

# Oefeningen Wiskunde II 2013, 1e Bach Informtica

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## Hoofdstuk 1

### 1.1

a)

$$\begin{pmatrix} 1 & 1 & -2 & 10 \\ 4 & 2 & 2 & 1 \\ 6 & 2 & 3 & 4 \end{pmatrix} \begin{array}{l} R2 = R2 - 4R1 \\ R3 = R3 - 6R1 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 1 & -2 & 10 \\ 0 & -2 & 10 & -39 \\ 0 & -4 & 15 & -56 \end{pmatrix} \begin{array}{l} R2 = -\frac{1}{2}R2 \\ R3 = -\frac{3}{4}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 1 & -2 & 10 \\ 0 & 1 & -5 & \frac{39}{2} \\ 0 & 1 & -\frac{15}{4} & 14 \end{pmatrix}$$
$$\begin{array}{l} R1 = R1 - R2 \\ R3 = R3 - R2 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 3 & -\frac{19}{2} \\ 0 & 1 & 3 & \frac{39}{2} \\ 0 & 0 & \frac{5}{4} & -\frac{11}{2} \end{pmatrix} \begin{array}{l} R3 = \frac{4}{5}R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 3 & -\frac{19}{2} \\ 0 & 1 & 3 & \frac{39}{2} \\ 0 & 0 & 1 & -\frac{22}{5} \end{pmatrix} \begin{array}{l} R1 = R1 - 3R3 \\ R2 = R2 - 3R3 \\ \rightarrow \end{array} \begin{pmatrix} 1 & 0 & 0 & \frac{37}{10} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{22}{5} \end{pmatrix}$$

Antwoord:

$$V = \left\{ \left( \frac{37}{10}, -\frac{5}{2}, -\frac{22}{5} \right) \right\}$$

b)

$$\left( \begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array} \right) \begin{array}{l} R2 = R2 - 3R1 \\ R3 = R3 - 5R1 \\ \rightarrow \end{array} \left( \begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 0 & -7 & 11 & 10 \\ 0 & -7 & 11 & 7 \end{array} \right)$$

Antwoord:

$$V = \emptyset$$

## 1.2

a)

$$\left( \begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 3 & 1 & -2 & 1 \\ 1 & -3 & 4 & k \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & -\frac{7}{5} & -\frac{4}{5} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Antwoord:

$$V = \left\{ \left( \frac{3}{5} + \frac{1}{5}\lambda, -\frac{4}{5} + \frac{7}{5}\lambda, \lambda \right) \mid \lambda \in \mathbb{R} \right\}$$

## 1.4

$$\begin{aligned} & (Ax + B)(x^2 - 1) + C(x^2 + 4)(x + 1) + D(x^2 + 4)(x - 1) \\ &= Ax^3 - Ax + Bx^2 - B + cx^3 + cx^2 + 4Cx + 4C + Dx^3 - Dx^2 + 4Dx - 4D \end{aligned}$$

$$\left\{ \begin{array}{l} A + C + D = 0 \\ B + C - D = 0 \\ -A + 4C + 4D = 0 \\ -B + 4C - 4D = 1 \end{array} \right. \rightarrow \left( \begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 4 & 4 & 0 \\ 0 & -1 & 4 & -4 & 1 \end{array} \right) \xrightarrow{GRM} \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 0 & 1 & -\frac{1}{10} \end{array} \right)$$

Antwoord:

$$\left\{ \begin{array}{l} A = 0 \\ B = -\frac{1}{5} \\ C = \frac{1}{10} \\ D = -\frac{1}{10} \end{array} \right.$$

## 1.5

$$\begin{aligned} & \frac{1}{x(x^2 - b^2)} = \frac{A}{x} + \frac{B}{x + b} + \frac{C}{x - b} \\ \Leftrightarrow & \frac{Ax^2 - Ab^2 + Bx^2 + Bxb + Cx^2 - Cxb}{x(x^2 - b^2)} \\ \Leftrightarrow & \left\{ \begin{array}{l} A + B + C = 0 \\ Bb - Cb = 0 \\ -Ab^2 = 1 \end{array} \right. \\ \Leftrightarrow & \left\{ \begin{array}{l} A = -\frac{1}{b^2} \\ B = \frac{1}{2b^2} \\ C = \frac{1}{2b^2} \end{array} \right. \\ \Leftrightarrow & \frac{1}{x(x^2 - b^2)} = -\frac{1}{b^2x^2} + \frac{1}{2b^2(x + b)} + \frac{1}{2b^2(x - b)} \end{aligned}$$

## 1.6

a)

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & a \end{pmatrix} \xrightarrow{\substack{R2 = R2 - 2R1 \\ R3 = R3 - 4R1}} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & (a-24) \end{pmatrix} \xrightarrow{\substack{R2 = -\frac{1}{5}R2 \\ R3 = -\frac{1}{5}R3}} \begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & -\frac{(a-24)}{5} \end{pmatrix}$$

Antwoord:

Voor  $a \neq 14$  zal het stelsel geen oplossingen hebben. Voor  $a = 14$  zal het stelsel oneindig veel oplossingen hebben.

(voor  $a = 14$ )

$$\begin{pmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Wat we nog kunnen rijreducen tot:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We constateren dat  $z$  hier een vrije variabele is en wijzen haar de waarde  $\lambda$  toe. Dan is onze oplossingsverzameling de volgende:

$$V = \{(2 - \lambda, 2 + 2\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$$

b)

$$\left( \begin{array}{cc|c} 1 & a & (a+1) \\ a & 1 & 2 \end{array} \right) \xrightarrow{R2 = R2 - aR1} \left( \begin{array}{cc|c} 1 & a & (a+1) \\ 0 & (1-a^2) & (-a^2 - a + 2) \end{array} \right)$$

Antwoord:

Voor  $a = 1$ : oneindig veel oplossingen,  $\begin{cases} x = 2 - \lambda \\ y = \lambda \end{cases}$

Voor  $a = -1$ : geen oplossingen.

Voor  $a \neq 1 \wedge a \neq -1$ : precies 1 oplossing,  $\begin{cases} x = \frac{1}{a+1} \\ y = \frac{a+2}{a+1} \end{cases}$

c)

$$\left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ a & 1 & (a+1) & 0 \\ 2a & 1 & 1 & (a+1) \end{array} \right) \xrightarrow{\substack{R2 = R2 - R1 \\ R3 = R3 - 2R1}} \left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & -a & a & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array} \right)$$

$$\begin{aligned}
R2 &= -\frac{1}{a}R2 \left( \begin{array}{ccc|c} a & (a+1) & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -(2a+1) & -1 & (a+1) \end{array} \right) & R1 &= R1 - (a+1)R2 \\
&\xrightarrow{\quad} & R3 &= R3 + (2a+1)R2 \left( \begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2(a+1) & (a+1) \end{array} \right) \\
R3 &= -\frac{1}{2(a+1)}R3 \left( \begin{array}{ccc|c} a & 0 & (a+2) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right) & R1 &= R1 - (a+2)R3 \\
&\xrightarrow{\quad} & R2 &= R2 + R3 \left( \begin{array}{ccc|c} a & 0 & 0 & \frac{a+2}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right) \\
&& R1 &= \frac{1}{a}R1 \left( \begin{array}{ccc|c} 1 & 0 & 0 & \frac{a+2}{2a} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right)
\end{aligned}$$

Antwoord:

Voor  $a = 0$ : geen oplossingen.

Voor  $a = -1$ : oneindig veel oplossingen:  $\begin{cases} x = \lambda \\ y = \lambda \\ z = \lambda \end{cases}$

Voor  $a \neq -1 \wedge a \neq 0$ : precies 1 oplossing:  $\begin{cases} x = \frac{a+2}{2a} \\ y = -\frac{1}{2} \\ z = -\frac{1}{2} \end{cases}$

## 1.8

$$\begin{aligned}
&\left( \begin{array}{ccc} k & 5 & 3 \\ 5 & 1 & -1 \\ k & 2 & 1 \end{array} \right) & R1 &= R1 - \frac{k}{5}R2 \\
&& R3 &= R3 - \frac{k}{5}R2 \xrightarrow{\quad} \left( \begin{array}{ccc} 0 & 5 - \frac{k}{5} & 3 + \frac{k}{5} \\ 5 & 1 & -1 \\ 0 & 2 - \frac{k}{5} & 1 + \frac{k}{5} \end{array} \right)
\end{aligned}$$

Antwoord:

Enkel als  $k = 1$  zullen R1 en R3 lineair afhankelijk zijn en zullen er niet-triviale oplossingen zijn, anders is er enkel de triviale nuloplossing.

