LECTURE NOTES

Noise in continuous-time analog circuits

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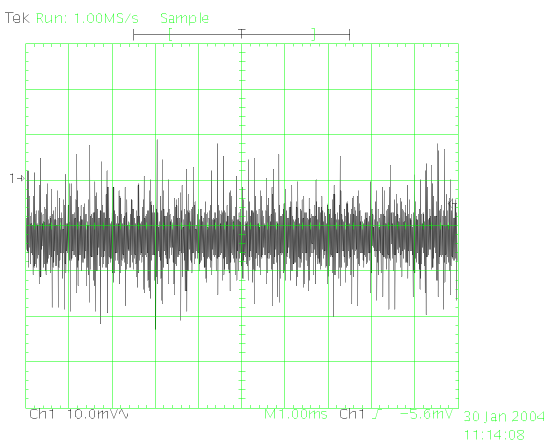
4 Noise in continuous-time analog circuits

The analog circuits designer cannot avoid a sufficient understanding of noise in devices, circuits and systems. Noise can be defined as [4.1] *"any unwanted perturbation that adversely affect or interfere with the wanted signal"*. Perturbation usually comes from external sources, such as electromagnetic coupling between a sensitive circuit and the mains (50Hz), domestic appliances or any other "power" device. It is worth noting that in any case, it is often possible to cancel perturbation effects using proper circuit techniques, adequate signal routing and shielding. Another source of perturbation is the device intrinsic noise. Unlike most external perturbation, it can't neither be exactly predicted nor totally cancelled. It can however be minimized using proper design techniques.
This section presents the fundamental aspects of noise and gives guidelines to perform simple calculations on practical circuits.

4.1• Noise in the time domain: what does it look like?



Noise is a fully random process. It is made of spectral components of random amplitude and phase, which values at instant t+Δt cannot be predicted even if we exactly know the values at instant t. Figure 4.1 is an oscilloscope screenshot showing the variations of the voltage measured at the terminals of a zener diode due to noise.

Figure 4.1: Noise voltage measured across a zener diode

4.1.1: Noise amplitude

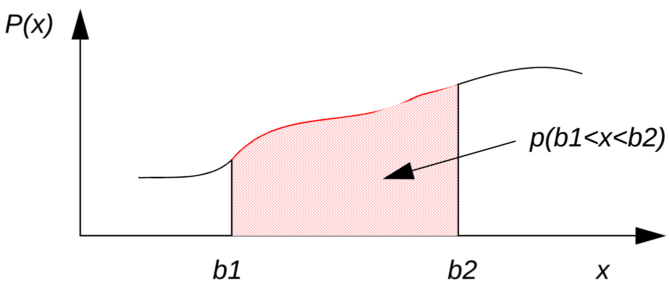


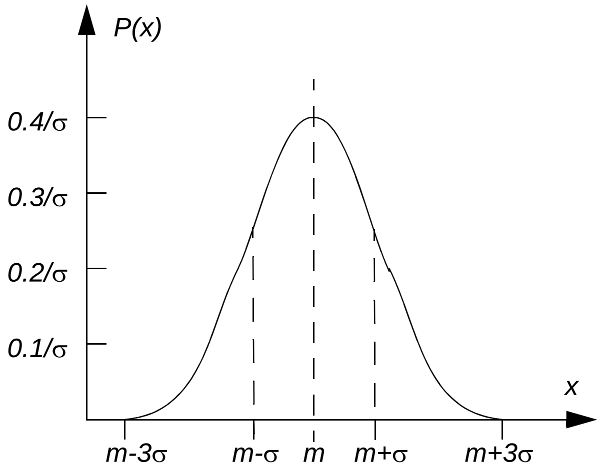
Figure 4.2: Calculation of the probability that a particular value appears knowing the probability density.

The instantaneous noise amplitude is not predictable. Therefore, determining noise characteristics involves mathematical theories like probability calculations. Assuming that the noise is characterised by a probability law described by a particular probability density P(x), the probability *p(b1<x<b2)* that a particular amplitude value *x* occurs between two limits *b1* and *b2* is calculated by integrating the probability density P(x) between *b1* and *b2*, which corresponds to the area delimited by the curve itself and the boundary values *b1* and

b2. This probability being in the 0 to 1 range, the total area under the curve equals 1. This method is useful to calculate particular amplitude values such as peak and root-mean-square values.

4.1.2: Noise peak value

Most of the usual noise sources are characterised by a Gaussian distribution (also known as normal distribution) [4.2] [4.3] which probability density P(x) is depicted in figure 4.3.



k	Area under the curve between m+kσ and m-kσ
0.4	0,311
1	0,683
1.4	0,838
2	0,954
2.4	0,984
3	0,997

Figure 4.3: Probability density of the Gaussian distribution

Usually, noise sources have a zero mean value *m*. Because of the symmetry of the distribution around the mean value *m*, noise has an equal probability to have either a positive or a negative amplitude. Calculation furthermore shows the area under the curve between $m\text{-}3\sigma$ and $m+3\sigma$ (σ is the standard deviation) equals 99,7% of the total area. It is a good practice to consider that the boundaries $+3\sigma$ and -3σ represent the extreme values of the noise amplitude, in other words, 3σ is the <u>peak</u> amplitude of the noise.

4.1.3: Noise rms value

The root mean square (rms) value of a signal is related to its thermal effects. Assuming we apply a time-dependent voltage to a resistor R, a certain amount of power will be dissipated. Then, if a DC voltage is applied to the same resistor, it is possible to find a particular value Vrms of this voltage giving the same power dissipation. This particular value Vrms is by definition the rms value of the time-dependent voltage. The root mean square value of a signal *x* is calculated as follows:

$$x_{rms} = \sqrt{\frac{1}{T}\int_0^T x^2 dt} \qquad \text{Eq. 4.1}$$

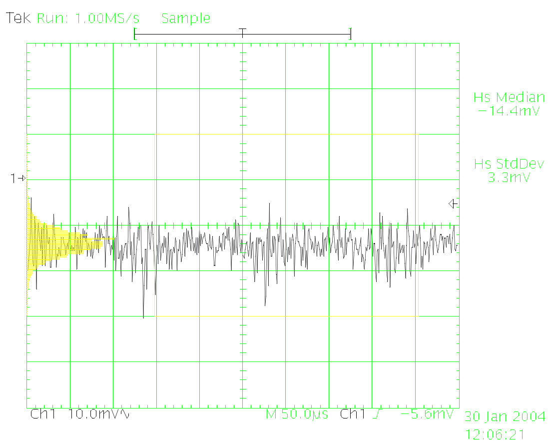


Figure 4.4: Noise voltage measured across a zener diode and the corresponding histogram, showing a Gaussian distribution.

In the case of a Gaussian noise, the rms value equals the standard deviation σ, the noise peak value x_{peak} is therefore:

$$x_{peak} = 3x_{rms}$$ Eq. 4.2

For example, figure 4.4 shows the voltage noise measured across a zener diode, as well as the corresponding histogram. The vertical scale is 10mV per division. The histogram exhibits a quasi-gaussian distribution which extends over 2 divisions, that is +/- 10mV, which represents the peak-to-peak value of the noise voltage. The corresponding rms value is therefore Vrms ≈ 3,3mV.

4.1.4: Signal to noise ratio (SNR)

Noise is inevitably present in any signal processing chain and will be processed the same way the signal is. For a given accuracy level, noise will be more or less acceptable. Therefore, the absolute noise power doesn't really matter. It is the signal-to-noise ratio, which is the ratio between the signal power and the total noise power, that is of utmost importance. The signal-to-noise ratio is therefore a measure of the quality of the signal processing. SNR is usually expressed in dB as follows:

$$SNR_{dB} = 10 \cdot \log\left(\frac{P_{signal}}{P_{noise}}\right)$$ Eq. 4.3

4.1.5: Noise summation

There is usually much more than a single noise source in a real circuit. The total noise of the circuit is of course a function of the individual sources, each producing a fraction of the total noise. If the sources are not correlated, the total noise is the sum of the individual noise contributions. The sources are not correlated if they are related to physically distinct components. For example, two identical resistors R1 and R2 produce the same noise power, but noise due to R1 is not correlated to noise due to R2.

