# Oefeningen Wiskunde II 2013, 1e Bach Informtica

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April 8, 2013

## Hoofdstuk 1

### 1.1

#### a)

$$\begin{pmatrix}1 & 1 & -2 & 10\\ 4 & 2 & 2 & 1\\ 6 & 2 & 3 & 4\end{pmatrix} \xrightarrow[R3 = R3 - 6R1]{R2 = R2 - 4R1} \begin{pmatrix}1 & 1 & -2 & 10\\ 0 & -2 & 10 & -39\\ 0 & -4 & 15 & -56\end{pmatrix} \xrightarrow[R3 = -\frac{3}{4}R3]{R2 = -\frac{1}{2}R2} \begin{pmatrix}1 & 1 & -2 & 10\\ 0 & 1 & -5 & \frac{39}{2}\\ 0 & 1 & -\frac{15}{4} & 14\end{pmatrix}$$

$$\xrightarrow[R3 = R3 - R2]{R1 = R1 - R2} \begin{pmatrix}1 & 0 & 3 & -\frac{19}{2}\\ 0 & 1 & 3 & \frac{39}{2}\\ 0 & 0 & \frac{5}{4} & -\frac{11}{2}\end{pmatrix} \xrightarrow{R3 = \frac{4}{5}R3} \begin{pmatrix}1 & 0 & 3 & -\frac{19}{2}\\ 0 & 1 & 3 & \frac{39}{2}\\ 0 & 0 & 1 & -\frac{22}{5}\end{pmatrix} \xrightarrow[R2 = R2 - 3R3]{R1 = R1 - 3R3} \begin{pmatrix}1 & 0 & 0 & \frac{37}{10}\\ 0 & 1 & 0 & -\frac{5}{2}\\ 0 & 0 & 1 & -\frac{22}{5}\end{pmatrix}$$

Antwoord:
$V = \left\{\left(\frac{37}{10}, -\frac{5}{2}, -\frac{22}{5}\right)\right\}$

#### b)

$$\left(\begin{array}{ccc|c}1 & 2 & -3 & -1\\ 3 & -1 & 2 & 7\\ 5 & 3 & -4 & 2\end{array}\right) \xrightarrow[R3 = R3 - 5R1]{R2 = R2 - 3R1} \left(\begin{array}{ccc|c}1 & 2 & -3 & -1\\ 0 & -7 & 11 & 10\\ 0 & -7 & 11 & 7\end{array}\right)$$

Antwoord:
$V = \emptyset$

## 1.2

### a)

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 1 & -3 & 4 & k \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{7}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Antwoord:
$V = \left\{(\frac{3}{5} + \frac{1}{5}\lambda, -\frac{4}{5} + \frac{7}{5}\lambda, \lambda) | \lambda \in \mathbb{R}\right\}$

## 1.4

$(Ax + B)(x^2 - 1) + C(x^2 + 4)(x + 1) + D(x^2 + 4)(x - 1)$
$= Ax^3 - Ax + Bx^2 - B + cx^3 + cx^2 + 4Cx + 4C + Dx^3 - Dx^2 + 4Dx - 4D$

$$\begin{cases} A + C + D = 0 \\ B + C - D = 0 \\ -A + 4C + 4D = 0 \\ -B + 4C - 4D = 1 \end{cases} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 4 & 4 & 0 \\ 0 & -1 & 4 & -4 & 1 \end{array}\right) \overset{GRM}{\rightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & -\frac{1}{10} \end{array}\right)$$

Antwoord:

$$\begin{cases} A = 0 \\ B = -\frac{1}{5} \\ C = \frac{1}{10} \\ D = -\frac{1}{10} \end{cases}$$

## 1.5

$$\frac{1}{x(x^2 - b^2} = \frac{A}{x} + \frac{B}{x+b} + \frac{C}{x-b}$$

$$\Leftrightarrow \frac{Ax^2 - Ab^2 + Bx^2 + Bxb + Cx^2 - Cxb}{x(x^2 - b^2)}$$

$$\Leftrightarrow \begin{cases} A + B + C = 0 \\ Bb - Cb = 0 \\ -Ab^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} A = -\frac{1}{b^2} \\ B = \frac{1}{2b^2} \\ C = \frac{1}{2b^2} \end{cases}$$

$$\Leftrightarrow \frac{1}{x(x^2 - b^2} = -\frac{1}{b^2x^2} + \frac{1}{2b^2(x+b)} + \frac{1}{2b^2(x-b)}$$

## 1.6

### a)

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & a \end{pmatrix} \overset{\substack{R2 = R2 - 2R1 \\ R3 = R3 - 4R1}}{\rightarrow} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & (a-24) \end{pmatrix} \overset{\substack{R2 = -\frac{1}{5}R2 \\ R3 = -\frac{1}{5}R3}}{\rightarrow} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & -\frac{(a-24)}{5} \end{pmatrix}$$

Antwoord:
Voor $a \neq 14$ zal het stelsel geen oplossingen hebben. Voor $a = 14$ zal het stelsel oneindig veel oplossingen hebben.

(voor $a = 14$)

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Wat we nog kunnen rijreduceren tot:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We constateren dat $z$ hier een vrije variabele is en wijzen haar de waarde $\lambda$ toe. Dan is onze oplossingsverzameling de volgende:
$V = \{(2-\lambda, 2+2\lambda, \lambda) \,|\lambda \in \mathbb{R}\}$

### b)

$$\left(\begin{array}{cc|c} 1 & a & (a+1) \\ a & 1 & 2 \end{array}\right) \overset{R2 = R2 - aR1}{\rightarrow} \left(\begin{array}{cc|c} 1 & a & (a+1) \\ 0 & (1-a^2) & (a^2-a+2) \end{array}\right)$$

Antwoord:
Voor $a = 1$: oneindig veel oplossingen, $\begin{cases} x = 2\lambda \\ y = \lambda \end{cases}$
Voor $a = -1$: geen oplossingen.
Voor $a \neq 1 \wedge a \neq -1$: precies 1 oplossing, $\begin{cases} x = \frac{1}{a+1} \\ y = \frac{a+2}{a+1} \end{cases}$

### c)

$$\left(\begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ a & 1 & (a+1) & 0 \\ 2a & 1 & 1 & (a+1) \end{array}\right) \overset{\substack{R2 = R2 - R1 \\ R3 = R3 - 2R1}}{\rightarrow} \left(\begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & -a & a & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array}\right)$$

$$R2 = -\frac{1}{a}R2 \atop \rightarrow \left(\begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array}\right) \begin{array}{c} R1 = R1-(a+1)R2 \\ R3 = R3+(2a+1)R2 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2(a+1) & (a+1) \end{array}\right)$$

$$R3 = -\frac{2}{(a+1)}R3 \atop \rightarrow \left(\begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right) \begin{array}{c} R1 = R1-(a+2)R3 \\ R2 = R2+R3 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} a & 0 & 0 & \frac{a+2}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$R1 = \frac{1}{a}R1 \atop \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a+2}{2a} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

Antwoord:
Voor $a = 0$: geen oplossingen.

Voor $a = -1$: oneindig veel oplossingen: $\begin{cases} x = \lambda \\ y = \lambda \\ z = \lambda \end{cases}$

Voor $a \neq -1 \wedge a \neq 0$: precies 1 oplossing: $\begin{cases} x = \frac{a+2}{2a} \\ y = -\frac{1}{2} \\ z = -\frac{1}{2} \end{cases}$

## 1.8

$$\begin{pmatrix} k & 5 & 3 \\ 5 & 1 & -1 \\ k & 2 & 1 \end{pmatrix} \begin{array}{c} R1 = R1-\frac{k}{5}R2 \\ R3 = R3-\frac{k}{5}R2 \\ \rightarrow \end{array} \begin{pmatrix} 0 & 5-\frac{k}{5} & 3+\frac{k}{5} \\ 5 & 1 & -1 \\ 0 & 2-\frac{k}{5} & 1+\frac{k}{5} \end{pmatrix}$$

Antwoord:
Enkel als $k = 1$ zullen R1 en R3 lineair afhankelijk zijn en zullen er niet-triviale oplossingen zijn, anders is er enkel de triviale nuloplossing.

