Tannakian symbols and multiplicative functions

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Intro

We have many new results in number theory - here we hope to give a flavour of the most important ideas.

Outline:

1. Lambda-rings
2. Tannakian symbols
3. Multiplicative functions
4. Motives
5. Miracles

Crash course on lambda-rings

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The operations ψ^1, ψ^2, ... are called **Adams operations**, and we will denote them by Ad^1, Ad^2, etc.

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For a general commutative ring (i.e. with possible torsion), we must use the **lambda-operations** (not the Adams operations) as starting point for the definition, and the axioms become more complicated.

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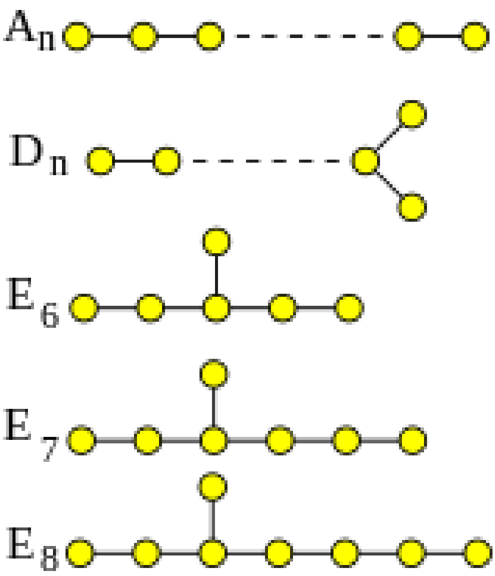
The Grothendieck ring is generated **as a lambda-ring** by elements in one-to-one correspondence with the *arms* of the Dynkin diagram.

Crash course on lambda-rings



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- Binomial rings
- Borger's lambda-algebraic geometry ("over $\mathbb{F}_1$")

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Example: $\frac{\{2,2,5,5\}}{\{1,1,1\}}$

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$$\mathrm{Ad}^2_{\bigcirc}\left(\frac{\{-1,-1,2,5\}}{\{1,-2,7\}}\right) = \frac{\{\cancel{(-1)^2},(-1)^2,\cancel{2^2},5^2\}}{\{\cancel{1^2},\cancel{(-2)^2},7^2\}} = \frac{\{1,25\}}{\{49\}}$$

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Let U be a set. We write $TS(M)^U$ for the set of functions from U to $TS(M)$ (think of this as vectors of symbols, indexed by U).

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Today: Focus on a single application, namely **new lambda-ring structures on multiplicative functions**.

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Super-important point: **These axioms imply that the function values $f(p^e)$ at prime power arguments completely determine the function.**

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Remark: Almost all multiplicative functions appearing in nature are rational.

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Let's focus on the Euler φ function.

Definition: $\varphi(n)$ is the number of positive integers $x \leq n$ that are coprime to n.

Values:

n	1	2	3	4	5	6	7	8	9	10	11	12	...	20
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	...	8

However, this display hides many of the patterns in the function values, and it is often better to tabulate only the function values $f(p^e)$ in a two-dimensional array indexed by the prime p and the exponent e. Here is what happens for the Euler function:

Crash course on multiplicative functions

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	1	2	4	8	16	32
$p = 3$	2	6	18	54	162	486
$p = 5$	4	20	100	500	2500	12500
$p = 7$	6	42	294	2058	14406	100842

We call this the **Bell table** of the multiplicative function φ.

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At $p = 5$:

$$1 + 4t + 20t^2 + 100t^3 + 500t^4 + \ldots = \frac{1-t}{1-5t}$$

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General computation:

$$1 + (p-1)t + (p^2 - p)t^2 + (p^3 - p^2)t^3 + \ldots =$$
$$= (1 + pt + p^2t^2 + p^3t^3 + \ldots) - (t + pt^2 + p^2t^3 + \ldots) =$$
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Tannakian symbol of the Euler φ function: $\frac{\{p\}}{\{1\}}$

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Look at divisors of 20. Note:

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Essentially all identities in the number theory literature can be proven like this, and generalized.

Example: **1**

Values: 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	1	1	1	1	1	1
$p = 3$	1	1	1	1	1	1
$p = 5$	1	1	1	1	1	1
$p = 7$	1	1	1	1	1	1
$p = 11$	1	1	1	1	1	1

Tannakian symbol:

$$\frac{\{1\}}{\varnothing}$$

Example: $\mathbf{0}$

Values: 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	0	0	0	0	0	0
$p = 3$	0	0	0	0	0	0
$p = 5$	0	0	0	0	0	0
$p = 7$	0	0	0	0	0	0
$p = 11$	0	0	0	0	0	0

Tannakian symbol:

$$\frac{\varnothing}{\varnothing}$$

Example: *Id*

Values: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19. . .

