

# Oefeningen Wiskunde II 2013, 1e Bach Informtica

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## Hoofdstuk 1

### 1.1

a)

$$\begin{pmatrix} 1 & 1 & -2 & 10 \\ 4 & 2 & 2 & 1 \\ 6 & 2 & 3 & 4 \end{pmatrix} \begin{array}{l} R2 = R2 - 4R1 \\ R3 = R3 - 6R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 1 & -2 & 10 \\ 0 & -2 & 10 & -39 \\ 0 & -4 & 15 & -56 \end{pmatrix} \begin{array}{l} R2 = -\frac{1}{2}R2 \\ R3 = -\frac{3}{4}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 1 & -2 & 10 \\ 0 & 1 & -5 & \frac{39}{2} \\ 0 & 1 & -\frac{15}{4} & 14 \end{pmatrix}$$
$$\begin{array}{l} R1 = R1 - R2 \\ R3 = R3 - R2 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 3 & -\frac{19}{2} \\ 0 & 1 & 3 & \frac{39}{2} \\ 0 & 0 & \frac{5}{4} & -\frac{11}{2} \end{pmatrix} \begin{array}{l} R3 = \frac{4}{5}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 3 & -\frac{19}{2} \\ 0 & 1 & 3 & \frac{39}{2} \\ 0 & 0 & 1 & -\frac{22}{5} \end{pmatrix} \begin{array}{l} R1 = R1 - 3R3 \\ R2 = R2 - 3R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & \frac{37}{10} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{22}{5} \end{pmatrix}$$

Antwoord:

$$V = \left\{ \left( \frac{37}{10}, -\frac{5}{2}, -\frac{22}{5} \right) \right\}$$

b)

$$\left( \begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array} \right) \begin{array}{l} R2 = R2 - 3R1 \\ R3 = R3 - 5R1 \\ \rightarrow \end{array} \left( \begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array} \right)$$

Antwoord:

$$V = \emptyset$$

#### 1.4

$$(Ax + B)(x^2 - 1) + C(x^2 + 4)(x + 1) + D(x^2 + 4)(x - 1) \\ = Ax^3 - Ax + Bx^2 - B + cx^3 + cx^2 + 4Cx + 4C + Dx^3 - Dx^2 + 4Dx - 4D$$

$$\begin{cases} A + C + D = 0 \\ B + C - D = 0 \\ -A + 4C + 4D = 0 \\ -B + 4C - 4D = 1 \end{cases} \rightarrow \left( \begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 4 & 4 & 0 \\ 0 & -1 & 4 & -4 & 1 \end{array} \right) \xrightarrow{GRM} \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & -\frac{1}{10} \end{array} \right)$$

Antwoord:

$$\begin{cases} A = 0 \\ B = -\frac{1}{5} \\ C = \frac{1}{10} \\ D = -\frac{1}{10} \end{cases}$$

#### 1.6

a)

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & a \end{pmatrix} \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 - 4R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & (a-24) \end{pmatrix} \begin{array}{l} R2 = -\frac{1}{5}R2 \\ R3 = -\frac{1}{5}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & -\frac{(a-24)}{5} \end{pmatrix}$$

Antwoord:

Voor  $a \neq 14$  zal het stelsel geen oplossingen hebben. Voor  $a = 14$  zal het stelsel oneindig veel oplossingen hebben.

(voor  $a = 14$ )

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Wat we nog kunnen rijreducen tot:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We constateren dat  $z$  hier een vrije variabele is en wijzen haar de waarde  $\lambda$  toe. Dan is onze oplossingsverzameling de volgende:

$$V = \{(2 - \lambda, 2 + 2\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$$

b)

$$\left( \begin{array}{cc|c} 1 & a & (a+1) \\ a & 1 & 2 \end{array} \right) \xrightarrow{R2 = R2 - aR1} \left( \begin{array}{cc|c} 1 & a & (a+1) \\ 0 & (1-a^2) & (a^2-a+2) \end{array} \right)$$

Antwoord:

Voor  $a = 1$ : oneindig veel oplossingen,  $\begin{cases} x = 2\lambda \\ y = \lambda \end{cases}$

Voor  $a = -1$ : geen oplossingen.