## 1.10

### a)

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ b1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \begin{array}{c} R1 = R1-3R2 \\ R3 = R3-R2 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 0 & -8 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \begin{array}{c} R1 = R2 \\ R2 = -\frac{1}{2}R3 \\ R3 = -\frac{1}{2}R1 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right)$$

$$\begin{array}{c} R1 = R1 - 3R2 \\ R3 = R3 - 4R2 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \begin{array}{c} R3 = \frac{1}{5}R3 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{c} R1 = R1 - 4R3 \\ R2 = R2 + R3 \\ \rightarrow \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Antwoord:
Enkel de tiriviale oplossing $\vec{x} = \vec{0}$, want $Rang(A) = n$

**c)**

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 2 & -1 & 7 \\ 5 & 5 & 4 \\ 9 & 7 & 6 \end{pmatrix} \begin{array}{c} R2 = R2 - 2R1 \\ R3 = R3 - 2R1 \\ R4 = R4 - 5R1 \\ R5 = R5 - 9R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -5 & 5 \\ 0 & -5 & 9 \\ 0 & -11 & 15 \end{pmatrix} \begin{array}{c} R2 = -\frac{1}{3}R2 \\ R2\ en\ R3\ zijn\ lineair \\ afhankelijk,\ schrap\ R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & 9 \\ 0 & -11 & 15 \end{pmatrix}$$

$$\begin{array}{c} R1 = R1 - 2R2 \\ R3 = R3 + 5R2 \\ R4 = R4 + 11R2 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{c} R3 = \frac{1}{4}R3 \\ R3\ en\ R4\ zijn\ lineair \\ afhankelijk,\ schrap\ R4 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} R1 = R1 - R3 \\ R2 = R2 + R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Antwoord:

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

## 1.12

**a)**

$$\begin{pmatrix} 1 & 5 & 2 & 3 \\ 4 & 2 & -1 & -6 \\ -5 & 1 & 3 & 11 \end{pmatrix} \begin{array}{c} R2 = R2 - 4R1 \\ R3 = R3 + 5R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & -18 & -9 & -18 \\ 0 & 26 & 13 & 26 \end{pmatrix} \begin{array}{c} R2 = -\frac{1}{9} \\ R2\ en\ R3\ zijn\ lineair \\ afhankelijk,\ schrap\ R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 5 & 2 & 3 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Antwoord:
Lineair afhankelijk. Een zo groot mogelijke lineair onafhankelijke deelverzamelijk wordt bv. gegeven door de eerste 2 vectoren.
$\lambda_3 = pmetp$ willekeurig en $\lambda_4 = t$ met t willekeurig.

$2\lambda_2 = -p - 2t \Rightarrow \lambda_2 = \frac{1}{2}p - t$
$\lambda_1 = -5(-\frac{1}{2}p - t) - 2p - 3t = \frac{5}{2}p + 5t - 2p = \frac{1}{2}p - 2t$

$$(\frac{1}{2} - 2t)\begin{pmatrix}1\\4\\-5\end{pmatrix} + (\frac{1}{2}p - t)\begin{pmatrix}5\\2\\1\end{pmatrix} + p\begin{pmatrix}2\\-1\\3\end{pmatrix} + t\begin{pmatrix}3\\-6\\11\end{pmatrix} = \vec{0}$$

## 1.13

### a)

$$\begin{pmatrix}1 & -4 & 2\\-4 & 1 & -2\\2 & -2 & -2\end{pmatrix} \overset{\substack{R2=R2+4R1\\R3=R3-2R1}}{\rightarrow} \begin{pmatrix}1 & -4 & 2\\0 & -15 & 6\\0 & 6 & -6\end{pmatrix} \overset{R3=\frac{6}{15}R2}{\rightarrow} \begin{pmatrix}1 & -4 & 2\\0 & -15 & 6\\0 & 0 & -\frac{36}{10}\end{pmatrix}$$

Antwoord:
Deze drie vectoren zijn lineair onafhankelijk.

### b)

$$\begin{pmatrix}1 & 1 & -1\\2 & -1 & -5\\-5 & 4 & 14\\3 & -1 & -17\end{pmatrix} \overset{\substack{R2=R2-2R1\\R3=R3+5R1\\R4=R4-3R1}}{\rightarrow} \begin{pmatrix}1 & 1 & -1\\0 & -3 & -3\\0 & 9 & 9\\0 & -4 & 14\end{pmatrix}$$

Antwoord:
Lineair afhankelijk, een zo groot mogelijk lineair onafhakelijke deelverzameling wordt bv. gegeven door de eerste 3 vectoren.

### c)

$$\begin{pmatrix}1 & -1 & 1 & 2\\-2 & 5 & 0 & -15\\3 & 3 & 2 & 1\end{pmatrix} \overset{\substack{R2=R2+2R1\\R3=R3-3R1}}{\rightarrow} \begin{pmatrix}1 & -1 & 1 & 2\\0 & 3 & 2 & -11\\0 & 6 & -1 & -5\end{pmatrix} \overset{R3=R3-2R2}{\rightarrow} \begin{pmatrix}1 & -1 & 1 & 2\\0 & 3 & 2 & -11\\0 & 0 & -5 & 17\end{pmatrix}$$

Antwoord:
Lineair onafhankelijk?

### 1.14

$$\begin{pmatrix} cos(\theta) & 1 \\ 1 & 2cos(\theta) \end{pmatrix} \overset{R1 = R1 - cos(\theta)R2}{\rightarrow} \begin{pmatrix} 0 & 1 - 2cos^2(\theta) \\ 1 & 2cos(\theta) \end{pmatrix}$$

Lineair afhankelijk asa

$$\begin{aligned} 1 - 2cos^2(\theta) &= 0 \\ cos^2(\theta) &= \frac{1}{2} \\ cos(\theta) &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \pm\frac{\pi}{4} \vee \theta = \pm\frac{3\pi}{4} \end{aligned}$$

## Hoofdstuk 2

### 2.1

#### a)

$$A + B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$$

#### f)

$$2P - Q = 2\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 0 & 7 \end{pmatrix}$$

#### k)

$$P.C = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} . \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} Onmogelijk$$

#### p)

$$\vec{b}.C = \begin{pmatrix} 2 & 5 & -2 \end{pmatrix} . \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} (-10 + 20 - 4) & (6 - 5 + 2) \end{pmatrix} = \begin{pmatrix} 6 & 3 \end{pmatrix}$$

**2.3**

**a)**

$$A.B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} = A.B \ (toevallig)$$

**b)**

$$A.B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$A.B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

**2.4**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \rightarrow \left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right) \overset{GRM}{\rightarrow} \begin{cases} a = 0 \\ b = 1 \\ c = -10 \\ d = 7 \end{cases}$$

**2.5**

**a)**

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

**b)**

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

**2.6**

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}^2+\begin{pmatrix}0&-i\\i&0\end{pmatrix}^2+\begin{pmatrix}1&0\\0&-1\end{pmatrix}^2$$
$$=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}3&0\\0&3\end{pmatrix}$$

**2.7**

**b)**

$$\begin{pmatrix}1&2\\2&0\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}1&2\\2&0\end{pmatrix}\Leftrightarrow\begin{cases}2c=2b\\b+2d=2a\\2a=c+2d\\2b=2c\end{cases}\leftrightarrow\begin{cases}a=d\\b=2a-2d\\c=2d-2a\\d=a\end{cases}$$

**2.8**

**a)**

Neen, want $2AB$ komt van $AB+BA$ en $AB\neq BA$.

**b)**

Ja.

