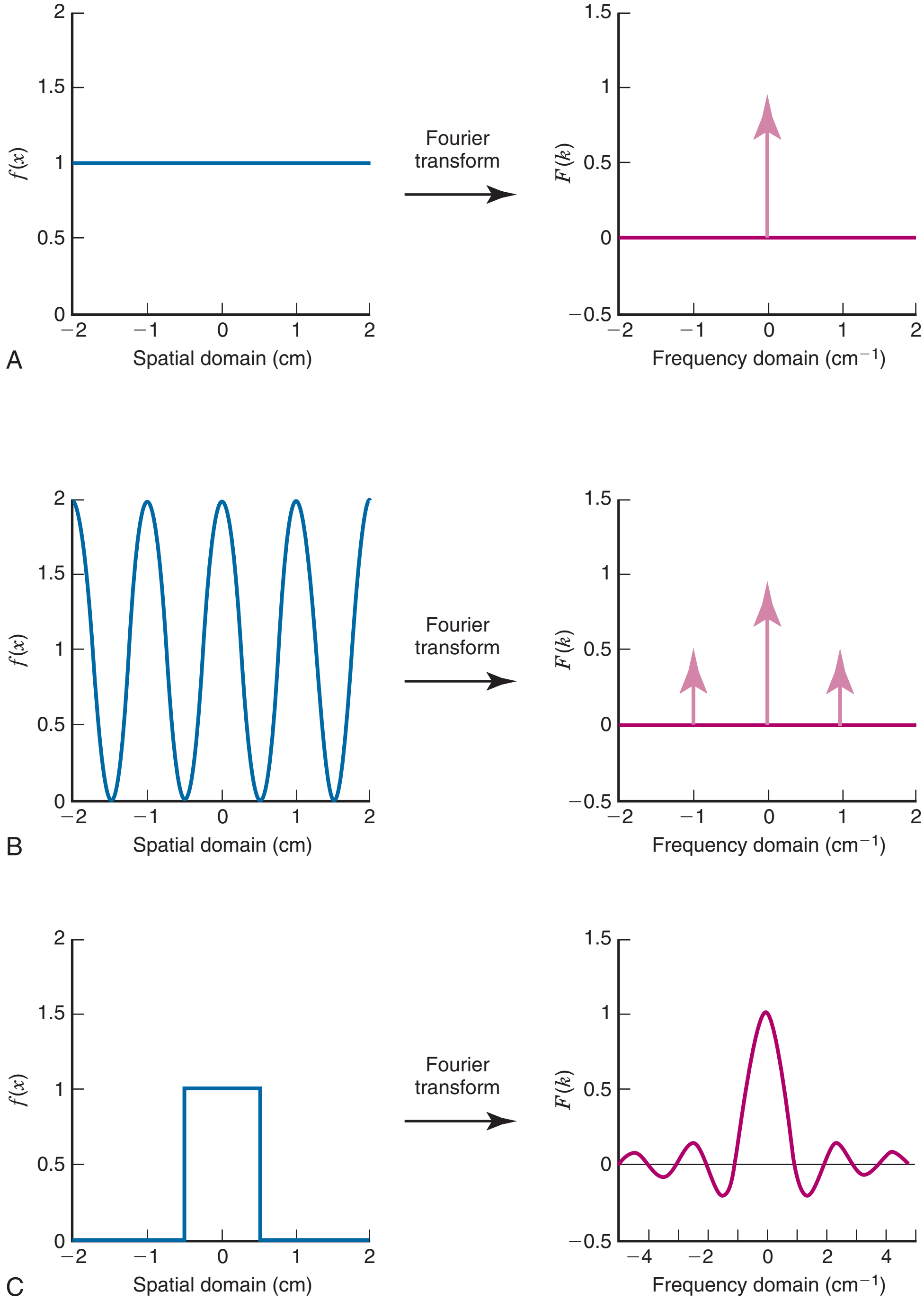

FIGURE F-5 Three functions and their Fourier transforms (FTs). *A*, $f(x) = 1.0$. Its FT has only one value, at $k = 0$. *B*, $f(x) = \cos(2\pi x) + 1$. Its FT has values at $k = 0$ and at $k = \pm 1$ cm^{-1}, corresponding to the spatial frequency of the cosine function. *C*, A boxcar function, $f(x) = 1$ when $-0.5 < x < 0.5$, $f(x) = 0$ otherwise. Its FT is a sinc