# TD 8/12/2020

1\. b) $\begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \end{cases}$ $\quad t \in [0, 2\pi]$

Calculer : longueur, abscisse curviligne $s(t)$, repère mobile $\vec{T}$, $\vec{N}$, courbure $c(t)$, rayon $R(t)$

$$\vec{M}(t) = \begin{bmatrix} \cos^3 t \\ \sin^3 t \end{bmatrix}.$$

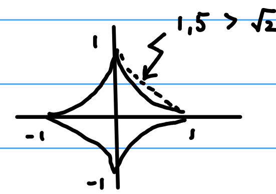
$$\frac{d\vec{M}}{dt} = \begin{bmatrix} -3\cos^2 t \sin t \\ 3 \sin^2 t \cos t \end{bmatrix} = \frac{3^{\cdot 2}}{2}\cos t \sin t \begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\ t \\ \overset{\sin}{\cancel{\cos}}\ t \end{bmatrix}$$

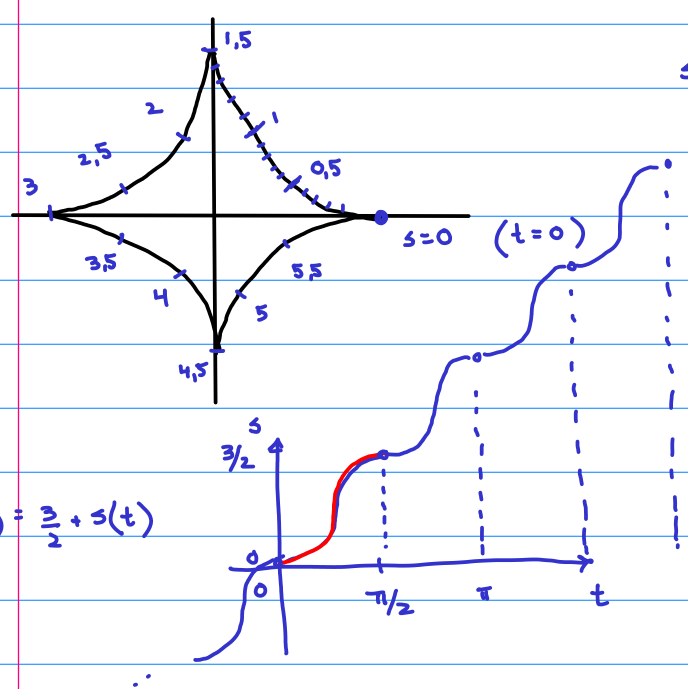
$$= \frac{3}{2}\sin(2t)\underbrace{\begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\ t \\ \overset{\sin}{\cancel{\cos}}\ t \end{bmatrix}}_{\text{unitaire}} \qquad v = \left\| \frac{d\vec{M}}{dt} \right\| = \frac{3}{2}|\sin 2t|$$


longueur totale :

$$\ell = \int_0^{2\pi} \left\| \frac{d\vec{M}}{dt} \right\| dt = \int_0^{2\pi} \frac{3}{2}|\sin 2t|\, dt = \frac{3}{2}\cdot 4\int_0^{\pi/2} \sin 2t\, dt$$

$$= \frac{6}{2}\cdot\Big[-\cos 2t\Big]_0^{\pi/2} = 3\,(1+1) = 6$$




$s(t+\pi/2) = \frac{3}{2} + s(t)$

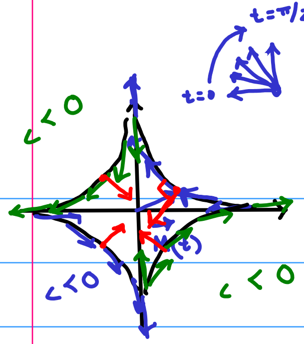
$$s(t) = \int_0^t \overbrace{\left\| \frac{d\vec{M}(u)}{du} \right\|}^{v(u)} du$$