Bell table:

	$e=1$	$e=2$	$e=3$	$e=4$	$e=5$	$e=6$
$p=2$	2	4	8	16	32	64
$p=3$	3	9	27	81	243	729
$p=5$	5	25	125	625	3125	15625
$p=7$	7	49	343	2401	16807	117649
$p=11$	11	121	1331	14641	161051	1771561

Tannakian symbol:

$$\frac{\{p\}}{\varnothing}$$

Example: Id_2

Values: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	
$p = 2$	4	16	64	256	1024	
$p = 3$	9	81	729	6561	59049	5
$p = 5$	25	625	15625	390625	9765625	244
$p = 7$	49	2401	117649	5764801	282475249	1384
$p = 11$	121	14641	1771561	214358881	25937424601	31384

Tannakian symbol:

$$\frac{\{p^2\}}{\varnothing}$$

Example: τ

Definition: $\tau(n)$ is the number of ordered factorizations of n.

Values: 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	2	3	4	5	6	7
$p = 3$	2	3	4	5	6	7
$p = 5$	2	3	4	5	6	7
$p = 7$	2	3	4	5	6	7
$p = 11$	2	3	4	5	6	7

Tannakian symbol:

$$\frac{\{1, 1\}}{\varnothing}$$

Example: τ_4

Let $\tau_k(n)$ be the number of ordered factorizations of n into k factors. Then τ_k is a multiplicative function. For $k = 2$ we recover the τ function from earlier.

For $k = 4$, the values are: 1, 4, 4, 10, 4, 16, 4, 20, 10, 16, 4, 40, 4, 16, 16, 35, 4, 40, 4...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	4	10	20	35	56	84
$p = 3$	4	10	20	35	56	84
$p = 5$	4	10	20	35	56	84
$p = 7$	4	10	20	35	56	84
$p = 11$	4	10	20	35	56	84

Tannakian symbol:

$$\frac{\{1, 1, 1, 1\}}{\varnothing}$$

Crash course on multiplicative functions

Definition: $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ is the number of prime factors of n (with multiplicity).

The Bell table of the Liouville function:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	-1	1	-1	1	-1
$p = 3$	-1	1	-1	1	-1
$p = 5$	-1	1	-1	1	-1
$p = 7$	-1	1	-1	1	-1

Tannakian symbol:

$$\frac{\{-1\}}{\varnothing}$$

Crash course on multiplicative functions

Definition: $\Theta(n)$ is the number of square-free divisors of n.

The Bell table of the Θ function:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	2	2	2	2	2
$p = 3$	2	2	2	2	2
$p = 5$	2	2	2	2	2
$p = 7$	2	2	2	2	2

Example: The core function

Values: 1, 2, 3, 2, 5, 6, 7, 2, 3, 10, 11, 6, 13, 14, 15, 2, 17, 6, 19...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	2	2	2	2	2	2
$p = 3$	3	3	3	3	3	3
$p = 5$	5	5	5	5	5	5
$p = 7$	7	7	7	7	7	7
$p = 11$	11	11	11	11	11	11

Tannakian symbol:

$$\frac{\{1\}}{\{-p+1\}}$$

Example: β

Values: 1, 1, 2, 3, 4, 2, 6, 5, 7, 4, 10, 6, 12, 6, 8, 11, 16, 7, 18...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	1	3	5	11	21	43
$p = 3$	2	7	20	61	182	547
$p = 5$	4	21	104	521	2604	13021
$p = 7$	6	43	300	2101	14706	102943
$p = 11$	10	111	1220	13421	147630	1623931

Tannakian symbol:

$$\frac{\{p, -1\}}{\varnothing}$$

Example: ψ

Values: 1, 3, 4, 6, 6, 12, 8, 12, 12, 18, 12, 24, 14, 24, 24, 24, 18, 36, 20. . .