## 1.10

a)

$$\begin{aligned}
&\left( \begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array} \right) & R1 &= R1 - 3R2 \\
&& R3 &= R3 - R2 \xrightarrow{\quad} \left( \begin{array}{ccc|c} 0 & -8 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array} \right) & R1 &= R2 \\
&&&& R2 &= -\frac{1}{2}R3 \\
&&&& R3 &= -\frac{1}{2}R1 \xrightarrow{\quad} \left( \begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array} \right)
\end{aligned}$$

$$\begin{array}{l}
R1 = R1 - 3R2 \\
R3 = R3 - 4R2 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 0 \end{array} \right) \\
R3 = \frac{1}{5}R3 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \\
R1 = R1 - 4R3 \\
R2 = R2 + R3 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)
\end{array}$$

Antwoord:

Enkel de triviale oplossing  $\vec{x} = \vec{0}$ , want  $\text{Rang}(A) = n$

c)

$$\begin{array}{l}
R2 = R2 - 2R1 \\
R3 = R3 - 2R1 \\
R4 = R4 - 5R1 \\
R5 = R5 - 9R1 \rightarrow \left( \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & -11 & 15 & 0 \end{array} \right) \\
R2 = -\frac{1}{3}R2 \\
R2 \text{ en } R3 \text{ zijn lineair} \\
\text{afhankelijk, schrap } R3 \rightarrow \left( \begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 9 & 0 \\ 0 & -11 & 15 & 0 \end{array} \right)
\end{array}$$

$$\begin{array}{l}
R1 = R1 - 2R2 \\
R3 = R3 + 5R2 \\
R4 = R4 + 11R2 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right) \\
R3 = \frac{1}{4}R3 \\
R3 \text{ en } R4 \text{ zijn lineair} \\
\text{afhankelijk, schrap } R4 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \\
R1 = R1 - R3 \\
R2 = R2 + R3 \rightarrow \left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)
\end{array}$$

Antwoord:

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$

## 1.12

a)

$$\begin{array}{l}
R2 = R2 - 4R1 \\
R3 = R3 + 5R1 \rightarrow \left( \begin{array}{cccc|c} 1 & 5 & 2 & 3 & 0 \\ 0 & -18 & -9 & -18 & 0 \\ 0 & 26 & 13 & 26 & 0 \end{array} \right) \\
R2 = -\frac{1}{9}R2 \\
R2 \text{ en } R3 \text{ zijn lineair} \\
\text{afhankelijk, schrap } R3 \rightarrow \left( \begin{array}{cccc|c} 1 & 5 & 2 & 3 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)
\end{array}$$

Antwoord:

Lineair afhankelijk. Een zo groot mogelijke lineair onafhankelijke deelverzameling wordt bv. gegeven door de eerste 2 vectoren.

$\lambda_3 = pmetp$  willekeurig en  $\lambda_4 = t$  met  $t$  willekeurig.

$$2\lambda_2 = -p - 2t \Rightarrow \lambda_2 = \frac{1}{2}p - t$$

$$\lambda_1 = -5(-\frac{1}{2}p - t) - 2p - 3t = \frac{5}{2}p + 5t - 2p = \frac{1}{2}p - 2t$$

$$(\frac{1}{2} - 2t) \begin{pmatrix} 1 \\ 4 \\ -5 \end{pmatrix} + (\frac{1}{2}p - t) \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + p \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -6 \\ 11 \end{pmatrix} = \vec{0}$$

### 1.13

a)

$$\begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{matrix} R2 = R2 + 4R1 \\ R3 = R3 - 2R1 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -4 & 2 \\ 0 & -15 & 6 \\ 0 & 6 & -6 \end{pmatrix} \begin{matrix} R3 = \frac{6}{15}R2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -4 & 2 \\ 0 & -15 & 6 \\ 0 & 0 & -\frac{36}{10} \end{pmatrix}$$

Antwoord:

Deze drie vectoren zijn lineair onafhankelijk.

b)

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & -5 \\ -5 & 4 & 14 \\ 3 & -1 & -17 \end{pmatrix} \begin{matrix} R2 = R2 - 2R1 \\ R3 = R3 + 5R1 \\ R4 = R4 - 3R1 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & 9 & 9 \\ 0 & -4 & 14 \end{pmatrix}$$

Antwoord:

Deze vier vectoren zijn lineair onafhankelijk.

c)

$$\begin{pmatrix} 1 & -1 & 1 & 2 \\ -2 & 5 & 0 & -15 \\ 3 & 3 & 2 & 1 \end{pmatrix} \begin{matrix} R2 = R2 + 2R1 \\ R3 = R3 - 3R1 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 2 & -11 \\ 0 & 6 & -1 & -5 \end{pmatrix} \begin{matrix} R3 = R3 - 2R2 \\ \rightarrow \end{matrix} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 3 & 2 & -11 \\ 0 & 0 & -5 & 17 \end{pmatrix}$$

Antwoord:

Lineair afhankelijk: Een zo groot mogelijke lineair onafhankelijke deelverzameling wordt bv. gegeven door de eerste 3 vectoren.

## 1.14

$$\begin{pmatrix} \cos(\theta) & 1 \\ 1 & 2\cos(\theta) \end{pmatrix} R1 = R1 - \cos(\theta)R2 \xrightarrow{\quad} \begin{pmatrix} 0 & 1 - 2\cos^2(\theta) \\ 1 & 2\cos(\theta) \end{pmatrix}$$

Lineair afhankelijk asa

$$\begin{aligned} 1 - 2\cos^2(\theta) &= 0 \\ \cos^2(\theta) &= \frac{1}{2} \\ \cos(\theta) &= \pm \frac{\sqrt{2}}{2} \\ \theta &= \pm \frac{\pi}{4} \vee \theta = \pm \frac{3\pi}{4} \end{aligned}$$

## Hoofdstuk 2

### 2.1

a)

$$A + B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -4 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 4 \end{pmatrix}$$

f)

$$2P - Q = 2 \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ 0 & 7 \end{pmatrix}$$

k)

$$P.C = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} \text{ Onmogelijk}$$

p)

$$\vec{b}.C = (2 \ 5 \ -2) \cdot \begin{pmatrix} -5 & 3 \\ 4 & -1 \\ 2 & -1 \end{pmatrix} = ((-10 + 20 - 4) \ (6 - 5 + 2)) = (6 \ 3)$$

### 2.3

a)

$$A.B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} = B.A \text{ (toevallig)}$$

b)

$$A.B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$
$$B.A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

### 2.4

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \rightarrow \left( \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 4 \end{array} \right) \xrightarrow{GRM} \begin{cases} a = 0 \\ b = 1 \\ c = -10 \\ d = 7 \end{cases}$$

### 2.5

a)

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} -5 & 4 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$



2.6

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

2.7

b)

$$\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2c = 2b \\ b + 2d = 2a \\ 2a = c + 2d \\ 2b = 2c \end{cases} \Leftrightarrow \begin{cases} a = d \\ b = 2a - 2d \\ c = 2d - 2a \\ d = a \end{cases}$$

2.8

a)

Nee, want  $2AB$  komt van  $AB + BA$  en  $AB \neq BA$ .

b)

Ja.