4.2• Noise in the frequency domain

As mentioned before, due to the random nature of noise, the usual parameters used to describe a deterministic signal (amplitude, phase, frequency...) do not really make sense to describe the noise. The only significant parameter is the power which is not very convenient for calculations since we usually prefer to deal with voltage and current. Power can be expressed in terms of root mean square value of voltage or current as far as the value of the resistor in which the power is dissipated is known. Last (but not least), noise power also depends on the bandwidth of the system. In order to avoid making the calculation impedance and bandwidth dependent, the concepts of power spectral density and voltage (or current) spectral density will be introduced.

4.2.1: Power spectral density

Due to its random nature, noise is a wide band "signal" and its power is spread over frequency. When noise power is measured, the measurement system can be modelled as a band-pass filter, which represents the frequency band in which the system operates, followed by an ideal wattmeter as depicted in figure 4.5. Therefore, the larger the measurement bandwidth Δf, the larger the number of spectral components taken into account and hence the higher the measured power. Figure 4.6 shows the spectrum of noise measured with a spectrum analyser for two different values of Δf. A spectrum analyser displays the power of a signal measured using a band-pass filter of width Δf (called RBW -Resolution BandWidth- on the display) which is swept over a frequency band that extends between the START and STOP frequencies given by the user. This is similar to the arrangement presented in figure 4.5, except that the centre frequency of the filter can be varied in the START to STOP range. Changing RBW by a factor of 10 changes the displayed noise level by 10dB, which is equivalent to a factor of 10 in linear, so we can infer that noise power is directly proportional to bandwidth.

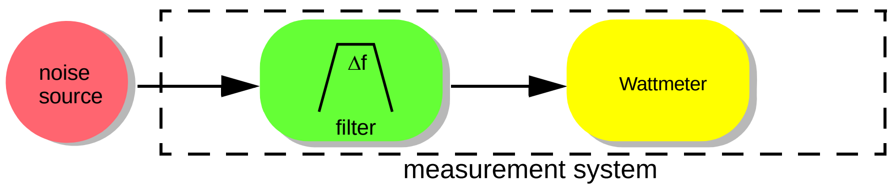


Figure 4.5: Noise measurement chain

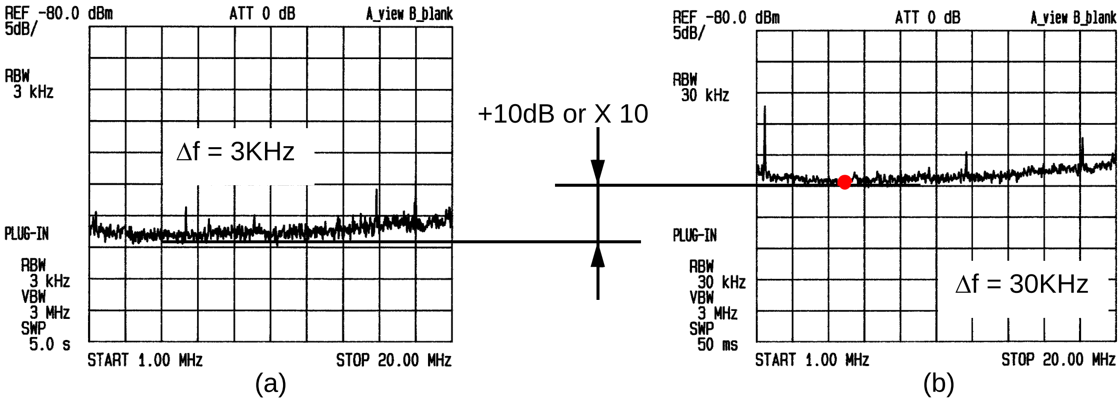


Figure 4.6: Noise displayed on a spectrum analyser.

The problem with the bandwidth dependence of the noise power is that giving the value of power is of no use if the measurement bandwidth is not given as well. Therefore, to avoid specifying the bandwidth, the noise power density is used instead of noise power for specifications and calculations. The noise power density is the noise power divided by the measurement bandwidth. Noise power density (usually denoted S_n) is expressed in watt per hertz (W/Hz), in other words, it is a normalisation of noise power over a 1Hz bandwidth.
For example, in figure 4.6b, the red dot is at -105 dBm (REF = -80dBm, 5 dB/div), that is a power level of:

$$P_n = 10^{\frac{-105}{10}} = 3,16 \cdot 10^{-11} mW = 3,16 \cdot 10^{-14} W \qquad \text{Eq. 4.4}$$

and the corresponding noise power density is (RBW = Δf = 30kHz):

$$S_n = \frac{P_n}{\Delta f} = 1,05 \cdot 10^{-18} W/Hz \qquad \text{Eq. 4.5}$$

The same calculation could be made in figure 4.6a with Δf = 3kHz and should lead to the same power density.

4.2.2: Voltage and current spectral density

To express the noise power spectral density in terms of voltage or current, it is mandatory to know the value of the resistor R in which the power was applied. The well-known relationship applies:

$$P_n = R \cdot I^2{}_{rms} = \frac{V^2{}_{rms}}{R} \quad \text{or} \quad S_n = R \cdot S_{ni} = \frac{S_{nv}}{R} \qquad \text{Eq. 4.6}$$

Where S_{nv} and S_{ni} are the noise voltage spectral density and noise current spectral density, expressed in V^2/Hz and A^2/Hz respectively.
For the spectrum analyser, the input impedance is 50Ω. Therefore, the power spectral density $S_n = 1{,}05.10^{-18}$ W/Hz corresponds to a noise voltage spectral density of:

$$S_{nv} = R \cdot S_n = 5,27 \cdot 10^{-17} V^2/Hz \qquad \text{Eq. 4.7}$$

or to a noise current spectral density of:

$$S_{ni} = \frac{S_n}{R} = 2,1 \cdot 10^{-20} A^2/Hz \qquad \text{Eq. 4.8}$$

The voltage or current spectral density is a normalisation of the power spectral density over a 1Ω resistor. It is worth noting that a noise source can be represented either as a voltage or a current source. Sometimes, and especially in data sheets, voltage or current spectral density is given in $V/\sqrt{Hz}$ or $A/\sqrt{Hz}$ which is simply the square root of the S_{nv} or S_{ni} value.

4.2.3: Characteristics of the most common noise sources [4.4]

4.2.3.1: White (or thermal, or Johnson, or Nyquist) noise

The white noise is a Gaussian noise which power spectral density is constant over frequency. In this case, calculating the noise power given its spectral density is simply a multiplication of the spectral density by the bandwidth of interest. This also applies for voltage or current spectral density. For example, if a noise source which voltage spectral density is $S_{nv} = 2.10^{-9}$ V/√Hz is connected to a wattmeter which input impedance is 50Ω, what is the power displayed in dBm for a 1KHz bandwidth? What peak value can we expect in the time domain?
Because of the white noise, the root mean square voltage is:

$$V^2{}_{rms} = (S_n)^2 \times \Delta f = (2 \cdot 10^{-9})^2 \times 1kHz = 4 \cdot 10^{-15} V^2{}_{rn} \quad \text{Eq. 4.9}$$

The corresponding power is:

$$P = \frac{V^2{}_{rms}}{R} = 8 \cdot 10^{-17} W \approx -131 dBm \quad \text{Eq. 4.10}$$

The peak voltage is:

$$V_{pk} = 3 \times \sqrt{V^2{}_{rms}} = 1,9 \cdot 10^{-7} V \quad \text{Eq. 4.11}$$

4.2.3.2: Flicker (or 1/f, or pink) noise

The flicker noise occurs in almost all electronic devices, and results from a variety of effects, such as impurities in a conductive channel, generation and recombination noise in a transistor due to base current, and so on. It is always related to a direct current. In electronic devices, it is a low-frequency phenomenon, as the higher frequencies are overshadowed by white noise from other sources. In oscillators, however, the low-frequency noise is mixed up to frequencies close to the carrier which results in oscillator phase noise. Its power spectral density decreases proportionally with frequency and can be expressed as:

$$S_n = \frac{k_f}{f/f_c} \quad \text{in W/Hz} \quad \text{Eq. 4.12}$$

Where f_c is called the corner frequency, a frequency at which thermal noise equals 1/f noise. In the case, of 1/f noise, calculating the noise power is no longer a multiplication of the spectral density by the bandwidth of interest but a true integration over the bandwidth:

$$P_n = \int_{fmin}^{fmax} S_n \cdot df = k_f \cdot f_c \cdot \ln\left[\frac{fmax}{fmin}\right] \quad \text{Eq. 4.13}$$

This also applies for voltage or current spectral density, the only restriction is that fmin cannot be zero. It is worth noting that 1/f noise exhibits a power that is constant per decade: there is the same power in the 1Hz..10Hz range than in the 100kHz..1MHz range, only the frequency ratio matters. This kind of noise is of course a problem at low frequencies, where thermal noise is much lower.

