(2)

THUIS **Oefeningen hoofdstuk 1**

1.1

a) $a_n = a_{n-1} + 3$ met $a_0 = 1$ of $a_n = 1 + n \cdot 3$

b) $a_n = a_{n-1} \cdot 3$ met $a_0 = 1$ of $a_n = 3^n$

c) $a_n = \left(\frac{-1}{5}\right)^n$ of $a_n = -\left(a_{n-1} \cdot \frac{1}{5}\right)$ met $a_0 = 1$

1.2

a) $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$

b) $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}$

c) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}$

d) $1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}$

e) $1, 3, 5, 11, 21, 43$

f) $1, 1+i, 2i, 2i-2, 4, 4+4i$

$(i-1)(1+i) = 1 - i^2 = 2$

$(1+i)^2 = 1 + 2i + i^2 = 2i$

1.3

a) $\lim_{n \to \infty} \frac{2n-1}{3n+4} = \frac{2}{3}$

c) $\lim_{n \to \infty} 1 - \frac{1}{n^2} = 1$

b) $\lim_{n \to \infty} \frac{n}{n^2+4} = 0$

d) $\lim_{n \to \infty} 4 + \left(\frac{1}{2}\right)^n = 4$

1.5

a) $a_{k+1} = a_k + 4$ met $a_0 = 1$

$\Rightarrow S_n = (n+1) + \frac{1}{2} n (n+1) \cdot 4$

$= n + 1 + 2n^2 + 2n$

$= 2n^2 + 3n + 1$

$\sum_{k=0}^{n-1} 1 + 4k$

$n + \frac{1}{2}(n-1) n \, 4$

$= n + 2n^2 - 2n$

$= 2n^2 - n$

FOUT → b) $a_{k+1} = 3 a_k$ of $a_k = 3^k$

pl: $\frac{3^n - 1}{2}$

$\Rightarrow S_n = \frac{1 - 3^{n+1}}{1 - 3} = \frac{1 - 3^{n+1}}{2} = \frac{1 - 3^n \cdot 3}{2}$

c) $a_k = \left(\frac{1}{3}\right)^k \Rightarrow S_n = \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} = \frac{3\left(1 - \left(\frac{1}{3}\right)^n\right)}{2}$

$n$ getallen $\Rightarrow$ 0 tot $n-1$

$\sum_{k=0}^{n-1} \left(\frac{1}{3}\right)^k$

1.11

a) ~~voor $x=0 \Rightarrow S_n = 0$~~ convergent voor $x$ in $]-\frac{1}{3}, \frac{1}{3}[$ met som $\frac{1}{1-3x}$, divergent elders

1.16

a) $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} (\sqrt{n} - \sqrt{n}) = 0$

$a_k = b_{k+1} - b_k$ met $b_k = \sqrt{k}$

$S_n = b_{n+1} - b_0$

$= \sqrt{n+1} - 0 = \sqrt{n+1}$

$\Rightarrow$ voor $n \to \infty$ : $\sqrt{n+1} \to \infty \Rightarrow$ divergent

1.17

a) $\sum_{n=2}^{\infty} \frac{1}{n} \to$ divergent

$\ln n < n$

$\frac{1}{\ln n} > \frac{1}{n} > 0$

$\Bigg\}$ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ : divergent

1.19

b) $f(x) = \frac{1}{x \ln x}$ voor $x \geq 2$

$\int_2^{\infty} \frac{1}{x \ln x} dx$

$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln|u| + c$

$= \ln|\ln x| + c$

$u = \ln x$

$du = \frac{1}{x} dx$

$\int_2^{\infty} \frac{1}{x \ln x} = [\ln|\ln x|]_2^{\infty}$

$= \ln|\ln \infty| - \ln|\ln 2|$

$= +\infty$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ : divergent

# OEFENZITTING

1.3

g) $\lim_{n\to\infty} \frac{10^n}{n!} = 0$

1.4

a) $a_{n+1} = a_n + 2a_{n-1}$ $\qquad 1, 3, \frac{5}{3}, \frac{11}{5}, \frac{21}{11}, \frac{44}{21}, \frac{86}{44}, \frac{174}{86}$

$a_n = a_{n-1} + 2a_{n-2}$

$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a_n}{a_{n-1}}$

$a_{n+1} = a_n + 2a_{n-1} \quad \downarrow \div a_n$

$\frac{a_{n+1}}{a_n} = 1 + 2\frac{a_{n-1}}{a_n}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + 2\frac{a_{n-1}}{a_n}\right)$

$L = 1 + 2 \underbrace{\lim_{n\to\infty} \frac{a_{n-1}}{a_n}}_{= 1/L}$

$L = 1 + \frac{2}{L}$

$L^2 - L - 2 = 0$

$D = b^2 - 4ac = 9$

$L = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{2}$ $\qquad L_1 = 2$ en $\cancel{L_2 = -1}$ verhouding van pos. getallen positief

1.6

a) $x^{(2n+3)} = x^3 x^{2n} \qquad n \geq 0$

$\sum_{i=0}^{n-1} x^k = \frac{1-x^n}{1-x}$

$\sum_{i=0}^{n-1} x^3 (x^2)^i = x^3 \frac{1-x^{2n}}{1-x^2}$

$1 - 3 + 2 - 4 + 3 - 5 + 4 - 6$

$1 + 2 + n - 1 + n$

1.8

$$\frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left(\frac{k+2-k}{k(k+2)}\right) = \frac{1}{2}\,\frac{2}{k(k+2)} = \frac{1}{k(k+2)}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+2)} = \sum_{k=1}^{n} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+2}\right)$$

$$= \frac{1}{2}\left(\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=3}^{n+2} \frac{1}{k}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{3}{4} - \frac{1}{2}\,\frac{1}{n+1} - \frac{1}{2}\,\frac{1}{n+2}$$

1.11

b) $\sum_{i=0}^{\infty} y^i = \frac{1}{1-y}$ als $|y| < 1$

$|-x^2| < 1 \Rightarrow x^2 < 1 \Rightarrow -1 < x < 1$

bij $x = 1$ en $x = -1$

$\sum_{k=0}^{\infty} (-1)^k = ?$ $\Rightarrow$ divergent

$$\sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1+x^2} \quad \text{als } -1 < x < 1$$
$$= \infty \quad \text{als } x > 1 \text{ en } x < -1$$

c) $|e^{-x}| < 1 \quad \Rightarrow \quad x > 0$

$$\sum_{k=0}^{\infty} (e^{-x})^k = \frac{1}{1-e^{-x}} \quad x > \cancel{0}\,1$$
$$= \text{divergent} \quad x \leq 0$$

1.13

a) $h k \nu / T > 0 \quad \Rightarrow$ Convergent

$$\sum_{k=0}^{\infty} (e^{-h\nu/T})^k = \frac{1}{1-e^{-h\nu/T}}$$

b) $\lim_{T \to 0+} q(T) = 1$

$\lim_{T \to +\infty} q(T) = \left(\frac{1}{0}\right) = +\infty$

1.16

b) $\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k}\right) = S_n$

$= \ln(1+1) + \ln\left(1+\frac{1}{2}\right) + \ln\left(1+\frac{1}{3}\right) + \dots \ln\left(1+\frac{1}{n}\right)$

$= \ln\left((1+1)\cdot\left(1+\frac{1}{2}\right)\cdot \dots \cdot \left(1+\frac{1}{n}\right)\right)$

$= \ln \prod_{k=1}^{n} \left(1+\frac{1}{k}\right) = \ln \prod_{k=1}^{n} \left(\frac{k+1}{k}\right)$

$= \ln\left(\frac{1+1}{1}\cdot\frac{2+1}{\cancel{2}}\cdot\frac{3+1}{\cancel{3}}\cdot \dots \cdot\frac{n+1}{\cancel{n}}\right)$

$= \ln(n+1)$

$\lim_{n\to\infty} S_n = \infty$

1.17

b) $\sum_{k=1}^{\infty} \frac{1}{k}$ : divergent

$\sum_{k=1}^{\infty} \frac{1}{k^n}$ : convergent $n>1$

$\ln n < n$ dus $\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ $\forall n > 1$

$\sum_{j=1}^{\infty} \frac{\ln j}{j^3} < \sum_{j=1}^{\infty} \frac{1}{n^2}$ -> convergent dus $\sum_{j=1}^{\infty} \frac{\ln j}{j^3}$ : convergent

a) $\ln n < n$ dus $\frac{1}{n} < \frac{1}{\ln n}$ $\forall n > 1$

$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \underbrace{\sum_{n=2}^{\infty} \frac{1}{n}}_{\text{divergent}}$ dus $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ : divergent

1.18

b) $\binom{2k}{k}^{-1} = \left( \frac{2k!}{k!(k!)} \right)^{-1} = \frac{k!k!}{(2k)!}$

$$\frac{(k+1)!\,(k+1)!}{\underset{2k+2}{(2(k+1))!}} \frac{(2k)!}{k!\;k!} = \frac{(k+1)\cancel{(k+1)}}{(2k+1)\underset{2\cancel{(k+1)}}{(2k+2)}}$$

$$= \frac{(k+1)}{(2k+1)\,2} = \frac{k+1}{4k+2}$$

$$\lim_{k\to\infty} \frac{k+1}{4k+2} = \frac{1}{4} \Rightarrow \text{absoluut convergent}$$

1.19

a) $f(x) = e^{-xt}$ $\quad \int e^{-xt}\,dx = \frac{-e^{-xt}}{t} + c$

$$\int_1^\infty e^{-xt}\,dx = \left[ \frac{-e^{-xt}}{t} \right]_1^\infty = \frac{e^{-t}}{t} \Rightarrow \text{Convergent}$$

$\Rightarrow$ reeks is ook convergent

②

THUIS

Oefeningen hoofdstuk 2

2.1

a) $R = \lim_{k\to\infty} \left| \frac{c_k}{c_{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k^3}{(k+1)^3} \right| = \lim_{k\to\infty} \frac{k^3}{(k+1)^3} = 1$

c) $R = \lim_{k\to\infty} \left| \frac{k!}{2^k} \frac{2^{k+1}}{(k+1)!} \right| = \lim_{k\to\infty} \left| \frac{2}{(k+1)} \right| = 0$

2.2

a) $R = \lim_{k\to\infty} \left| \frac{1}{4^k} \frac{4^{k+1}}{1} \right| = 4$

$$\sum_{m=0}^{\infty} \left(\frac{x}{4}\right)^m = \frac{1}{1-\frac{x}{4}} = \frac{1}{\frac{4-x}{4}} = \frac{4}{4-x}$$

2.5

a) $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

$f'(x) = \frac{-1}{(3+x)^2}$ $\qquad f'''(x) = \frac{-6}{(3+x)^4}$

$f''(x) = \frac{2}{(3+x)^3}$ $\qquad f^{(k)}(x) = \frac{(-1)^k k!}{(3+x)^{k+1}} \Rightarrow f^{(k)}(0) = \frac{(-1)^k k!}{3^{k+1}}$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{3^{k+1}} x^k = \frac{1}{3} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{3}\right)^k$$

d) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-3)^k}{k!} x^k$

2.6

a) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4$

2.8 Vertrek van $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ en leid 2 x af

$$\frac{+1}{(1-x)^2} = \sum_{k=0}^{\infty} k\,x^{k-1} = \sum_{k=1}^{\infty} k\,x^{k-1}$$

$$\frac{2}{(1-x)^3} = \sum_{k=1}^{\infty} k(k-1)\,x^{k-2}$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)\,x^{k-2}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} (k+1)k\,x^{k-1}$$

2.11

$$T(R) = \frac{3\ln(1+KR)}{K^3R^3} - \frac{3KR}{K^3R^3} + \frac{3K^2R^2}{2K^3R^3}$$

$$= \frac{3}{K^3}\frac{\ln(1+KR)}{R^3} - \frac{3}{K^2R^2} + \frac{3}{2}\frac{1}{KR}$$

$$T'(R) = \frac{3}{K^3}\left(\frac{\frac{1\cdot K}{1+KR}\cdot R^3 - \ln(1+KR)\cdot 3R^2}{R^6}\right) + \frac{6}{K^2}\frac{1}{R^3} - \frac{3}{2K}\frac{1}{R^2}$$

$$= \frac{3}{K^2R^3(1+KR)} - \frac{9\ln(1+KR)}{K^3R^4} + \frac{6}{K^2}\frac{1}{R^3} - \frac{3}{2K}\frac{1}{R^2}$$

$$T''(R) = \frac{-3(2K^2R+3K^3R^2)}{(K^2R^2+K^3R^3)^2} -$$

oplossing: $1 - \frac{3}{4}KR + \frac{3}{5}(KR)^2$

OEFENZITTING

2.1

b) $R = \frac{1}{L} = \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|$

$\left(\frac{2k}{k}\right) = \frac{(2k)!}{k!\,k!}$

$\frac{c_k}{c_{k+1}} = \frac{(2k!)}{k!\,k!} \cdot \frac{(k+1)!\,(k+1)!}{(2(k+1))!}$ $(2k+2)!$

$= \frac{(k+1)\,(k+1)}{(2k+1)\,(2(k+1))}$

$= \frac{k+1}{2\,(2k+1)} = \frac{k+1}{4k+2}$

$R = \lim_{k \to \infty} \frac{k+1}{4k+2} = \frac{1}{4}$

d) $\frac{c_k}{c_{k+1}} = \frac{1}{k^2} \cdot \frac{(k+1)^2}{1} = \frac{k^2+2k+1}{k^2}$