$$= \int_0^t \frac{3}{2}|\sin 2u|\, du \qquad (0 < t < \pi/2)$$

$$= \frac{3}{2}\int_0^t \sin 2u\, du$$

$$= \frac{3}{2}\left[-\frac{\cos 2u}{2}\right]_0^t = \frac{3}{4}(1 - \cos 2t)$$

$\rightsquigarrow$ on peut exprimer $t = t(s)$




$|x| = sg(x)\cdot x \quad \Leftrightarrow \quad x = sg(x)\,|x|$

$$\vec{T} = \frac{d\vec{M}}{dt} \Big/ \left\|\frac{d\vec{M}}{dt}\right\| = \frac{1}{v}\frac{d\vec{M}}{dt} = \frac{1}{\cancel{3/2}\,|\sin 2t|}\cdot \cancel{3/2}\,\sin 2t \begin{bmatrix} \overset{-\cos}{\cancel{-\sin}}\, t \\ \underset{\sin}{\cancel{\cos}}\, t \end{bmatrix}$$

$$= \underbrace{sg(\sin 2t)}_{+/-} \begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\, t \\ \underset{\sin}{\cancel{\cos}}\, t \end{bmatrix}$$

$\vec{N} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{T}$

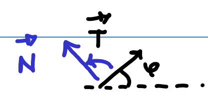
$$\vec{N} = \begin{cases} -\begin{bmatrix} \overset{\sin}{\cancel{\cos}}\, t \\ \underset{\cos}{\cancel{\sin}}\, t \end{bmatrix} \\ +\begin{bmatrix} \overset{\sin}{\cancel{\cos}}\, t \\ \underset{\cos}{\cancel{\sin}}\, t \end{bmatrix} \end{cases} = \begin{cases} +\begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\, t \\ \underset{\sin}{\cancel{\cos}}\, t \end{bmatrix} & 0 < t < \pi/2 \quad \text{ou} \quad \pi < t < 3\pi/2 \\ -\begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\, t \\ \underset{\sin}{\cancel{\cos}}\, t \end{bmatrix} & \pi/2 < t < \pi \quad \text{ou} \quad 3\pi/2 < t < 2\pi \end{cases}$$

<u>Rem.</u>: $\vec{T} = \frac{1}{v}\frac{d\vec{M}}{dt} = \frac{\cancel{dt}}{ds}\frac{d\vec{M}}{\cancel{dt}} = \frac{d\vec{M}}{ds}$ $\qquad s = \int_0^t v(u)\,du$

$$\boxed{\frac{ds}{dt} = v}$$

$c(t)$ : vitesse de rotation de $\vec{T}$

$\vec{T} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}$




$$\boxed{c(s) = \frac{d\varphi}{ds}}$$

$$\frac{d\vec{T}}{ds} = \begin{bmatrix} -\sin\varphi \cdot d\varphi/ds \\ \cos\varphi \cdot d\varphi/ds \end{bmatrix} = c(s) \underbrace{\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}}$$

$\vec{N} = \vec{T}$ tourné de $+\pi/2$

$$\boxed{\frac{d\vec{T}}{ds} = c\,\vec{N}}$$

$\vec{T} = \pm \begin{bmatrix} -\overset{\cos}{\cancel{\sin}}\, t \\ \underset{\sin}{\cancel{\cos}}\, t \end{bmatrix}$ $\qquad \frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\cdot\frac{dt}{ds} = \frac{1}{v}\frac{d\vec{T}}{dt}$

$$= \frac{1}{v}\cdot\left(\pm \begin{bmatrix} +\overset{\sin}{\cancel{\cos}}\, t \\ +\overset{\cos}{\cancel{\sin}}\, t \end{bmatrix}\right) = -\frac{1}{v}\left(\mp \underbrace{\begin{bmatrix} \overset{\sin}{\cancel{\cos}}\, t \\ \underset{\cos}{\cancel{\sin}}\, t \end{bmatrix}}_{\vec{N}}\right)$$