4.2.3.3: Shot noise

Shot noise is a type of electronic noise that occurs when the finite number of particles that carry energy, such as electrons in an electronic circuit or photons in an optical device, is small enough to give rise to detectable statistical fluctuations in a measurement. It is important in electronics, telecommunications, and fundamental physics.

The magnitude of this noise increases with the average magnitude of the current or intensity of the light. However, since the magnitude of the average signal increases more rapidly than that of the shot noise (its relative strength decreases with increasing signal), shot noise is often only a problem with small current or light intensity.

4.2.3.4: Burst noise

Burst noise is a type of electronic noise that occurs in semiconductors. It is also called popcorn noise, impulse noise, bi-stable noise, or random telegraph signal (RTS) noise. It consists of sudden step-like transitions between two or more discrete voltage or current levels, as high as several hundred microvolts, at random and unpredictable times. Each shift in offset voltage or current often lasts from several milliseconds to seconds, and sounds like popcorn popping if hooked up to an audio speaker.

4.3• Modelling noise in devices

4.3.1: Passive devices

4.3.1.1: The resistor

A noisy (real) resistor can be modelled either as an ideal (noiseless) resistor in series with a voltage noise source or as an ideal resistor in parallel with a current noise source, as shown in figure 4.7. The associated noise source depends not only on the resistor value but also on the operating temperature:

$$S_{nv} = 4kT \cdot R \qquad \text{in V}^2\text{/Hz} \qquad \text{Eq. 4.14}$$

$$S_{ni} = \frac{4kT}{R} \qquad \text{in A}^2\text{/Hz} \qquad \text{Eq. 4.15}$$

At room temperature (300K) 4kT = 1,6 10^{-20} Joule.
Noise in resistors is mostly thermal noise. However, depending on the material used to fabricate the resistor, other mechanism of noise can occur. For example, carbon resistors also exhibit 1/f noise, whereas metal film resistors do not [4.4].
Note that in the models, noise sources do no have an arrow because phase does not make sense.

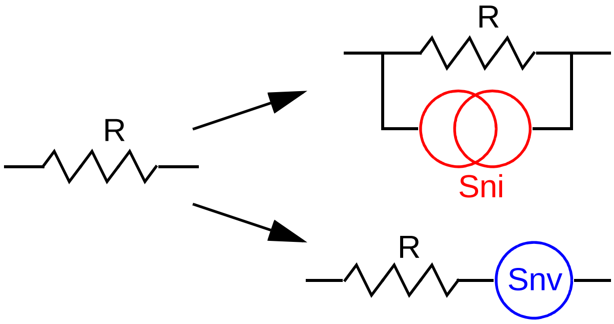


Figure 4.7: Equivalent representation of a resistor for noise calculation

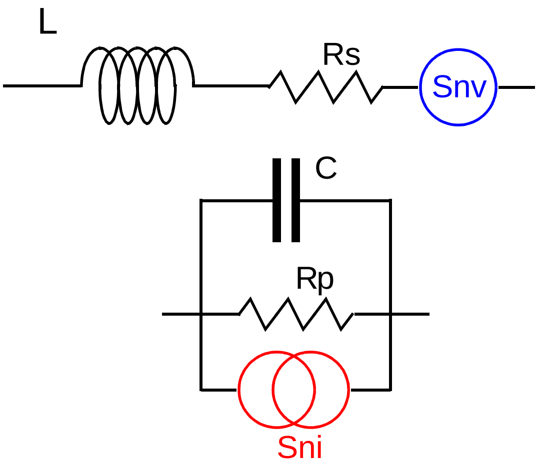


Figure 4.8: Noisy reactive components

4.3.1.2: Reactive components

Reactive components (capacitors and inductors) do not produce noise, at least as long as they are ideal, which assumption is correct in most low-frequency cases. Real components however exhibit losses that are modelled by resistors, as shown in figure 4.8. A real inductor also includes a series resistor which is due to the resistivity of the wire used to realise the inductor. This relatively low value resistor creates a noise S_{nv} = 4kT Rs. A real capacitor includes a parallel resistor which models the dielectric losses. This relatively large resistor creates a noise S_{ni} = 4kT/Rp.

4.3.2: The operational amplifier

The operational amplifier is the key element in analog signal conditioning and therefore, it is of utmost importance to provide an equivalent model for noise calculation. Operational amplifiers contain a large number of transistors, each contributing to the total noise. When an operational amplifier is connected as voltage follower (figure 4.9) the noise measured at the output can be modelled as a source S_{nve} of same value (because the unity gain) in series with any of the inputs.

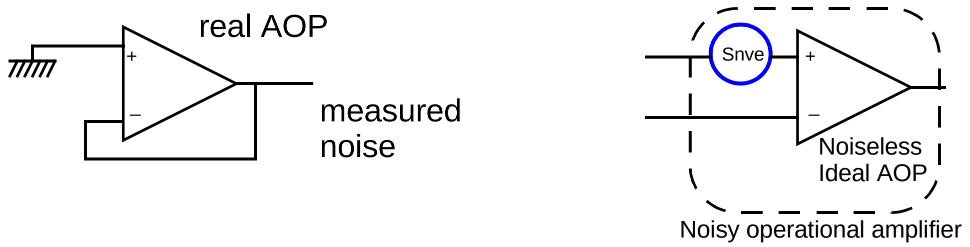


Figure 4.9: Noise voltage model for the operational amplifier

If the non-inverting input is connected to the ground through a resistor (figure 4.10) the noise at the output is expected to increase due to noise of the resistor itself. However, the measured noise is greater than the expected one, that is the AOP voltage noise plus the noise due to the resistor. Moreover, if the resistor value is varied, the extra noise changes accordingly. This suggests another current noise source at the non-inverting input of the amplifier (and at the inverting amplifier as well).

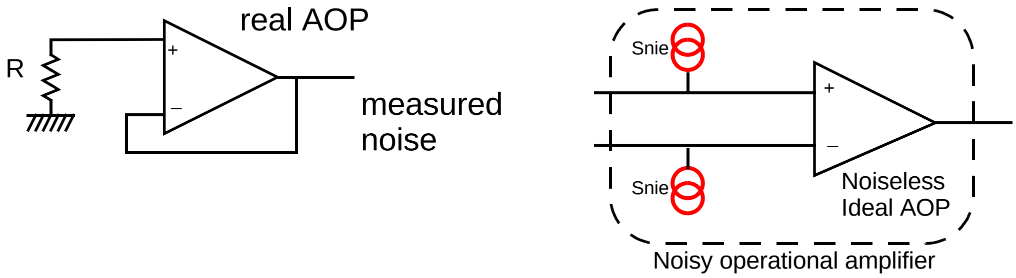


Figure 4.10: Current noise model for the operational amplifier

Finally, the noise model including all the contributions is shown in figure 4.11. Since these sources are due to semiconductors, 1/f noise is also present. Figures 4.12a and 4.12b show the noise spectrum for current and voltage noise respectively.