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	3	6	12	24	48	96
$p = 3$	4	12	36	108	324	972
$p = 5$	6	30	150	750	3750	18750
$p = 7$	8	56	392	2744	19208	134456
$p = 11$	12	132	1452	15972	175692	1932612

Tannakian symbol:

$$\frac{\{p\}}{\{-1\}}$$

Example: θ

Values: 1, 2, 2, 2, 2, 4, 2, 2, 2, 4, 2, 4, 2, 4, 4, 2, 2, 4, 2...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	2	2	2	2	2	2
$p = 3$	2	2	2	2	2	2
$p = 5$	2	2	2	2	2	2
$p = 7$	2	2	2	2	2	2
$p = 11$	2	2	2	2	2	2

Tannakian symbol:

$$\frac{\{1\}}{\{-1\}}$$

Example: μ

Values: 1, -1, -1, 0, -1, 1, -1, 0, 0, 1, -1, 0, -1, 1, 1, 0, -1, 0, -1...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	-1	0	0	0	0	0
$p = 3$	-1	0	0	0	0	0
$p = 5$	-1	0	0	0	0	0
$p = 7$	-1	0	0	0	0	0
$p = 11$	-1	0	0	0	0	0

Tannakian symbol:

$$\frac{\varnothing}{\{1\}}$$

Example: γ

Values: 1, -1, -1, -1, -1, 1, -1, -1, -1, 1, -1, 1, -1, 1, 1, -1, -1, 1, -1...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	-1	-1	-1	-1	-1	-1
$p = 3$	-1	-1	-1	-1	-1	-1
$p = 5$	-1	-1	-1	-1	-1	-1
$p = 7$	-1	-1	-1	-1	-1	-1
$p = 11$	-1	-1	-1	-1	-1	-1

Tannakian symbol:

$$\frac{\{1\}}{\{2\}}$$

Example: ξ

Values: 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 0, 1...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	1	0	0	0	0	0
$p = 3$	1	0	0	0	0	0
$p = 5$	1	0	0	0	0	0
$p = 7$	1	0	0	0	0	0
$p = 11$	1	0	0	0	0	0

Tannakian symbol:

$$\frac{\varnothing}{\{-1\}}$$

Example: ϵ_2

This is the characteristic function of square numbers.

Values: 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	0	1	0	1	0	1
$p = 3$	0	1	0	1	0	1
$p = 5$	0	1	0	1	0	1
$p = 7$	0	1	0	1	0	1
$p = 11$	0	1	0	1	0	1

Tannakian symbol:

$$\frac{\{1, -1\}}{\varnothing}$$

Example: ϵ_3

This is the characteristic function of cube numbers.

Values: 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0. . .

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	0	0	1	0	0	1
$p = 3$	0	0	1	0	0	1
$p = 5$	0	0	1	0	0	1
$p = 7$	0	0	1	0	0	1
$p = 11$	0	0	1	0	0	1

Tannakian symbol:

$$\frac{\{1, \omega, \omega^2\}}{\varnothing}$$

Example: σ_0

Values: 1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2...

Bell table:

	$e=1$	$e=2$	$e=3$	$e=4$	$e=5$	$e=6$
$p=2$	2	3	4	5	6	7
$p=3$	2	3	4	5	6	7
$p=5$	2	3	4	5	6	7
$p=7$	2	3	4	5	6	7
$p=11$	2	3	4	5	6	7

Tannakian symbol:

$$\frac{\{1,1\}}{\varnothing}$$

Example: σ_1 (or σ)

Values: 1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	$e = 6$
$p = 2$	3	7	15	31	63	127
$p = 3$	4	13	40	121	364	1093
$p = 5$	6	31	156	781	3906	19531
$p = 7$	8	57	400	2801	19608	137257
$p = 11$	12	133	1464	16105	177156	1948717

Tannakian symbol:

$$\frac{\{p, 1\}}{\varnothing}$$

Example: σ_2

Values: 1, 5, 10, 21, 26, 50, 50, 85, 91, 130, 122, 210, 170, 250, 260, 341, 290, 455, 362...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$	
$p = 2$	5	21	85	341	1365	
$p = 3$	10	91	820	7381	66430	5
$p = 5$	26	651	16276	406901	10172526	254
$p = 7$	50	2451	120100	5884901	288360150	1412
$p = 11$	122	14763	1786324	216145205	26153569806	31645

Tannakian symbol:

$$\frac{\{1, p^2\}}{\varnothing}$$

Example: σ_3

Values: 1, 9, 28, 73, 126, 252, 344, 585, 757, 1134, 1332, 2044, 2198, 3096, 3528, 4681, 4914, 6813, 6860...