2.9

a)

$$\left( \begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array} \right) \xrightarrow{R2 = R2 - 2R1} \left( \begin{array}{cc|cc} 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 1 \end{array} \right) \xrightarrow{R1 = R1 + \frac{3}{7}R2} \left( \begin{array}{cc|cc} 2 & 0 & \frac{1}{7} & \frac{3}{7} \\ 0 & 7 & -2 & 1 \end{array} \right)$$

$$\begin{matrix} R1 = \frac{1}{2}R1 \\ R2 = \frac{1}{7}R2 \end{matrix} \xrightarrow{} \left( \begin{array}{cc|cc} 1 & 0 & \frac{1}{14} & \frac{3}{14} \\ 0 & 1 & -\frac{2}{7} & \frac{1}{7} \end{array} \right) \quad \text{Antwoord: } \left( \begin{array}{cc|cc} \frac{1}{14} & \frac{3}{14} \\ -\frac{2}{7} & \frac{1}{7} \end{array} \right)$$

f)

$$\left( \begin{array}{cccc|cccc} 3 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R2 = R2 - 3R1}$$

$$\begin{pmatrix} 1 & 2 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & | & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & | & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R1 = R1 + R2} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & | & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & | & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R4 = R4 - \frac{4}{3}R3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & | & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{2}{3} & | & 0 & 0 & -\frac{4}{3} & 1 \end{pmatrix} \xrightarrow{R3 = R3 - \frac{3}{2}R4} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 & | & 1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & | & 0 & 0 & 3 & -\frac{3}{2} \\ 0 & 0 & 0 & \frac{2}{3} & | & 0 & 0 & -\frac{4}{3} & 1 \end{pmatrix} \xrightarrow{\begin{matrix} R2 = -\frac{1}{2}R2 \\ R3 = \frac{1}{3}R3 \\ R4 = \frac{3}{2}R4 \end{matrix}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & | & 0 & 0 & -2 & \frac{3}{2} \end{pmatrix}$$

Antwoord:

$$\begin{pmatrix} 1 & -2 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -2 & \frac{3}{2} \end{pmatrix}$$

## 2.10

$$\begin{pmatrix} 2 & 5 & 1 & | & 1 & 0 & 0 \\ 3 & 1 & 2 & | & 0 & 1 & 0 \\ -2 & 1 & 0 & | & 0 & 0 & 1 \end{pmatrix} \xrightarrow{GRM} \begin{pmatrix} 19 & 0 & 0 & | & 2 & -1 & -9 \\ 0 & 19 & 0 & | & 5 & -2 & 1 \\ 0 & 0 & 19 & | & -5 & 12 & 13 \end{pmatrix}$$

Antwoord:

$$\frac{1}{19} \begin{pmatrix} 2 & -1 & -9 \\ 4 & -2 & 1 \\ -5 & 12 & 13 \end{pmatrix}$$

## 2.14

a)

$$\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

## 2.17

a)

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{want} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (1*1 - 1*0) & (1*1 - 1*0) \\ (0*1 - 0*0) & (0*1 - 0*0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

## Hoofdstuk 3

### 3.1

$$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 5 \quad D1 = \begin{vmatrix} 11 & 1 \\ 12 & 2 \end{vmatrix} = 10 \quad D2 = \begin{vmatrix} 4 & 11 \\ 3 & 22 \end{vmatrix} = 15$$

$$x = 2 \text{ en } y = 3$$

### 3.2

a)

$$\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

b)

$$\begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix} = 2$$

c)

$$\begin{vmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{vmatrix} = \cos^2(n\theta) + \sin^2(n\theta) = 1$$

### 3.3

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 14 & 2 & 3 \\ 36 & 4 & 9 \end{vmatrix} = 2, \quad D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 14 & 3 \\ 1 & 36 & 9 \end{vmatrix} = 4, \quad D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 14 \\ 1 & 4 & 36 \end{vmatrix} = 6$$

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, x_3 = \frac{D_3}{D}$$

Antwoord:

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

### 3.4

a)

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 4 & -2 \\ -3 & 1 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 1 \\ -3 & 2 \end{vmatrix} = 10 - 2 + 33 = 41$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & -2 \end{vmatrix} = -2 + 11 + 32 = 41$$

c)

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a \cdot \begin{vmatrix} a & b \\ c & a \end{vmatrix} - c \cdot \begin{vmatrix} b & c \\ c & a \end{vmatrix} + b \cdot \begin{vmatrix} b & c \\ a & b \end{vmatrix} = a \cdot (a^2 - bc) - c \cdot (ab - c^2) + b \cdot (b^2 - ac) = a^3 + b^3 + c^3 + 3abc$$

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = -b \cdot \begin{vmatrix} c & b \\ b & a \end{vmatrix} + a \cdot \begin{vmatrix} a & c \\ b & a \end{vmatrix} - c \cdot \begin{vmatrix} a & c \\ c & b \end{vmatrix} = -b \cdot (ac - b^2) + a \cdot (a^2 - bc) - c \cdot (ab - c^2) = a^3 + b^3 + c^3 + 3abc$$

e)

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 0 & 1 \\ 6 & 0 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 0 \\ 2 & 6 \end{vmatrix} = 18$$

$$\begin{vmatrix} 0 & 3 & 2 \\ 2 & 0 & 1 \\ 2 & 6 & 0 \end{vmatrix} = -3 \cdot \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 6 \cdot \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = 18$$

### 3.6

b)

$$\begin{vmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix} = 3 \cdot 2 \cdot 2 - 1 \cdot 4 \cdot 2 = 4$$

c)

$$\begin{vmatrix} -2 & 6 & 17 & -5 \\ 0 & 3 & 22 & -12 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 0 & -6 \end{vmatrix} = (-2) \cdot 3 \cdot 4 \cdot (-6) = 144$$

d)

$$\begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{vmatrix} = -2 \begin{vmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -2 \cdot 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 16$$

### 3.8

b)

$$\begin{vmatrix} 1 & 2x & 3x^2 \\ 2x^3 & 3x^4 & 4x^5 \\ 3x^6 & 4x^7 & 6x^8 \end{vmatrix} \begin{matrix} R2 = R2 - 2x^3 R1 \\ R3 = R3 - 3x^6 R1 \\ = \end{matrix} \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & -x^4 & -x^5 \\ 0 & -2x^7 & -3x^8 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} -x^4 & -x^5 \\ -2x^7 & -3x^8 \end{vmatrix} = 0$$

$$\Leftrightarrow 3x^1 2 - 2x^1 2 = x^1 2 = 0 \Leftrightarrow x = 0$$

### 3.9

a)

$$\begin{vmatrix} 3 & 2 & -2 \\ 6 & 1 & 5 \\ -9 & 3 & 4 \end{vmatrix} \begin{matrix} R2 = R2 - 2R1 \\ R3 = R3 + 3R1 \\ = \end{matrix} \begin{vmatrix} 3 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & 9 & -2 \end{vmatrix} \begin{matrix} R3 = -\frac{1}{3}R3 \\ = \end{matrix} -3 \cdot \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 9 & -2 \end{vmatrix} \begin{matrix} R3 = R3 - 9R2 \\ = \end{matrix} -3 \cdot \begin{vmatrix} 3 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 25 \end{vmatrix}$$

$$= (-3) * 3 * 25 = -225$$

b)

$$\begin{array}{l}
 \left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array} \right| \begin{array}{l} R3 = R3 - R1 \\ R4 = R4 - R1 \\ = \end{array} \left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right| \begin{array}{l} R4 = R4 - R2 \\ = \end{array} \\
 \\
 \left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{array} \right| \begin{array}{l} R4 = R4 - R3 \\ = \end{array} \left| \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right| = 1
 \end{array}$$

### 3.11

$$A^T = A^{-1} \quad (1. \text{ Definitie orthogonale matrix})$$

$$A.A^T = I = A^T.A \quad (2. \text{ Direct gevolg van definitie})$$

$$\det(A^T) = \det(A^{-1}) \quad (3. \text{ (uit 1.)})$$

$$\frac{\det(A^{-1})}{\det(A)} = 1 \quad (4. \text{ (uit 3.)})$$

$$\det(A.A^{-1}) = \det(I) = 1 \quad (5. \text{ (uit 2.)})$$

$$\det(A.A^{-1}) = \det(A).\det(A^{-1}) \quad (6. \text{ eigenschap determinant})$$

$$\frac{\det(A^{-1})}{\det(A)} = \det(A).\det(A^{-1}) \quad (7. \text{ (uit 4. en 5.)})$$

$$\frac{1}{\det(A)} = \det(A) \quad (8.)$$

$$\det(A) = 1 \vee \det(A) = -1 \quad (9.)$$

## Hoofdstuk 4

### 4.1

$$\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

d)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0-1 \\ 1-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