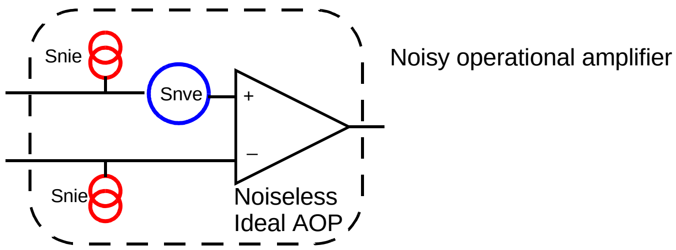


Figure 4.11: Noise model for an operational amplifier

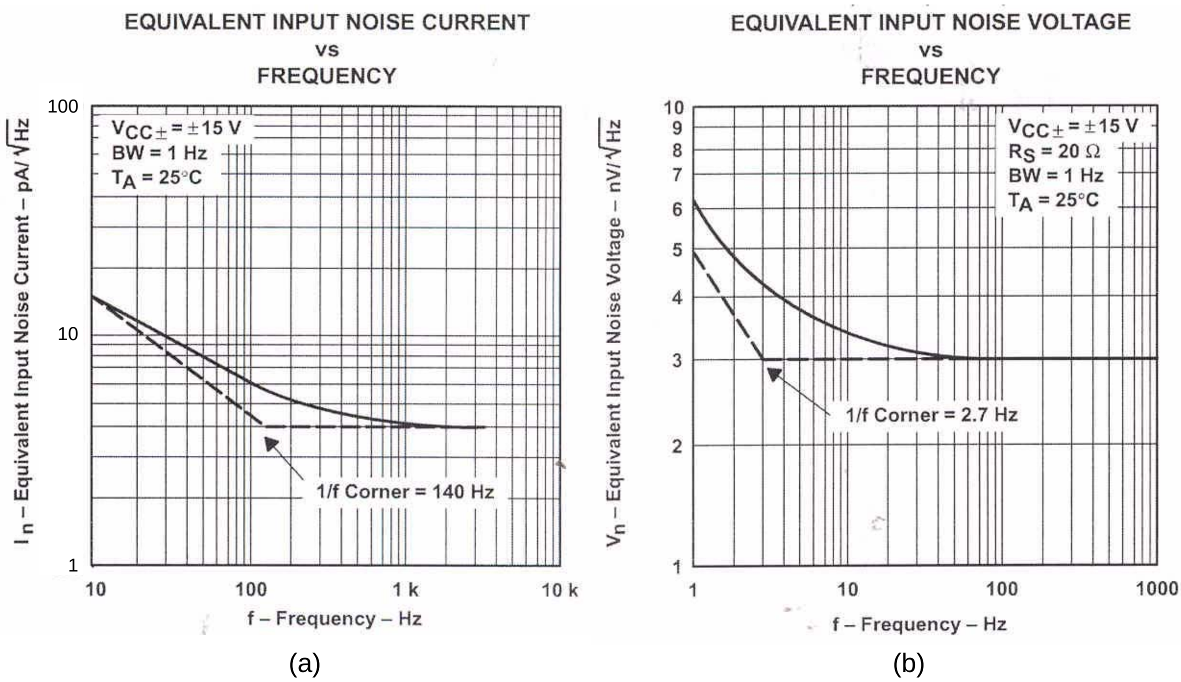


Figure 4.12: Noise characteristics for an operational amplifier

These characteristic curves allow the designer to determine both the thermal (white) noise component as well as the 1/f one. White noise spectral density is estimated at high frequencies, where white noise dominates. In figure 4.12b, we find S_{nveth} = 3 nV/√Hz or $S_{nveth} = 9\ 10^{-18}\ V^2/Hz$ and similarly, in figure 4.12a, $S_{nieth} = 16\ 10^{-24}\ A^2/Hz$. The constant term in the model for 1/f noise (confer to equation 4.12), denoted k_{fv} and k_{fi} for voltage and current spectral density respectively, can be determined with a simple observation: at the given 1/f corner, the 1/f noise equals the thermal noise and hence, at $f = f_c$, for the noise voltage:

$$S_{nveth} = \frac{k_{fv}}{f_c/f_c} = 9 \cdot 10^{-18} \qquad V^2/(Hz) \qquad \text{Eq. 4.16}$$

yielding the value of the constant: $k_{fv} = 9\ 10^{-18}\ V^2/Hz$. The same reasoning applies for k_{fi}. Obviously, the constants exactly correspond to the thermal noise level.

4.4• Noise in circuits

The previous sections have detailed the different mechanisms for noise and models for the most commonly used components were given. This section aims at giving the necessary tools in order to be able to calculate the noise at the output of a circuit containing many components. The particular case of filtered noise will also be investigated.

4.4.1: Basic noise analysis tools

There is no particular analysis tool dedicated to noise calculation but rather a set of rules that must be followed. Unlike classical signal analysis where there is usually only one source, noise analysis deals with many sources, since every component in the circuit is a noise source by itself (ideal reactive elements excepted) and some others, like operational amplifiers, are modelled by three sources. Therefore, noise analysis is a time consuming process, although there is no particular difficulties in the calculation.

To perform a noise analysis, the following "Golden Rules" must be followed:

• Noise analysis is by essence a small-signal analysis, therefore, all constant sources are set to zero (i.e. a short circuit for voltage sources, an open circuit for current sources).

- **Always** express noise sources as power spectral densities in V^2/Hz or A^2/Hz. Choose either the voltage or current model, the result should be the same! (but calculations may be more or less easy depending on the the circuit topology and the model used...).

- Noise analysis is, by definition, performed with no signal: all signal sources are also set to zero. However, their internal impedance, if any, remains. It is worth noting that this internal impedance **does not produce** a 4kTR or 4kT/R noise [note 1].

- If sources are uncorrelated, the superposition theorem applies and then, the total contribution of different spectral densities is the spectral density S_n such as:

$$S_n = \sum_j S_{nj} \qquad \text{Eq. 4.17}$$

- The Ohm law applies for spectral densities and:

$$S_{nv} = R^2 \cdot S_{ni} \qquad \text{Eq. 4.18}$$

- When a transfer function is involved (figure 4.13), the noise is affected like any other signal. However, since noise is a random process, the phase concept doesn't make sense, and only the amplitude matters. In the general case when the transfer function is complex, the square of the modulus is used as follows:

$$S_{nout} = |G|^2 \cdot S_{nin} \qquad \text{Eq. 4.19}$$

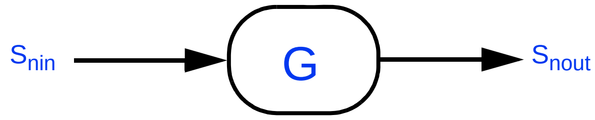


Figure 4.13: Transfer function definition

Below are some simple examples to illustrate these concepts. First, let calculate the noise produced by two resistors R1 and R2 in series as shown in figure 4.14a:

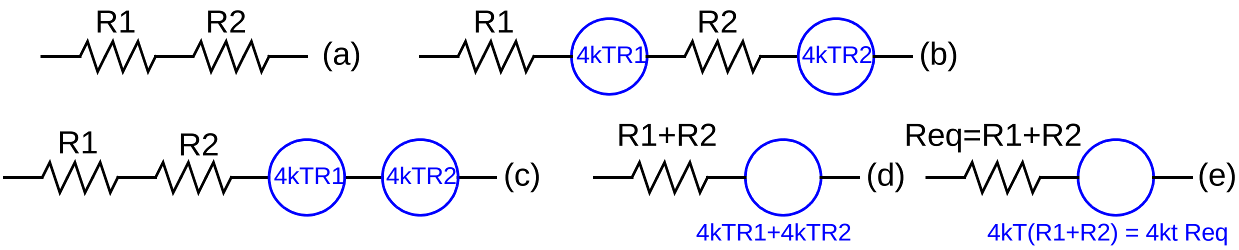


Figure 4.14: Noise of two resistors in series

The noise sources of each resistor are added (b), then the circuit can be re-arranged (c) because the order of elements doesn't matter in a series arrangement. The equivalent resistor is found as R1+R2 and since the sources come from distinct devices, they are independent and can be added provided they are expressed as power spectral densities (d). Finally, the noise produced by two resistors in series is found to be 4kT(R1+R2) (e), that is the noise of the equivalent resistor.