Voor  $a \neq 1 \wedge a \neq -1$ : precies 1 oplossing,  $\begin{cases} x = \frac{1}{a+1} \\ y = \frac{a+2}{a+1} \end{cases}$

c)

$$\left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ a & 1 & (a+1) & 0 \\ 2a & 1 & 1 & (a+1) \end{array} \right) \xrightarrow{\begin{matrix} R2 = R2 - R1 \\ R3 = R3 - 2R1 \end{matrix}} \left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & -a & a & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array} \right)$$

$$\xrightarrow{R2 = -\frac{1}{a}R2} \left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array} \right) \xrightarrow{\begin{matrix} R1 = R1 - (a+1)R2 \\ R3 = R3 + (2a+1)R2 \end{matrix}} \left( \begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2(a+1) & (a+1) \end{array} \right)$$

$$\xrightarrow{R3 = -\frac{2}{(a+1)}R3} \left( \begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right) \xrightarrow{\begin{matrix} R1 = R1 - (a+2)R3 \\ R2 = R2 + R3 \end{matrix}} \left( \begin{array}{ccc|c} a & 0 & 0 & \frac{a+2}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right)$$

$$\xrightarrow{R1 = \frac{1}{a}R1} \left( \begin{array}{ccc|c} 1 & 0 & 0 & \frac{a+2}{2a} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right)$$

Antwoord:

Voor  $a = 0$ : geen oplossingen.

Voor  $a = -1$ : oneindig veel oplossingen:  $\begin{cases} x = \lambda \\ y = \lambda \\ z = \lambda \end{cases}$

Voor  $a \neq -1 \wedge a \neq 0$ : precies 1 oplossing:  $\begin{cases} x = \frac{a+2}{2a} \\ y = -\frac{1}{2} \\ z = -\frac{1}{2} \end{cases}$

1.8

$$\begin{pmatrix} k & 5 & 3 \\ 5 & 1 & -1 \\ k & 2 & 1 \end{pmatrix} \begin{array}{l} R1 = R1 - \frac{k}{5}R2 \\ R3 = R3 - \frac{k}{5}R2 \\ \rightarrow \end{array} \begin{pmatrix} 0 & 5 - \frac{k}{5} & 3 + \frac{k}{5} \\ 5 & 1 & -1 \\ 0 & 2 - \frac{k}{5} & 1 + \frac{k}{5} \end{pmatrix}$$

Antwoord:

Enkel als  $k = 1$  zullen R1 en R3 lineair afhankelijk zijn en zullen er niet-triviale oplossingen zijn, anders is er enkel de triviale nuloplossing.

1.10

a)

$$\begin{pmatrix} 3 & 1 & 1 & | & 0 \\ 1 & 3 & 1 & | & 0 \\ 1 & 1 & 3 & | & 0 \end{pmatrix} \begin{array}{l} R1 = R1 - 3R2 \\ R3 = R3 - R2 \\ \rightarrow \end{array} \begin{pmatrix} 0 & -8 & -2 & | & 0 \\ 1 & 3 & 1 & | & 0 \\ 0 & -2 & 2 & | & 0 \end{pmatrix} \begin{array}{l} R1 = R2 \\ R2 = -\frac{1}{2}R3 \\ R3 = -\frac{1}{2}R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 3 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 4 & 1 & | & 0 \end{pmatrix}$$

$$\begin{array}{l} R1 = R1 - 3R2 \\ R3 = R3 - 4R2 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 5 & | & 0 \end{pmatrix} \begin{array}{l} R3 = \frac{1}{5}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix} \begin{array}{l} R1 = R1 - 4R3 \\ R2 = R2 + R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$

Antwoord:

Enkel de triviale oplossing  $\vec{x} = \vec{0}$ , want  $\text{Rang}(A) = n$

c)

