TD Longueurs de courbes

1. d) $\begin{cases} x(t) = \cos t\,(1+\cos t) \\ y(t) = \sin t\,(1+\cos t) \end{cases} \qquad t \in [0, 2\pi]$

$$\ell = \int_0^{2\pi} \left\| \frac{d\vec{OM}(t)}{dt} \right\| dt$$

$$\vec{OM}(t) = \begin{bmatrix} \cos t\,(1+\cos t) \\ \sin t\,(1+\cos t) \end{bmatrix} = \overbrace{(1+\cos t)}^{2\cos^2(t/2)} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\frac{d\vec{OM}}{dt} = (1+\cos t)' \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + (1+\cos t) \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}'$$

$$= \underbrace{-\sin t}_{-2\cdot\sin(t/2)\cos(t/2)} \underbrace{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}_{\vec{u}} + \underbrace{(1+\cos t)}_{2\cos^2(t/2)} \underbrace{\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}_{\vec{v}} \qquad \vec{u}\cdot\vec{v} = 0$$

$2\cos t/2 \left(-\sin t/2\, \vec{u} + \underbrace{\cos t/2\, \vec{v}}_{\text{unitaire}} \right)$

$$\left\| \frac{d\vec{OM}}{dt} \right\| = \sqrt{(-\sin t)^2 + (1+\cos t)^2} \qquad (\vec{u}, \vec{v}) \text{ base orthonormée}$$

$2|\cos t/2|$

$$= \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}$$

$\frac{d\vec{OM}}{dt} = 0 \Leftrightarrow \cos t = -1 \Leftrightarrow t = \pi$

$$= \sqrt{2 + 2\cos t} = \sqrt{2(1+\cos t)}$$

$$\ell = \sqrt{2}\int_0^{2\pi} \sqrt{1+\cos t}\, dt$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

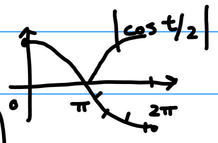
$t = 2\theta \qquad = 2\cos^2\theta - 1$

$\theta = t/2 \qquad = 1 - 2\sin^2\theta$

$(1 + \cos 2\theta = 2\cos^2\theta$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2(t/2)}\, dt$$

$$= 2\int_0^{2\pi} |\cos(t/2)|\, dt$$

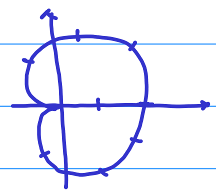


$$= 2\left(\int_0^{\pi} \cos t/2\, dt + \int_{\pi}^{2\pi} -\cos t/2\, dt \right)$$

ou

$$= 4\int_0^{\pi} \cos t/2\, dt = 8\left[\sin t/2\right]_0^{\pi}$$

$$= 8\,(\sin \pi/2 - 0) = \underline{8}$$



$$\rho = e^{-t/10}$$

e) $\overrightarrow{OM}(t) = e^{-t/10} \cdot \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

$$\frac{d\overrightarrow{OM}}{dt} = -\frac{1}{10} e^{-t/10} \underbrace{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}_{\vec{u}} + e^{-t/10} \underbrace{\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}}_{\vec{v}}$$

$\vec{u} \cdot \vec{v} = 0$

unitaire unitaire

$$\left\| \frac{d\overrightarrow{OM}}{dt} \right\| = e^{-t/10} \sqrt{\left(\frac{-1}{10}\right)^2 + 1^2}$$

$$= e^{-t/10} \sqrt{\frac{1}{100} + 1} = \frac{\sqrt{101}}{10} e^{-t/10}.$$

$$\ell = \int_0^{+\infty} \left\| \frac{d\overrightarrow{OM}}{dt} \right\| dt = \int_0^{\infty} \frac{\sqrt{101}}{10} e^{-t/10} dt$$

$$= \frac{\sqrt{101}}{10} \int_0^{\infty} e^{-t/10} dt = \frac{\sqrt{101}}{\cancel{10}} \lim_{R \to \infty} \left[\frac{e^{-t/10}}{-1/\cancel{10}} \right]_0^R$$

$$= \sqrt{101} \lim_{R \to \infty} \left(1 - e^{-R/10} \right) = \sqrt{101}$$