**2.9**

**a)**

$$\left(\begin{array}{cc|cc}2&-3&1&0\\4&1&0&1\end{array}\right)\overset{R2=R2-2R1}{\rightarrow}\left(\begin{array}{cc|cc}2&-3&1&0\\0&7&-2&1\end{array}\right)\overset{R1=R1+\frac{3}{7}R2}{\rightarrow}\left(\begin{array}{cc|cc}2&0&\frac{1}{7}&\frac{3}{7}\\0&7&-2&1\end{array}\right)$$

$$\overset{\substack{R1=\frac{1}{2}R1\\R2=\frac{1}{7}R2}}{\rightarrow}\left(\begin{array}{cc|cc}1&0&\frac{1}{14}&\frac{3}{14}\\0&1&-\frac{2}{7}&\frac{1}{7}\end{array}\right)\quad Antwoord:\begin{pmatrix}\frac{1}{14}&\frac{3}{14}\\-\frac{2}{7}&\frac{1}{7}\end{pmatrix}$$

**f)**

$$\left(\begin{array}{cccc|cccc}3&4&0&0&1&0&0&0\\1&2&0&0&0&1&0&0\\0&0&3&1&0&0&1&0\\0&0&4&2&0&0&0&1\end{array}\right)\rightarrow\left(\begin{array}{cccc|cccc}1&2&0&0&0&1&0&0\\3&4&0&0&1&0&0&0\\0&0&3&1&0&0&1&0\\0&0&4&2&0&0&0&1\end{array}\right)\overset{R2=R2-3R1}{\rightarrow}$$

$$\left(\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \overset{R1=R1+R2}{\rightarrow} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \overset{R4=R4-\frac{4}{3}R3}{\rightarrow}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{4}{3} & 1 \end{array}\right) \overset{R3=R3-\frac{3}{2}R4}{\rightarrow} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 3 & -\frac{3}{2} \\ 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & -\frac{4}{3} & 1 \end{array}\right) \overset{\substack{R2=-\frac{1}{2}R2 \\ R3=\frac{1}{3}R3 \\ R4=\frac{3}{2}R4}}{\rightarrow}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & \frac{3}{2} \end{array}\right)$$

Antwoord:

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & \frac{3}{2} \end{pmatrix}$$

## 2.10

$$\left(\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{GRM}{\rightarrow} \left(\begin{array}{ccc|ccc} 19 & 0 & 0 & 2 & -1 & -9 \\ 0 & 19 & 0 & 5 & -2 & 1 \\ 0 & 0 & 19 & -5 & 12 & 13 \end{array}\right)$$

Antwoord:

$$\frac{1}{19}\begin{pmatrix} 2 & -1 & -9 \\ 4 & -2 & 1 \\ -5 & 12 & 13 \end{pmatrix}$$

## 2.14

### a)

$$\begin{pmatrix} cos(\theta_1) & -sin(\theta_1) \\ sin(\theta_1) & cos(\theta_1) \end{pmatrix} \cdot \begin{pmatrix} cos(\theta_2) & -sin(\theta_2) \\ sin(\theta_2) & cos(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2) & -(cos(\theta_1)sin(\theta_2) + cos(\theta_2)sin(\theta_1)) \\ sin(\theta_1)cos(\theta_2) + cos(\theta_1)sin(\theta_2) & cos(\theta_1)cos(\theta_2) - -sin(\theta_1)sin(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} cos(\theta_1+\theta_2) & -sin(\theta_1+\theta_2) \\ sin(\theta_1+\theta_2) & cos(\theta_1+\theta_2) \end{pmatrix}$$

### 2.17

#### a)

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad want \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (1*1-1*0) & (1*1-1*0) \\ (0*1-0*0) & (0*1-0*0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

## Hoofdstuk 3

### 3.1

$$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 5 \; D1 = \begin{vmatrix} 11 & 1 \\ 12 & 2 \end{vmatrix} = 10 \; D2 = \begin{vmatrix} 4 & 11 \\ 3 & 22 \end{vmatrix} = 15$$

$$x = 2 \; en \; y = 3$$

### 3.2

#### a)

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

#### b)

$$\begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 2$$

#### c)

$$\begin{vmatrix} cos(n\theta) & -sin(n\theta) \\ sin(n\theta) & cos(n\theta) \end{vmatrix} = cos^2(n\theta) + sin^2(n\theta) = 1$$

### 3.3

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 36 & 4 & 9 \end{vmatrix} = 2, \; D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 36 & 9 \end{vmatrix} = 4, \; D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & 4 & 36 \end{vmatrix} = 6$$

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ x_3 = \frac{D_3}{D}$$

Antwoord:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

**3.4**

**a)**

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = 2.\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1).\begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} + 3.\begin{vmatrix} 4 & 1 \\ -3 & 2 \end{vmatrix} = 10 - 2 + 33 = 41$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = -(-1).\begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} + 1.\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -2 + 11 + 32 = 41$$

**c)**

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a.\begin{vmatrix} a & b \\ c & a \end{vmatrix} - c.\begin{vmatrix} b & c \\ c & a \end{vmatrix} + b.\begin{vmatrix} b & c \\ a & b \end{vmatrix} = a.(a^2 - bc) - c.(ab - c^2) + b.(b^2 - ac) = a^3 + b^3 + c^3 + 3abc$$

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = -b.\begin{vmatrix} c & b \\ b & a \end{vmatrix} + a.\begin{vmatrix} a & c \\ b & a \end{vmatrix} - c.\begin{vmatrix} a & c \\ c & b \end{vmatrix} = -b.(ac - b^2) + a.(a^2 - bc) - c.(ab - c^2) = a^3 + b^3 + c^3 + 3abc$$

**e)**

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = 0.\begin{vmatrix} 0 & 1 \\ 6 & 0 \end{vmatrix} - 3.\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 2.\begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} = 18$$

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = -3.\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 0.\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 6.\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 18$$

## 3.6

### b)

$$\begin{vmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix} = 3.2.2 - 1.4.2 = 4$$

### c)

$$\begin{vmatrix} -2 & 6 & 17 & -5 \\ 0 & 3 & 22 & -12 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 0 & -6 \end{vmatrix} = (-2).3.4.(-6) = 144$$

### d)

$$\begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -2.2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 16$$

## 3.8

### b)

$$\begin{vmatrix} 1 & 2x & 3x^2 \\ 2x^3 & 3x^4 & 4x^5 \\ 3x^6 & 4x^7 & 6x^8 \end{vmatrix} \overset{\substack{R2 = R2 - 2x^3 R1 \\ R3 = R3 - 3x^6 R1}}{=} \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & -x^4 & -x^5 \\ 0 & -2x^7 & -3x^8 \end{vmatrix} = 0 \;\Leftrightarrow\; \begin{vmatrix} -x^4 & -x^5 \\ -2x^7 & -3x^8 \end{vmatrix} = 0$$

$$\Leftrightarrow 3x^12 - 2x^12 = x^12 = 0 \;\Leftrightarrow\; x = 0$$

## 3.9

### a)

$$\begin{vmatrix} 3 & 2 & -2 \\ 6 & 1 & 5 \\ -9 & 3 & 4 \end{vmatrix} \overset{\substack{R2 = R2 - 2R \\ R3 = R3 + 3R1}}{=} \begin{vmatrix} 3 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 9 & -2 \end{vmatrix} \overset{R3 = -\frac{1}{3}R3}{=} -3. \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 9 & -2 \end{vmatrix} \overset{R3 = R3 - 9R2}{=} -3. \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 25 \end{vmatrix}$$

$$= (-3) * 3 * 25 = -225$$

**b)**

$$\begin{vmatrix}1&0&1&1\\0&1&1&0\\1&0&0&1\\1&1&1&0\end{vmatrix} \underset{=}{\substack{R3=R3-R1\\R4=R4-R1}} \begin{vmatrix}1&0&1&1\\0&1&1&0\\0&0&-1&0\\0&1&0&-1\end{vmatrix} \underset{=}{R4=R4-R2}$$

$$\begin{vmatrix}1&0&1&1\\0&1&1&0\\0&0&-1&0\\0&0&-1&-1\end{vmatrix} \underset{=}{R4=R4-R3} \begin{vmatrix}1&0&1&1\\0&1&1&0\\0&0&-1&0\\0&0&0&-1\end{vmatrix} = 1$$