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & 7 \\ 5 & 5 & 4 \\ 9 & 7 & 6 \end{pmatrix} \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 - 2R1 \\ R4 = R4 - 5R1 \\ R5 = R5 - 9R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -5 & 5 \\ 0 & -5 & 9 \\ 0 & -11 & 15 \end{pmatrix} \begin{array}{l} R2 = -\frac{1}{3}R2 \\ R2 \text{ en } R3 \text{ zijn lineair} \\ \text{afhankelijk, schrap } R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & 9 \\ 0 & -11 & 15 \end{pmatrix}$$

$$\begin{array}{l} R1 = R1 - 2R2 \\ R3 = R3 + 5R2 \\ R4 = R4 + 11R2 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{l} R3 = \frac{1}{4}R3 \\ R3 \text{ en } R4 \text{ zijn lineair} \\ \text{afhankelijk, schrap } R4 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R1 = R1 - R3 \\ R2 = R2 + R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Antwoord:

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

## 1.12

a)

$$\begin{pmatrix} 1 & 5 & 2 & 3 \\ 4 & 2 & -1 & -6 \\ -5 & 1 & 3 & 11 \end{pmatrix} \begin{array}{l} R2 = R2 - 4R1 \\ R3 = R3 + 5R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -18 & -9 & -18 \\ 0 & 26 & 13 & 26 \end{pmatrix} \begin{array}{l} R2 = -\frac{1}{9} \\ R2 \text{ en } R3 \text{ zijn lineair} \\ \text{afhankelijk, schrap } R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Antwoord:

Lineair afhankelijk. Een zo groot mogelijke lineair onafhankelijke deelverzameling wordt bv. gegeven door de eerste 2 vectoren.

$\lambda_3 = p$  met  $p$  willekeurig en  $\lambda_4 = t$  met  $t$  willekeurig.

$$2\lambda_2 = -p - 2t \Rightarrow \lambda_2 = -\frac{1}{2}p - t$$

$$\lambda_1 = -5(-\frac{1}{2}p - t) - 2p - 3t = \frac{5}{2}p + 5t - 2p = \frac{1}{2}p + 5t$$

$$\left(\frac{1}{2} + 5t\right) \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} + \left(-\frac{1}{2}p - t\right) \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + p \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -6 \\ 11 \end{pmatrix} = \vec{0}$$

## 1.13

a)

$$\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{array}{l} R2 = R2 + 4R1 \\ R3 = R3 - 2R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & -4 & 2 \\ 0 & -15 & 6 \\ 0 & 6 & -6 \end{pmatrix} R3 = \frac{6}{15}R2 \rightarrow \begin{pmatrix} 1 & -4 & 2 \\ 0 & -15 & 6 \\ 0 & 0 & -\frac{36}{10} \end{pmatrix}$$

Antwoord:

Deze drie vectoren zijn lineair onafhankelijk.

b)

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & -5 \\ -5 & 4 & 14 \\ 3 & -1 & -17 \end{pmatrix} \begin{array}{l} R2 = R2 - 2R1 \\ R3 = R3 + 5R1 \\ R4 = R4 - 3R1 \end{array} \xrightarrow{\quad} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & 9 & 9 \\ 0 & -4 & 14 \end{pmatrix}$$

Antwoord:

Lineair afhankelijk, een zo groot mogelijk lineair onafhankelijke deelverzameling wordt bv. gegeven door de eerste 3 vectoren.

c)

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ -2 & 5 & 0 & -15 \\ 3 & 3 & 2 & 1 \end{pmatrix} \begin{array}{l} R2 = R2 + 2R1 \\ R3 = R3 - 3R1 \end{array} \xrightarrow{\quad} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 2 & -11 \\ 0 & 6 & -1 & -5 \end{pmatrix} \begin{array}{l} R3 = R3 - 2R2 \end{array} \xrightarrow{\quad} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 2 & -11 \\ 0 & 0 & -5 & 17 \end{pmatrix}$$

Antwoord:

Lineair onafhankelijk?