Bell table:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e =$
$p = 2$	9	73	585	4681	374
$p = 3$	28	757	20440	551881	1490
$p = 5$	126	15751	1968876	246109501	307636
$p = 7$	344	117993	40471600	13881758801	4761443
$p = 11$	1332	1772893	2359720584	3140788097305	41803889

Tannakian symbol:

$$\frac{\{1, p^3\}}{\varnothing}$$

Crash course on multiplicative functions

Main goal: Understand **all** operations on multiplicative functions!

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Methodology: We have combed the literature for known operations on multiplicative functions.

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In addition, we have "extended" known operations on varieties and on L-functions to all multiplicative functions.

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Main goal: Understand **all** operations on multiplicative functions!

Methodology: We have combed the literature for known operations on multiplicative functions.

In addition, we have "extended" known operations on varieties and on L-functions to all multiplicative functions.

All these operations can be organized into two lambda-ring structures on $Mult(\mathbb{C})$ (the set of all multiplicative functions).

Motivation

Motivation for looking at multiplicative functions: Understand the Grothendieck ring of motives (the "ultimate representation ring"). Basic examples of motives:

- Any variety V over $\mathbb{Q}$ or $\mathbb{F}_q$ gives a motive. Examples: Number fields, the Fermat equation, elliptic curves, higher-genus curves, Calabi-Yau varieties, ...
- For any integer i and any variety V, we have the motive $h^i(V)$, which captures the i'th cohomology of V. This motive can be "twisted" and sometimes decomposed further.

Motivation

Let $K_0(Mot)$ be "the Grothendieck ring of motives over $\mathbb{Z}$".

- Instead of motives, we could also use Galois representations or automorphic representations.
- Augmentation on motives is just the Euler characteristic.

Motivation

We have a diagram, in which all arrows (except from Varieties) are homomorphisms of lambda-rings!

$$\begin{tikzcd}
K_0(\mathit{Mot}) \arrow[r] \arrow[rrr, bend left, dotted, "\text{injective?}"] & \text{L-functions} \arrow[r] \arrow[d] & \mathit{Mult}_{rat}(\mathbb{C}) \arrow[r, hook, "\text{dense}"] \arrow[d, "\simeq"] & \mathit{Mult}(\mathbb{C})^{\bigcirc} \arrow[d, "\simeq\ (\text{Bell der.})"] \\
\text{Varieties} \arrow[u, "h"] \arrow[rrd, bend right, dotted, "\text{Euler char.}"] & \text{Satake param's} \arrow[r] & \text{Tann. sym's} \arrow[d, "\varepsilon\ (\text{augm.})"] & \mathit{Mult}(\mathbb{C})^{\square} \\
 & & \mathbb{Z} &
\end{tikzcd}$$

Motivation

Notes:

- By "L-functions" we mean L-functions with Euler products.
- By "Varieties" we mean smooth projective varieties over $\mathbb{Q}$ (think "nonsingular", "compact", "defined as the zero locus of some polynomials").
- Special cases of Satake parameters are **Hecke eigenvalues** and **Frobenius eigenvalues**.
- We have glossed over a subtlety related to the definition of ε.

Motivation

The L-function of an elliptic curve may look like this:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	1	-1	-3	-1	5
$p = 3$	0	-3	0	9	0
$p = 5$	-3	4	3	-29	72
$p = 7$	-1	-6	13	29	-120
$p = 11$	-1	1	-1	1	-1

Motivation

The Hasse-Weil zeta function of the elliptic curve $y^2 + y = x^3 - x^2$ gives:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	5	5	5	25	25
$p = 3$	5	15	20	75	275
$p = 5$	5	35	140	595	3025
$p = 7$	10	60	310	2400	

Motivation

The Hasse-Weil zeta function of the elliptic curve $y^2 + y = x^3 - x^2$ gives:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	5	5	5	25	25
$p = 3$	5	15	20	75	275
$p = 5$	5	35	140	595	3025
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One of the numbers appearing in the Birch-Swinnerton-Dyer conjecture for this curve is 5.

Motivation

For many classical functions, our explicit Tannakian symbols completely determine the function.

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Analogy: The rows of a motivic Bell table are like the decimals of a real number

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For many classical functions, our explicit Tannakian symbols completely determine the function.

For functions which are less nice, but motivic, there is a *metric* such that the first rows determine the zeta type up to some small error.