**3.11**

$$\begin{aligned}
A^\tau &= A^{-1} && \text{(1. Definitie orthogonale matrix)}\\
A.A^\tau &= I = A^\tau.A && \text{(2. Direct gevolg van definitie)}\\
det(A^\tau) &= det(A^{-1}) && \text{(3. (uit 1.))}\\
\frac{det(A^{-1})}{det(A)} &= 1 && \text{(4. (uit 3.))}\\
det(A.A^{-1}) &= det(I) = 1 && \text{(5. (uit 2.))}\\
det(A.A^{-1}) &= det(A).det(A^{-1}) && \text{(6. eigenschap determinant)}\\
\frac{det(A^{-1})}{det(A)} &= det(A).det(A^{-1}) && \text{(7. (uit 4. en 5.))}\\
\frac{1}{det(A)} &= det(A) && \text{(8.)}\\
det(A) &= 1 \vee det(A) = -1 && \text{(9.)}
\end{aligned}$$

# Hoofdstuk 4

## 4.1

$$\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

**b)**

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\0\\0\end{pmatrix}$$

**d)**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0-1 \\ 1-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

## 4.2

$$V : \vec{x} = \vec{p_0} + t_1(\vec{p_1} - \vec{p_0}) + t_2(\vec{p_2} - \vec{p_0})$$

**a)**

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

**b)**

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

## 4.4

**a)**

We berekenen een normaalvector:

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

Er bestaat een a zodat dit een eenheidsnormaal is:

$$a. \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

$$\sqrt{(-4a)^2 + (5a)^2 + (-3a)^2} = \sqrt{50a^2} = 1$$

Antwoord:

$$\begin{pmatrix} \frac{-4}{\sqrt{50}} \\ \frac{5}{\sqrt{50}} \\ \frac{-3}{\sqrt{50}} \end{pmatrix}$$

### 4.5

Stel:

$$Normaal: \ \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \ Positievector: \ \vec{p} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \ en \ \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Carthesische vergelijking van een vlak:

$$V: \ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Parametervergelijking van een vlak:

$$V: \ \vec{x} = \vec{p} + t_1\vec{r_1} + t_2\vec{r_2}$$

waarbij:

$$\vec{r_1} = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}, \ en \ \vec{r_2} = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

**a)**

$$V: \ 2(x - 1) + 1(y - 1) + 3(z - 2) = 0$$

$$\Leftrightarrow 2x + y + 3z = 9$$

$$V: \ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t2 \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

**b)**

$$V: \ -1(x - 2) + 4(y + 1) + 5(z - 3) = 0$$

$$\Leftrightarrow -x + 4y + 5z = 9$$

$$V: \ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t1 \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + t2 \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}$$

**4.6**

$$\begin{vmatrix} x & 1 & 1 & 0 \\ y & 0 & 2 & 1 \\ z & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow x. \begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} - y. \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} + z. \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 0$$

$$\Leftrightarrow 5x + y + 2z = 7$$

**4.7**

$$r_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \ richtvector: \ r_2 = \begin{pmatrix} 2-1 \\ 4-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$normaal: \ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$V: \ -(x-1) - (z-3) = 0$$

$$\Leftrightarrow x + z = 4$$

**4.8**

Stel:

$$Positievectoren: \ \vec{p}, \ en \ \vec{q}\ , \ \ Richtvector: (\vec{q} - \vec{p}), \ en \ Normalen: \ n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \ en \ n_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

Carthesische vergelijking van een rechte:

$$L: \ \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

Parametervergelijking van een rechte:

$$L: \ \vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

**a)**

$$L:\ \vec{x} = \begin{pmatrix}1\\2\\8\end{pmatrix} + t\begin{pmatrix}3\\-1\\-4\end{pmatrix}$$

$$\begin{pmatrix}a\\3\\b\end{pmatrix} = \begin{pmatrix}1\\2\\8\end{pmatrix} + t\begin{pmatrix}3\\-1\\-4\end{pmatrix} \Leftrightarrow \begin{cases} a = 1+3t \\ 3 = 2+(-1)t \\ b = 8-4t \end{cases} \Leftrightarrow \begin{cases} a = 1+3(-\frac{1}{3}) \\ t = -\frac{1}{3} \\ b = 8-4(-\frac{1}{3}) \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{3} \\ t = -\frac{1}{3} \\ b = 12 \end{cases}$$

**b)**

$$V:\ 3(x+4) - 2(y+0) + 6(z-3) = 0$$
$$\Leftrightarrow 3x - 2y + 6z = 6$$
$$\Leftrightarrow 3(1+3t) - 2(2-t) + 6(8-4t) = 6$$
$$\Leftrightarrow 9t + 2t - 24t + 3 - 4 + 48 = 6 \ \Leftrightarrow\ 9t + 2t - 24t + 3 - 4 + 48 = 6$$
$$\Leftrightarrow t = \frac{41}{13}$$

Antwoord:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\8\end{pmatrix} + \frac{41}{13}\begin{pmatrix}3\\-1\\-4\end{pmatrix} = \begin{pmatrix}\frac{136}{13}\\-\frac{15}{13}\\-\frac{60}{13}\end{pmatrix}$$

## 4.10

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Is een richtvector van de rechte. Dus:

$$L:\ \begin{pmatrix}1\\3\\2\end{pmatrix} + t.\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

Of:

$$L:\begin{cases} y &= 3 \\ x+z &= 3 \end{cases}$$

**4.18**

**a)**

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} \times \begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}9\\-1\\3\end{pmatrix}$$

$$\begin{pmatrix}0\\3\\1\end{pmatrix} \times \begin{pmatrix}1\\3\\-2\end{pmatrix} = \begin{pmatrix}-9\\1\\-3\end{pmatrix}$$

**d)**

$$\left(\begin{pmatrix}1\\3\\-2\end{pmatrix} \times \begin{pmatrix}0\\-1\\2\end{pmatrix}\right) . \begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}8\\-2\\1\end{pmatrix} . \begin{pmatrix}0\\3\\1\end{pmatrix} = -7$$

**e)**

$$\begin{pmatrix}1\\3\\-2\end{pmatrix} \times \left(\begin{pmatrix}0\\3\\1\end{pmatrix} \times \begin{pmatrix}0\\-1\\2\end{pmatrix}\right) = \begin{pmatrix}1\\3\\-2\end{pmatrix} \times \begin{pmatrix}7\\0\\0\end{pmatrix} = \begin{pmatrix}0\\14\\-21\end{pmatrix}$$

**4.22**

**a)**

$$d(\vec{p}, \vec{v})) = \frac{|10 - 2.1 - 3.1 - (-1).2|}{\sqrt{2^2 + 3^3 + 1^2}} = \frac{7}{\sqrt{14}}$$

**b)**

$$\vec{r_1} = \begin{pmatrix}-1\\0\\1\end{pmatrix}, \quad \vec{r_2} = \begin{pmatrix}-1\\1\\0\end{pmatrix}, \quad \vec{p} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\vec{r_1} \times \vec{r_2} = \begin{pmatrix}-1\\0\\1\end{pmatrix} \times \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix}$$

$$-(x-1) - y - z = 0$$

$$x + y + z = 1$$

afstand tot $(2,1,1)^\tau$:

$$d(\vec{p},v)=\frac{|1-2-1-1|}{\sqrt{3}}=\sqrt{3}$$

### 4.30

**a)**

$$\left|det\begin{pmatrix}1&-1&2\\1&4&2\\0&0&2\end{pmatrix}\right|$$

# Hoofdstuk 5

### 5.1

**a)**

$$\begin{vmatrix}2-\lambda&2\\1&3-\lambda\end{vmatrix}=0\quad\longrightarrow\quad\lambda^2-(2+3)\lambda+4=0\quad\longrightarrow\quad\lambda=4\ \vee\ \lambda=1$$