$\lim_{k \to \infty} 1 + \frac{2}{k} + \frac{1}{k^2} = 1 = R$

2.2

d) $\frac{c_k}{c_{k+1}} = \frac{2^k}{2^{k+1}} = \frac{1}{2}$

$R = \lim_{k \to \infty} \frac{1}{2} = \frac{1}{2}$

$\sum_{k=0}^{\infty} a x^k = \frac{a}{1-x}$ $\quad (|x| < 1)$

$\sum_{k=\boxed{2}}^{\infty} (2x)^k = \frac{1}{1-2x} - 1 - 2x$

2.5

b) $\frac{1}{1+x^2} = (1+(x^2))^{-1}$

$= 1 - x^2 + \frac{2}{2}x^4 - \frac{6}{6}x^6 + \frac{24}{24}x^8 - \dots$

$= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots$

d) $e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \frac{81x^4}{24} + \dots$

e) $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$

2.6

b) $(1-x)^{-1/2} = 1 - \frac{1}{2}(-x) + \frac{3}{4}\,\frac{1}{2}(-x)^2 + \frac{-15}{8 \cdot 3!}(-x)^3$

$+ \frac{105}{16 \cdot 4!}(-x)^4$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$

d) $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}$

2.7

a) $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$

$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{k=0}^{\infty} x^k$

$\frac{+1}{(1-x)^2} = \sum_{k=0}^{\infty} k\,x^{k-1} = \sum_{k=0}^{\infty} k\,x^{k-1}\,\frac{x}{x}$

$= \sum_{k=0}^{\infty} \frac{k\,x^k}{x}$

$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} k\,x^k = \sum_{k=1}^{\infty} k\,x^k$

2.13

a) $$pV + -pnb + \frac{n^2 a}{V} - \frac{n^3 ba}{V^2} = nRT$$

$$pV = nRT + pnb - \frac{n^2 a}{V} + \frac{n^3 ba}{V^2}$$

$$= nRT\left(1 + \frac{pb}{RT} - \frac{na}{VRT} + \frac{n^2 ba}{V^2 RT}\right)$$

$$= nRT\left(\underbrace{\left(1 + \frac{pb}{RT}\right)}_{B_0}\left(\frac{n}{V}\right)^0 - \underbrace{\frac{a}{RT}}_{B_1}\left(\frac{n}{V}\right)^1 + \underbrace{\frac{ba}{RT}}_{B_2}\left(\frac{n}{V}\right)^2\right)$$

b) $$p(V - nb) = nRT\, e^{-an/RTV}$$

$$pV = nRT\, e^{\frac{-an}{RTV}} + pnb$$

$$pV = nRT\left(e^{-\frac{an}{RTV}} + \frac{pb}{RT}\right)$$

$$e^{-\frac{a}{RT}\frac{n}{V}} = 1 + \frac{-a}{RT}\frac{n}{V} + \frac{\frac{a^2}{R^2T^2}\frac{n^2}{V^2}}{2!}$$

$$pV = nRT\left(\underbrace{\frac{pb}{RT} + 1}_{B_0} - \underbrace{\frac{a}{RT}}_{B_1}\frac{n}{V} + \underbrace{\frac{a^2}{2\cdot R^2T^2}}_{B_2}\frac{n^2}{V^2}\right)$$

2.15

a) $$\frac{1}{e^x - 1} \stackrel{?}{=} \sum_{n=0}^{\infty} e^{-x(n+1)} = \sum_{n=0}^{\infty} e^{-x}\left(e^{-x}\right)^n$$

$$\sum_{k=0}^{\infty} a x^k = \frac{a}{1-x} \quad \Rightarrow \quad a = e^{-x}, \quad x = e^{-x}$$

$$= \frac{\cancel{e^{-x}}}{1 - \cancel{e^{-x}}}$$

$$e^{-x}$$

$$= \frac{1}{\frac{1}{e^{-x}} - 1} = \frac{1}{e^x - 1}$$

b) $I = \sum_{n=0}^{\infty} \int_0^{\infty} x e^{-x(n+1)} \, dx$

$u = x \qquad du = dx$

$dv = e^{-x(n+1)} dx \qquad v = \frac{e^{-x(n+1)}}{-(n+1)}$

$$= \sum_{n=0}^{\infty} \left( \left[ \frac{x e^{-x(n+1)}}{-(n+1)} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-x(n+1)}}{-(n+1)} dx \right)$$

$$= \sum_{n=0}^{\infty} \left( \left[ \frac{x}{-(n+1) e^{x(n+1)}} \right]_0^{\infty} - \left[ \frac{e^{-x(n+1)}}{(n+1)^2} \right]_0^{\infty} \right)$$

$$= \sum_{n=0}^{\infty} \left( \underbrace{\frac{\infty}{-(n+1)\cdot\infty}}_{\text{hôpital}} - 0 - 0 + \frac{1}{(n+1)^2} \right)$$

$$\cancel{= \sum_{n=0}^{\infty} \left( \left[ \frac{1}{-(n+1)^2 e^{x(n+1)}} \right]_0^{\infty} + \frac{1}{(n+1)^2} \right)}$$

$$\cancel{= \sum_{n=0}^{\infty} \left( - \frac{1}{-(n+1)^2} + \frac{1}{(n+1)^2} \right)}$$

$$\cancel{= \sum_{n=1}^{\infty} \left( \frac{1}{n^2} + \frac{1}{n^2} \right)}$$

enkel hopital voor naar oneindig
direct voor naar nul

$$= \sum_{n=1}^{\infty} \underbrace{\lim_{x\to\infty} \frac{1}{-(n)^2 e^{x(n)}}}_{=0} - 0 - 0 + \frac{1}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

2

THUIS **Oefeningen hoofdstuk 3**

3.1

a) $\int_{-1}^{1} x^{k+l}\,dx = \left[\frac{x^{k+l+1}}{k+l+1}\right]_{-1}^{1} = \frac{1}{k+l+1} - \frac{(-1)^{k+l+1}}{k+l+1}$

$\frac{1+(-1)^{k+l}}{k+l+1} = \begin{cases} \text{stel } k+l+1 = \text{oneven}: \int_{-1}^{1} x^{k+l}\,dx = \frac{2}{k+l+1} \\ \quad \to k \text{ en } l \text{ even of } k \text{ en } l \text{ oneven} \\ \text{stel } k+l+1 = \text{even}: \int_{-1}^{1} x^{k+l}\,dx = 0 \\ \quad \to k \text{ even en } l \text{ oneven of } l \text{ even en } k \text{ oneven} \end{cases}$

b) $\int \cos(\pi x)\sin(\pi x)\,dx = \frac{1}{\pi}\int u\,du = \frac{1}{\pi}\frac{u^2}{2} + c = \frac{1}{2\pi}\sin^2 \pi x$

$u = \sin(\pi x)$

$du = \cos(\pi x)\pi\,dx$

$\int_{-1}^{1} \cos(\pi x)\sin(\pi x)\,dx = \left[\frac{1}{2\pi}\sin^2(\pi x)\right]_{-1}^{1}$

$= \frac{1}{2\pi}\sin^2 \pi - \frac{1}{2\pi}\sin^2(-\pi)$

$= 0 - 0 = 0$

3.5

$c_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(x) f(x)\,dx \quad (p\ 42)$

$x^5 = c_0P_0(x) + c_1P_1(x) + c_2P_2(x) + c_3P_3(x) + c_4P_4(x) + c_5P_5(x)$

$c_0 = \frac{1}{2}\int_{-1}^{1} x^5\,dx = 0 \qquad \left[\frac{x^6}{6}\right]_{-1}^{1} = \frac{1}{6} - \frac{1}{6} = 0$

$c_1 = \frac{3}{4}\int_{-1}^{1} 5x^4(x^2-1)\,dx = \cancel{-\frac{18}{7}}\ \frac{3}{7}$

$\int 5x^6 - 5x^4\,dx = \left[\frac{5}{7}x^7\right]_{-1}^{1} + \left[x^5\right]_{-1}^{1} = \frac{10}{7} + 2 = \frac{24}{7}$

$c_2 = \frac{5}{8\cdot 2!}\int_{-1}^{1} 20x^3(x^2-1)^2\,dx = 0$

$\int 20x^7 - 40x^5 + 20x^3\,dx = \left[\frac{20}{8}x^8\right]_{-1}^{1} - \left[\frac{40}{6}x^6\right]_{-1}^{1} + \left[\frac{20}{4}x^4\right]_{-1}^{1} = 0$

$c_3 = \frac{7}{16\cdot 3!}\int_{-1}^{1} 60x^2(x^2-1)^3\,dx = \cancel{8\ldots}\ 4/9$

$\int 60x^8 - 180x^6 + 180x^4 - 60x^2 = \left[\frac{20}{3}x^9\right]_{-1}^{1} - \left[\frac{180}{7}x^7\right]_{-1}^{1} + \left[35x^5\right]_{-1}^{1} - \left[20x^3\right]$

$c_4 = \frac{9}{32\cdot 4!}\int_{-1}^{1} 120x(x^2-1)^4\,dx = 0$

$\int 120x^9 - 480x^7 + 720x^5 - 480x^3 + 120x = 0$

$c_5 = \frac{11}{64\cdot 5!}\int_{-1}^{1} 120(x^2-1)^5\,dx = \frac{8}{63}$

$c_k = 0$ voor $k > 5$

$\int 120x^{10} - 600x^8 + 1200x^6 - 1200x^4 + 600x^2 - 120\,dx$

$= \left[\frac{120}{11}x^{11}\right]_{-1}^{1} - \left[\frac{200}{3}x^9\right]_{-1}^{1} + \left[\frac{1200}{7}x^7\right]_{-1}^{1} - \left[240x^5\right]_{-1}^{1} + \left[200x^3\right]_{-1}^{1} - \left[120x\right]_{-1}^{1}$

$= -\frac{20480}{231}$

3.10

$$\left(\begin{array}{cccc|cccc} 2 & 0 & \frac{2}{3} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_4 \mapsto \frac{2}{3} R_4 - \frac{2}{5} R_2]{R_3 \mapsto R_3 - \frac{1}{3} R_1}$$

$$\left(\begin{array}{cccc|cccc} 2 & 0 & 2/3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 2/5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 8/45 & 0 & -1/3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 16/525 & 0 & -2/5 & 0 & 2/3 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/5 & 0 & 3/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -15/8 & 0 & 45/8 & 0 \\ 0 & 0 & 0 & 1 & 0 & -105/8 & 0 & 175/8 \end{array}\right) \begin{array}{l} R_2 \mapsto R_2 - \frac{3}{5} R_4 \\ R_1 \mapsto R_1 - \frac{1}{3} R_3 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 9/8 & 0 & -15/8 & 0 \\ 0 & 1 & 0 & 0 & 0 & 75/8 & 0 & -105/8 \\ 0 & 0 & 1 & 0 & -15/8 & 0 & 45/8 & 0 \\ 0 & 0 & 0 & 1 & 0 & -105/8 & 0 & 175/8 \end{array}\right)$$

$\Rightarrow$ De matrix is inverteerbaar en de inverse is

$$A^{-1} = \begin{pmatrix} 9/8 & 0 & -15/8 & 0 \\ 0 & 75/8 & 0 & -105/8 \\ -15/8 & 0 & 45/8 & 0 \\ 0 & -105/8 & 0 & 175/8 \end{pmatrix}$$

## OEFENZITTING

3.2

$$\int_{-1}^{1} P_3(x)\,dx = \int_{-1}^{1} (ax^3 + bx^2 + cx + d)\,dx = 0$$

$$\Rightarrow \left[\frac{a}{4}x^4\right]_{-1}^{1} + \left[\frac{b}{3}x^3\right]_{-1}^{1} + \left[\frac{c}{2}x^2\right]_{-1}^{1} + \left[dx\right]_{-1}^{1} = 0$$

$$\Rightarrow \frac{b}{3} + \frac{b}{3} + d + d = 0 \Rightarrow \frac{2}{3}b + 2d = 0$$

$$\int_{-1}^{1} (ax^4 + bx^3 + cx^2 + dx)\,dx = 0$$

$$\Rightarrow \left[\frac{a}{5}x^5\right]_{-1}^{1} + \left[\frac{b}{4}x^4\right]_{-1}^{1} + \left[\frac{c}{3}x^3\right]_{-1}^{1} + \left[\frac{d}{2}x^2\right]_{-2}^{1} = 0$$

$$\Rightarrow \frac{a}{5} + \frac{a}{5} + \frac{c}{3} + \frac{c}{3} = 0 \Rightarrow \frac{2}{5}a + \frac{2}{3}c = 0$$

$$\int_{-1}^{1} (ax^5 + bx^4 + cx^3 + dx^2)\,dx = 0$$

$$\Rightarrow \left[\frac{a}{6}x^6\right]_{-1}^{1} + \left[\frac{b}{5}x^5\right]_{-1}^{1} + \left[\frac{c}{4}x^4\right]_{-1}^{1} + \left[\frac{d}{3}x^3\right]_{-1}^{1} = 0$$

$$\Rightarrow \frac{2b}{5} + \frac{2}{3}d = 0$$

$$\begin{cases} \frac{2}{3}b + 2d = 0 \\ \frac{2}{5}a + \frac{2}{3}c = 0 \\ \frac{2b}{5} + \frac{2}{3}d = 0 \\ a + b + c + d = 1 \end{cases}$$