Analogy: The rows of a motivic Bell table are like the decimals of a real number

Geometry of motives? All motives seem to sit as lattice points inside a certain Hilbert space, which sits inside $Mult(\mathbb{C})$.

Operations on multiplicative functions

$$\text{Dirichlet convolution:} \quad (f \oplus g)(n) = \sum_{ab=n} f(a)g(b)$$

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Tensor product: $(f \otimes g)(n) = (\textit{complicated})$

Operations on multiplicative functions

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$$\text{Natural product:} \quad (f \boxtimes g)(n) = f(n) \cdot g(n)$$

$$\text{Tensor product:} \quad (f \otimes g)(n) = (\textit{complicated})$$

The last operation corresponds to Cartesian product of varieties, tensor product of motives, and (almost) Rankin-Selberg product of L-functions.

Operations on multiplicative functions

$$\mathrm{Ad}_{\bigcirc}^{k}(f)(n) = \sum_{a_1 a_2 \cdots a_k = n^k} f(a_1) f(a_2) \cdots f(a_k) \omega_k^{\Omega(a_1) + 2\Omega(a_2) + \ldots + k\Omega(a_k)}$$

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$$\mathrm{Ad}^k_{\square}(f)(n) = f(n^k)$$

Operations on multiplicative functions

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Operations on multiplicative functions

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The last operation here is defined so that it corresponds to an operation called *base change* on varieties over finite fields.

Operations on multiplicative functions

Main results on lambda-ring structures:

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Miracle 1: Using these eight fundamental operations, we can express all operations on $Mult(\mathbb{C})$ that have been studied in the mathematical literature.

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Miracle 2: The set $Mult(\mathbb{C})$ together with operations $\oplus, \otimes, \mathrm{Ad}^k_{\bigcirc}$ is a lambda-ring. Furthermore, $\widehat{\mathrm{Ad}}^k_{\bigcirc}$ are "non-unital" Adams operations, and $\mathrm{Ad}^k_{\bigcirc} \circ \widehat{\mathrm{Ad}}^k_{\bigcirc} = id$.

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Miracle 3: Exactly the same thing is true for the box operations.

Miracle 4: There is an operation from $Mult(\mathbb{C}$ to $Mult(\mathbb{C})$ which is an isomorphism from the first lambda-ring to the second. We call this the Bell derivative.

Operations on multiplicative functions

Definition: The **Bell derivative** of a multiplicative function f is the function $(f \boxtimes \tau) \ominus f$

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Theorem: The fixed points of the Bell derivative are precisely the completely multiplicative functions.

Operations on multiplicative functions

Important point: All operations can be implemented in a computer.

Operations on multiplicative functions

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Miracle 6: Almost all operations can be expressed explicitly using Tannakian symbols. In particular, this is true for the two examples which were "complicated".

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is also multiplicative.

In our language, we have $g = f \oplus \varphi$

Operations on multiplicative functions

Recall the diagram, in which all arrows (except from Varieties) are homomorphisms of lambda-rings!

$$
\begin{tikzcd}
K_0(\mathit{Mot}) \arrow[r] \arrow[rrr, dotted, bend left, "\text{injective?}"] & \text{L-functions} \arrow[r] \arrow[d] & \mathit{Mult}_{rat}(\mathbb{C}) \arrow[r, hook, "\text{dense}"] \arrow[d, "\simeq"] & \mathit{Mult}(\mathbb{C})^{\bigcirc} \arrow[d, "\simeq\ \text{(Bell der.)}"] \\
\text{Varieties} \arrow[u, "h"] \arrow[drr, dotted, bend right, "\text{Euler char.}"'] & \text{Satake param's} \arrow[r] & \text{Tann. sym's} \arrow[d, "\varepsilon\ \text{(augm.)}"] & \mathit{Mult}(\mathbb{C})^{\square} \\
 & & \mathbb{Z} &
\end{tikzcd}
$$

The end

Thank you!

The end

Thank you!

Next talk: More examples and applications, and code.

Examples and applications

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Recap:

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The next few slides will give just **a few examples** of what we can do.

Example: Illustrating the Weil conjectures

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The idea is:

- Define a multiplicative function N_V using point counts in finite fields.
- Define H_V to be the Bell antiderivative of N_V.
- Pick a good prime and compute the Tannakian symbol of H_V.
- Plot the Tannakian symbol.