Voor $\lambda=4$:

$$\left(\begin{array}{cc|c}-2&2&0\\1&-1&0\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}0&0&0\\1&-1&0\end{array}\right)\longrightarrow eigenvector:\ c\begin{pmatrix}1\\1\end{pmatrix}$$

Voor $\lambda=1$:

$$\left(\begin{array}{cc|c}1&2&0\\1&2&0\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right)\longrightarrow eigenvector:\ c\begin{pmatrix}-2\\1\end{pmatrix}$$

**b)**

$$\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}=0\quad\longrightarrow\quad\lambda^2-(3+3)\lambda+8=0\quad\longrightarrow\quad\lambda=4\ \vee\ \lambda=2$$

Voor $\lambda=4$:

$$\left(\begin{array}{cc|c}-1&1&0\\1&-1&0\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}-1&1&0\\0&0&0\end{array}\right)\longrightarrow eigenvector:\ c\begin{pmatrix}1\\1\end{pmatrix}$$

Voor $\lambda=2$:

$$\left(\begin{array}{cc|c}1&1&0\\1&1&0\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1&1&0\\0&0&0\end{array}\right)\longrightarrow eigenvector:\ c\begin{pmatrix}-1\\1\end{pmatrix}$$

**e)**

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (1-\lambda)(\lambda^2-1^2)-2(2(-1-\lambda)) = 0 \quad \longrightarrow \quad \lambda = -1 \vee \lambda = 3$$

Voor $\lambda = -1$:

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Voor $\lambda = 3$:

$$\left(\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

**f)**

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad -\lambda(\lambda^2-4)-2(-2\lambda-4)+2(4+2\lambda) = 0 \quad \longrightarrow \quad \lambda = -2 \vee \lambda = 4$$

Voor $\lambda = -2$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Voor $\lambda = 4$

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

**g)**

$$\begin{vmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad -\lambda(\lambda^2-9)-3(-3\lambda) = 0 \quad \longrightarrow \quad \lambda = 0 \vee \lambda = 3\sqrt{2} \vee \lambda = -3\sqrt{2}$$

Voor $\lambda = 0$

$$\left(\begin{array}{ccc|c} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Voor $\lambda = 3\sqrt{2}$

$$\left(\begin{array}{ccc|c} -3\sqrt{2} & 3 & 0 & 0 \\ 3 & -3\sqrt{2} & 3 & 0 \\ 0 & 3 & -3\sqrt{2} & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector:\ c\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Voor $\lambda = -3\sqrt{2}$

$$\left(\begin{array}{ccc|c} 3\sqrt{2} & 3 & 0 & 0 \\ 3 & 3\sqrt{2} & 3 & 0 \\ 0 & 3 & 3\sqrt{2} & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector:\ c\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

## 5.2

### a)

TODO

### a) voor 5.1g

$$c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \overset{\sqrt{(-1a)^2+(0a)^2+a^2}=1}{\rightarrow} c\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$c\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \overset{\sqrt{a^2+(\sqrt{2}a)^2+a^2}=1}{\rightarrow} c\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$c\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \overset{\sqrt{a^2+(-\sqrt{2}a)^2+a^2}=1}{\rightarrow} c\begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}$$

### b) voor 5.1g

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}^2 \overset{!}{\neq} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TODO

## 5.4

$$det(A - \lambda I) = det(A) = 0$$

**5.5**

$$\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \lambda I\right)\vec{x} = \vec{0}$$

$$\Leftrightarrow \begin{vmatrix} a-\lambda & b \\ 0 & d-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (a-\lambda)(d-\lambda) = 0$$

$$\Leftrightarrow \lambda = a \vee \lambda = d$$

TODO BEWIJS DOOR INDUCTIE

**5.6**

**a)**

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

Diagonaliseerbaar.

**5.7**

**a)**

TODO

**b)**

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (1-\lambda)((1-\lambda)^2-1) = 0 \quad \longrightarrow \quad \lambda = 0 \vee \lambda = 1 \vee \lambda = 2$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Voor $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector:\ c\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Voor $\lambda = 1$:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector:\ c\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Voor $\lambda = 2$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector:\ c\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonaliseerbaar.

**c)**

Niet diagonaliseerbaar.

**e)**

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (-\lambda)^3+1 = 0 \quad \longrightarrow \quad \lambda = 1 \vee \lambda = -\frac{1}{2}-\frac{\sqrt{3}}{2}i \vee \lambda = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}$$

TODO

**e)**

Zie toledo
`https://cygnus.cc.kuleuven.be/bbcswebdav/pid-11087333-dt-content-rid-10466789_2/courses/a-G0017a-1213/Oefening%205_7_e.pdf`

## 5.8

### a)

$$\begin{vmatrix} 1+i-\lambda & 2-i \\ 3+i & -i-\lambda \end{vmatrix} = 0 \quad \longrightarrow$$

$$(1+i-\lambda)(-i-\lambda)-(2-i)(3+i)=0 \quad \longrightarrow \quad \lambda = 3 \ \vee \ \lambda = -2$$

Voor $\lambda = 3$:

$$\left(\begin{array}{cc|c} 4+i & 2-i & 0 \\ 3+i & -3-i & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c\begin{pmatrix}1\\1\end{pmatrix}$$

Voor $\lambda = -2$:

$$\left(\begin{array}{cc|c} 3+i & 2-i & 0 \\ 3+i & 2-i & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 3+i & 2+i & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c\begin{pmatrix}\frac{-i-7}{10}\\1\end{pmatrix}$$

### b)

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad -\lambda((-\lambda)^2+i^2)-i(i\lambda) \quad \longrightarrow \quad \lambda = 0 \vee \lambda = \sqrt{2} \vee \lambda = -\sqrt{2}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Voor $\lambda = 0$:

$$\left(\begin{array}{ccc|c} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c\begin{pmatrix}1\\0\\1\end{pmatrix}$$

Voor $\lambda = \sqrt{2}$:

$$\left(\begin{array}{ccc|c} -\sqrt{2} & -i & 0 & 0 \\ i & -\sqrt{2} & -i & 0 \\ 0 & i & -\sqrt{2} & 0 \end{array}\right) \quad \longrightarrow \quad TODO$$

Voor $\lambda = -\sqrt{2}$:

$$\left(\begin{array}{ccc|c} \sqrt{2} & -i & 0 & 0 \\ i & \sqrt{2} & -i & 0 \\ 0 & i & \sqrt{2} & 0 \end{array}\right) \quad \longrightarrow \quad TODO$$

**c)**

$$\begin{vmatrix} 2-\lambda & i \\ i & 2-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (2-\lambda)(1-\lambda)+1 = 0 \quad \longrightarrow \quad \lambda = \frac{3+i\sqrt{3}}{2} \vee \lambda = \frac{3-i\sqrt{3}}{2}$$

$$D = \begin{pmatrix} \frac{3+i\sqrt{3}}{2} & 0 \\ 0 & \frac{3-i\sqrt{3}}{2} \end{pmatrix}$$

Voor $\lambda = \frac{3+i\sqrt{3}}{2}$:

$$\left(\begin{array}{cc|c} 2-\frac{3+i\sqrt{3}}{2} & i & 0 \\ i & 1-\frac{3+i\sqrt{3}}{2} & 0 \end{array}\right) \quad \longrightarrow \quad TODO$$

Voor $\lambda = \frac{3-i\sqrt{3}}{2}$:

$$\left(\begin{array}{cc|c} 2-\frac{3-i\sqrt{3}}{2} & i & 0 \\ i & 1-\frac{3-i\sqrt{3}}{2} & 0 \end{array}\right) \quad \longrightarrow \quad TODO$$

## 5.11

**a)**

$$\begin{vmatrix} \alpha-\beta & \beta & \beta \\ \beta & \alpha-\beta & \beta \\ \beta & \beta & \alpha-\beta \end{vmatrix} = 0 \longrightarrow (-\lambda-\alpha-2\beta)(\lambda-\alpha+\beta)^2 = 0$$