## 1.14

$$\begin{pmatrix} \cos(\theta) & 1 \\ 1 & 2\cos(\theta) \end{pmatrix} \begin{array}{l} R1 = R1 - \cos(\theta)R2 \end{array} \xrightarrow{\quad} \begin{pmatrix} 0 & 1 - 2\cos^2(\theta) \\ 1 & 2\cos(\theta) \end{pmatrix}$$

Lineair afhankelijk als

$$\begin{aligned} 1 - 2\cos^2(\theta) &= 0 \\ \cos^2(\theta) &= \frac{1}{2} \\ \cos(\theta) &= \pm \frac{\sqrt{2}}{2} \\ \theta &= \pm \frac{\pi}{4} \vee \theta = \pm \frac{3\pi}{4} \end{aligned}$$

## Hoofdstuk 2

### 2.1

a)

$$A + B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$$

f)

$$2P - Q = 2 \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 0 & 7 \end{pmatrix}$$

k)

$$P.C = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} \text{ Onmogelijk}$$

p)

$$\vec{b}.C = (2 \ 5 \ -2) \cdot \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} = ((-10 + 20 - 4) \ (6 - 5 + 2)) = (6 \ 3)$$

## 2.3

a)

$$A.B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} = A.B \text{ (toevallig)}$$

b)

$$A.B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$
$$A.B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

## 2.4

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \rightarrow \left( \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \end{array} \right) \xrightarrow{GRM} \begin{cases} a = 0 \\ b = 1 \\ c = -10 \\ d = 7 \end{cases}$$

## 2.7

b)

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2c = 2b \\ b + 2d = 2a \\ 2a = c + 2d \\ 2b = 2c \end{cases} \leftrightarrow \begin{cases} a = d \\ b = 2a - 2d \\ c = 2d - 2a \\ d = a \end{cases}$$

## 2.8

a)

Neen, want  $2AB$  komt van  $AB + BA$  en  $AB \neq BA$ .

b)

Ja.

## 2.9

a)

$$\left( \begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array} \right) \xrightarrow{R2 = R2 - 2R1} \left( \begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array} \middle| \begin{array}{c} 3 \\ 2 \\ 1 \\ 4 \end{array} \right) \xrightarrow{R1 = R1 + \frac{3}{7}R2} \left( \begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 1 \end{array} \right)$$

$$\begin{array}{l} R1 = \frac{1}{2}R1 \\ R2 = \frac{1}{7}R2 \end{array} \xrightarrow{\rightarrow} \left( \begin{array}{cc|cc} 1 & 0 & \frac{1}{14} & \frac{3}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \quad \text{Antwoord: } \left( \begin{array}{cc|cc} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

## 2.10

$$\left( \begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{GRM} \left( \begin{array}{ccc|ccc} 19 & 0 & 0 & 2 & -1 & -9 \\ 0 & 19 & 0 & 5 & -2 & 1 \\ 0 & 0 & 19 & -5 & 12 & 13 \end{array} \right)$$

Antwoord:

$$\frac{1}{19} \begin{pmatrix} 2 & -1 & -9 \\ 4 & -2 & 1 \\ -5 & 12 & 13 \end{pmatrix}$$



## 2.14

a)

$$\begin{aligned} & \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -(\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} \end{aligned}$$

## 2.17

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ want } \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (1 * 1 - 1 * 0) & (1 * 1 - 1 * 0) \\ (0 * 1 - 0 * 0) & (0 * 1 - 0 * 0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

## Hoofdstuk 3

### 3.2

a)

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

b)

$$\begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 2$$

c)

$$\begin{vmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{vmatrix} = \cos^2(n\theta) + \sin^2(n\theta) = 1$$

### 3.3

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 36 & 4 & 9 \end{vmatrix} = 2, \quad D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 36 & 9 \end{vmatrix} = 4, \quad D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & 4 & 36 \end{vmatrix} = 6$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

Antwoord:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

### 3.4

a)

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 1 \\ -3 & 2 \end{vmatrix} = 10 - 2 + 33 = 41$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -2 + 11 + 32 = 41$$

e)

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 0 & 1 \\ 6 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} = 18$$

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = -3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 6 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 18$$

### 3.6

c)

$$\begin{vmatrix} -2 & 6 & 17 & -5 \\ 0 & 3 & 22 & -12 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 0 & -6 \end{vmatrix} = (-2) \cdot 3 \cdot 4 \cdot (-6) = 144$$

d)

$$\begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -2 \cdot 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 16$$