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$$y^2z^7 = x^9 + x^3z^6 + z^9$$

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The next slide shows a screenshot from our Sage code, where we see the values of N_V and for H_V for $q = 1, 5, 25, 125, 625, \ldots$

Example: Illustrating the Weil conjectures

```
K = GF(5)
R.<t> = PolynomialRing(K)
V = HyperellipticCurve(t^9 + t^3 + 1)
N_V_values = [1] + H.count_points_exhaustive(n=20)
print N_V_values
```

```
[1, 9, 27, 108, 675, 3069, 16302, 78633, 389475, 1954044, 9768627,
48814533, 244072650, 1220693769, 6103414827, 30517927308, 152587347075,
762939337869, 3814712047902, 19073477713833, 95367442165875]
```

```
H_V_values = BellAntiderivative(N_V_values)
print H_V_values
```

```
[1, 9, 54, 279, 1404, 7029, 35279, 176904, 885654, 4429404, 22148154,
110741904, 553710654, 2768554404, 13842773154, 69213866904, 346069335654,
1730346679404, 8651733398154, 43258666991904, 216293334960654]
```

```
X = TSC.getTSFromBellCoeffs(H_V_values)
```

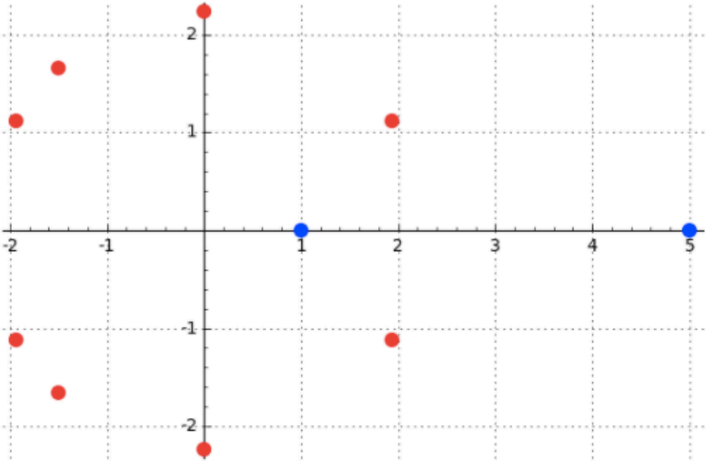
```
X.showplot()
```

Example: Illustrating the Weil conjectures

The Tannakian symbol of H_V at $p = 5$ is illustrated here:

```
X = TSC.getTSFromBellCoeffs(bc)
```

```
X.showplot()
```



Example: Illustrating the Weil conjectures

We make four observations - the unbelievable thing is that these observations hold for any variety in any dimension, provided the variety is projective and nonsingular.

1. The Bell antiderivative of N_V is rational (otherwise our Tannakian symbol wouldn't exist).
2. All elements of the Tannakian symbol lie on circles with radii 1, $\sqrt{5}$ and 5.
3. There is a certain **symmetry with respect to reflection in the middle circle**, not really visible in the case of curves.
4. The number of points on the m'th circle is the m'th Betti number.

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These are equivalent to our observations made above.

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Here we compute the Tannakian symbol (at the prime $p = 5$) of a product of two curves (of genus 2 and genus 4).

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Here we compute the Tannakian symbol (at the prime $p = 5$) of a product of two curves (of genus 2 and genus 4).

Let's try also a 4-fold (a product of a genus 2 curve with a genus 4 curve):

```
W = HyperellipticCurve(t^5+t^2+1)
N_W_values = [1] + W.count_points_exhaustive(n=20)
H_W_values = BellAntiderivative(N_W_values)
Y = TSC.getTSFromBellCoeffs(H_W_values)
(X*Y).showplot()
```

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Note the last line, where we take the tensor product of two Tannakian symbols and ask Sage to plot it.

Example: Illustrating the Weil conjectures

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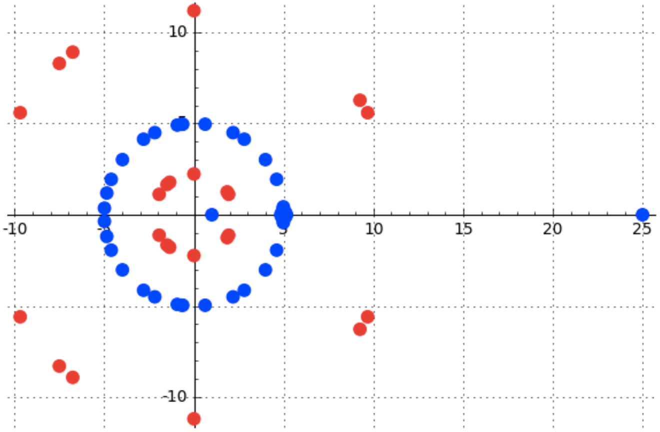