4.2

$$V : \vec{x} = \vec{p}_0 + t_1(\vec{p}_1 - \vec{p}_0) + t_2(\vec{p}_2 - \vec{p}_0)$$

a)

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

b)

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

4.4

a)

We berekenen een normaalvector:

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

Er bestaat een  $a$  zodat dit een eenheidsnormaal is:

$$a \cdot \begin{pmatrix} -4 \\ 5 \\ -3 \end{pmatrix}$$

$$\sqrt{(-4a)^2 + (5a)^2 + (-3a)^2} = \sqrt{50a^2} = 1$$

Antwoord:

$$\begin{pmatrix} \frac{-4}{\sqrt{50}} \\ \frac{5}{\sqrt{50}} \\ \frac{-3}{\sqrt{50}} \end{pmatrix}$$

## 4.5

Stel:

$$\text{Normaal : } \vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \text{ Positievector : } \vec{p} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \text{ en } \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Cartesische vergelijking van een vlak:

$$V : a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Parametervergelijking van een vlak:

$$V : \vec{x} = \vec{p} + t_1 \vec{r}_1 + t_2 \vec{r}_2$$

waarbij:

$$\vec{r}_1 = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}, \text{ en } \vec{r}_2 = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

a)

$$V : 2(x - 1) + 1(y - 1) + 3(z - 2) = 0$$

$$\Leftrightarrow 2x + y + 3z = 9$$

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + t_1 \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

b)

$$V : -1(x - 2) + 4(y + 1) + 5(z - 3) = 0$$

$$\Leftrightarrow -x + 4y + 5z = 9$$

$$V : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} -4 \\ -1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}$$



#### 4.6

$$\begin{vmatrix} x & 1 & 1 & 0 \\ y & 0 & 2 & 1 \\ z & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow x \cdot \begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} - y \cdot \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} + z \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 0$$

$$\Leftrightarrow 5x + y + 2z = 7$$

#### 4.7

$$r_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \text{ richtvector : } r_2 = \begin{pmatrix} 2-1 \\ 4-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\text{normaal : } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$V : -(x-1) - (z-3) = 0$$

$$\Leftrightarrow x + z = 4$$

#### 4.8

Stel:

$$\text{Positievectoren : } \vec{p}, \text{ en } \vec{q}, \text{ Richtvector : } (\vec{q}-\vec{p}), \text{ en Normalen : } n_1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \text{ en } n_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

Cartesische vergelijking van een rechte:

$$L : \begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases}$$

Parametervergelijking van een rechte:

$$L : \vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

a)

$$L : \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} a \\ 3 \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} \Leftrightarrow \begin{cases} a = 1 + 3t \\ 3 = 2 + (-1)t \\ b = 8 - 4t \end{cases} \Leftrightarrow \begin{cases} a = 1 + 3(-\frac{1}{3}) \\ t = -\frac{1}{3} \\ b = 8 - 4(-\frac{1}{3}) \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{3} \\ t = -\frac{1}{3} \\ b = 12 \end{cases}$$

b)

$$V : 3(x+4) - 2(y+0) + 6(z-3) = 0$$

$$\Leftrightarrow 3x - 2y + 6z = 6$$

$$\Leftrightarrow 3(1+3t) - 2(2-t) + 6(8-4t) = 6$$

$$\Leftrightarrow 9t + 2t - 24t + 3 - 4 + 48 = 6 \Leftrightarrow 9t + 2t - 24t + 3 - 4 + 48 = 6$$

$$\Leftrightarrow t = \frac{41}{13}$$

Antwoord:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + \frac{41}{13} \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{136}{13} \\ -\frac{15}{13} \\ -\frac{60}{13} \end{pmatrix}$$

4.10

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Is een richtvector van de rechte. Dus:

$$L : \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Of:

$$L : \begin{cases} y = 3 \\ x + z = 3 \end{cases}$$

4.18

a)

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ -3 \end{pmatrix}$$

d)

$$\left( \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = -7$$

e)

$$\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \left( \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 14 \\ -21 \end{pmatrix}$$

4.22

a)

$$d(\vec{p}, \vec{v}) = \frac{|10 - 2 \cdot 1 - 3 \cdot 1 - (-1) \cdot 2|}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{7}{\sqrt{14}}$$

b)

$$\vec{r}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{r}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$-(x-1) - y - z = 0$$

$$x + y + z = 1$$

afstand tot  $(2, 1, 1)^T$ :

$$d(\vec{p}, v) = \frac{|1 - 2 - 1 - 1|}{\sqrt{3}} = \sqrt{3}$$

#### 4.30

a)

$$\left| \det \begin{pmatrix} 1 & -1 & 2 \\ 1 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \right|$$

### Hoofdstuk 5

#### 5.1

a)

$$\left| \begin{array}{cc|c} 2-\lambda & 2 & 0 \\ 1 & 3-\lambda & 0 \end{array} \right| = 0 \quad \longrightarrow \quad \lambda^2 - (2+3)\lambda + 4 = 0 \quad \longrightarrow \quad \lambda = 4 \vee \lambda = 1$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

b)

$$\left| \begin{array}{cc|c} 3-\lambda & 1 & 0 \\ 1 & 3-\lambda & 0 \end{array} \right| = 0 \quad \longrightarrow \quad \lambda^2 - (3+3)\lambda + 8 = 0 \quad \longrightarrow \quad \lambda = 4 \vee \lambda = 2$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

e)

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0 \longrightarrow (1-\lambda)(\lambda^2-1^2)-2(2(-1-\lambda)) = 0 \longrightarrow \lambda = -1 \vee \lambda = 3$$

Voor  $\lambda = -1$ :

$$\begin{pmatrix} 2 & 2 & 0 & | & 0 \\ 2 & 2 & 0 & | & 0 \\ 0 & 2 & 0 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{pmatrix} \longrightarrow \text{eigenvector} : c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 3$ :

$$\begin{pmatrix} -2 & 2 & 0 & | & 0 \\ 2 & -2 & 0 & | & 0 \\ 0 & 2 & -4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 1 & -2 & | & 0 \end{pmatrix} \longrightarrow \text{eigenvector} : c \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

f)

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \longrightarrow -\lambda(\lambda^2-4)-2(-2\lambda-4)+2(4+2\lambda) = 0 \longrightarrow \lambda = -2 \vee \lambda = 4$$

Voor  $\lambda = -2$

$$\begin{pmatrix} 2 & 2 & 2 & | & 0 \\ 2 & 2 & 2 & | & 0 \\ 2 & 2 & 2 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \longrightarrow \text{eigenvector} : c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 4$

$$\begin{pmatrix} -4 & 2 & 2 & | & 0 \\ 2 & -4 & 2 & | & 0 \\ 0 & 2 & -4 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

g)

$$\begin{vmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{vmatrix} = 0 \longrightarrow -\lambda(\lambda^2-9)-3(-3\lambda) = 0 \longrightarrow \lambda = 0 \vee \lambda = 3\sqrt{2} \vee \lambda = -3\sqrt{2}$$

Voor  $\lambda = 0$

$$\begin{pmatrix} 0 & 3 & 0 & | & 0 \\ 3 & 0 & 3 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 3\sqrt{2}$

$$\left( \begin{array}{ccc|c} -3\sqrt{2} & 3 & 0 & 0 \\ 3 & -3\sqrt{2} & 3 & 0 \\ 0 & 3 & -3\sqrt{2} & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Voor  $\lambda = -3\sqrt{2}$

$$\left( \begin{array}{ccc|c} 3\sqrt{2} & 3 & 0 & 0 \\ 3 & 3\sqrt{2} & 3 & 0 \\ 0 & 3 & 3\sqrt{2} & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

## 5.2

a)

TODO

a) voor 5.1g

$$\begin{aligned} c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{(-1a)^2 + (0a)^2 + a^2} = 1} c \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix} \\ c \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{a^2 + (\sqrt{2}a)^2 + a^2} = 1} c \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \\ c \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} & \xrightarrow{\sqrt{a^2 + (-\sqrt{2}a)^2 + a^2} = 1} c \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

b) voor 5.1g

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}^2 \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TODO

## 5.4

$$\det(A - \lambda I) = \det(A) = 0$$

### 5.5

$$\begin{aligned} & \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \lambda I \right) \vec{x} = \vec{0} \\ & \Leftrightarrow \begin{vmatrix} a - \lambda & b \\ 0 & d - \lambda \end{vmatrix} = 0 \\ & \Leftrightarrow (a - \lambda)(d - \lambda) = 0 \\ & \Leftrightarrow \lambda = a \vee \lambda = d \end{aligned}$$

TODO BEWIJS DOOR INDUCTIE

### 5.6

a)

$$\begin{aligned} D &= \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \\ & \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Diagonaliseerbaar.