### 3.8

b)

$$\begin{vmatrix} 1 & 2x & 3x^2 \\ 2x^3 & 3x^4 & 4x^5 \\ 3x^6 & 4x^7 & 6x^8 \end{vmatrix} \begin{matrix} R2 = R2 - 2x^3 R1 \\ R3 = R3 - 3x^6 R1 \\ = \end{matrix} \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & -x^4 & -x^5 \\ 0 & -2x^7 & -3x^8 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} -x^4 & -x^5 \\ -2x^7 & -3x^8 \end{vmatrix} = 0$$

$$\Leftrightarrow 3x^1 2 - 2x^1 2 = x^1 2 = 0 \Leftrightarrow x = 0$$

### 3.9

a)

$$\begin{vmatrix} 3 & 2 & -2 \\ 6 & 1 & 5 \\ -9 & 3 & 4 \end{vmatrix} \begin{matrix} R2 = R2 - 2R1 \\ R3 = R3 + 3R1 \\ = \end{matrix} \begin{vmatrix} 3 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 9 & -2 \end{vmatrix} \begin{matrix} R3 = -\frac{1}{3}R3 \\ = -3 \cdot \end{matrix} \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 9 & -2 \end{vmatrix} \begin{matrix} R3 = R3 - 9R2 \\ = -3 \cdot \end{matrix} \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 25 \end{vmatrix}$$

$$= (-3) * 3 * 25 = -225$$

b)

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \begin{matrix} R3 = R3 - R1 \\ R4 = R4 - R1 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} \begin{matrix} R4 = R4 - R2 \\ = \end{matrix}$$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{vmatrix} \begin{matrix} R4 = R4 - R3 \\ = \end{matrix} \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1$$

### 3.11

$$A^T = A^{-1} \quad (1. \text{ Definitie orthogonale matrix})$$

$$A.A^T = I = A^T.A \quad (2. \text{ Direct gevolg van definitie})$$

$$\det(A^T) = \det(A^{-1}) \quad (3. \text{ (uit 1.)})$$

$$\frac{\det(A^{-1})}{\det(A)} = 1 \quad (4. \text{ (uit 3.)})$$

$$\det(A.A^{-1}) = \det(I) = 1 \quad (5. \text{ (uit 2.)})$$

$$\det(A.A^{-1}) = \det(A).\det(A^{-1}) \quad (6. \text{ eigenschap determinant})$$

$$\frac{\det(A^{-1})}{\det(A)} = \det(A).\det(A^{-1}) \quad (7. \text{ (uit 4. en 5.)})$$

$$\frac{1}{\det(A)} = \det(A) \quad (8.)$$

$$\det(A) = 1 \vee \det(A) = -1 \quad (9.)$$

## Hoofdstuk 4

### 4.1

$$\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

d)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0-1 \\ 1-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

### 4.2

a)

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

#### 4.4

a)

We berekenen een normaalvector:

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

Er bestaat een  $a$  zodat dit een eenheidsnormaal is:

$$a \cdot \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

$$\sqrt{(-4a)^2 + (5a)^2 + (-3a)^2} = \sqrt{50a^2} = 1$$

Antwoord:

$$\begin{pmatrix} \frac{-4}{\sqrt{50}} \\ \frac{5}{\sqrt{50}} \\ \frac{-3}{\sqrt{50}} \end{pmatrix}$$

#### 4.5

Stel:

$$\text{Normaal : } \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ Positievector : } \vec{p} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \text{ en } \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Cartesische vergelijking van een vlak:

$$V : a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Parametervergelijking van een vlak:

$$V : \vec{x} = \vec{p} + t_1 \vec{r}_1 + t_2 \vec{r}_2$$

waarbij:

$$\vec{r}_1 = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}, \text{ en } \vec{r}_2 = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

a)

$$\begin{aligned} V : 2(x-1) + 1(y-1) + 3(z-2) &= 0 \\ \Leftrightarrow 2x + y + 3z &= 9 \\ V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t2 \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

4.5

b)

$$\begin{aligned} V : -1(x-2) + 4(y+1) + 5(z-3) &= 0 \\ \Leftrightarrow -x + 4y + 5z &= 9 \\ V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t1 \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + t2 \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