## 5.13

1. Basisstap Het geldt voor $k = 1$ want (gegeven)

$$A^k = XD^kX^{-1}$$

2. Inductiestap: tel dat het klopt voor $k = n$ met $n > 1$ en $n \in$ N 3. Te bewijzen:

$$A^{k+1} = X.D^{k+1}.X^{-1}$$

$$A.A^k = X.D.D^k.X^{-1}$$

$$A.A^k = X.D.X^{-1} \;.\; XD^kX^{-1} = X.D.D^k.X^{-1}$$

QED

## 5.15

TODO

# Hoofdstuk 6

## 6.1

$$\frac{d\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}}{dt} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} \quad \frac{d\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}}{dt} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

OK

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$

OK

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (4-\lambda)(1-\lambda)+2=0 \quad \longrightarrow \quad \lambda = 2 \quad \vee \quad \lambda = 3$$

Voor $\lambda = 2$:

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Voor $\lambda = 3$:

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## 6.2

Stabiliteit:

$$\lambda_1 \geq 0 \quad of \quad \lambda_2 \geq 0$$

### a)

$$\vec{x'} = \begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix}\vec{x}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (5-\lambda)(-\lambda)+4=0 \quad \longrightarrow \quad \lambda = 4 \quad \vee \quad \lambda = 1$$

Voor $\lambda = 4$:

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector: \ c\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Voor $\lambda = 1$:

$$\left(\begin{array}{cc|c} 4 & 4 & 0 \\ -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector: \ c\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{4t}\begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Evenwichten:

$$5x + 4y = 0 \quad en \quad -x = 0$$

$$x = 0 \quad en \quad y = 0$$

De oorsprong is een onstabiel evenwicht.

**b)**

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = -1 \quad \vee \quad \lambda = 3$$

Voor $\lambda = -1$:

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector: \ c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Voor $\lambda = 3$:

$$\left(\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow eigenvector: \ c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x} = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Evenwichten:

$$x + y = 0 \quad en \quad x + y = 0$$

$$x = 0 \quad en \quad y = 0$$

De oorsprong is een onstabie evenwicht.

**c)**

$$\vec{x'} = \begin{pmatrix} 4 & -2 \\ 5 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 5 & -2-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \pm i$$

voor $\lambda = 1 + i$:

$$\left(\begin{array}{cc|c} 3-i & -2 & 0 \\ 5 & -3-i & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} 2 \\ 3+i \end{pmatrix}$$

voor $\lambda = 1ii$:

$$\left(\begin{array}{cc|c} 3+i & -2 & 0 \\ 5 & -3+i & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} 2 \\ 3-i \end{pmatrix}$$

$$x(t) = c_1 e^t (cos(t) + isin(t)) \begin{pmatrix} 2 \\ 3+i \end{pmatrix} + c_2 e^t (cos(-t) + isin(-t)) \begin{pmatrix} 2 \\ 3-i \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 e^{(1-i)t} \begin{pmatrix} 2 \\ 3+i \end{pmatrix} + c_2 e^{(1+i)t} \begin{pmatrix} 2 \\ 3-i \end{pmatrix} \longrightarrow c_1 = \frac{1}{2} \quad en \quad c_2 = \frac{1}{2}$$

$$\vec{x'} = e^t \begin{pmatrix} cos(t) - sin(t) \\ 2cos(t) - sin(t) \end{pmatrix}$$

## 6.3

**a)**

$$\vec{x'} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (4-\lambda)(3-\lambda) - 6 = 0 \quad \longrightarrow \quad \lambda = 6 \quad \vee \quad \lambda = 1$$

Voor $\lambda = 6$:

$$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor $\lambda = 1$:

$$\left(\begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad eigenvector: \ c \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\vec{x} = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 2 - 3 \end{pmatrix}$$

**c)**

$$\vec{x'} = \begin{pmatrix} 6 & -8 \\ 4-6 & \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 6-\lambda & -8 \\ 4 & -6-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm 2$$

voor $\lambda = 2$:

$$\left(\begin{array}{cc|c} 4 & -8 & 0 \\ 4 & -8 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

voor $\lambda = -2$:

$$\left(\begin{array}{cc|c} 8 & -8 & 0 \\ 4 & -4 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \longrightarrow \quad c_1 = 1 \quad en \quad c_2 = -2$$

$$\vec{x} = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## 6.5

**a)**

$$-k_1 a + k_2 b + k_1 a - (k_2 + k_3) b + k_3 b = 0$$

OK

**b)**

$$\vec{x'} = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -3 & 0 \\ 0 & 2 & 0 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 1 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} \longrightarrow \lambda = 0 \vee \lambda = -1 \vee \lambda = -4$$

voor $\lambda = 0$:

$$c\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

voor $\lambda = -1$:

$$c\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

voor $\lambda = -4$:

$$c\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

**6.6**

$$\begin{cases} x' = 2x + y \\ y' = 3x + 4y \\ z' = 5x - 6y + 32 \end{cases}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 3 & 4-\lambda & 0 \\ 5 & -6 & 3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \vee \lambda = 3 \vee \lambda = 5$$

voor $\lambda = 1$:

$$c\begin{pmatrix}-2\\2\\11\end{pmatrix}$$

voor $\lambda = 3$:

$$c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

voor $\lambda = 5$:

$$c\begin{pmatrix}2\\6\\-13\end{pmatrix}$$

$$\begin{pmatrix}5\\4\\3\end{pmatrix} = c_1\begin{pmatrix}-2\\2\\11\end{pmatrix} + c_2\begin{pmatrix}0\\0\\1\end{pmatrix} + c_3\begin{pmatrix}2\\6\\-13\end{pmatrix}$$

$$c_1 = -\frac{3}{2} \wedge c_2 = \frac{67}{2} \wedge c_3 = 1$$

$$\vec{x} = \frac{67}{2}e^{3t}\begin{pmatrix}0\\0\\1\end{pmatrix} + e^{5t}\begin{pmatrix}2\\6\\-13\end{pmatrix} - \frac{3}{2}e^{t}\begin{pmatrix}-2\\2\\11\end{pmatrix}$$

**6.7**

**a)**

$$x' = y \quad en \quad y' = -3y - 2x$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = -1 \vee \lambda = -2$$

voor $\lambda = -1$:

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad c\begin{pmatrix} -1 & 1 \end{pmatrix}$$

voor $\lambda = -2$:

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right) \quad \longrightarrow \quad \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \longrightarrow \quad c\begin{pmatrix} -1 & 2 \end{pmatrix}$$

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

**6.8**

**a)**

$$\vec{x'} = \begin{pmatrix} -2 & 4 \\ 1 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (-2-\lambda)(-4-\lambda)-4 = 0 \quad \longrightarrow \quad \lambda = \frac{-6+\sqrt{20}}{2} \quad \vee \quad \lambda = \frac{-6-\sqrt{20}}{2}$$

Evenwichten:

$$-2x + 4y = 0 \quad en \quad -4y + x = 0$$

$$x = 0 \quad en \quad y = 0$$

Stabiliteit:

$$\lambda_1 < 0 \quad en \quad \lambda_2 < 0$$

De oorsprong is een stabiel evenwicht.