Another frequent problem is to determine the noise due to a voltage divider made of two resistors R1

Note 1: internal impedance does not create noise **unless it is made using a classical resistor**. This is because the concept of internal impedance, although modelled by a resistor, is only the description of the circuit behaviour, that is a linearisation of the real I/V characteristic around the operating point. Since the circuit is certainly made of semiconductor devices, the noise mechanisms involved are not comparable to the noise mechanism of the resistor itself.

and R2 as shown in figure 4.15a:

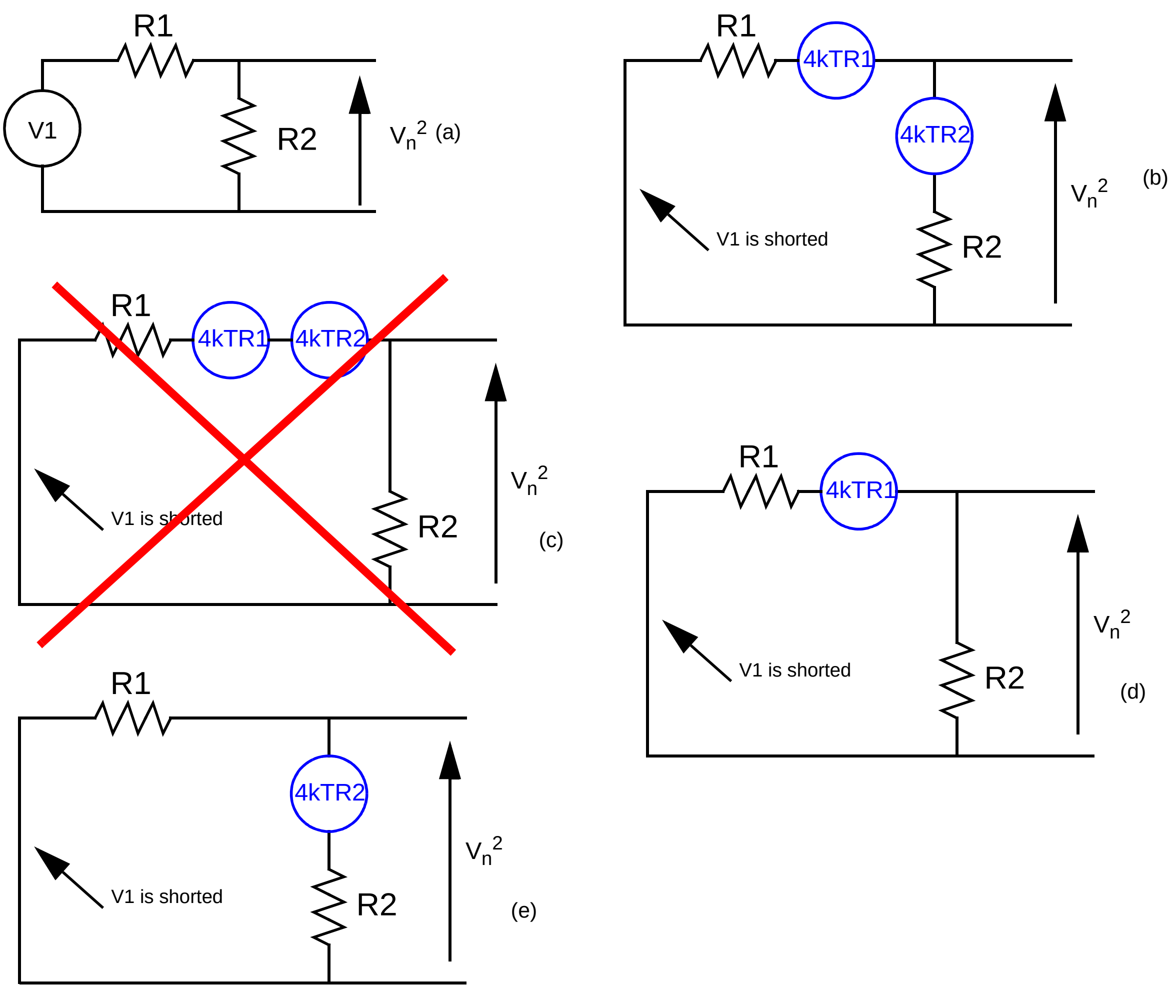


Figure 4.15: Noise of a voltage divider

The first step is to associate a noise source to each resistor, voltage V1 is set to zero (shorted) since it is not part of the voltage divider and therefore does not play a role in noise analysis (b). Unlike the previous example, noise source 4kTR2 cannot be moved (c) because the noise source is part of the resistor: measuring noise across a resistor must include its own noise source. The problem can be solved using the superposition theorem. The first contribution (d) is therefore:

$$v_{na}{}^2 = 4kTR1 \cdot \left(\frac{R2}{R1 + R2}\right)^2 \qquad \text{Eq. 4.20}$$

Similarly, the second contribution (e) is:

$$v_{nb}{}^2 = 4kTR2 \cdot \left(\frac{R1}{R1 + R2}\right)^2 \qquad \text{Eq. 4.21}$$

In equations 4.20 and 4.21, the transfer function between the noise source and the output is those of a voltage divider, which modulus is squared. Finally, the total output noise is the sum of the two contributions:

$$v_n{}^2 = v_{na}{}^2 + v_{nb}{}^2 = \left(\frac{4kT}{R2} + \frac{4kT}{R1}\right) \cdot \left(\frac{R1 \cdot R2}{R1 + R2}\right)^2 = 4kT \cdot \left(\frac{R1 \cdot R2}{R1 + R2}\right) \qquad \text{Eq. 4.22}$$

Equation 4.22 gives the noise of two resistors in parallel as expected, since the equivalent Thevenin resistance for a voltage divider is R1//R2.

4.4.2: Filtered noise

When a source noise is applied to a filter, the output noise is of course influenced by its frequency-dependent transfer function. In particular, a low-pass filter with the appropriate cut-off frequency can be used to eliminate high frequency noise in order to improve the signal-to-noise ratio of a circuit by reducing out-of-band noise without altering the signal. Therefore, the analysis of low-pass filtered noise is of great interest, especially in the case of a first-order low-pass filter which gives a widely used result. Consider a white noise source at the input of a first order **noiseless** low-pass filter, as shown in figure 4.16:

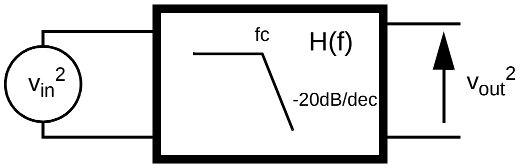


Figure 4.16: White noise filtered by a first order low-pass filter

By definition, the output voltage spectral density v_{out}^2 is:

$$v_{out}^2 = |H(f)|^2 \cdot v_{in}^2 \qquad \text{Eq. 4.23}$$

And the output power is obtained by integrating v_{out}^2 from f = 0 to f $\rightarrow\infty$

$$P_{out} = \int_0^\infty v_{out}^2 \cdot df = \int_0^\infty |H(f)|^2 \cdot v_{in}^2 \cdot df = v_{in}^2 \cdot \int_0^\infty \frac{1}{1 + \left(\frac{f}{fc}\right)^2} \cdot df = v_{in}^2 \cdot \frac{\pi}{2} \cdot fc = v_{in}^2 \cdot f'c \qquad \text{Eq. 4.24}$$

The above result shows that when a white noise of power spectral density v_{in}^2 is filtered with a first-order low-pass filter, the output power is directly proportional to the bandwidth f'c = π/2 fc **which is different** from the signal bandwidth fc. Bandwidth f'c is called the equivalent noise bandwidth. This can be explained graphically as shown in figure 4.17. When the dashed surface of the ideal filter (in red) equals the dashed surface of the real filter (in blue), the area under the two transfer functions is the same, that is the output noise power is the same. The noise equivalent bandwidth is therefore the cut-off frequency f'c of an ideal low-pass filter though which the input noise source will produce the same noise power than through the real low-pass filter. In general, the equivalent noise bandwidth is determined using the following equation:

$$\int_0^\infty |H(f)|^2 \cdot df = f'c \qquad \text{Eq. 4.25}$$

Of course, f'_c strongly depends on the type and order of the filter (that's why the result in equation 4.24 is **only valid** for first-order low-pass filters). However, it is clear from figure 4.17 that the higher the filter order, the sharper the roll-off above fc and therefore, when higher order filters are involved, the noise equivalent bandwidth f'c is very close from the signal bandwidth fc.

