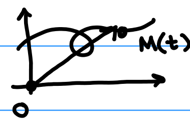
TD 2020/12/4 13h30

# Longueur de courbes

1. a) $\begin{cases} x(t) = t - \sin t \\ y(t) = 1 - \cos t \end{cases}$ $\qquad \vec{OM}(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$

$\vec{OM}(t+2\pi) = \vec{OM}(t) + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$

$t \in \mathbb{R}$.



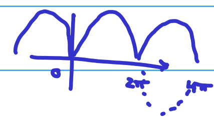

$\theta = t/2$

$$\frac{d\vec{OM}}{dt} = \begin{bmatrix} 1 - \cos t \\ \underset{\sin 2\theta}{\sin t} \end{bmatrix} = \begin{bmatrix} 2\sin^2(t/2) \\ 2\sin t/2 \cos t/2 \end{bmatrix} = 2\sin t/2 \underbrace{\begin{bmatrix} \sin t/2 \\ \cos t/2 \end{bmatrix}}_{\|\cdot\| = 1}$$

$$\left\| \frac{d\vec{OM}}{dt} \right\| = 2 \left| \sin t/2 \right|$$

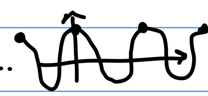
$$\left\| \frac{d\vec{OM}}{dt} \right\| = \sqrt{(1-\cos t)^2 + \sin^2 t}$$

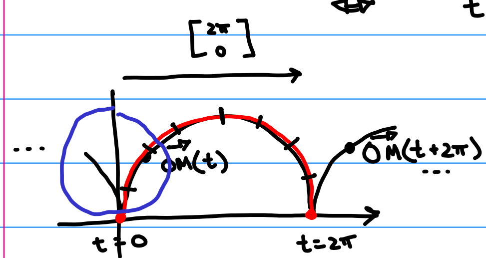
$$= \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}}$$




$$= \sqrt{2 - 2\cos t} = \sqrt{2}\sqrt{1 - \cos t}$$

$$\left( \left\| \frac{d\vec{OM}}{dt} \right\| = 0 \iff \cos t = 1 \iff t = 2k\pi \quad k \in \mathbb{Z} \right)$$






cycloïde

$$\ell = \text{longueur d'une arche} = \int_0^{2\pi} \left\| \frac{d\vec{OM}}{dt} \right\| dt = \sqrt{2} \int_0^{2\pi} \sqrt{\underbrace{1 - \cos t}_{\geq 0}}\, dt$$

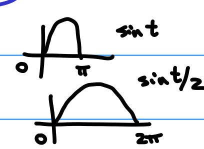
$$1 - \cos t = 1 - \cos 2 \cdot \underbrace{t/2}_{\theta}$$

$$\begin{cases} \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \quad = 1 - 2\sin^2\theta \\ \quad = 2\cos^2\theta - 1 \end{cases}$$

$$= 1 - (1 - 2\sin^2(t/2))$$

$$= 2\sin^2(t/2)$$

$$\ell = \sqrt{2} \int_0^{2\pi} \sqrt{2\sin^2(t/2)}\, dt = 2 \int_0^{2\pi} |\sin(t/2)|\, dt$$




$$= 2 \int_0^{2\pi} \sin t/2\, dt = 4 \Big[ -\cos t/2 \Big]_0^{2\pi}$$

$$= 4(-\cos\pi + \cos 0) = 4(+1+1) = 8$$

a) $\vec{OM}(t) = \begin{bmatrix} 2t^2 \\ t^3 \end{bmatrix}$ , $t \in [0, 2\pi]$ $\qquad \dfrac{d\vec{OM}}{dt} = \begin{bmatrix} 4t \\ 3t^2 \end{bmatrix} = t \begin{bmatrix} 4 \\ 3t \end{bmatrix}$

$$\left\| \frac{d\vec{OM}}{dt} \right\| = |t| \sqrt{4^2 + 9t^2} \underset{t \in [0,2\pi]}{=} t \sqrt{16 + 9t^2}$$

$$l = \frac{1}{18} \int_0^{2\pi} 18t \sqrt{\underbrace{16 + 9t^2}} \, dt \qquad \begin{aligned} u &= 16 + 9t^2 \\ u' &= 18t \end{aligned}$$

$$= \frac{1}{18} \left[ \frac{(16 + 9t^2)^{3/2}}{3/2} \right]_0^{2\pi} = \frac{\cancel{2}^{\,1}}{\underset{27}{3 \cdot \cancel{18}}} \left( (16 + 36\pi^2)^{3/2} - 64 \right)$$

b) $\vec{OM}(t) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}$

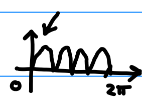
$$\frac{d\vec{OM}}{dt} = \begin{bmatrix} -3\cos^2 t \ \sin t \\ 3 \ \sin^2 t \ \cos t \end{bmatrix} = 3 \cos t \sin t \cdot \overbrace{\begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}}^{\text{unitaire}}$$

$$\left\| \frac{d\vec{OM}}{dt} \right\| = 3 \left| \cos t \sin t \right| = 3/2 \left| \sin 2t \right|$$

pt singulier $\iff \sin 2t = 0 \iff 2t = k\pi \quad k \in \mathbb{Z}$

$\iff t = k\pi/2 \quad k \in \mathbb{Z}$

$$l_{tot} = \int_0^{2\pi} \left\| \frac{d\vec{OM}}{dt} \right\| dt = \frac{3}{2} \int_0^{2\pi} |\sin 2t| \, dt$$




$$= \frac{3}{2} \times 4 \times \int_0^{\pi/2} \sin 2t \, dt$$

$$= -6 \cdot \left[ \frac{\cos 2t}{2} \right]_0^{\pi/2} = -3(-1-1) = 6$$