### 5.7

a)

TODO

b)

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (1 - \lambda)((1 - \lambda)^2 - 1) = 0 \quad \longrightarrow \quad \lambda = 0 \vee \lambda = 1 \vee \lambda = 2$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Voor  $\lambda = 0$ :

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonaliseerbaar.

c)

Niet diagonaliseerbaar.

e)

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \longrightarrow (-\lambda)^3 + 1 = 0 \longrightarrow \lambda = 1 \vee \lambda = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \vee \lambda = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}$$

TODO

e)

Zie toledo

[https://cygnus.cc.kuleuven.be/bbcswebdav/pid-11087333-dt-content-rid-10466789\\_2/courses/a-G0017a-1213/0efening%205\\_7\\_e.pdf](https://cygnus.cc.kuleuven.be/bbcswebdav/pid-11087333-dt-content-rid-10466789_2/courses/a-G0017a-1213/0efening%205_7_e.pdf)



## 5.8

a)

$$\begin{vmatrix} 1+i-\lambda & 2-i \\ 3+i & -i-\lambda \end{vmatrix} = 0 \longrightarrow$$

$$(1+i-\lambda)(-i-\lambda) - (2-i)(3+i) = 0 \longrightarrow \lambda = 3 \vee \lambda = -2$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{cc|c} 4+i & 2-i & 0 \\ 3+i & -3-i & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = -2$ :

$$\left( \begin{array}{cc|c} 3+i & 2-i & 0 \\ 3+i & 2-i & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 3+i & 2+i & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} \frac{-i-7}{10} \\ 1 \end{pmatrix}$$

b)

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & -i \\ 0 & i & -\lambda \end{vmatrix} = 0 \longrightarrow -\lambda((- \lambda)^2 + i^2) - i(i\lambda) \longrightarrow \lambda = 0 \vee \lambda = \sqrt{2} \vee \lambda = -\sqrt{2}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

Voor  $\lambda = 0$ :

$$\left( \begin{array}{ccc|c} 0 & -i & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & i & 0 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Voor  $\lambda = \sqrt{2}$ :

$$\left( \begin{array}{ccc|c} -\sqrt{2} & -i & 0 & 0 \\ i & -\sqrt{2} & -i & 0 \\ 0 & i & -\sqrt{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

Voor  $\lambda = -\sqrt{2}$ :

$$\left( \begin{array}{ccc|c} \sqrt{2} & -i & 0 & 0 \\ i & \sqrt{2} & -i & 0 \\ 0 & i & \sqrt{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

c)

$$\begin{vmatrix} 2-\lambda & i \\ i & 2-\lambda \end{vmatrix} = 0 \longrightarrow (2-\lambda)(1-\lambda)+1 = 0 \longrightarrow \lambda = \frac{3+i\sqrt{3}}{2} \vee \lambda = \frac{3-i\sqrt{3}}{2}$$

$$D = \begin{pmatrix} \frac{3+i\sqrt{3}}{2} & 0 \\ 0 & \frac{3-i\sqrt{3}}{2} \end{pmatrix}$$

Voor  $\lambda = \frac{3+i\sqrt{3}}{2}$ :

$$\left( \begin{array}{cc|c} 2 - \frac{3+i\sqrt{3}}{2} & i & 0 \\ i & 1 - \frac{3+i\sqrt{3}}{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

Voor  $\lambda = \frac{3-i\sqrt{3}}{2}$ :

$$\left( \begin{array}{cc|c} 2 - \frac{3-i\sqrt{3}}{2} & i & 0 \\ i & 1 - \frac{3-i\sqrt{3}}{2} & 0 \end{array} \right) \longrightarrow \text{TODO}$$

## 5.11

a)

$$\begin{vmatrix} \alpha - \beta & \beta & \beta \\ \beta & \alpha - \beta & \beta \\ \beta & \beta & \alpha - \beta \end{vmatrix} = 0 \longrightarrow (-\lambda - \alpha - 2\beta)(\lambda - \alpha + \beta)^2 = 0$$

## 5.13

1. Basisstap Het geldt voor  $k = 1$  want (gegeven)

$$A^k = XD^kX^{-1}$$

2. Inductiestap: tel dat het klopt voor  $k = n$  met  $n > 1$  en  $n \in \mathbb{N}$  3. Te bewijzen:

$$A^{k+1} = X.D^{k+1}.X^{-1}$$

$$A.A^k = X.D.D^k.X^{-1}$$

$$A.A^k = X.D.X^{-1} . XD^kX^{-1} = X.D.D^k.X^{-1}$$

QED

## 5.15

TODO

## Hoofdstuk 6

### 6.1

$$\frac{d \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}}{dt} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix} \quad \frac{d \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}}{dt} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$
$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} = \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

OK

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 3e^{3t} \end{pmatrix}$$

OK

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (4-\lambda)(1-\lambda) + 2 = 0 \quad \longrightarrow \quad \lambda = 2 \quad \vee \quad \lambda = 3$$

Voor  $\lambda = 2$ :

$$\left( \begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array} \right) \quad \longrightarrow \quad \left( \begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array} \right) \quad \longrightarrow \quad \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \longrightarrow \quad \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

### 6.2

Stabiliteit:

$$\lambda_1 \geq 0 \quad \text{of} \quad \lambda_2 \geq 0$$

a)

$$\vec{x}' = \begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 5-\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \longrightarrow \quad (5-\lambda)(-\lambda) + 4 = 0 \quad \longrightarrow \quad \lambda = 4 \quad \vee \quad \lambda = 1$$

Voor  $\lambda = 4$ :

$$\left( \begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 4 & 4 & 0 \\ -1 & -1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x} = c_1 e^{4t} \begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Evenwichten:

$$5x + 4y = 0 \quad \text{en} \quad -x = 0$$

$$x = 0 \quad \text{en} \quad y = 0$$

De oorsprong is een onstabiel evenwicht.

b)

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = -1 \quad \vee \quad \lambda = 3$$

Voor  $\lambda = -1$ :

$$\left( \begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Voor  $\lambda = 3$ :

$$\left( \begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Evenwichten:

$$x + y = 0 \quad \text{en} \quad x + y = 0$$

$$x = 0 \quad \text{en} \quad y = 0$$

De oorsprong is een onstabiel evenwicht.

c)

$$\vec{x}' = \begin{pmatrix} 4 & -2 \\ 5 & -2 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4 - \lambda & -2 \\ 5 & -2 - \lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \pm i$$

voor  $\lambda = 1 + i$ :

$$\left( \begin{array}{cc|c} 3-i & -2 & 0 \\ 5 & -3-i & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} 2 \\ 3+i \end{pmatrix}$$

voor  $\lambda = 1 - i$ :

$$\left( \begin{array}{cc|c} 3+i & -2 & 0 \\ 5 & -3+i & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} 2 \\ 3-i \end{pmatrix}$$

$$x(t) = c_1 e^t (\cos(t) + i \sin(t)) \begin{pmatrix} 2 \\ 3+i \end{pmatrix} + c_2 e^t (\cos(-t) + i \sin(-t)) \begin{pmatrix} 2 \\ 3-i \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 e^{(1-i)t} \begin{pmatrix} 2 \\ 3+i \end{pmatrix} + c_2 e^{(1+i)t} \begin{pmatrix} 2 \\ 3-i \end{pmatrix} \longrightarrow c_1 = \frac{1}{2} \quad \text{en} \quad c_2 = \frac{1}{2}$$

$$\vec{x}' = e^t \begin{pmatrix} \cos(t) - \sin(t) \\ 2\cos(t) - \sin(t) \end{pmatrix}$$