$a = 2{,}5$

$b = 0$

$c = -1{,}5$

$d = 0$

$$\frac{5}{2}x^3 - \frac{3}{2}x = 0$$

3.6

$$\cancel{c_k = \frac{(-1)^k (2k+1)}{2^{k+1}\, k!} \int_{-1}^{1} \left(\frac{d^k}{dx^k} f(x)\right)(x^2-1)^k \, dx}$$

$$c_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) f(x)\, dx$$

$$c_0 = \frac{1}{2}\left(\int_{-1}^{0} 0\, dx + \int_0^1 x\, dx\right)$$

$$= \frac{1}{2}\ \frac{1}{2} = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\left(\int_{-1}^{0} x\cdot 0\, dx + \int_0^1 x^2\, dx\right)$$

$$= \frac{3}{2}\ \frac{1}{3} = \frac{1}{2}$$

$$\frac{3}{2}x^3 - \frac{1}{2}x$$

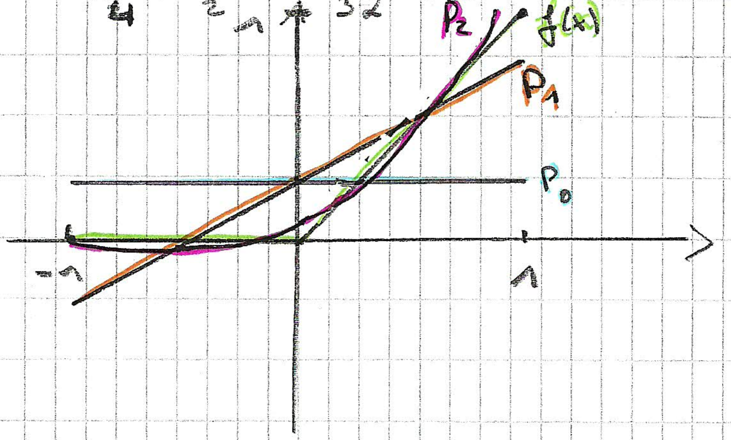
$$c_2 = \frac{5}{2}\left(\int_{-1}^{0} \frac{1}{2}(3x^2-1)\,0\, dx + \int_0^1 \frac{1}{2}(3x^2-1)\cdot x\, dx\right)$$

$$= \frac{5}{2}\left(\left[\frac{3}{8}x^4\right]_0^1 - \left[\frac{1}{4}x^2\right]_0^1\right)$$

$$= \frac{5}{2}\left(\frac{3}{8} - \frac{1}{4}\right) = \frac{5}{2}\ \frac{1}{8} = \frac{5}{16}$$

$$f(x) = \frac{1}{4} + \frac{1}{2}x + \frac{5}{16}\ \frac{1}{2}(3x^2-1) + \dots$$

$$= \frac{1}{4} + \frac{1}{2}x + \frac{5}{32}(3x^2-1) + \dots$$



3.7

$$c_0 = \frac{1}{2}\int_{-1}^{1} 1\, e^{itx}\,dx = \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_{-1}^{1}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad = \frac{1}{2}\left(\frac{e^{it}}{it} - \frac{e^{-it}}{it}\right)$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \qquad = \frac{1}{2it}\left(e^{it}-e^{-it}\right)$$

$$= \frac{1}{t}\sin t$$

$$(2\cdot 0+1)\, i^0\, j_0(t) = \frac{1}{t}\sin t$$

$$\Rightarrow j_0(t) = \frac{1}{t}\sin t$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} x\, e^{itx}\,dx$$

$$u = x \qquad du = dx$$
$$dv = e^{itx}dx \qquad v = \frac{e^{itx}}{it}$$

$$\frac{3}{2}\left[\frac{x\, e^{itx}}{it}\right]_{-1}^{1} - \int_{-1}^{1}\frac{e^{itx}}{it}\,dx$$

$$= \frac{3}{2}\left(-\left[\frac{e^{itx}}{(it)^2}\right]_{-1}^{1} + \frac{e^{it}}{it} + \frac{e^{-it}}{it}\right)$$

$$= \frac{3}{2}\left(\frac{e^{it}}{t^2} - \frac{e^{-it}}{t^2} + \frac{2}{it}\cos t\right)$$

$$\frac{3}{2}\left(\frac{2i}{t^2}\sin t + \frac{2}{it}\cos t\right)$$

$$\frac{3}{2}\left(\frac{2}{t}\left(\frac{i}{t}\sin t + \frac{1}{i}\cos t\right)\right)$$

$$(2+1)i \quad j_1(t) = \frac{3}{\cancel{2}}\,\frac{\cancel{2}}{t}\left(\frac{i}{t}\sin t + \frac{1}{i}\cos t\right)$$

$$j_1(t) = \frac{1}{ti}\left(\frac{i}{t}\sin t + \frac{1}{i}\cos t\right)$$

$$j_1(t) = \frac{1}{t}\left(\frac{1}{t}\sin t - \cos t\right)$$

3.11

a) $A = \begin{pmatrix} \int dx & \int x\,dx \\ \int x\,dx & \int x^2 dx \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} e - e^{-1} \\ 2e^{-1} \end{pmatrix}$ $\quad \int_{-1}^{1} e^x dx = [e^x]_{-1}^{1} = e - e^{-1}$

$\int_{-1}^{1} e^x x\,dx = [e^x x]_{-1}^{1} - \int_{-1}^{1} e^x dx$

$u = x \qquad du = dx$

$dv = e^x dx \qquad v = e^x$

$= e + e^{-1} - e + e^{-1} = 2e^{-1}$

$\to$ opl: $\frac{1}{2}(e - \frac{1}{e}) + \frac{3}{e}x = p_1(x)$

d) gebruik stelling 3.7.2 om $p_1$ en $p_2$ te berekenen mbv Legendre veelterm

$c_0 = \frac{1}{2}\int_{-1}^{1} e^x dx = \frac{1}{2}(e^1 - e^{-1})$

$c_1 = \frac{3}{2}\int_{-1}^{1} e^x x\,dx = 3e^{-1}$

$\to$ opl: $\frac{1}{2}(e - \frac{1}{e}) + 3\frac{1}{e}x$

b) $p_2(x) = \left(\frac{-3}{4}e + \frac{33}{4}e^{-1}\right) + \frac{3}{e}x + \left(\frac{15}{4}e - \frac{105}{4}e^{-1}\right)x^2$

$\Rightarrow$ ook met methode (a) en d))

# Fourierreeksen

## Ter herinnering: formules van Simpson

Bij het uitrekenen van Fouriercoëfficiënten kunnen de volgende formules van pas komen:

$$\begin{aligned}
\sin(x+y)+\sin(x-y) &= 2\sin x\cos y\\
\sin(x+y)-\sin(x-y) &= 2\cos x\sin y\\
\cos(x+y)+\cos(x-y) &= 2\cos x\cos y\\
\cos(x+y)-\cos(x-y) &= -2\sin x\sin y
\end{aligned}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

## Fourierreeks van een functie

Zij $f$ een functie op $]-l,l[$. Dan is de Fourierreeks van $f$ gegeven door

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right).$$

De Fouriercoëfficienten $a_n$ en $b_n$ zijn gelijk aan

$$\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx & n&=0,1,2,\ldots\\
b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx & n&=1,2,\ldots
\end{aligned}$$

Indien $f$ en $f'$ stuksgewijs continu zijn op $]-l,l[$, dan convergeert de Fourierreeks van $f$ naar $f(x)$ in elk punt $x$ waar $f$ continu is. In een punt waar $f$ niet continu is convergeert de Fourierreeks naar

$$\frac{f(x+)+f(x-)}{2},$$

waarbij $f(x-)$ en $f(x+)$ de linker-en rechterlimiet van $f$ in $x$ zijn.

## Fourierreeksen van even en oneven functies

Het kan vaak veel rekenwerk besparen om voor je begint even te controleren of een functie even (dwz. $f(x)=f(-x)$) of oneven (dwz. $f(-x)=-f(x)$) is. In dat geval vereenvoudigen de formules voor de coëfficiënten immers tot

| even functie op $]-l,l[$ | oneven functie op $]-l,l[$ |
|---|---|
| $a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$ | $a_n=0$ |
| $b_n=0$ | $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ |

Dergelijke Fourierreeksen met enkel cosinus -of sinustermen noemen we ook wel **Fourier-cosinus**, respectievelijk **Fourier-sinus** reeksen.

2

THUIS

## Oefeningen hoofdstuk 4

4.1

$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$

$$\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \frac{1}{2}\left(\cos\left(\frac{n\pi x}{l} - \frac{m\pi x}{l}\right) - \cos\left(\frac{n\pi x}{l} + \frac{m\pi x}{l}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left((n-m)\frac{\pi x}{l}\right) - \cos\left((n+m)\frac{\pi x}{l}\right)\right)$$

$$\frac{1}{2}\int_{-l}^{l}\cos\left((n-m)\frac{\pi x}{l}\right)dx - \frac{1}{2}\int_{-l}^{l}\cos\left((n+m)\frac{\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\frac{l}{\pi(n-m)}\sin\left((n-m)\frac{\pi x}{l}\right)\right]_{-l}^{l} - \frac{1}{2}\left[\frac{l}{\pi(n+m)}\sin\left((n+m)\frac{\pi x}{l}\right)\right]_{-l}^{l}$$

sin$(n\pi) = 0$ met $n \in \mathbb{Z}$

$$= \frac{1}{2}\left(\frac{l}{\pi(n-m)}\underbrace{\sin\left((n-m)\pi\right)}_{=0} - \frac{l}{\pi(n-m)}\underbrace{\sin\left((n-m)(-\pi)\right)}_{=0}\right)$$

$$-\frac{1}{2}\left(\frac{l}{\pi(n+m)}\underbrace{\sin\left((m+n)\pi\right)}_{=0} - \frac{l}{\pi(n-m)}\underbrace{\sin\left((m+n)(-\pi)\right)}_{=0}\right)$$

$$= 0$$

4.2

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$

$$\cos\frac{n\pi x}{l}\sin\frac{mx\pi}{l} = \frac{1}{2}\left(\sin\left(\frac{(m+n)x\pi}{l}\right) + \sin\left(\frac{(m-n)x\pi}{l}\right)\right)$$

$$\frac{1}{2}\int_{-l}^{l}\sin\left(\frac{(n+m)x\pi}{l}\right)dx + \frac{1}{2}\int_{-l}^{l}\sin\left(\frac{(m-n)x\pi}{l}\right)dx$$

$$= \frac{1}{2}\left[-\frac{l}{(n+m)\pi}\cos\frac{(n+m)x\pi}{l}\right]_{-l}^{l} + \frac{1}{2}\left[\frac{-l}{(m-n)\pi}\cos\frac{(m-n)x\pi}{l}\right]_{-l}^{l}$$

$\cos(n\pi) = \cos(-n\pi)$

$$= \frac{1}{2}\left(\frac{-l}{(n+m)\pi}\cos\left((n+m)\pi\right) + \frac{l}{(n+m)\pi}\cos\left((n+m)(-\pi)\right)\right)$$

$$+\frac{1}{2}\left(\frac{-l}{(m-n)\pi}\cos\left((m-n)\pi\right) + \frac{l}{(m-n)\pi}\cos\left((m-n)(-\pi)\right)\right)$$

$$= 0$$

4.11

b) fourierreeks op interval $[-\pi,\pi]$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\underbrace{\cos(0\cdot x)}_{=1}dx$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}\left(-(-1)-(-(1))\right)$
$= 2/\pi$

$a_n = \frac{1}{\pi}\int_0^{\pi}\underbrace{\sin x\cos(nx)}_{=\frac{1}{2}(\sin((n+1)x)+\sin((1-n)x))}dx$

$= \frac{1}{2\pi}\left(\left[\frac{-1}{n+1}\cos((n+1)x)\right]_0^{\pi} + \left[\frac{-1}{(1-n)}\cos((1-n)x)\right]_0^{\pi}\right)$

$= \frac{1}{2\pi}\left(\frac{-1}{n+1}\underbrace{\cos((n+1)\pi)}_{\substack{n=\text{oneven}:\,1\\ n=\text{even}:\,-1}} + \frac{1}{n+1}\underbrace{\cos((n+1)(0))}_{=1} - \frac{1}{(1-n)}\underbrace{\cos((1-n)\pi)}_{\substack{n=\text{even}:\,-1\\ n=\text{oneven}:\,1}} + \frac{1}{(1-n)}\underbrace{\cos((1-n)(0))}_{=1}\right)$

$= \frac{1}{2\pi}\left(\frac{+1}{n+1} + \frac{1}{n-1} - (-1)^{n+1}\frac{1}{n+1} - (-1)^{n+1}\frac{1}{1-n}\right) = \frac{1}{2\pi}\left(\frac{1}{n+1}\left(1-(-1)^{n+1}\right) + \frac{1}{1-n}\left(1-(-1)^{n+1}\right)\right)$

$a_n = \frac{2n}{\pi(1-(2n)^2)} = \frac{2n}{\pi(1-4n^2)}$

stel $n$ = oneven : $a_n = 0$

$n$ = even: $a_n = \frac{n}{\pi(1-n^2)}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \frac{1}{\pi}\int_0^{\pi}\underbrace{\sin x\sin nx}_{=\frac{1}{2}(\cos((1-n)x)-\cos((1+n)x))}dx$

