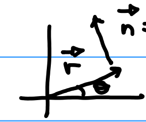
TD 8/12/2020

1\. e) $\begin{cases} x(t) = e^{-t/10} \cos t \\ y(t) = e^{-t/10} \sin t \end{cases}$ $t \in [0, \infty[$

① longueur totale de la courbe ?

② rayon de courbure en chaque point.

① $\ell = \displaystyle\int_0^\infty \left\| \frac{d\vec{M}}{dt} \right\| dt$



$\vec{n} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

$$M(t) = \vec{OM}(t) = \begin{bmatrix} e^{-t/10} \cos t \\ e^{-t/10} \sin t \end{bmatrix} = e^{-t/10} \underbrace{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}_{\vec{r}(t)}$$

$$\frac{d\vec{M}}{dt} = \left(e^{-t/10}\right)' \vec{r} + e^{-t/10} \frac{d}{dt} \vec{r} = -\frac{1}{10} e^{-t/10} \vec{r} + e^{-t/10} \vec{n}$$

$$= e^{-t/10} \left(-\frac{1}{10} \vec{r}(t) + \vec{n}(t) \right)$$

$$v = \left\| \frac{d\vec{M}}{dt} \right\| = e^{-t/10} \left\| -\frac{1}{10} \vec{r}(t) + \vec{n}(t) \right\| = e^{-t/10} \sqrt{\left(-\frac{1}{10}\right)^2 + 1^2}$$

base orthonormée

$$= \frac{\sqrt{101}}{10} e^{-t/10}$$

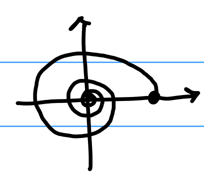
$$\ell = \int_0^\infty \frac{\sqrt{101}}{10} e^{-t/10} dt = \frac{\sqrt{101}}{10} \lim_{R \to \infty} \int_0^R e^{-t/10} dt$$

$$= \frac{\sqrt{101}}{\cancel{10}} \lim_{R \to \infty} \left[\frac{e^{-t/10}}{-1/\cancel{10}} \right]_0^R = \sqrt{101} \lim_{R \to \infty} \left(1 - e^{-R/10} \right) = \underline{\sqrt{101}}$$

$$\left(e^{\lambda t}\right)' = \lambda e^{\lambda t}$$

$$\int e^{\lambda t} dt = \frac{e^{\lambda t}}{\lambda} + C$$

$\rho(\theta) = e^{-\theta/10}$



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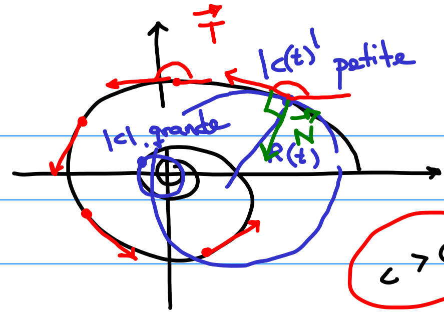
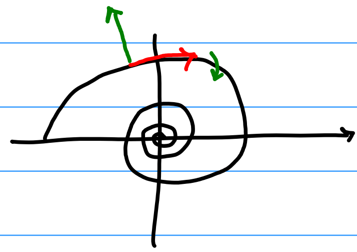


$R(t) = \frac{1}{|c(t)|}$

$c > 0$ tourne dans le sens trigo



$c < 0$ sens antitrigo.

$\vec{T} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ α angle directeur

$\boxed{c = \frac{d\alpha}{ds}} = \frac{d\alpha}{dt} \cdot \frac{dt}{ds} = \frac{1}{v}\frac{d\alpha}{dt}$

abscisse curviligne

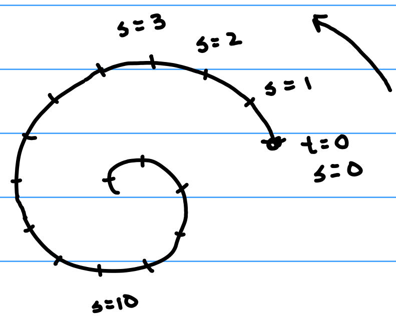
$\vec{N} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}$

$$\frac{d\vec{T}}{ds} = \begin{bmatrix} -\sin\alpha \cdot d\alpha/ds \\ \cos\alpha \cdot d\alpha/ds \end{bmatrix} = c\,\vec{N}$$