## 6.3

a)

$$\vec{x}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 4 - \lambda & 2 \\ 3 & 3 - \lambda \end{vmatrix} = 0 \longrightarrow (4 - \lambda)(3 - \lambda) - 6 = 0 \longrightarrow \lambda = 6 \quad \vee \quad \lambda = 1$$

Voor  $\lambda = 6$ :

$$\left( \begin{array}{cc|c} -2 & 2 & 0 \\ 3 & -3 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Voor  $\lambda = 1$ :

$$\left( \begin{array}{cc|c} 3 & 2 & 0 \\ 3 & 2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \text{eigenvector} : c \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\vec{x} = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

c)

$$\vec{x}' = \begin{pmatrix} 6 & -8 \\ 4-6 & \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} 6-\lambda & -8 \\ 4 & -6-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm 2$$

voor  $\lambda = 2$ :

$$\left( \begin{array}{cc|c} 4 & -8 & 0 \\ 4 & -8 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

voor  $\lambda = -2$ :

$$\left( \begin{array}{cc|c} 8 & -8 & 0 \\ 4 & -4 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \longrightarrow c_1 = 1 \text{ en } c_2 = -2$$

$$\vec{x} = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## 6.5

a)

$$-k_1a + k_2b + k_1a - (k_2 + k_3)b + k_3b = 0$$

OK

b)

$$\vec{x}' = \begin{pmatrix} -2 & 1 & 0 \\ 2 & -3 & 0 \\ 0 & 2 & 0 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 1 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} \longrightarrow \lambda = 0 \vee \lambda = -1 \vee \lambda = -4$$

voor  $\lambda = 0$ :

$$c \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

voor  $\lambda = -1$ :

$$c \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

voor  $\lambda = -4$ :

$$c \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

## 6.6

$$\begin{cases} x' = 2x + y \\ y' = 3x + 4y \\ z' = 5x - 6y + 32 \end{cases}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 3 & 4-\lambda & 0 \\ 5 & -6 & 3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \vee \lambda = 3 \vee \lambda = 5$$

voor  $\lambda = 1$ :

$$c \begin{pmatrix} -2 \\ 2 \\ 11 \end{pmatrix}$$

voor  $\lambda = 3$ :

$$c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

voor  $\lambda = 5$ :

$$c \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 2 \\ 11 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix}$$

$$c_1 = -\frac{3}{2} \wedge c_2 = \frac{67}{2} \wedge c_3 = 1$$

$$\vec{x} = \frac{67}{2} e^{3t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + e^{5t} \begin{pmatrix} 2 \\ 6 \\ -13 \end{pmatrix} - \frac{3}{2} e^t \begin{pmatrix} -2 \\ 2 \\ 11 \end{pmatrix}$$

## 6.7

a)

$$x' = y \quad \text{en} \quad y' = -3y - 2x$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = -1 \vee \lambda = -2$$

voor  $\lambda = -1$ :

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

voor  $\lambda = -2$ :

$$\left( \begin{array}{cc|c} 2 & 1 & 0 \\ -2 & 1 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\vec{x} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

## 6.8

a)

$$\vec{x}' = \begin{pmatrix} -2 & 4 \\ 1 & -4 \end{pmatrix} \vec{x}$$

$$\begin{vmatrix} -2-\lambda & 4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \longrightarrow (-2-\lambda)(-4-\lambda)-4 = 0 \longrightarrow \lambda = \frac{-6+\sqrt{20}}{2} \vee \lambda = \frac{-6-\sqrt{20}}{2}$$

Evenwichten:

$$-2x + 4y = 0 \quad \text{en} \quad -4y + x = 0$$

$$x = 0 \quad \text{en} \quad y = 0$$

Stabiliteit:

$$\lambda_1 < 0 \quad \text{en} \quad \lambda_2 < 0$$

De oorsprong is een stabiel evenwicht.

b)

$$\begin{vmatrix} 2-\lambda & 4 \\ 1 & -4-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-2 \pm \sqrt{52}}{2} \longrightarrow \text{instabiel}$$



c)

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm i\sqrt{2} \longrightarrow \text{geen uitspraak mogelijk}$$

## 6.9

a)

$$\vec{F} = m \cdot \vec{a}$$

$$\begin{pmatrix} 36x \\ 12x + 16y \end{pmatrix} = 4 \begin{pmatrix} x'' \\ y'' \end{pmatrix} \longrightarrow \begin{cases} x'' = 9 \\ y'' = 3x + 4y \end{cases}$$

b)

$$\begin{cases} x'' = 9 \\ y'' = 3x + 4y \\ x' = a \\ y' = b \end{cases} \longrightarrow \begin{cases} a = x' \\ b = y' \\ a' = 9x \\ b' = 3x + 4y \end{cases} \longrightarrow A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{pmatrix}$$

c)

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 9 & 0 & -\lambda & 0 \\ 3 & 4 & 0 & -\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \pm 2 \vee \lambda = \pm 3$$

voor  $\lambda = 3$ :

$$\left( \begin{array}{cccc|c} -3 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 9 & 0 & -3 & 0 & 0 \\ 3 & 4 & 0 & -3 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{5}{9} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} 5 \\ 3 \\ 15 \\ 9 \end{pmatrix}$$

voor  $\lambda = -3$ :

$$\left( \begin{array}{cccc|c} 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 9 & 0 & 3 & 0 & 0 \\ 3 & 4 & 0 & 3 & 0 \end{array} \right) \longrightarrow \left( \begin{array}{cccc|c} 1 & 0 & 0 & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow c \begin{pmatrix} -5 \\ -3 \\ 15 \\ 9 \end{pmatrix}$$

voor  $\lambda = 2$ :

$$\left( \begin{array}{cccc|c} -2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 9 & 0 & -2 & 0 & 0 \\ 3 & 4 & 0 & -2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \rightarrow c \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

voor  $\lambda = -2$ :

$$\left( \begin{array}{cccc|c} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 9 & 0 & 2 & 0 & 0 \\ 3 & 4 & 0 & 2 & 0 \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \rightarrow c \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

## 6.10

a)

Evenwichten:

$$\begin{aligned} x + xy = 0 \quad \text{en} \quad 2y - xy = 0 \\ x = 0 \quad \text{en} \quad y = 0 \quad \text{of} \quad x = 2 \quad \text{en} \quad y = -1 \end{aligned}$$

A:

$$A = \begin{pmatrix} 1+y & x \\ -y & 2-x \end{pmatrix}$$

voor  $(0,0)$ :

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda_1 = 1 \quad \text{en} \quad \lambda_2 = 2$$

$$\lambda_1 \geq 0 \quad \text{of} \quad \lambda_2 \geq 0$$

$(0,0)$  is een onstabiel evenwicht.

voor  $(-1,2)$ :

$$\begin{pmatrix} 3 & -1 \\ -2 & 3 \end{pmatrix}$$

$$\lambda_1 = 7 \quad \text{en} \quad \lambda_2 = -1$$

$$\lambda_1 \geq 0 \quad \text{of} \quad \lambda_2 \geq 0$$

$(-1,2)$  is een onstabiel evenwicht.

c)

$$\begin{cases} x' = x - 2xy + xy^2 = 0 \\ y' = y + xy = 0 \end{cases}$$

evenwichten:  $(0, 0)$  en  $(-1, 1)$ :

$$A = \begin{pmatrix} 1 - 2y + y^2 & -2x + 2xy \\ y & 1 + x \end{pmatrix}$$

voor  $(0, 0)$ :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \lambda = 1 \longrightarrow \textit{instabiel}$$

voor  $(-1, 1)$ :

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longrightarrow \textit{geen } \lambda \longrightarrow \textit{geen uitspraak}$$

## 6.11

a)

Lotka Voltura roofdier/prooidier model

$$x' = x(a - bx + cy) \quad \text{en} \quad y' = y(-k + lx)$$

hier:  $a = 1$   $b = \frac{1}{4}$   $c = \frac{1}{4}$   $k = 1$   $p = 1$ .  $x$  is roofdier,  $y$  is prooidier.