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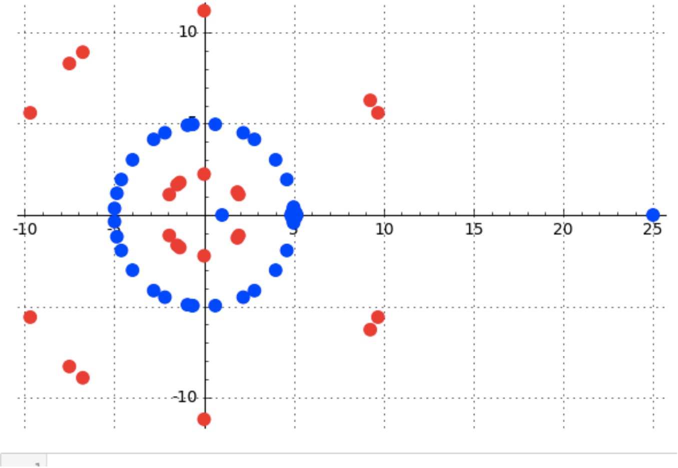
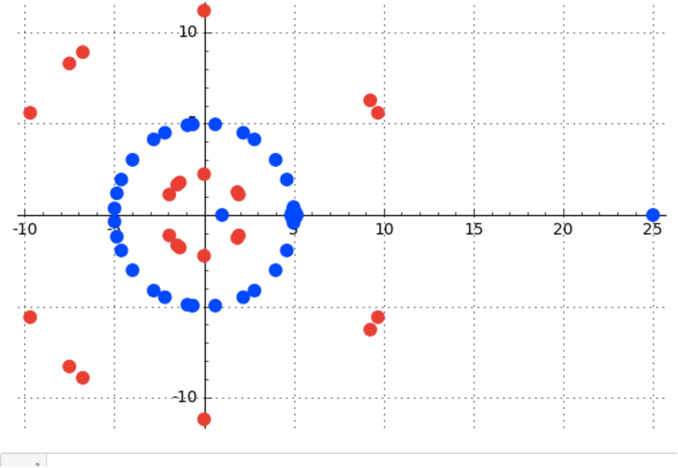


The red points contribute to the odd Betti numbers, and the big red circle has 12 points, meaning the third Betti number is 12.

Example: Illustrating the Weil conjectures



The red points contribute to the odd Betti numbers, and the big red circle has 12 points, meaning the third Betti number is 12. This is of course consistent with the Künneth formula.

Example: Illustrating the Weil conjectures

Here's some code for computing the second and third Betti number, in case the points are too close to each other for manual counting.

```
In [136]:  1 (X*Y).filter(lambda x: 5.1 < abs(x) < 24.9).odddimension()
                                                    less than a minute ago
                                                    0.029 seconds
Out[136]: 12

In [137]:  1 (X*Y).filter(lambda x: 4.9 < abs(x) < 5.1).evendimension()
                                                    less than a minute ago
                                                    0.025 seconds
Out[137]: 34
```

The second Betti number is 34 in this case.

Application: New observations on point counting

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Here $q = p^e$ is any prime power.

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Here is the Bell table of N_V:

	$e=1$	$e=2$	$e=3$	$e=4$	$e=5$	$e=6$
$p=2$	1	3	7	15	31	63
$p=3$	2	8	26	80	242	728
$p=5$	4	24	124	624	3124	15624
$p=7$	6	48	342	2400	16806	117648
$p=11$	10	120	1330	14640	161050	1771560

Application: New observations on point counting

Here is the Bell table of I_V:

	$e=1$	$e=2$	$e=3$	$e=4$	$e=5$	$e=6$
$p=2$	1	2	4	8	16	32
$p=3$	2	6	18	54	162	486
$p=5$	4	20	100	500	2500	12500
$p=7$	6	42	294	2058	14406	100842
$p=11$	10	110	1210	13310	146410	1610510

(Remark: This happens to be the Euler φ function.)

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Note that the finite field $\mathbb{F}_q$ and the finite ring $\mathbb{Z}/q\mathbb{Z}$ are very different objects.