**b)**

$$\begin{vmatrix} 2-\lambda & 4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-2 \pm \sqrt{52}}{2} \longrightarrow instabiel$$

**c)**

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm i\sqrt{2} \longrightarrow \textit{geen uitspraak mogelijk}$$

## 6.9

**a)**

$$\vec{F} = m.\vec{a}$$

$$\begin{pmatrix} 36x \\ 12x + 16y \end{pmatrix} = 4 \begin{pmatrix} x'' \\ y'' \end{pmatrix} \longrightarrow \begin{cases} x'' = 9 \\ y'' = 3x + 4y \end{cases}$$

**b)**

$$\begin{cases} x'' = 9 \\ y'' = 3x + 4y \\ x' = a \\ y' = b \end{cases} \longrightarrow \begin{cases} a = x' \\ b = y' \\ a' = 9x \\ b' = 3x + 4y \end{cases} \longrightarrow A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{pmatrix}$$

**c)**

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 9 & 0 & -\lambda & 0 \\ 3 & 4 & 0 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm 2 \vee \lambda = \pm 3$$

voor $\lambda = 3$:

$$\left(\begin{array}{cccc|c} -3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 9 & 0 & -3 & 0 & 0 \\ 3 & 4 & 0 & -3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{9} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} 5 \\ 3 \\ 15 \\ 9 \end{pmatrix}$$

voor $\lambda = -3$:

$$\left(\begin{array}{cccc|c} 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 9 & 0 & 3 & 0 & 0 \\ 3 & 4 & 0 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} -5 \\ -3 \\ 15 \\ 9 \end{pmatrix}$$

voor $\lambda = 2$:

$$\left(\begin{array}{cccc|c} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 9 & 0 & -2 & 0 & 0 \\ 3 & 4 & 0 & -2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

voor $\lambda = -2$:

$$\left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 9 & 0 & 2 & 0 & 0 \\ 3 & 4 & 0 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow c \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

## 6.10

### a)

Evenwichten:

$$x + xy = 0 \quad en \quad 2y - xy = 0$$

$$x = 0 \quad en \quad y = 0 \quad of \quad x = 2 \quad en \quad y = -1$$

A:

$$A = \begin{pmatrix} 1+y & x \\ -y & 2-x \end{pmatrix} [$$

voor $(0,0)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda_1 = 1 \quad en \quad \lambda_2 = 2$$

$$\lambda_1 \geq 0 \quad of \quad \lambda_2 \geq 0$$

$(0,0)$ is een onstabiel evenwicht.
voor $(-1,2)$:

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\lambda_1 = 7 \quad en \quad \lambda_2 = -1$$

$$\lambda_1 \geq 0 \quad of \quad \lambda_2 \geq 0$$

$(-1,2)$ is een onstabiel evenwicht.

**c)**

$$\begin{cases} x' = x - 2xy + xy^2 = 0 \\ y' = y + xy = 0 \end{cases}$$

evenwichten: $(0,0)$ en $(-1,1)$:

$$A = \begin{pmatrix} 1 - 2y + y^2 & -2x + 2xy \\ y & 1 + x \end{pmatrix}$$

voor $(0,0)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \lambda = 1 \longrightarrow instabiel$$

voor $(-1,1)$:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longrightarrow geen\ \lambda \longrightarrow geen\ uitspraak$$

## 6.11

**a)**

Lotka Voltura roofdier/prooidier model

$$x' = x(a - bx + cy) \quad en \quad y' = y(-k + lx)$$

hier: $a = 1$ $b = \frac{1}{4}$ $c = \frac{1}{4}$ $k = 1$ $p = 1$. x is roofdier, y is prooidier.

**b)**

$$-x + pxy = 0 \quad en \quad y - \frac{1}{4}y^2 - \frac{1}{4} = 0$$

$$\longrightarrow (0,0)\ (0,4)\ (\frac{1-4p}{p}, \frac{1}{p})$$

**c)**

$$A = \begin{pmatrix} -1 + py & px \\ \frac{y}{4} & 1 - \frac{y}{2} - \frac{x}{4} \end{pmatrix}$$

voor $(0,0)$:

$$\begin{vmatrix} p - 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \vee \lambda = p - 1 \quad instabiel$$

voor $(0,4)$:

$$\begin{vmatrix} 4p-1-\lambda & 0 \\ 4 & -1-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-4p-2 \pm \sqrt{16p^2+8p}}{2} \quad \textit{stabiel als } p < \frac{1}{4}$$

voor $(\frac{1-4p}{p}, \frac{1}{p})$

$$\begin{vmatrix} 0 & 1-4p \\ \frac{1}{4p^2} & 1-\frac{1}{2p}-\frac{1-4p}{4p} \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-\frac{3}{4p} \pm \sqrt{\frac{9-64p}{16p^2}}}{2} \quad \textit{stabiel als } p > \frac{1}{4}$$

## 13

### a)

$$\begin{cases} x' = x(a-bx-cy) \\ y' = y(p-qx-ry) \end{cases}$$

evenwichten:

$$(0,0)(0,\frac{p}{r})(\frac{a}{b},0)(\frac{cp-ar}{qc-rb},\frac{qa-pb}{qc-rb})$$

### b)

invullen:

$$(0,0)(0,2)(2,0)(\frac{2}{3},\frac{2}{3})$$

$$A = \begin{pmatrix} a-2xb-cy & -cx \\ -qy & p-qxy-2r \end{pmatrix} \longrightarrow \begin{pmatrix} -2x-2y+2 & -2x \\ -2y & -2x-2y+2 \end{pmatrix}$$

voor $(0,0)$:

$$\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \longrightarrow \textit{instabiel}$$

voor $(0,2)$:

$$\begin{pmatrix} -2-\lambda & 0 \\ -4 & -2-\lambda \end{pmatrix} \longrightarrow \lambda = -\frac{1}{2} \longrightarrow \textit{stabiel}$$

voor $(2,0)$:

$$\begin{pmatrix} -2-\lambda & -4 \\ 0 & -2-\lambda \end{pmatrix} \longrightarrow \lambda = -\frac{1}{2} \longrightarrow \textit{stabiel}$$

voor $(\frac{2}{3},\frac{2}{3})$:

$$\begin{pmatrix} -\frac{8}{3}+2 & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{8}{3}+2 \end{pmatrix} \longrightarrow \lambda = \frac{2}{3} \vee \lambda = -2 \longrightarrow \textit{instabiel}$$

# Hoofdstuk 7

## 7.11

### a)

convergentienterval

$$\sum_{k=0}^{\infty} 3^k x^k$$

$$L = \lim_{n\to\infty} \left| \frac{3^{k+1}x^{k+1}}{3^k x^k} \right| = \lim_{n\to\infty} |3x|$$

$$-1 < 3x < 1 \to -\frac{1}{3} < x < \frac{1}{3}$$

$$x \in \left] -\frac{1}{3}, \frac{1}{3} \right[$$

som $a = 1$ en $x = 3x$

$$\frac{1}{1-3x}$$

### b)

$$\sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{k+1}x^{2k+2}}{(-1)^k x^{2k}} \right| = |-x^2|$$

$$-1 < -x^2 < 1 \quad \longrightarrow \quad 1 > x^2 > -1 \quad \longrightarrow \quad 1 > x > -1$$

$$x \in ]-1, 1[$$

som $a = 1$ en $x = -x^2$

$$\frac{1}{1+x^2}$$

## 7.13

### a)

$$s_n = \frac{1}{1 - e^{-\frac{hv}{T}}}$$

**b)**

$$\lim_{T \to 0+} q(T) = \lim_{T \to 0+} \sum_{k=0}^{\infty} e^{-\frac{hkv}{T}} = 1$$

$$\lim_{T \to \infty} q(T) = \lim_{T \to \infty} \sum_{k=0}^{\infty} e^{-\frac{hkv}{T}} = \infty$$

## 7.16

**b)**

$$\lim_{k \to \infty} \ln\left(1 + \frac{1}{k}\right) = \ln(1) = 0$$

$$\begin{aligned}\sum_{k+1}^{\infty} \ln\left(1 + \frac{1}{n}\right) &= \ln(1+1) + \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{3}\right) + ... \\ &= \ln\left((1+1)(1+\frac{1}{2})(1+\frac{1}{3})...\right) \\ &= \ln\left(2(1+\frac{1}{2})(1+\frac{1}{3})...\right) \\ &= \ln\left(3(1+\frac{1}{3})...\right) \\ &= \ln(k+1)\end{aligned}$$

## 7.17

**a)**

$$\ln n < n \to \frac{1}{\ln n} > \frac{1}{n}$$

$$0 \leq \frac{1}{\ln n} \leq \frac{1}{n}$$

$$\sum \frac{1}{n} \textit{ is divergent} \to \sum \frac{1}{\ln n} \textit{ is ook divergent}$$