b)

$$\begin{aligned} -x + pxy &= 0 \quad \text{en} \quad y - \frac{1}{4}y^2 - \frac{1}{4} = 0 \\ &\longrightarrow (0, 0) \quad (0, 4) \quad \left(\frac{1-4p}{p}, \frac{1}{p}\right) \end{aligned}$$

c)

$$A = \begin{pmatrix} -1 + py & px \\ \frac{y}{4} & 1 - \frac{y}{2} - \frac{x}{4} \end{pmatrix}$$

voor  $(0, 0)$ :

$$\begin{vmatrix} p - 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \longrightarrow \lambda = 1 \vee \lambda = p - 1 \quad \textit{instabiel}$$

voor  $(0, 4)$ :

$$\begin{vmatrix} 4p-1-\lambda & 0 \\ 4 & -1-\lambda \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-4p-2 \pm \sqrt{16p^2+8p}}{2} \quad \text{stabiel als } p < \frac{1}{4}$$

voor  $(\frac{1-4p}{p}, \frac{1}{p})$

$$\begin{vmatrix} 0 & 1-4p \\ \frac{1}{4p^2} & 1-\frac{1}{2p}-\frac{1-4p}{4p} \end{vmatrix} = 0 \longrightarrow \lambda = \frac{-\frac{3}{4p} \pm \sqrt{\frac{9-64p}{16p^2}}}{2} \quad \text{stabiel als } p > \frac{1}{4}$$

### 13

a)

$$\begin{cases} x' = x(a - bx - cy) \\ y' = y(p - qx - ry) \end{cases}$$

evenwichten:

$$(0, 0) (0, \frac{p}{r}) (\frac{a}{b}, 0) (\frac{cp-ar}{qc-rb}, \frac{qa-pb}{qc-rb})$$

b)

invullen:

$$(0, 0) (0, 2) (2, 0) (\frac{2}{3}, \frac{2}{3})$$

$$A = \begin{pmatrix} a-2xb-cy & -cx \\ -qy & p-qxy-2r \end{pmatrix} \longrightarrow \begin{pmatrix} -2x-2y+2 & -2x \\ -2y & -2x-2y+2 \end{pmatrix}$$

voor  $(0, 0)$ :

$$\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} \longrightarrow \text{instabiel}$$

voor  $(0, 2)$ :

$$\begin{pmatrix} -2-\lambda & 0 \\ -4 & -2-\lambda \end{pmatrix} \longrightarrow \lambda = -\frac{1}{2} \longrightarrow \text{stabiel}$$

voor  $(2, 0)$ :

$$\begin{pmatrix} -2-\lambda & -4 \\ 0 & -2-\lambda \end{pmatrix} \longrightarrow \lambda = -\frac{1}{2} \longrightarrow \text{stabiel}$$

voor  $(\frac{2}{3}, \frac{2}{3})$ :

$$\begin{pmatrix} -\frac{8}{3}+2 & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{8}{3}+2 \end{pmatrix} \longrightarrow \lambda = \frac{2}{3} \vee \lambda = -2 \longrightarrow \text{instabiel}$$

## Hoofdstuk 7

### 7.1

a)

$$T_n = 3n + 1 \text{ en } T_n = \begin{cases} 1 & \text{als } n = 0 \\ T_{n-1} + 3 & \text{als } n > 0 \end{cases}$$

b)

$$T_n = 3^n \text{ en } T_n = \begin{cases} 1 & \text{als } n = 0 \\ 3T_{n-1} & \text{als } n > 0 \end{cases}$$

c)

$$T_n = \left(\frac{-1}{5}\right)^n \text{ en } T_n = \begin{cases} 1 & \text{als } n = 0 \\ \frac{-T_{n-1}}{5} & \text{als } n > 0 \end{cases}$$

### 7.2

a)

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{1}{2} \\ a_3 &= 1 \\ a_4 &= \frac{3}{2} \\ a_5 &= 2 \\ a_6 &= \frac{5}{2} \end{aligned}$$

b)

$$\begin{aligned} b_1 &= 1 \\ b_2 &= \frac{2}{3} \\ b_3 &= \frac{4}{9} \\ b_4 &= \frac{8}{27} \\ b_5 &= \frac{16}{81} \\ b_6 &= \frac{32}{243} \end{aligned}$$

c)

$$\begin{aligned} c_1 &= \frac{1}{3} \\ c_2 &= \frac{1}{6} \\ c_3 &= \frac{1}{12} \\ c_4 &= \frac{1}{20} \\ c_5 &= \frac{1}{36} \\ c_6 &= \frac{1}{42} \end{aligned}$$

d)

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 1 \\ u_3 &= \frac{1}{2} \\ u_4 &= \frac{1}{6} \\ u_5 &= \frac{1}{24} \\ u_6 &= \frac{1}{120} \end{aligned}$$

7.7

$$(a_k = b_{k+1} - b_k)$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad (b_k = -\frac{1}{k})$$

$$\sum_{k=1}^{10} \frac{1}{k(k+1)} = b_{n+1} - b_1 = -\frac{1}{n+1} + 1 = \frac{10}{11}$$

7.8

$$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{K+2} = \frac{AK + 2A + BK}{k(k+2)}$$

$$\begin{cases} A+B &= 0 \\ 2A &= 1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

$$\frac{1}{k(k+2)} = \left( \frac{1}{2k} - \frac{1}{2k+4} \right)$$

$$\sum_{k=1}^{10} \frac{1}{k(k+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

## 7.11

a)

convergent interval

$$\sum_{k=0}^{\infty} 3^k x^k$$
$$L = \lim_{n \rightarrow \infty} \left| \frac{3^{k+1} x^{k+1}}{3^k x^k} \right| = \lim_{n \rightarrow \infty} |3x|$$
$$-1 < 3x < 1 \rightarrow -\frac{1}{3} < x < \frac{1}{3}$$
$$x \in \left] -\frac{1}{3}, \frac{1}{3} \right[$$

som  $a = 1$  en  $x = 3x$

$$\frac{1}{1 - 3x}$$

b)

$$\sum_{k=0}^{\infty} (-1)^k x^{2k}$$
$$L = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{k+1} x^{2k+2}}{(-1)^k x^{2k}} \right| = | -x^2 |$$
$$-1 < -x^2 < 1 \rightarrow 1 > x^2 > -1 \rightarrow 1 > x > -1$$
$$x \in ]-1, 1[$$

som  $a = 1$  en  $x = -x^2$

$$\frac{1}{1 + x^2}$$

## 7.13

a)

$$s_n = \frac{1}{1 - e^{-\frac{h\nu}{T}}}$$

b)

$$\lim_{T \rightarrow 0+} q(T) = \lim_{T \rightarrow 0+} \sum_{k=0}^{\infty} e^{-\frac{hkv}{T}} = 1$$

$$\lim_{T \rightarrow \infty} q(T) = \lim_{T \rightarrow \infty} \sum_{k=0}^{\infty} e^{-\frac{hkv}{T}} = \infty$$

**7.16**

b)

$$\lim_{k \rightarrow \infty} \ln \left( 1 + \frac{1}{k} \right) = \ln(1) = 0$$

$$\sum_{k=1}^{\infty} \ln \left( 1 + \frac{1}{n} \right) = \ln(1+1) + \ln \left( 1 + \frac{1}{2} \right) + \ln \left( 1 + \frac{1}{3} \right) + \dots$$

$$= \ln \left( (1+1) \left( 1 + \frac{1}{2} \right) \left( 1 + \frac{1}{3} \right) \dots \right)$$

$$= \ln \left( 2 \left( 1 + \frac{1}{2} \right) \left( 1 + \frac{1}{3} \right) \dots \right)$$

$$= \ln \left( 3 \left( 1 + \frac{1}{3} \right) \dots \right)$$

$$= \ln(k+1)$$

**7.17**

a)

$$\ln n < n \rightarrow \frac{1}{\ln n} > \frac{1}{n}$$

$$0 \leq \frac{1}{\ln n} \leq \frac{1}{n}$$

$$\sum \frac{1}{n} \text{ is divergent} \rightarrow \sum \frac{1}{\ln n} \text{ is ook divergent}$$



b)

$$\ln j < j \rightarrow \frac{\ln j}{j^3} < \frac{1}{j^2}$$

$$\lim_{j \rightarrow \infty} \frac{\frac{\ln j}{j^3}}{\frac{1}{j^2}} = \lim_{j \rightarrow \infty} \frac{\ln j}{j} = 0$$

$$\sum_{j=1}^{\infty} \frac{1}{j^2} \text{ convergeert naar } 1$$

Eig. 7.6.6.:  $\sum_{j=1}^{\infty} \frac{\ln j}{j^3}$  is convergent

**7.18**

a)