(abscisse curviligne ? $s(t) = \int_0^t \left\| \frac{d\vec{M}}{du}(u) \right\| du$

$$= \int_0^t \frac{\sqrt{101}}{10} e^{-u/10}\, du$$

$$= \sqrt{101}\left(1 - e^{-t/10}\right)$$



$$\frac{ds}{dt} = \sqrt{101} \cdot \left(\frac{1}{10}\right) e^{-t/10} = v = \left\| \frac{d\vec{M}}{dt} \right\|$$

$$v = \frac{ds}{dt} = \left\| \frac{d\vec{M}}{dt} \right\|$$

$$s = \sqrt{101}\left(1 - e^{-t/10}\right)$$

$$1 - \frac{s}{\sqrt{101}} = e^{-t/10}$$

$$-10 \ln\left(1 - \frac{s}{\sqrt{101}}\right) = t(s)$$ ~ paramétrisation naturelle : $\vec{M}(t(s))$

$$\frac{d\vec{M}}{ds} = \frac{d\vec{M}}{dt} \cdot \frac{1}{v} = \vec{T}$$

$$\left\| \frac{d\vec{M}}{ds} \right\| = \left\| \frac{d\vec{M}}{dt} \cdot \frac{dt}{ds} \right\|$$

$$\frac{d\vec{M}}{dt} = v\,\vec{T}$$

$$= \left\| \frac{d\vec{M}}{dt} \cdot \frac{1}{v} \right\| = \left\| \frac{d\vec{M}}{dt} \right\| \cdot \frac{1}{v} = 1$$

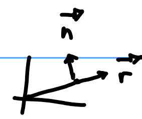
$$\vec{M} = e^{-t/10}\,\vec{r}(t) . \qquad \frac{d\vec{M}}{dt} = e^{-t/10}\left(\frac{-1}{10}\vec{r} + \vec{n}\right) = v\,\vec{T}$$

$$v = \frac{ds}{dt} = \left\| \frac{d\vec{M}}{dt} \right\| = \frac{\sqrt{101}}{10}\, e^{-t/10}$$

$$\vec{T} = \frac{1}{v}\frac{d\vec{M}}{dt} = \frac{1}{\sqrt{101}/10\ \cancel{e^{-t/10}}} \cdot \cancel{e^{-t/10}} \left(\frac{-1}{10}\vec{r} + \vec{n}\right)$$

$$= \frac{+1}{\sqrt{101}}\left(-\vec{r} + 10\,\vec{n}\right) = \frac{1}{\sqrt{101}}\begin{pmatrix} -\cos t - 10\sin t \\ -\sin t + 10\cos t \end{pmatrix}$$

$$\frac{d\vec{T}}{dt} = \frac{1}{\sqrt{101}}\left(-\vec{n} - 10\,\vec{r}\right) = \frac{-1}{\sqrt{101}}\left(10\,\vec{r} + \vec{n}\right) = \vec{N}$$



$$\frac{d\vec{T}}{ds} = \frac{dt}{ds}\cdot\frac{d\vec{T}}{dt} = \underbrace{\left(\frac{1}{v}\right)}_{c(t)}\vec{N}$$

$$\boxed{c(t) = \frac{10}{\sqrt{101}}\, e^{t/10}} \quad > 0$$

croissante ✓

$$R(t) = \frac{1}{c(t)} = \frac{\sqrt{101}}{10}\, e^{-t/10} \quad > 0 \quad \text{décroissant} \quad \to 0$$

$$\frac{d\vec{M}}{dt} = v\,\vec{T}$$

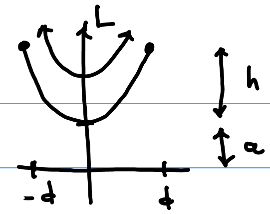
$$\frac{d^2\vec{M}}{dt^2} = \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{dt} = v'\,\vec{T} + v^2 c\,\vec{N}$$

$$\frac{d\vec{M}}{dt} \wedge \frac{d^2\vec{M}}{dt^2} = v\vec{T} \wedge \left(v'\,\vec{T} + v^2 c\,\vec{N}\right)$$

$$= \vec{0} + v^3 c\,\underbrace{\vec{T}\wedge\vec{N}}_{\vec{k}} = v^3 c\ \vec{k}$$

$$\Rightarrow \quad v^3 c = \text{dét}\left(\frac{d\vec{M}}{dt}, \frac{d^2\vec{M}}{dt^2}\right) = \begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix} = x'y'' - y'x''$$

5. $\vec{M}(t) = \begin{bmatrix} t \\ a\,\mathrm{ch}(t/a) \end{bmatrix}$ $\quad t \in [-d, d]$.



$$\frac{d\vec{M}}{dt} = \begin{bmatrix} 1 \\ \cancel{a}\ \mathrm{sh}(t/a)\ 1/\cancel{a} \end{bmatrix}$$

$$1 = \mathrm{ch}^2 t - \mathrm{sh}^2 t$$

$$v = \left\| \frac{d\vec{M}}{dt} \right\| = \sqrt{1 + \mathrm{sh}^2(t/a)}$$

$$= \sqrt{\mathrm{ch}^2(t/a)} = \left| \mathrm{ch}(t/a) \right| = \mathrm{ch}(t/a)$$

$$L = 2\int_{\cancel{-d}\ 0}^{d} \mathrm{ch}(t/a)\, dt = 2\Big[a\,\mathrm{sh}(t/a) \Big]_{\cancel{-d}\ 0}^{d} = 2\,a\,\mathrm{sh}(d/a)$$

$$a + h = a\,\mathrm{ch}(d/a) \quad \Rightarrow \quad h = a\left(\mathrm{ch}(d/a) - 1\right)$$