Although not very efficient as far as attenuation is concerned (only 20dB per decade!), the first order low-pass filter is widely used in the different stages of a signal conditioning chain (for example for anti-aliasing filter when oversampling is used). In this case, it may be interesting to have some guidelines to determine the value of R and C of the filter. Usually, you know the value of the required signal cut-off frequency, but only one equation cannot give a single solution if two variables are involved. What could be the second equation? The noise analysis has proven to be helpful in this matter. Since the RC filter uses a noisy resistor, the filter itself generates noise. This situation is depicted in figure 4.18.

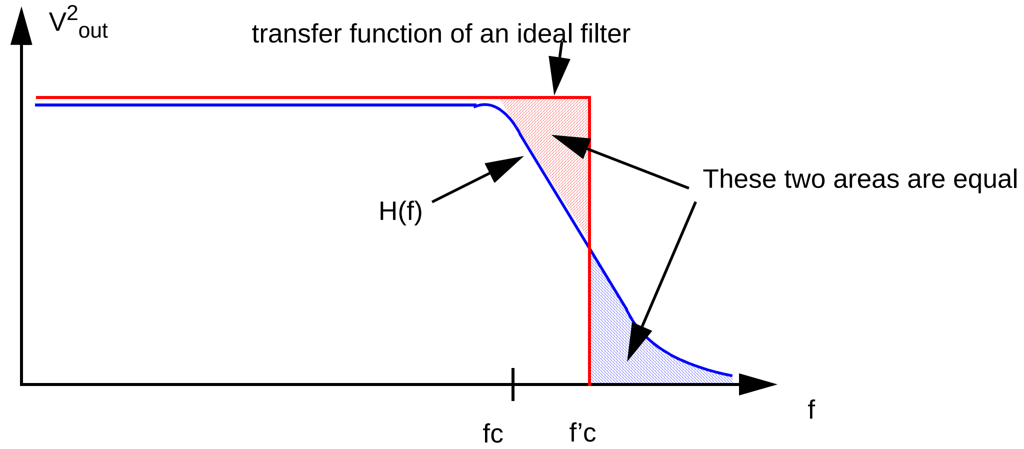


Figure 4.17: Equivalent noise bandwidth concept

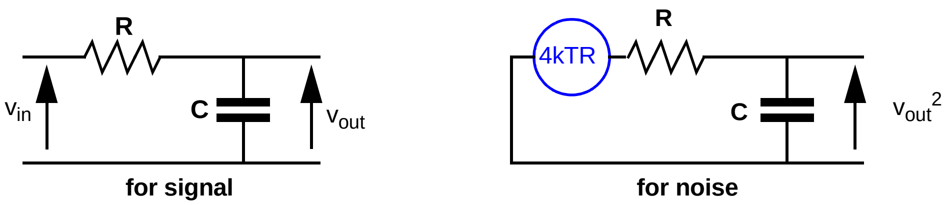


Figure 4.18: Noise power calculation for the 1st order RC low-pass filter

It is clear that this problem is similar to that of figure 4.16 if you consider that v_{in}^2 = 4kTR. The noise power is therefore given by equation 4.24 and recalling that $f_c = (2\,\pi\,R\,C)^{-1}$:

$$P_{out} = v_{in}^2 \cdot \frac{\pi}{2} \cdot fc = 4kTR \cdot \frac{\pi}{2} \cdot \frac{1}{2\pi RC} = k\frac{T}{C} \qquad \text{Eq. 4.26}$$

Oops! What's all this stuff? Equation 4.26 shows that noise power only depends on the value of the **noiseless** device! Well, nothing's wrong indeed. This can even be easily explained: for a given capacitor value, if you increase R, v_{in}^2 increases proportionally but f_c decreases in the same proportion. Therefore, the product v_{in}^2 x fc, which is the noise power, remains constant and the result is independent from R. So, when designing a RC low-pass filter, the first question is: what noise power is allowed? This yields the minimum value of C, and then the required cut-off frequency f_c yields the value of R.

4.5• Noise in systems

4.5.1: Gain sharing in a multi-stage amplifier

Consider the conventional multi-stage amplifier shown in figure 4.19. The use of more than one stage is mandatory since in practice, there is little chance to achieve both gain, bandwidth and noise requirements with a single amplifier stage. So, the gain is shared between the different stages, each of them having their own input equivalent noise source (the b_i terms are in V^2/Hz).

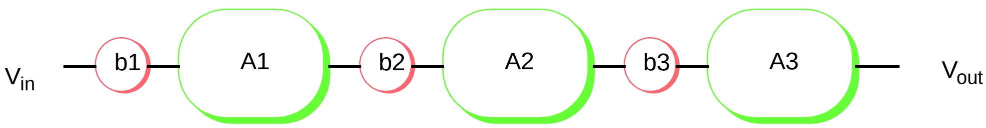


Figure 4.19: A multi-stage amplifier

The output noise voltage spectral density is simply:

$$v_{outnoise}^2 = b1 \cdot |A1|^2 \cdot |A2|^2 \cdot |A3|^2 + b2 \cdot |A2|^2 \cdot |A3|^2 + b3 \cdot |A3|^2 \qquad \text{Eq. 4.27}$$

For the signal, the output power is:

$$v_{outsignal}^2 = v_{insignal}^2 \cdot |A1|^2 \cdot |A2|^2 \cdot |A3|^2 \qquad \text{Eq. 4.28}$$

The signal-to-noise ratio at the output is thus:

$$\frac{v_{outsignal}^2}{v_{outnoise}^2} = \frac{v_{insignal}^2 \cdot |A1|^2 \cdot |A2|^2 \cdot |A3|^2}{b1 \cdot |A1|^2 \cdot |A2|^2 \cdot |A3|^2 + b2 \cdot |A2|^2 \cdot |A3|^2 + b3 \cdot |A3|^2} = \frac{v_{insignal}^2}{b1 + \frac{b2}{|A1|^2} + \frac{b3}{|A1|^2 \cdot |A2|^2}} \qquad \text{Eq. 4.29}$$

To maximise the signal-to-noise ratio, several conditions must be met:

- The noise of the first stage (b1) must be minimised since it remains unchanged
- The gain of the first stage (A1) must be maximised since it attenuates the noise of the subsequent stages (b2, b3...). As a general rule, if A1 is sufficiently large (say ten or more), the overall noise is dominated by b1, the other noise sources only play a marginal role and can be neglected.

So, the designer must pay a particular attention to the first stage and the task isn't easy.

4.5.2: Determination of the optimal source resistance

At the input of the data acquisition chain, a source will be connected. When dealing with high frequencies, it is of good practice to match the source impedance with the input impedance of the chain in order to maximise the signal transfer. The source impedance produces a noise expressed as 4kTRs (see appendix 1 for explanation) and the situation is as follows:

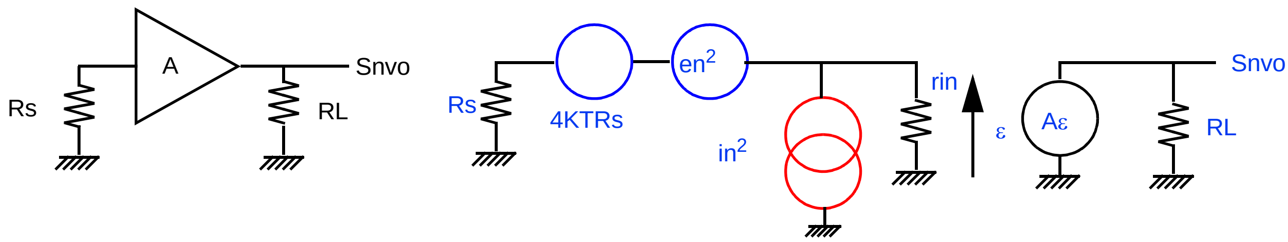


Figure 4.20: Influence of the source resistance

The output noise spectral density is written as follows:

$$S_{nvo} = (4kT \cdot Rs + en^2) \cdot |A|^2 \cdot \left(\frac{rin}{rin + Rs}\right)^2 + in^2 \cdot |A|^2 \cdot (rin \parallel Rs)^2 \qquad \text{Eq. 4.30}$$