**b)**

$$\ln j < j \to \frac{\ln j}{j^3} < \frac{1}{j^2}$$

$$\lim_{j\to\infty} \frac{\frac{\ln j}{j^3}}{\frac{1}{j^2}} = \lim_{j\to\infty} \frac{\ln j}{j} = 0$$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} \text{ convergeert naar 1}$$

$$\text{Eig. 7.6.6.:} \sum_{j=1}^{\infty} \frac{\ln j}{j^3} \text{ is convergent}$$

## 7.18

**a)**

$$\lim_{k\to\infty} \frac{\binom{2k+2}{k+1}}{\binom{2k}{k}} = \lim_{k\to\infty} \frac{\frac{(2k+2)!}{(k+1)!(k+1)!}}{\frac{(2k)!}{k!k!}} = \lim_{k\to\infty} \frac{(2k+2)!}{(2k)!(k+1)(k+1)}$$

$$= \lim_{k\to\infty} \frac{2(k+1)(2k+1)(2k)!}{(2k)!(k+1)(k+1)}$$

$$= \lim_{k\to\infty} \frac{4k+2}{k+1} = 4 > 1 \to \text{divergent}$$

**b)**

$$\lim_{k\to\infty} \frac{\binom{2k+2}{k+1}^{-1}}{\binom{2k}{k}^{-1}} = \lim_{k\to\infty} \frac{\frac{(2k)!}{k!k!}}{\frac{(2k+2)!}{(k+1)!(k+1)!}} = \lim_{k\to\infty} \frac{(2k)!(k+1)(k+1)}{(2k+2)!}$$

$$= \lim_{k\to\infty} \frac{(2k)!(k+1)(k+1)}{2(k+1)(2k+1)(2k)!}$$

$$= \lim_{k\to\infty} \frac{k+1}{4k+2} = \frac{1}{4} < 1 \to \text{convergent}$$

**c)**

$$\text{Tip: } \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e^n}$$

# Hoofdstuk 8

## 8.1

$$R = \lim_{k\to\infty} \frac{C_k}{C_{k+1}}$$

**a)**

$$\lim_{k\to\infty} \frac{k^3}{(k+1)^3} = 1$$

**b)**

$$R = \lim_{k\to\infty} \frac{\binom{2k}{k}}{\binom{2k+2}{k+1}} = \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} = \frac{1}{4}$$

**d)**

$$R = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{(n-1)^2}} = R = \lim_{n\to\infty} \frac{(n-1)^2}{n^2} = 1$$

## 8.2

**d)**

convergentiestraal:

$$R = \lim_{k\to\infty} \frac{2^k}{2^{k+1}} = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2}$$

som: $a = 4x$ en $x = 2x$:

$$\frac{4x}{1-2x}$$

**8.3**

zie voorbeeild 7.2.3 p 133

$$R = \lim_{k\to\infty} \frac{\phi_n}{\phi_{n+1}} = \frac{1}{\phi} = \frac{2}{1+\sqrt{5}}$$

**8.5**

$$\sum_{k=0}^{n} \frac{f^{(n)}(0)}{k!} x^k$$

**a)**

$$f(x) = \frac{1}{3+0} = \frac{1}{3} \quad f'(x) = \frac{-1}{(3+0)^2} = -\frac{1}{9}$$
$$f''(x) = \frac{2}{3+0)^3} = \frac{6}{27} \quad f'''(x) = \frac{-6}{(3+x)^3} = -\frac{6}{81}$$

dus

$$\sum_{k=0}^{n} \frac{\frac{(-1)^k}{3^k} k!}{k!} x^k = \frac{1}{3} \sum_{k=0}^{n} (-1)^k \left(\frac{x}{3}\right)^k$$

**c)**

Ga dit niet manueel uitrekenen. Er is een truk voor. (zie p 158)

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!}$$

dus

$$\sin(2x^2) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{(2x^2)^{2k+1}}{(2k+1)!}$$

**d)**

zie c)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

dus

$$e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!}$$

**e)**

zie a)

$$\sin(x) = \sum_{k=0}^{\infty}(-1)^{2k}\frac{x^{2k+1}}{(2k+1)!}$$

dus

$$\frac{\sin(x)}{x} = \frac{\sum_{k=0}^{\infty}(-1)^{2k}\frac{x^{2k+1}}{(2k+1)!}}{x} = \sum_{k=0}^{\infty}(-1)^{2k}\frac{x^{2k}}{(2k+1)!}$$

## 8.7

**a)**

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{d\sum_{k=0}^{\infty} x^k}{dx} = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

**b)**

neem de afgeleide van het antwoord van a)

$$\frac{d(\sum_{k=0}^{\infty} kx^k)}{dx} = \sum_{k=0}^{\infty} k^2x^{k-1} = \frac{d(\frac{x}{(1-x)^2})}{dx} = \frac{1+x}{(1-x)^3}$$

dus

$$\sum_{k=0}^{\infty} k^2x^k = \frac{(1+x)x}{(1-x)^3}$$

## 8.10

Taylorreeks:

$$\sum_{n=0}^{\infty}\frac{f(a)^{(n)}}{n!}(x-a)^n$$

**a)**

$$f(x) = \ln(x) \quad f^{(n)}(x) = (-1)^{(n+1)} \frac{n!}{x^n} \text{ voor } n > 0$$

Taylorreeks:

$$\ln(a) + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n} \frac{(x-a)}{a}^n$$

# Hoofdstuk 9

## 9.1

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = 0$$

$$= \frac{1}{2} \int_{-l}^{l} \cos\left(\frac{n\pi x - m\pi x}{l}\right) dx - \frac{1}{2} \int_{-l}^{l} \cos\left(\frac{n\pi x + m\pi x}{l}\right) dx$$

$$= \frac{1}{2} \left[\frac{l \sin\left(\frac{n\pi - m\pi}{l} x\right]}{ln\pi - m\pi}\right]_{-l}^{l} - \frac{1}{2} \left[\frac{l \sin\left(\frac{n\pi + m\pi}{l} x\right]}{ln\pi + m\pi}\right]_{-l}^{l}$$

$$= \frac{1}{2} \left(-\frac{l}{n\pi - m\pi} \left(\sin(n\pi - m\pi) - \sin(m\pi - n\pi)\right)\right)$$

$$-\frac{1}{2} \left(-\frac{l}{n\pi + m\pi} \left(\sin(n\pi + m\pi) - \sin(-(m\pi + n\pi))\right)\right)$$

$$= 0$$

## 9.5

**b)**

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$= \frac{1}{\pi} \int_{o}^{\pi} \cos(nx) dx$$

$$= \frac{1}{n\pi} \left[\sin(nx)\right]_0^{\pi}$$

$$= \frac{1}{n\pi} (\sin(n\pi) - \sin(0)) = 0$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)dx \\
&= -\frac{1}{n\pi}\left[\cos(nx)\right]_0^{\pi} \\
&\frac{1}{n\pi}(\cos(n\pi) - \cos(0)) \\
&= \frac{\cos(n\pi) - 1}{n\pi}
\end{aligned}
$$

$$
f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1} sin((2n+1)x)
$$

**9.6**

$$
\begin{aligned}
\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1} sin((2n+1)x) &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} \\
\iff \sin((2n+1)x) &= (-1)^n \\
\iff \begin{cases} \sin((2n+1)x) = 1 \text{ als n even} \\ \sin((2n+1)x) = -1 \text{ als n oneven} \end{cases} &\iff x = \frac{\pi}{2}
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} \\
\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} &= (f(\frac{\pi}{2}) - \frac{1}{2})\frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}
$$

**9.8**

$$b_n = 0$$

$$a_n = \frac{2\omega}{\pi} \int_{\frac{-\pi}{2\omega}}^{\frac{\pi}{2\omega}} |\sin(\omega x)| \cos(nx) dx$$

$$= \frac{2\omega}{\pi} \int_0^{\frac{\pi}{2\omega}} \sin(\omega x) \cos(nx) dx$$

$$= \frac{4}{\pi(1 - 4n^2)}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\omega x)}{4n^2 - 1}$$