$= \frac{1}{2\pi}\left(\left[\frac{1}{1-n}\sin(1-n)x\right]_0^{\pi} - \left[\frac{1}{1+n}\sin(1+n)x\right]_0^{\pi}\right)$

$= \frac{1}{2\pi}\left(\frac{1}{1-n}\underbrace{\sin(1-n)\pi}_{=0} - \frac{1}{1-n}\underbrace{\sin 0}_{=0} - \frac{1}{1+n}\underbrace{\sin(1+n)\pi}_{=0} + \frac{1}{1+n}\underbrace{\sin 0}_{=0}\right)$

$= 0$ enkel voor $n \neq 1$

als $n=1$ : $\frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx$
$= \frac{1}{\pi}\left[\frac{1}{2}x - \frac{\sin 2x}{4}\right]_0^{\pi}$
$= \frac{1}{2}$

fourierreeks: $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

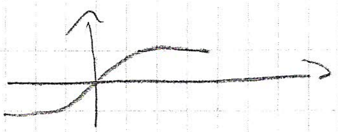
$\cancel{\frac{1}{\pi} + \sum^{\infty}\frac{(1-(-1)^{n+1})\,2n}{\pi(1-n^2)}\cos nx}$

oplossing: $\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{4k^2-1}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

4.14

$f$ = oneven

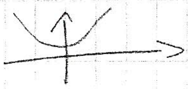


$f(-x) = -f(x)$

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

$$= -\int_{0}^{l} f(x)\,dx + \int_{0}^{l} f(x)\,dx = 0$$

$f$ = even



$f(-x) = f(x)$

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx$$

$$= \int_{0}^{l} f(x)\,dx + \int_{0}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx$$

4.15

stel $f_e = \frac{1}{2}\left[f(x) + f(-x)\right]$ en $f_o = \frac{1}{2}\left[f(x) - f(-x)\right]$

bij even: $f(-x) = f(x) \Rightarrow \frac{1}{2}\left(f(x) + f(-x)\right) = f(x)$ waarvoor even

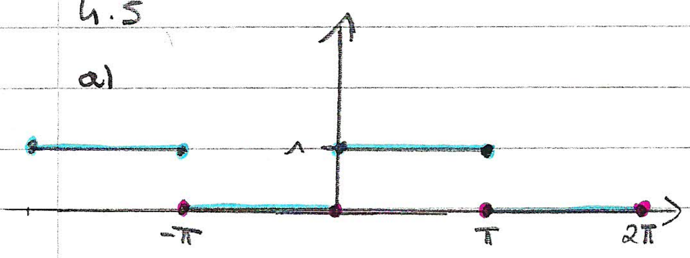
bij oneven: $f(-x) = -f(x) \Rightarrow \frac{1}{2}\left(f(x) - f(-x)\right) = f(x)$ waarvoor oneven

$\Rightarrow f_e + f_o = f(x)$

OEFENZITTING

4.5

a)



b) $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 0 \, dx$

$= \frac{1}{\pi} \int_0^{\pi} 1 \, dx = \frac{1}{\pi} [x]_0^{\pi} = 1$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n \pi x}{\pi} dx$

$= \frac{1}{\pi} \int_0^{\pi} 1 \cos nx \, dx$

$= \frac{1}{\pi} \left[ \frac{1}{n} \sin nx \right]_0^{\pi} = \frac{1}{\pi n} \overbrace{\sin n\pi}^{=0} = 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{\pi} dx$

$= \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx$

$\hookrightarrow n = \text{even} : \cos n\pi = 1$
$n = \text{oneven} : \cos n\pi = -1$

$= \frac{1}{\pi} \left[ -\frac{1}{n} \cos nx \right]_0^{\pi} = \frac{-1}{\pi n} \cos n\pi + \frac{1}{\pi n} \underbrace{\cos 0}_{1}$

$= (-1)^n \cdot \frac{-1}{\pi n} + \frac{1}{\pi n}$

$f(x) \approx \frac{1}{2} + \sum_{n=1}^{\infty} \left( (-1)^n \cdot \frac{-1}{\pi n} + \frac{1}{\pi n} \right) \sin nx$

stel $n = \text{even} : = 0$

$n = \text{oneven} : \frac{2}{\pi n}$

$f(x) \approx \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{2}{\pi (2n+1)} \sin((2n+1)x) \right)$

$\approx \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin((2n+1)x)$

c) $\frac{f(0+)+f(0-)}{2} = \frac{1}{2}$

$\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \underbrace{\sin(2n+1)\cdot 0}_{=0} = \frac{1}{2}$

klopt

$\frac{f(\pi+)+f(\pi-)}{2} = \frac{1}{2}$

$\frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{2n+1} \underbrace{\sin(2n+1)\cdot\pi}_{=0} = \frac{1}{2}$

klopt

4.6

$x = \frac{\pi}{2}$ $\quad \sin\left((2n+1)\,\pi/2\right)$ $\quad n =$ even: $= 1$

$n =$ oneven: $= -1$

4.7

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2\,dx = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi(2n+1)}\right)^2$$

$$\frac{1}{\pi}\int_0^{\pi} 1\,dx = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{1}{\pi}[x]_0^{\pi} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$$

$$\left(1-\frac{1}{2}\right)\cdot\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$$

4.11

c) $\cos^2 x = \frac{1+\cos 2x}{2}$

= fourierreeks : som van cos en sin

4.17

a) $f(x) = l - x \qquad 0 \leq x \leq l$

$f(x) = -l + x \qquad -l \leq x \leq 0$

oneven functie : $a_n = 0$

$b_n = \frac{2}{l} \int_0^l (l-x) \sin \frac{n\pi x}{l} dx$

$u = l - x \qquad du = -dx$

$dv = \sin \frac{n\pi x}{l} dx \qquad v = -\frac{l}{n\pi} \cos \frac{n\pi x}{l}$

$= \frac{2}{l} \left( \left[ (l-x)\left(\frac{-l}{n\pi} \cos \frac{n\pi x}{l}\right) \right]_0^l - \int_0^l \frac{l}{n\pi} \cos \frac{n\pi x}{l} dx \right.$

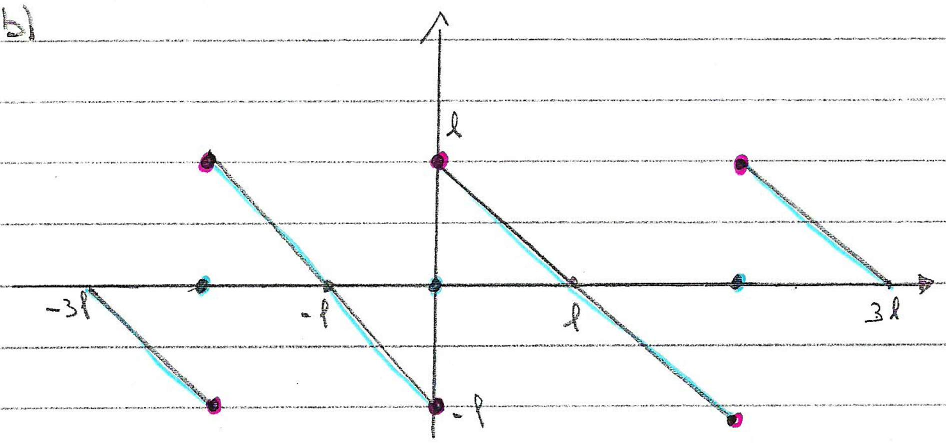
$= \frac{2}{l} \left( -l \frac{-l}{n\pi} \cos 0 \right) - \frac{l}{n\pi} \left[ \frac{l}{n\pi} \sin \frac{n\pi x}{l} \right]_0^l \Big)$

$= \frac{2}{l} \left( \frac{l^2}{n\pi} - \frac{l}{n\pi} \left( \frac{l}{n\pi} \underbrace{\sin n\pi}_{=0} - \frac{l}{n\pi} \underbrace{\sin 0}_{=0} \right) \right)$

$= \frac{2l}{n\pi}$

$f(x) \approx \sum_{n=1}^{\infty} \frac{2l}{n\pi} \sin \frac{n\pi x}{l}$

b)



4.8

$f(x) = \text{even} \Rightarrow b_n = 0$

$$a_0 = \frac{2\omega}{\pi}\int_0^{\pi/\omega} |\sin \omega x| \cos \frac{0\pi\omega x}{\pi}\, dx$$

$0 \leq \sin 0$ tot $\sin \pi$
$\Rightarrow$ altijd positief

$$a_0 = \frac{2\omega}{\pi}\int_0^{\pi/\omega} \sin \omega x \underbrace{\cos 0\omega x}_{1}\, dx$$

$$= \frac{2\omega}{\pi}\left[-\frac{1}{\omega}\cos \omega x\right]_0^{\pi/\omega}$$

$$= \frac{2\omega}{\pi}\left[-\frac{1}{\omega}\underbrace{\cos \pi}_{-1} + \frac{1}{\omega}\underbrace{\cos 0}_{1}\right]$$

$$= \frac{2\omega}{\pi}\left(\frac{2}{\omega}\right) = \frac{4}{\pi}$$

$$a_n = \frac{2\omega}{\pi}\int_0^{\pi/\omega} \sin \omega x \cos n\omega x$$

$$= \frac{\omega}{\pi}\int_0^{\pi/\omega}\Big(\underbrace{\sin(\omega x + n\omega x)}_{\sin((1+n)\omega x)} + \underbrace{\sin(\omega x - n\omega x)}_{\sin((1-n)\omega x)}\Big)\, dx$$

$$= \frac{\omega}{\pi}\left(\left[\frac{-1}{(1+n)\omega}\cos((1+n)\omega x)\right]_0^{\pi/\omega} + \left[\frac{-1}{(1-n)\omega}\cos((1-n)\omega x)\right]_0^{\pi/\omega}\right)$$

$\cos 0 = 1$

$\cos(1+n)\pi$ $n = \text{even}: -1$
$n = \text{oneven}: 1$

$$= \frac{\omega}{\pi}\left(\frac{(-1)^n}{(1+n)\omega} + \frac{1}{(1+n)\omega} + \frac{(-1)^n}{(1-n)\omega} + \frac{1}{(1-n)\omega}\right)$$

$n = \text{even}$ $\quad \frac{1}{(1+n)\omega} \rightarrow \frac{2}{(1+n)\omega}$

$n = \text{oneven}$ $\quad \frac{-1}{(1+n)\omega} \rightarrow = 0$

$$\begin{cases} = 0 : n = \text{oneven} \\ = \frac{\omega}{\pi}\left(\frac{2}{(1+n)\omega} + \frac{2}{(1-n)\omega}\right) = \frac{2}{\pi}\left(\frac{1}{1+n} + \frac{1}{1-n}\right) \end{cases}$$

$$= \frac{2}{\pi}\left(\frac{1-n+1+n}{1^2 - n^2}\right) = \frac{4}{\pi(1-n^2)}$$

$n$ vervangen door $2n$ $\Rightarrow$ altijd even

$$= \frac{4}{\pi(1-4n^2)}$$

$$f(x) \approx \frac{4}{2\pi} + \sum_{n=1}^{\infty}\frac{4}{\pi(1-4n^2)}\cos\frac{2n\pi x\omega}{\pi}$$

$$\approx \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\omega x}{1-4n^2}$$

4.9

Stel $x = 0$ $\qquad \cos 0 = 1$

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = f(0) = |\sin 0| = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{2}{\pi}\frac{\pi}{4} = \frac{1}{2}$$

4.10

$$\frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} \sin^2 \omega x \, dx = \frac{16}{2\pi^2} + \sum_{n=1}^{\infty} \left( \frac{4^2}{\pi^2 (1-4n^2)^2} \right)$$

hoofdstuk 5

# Fouriertransformaties

## Fouriertransformatie, goniometrische vorm

Zij $f$ een functie die 'voor grote waarden snel genoeg naar nul gaat' (een voldoende voorwaarde is $\int_{-\infty}^{\infty} |f(x)|dx < \infty$).
De **Fouriertransformatie (in de goniometrische vorm)** is de overgang van $f(x)$ naar de twee functies $u(y)$ en $v(y)$ voor $y > 0$ door

$$u(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(xy)dx; \qquad v(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(xy)dx$$

De **inverse Fouriertransformatie** vertelt ons hoe we $f(x)$ weer terug kunnen vinden uit de functies $u(y)$ en $v(y)$:

$$f(x) = \int_0^{\infty} u(y)\cos(xy)dy + \int_0^{\infty} u(y)\cos(xy)dy,$$

met het gebruikelijke voorbehoud dat we $f(x)$ moeten vervangen door $\frac{f(x+)+f(x-)}{2}$ in punten waar $f$ niet continu is.