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\binom{2k+2}{k+1}}{\binom{2k}{k}} &= \lim_{k \rightarrow \infty} \frac{\frac{(2k+2)!}{(k+1)!(k+1)!}}{\frac{(2k)!}{k!k!}} = \lim_{k \rightarrow \infty} \frac{(2k+2)!}{(2k)!(k+1)(k+1)} \\ &= \lim_{k \rightarrow \infty} \frac{2(k+1)(2k+1)(2k)!}{(2k)!(k+1)(k+1)} \\ &= \lim_{k \rightarrow \infty} \frac{4k+2}{k+1} = 4 > 1 \rightarrow \text{divergent} \end{aligned}$$

b)

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\binom{2k+2}{k+1}^{-1}}{\binom{2k}{k}^{-1}} &= \lim_{k \rightarrow \infty} \frac{\frac{(2k)!}{k!k!}}{\frac{(2k+2)!}{(k+1)!(k+1)!}} = \lim_{k \rightarrow \infty} \frac{(2k)!(k+1)(k+1)}{(2k+2)!} \\ &= \lim_{k \rightarrow \infty} \frac{(2k)!(k+1)(k+1)}{2(k+1)(2k+1)(2k)!} \\ &= \lim_{k \rightarrow \infty} \frac{k+1}{4k+2} = \frac{1}{4} < 1 \rightarrow \text{convergent} \end{aligned}$$

c)

$$\begin{aligned}\text{Tip: } \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n &= e \\ \lim_{n \rightarrow \infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \lim_{n \rightarrow \infty} \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \lim_{n \rightarrow \infty} \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = \frac{1}{e}\end{aligned}$$

## Hoofdstuk 8

### 8.1

$$R = \lim_{k \rightarrow \infty} \frac{C_k}{C_{k+1}}$$

a)

$$\lim_{k \rightarrow \infty} \frac{k^3}{(k+1)^3} = 1$$

b)

$$R = \lim_{k \rightarrow \infty} \frac{\binom{2k}{k}}{\binom{2k+2}{k+1}} = \lim_{k \rightarrow \infty} \frac{k^2 + 2k + 1}{4k^2 + 6k + 2} = \frac{1}{4}$$

d)

$$R = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2}}{\frac{1}{(n-1)^2}} = R = \lim_{n \rightarrow \infty} \frac{(n-1)^2}{n^2} = 1$$

### 8.2

d)

convergentiestraal:

$$R = \lim_{k \rightarrow \infty} \frac{2^k}{2^{k+1}} = \lim_{k \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

som:  $a = 4x$  en  $x = 2x$ :

$$\frac{4x}{1-2x}$$

### 8.3

zie voorbeeld 7.2.3 p 133

$$R = \lim_{k \rightarrow \infty} \frac{\phi_n}{\phi_{n+1}} = \frac{1}{\phi} = \frac{2}{1 + \sqrt{5}}$$

### 8.5

$$\sum_{k=0}^n \frac{f^{(n)}(0)}{k!} x^k$$

a)

$$f(x) = \frac{1}{3+x} = \frac{1}{3} \quad f'(x) = \frac{-1}{(3+x)^2} = -\frac{1}{9}$$
$$f''(x) = \frac{2}{(3+x)^3} = \frac{6}{27} \quad f'''(x) = \frac{-6}{(3+x)^3} = -\frac{6}{81}$$

dus

$$\sum_{k=0}^n \frac{\frac{(-1)^k}{3^k} k!}{k!} x^k = \frac{1}{3} \sum_{k=0}^n (-1)^k \left(\frac{x}{3}\right)^k$$

c)

Ga dit niet manueel uitrekenen. Er is een truk voor. (zie p 158)

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!}$$

dus

$$\sin(2x^2) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{(2x^2)^{2k+1}}{(2k+1)!}$$

d)

zie c)

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

dus

$$e^{-3x} = \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!}$$

e)

zie a)

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!}$$

dus

$$\frac{\sin(x)}{x} = \frac{\sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k+1}}{(2k+1)!}}{x} = \sum_{k=0}^{\infty} (-1)^{2k} \frac{x^{2k}}{(2k+1)!}$$

## 8.7

a)

$$\begin{aligned} \frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \\ \frac{d \sum_{k=0}^{\infty} x^k}{dx} &= \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \\ \sum_{k=0}^{\infty} kx^k &= \frac{x}{(1-x)^2} \end{aligned}$$

b)

neem de afgeleide van het antwoord van a)

$$\frac{d(\sum_{k=0}^{\infty} kx^k)}{dx} = \sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{d(\frac{x}{(1-x)^2})}{dx} = \frac{1+x}{(1-x)^3}$$

dus

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{(1+x)x}{(1-x)^3}$$

## 8.10

Taylorreeks:

$$\sum_{n=0}^{\infty} \frac{f(a)^{(n)}}{n!} (x-a)^n$$

a)

$$f(x) = \ln(x) \quad f^{(n)}(x) = (-1)^{(n+1)} \frac{n!}{x^n} \text{ voor } n > 0$$

Taylorreeks:

$$\ln(a) + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n} \frac{(x-a)^n}{a}$$

## Hoofdstuk 9

### 9.1

$$\begin{aligned} & \int_{-l}^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = 0 \\ &= \frac{1}{2} \int_{-l}^l \cos\left(\frac{n\pi x - m\pi x}{l}\right) dx - \frac{1}{2} \int_{-l}^l \cos\left(\frac{n\pi x + m\pi x}{l}\right) dx \\ &= \frac{1}{2} \left[ \frac{l \sin\left(\frac{n\pi - m\pi}{l} x\right)}{n\pi - m\pi} \right]_{-l}^l - \frac{1}{2} \left[ \frac{l \sin\left(\frac{n\pi + m\pi}{l} x\right)}{n\pi + m\pi} \right]_{-l}^l \\ &= \frac{1}{2} \left( -\frac{l}{n\pi - m\pi} (\sin(n\pi - m\pi) - \sin(m\pi - n\pi)) \right) \\ &\quad - \frac{1}{2} \left( -\frac{l}{n\pi + m\pi} (\sin(n\pi + m\pi) - \sin(-(m\pi + n\pi))) \right) \\ &= 0 \end{aligned}$$

### 9.5

b)

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx \\ &= \frac{1}{n\pi} [\sin(nx)]_0^{\pi} \\ &= \frac{1}{n\pi} (\sin(n\pi) - \sin(0)) = 0 \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\
&= \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx \\
&= -\frac{1}{n\pi} [\cos(nx)]_0^{\pi} \\
&= \frac{1}{n\pi} (\cos(n\pi) - \cos(0)) \\
&= \frac{\cos(n\pi) - 1}{n\pi}
\end{aligned}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin((2n+1)x)$$

9.6

$$\begin{aligned}
\frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin((2n+1)x) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \\
&\iff \sin((2n+1)x) = (-1)^n \\
\iff \begin{cases} \sin((2n+1)x) = 1 \text{ als } n \text{ even} \\ \sin((2n+1)x) = -1 \text{ als } n \text{ oneven} \end{cases} &\iff x = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} \\
\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} &= \left(f\left(\frac{\pi}{2}\right) - \frac{1}{2}\right) \frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

## 9.8

$$\begin{aligned}
 b_n &= 0 \\
 a_n &= \frac{2\omega}{\pi} \int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} |\sin(\omega x)| \cos(nx) dx \\
 &= \frac{2\omega}{\pi} \int_0^{\frac{\pi}{2\omega}} \sin(\omega x) \cos(nx) dx \\
 &= \frac{4}{\pi(1-4n^2)}
 \end{aligned}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n\omega x)}{4n^2 - 1}$$