4.7

$$\begin{aligned} r_1 &= \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \text{ richtvector : } r_2 = \begin{pmatrix} 2-1 \\ 4-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \\ \text{normaal : } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \\ V : -(x-1) - (z-3) &= 0 \\ \Leftrightarrow x + z &= 4 \end{aligned}$$

4.8

Stel:

$$\text{Positievectoren : } \vec{p}, \text{ en } \vec{q}, \text{ Richtvector : } (\vec{q}-\vec{p}), \text{ en Normalen : } n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \text{ en } n_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

Cartesische vergelijking van een rechte:

$$L : \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

Parametervergelijking van een rechte:

$$L : \vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

a)

$$L : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} a \\ 3 \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} \Leftrightarrow \begin{cases} a = 1 + 3t \\ 3 = 2 + (-1)t \\ b = 8 - 4t \end{cases} \Leftrightarrow \begin{cases} a = 1 + 3(-\frac{1}{3}) \\ t = -\frac{1}{3} \\ b = 8 - 4(-\frac{1}{3}) \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{3} \\ t = -\frac{1}{3} \\ b = 12 \end{cases}$$

b)

$$V : 3(x+4) - 2(y+0) + 6(z-3) = 0$$

$$\Leftrightarrow 3x - 2y + 6z = 6$$

$$\Leftrightarrow 3(1+3t) - 2(2-t) + 6(8-4t) = 6$$

$$\Leftrightarrow 9t + 2t - 24t + 3 - 4 + 48 = 6 \Leftrightarrow 9t + 2t - 24t + 3 - 4 + 48 = 6$$

$$\Leftrightarrow t = \frac{41}{13}$$

Antwoord:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + \frac{41}{13} \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{136}{13} \\ \frac{15}{13} \\ -\frac{60}{13} \end{pmatrix}$$

4.18

a)

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ -3 \end{pmatrix}$$

d)

$$\left( \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = -7$$

e)

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \left( \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ -21 \end{pmatrix}$$

#### 4.22

a)

$$d(\vec{p}, \vec{v}) = \frac{|10 - 2 \cdot 1 - 3 \cdot 1 - (-1) \cdot 2|}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{7}{\sqrt{14}}$$

b)

$$\vec{r}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$-(x-1) - y - z = 0$$

$$x + y + z = 1$$

afstand tot  $(2, 1, 1)^T$ :

$$d(\vec{p}, v) = \frac{|1 - 2 - 1 - 1|}{\sqrt{3}} = \sqrt{3}$$

#### 4.30

a)

$$\left| \det \begin{pmatrix} 1 & -1 & 2 \\ 1 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \right|$$



## Hoofdstuk 5

### 5.1

a)

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda^2 - (2+3)\lambda + 4 = 0 \longrightarrow \lambda = 4 \vee \lambda = 1$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

b)

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda^2 - (3+3)\lambda + 8 = 0 \longrightarrow \lambda = 4 \vee \lambda = 2$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

e)

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0 \longrightarrow (1-\lambda)(\lambda^2-1^2)-2(2(-1-\lambda)) = 0 \longrightarrow \lambda = -1 \vee \lambda = 3$$

Voor  $\lambda = -1$ :

$$\left( \begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 2 & -4 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

g)

e)

$$\begin{vmatrix} 1\lambda & 3 & 0 \\ 3 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{vmatrix} = 0 \longrightarrow -\lambda(\lambda^2 - 9) - 3(-3\lambda) = 0 \longrightarrow \lambda = 0 \vee \lambda = 3\sqrt{2} \vee \lambda = -3\sqrt{2}$$

Voor  $\lambda = 0$

$$\left( \begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 3\sqrt{2}$

$$\left( \begin{array}{ccc|c} -3\sqrt{2} & 3 & 0 & 0 \\ 3 & -3\sqrt{2} & 3 & 0 \\ 0 & 3 & -3\sqrt{2} & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Voor  $\lambda = -3\sqrt{2}$

$$\left( \begin{array}{ccc|c} 3\sqrt{2} & 3 & 0 & 0 \\ 3 & 3\sqrt{2} & 3 & 0 \\ 0 & 3 & 3\sqrt{2} & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