### Even en oneven functies

Net als bij Fourierreeksen is het ook in het geval van Fouriertransformaties nuttig om even na te gaan of je met een even of een oneven functie werkt. In deze gevallen vereenvoudigen de formules immers tot

| $f$ is een even functie | $f$ is een oneven functie |
|---|---|
| Fouriergetransformeerde: $u(y) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos(xy)dx$ <br> $v(y) = 0$ | Fouriergetransformeerde: $u(y) = 0$ <br> $v(y) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin(xy)dx$ |
| Inverse Fouriertransformatie: $f(x) = \int_0^{\infty} u(y)\cos(xy)dy$ | Inverse Fouriertransformatie: $f(x) = \int_0^{\infty} v(y)\sin(xy)dy$ |

(Probeer dit zelf aan te tonen, zie ook oef. 5.3)
Deze integraalvoorstelling voor $f(x)$ noemen we ook wel een **Fouriercosinusintegraal**, resp. **Fouriersinusintegraal**

*Merk op:* we kunnen dit gebruiken om een fouriertransformatie te geven voor functies die enkel gedefinieerd zijn voor $x \geqslant 0$. Concreet, gegeven een functie $f(x)$, gedefinieerd voor $x \geqslant 0$, kunnen we deze uitbreiden tot

$$f^{\text{even}} = \begin{cases} f(x) & x \geqslant 0 \\ f(-x) & x < 0 \end{cases} \quad \text{of} \quad f^{\text{oneven}} = \begin{cases} f(x) & x \geqslant 0 \\ -f(-x) & x < 0 \end{cases}$$

Dit zijn even of oneven functies, dus de integraalvoorstelling van $f^{\text{even}}$ en $f^{\text{oneven}}$ is een Fouriercosinusintegraal of een Fouriersinusintegraal.

## Fouriertransformaties, exponentiële vorm

Zij $f$ een continue functie waarvoor $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. Dan geldt (onder bepaalde voorwaarden)

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)e^{ixy}dy, \\ g(y) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ixy}dx. \end{aligned}$$

De functie $g(y)$ heet de **Fourier-getransformeerde** van $f(x)$, en $f(x)$ is de **invers Fourier-getransformeerde** van $g(y)$.

(2)

THUIS

Oefeningen hoofdstuk 5

5.1

$u(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(xy) dx \qquad y = \frac{n\pi}{l}$

$u(y) = \frac{1}{\pi} \int_{-1}^{1} \underbrace{\overbrace{(1-|x|)}^{\text{even}} \overbrace{\cos xy}^{\text{even}} dx}_{\text{even}}$

$f(x) = 1 - |x|$

$f(-x) = 1 - |-x| = 1 - |x|$

$\Rightarrow$ even

$= \frac{2}{\pi} \int_0^1 (1-|x|) \cos xy \, dx$

$|x|$ met $0 \leq x \leq 1 \Rightarrow |x| = x$

$= \frac{2}{\pi} \int_0^1 \cos xy \, dx - \frac{2}{\pi} \int_0^1 x \cos xy \, dx$

$u = x \qquad du = dx$

$dv = \cos xy\, dx \qquad v = \frac{1}{y} \sin xy$

$= \frac{2}{\pi} \left[\frac{1}{y} \sin xy\right]_0^1 - \frac{2}{\pi}\left[\frac{x}{y} \sin xy\right]_0^1 + \frac{2}{\pi} \int_0^1 \frac{1}{y} \sin xy \, dx$

$= \cancel{\frac{2}{\pi y} \sin y} - \frac{2}{\pi y} \underbrace{\sin 0}_{=0} - \cancel{\frac{2}{\pi y} \sin y} + 0 + \frac{2}{\pi y}\left[-\frac{1}{y} \cos xy\right]_0^1$

$= -\frac{2}{\pi y^2} \cos y + \frac{2}{\pi y^2} \underbrace{\cos 0}_{=1}$

$= \frac{2}{\pi y^2} - \frac{2}{\pi y^2} \cos y = \frac{2}{\pi y^2}(1 - \cos y)$ met $y = \frac{n\pi}{l}$

$v(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(xy) dx$

$v(y) = \frac{1}{\pi} \int_{-1}^{1} \underbrace{\overbrace{(1-|x|)}^{\text{even}} \overbrace{\sin(xy)}^{\text{oneven}} dx}_{\text{oneven}}$

$= 0$

5.3

a) $v(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\overbrace{f(x)}^{\text{even}} \overbrace{\sin(xy)}^{\text{oneven}}}_{\text{oneven}} dx$ oneven functie over interval $[-l, l] = 0$

$= 0$

$f(x) = \int_0^{\infty} u(y) \cos(xy) dx + \int_0^{\infty} \underbrace{v(y)}_{=0} \sin(xy) dx$

$= \int_0^{\infty} u(y) \cos(xy) dx$

b) $u(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \underbrace{\underbrace{f(x)}_{\text{oneven}} \underbrace{\cos(xy)}_{\text{even}}}_{\text{oneven}} dx = 0$

$f(x) = \int_0^{\infty} v(y) \sin(xy) dx$

5.9

a) $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{ixy}\, dy$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{C}{\sqrt{2\pi}\,(a+iy)}\, e^{ixy}\, dy$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C\, e^{ixy}}{(a+iy)}\, dy$

voor $x > 0 : |x| = x$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \underbrace{\int_{-\infty}^{0} 0^2\, dx}_{=0} + \int_0^{\infty} |C\, e^{-a|x|}|^2\, dx$$

$$= \int_0^{\infty} |C^2 e^{-ax2}|\, dx \quad C^2 > 0 \text{ en } e^{-ax2} > 0$$

$$= \int_0^{\infty} C^2 e^{-ax2}\, dx = C^2 \int_0^{\infty} e^{-2ax}\, dx$$

$$= C^2 \left[ \frac{-1}{2a} e^{-2ax} \right]_0^{\infty}$$

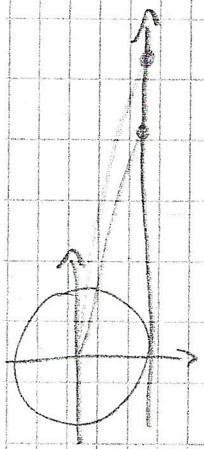
$$= C^2 \left( -\frac{1}{2a} \underbrace{e^{-\infty}}_{=0} + \frac{1}{2a} \underbrace{e^{0}}_{=1} \right)$$

$$= C^2 \frac{1}{2a} = 1$$

$$\Rightarrow C^2 = 2a \quad \Rightarrow C = \pm\sqrt{2a}$$

$$C > 0 : \; C = \sqrt{2a}$$

$\text{bgtan}\, \infty = \frac{\pi}{2}$



b) $g(y) = \frac{\sqrt{2a}}{\sqrt{2\pi}\,(a+iy)} = \frac{\sqrt{a}}{\sqrt{\pi}} \frac{1}{(a+iy)}$

$$\int_{-\infty}^{\infty} |g(y)|^2\, dy = \int_{-\infty}^{\infty} \left| \frac{\sqrt{a}}{\sqrt{\pi}} \frac{1}{(a+iy)} \right|^2 dy$$

$$= \int_{-\infty}^{\infty} \frac{a}{\pi} \left| \frac{1}{(a+iy)} \right|^2 dy = \int_{-\infty}^{\infty} \frac{a}{\pi} \left( \frac{a^2+y^2}{(a^2+y^2)^2} \right) dy$$

$$\frac{1}{a+iy} \cdot \frac{a-iy}{a-yi} = \frac{a-iy}{a^2-y^2i^2} = \frac{a-iy}{a^2+y^2}$$

$$= \frac{a}{a^2+y^2} - i \frac{y}{a^2+y^2}$$

$$\left| \frac{a}{a^2+y^2} - i \frac{y}{a^2+y^2} \right|^2 = \left( \sqrt{\frac{a^2}{(a^2+y^2)^2} + \frac{y^2}{(a^2+y^2)^2}} \right)^2$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2+y^2}\, dy$$

$$= \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2} \frac{1}{1+\left(\frac{y}{a}\right)^2}\, dy$$

$t = \frac{y}{a} \quad dt = \frac{1}{a}\, dy$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt$$

$$= \frac{1}{\pi} \left[ \text{bgtan} \frac{y}{a} \right]_{-\infty}^{\infty} = \frac{1}{\pi} \left( \frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$= \frac{1}{\pi} \left( \frac{2\pi}{2} \right) = 1$$

OEFENZITTING

5.4

even: $f(x) = f(-x)$

$$U(y) = \frac{2}{\pi} \int_0^\infty f(x) \cos(xy)\,dx$$

$$= \frac{2}{\pi} \overbrace{\int_0^\infty e^{-x} \cos(xy)\,dx}^{I}$$

$u = e^{-x}$ $du = -e^{-x}dx$

$dv = \cos(xy)$ $v = \frac{\sin(xy)}{y}$

$$= \frac{2}{\pi} \left( \overbrace{\left[ e^{-x} \frac{\sin(xy)}{y} \right]_0^\infty}^{=0} + \int_0^\infty \frac{\sin(xy)}{y} e^{-x} dx \right)$$

$e^{-\infty} = 0$ $e^0 = 1$ $\sin 0 = 0$

$u = e^{-x}$ $du = -e^{-x}$

$dv = \sin(xy)$ $v = \frac{-\cos xy}{y}$

$$= \frac{2}{\pi} \frac{1}{y} \left( \left[ e^{-x} \frac{-\cos(xy)}{y} \right]_0^\infty - \int_0^\infty e^{-x} \frac{\cos(xy)}{y} dx \right.$$

$\cos(0) = 1$

$$= \frac{2}{\pi y} \left( \left( +\frac{1}{y} \right) - \frac{1}{y} I \right)$$

$$= \frac{2}{\pi y^2} (1 - I) = \frac{2}{\pi} I$$

$$\Rightarrow \frac{2}{\pi y^2} = \frac{2}{\pi} I + \frac{2}{\pi y^2} I$$

$$\frac{2}{\pi y^2} = \left( \frac{2}{\pi} + \frac{2}{\pi y^2} \right) I$$

$$\frac{2}{\pi y^2} = \frac{2}{\pi} \left( 1 + \frac{1}{y^2} \right) I$$

$$\frac{2}{\pi y^2} \frac{\pi}{2} = \frac{y^2 + 1}{y^2} I$$

$$I = \frac{1}{\cancel{y^2}} \frac{\cancel{y^2}}{y^2 + 1}$$

$$U(y) = \frac{2}{\pi (y^2 + 1)}$$

$$f(x) = \int_0^\infty \frac{2}{\pi} \frac{1}{y^2+1} \cos(xy)\,dy$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos(xy)}{y^2+1} dy$$

5.6

bÿ ~~of~~ 5.4: $e^{-x} = f(x) = \int_0^\infty \frac{2}{\pi} \frac{\cos(xy)}{y^2+1}\, dy$

$$v(y) = \frac{2}{\pi}\frac{1}{y^2+1}$$

5.7

$$v(y) = \frac{2}{\pi}\int_0^\infty f(x)\sin(xy)\,dx$$

$$= \frac{2}{\pi}\int_0^1 \sin(xy)\,dx + \frac{2}{\pi}\int_1^\infty 0\,dx$$

$$= \frac{2}{\pi}\left[\frac{-\cos(xy)}{y}\right]_0^1 = \frac{2}{\pi}\frac{-\cos y}{y} + \frac{2}{\pi}\frac{\cos 0}{y}$$

$$= \frac{2}{\pi}\left(\frac{-\cos y}{y}+\frac{1}{y}\right) = \frac{2}{\pi y}(-\cos y + 1)$$

5.8

$$g(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-ixy}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \overbrace{\int_{-\pi/2}^{\pi/2}\cos(x)\, e^{-ixy}\,dx}^{I}$$

$du = \cos x\,dx \quad u = \sin(x)$
$v = e^{-ixy} \quad dv = -iy\,e^{-ixy}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\left(\left[e^{-ixy}\sin x\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\sin(x)\cdot iy\, e^{-ixy}\,dx\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(e^{-i\frac{\pi}{2}y} + e^{+i\frac{\pi}{2}y} + iy\int_{-\pi/2}^{\pi/2}\sin x\; e^{-ixy}\,dx\right)$$

$du = \sin x\,dx \quad u = -\cos x$
$v = e^{-xyi} \quad dv = -iy\,e^{-xyi}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\left(\cancel{1}\, e^{-i\frac{\pi}{2}y} + e^{iy\frac{\pi}{2}} + iy\left(\underbrace{\left[-\cos x\, e^{-xyi}\right]_{-\pi/2}^{\pi/2}}_{=0} - \int_{-\pi/2}^{\pi/2}\cos x\; iy\, e^{-xyi}\,dx\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(1 e^{-i\frac{\pi}{2}y} + e^{iy\frac{\pi}{2}} - i^2y^2 \underbrace{\int_{-\pi/2}^{\pi/2}\cos x\; e^{-xyi}\,dx}_{=I}\right)$$

$$\Rightarrow \cancel{\frac{1}{\sqrt{2\pi}}}\, I = \cancel{\frac{1}{\sqrt{2\pi}}}\left(e^{iy\frac{\pi}{2}} + e^{-i\frac{\pi}{2}y} + y^2 I\right)$$

$$\cos t = \frac{e^{it}+e^{-it}}{2}$$

$$(1-y^2)\,I = e^{-i\frac{\pi}{2}y} + e^{i\frac{\pi}{2}y}$$

$$I = \frac{e^{-i\frac{\pi}{2}y}+e^{i\frac{\pi}{2}y}}{1-y^2} = \frac{2\cos\frac{\pi}{2}y}{1-y^2}$$

$$g(y) = \frac{1\,\sqrt{2}\;\cancel{2}\cos\frac{\pi}{2}y}{\sqrt{\cancel{2}}\sqrt{\pi}\,\sqrt{2}\,(1-y^2)} = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\cos(\frac{\pi}{2}y)}{1-y^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{ixy}\, dy$$ $x =$ constante: kies $x = 0$