The source resistor is noisy but if the amplifier is ideal, it makes no noise and the signal-to-noise ratio (SNR) at its output is the same than the SNR at its input. For a real amplifier, the SNR at the output will be deteriorated due to additional noise brought by the amplifier itself. To measure to what extent the SNR is affected, the noise factor F is introduced:

$$F = \frac{\text{total noise at the output}}{\text{noise at the output due to the source}} = \frac{\text{noise at the output due to the source } + \text{noise at the output due to the amplifier}}{\text{noise at the output due to the source}}$$

Eq. 4.31

$$F = \frac{S_{nvo}}{4kT \cdot Rs \cdot |A|^2 \cdot \left(\frac{rin}{rin + Rs}\right)^2} = 1 + \frac{en^2}{4kT \cdot Rs} + \frac{in^2}{4kT} \cdot Rs$$ Eq. 4.32

Noise factor F reaches a minimum for an optimal value of Rs, as shown in figure 4.21:

$$Rsopt^2 = \frac{en^2}{in^2}$$ Eq. 4.33

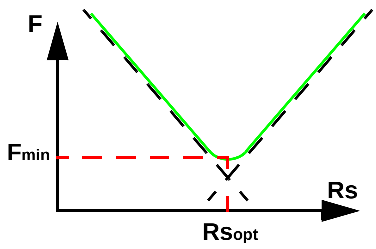


Figure 4.21: Noise factor F versus Rs value

It is worth noting that reaching Fmin means that as far as noise is concerned, the amplifier is well matched to the source and adds the lowest possible noise to the system. Noise factor F is often expressed in decibel as follows and is then called the noise figure NF:

$$NF(dB) = 10 \cdot \log(F)$$ Eq. 4.34

4.6• A practical example: the inverting amplifier using an operational amplifier

The practical use of the concepts presented in the preceding sections will be illustrated on the example of figure 4.22a. The input signal voltage is applied to resistor R1, of which the other terminal voltage is zero due to the operational amplifier. The input current flows into R1, then into R2 and produces the corresponding voltage at the output of the amplifier. The low-pass filter R3-C limits the bandwidth of the system to 100kHz, thus reducing the noise power at output vout (this bandwidth value is for illustration purpose only. In real designs, it will be determined from signal characteristics).

4.6.1: Equivalent schematic for noise calculation

For noise analysis (figure 4.22b), the input signal must be set to zero. In our case, the signal is a voltage and the input is therefore shorted (if the input signal were a current, the input would have been left open). Noise contributions of the operational amplifier ($e_n^{\mathbf{2}}$, i_{np}^2, i_{nn}^2) are added, as well as the noise of resistor R1, R2 and R3. For resistors R1and R2, a current noise model was chosen, but the noise voltage model could have been used instead as well without altering the final result, as will be shown later. This remark also applies for R3 and more generally for any resistor, whatever the circuit is. Capacitor C is considered ideal and does not therefore produce noise.

The total noise at the output is the sum of the filtered noise from the inverting amplifier (R1, R2, AOP) and the noise from the R3-C filter itself.

4.6.2: Noise at node v1

We will first calculate noise at node V1. Since the operational amplifier is an ideal voltage source (i.e. with a negligible internal impedance), noise from R3 does not influence the noise voltage at node V1. Furthermore, current noise source i_{nn}^2 is shorted: all the noise current flows into the ground and does therefore not produce any contribution to the total noise at node V1 (the current source is not zero but its contribution is zero). Let's begin with source i_{np}^2: the other sources are removed (superposition theorem). For a current (either signal or noise), the problem is to find out where and through what device it flows in order to be able to determine the induced voltage (figure 4.23a).

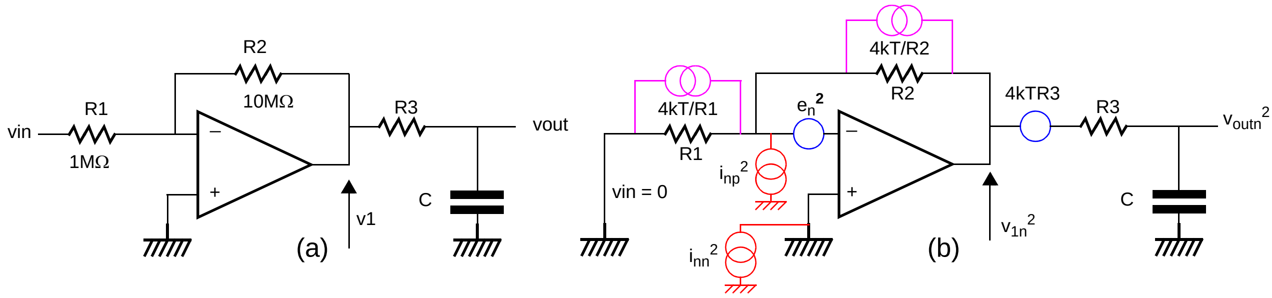


Figure 4.22: The high input impedance inverting amplifier

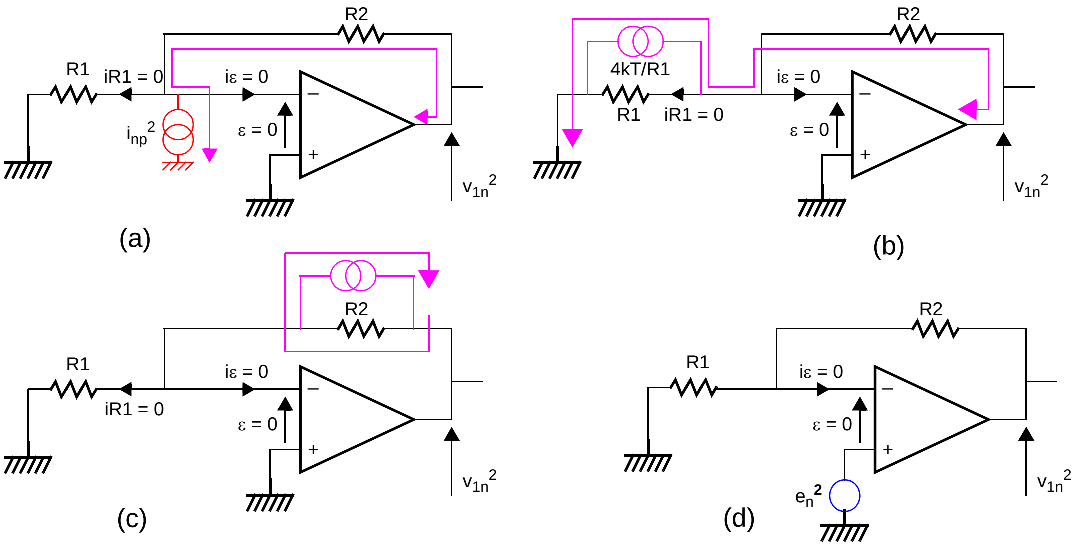


Figure 4.23: The different steps of noise calculation

Since the operational amplifier uses feedback, usual approximations (ε = 0, iε = 0) apply. Noise current cannot flow through R1 because the voltage across its terminals is zero: the left terminal is tied to the ground and the right terminal is held at the ground potential due to the operational amplifier. Noise current cannot flow through the inverting input of the operational amplifier either. Therefore, the only path is through resistor R2 and then through the output of the operational amplifier which closes the loop. The noise voltage across resistor R2 is thus:

$$v_{R2n}^2 = inp^2 \cdot R2^2 \qquad \text{Eq. 4.35}$$

Because ε = 0, the voltage across resistor R2 is also the voltage at node V1 (in other words, the voltage gain between the voltage across resistor R2 and voltage at node V1 is unity). Then:

$$v_{1na}^2 = inp^2 \cdot R2^2 \qquad \text{Eq. 4.36}$$

A similar reasoning holds for noise source 4kT/R1 (figure 4.23b) and 4kT/R2 (figure 4.23c):

$$v_{1nb}^2 = 4k\frac{T}{R1} \cdot R2^2 \qquad \text{Eq. 4.37}$$

$$v_{1nc}^2 = 4k\frac{T}{R2} \cdot R2^2 = 4kT \cdot R2 \qquad \text{Eq. 4.38}$$

For noise source e_n^2 (figure 4.23d), the idea is to move it to the non-inverting input because it does not make any change for noise [note 2] but eases the calculation: the operational amplifier is now a non-inverting amplifier [note 3] for the noise source with a gain of $1 + \frac{R2}{R1}$:

$$v_{1nd}^2 = en^2 \cdot \left(1 + \frac{R2}{R1}\right)^2$$ Eq. 4.39

Finally, the noise voltage spectral density V_{1n}^2 at node V1 is the sum of the four individual contributions:

$$v_{1n}^2 = inp^2 \cdot R2^2 + 4k\frac{T}{R1} \cdot R2^2 + 4kTR2 + en^2 \cdot \left(1 + \frac{R2}{R1}\right)^2$$ Eq. 4.40