## 5.2

a)

TODO

a) voor 5.1g

$$\begin{aligned} c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{(-1a)^2 + (0a)^2 + a^2} = 1} c \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \\ c \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{a^2 + (\sqrt{2}a)^2 + a^2} = 1} c \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \\ c \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{a^2 + (-\sqrt{2}a)^2 + a^2} = 1} c \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

**b) voor 5.1g**

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}^2 \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TODO

**5.4**

$$\det(A - \lambda I) = \det(A) = 0$$

**5.5**

$$\begin{aligned} & \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \lambda I \right) \vec{x} = \vec{0} \\ & \Leftrightarrow \begin{vmatrix} a - \lambda & b \\ 0 & d - \lambda \end{vmatrix} = 0 \\ & \Leftrightarrow (a - \lambda)(d - \lambda) = 0 \\ & \Leftrightarrow \lambda = a \vee \lambda = d \end{aligned}$$

TODO BEWIJS DOOR INDUCTIE

**5.7**

**a)**

TODO

**b)**

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0 \longrightarrow (1 - \lambda)((1 - \lambda)^2 - 1) = 0 \longrightarrow \lambda = 0 \vee \lambda = 1 \vee \lambda = 2$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Voor  $\lambda = 0$ :

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonaliseerbaar.

c)

Niet diagonaliseerbaar.

e)

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \longrightarrow (-\lambda)^3 + 1 = 0 \longrightarrow \lambda = 1 \vee \lambda = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \vee \lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}$$

TODO

e)

Zie toledo

[https://cygnus.cc.kuleuven.be/bbcswebdav/pid-11087333-dt-content-rid-10466789\\_2/courses/a-G0017a-1213/0efening%205\\_7\\_e.pdf](https://cygnus.cc.kuleuven.be/bbcswebdav/pid-11087333-dt-content-rid-10466789_2/courses/a-G0017a-1213/0efening%205_7_e.pdf)

## 5.8

a)

$$\begin{vmatrix} 1+i-\lambda & 2-i \\ 3+i & -i-\lambda \end{vmatrix} = 0 \longrightarrow$$

$$(1+i-\lambda)(-i-\lambda) - (2-i)(3+i) = 0 \longrightarrow \lambda = 3 \vee \lambda = -2$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{cc|c} 4+i & 2-i & 0 \\ 3+i & -3-i & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = -2$ :

$$\left( \begin{array}{cc|c} 3+i & 2-i & 0 \\ 3+i & 2-i & 0 \end{array} \right) \longrightarrow \text{TODO} \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b)

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = 0 \longrightarrow -\lambda((- \lambda)^2 + i^2) - i(i\lambda) \longrightarrow \lambda = 0 \vee \lambda = \sqrt{2} \vee \lambda = -\sqrt{2}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Voor  $\lambda = 0$ :

$$\left( \begin{array}{ccc|c} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = \sqrt{2}$ :

$$\left( \begin{array}{ccc|c} -\sqrt{2} & -i & 0 & 0 \\ i & -\sqrt{2} & -i & 0 \\ 0 & i & -\sqrt{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

Voor  $\lambda = -\sqrt{2}$ :

$$\left( \begin{array}{ccc|c} \sqrt{2} & -i & 0 & 0 \\ i & \sqrt{2} & -i & 0 \\ 0 & i & \sqrt{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

c)

$$\begin{vmatrix} 2-\lambda & i \\ i & 2-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (2-\lambda)(1-\lambda)+1 = 0 \quad \longrightarrow \quad \lambda = \frac{3+i\sqrt{3}}{2} \vee \lambda = \frac{3-i\sqrt{3}}{2}$$

$$D = \begin{pmatrix} \frac{3+i\sqrt{3}}{2} & 0 \\ 0 & \frac{3-i\sqrt{3}}{2} \end{pmatrix}$$

Voor  $\lambda = \frac{3+i\sqrt{3}}{2}$ :

$$\left( \begin{array}{cc|c} 2 - \frac{3+i\sqrt{3}}{2} & i & 0 \\ i & 1 - \frac{3+i\sqrt{3}}{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