function, $\text{sinc}(x) = [\sin(\pi x)]/(\pi x)$. Note that this also is the modulation transfer function of an aperture or a detector element of unit width.

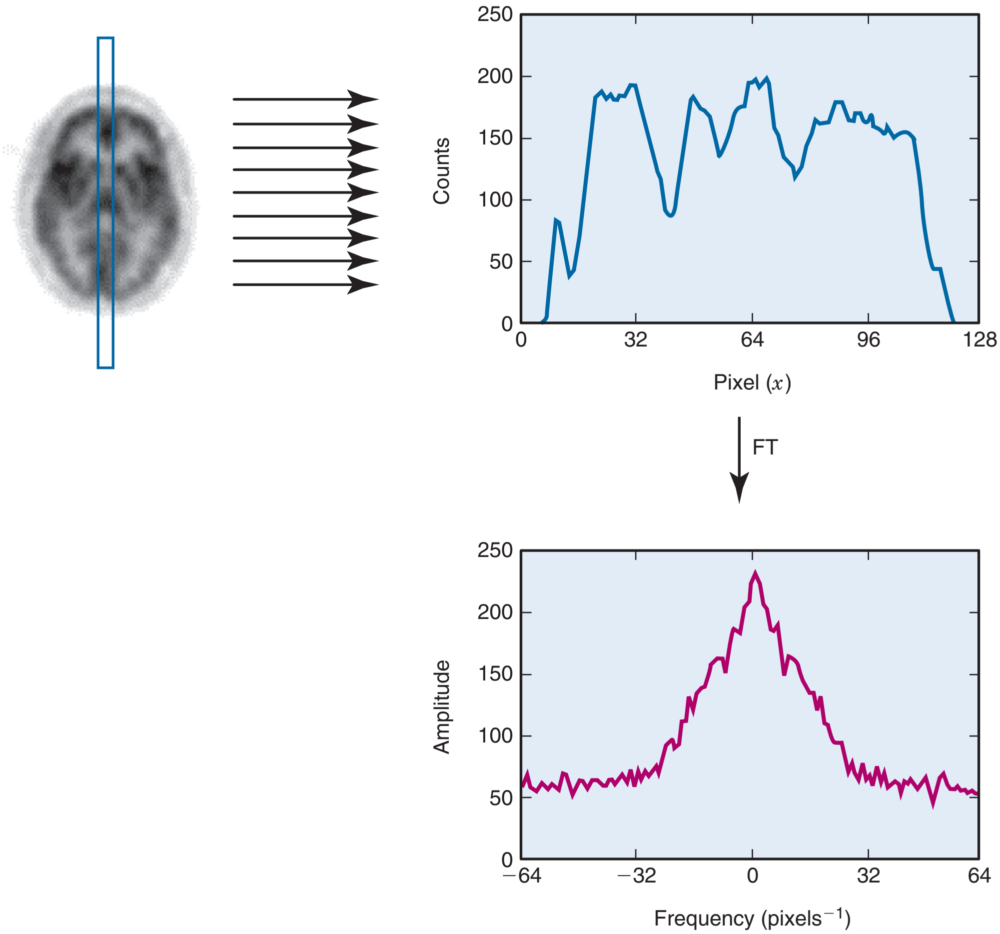

FIGURE F-6 Frequency spectrum for a profile recorded through a brain image. The box on the image indicates the sampling line for the profile (i.e., the direction x is vertical).

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