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{\cos(\frac{\pi}{2} y)}{1 - y^2}\, e^{0}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\cos(\pi/2\, y)}{1 - y^2}\, dy$$

$$\cos(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos(\frac{\pi}{2} y)}{1 - y^2}\, dy$$

$$\pi = \int_{-\infty}^{\infty} \frac{\cos(\frac{\pi}{2} y)}{1 - y^2}\, dy$$

②

THUIS

## Oefeningen hoofdstuk 6

6.2

$$\int_0^{\pi}\int_0^{R} e^{-r}\cos^2\theta\,\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi}\left[-\cos^2\theta\,\sin\theta\;e^{-r}\right]_0^R d\theta$$

$$= \int_0^{\pi}\left(-\cos^2\theta\,\sin\theta\;e^{-R} + \cos^2\theta\,\sin\theta\;\underbrace{e^{-0}}_{=1}\right)d\theta$$

$$= -e^{-R}\int_0^{\pi}\cos^2\theta\,\sin\theta\,d\theta + \int\cos^2\theta\,\sin\theta\,d\theta$$

$$= (1-e^{-R})\int_0^{\pi}\cos^2\theta\,\sin\theta\,d\theta \qquad u=\cos\theta \quad du=-\sin\theta\,d\theta$$

$$\int\cos^2\theta\,\sin\theta\,d\theta = -\int u^2\,du = -\frac{1}{3}u^3 = -\frac{1}{3}\cos^3\theta$$

$$= (1-e^{-R})\left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$\cos\pi = -1$ $\quad\cos 0 = 1$

$$= (1-e^{-R})\left(-\frac{1}{3}\cos^3(\pi) + \frac{1}{3}\cos^3 0\right)$$

$$= (1-e^{-R})\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{3}(1-e^{-R})$$

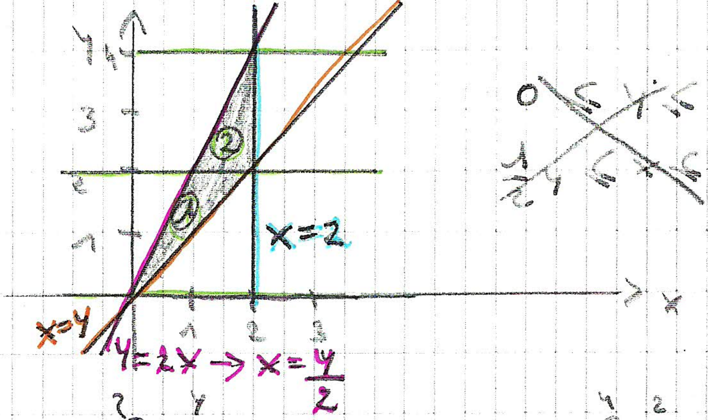
6.3

a) $0 \leq x \leq 2 \qquad x \leq y \leq 2x$

$$\int_0^4\int_{\frac{1}{2}y}^{y} f(x,y)\,dx\,dy$$

~~$0 \leq y \leq 4$~~

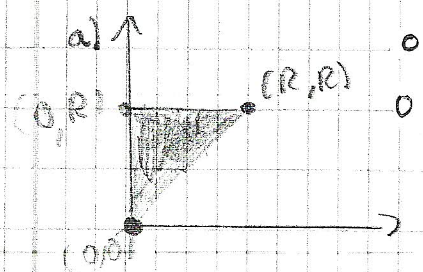
~~$\frac{1}{2}y \leq x \leq y$~~




$$\int_0^2\int_{\frac{y}{2}}^{y} f(x,y)\,dx\,dy \;① + \int_2^4\int_{\frac{y}{2}}^{2} f(x,y)\,dx\,dy \;②$$

6.4

a)




$0 \leqslant \cancel{x}\,y \leqslant R$

$0 \leqslant x \leqslant y$

$$\int_0^R \int_0^y e^{x/y}\,dx\,dy$$

$$= \int_0^R \left[ y e^{x/y} \right]_0^y dy$$

$$= \int_0^R \left( y e^{1} - y e^{0} \right) dy$$

$$= \int_0^R y\,(e-1)\,dy$$

$$= \left[ \frac{1}{2}(e-1)\,y^2 \right]_0^R$$

$$= \frac{1}{2}(e-1)R^2 - 0$$

$$= \frac{R^2}{2}(e-1)$$

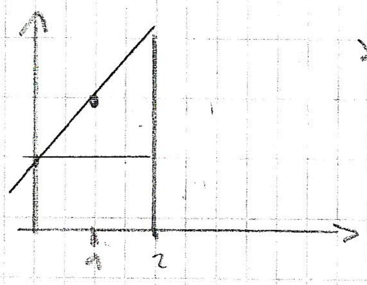
6.6

a) grondvlak: $0 \leqslant x \leqslant 2$ en $0 \leqslant y \leqslant 1$

$x - y = -1$

$x = -1 + y$

$$z = 2 + \frac{x}{2} - \frac{y}{2}$$




$$\int_0^2 \int_0^1 \left(2 + \frac{x}{2} - \frac{y}{2}\right) dy\,dx$$

$$= \int_0^2 \left[ 2y + \frac{xy}{2} - \frac{y^2}{4} \right]_0^1 dx$$

$$= \int_0^2 \left(2 + \frac{x}{2} - \frac{1}{4}\right) dx = \left[ 2x + \frac{x^2}{4} - \frac{1}{4}x \right]_0^2$$

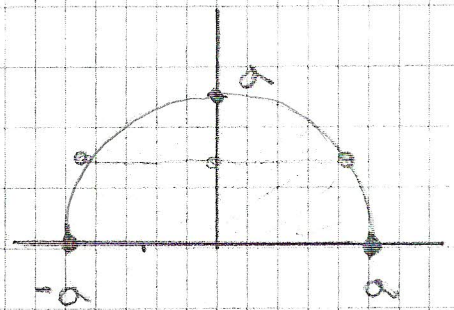
$$= 4 + 1 - \frac{1}{2} = \frac{9}{2}$$

OEFENZITTING

$$y = \sqrt{a^2 - x^2} \to x = \pm\sqrt{a^2 - y^2}$$

6.3

b)



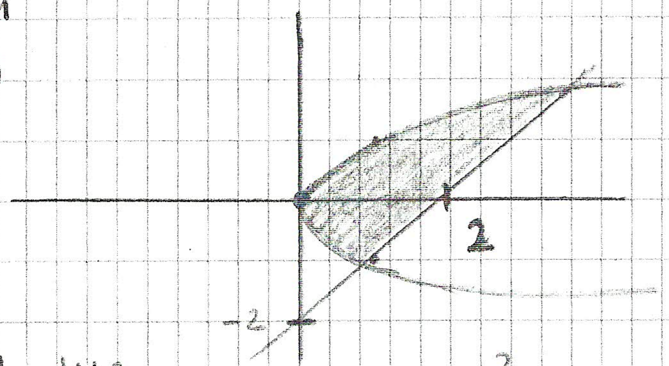

$$0 < y < a$$

$$-\sqrt{a^2-y^2} < x < \sqrt{a^2-y^2}$$

$$\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\, dx\, dy$$

6.4

e)




$$-1 < y < 2$$

$$y^2 < x < y+2$$

$$\int_{-1}^{2} \int_{y^2}^{y+2} xy\, dx\, dy = \int_{-1}^{2} \left[\frac{yx^2}{2}\right]_{y^2}^{y+2} dy$$

$$= \int_{-1}^{2} \left( \frac{y(y+2)^2}{2} - \frac{y y^4}{2} \right) dy$$

$$= \int_{-1}^{2} \frac{y}{2}(y^2+4y+4) - \frac{y^5}{2} ) dy$$

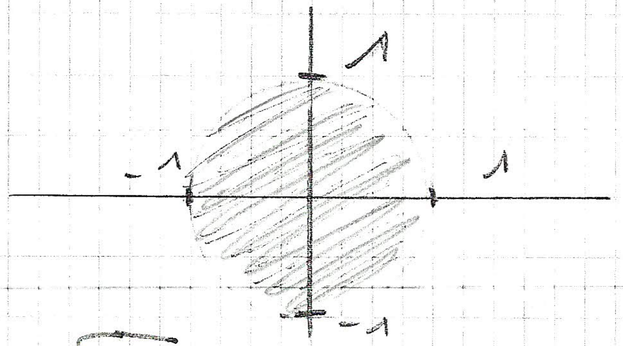
$$= \int_{-1}^{2} \left( \frac{y^3}{2} + 2y^2 + 2y - \frac{y^5}{2} \right) dy$$

$$= \left[\frac{y^4}{8}\right]_{-1}^{2} + \left[\frac{2}{3}y^3\right]_{-1}^{2} + \left[\frac{2}{2}y^2\right]_{-1}^{2} - \left[\frac{y^6}{12}\right]_{-1}^{2}$$

$$= 2 - \frac{1}{8} + \frac{16}{3} + \frac{2}{3} + 4 - 1 - \frac{16}{3} + \frac{1}{12} = \frac{45}{8} = 5{,}625$$

6.6

b)




$z = \pm\sqrt{1-x^2}$

$-1 < x < 1$

$-\sqrt{1-x^2} < y < \sqrt{1-x^2}$

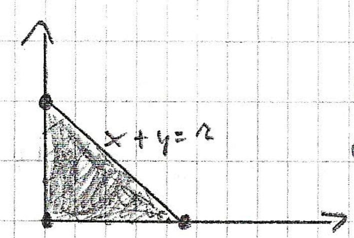
$$2 \cdot \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\, dx = 2 \int_{-1}^{1} \left[ y \cdot \sqrt{1-x^2} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 2 \int_{-1}^{1} \left( 1-x^2 + 1-x^2 \right) dx = 2 \int_{-1}^{1} (2-2x^2)\, dx$$

$$= 2 \left[ 2x \right]_{-1}^{1} - 2 \left[ \frac{2}{3} x^3 \right]_{-1}^{1}$$

$$= 4 + 4 - \frac{4}{3} - \frac{4}{3} = \frac{16}{3}$$

6.7




$0 < x < r$

$0 < y < r-x$

$$M = \int_0^r \int_0^{r-x} xy\, dy\, dx = \int_0^r \left[ \frac{x}{2} y^2 \right]_0^{r-x} dx$$

$$= \int_0^r \left( \frac{x}{2} (r-x)^2 \right) dx = \int_0^r \frac{x}{2} (r^2 - 2rx + x^2)\, dx$$

$$= \int_0^r \left( \frac{r^2 x}{2} - \frac{\cancel{2} r x^2}{\cancel{2}} + \frac{x^3}{2} \right) dx$$

$$= \left[ \frac{r^2}{2} \cdot \frac{x^2}{2} \right]_0^r - \left[ \frac{r}{3} x^3 \right]_0^r + \left[ \frac{x^4}{2 \cdot 4} \right]_0^r$$

$$= \frac{r^4}{4} - \frac{r^4}{3} + \frac{r^4}{8} = \frac{1}{24} r^4$$

6.7 $\bar{x} = \dfrac{24}{r^4} \displaystyle\int_0^r \int_0^{r-x} x^2 y \, dy \, dx$

$\bar{y} = \dfrac{24}{r^4} \displaystyle\int_0^r \int_0^{r-x} y^2 x \, dy \, dx$

6.13

a) $M = \displaystyle\int_0^r \int_0^r \rho_0 \, dy \, dx$

$= \displaystyle\int_0^r r \rho_0 \, dx$

$= r^2 \rho_0$

b) afstand tot de y-as is $|x|$ voor het punt $(x, y)$

$I = \displaystyle\int_0^r \int_0^r x^2 \rho_0 \, dy \, dx$

$= \displaystyle\int_0^r [x^2 y \rho_0]_0^r \, dx$

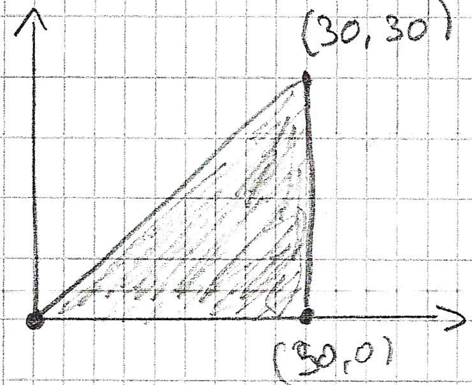
$= \displaystyle\int_0^r x^2 r \rho_0 \, dx$

$= [\frac{x^3}{3} r \rho_0]_0^r$

$= \dfrac{r^3 r \rho_0}{3} = \dfrac{r^4}{3} \rho_0$

c) $I_{y=x} = \displaystyle\int_0^r \int_0^r \frac{(x-y)^2}{2} \rho_0 \, dx \, dy = \rho_0 \frac{r^4}{12}$