Voor  $\lambda = \frac{3-i\sqrt{3}}{2}$ :

$$\left( \begin{array}{cc|c} 2 - \frac{3-i\sqrt{3}}{2} & i & 0 \\ i & 1 - \frac{3-i\sqrt{3}}{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

## 5.11

TODO

## 5.13

1. Basisstap Het geldt voor  $k = 1$  want (gegeven)

$$A^k = XD^kX^{-1}$$

2. Inductiestap: tel dat het klopt voor  $k = n$  met  $n > 1$  en  $n \in \mathbb{N}$  3. Te bewijzen:

$$A^{k+1} = X.D^{k+1}.X^{-1}$$

$$A.A^k = X.D.D^k.X^{-1}$$

$$A.A^k = X.D.X^{-1} . XD^kX^{-1} = X.D.D^k.X^{-1}$$

QED

## 5.15

TODO

## Hoofdstuk 6

### 6.1

$$\frac{d \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}}{dt} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} \quad \frac{d \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}}{dt} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$
$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

OK

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$

OK

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (4-\lambda)(1-\lambda) + 2 = 0 \quad \longrightarrow \quad \lambda = 2 \quad \vee \quad \lambda = 3$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array} \right) \quad \longrightarrow \quad \left( \begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array} \right) \quad \longrightarrow \quad \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

### 6.2

a)

$$\vec{x}' = \begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (5-\lambda)(-\lambda) + 4 = 0 \quad \longrightarrow \quad \lambda = 4 \quad \vee \quad \lambda = 1$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array} \right) \quad \longrightarrow \quad \left( \begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 4 & 4 & 0 \\ -1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{4t} \begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Evenwichten:

$$5x + y = 0 \quad \text{en} \quad -x = 0$$

$$x = 0 \quad \text{en} \quad y = 0$$

De oorsprong is een evenwicht.

Stabiliteit:

$$\lambda_1 \geq 0 \quad \text{of} \quad \lambda_2 \geq 0$$

De oorsprong is een onstabiel evenwicht.

## 6.3

a)

$$\vec{x}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \vec{x}$$

$$\left| \begin{array}{cc} 4 - \lambda & 2 \\ 3 & 3 - \lambda \end{array} \right| = 0 \longrightarrow (4 - \lambda)(3 - \lambda) - 6 = 0 \longrightarrow \lambda = 6 \quad \vee \quad \lambda = 1$$

Voor  $\lambda = 6$ :

$$\left( \begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\vec{x} = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t (2 - 3)$$



## 6.8

a)

$$\vec{x}' = \begin{pmatrix} -2 & 4 \\ 1 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \longrightarrow (-2-\lambda)(-4-\lambda)-4=0 \longrightarrow \lambda = \frac{-6+\sqrt{20}}{2} \vee \lambda = \frac{-6-\sqrt{20}}{2}$$

Evenwichten:

$$\begin{aligned} -2x + 4y &= 0 & en & & -4y + x &= 0 \\ x &= 0 & en & & y &= 0 \end{aligned}$$

Stabiliteit:

$$\lambda_1 < 0 \quad en \quad \lambda_2 < 0$$

De oorsprong is een stabiel evenwicht.

## 6.10

a)

Evenwichten:

$$\begin{aligned} x + xy &= 0 & en & & 2y - xy &= 0 \\ x &= 0 & en & & y &= 0 & of & & x &= 2 & en & & y &= -1 \end{aligned}$$

A:

$$A = \begin{pmatrix} 1+y & x \\ -y & 2-x \end{pmatrix} [$$

voor  $(0,0)$ :

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ \lambda_1 &= 1 & en & & \lambda_2 &= 2 \\ \lambda_1 &\geq 0 & of & & \lambda_2 &\geq 0 \end{aligned}$$

$(0,0)$  is een onstabiel evenwicht.

voor  $(-1,2)$ :

$$\begin{aligned} &\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix} \\ \lambda_1 &= 7 & en & & \lambda_2 &= -1 \\ \lambda_1 &\geq 0 & of & & \lambda_2 &\geq 0 \end{aligned}$$

$(-1,2)$  is een onstabiel evenwicht.