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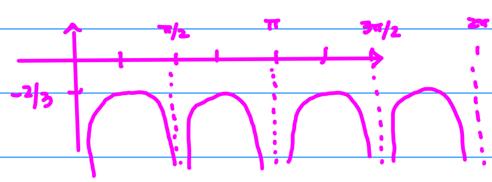
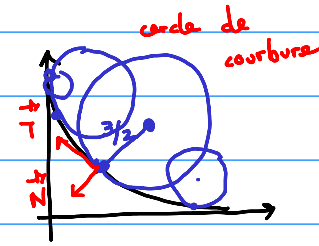
$$\Rightarrow c = \frac{-1}{v} = \frac{-1}{3/2\,|\sin(2t)|} = \frac{-2}{3\,|\sin(2t)|} < 0 \quad ✓$$




$$R(t) = \frac{1}{c(t)} = -\frac{3}{2}\,|\sin 2t|$$



cas part.

$$\left( \begin{aligned} \frac{d\vec{M}}{ds} &= \vec{T} \\ \frac{d^2\vec{M}}{ds^2} &= c\,\vec{N} \end{aligned} \right. \qquad (v=1)$$

<u>Rem.</u>:

$$\left\{ \begin{aligned} \frac{d\vec{M}}{dt} &= v\,\vec{T} \\ \frac{d^2\vec{M}}{dt^2} &= \frac{dv}{dt}\,\vec{T} + v\,\frac{d\vec{T}}{dt} = \underbrace{\frac{dv}{dt}}\,\vec{T} + \underbrace{v^2 c}\,\vec{N} \end{aligned} \right.$$

$$\Rightarrow \frac{d\vec{M}}{dt} \wedge \frac{d^2\vec{M}}{dt^2} = v\,\vec{T} \wedge \left( v'\,\vec{T} + v^2 c\,\vec{N} \right)$$

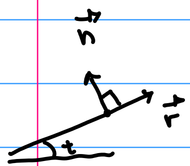
$$= v\,\underset{\vec{0}}{\cancel{\vec{T}\wedge\vec{T}}} + v^3 c\,\underbrace{\vec{T}\wedge\vec{N}}_{\vec{k}} = v^3 c\,\vec{k}$$

$$\begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix} = \det\left( \frac{d\vec{M}}{dt}, \frac{d^2\vec{M}}{dt^2} \right) = v^3 c \quad \Rightarrow \left( c = \frac{x'y'' - y'x''}{v^3} \right.$$

7)

$$\vec{M} = \begin{bmatrix} \sqrt{t}\cos t \\ \sqrt{t}\sin t \\ t \end{bmatrix} = \sqrt{t}\underbrace{\begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix}}_{\vec{r}(t)} + t\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{k}}$$

$$\frac{d\vec{M}}{dt} = \frac{1}{2\sqrt{t}}\vec{r}(t) + \sqrt{t}\underbrace{\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}_{\vec{n}(t)} + \vec{k}$$




$(\vec{r}, \vec{n}, \vec{k})$ base orthonormée

$$\left\| \frac{d\vec{M}}{dt} \right\| = \sqrt{\frac{1}{4t} + t + 1}$$

$$= \sqrt{\frac{1 + 4t^2 + 4t}{4t}} = \sqrt{\frac{(2t+1)^2}{4t}} = \frac{|2t+1|}{2\sqrt{t}}$$

sur $[0, +\infty[$ : $\quad \left\| \frac{d\vec{M}}{dt} \right\| = \frac{2t+1}{2\sqrt{t}}$

$$\ell = \int_0^{12} \frac{2t+1}{2\sqrt{t}}\,dt = \int_0^{12} \left( \sqrt{t} + \frac{1}{2\sqrt{t}} \right) dt$$

$$= \left[ \frac{t^{3/2}}{3/2} + \sqrt{t} \right]_0^{12} = m\sqrt{3}$$