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- $\mathbb{F}_9$ is described by $\{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$ (is a field of characteristic 3) (where $i^2 = -1$)

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Think carefully about what happens here. N_V comes from point counting in finite fields, and I_V comes from point counting using modular arithmetic.

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The two functions N_V and I_V really should not have any relation beyond the first column!!!

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However, we have found that...

- **The Bell derivative of I_V is equal to N_V (!)**
- **This works in general for cellular varieties, e.g. projective spaces and Grassmannians.**

Application: New observations on point counting

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For varieties which are not cellular, this is not true, but it could be interesting to investigate other possible relations between N_V and I_V.

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Let us look at a single example - we take the affine variety V defined by

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Let us look at a single example - we take the affine variety V defined by

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This is a compact genus 3 curve with one point removed.

Note again: **No-one expects any relation beyond the first column.**

Application: New observations on point counting

The Bell table of N_V:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	**2**	4	8	16	32
$p = 3$	**3**	27	27	27	243
$p = 5$	**3**	19	147	667	3043
$p = 7$	**7**	91	343	2107	16807

The Bell table of I_V:

	$e = 1$	$e = 2$	$e = 3$	$e = 4$	$e = 5$
$p = 2$	**2**	2	4	8	16
$p = 3$	**3**	9	27	81	243
$p = 5$	**3**	15	75	375	1875
$p = 7$	**7**	49	343	2401	16807

The first columns must be identical because $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

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For $p=2$, one row is a shift of the other. Why?

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For $p = 5$, one row is chaotic and the other one is a geometric sequence. Why?

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For $p=3$ and $p=7$, every second point count agree. Why?

Application: New observations on point counting

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What happens for bigger primes?

Application: New observations on point counting

What happens for bigger primes?

What happens for other curves, or higher-dimensional varieties?

Application: New observations on point counting

What happens for bigger primes?

What happens for other curves, or higher-dimensional varieties?

To the best of our knowledge, no-one has observed these patterns before

Ramanujan's lambda-ring operation

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However, the lambda operations do not appear in the number theory literature, with only two exceptions:

- The notion of a **central character** of an L-function. This is analogous to the top exterior power of a vector bundle.
- An **old theorem of Ramanujan** for the sigma function, generalized by Busche.

Ramanujan's lambda-ring operation

The Ramanujan-Busche identities are usually stated as follows (from the book of McCarthy: Introduction to Arithmetical functions).

Theorem 1.12. If f is a multiplicative function then the following statements are equivalent:

(1) f is a convolution of two completely multiplicative functions.

(2) There is a multiplicative function F such that for all m and n,

$$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)F(d) .$$

Ramanujan's lambda-ring operation

(3) There is a completely multiplicative function B such that for all m and n,

$$f(m)f(n) = \sum_{d \mid (m,n)} f(mn/d^2)B(d) .$$

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$$F = \lambda^2(f) \otimes \mu$$

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where λ^2 is the second lambda-operation and μ is the Möbius function.

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and

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where λ^2 is the second lambda-operation and μ is the Möbius function.

This is a nice reformulation of the Ramanujan-Busche identities, but it also opens the door to generalizations of this theorem and other similar ones.

Solving multiplicative functional equations

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$$\sum_{d|n^2} (-1)^{\Omega(d)} \tau(d) \tau\left(\frac{n^2}{d}\right) = \tau(n)$$

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But here is a challenge: Can you find all multiplicative functions f such that the identity

$$\sum_{d|n^2}(-1)^{\Omega(d)}f(d)f\left(\frac{n^2}{d}\right)=f(n)$$

holds?

Solving multiplicative functional equations

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As examples, solutions here include:

- All the functions τ_k (counting ordered factorizations into k factors)
- The characteristic functions of cubes, of 7'th powers, of 15'th powers, etc.
- None of the other functions seen in examples above.

Proving identities between multiplicative functions

Another example, to illustrate the use of Tannakian symbols in proofs. We want to prove that

$$n \cdot \tau(n) = \sum_{r=1}^{n} \sigma\left(gcd(n, r)\right)$$

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We can prove the statement for all n, using Tannakian symbols.

Computer code and automatic generation of new identities

Before finishing, we show some work in progress on implementing all our operations in CoCalc (formerly SageMathCloud), and automating the proofs of identities like the previous one.

All code is openly available and can be run online at CoCalc without installing anything locally.

Thanks for your attention :-)