This relation holds for any kind of noise, either thermal or 1/f. For thermal (white) noise, the characteristics in figure 4.12 yield $i_{np}^2 = 16\ 10^{-24}$ A^2/Hz, $e_n^2 = 9\ 10^{-18}$ V^2/Hz and:

$$v_{1nth}^2 = 16 \cdot 10^{-10} + 16 \cdot 10^{-13} + 16 \cdot 10^{-14} + 1,09 \cdot 10^{-15} \approx 16 \cdot 10^{-10} \qquad V^2/(Hz)$$ Eq. 4.41

For 1/f noise, we can consider that resistors are noiseless. As mentioned in section 4.3.2, the constant terms for 1/f noise equal the thermal noise level and then:

$$v_{1nf}^2 = \frac{inp^2 \cdot f_{ci}}{f} \cdot R2^2 + \frac{en^2 \cdot f_{cv}}{f} \cdot \left(1 + \frac{R2}{R1}\right)^2$$ Eq. 4.42

$$v_{1nf}^2 = (2240 \cdot 10^{-10} + 2940 \cdot 10^{-18}) \cdot \frac{1}{f} \approx \frac{2240 \cdot 10^{-10}}{f} \qquad V^2/(Hz)$$ Eq. 4.43

4.6.3: What if we choose voltage noise instead of current noise for resistors?

As mentioned earlier, the two models available for resistors are strictly equivalent. To be definitively convinced, let redraw figure 4.23c with the noise voltage model for resistor R2 (figure 4.24):

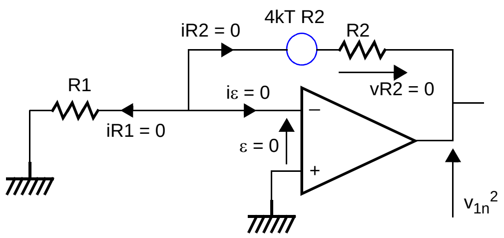


Figure 4.24: Circuit of figure 4.24 using the noise voltage model.

For the reason mentioned in a previous section, the current through R1 is zero, and since the inverting input current of the operational amplifier is also zero, no current can flow into R2, giving a zero voltage vR2 across resistor R2. As ε = 0, v_{1n}^2 is simply:

$$v_{1n}^2 = 4kT \cdot R2$$ Eq. 4.44

which is result yield by equation 4.38 using the noise current model. The loop is closed...

Note 2: remember that phase doesn't matter as far as noise is concerned, so the inverting input of the operational amplifier is equivalent to the non-inverting one in this case.

Note 3: We have seen that bandwidth is different for signal and for noise. Similarly, gain for signal is different from that of noise: for signal, this circuit is an inverting amplifier but it is an non-inverting amplifier for noise. That's why $1 + \frac{R2}{R1}$ is often referred as "noise gain".

4.6.4: Filtered thermal noise

The noise at node v1 is filtered by the low-pass filter and the corresponding noise power at vout is therefore:

$$P_{1nth} = v_{1nth}^2 \cdot \frac{\pi}{2} \cdot fc \approx 251,3 \cdot 10^{-6} \qquad V^2 \qquad \text{Eq. 4.45}$$

The R-C filter also induces noise. If we allow the filter noise to be, for example, at most 5% of the amplifier noise, we can calculate the minimum value of capacitor C with:

$$P_{filter} = k\frac{T}{C} = 0,05 \cdot P_{1nth} \qquad V^2 \qquad \text{Eq. 4.46}$$

$$C = \frac{kT}{0,05 \cdot P_{1nth}} \approx 0,32 \qquad fF \qquad \text{Eq. 4.47}$$

This value is of course not practical, and a much greater value must be selected, at least 10pF to be larger than the parasitic capacitance. With this value, the noise due to the filter itself is negligible. The value of R3 for C = 10pF is therefore:

$$R3 = \frac{1}{2\pi \cdot C \cdot fc} \approx 159 \qquad k\Omega \qquad \text{Eq. 4.48}$$

4.7• Summary

The key points for noise calculation are summarised below:

- **Signal sources** must be **zero** (short circuit or open circuit depending on the source).
- For resistors, noise current representation is **strictly equivalent** to noise voltage model.
- Any shorted current noise source does not contribute to noise.
- Noise voltage source of the operational amplifier can be connected either on the inverting or on the non-inverting input, whatever is the most convenient.
- Noise is considered as a (very) small signal: $\varepsilon = 0$ and $i_\varepsilon = 0$ for the operational amplifier. Other small-signal concepts hold.
- Noise current cannot flow in a resistor across which the voltage cannot change. This happens when both terminals are connected to fixed voltages (which are considered shorted to ground for the small-signal approximation).
- Filter bandwidth is greater for noise than it is for signal. Bandwidth depends on **noise type** and on **filter order**.

4.8• Glossary

Continuous-time (system): "temps continu", c’est à dire non échantillonné
DC: continu, pour une tension ou un courant
The mains: le secteur
Standard deviation: écart type

4.9• Appendix 1

The concept of source resistance is sometimes confusing. In the calculation made in section 4.5.2, the source resistance refers to a physical resistor since it produces a 4kTRs noise voltage. Therefore, if a current flows through it, the corresponding voltage appears across it. In textbooks dealing with radio frequencies (RF), the same source resistance concept appears, but here, the meaning is quite different. In the RF frequency range, matching the source to the load is mandatory. When you connect for example an antenna to a receiver, you will receive at least a noise power that corresponds to the ambient noise floor kTΔf, which is applied to the receiver input (figure 4.25a). The receivers input resistance is Rr. As far as noise is concerned, this arrangement is similar to figure 4.25b and because matching is mandatory Rs must equal the receiver input resistance Rr.

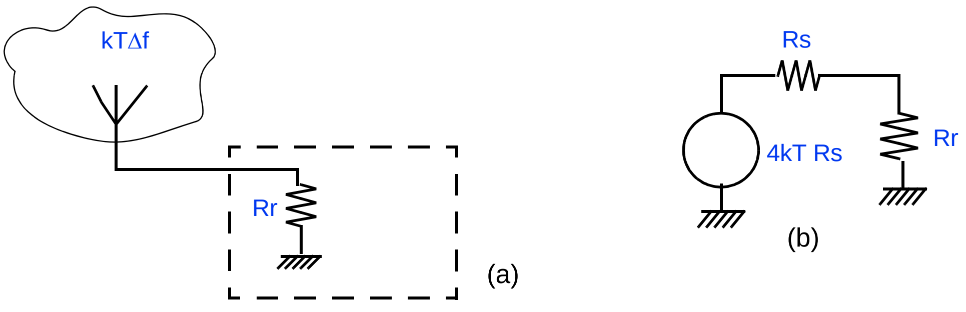


Figure 4.25: Receiver's input equivalent circuit

The noise voltage spectral density across Rr (when Rr = Rs) is:

$$v_n^2 = 4kTRs \cdot \left(\frac{Rr}{Rr + Rs}\right)^2 = 4kTRs \cdot \frac{1}{4} \qquad \text{Eq. 4.49}$$

The noise power dissipated by Rr is:

$$P_n = \frac{v_n^2}{Rs} \cdot \Delta f = kT\Delta f \qquad \text{Eq. 4.50}$$

In this case, the source resistance is not a real device but a model for the noise received at the antenna.

4.10• References

[4.1]: "Low-Noise Electronic Design", C.D. Mortenbacher, F.C. Fitchen,
John Wiley & Sons, ISBN 0-471-61950-7

[4.2]: "Electrical Noise", A.R. Bennett, McGraw-Hill

[4.3]: http://en.wikipedia.org/wiki/Normal_distribution

[4.4]: http://en.wikipedia.org/wiki/Electronic_noise