6.12




$0 < y < 30$

$y < x < 30$

$$\int_0^{10}\int_0^{30}\int_0^{x} 0,65t^2 (x^2+y^2-24x-16y+120)\,dy\,dx\,dt$$

$$\int_0^{10} 0,65t^2 \int_0^{30} [x^2 y]_0^x + \left[y^3 \tfrac{1}{3}\right]_0^x - [24xy]_0^x - \left[\tfrac{16y^2}{2}\right]_0^x$$

$$+ [120y]_0^x \, dx\, dt$$

$$= \int_0^{10} 0,65t^2 \int_0^{30} \left(x^3 + \frac{x^3}{3} - 24x^2 - 8x^2 + 120x\right) dx\, dt$$

$$= \int_0^{10} 0,65t^2 \int_0^{30} \left(\frac{4}{3}x^3 - 32x^2 + 120x\right) dx\, dt$$

$$= \int_0^{10} 0,65t^2 \left( \left[\frac{4}{4\cdot 3}x^4\right]_0^{30} - \left[\frac{32}{3}x^3\right]_0^{30} + \left[120\frac{x^2}{2}\right]_0^{30} \right) dt$$

$$= \int_0^{10} 0,65t^2 \,(270\,000 - 288\,000 + 54\,000)\, dt$$

$$= \int_0^{10} 0,65\, t^2 \cdot 36000\, dt$$

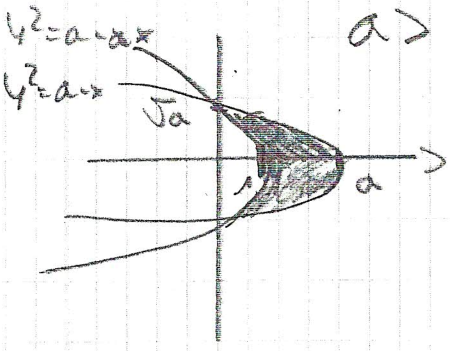
$$= 23400 \left[\frac{t^3}{3}\right]_0^{10}$$

$$= 7\,800\,000 = 7,8 \cdot 10^6$$

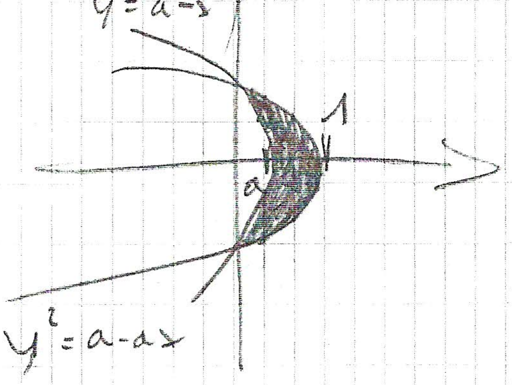
6.5

$y = \pm\sqrt{a-x}$

$a > 1$




$a < 1$




oppervlakte $\to f(x) = 1$

stel $a > 1$

$-\sqrt{a} < y < \sqrt{a}$

$\frac{-y^2}{a} + 1 < x < -y^2 + a$

$$\int_{-\sqrt{a}}^{\sqrt{a}} \int_{\frac{-y^2}{a}+1}^{-y^2+a} 1 \, dx\, dy$$

$$\int_{-\sqrt{a}}^{\sqrt{a}} -y^2 + a + \frac{y^2}{a} - 1 \quad dy$$

$$\int_{-\sqrt{a}}^{\sqrt{a}} y^2\left(\frac{1}{a} - 1\right) + (a-1)\, dy$$

$$\left[\left(\frac{1}{a} - 1\right)\frac{y^3}{3} + (a-1)y\right]_{-\sqrt{a}}^{\sqrt{a}}$$

$$\left(\frac{1}{a} - 1\right)\frac{(\sqrt{a})^3}{3} + (a-1)\sqrt{a} - \left(\frac{1}{a} - 1\right)\frac{(-\sqrt{a})^3}{3} - (a-1)(-\sqrt{a})$$

$$= 2\left(\frac{1}{a} - 1\right)\frac{(\sqrt{a})^3}{3} + 2(a-1)\sqrt{a} \qquad \sqrt{a}^3 = a\sqrt{a}$$

$$= 2(1-a)\frac{\sqrt{a}}{3} + 2(a-1)\sqrt{a}$$

$$= 2(a-1)\sqrt{a}\left(1 - \frac{1}{3}\right) = \frac{4}{3}(a-1)\sqrt{a}$$

stel $a < 1$

$-\sqrt{a} < y < \sqrt{a}$

$-y^2 + a < x < -\frac{y^2}{a} + 1$

$y^2 = a - ax \to x = \frac{y^2 - a}{-a}$

$y^2 = a - x \to x = \frac{y^2 - a}{-1}$

$\Rightarrow$ bij $a < 1$ : boven en ondergrens wisselen dus tegengesteld

$\Rightarrow \frac{4}{3}(1-a)\sqrt{a}$

6.14

a) integreren naar x en y => c is constante
mag afleiden binnen de integraal

$\cancel{\frac{d(x-c)^2}{dx}} =$

$$\frac{d}{dc} \iint_D (x-c)^2 \rho(x,y)\, dx\, dy$$

$$= \iint_D \frac{d(x-c)^2}{dc} \rho(x,y)\, dx\, dy$$

$$= \iint_D -2(x-c)\, \rho(x,y)\, dx\, dy = 0$$

$$= -2 \iint_D x\, \rho(x,y)\, dx\, dy + 2c \iint_D \rho(x,y)\, dx\, dy = 0$$

$$c = \frac{\iint_D x\, \rho(x,y)\, dx\, dy}{\iint_D \rho(x,y)\, dx\, dy} = \bar{x}$$

$\iint_D \rho(x,y)\, dx\, dy = M$ (massa)

b) $I = \iint_D \underbrace{[d_\ell(x,y)]^2}_{(d-y)^2} \rho(x,y)\, dA$

$$= \iint_D (d-y)^2 \rho(x,y)\, dA$$

=> minimaal : 1e afgel. = 0

$$DI = \iint_D \frac{d}{dd}(d-y)^2 \quad \rho(x,y)\, dA$$

$$= \iint_D -2(d-y)\, \rho(x,y)\, dA = 0$$

$$\Rightarrow -2d \iint_D \rho(x,y)\, dA + 2 \iint_D y\, \rho(x,y)\, dA = 0$$

$$d = \frac{\iint_D y\, \rho(x,y)\, dA}{\iint_D \rho(x,y)\, dA} \rightarrow \bar{y}$$

$\iint_D \rho(x,y)\, dA \rightarrow M$

=> minimaal bij y-coordinaat van massamiddelpunt

②

THUIS Oefeningen hoofdstuk 7

7.1

✓ a) $0 < r < a \qquad 0 \leq \theta \leq 2\pi$

$$e^{-(x^2+y^2)} \frac{y^2}{x^2+y^2} = e^{-r^2} \frac{r^2 \sin^2\theta}{r^2} = e^{-r^2}\sin^2\theta$$

$$\int_0^{2\pi}\int_0^a e^{-r^2}\sin^2\theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \underbrace{\sin^2\theta}_{=\frac{1}{2}(1-\cos 2\theta)} d\theta \int_0^a e^{-r^2} \cdot r \, dr \qquad \begin{aligned} u &= e^{-r^2} \\ du &= -2r e^{-r^2} dr \end{aligned}$$

$$= -\frac{1}{2}\int du = -\frac{1}{2}u = -\frac{1}{2}e^{-r^2}$$

$$= \left(\left[\frac{1}{2}\theta\right]_0^{2\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi}\right) \cdot \left(\left[-\frac{1}{2}e^{-r^2}\right]_0^a\right)$$

$\sin 4\pi = 0$

$\sin 0 = 0$

$$= \pi\left(-\frac{1}{2}e^{-a^2} + \frac{1}{2}e^0\right)$$

$$= \pi\left(-\frac{1}{2}e^{-a^2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{2}\left(1 - e^{-a^2}\right)$$

✓ b) $0 < r < \infty \qquad 0 \leq \theta < 2\pi$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\sin^2\theta \, r \, dr \, d\theta$$

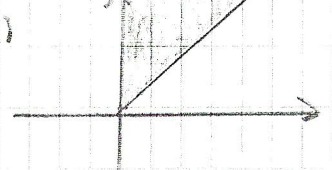
$$= \left(\left[\frac{1}{2}\theta\right]_0^{2\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi}\right)\left(\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty}\right)$$

$$= \pi\left(-\frac{1}{2}\underbrace{e^{-\infty^2}}_{=0} + \frac{1}{2}\right)$$

$$= \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

✓ 7.3

c)



$0 \leq x \leq \infty \qquad$ of $\qquad 0 \leq r \leq \infty$

$x \leq y \leq \infty \qquad\qquad \pi/4 \leq \theta \leq \pi/2$

$$e^{-(x^2+y^2)} = e^{-r^2}$$

$$\int_{\pi/4}^{\pi/2}\int_0^{\infty} e^{-r^2} r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} d\theta$$

$$= \int_{\pi/4}^{\pi/2}\left(-\frac{1}{2}\underbrace{e^{-\infty^2}}_{=0} + \frac{1}{2}\underbrace{e^0}_{=1}\right)d\theta = \left[\frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

(2)

THUIS Oefeningen hoofdstuk 7

7.1

✓ a) $0 < r < a \qquad 0 \leq \theta \leq 2\pi$

$$e^{-(x^2+y^2)} \frac{y^2}{x^2+y^2} = e^{-r^2} \frac{r^2 \sin^2\theta}{r^2} = e^{-r^2} \sin^2\theta$$

$$\int_0^{2\pi} \int_0^a e^{-r^2} \sin^2\theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \underbrace{\sin^2\theta}_{=\frac{1}{2}(1-\cos 2\theta)} d\theta \int_0^a e^{-r^2} \cdot r \, dr \qquad u = e^{-r^2} \quad du = -2r\, e^{-r^2} dr$$

$$= -\frac{1}{2}\int du = -\frac{1}{2} u = -\frac{1}{2} e^{-r^2}$$

$$= \left( \left[\frac{1}{2}\theta\right]_0^{2\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi} \right) \cdot \left( \left[-\frac{1}{2} e^{-r^2}\right]_0^a \right)$$

$\sin 4\pi = 0$

$\sin 0 = 0$

$$= \pi \left( -\frac{1}{2} e^{-a^2} + \frac{1}{2} e^0 \right)$$

$$= \pi \left( -\frac{1}{2} e^{-a^2} + \frac{1}{2} \right)$$

$$= \frac{\pi}{2}\left(1 - e^{-a^2}\right)$$

✓ b) $0 < r < \infty \qquad 0 \leq \theta < 2\pi$

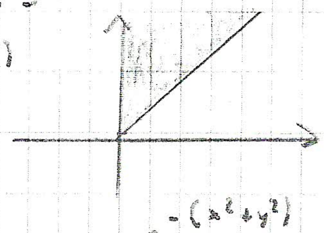
$$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} \sin^2\theta \, r \, dr \, d\theta$$

$$= \left( \left[\frac{1}{2}\theta\right]_0^{2\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi} \right) \left( \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} \right)$$

$$= \pi \left( -\frac{1}{2} \underbrace{e^{-\infty^2}}_{=0} + \frac{1}{2} \right)$$

$$= \pi \left(\frac{1}{2}\right) = \frac{\pi}{2}$$

✓ 7.3

c) 

$0 \leq x \leq \infty \qquad$ of $\qquad 0 \leq r \leq \infty$

$x \leq y \leq \infty \qquad\qquad \pi/4 \leq \theta \leq \pi/2$

$$e^{-(x^2+y^2)} = e^{-r^2}$$

$$\int_{\pi/4}^{\pi/2} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left( -\frac{1}{2} \underbrace{e^{-\infty^2}}_{=0} + \frac{1}{2} \underbrace{e^0}_{=1} \right) d\theta = \left[\frac{1}{2}\theta\right]_{\pi/4}^{\pi/2} = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8}$$

7.7

a) $e^{-x^2} \quad : \Rightarrow \quad f(x) = f(-x)$

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$$

$$\sqrt{\pi} \qquad \Rightarrow \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

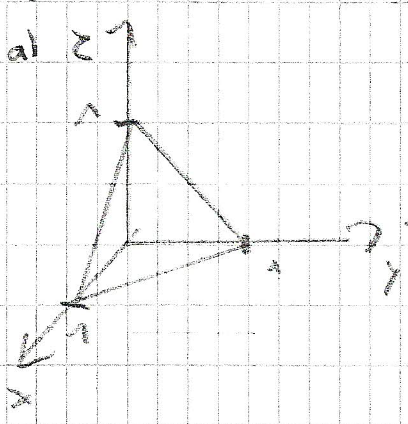
b) Stel $y = \frac{(x-\mu)}{\sqrt{2}\sigma}$ $\quad dy = \frac{1}{\sqrt{2}\sigma}\,dx$

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\infty}^{\infty} e^{-y^2} \cdot \sqrt{2}\sigma\,dy$$

$$= \sqrt{\pi}\sqrt{2}\sigma$$

7.8

a)



$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

$0 \leq z \leq 1-x-y$

$(1-x-y)^2 = (1-x)^2 - 2(1-x)y + y^2$

$= 1 - 2x + x^2 - 2y + 2xy + y^2$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left[xy\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \int_0^1\int_0^{1-x} \frac{xy}{2}(1-x-y)^2\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \frac{xy}{2} - x^2y + \frac{x^3y}{2} - xy^2 + x^2y^2 + \frac{xy^3}{2}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x} \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)y + (-x+x^2)y^2 + \left(\frac{x}{2}\right)y^3\,dy\,dx$$

$$= \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)\left[\frac{y^2}{2}\right]_{y=0}^{y=1-x} + (-x+x^2)\left[\frac{y^3}{3}\right]_0^{1-x} + \left(\frac{x}{2}\right)\left[\frac{y^4}{4}\right]_0^{1-x}$$

$(1-x)^2 = 1 - 2x + x^2$

$(1-x)^3 = 1 - 2x + x^2$
$\quad - x + 2x^2 - x^3$
$\quad = 1 - 3x + 3x^2 - x^3$

$(1-x)^4 = 1 - 3x + 3x^2 - x^3$
$\quad - x + 3x^2 - 3x^3 + x^4$
$\quad = 1 - 4x + 6x^2 - 4x^3 + x^4$

$$= \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)\frac{(1-x)^2}{2} + (-x+x^2)\frac{(1-x)^3}{3} + \left(\frac{x}{2}\right)\frac{(1-x)^4}{4}$$

$$= \int_0^1 \frac{x}{4} - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^2}{2} + x^3 - \frac{x^4}{2} + \frac{x^3}{4} - \frac{x^4}{4} + \frac{x^5}{4} - \frac{x}{3} + \frac{x^2}{3} + x^2 - x^3 - x^3 + x^4 + \frac{x^4}{3} - \frac{x^5}{3} + \frac{x}{8} - \frac{x^2}{2} + \frac{3x^3}{4} - \frac{x^4}{2} + \frac{x^5}{8}\,dx$$

$$= \int_0^1 \frac{1}{24}x - \frac{1}{6}x^2 + \frac{1}{4}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5\,dx$$

$$= \frac{1}{24}\left[\frac{x^2}{2}\right]_0^1 - \frac{1}{6}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1 - \frac{1}{6}\left[\frac{x^5}{5}\right]_0^1 + \frac{1}{24}\left[\frac{x^6}{6}\right]_0^1 = \frac{1}{720}$$

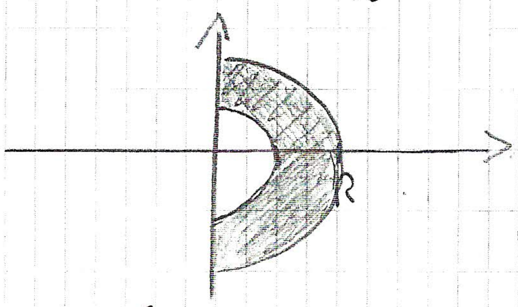
OEFENZITTING

7.3

b) omvormen tot poolcoördinaten

-> schijf $D = 1 \leq x^2+y^2 \leq R^2$ om verwarring te vermijden

$$\frac{xy^2}{(x^2+y^2)^2} = \frac{r\cos\theta \, r^2 \sin^2\theta}{r^4} = \frac{\cos\theta \overset{1-\cos^2}{\overbrace{\sin^2\theta}}}{r} = (\cos\theta - \cos^3\theta)\frac{1}{r}$$




$1 \leq r \leq R$

$-\pi/2 \leq \theta \leq \pi/2$

$dA = dx\,dy = r\,dr\,d\theta$

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{R} \cos\theta - \cos^3\theta \; dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} [r\cos\theta]_1^R - [r\cos^3\theta]_1^R \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (R\cos\theta - \cos\theta - R\cos^3\theta + \cos^3\theta)\, d\theta$$

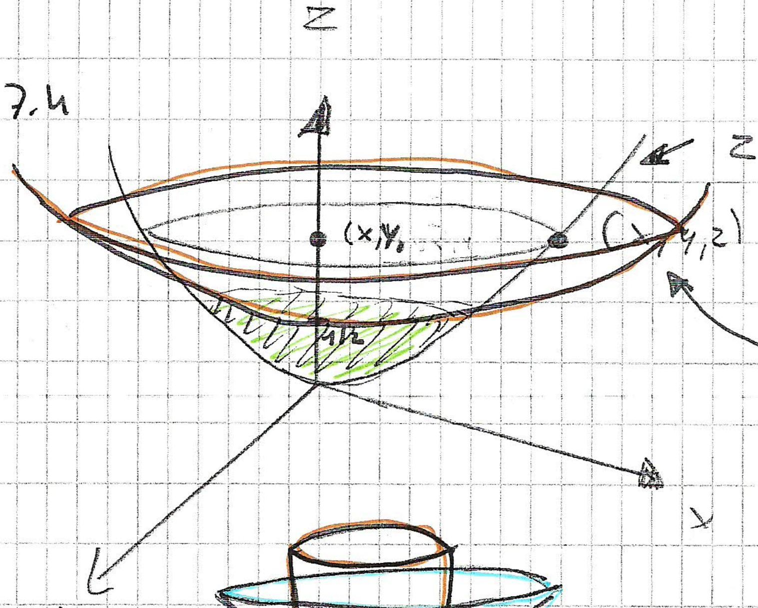
$$= \int_{-\pi/2}^{\pi/2} (R-1)\cos\theta - (R-1)\cos^3\theta \; d\theta$$

$$= \int_{-\pi/2}^{\pi/2} (R-1)(\cos\theta - \cos^3\theta)\, d\theta$$

$$= (R-1)\left( [-\sin\theta]_{-\pi/2}^{\pi/2} - \left[\frac{1}{4}\sin^4\theta\right]_{-\pi/2}^{\pi/2}\right)$$

$$= (R-1)\left(1+1-\frac{1}{4}-\frac{1}{4}\right) = \frac{3}{2}(R-1)$$

7.4

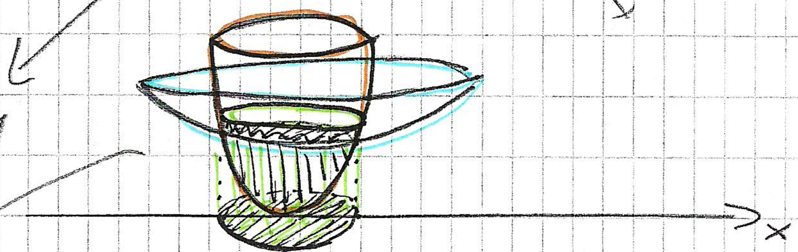



$z = x^2+y^2$

$2z = x^2+y^2+1$

$z = \frac{x^2+y^2+1}{2}$

begint bij $\frac{1}{2}$




snijkromme bepalen:

$$x^2+y^2 = \frac{x^2+y^2+1}{2}$$

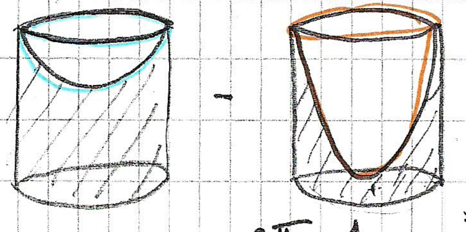
$$\frac{x^2}{2}+\frac{y^2}{2} = \frac{1}{2} \Rightarrow x^2+y^2 = 1 \rightarrow \text{ cirkel met straal } 1$$

$$2z = \boxed{x^2+y^2}+1 = 1+1$$

$$\Rightarrow z = 1$$

$$\begin{cases} z = 1 \\ x^2+y^2 = 1 \end{cases} \Rightarrow D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 1\}$$

volume:



poolcoordinaten: $\displaystyle\int_0^{2\pi}\int_0^1 \overbrace{\frac{r^2+1}{2}}^{\frac{x^2+y^2+1}{2}\,dx\,dy} r\,dr\,d\theta - \int_0^{2\pi}\int_0^1 \overbrace{r^2}^{x^2+y^2} r\,dr\,d\theta$

$= \frac{r^3}{2}+\frac{r}{2}$ ; $r^3$

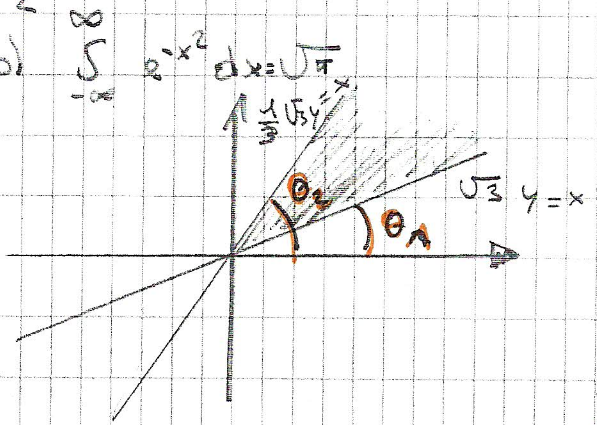
$$= \int_0^{2\pi} \left[\frac{r^4}{8}\right]_0^1 + \left[\frac{r^2}{4}\right]_0^1 d\theta - \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}+\frac{1}{4}\,d\theta - \int_0^{2\pi}\frac{1}{4}\,d\theta = \left[\frac{3}{8}\theta\right]_0^{2\pi} - \left[\frac{1}{4}\theta\right]_0^{2\pi}$$

$$= \frac{3}{8}2\pi - 2\pi\frac{1}{4} = \frac{3}{4}\pi - \frac{\pi}{2} = \frac{1}{4}\pi$$

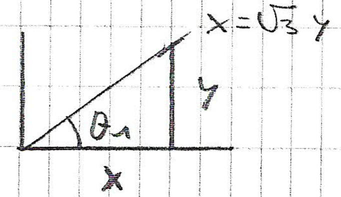
7.2

b) $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$




$\pi/6 \leq \theta \leq \pi/3$

$0 \leq r \leq \infty$




$\tan\theta_1 = \frac{y}{x} = \frac{y}{\sqrt{3}y} = \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\tan\theta_2 = \frac{y}{\frac{\sqrt{3}}{3}y} = \frac{3}{\sqrt{3}} = \frac{\pi}{3}$

poolcoördinaten:

$$\int_{\pi/6}^{\pi/3}\int_0^{\infty} e^{-2r^2} \frac{\cancel{r}\cos\theta + \cancel{r}\sin\theta}{\cancel{r^2}} \cdot \cancel{r}\, dr\, d\theta$$

$$= \int_{\pi/6}^{\pi/3} (\cos\theta + \sin\theta) \int_0^{\infty} e^{-2r^2}\, dr\, d\theta$$

stel $x = \sqrt{2}\,r$ $\quad dx = \sqrt{2}\,dr$

$$= \int_{\pi/6}^{\pi/3} (\cos\theta + \sin\theta) \frac{1}{\sqrt{2}} \frac{\sqrt{\pi}}{2}\, d\theta$$

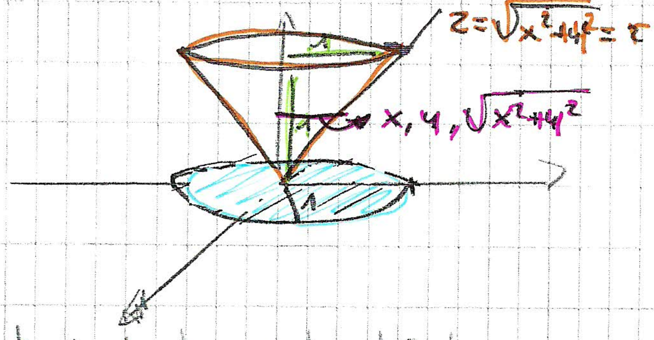
$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \left( \left[\sin\theta\right]_{\pi/6}^{\pi/3} + \left[-\cos\theta\right]_{\pi/6}^{\pi/3} \right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} (\sqrt{3} - 1) = \sqrt{\frac{\pi}{8}} (\sqrt{3} - 1)$$

7.8

d) $0 \leq z \leq 1$

$x^2 + y^2 \leq z^2$

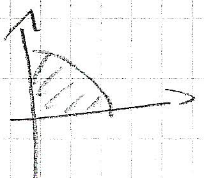



cilindercoördinaten:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

met $dx\,dy\,dz = r\,dr\,d\theta\,dz$

gebied in x, y- vlak



$$\iiint_D xy \, dx\, dy\, dz = \int_0^{\pi/2} \int_0^{1} \int_r^{1} r^3 \cos\theta \sin\theta \, dz\, dr\, d\theta$$

+ uitrekenen

$\Rightarrow 1/40$

7.8

c) bolcoordinaten

$\theta$: hoogte $\quad 0 \leq \theta \leq \pi$

$0 \leq \varphi \leq 2\pi$

$0 \leq r \leq R$

$z^2 = r^2 \cos^2\theta$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^2 \cos^2\theta \cdot r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

$$= \int_0^{2\pi} \int_0^{\pi} \cos^2\theta \sin\theta \left[\frac{r^5}{5}\right]_0^R d\theta\, d\varphi$$

$u = \cos\theta \rightarrow du = -\sin\theta \, d\theta$

$$= \int_0^{2\pi} \int_{-1}^{+1} + u^2 \frac{R^5}{5} \, du\, d\varphi$$

$$= \int_0^{2\pi} + \frac{R^5}{5} \left[\frac{u^3}{3}\right]_{-1}^{+1} d\varphi$$

$$= \int_0^{2\pi} + \frac{R^5}{5} \left(+\frac{1}{3} + \frac{1}{3}\right) d\varphi$$

$$= \int_0^{2\pi} + \frac{2R^5}{15} \, d\varphi$$

$$= + \frac{2R^5}{15} 2\pi = + \frac{4R^5\pi}{15}$$

7.10

a) integreren over heel de ruimte

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r \left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right)^2 \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\varphi}_{dV}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} r^3 \frac{1}{\pi a_0^3} e^{-2r/a_0} dr$$

$$= [\varphi]_0^{2\pi} \cdot [-\cos\theta]_0^{\pi} \quad \cdot \quad 3\times \text{ partieel integreren}$$

b) $$\int_0^{2\pi}\int_0^{\pi}\int_0^{a_0} \left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right)^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

c) $$\int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{3}{2}a_0} \left(\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\right)^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi$$