12.11.2024
- PS -

Defn: f is diffable at a if $\exists m \in \mathbb{R}$, a fn $E(h)$ s.t $f(a+h) = f(a) + mh + E(h)$ and $\underbrace{\lim_{h\to 0} \frac{E(h)}{h} = 0}_{\text{we say } E(h) \text{ is } o(h)}$

* $E(h)$ is called $o(h^n)$ if $\lim_{h\to 0} \frac{E(h)}{h^n} = 0$

* If $n>m$ and $E(h)$ is $o(h^n)$, then $E(h)$ is also $o(h^m)$

1) Let $n>m$. f is $o(h^m)$ and g is $o(h^n)$

 * Show $f+g$ is $o(h^m)$ and

 ** $f \cdot g$ is $o(h^{m+n})$

WTS (*) $\lim_{h\to 0} \frac{f+g}{h^m} = 0 = \lim_{h\to 0} \frac{f}{h^m} + \frac{g}{h^m} = \underbrace{\lim_{h\to 0} \frac{f}{h^m}}_{0} + \underbrace{\lim_{h\to 0} \frac{g}{h^m}}_{0} = 0$ ∎

since limits exist we can use sum rule

f is $o(h^m) \iff \lim_{h\to 0} \frac{f(h)}{h^m} = 0$

g is $o(h^n) \Rightarrow g$ is also $o(h^m)$ since $\lim_{h\to 0} \frac{g(h)}{h^m} = \lim_{h\to 0} \frac{g(h)\, h^{n-m}}{h^n} = 0$ so $g(h)$ is also $o(h^m)$

(**) $\lim_{h\to 0} \frac{(f \cdot g)(h)}{h^{m+n}} = \lim_{h\to 0} \frac{f(h)}{h^m} \cdot \frac{g(h)}{h^n} = \lim_{h\to 0} \frac{f}{h^m} \cdot \lim_{h\to 0} \frac{g}{h^n} = 0 \cdot 0 = 0$ ∎

since both limits exist, apply product rule

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2) Define the fn f by $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$, Show that f is diff. at $\forall x \in \mathbb{R}$ but $f'(x)$ is discontinuous at $x=0$.

So when $x \neq 0$, $f(x) = x^2 \sin(1/x)$ which is diffable and

$f'(x) = 2x\sin(1/x) - \cos(1/x)$

by Leibniz Rule

f is diffable at $x=0$. If there exist $m \in \mathbb{R}$, $E(h)$ which is $o(h)$ s.t $f(h) = f(0) + \underbrace{m}_{f'(0)}h + E(h)$, $f(h) = f(0+h)$

when $x=0$, by little o defn. we know that $\frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \frac{mh}{h} + \cancel{\frac{E(h)}{h}} \to 0$

observe that $\lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} \frac{\sin(1/h)}{1/h} = 0 \rightsquigarrow \boxed{m=0}$

Thus for $f(h) = E(h)$ and $f(h)$ is $o(h)$ we have

$f(0+h) = \underbrace{f(0)}_{0} + 0 \cdot h + \underbrace{f(h)}_{o(h)}$

$\Rightarrow f$ is diffable at zero and $f'(0) = 0$

is $f'(x)$ cts at zero ?

$\boxed{\text{derivative fn is not cts.}} \leftarrow f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases} \rightarrow \boxed{\text{does not exist}} \neq 0$

3) Let f be the fn given in ②. and let $g(x) = f(x) + \frac{x}{2}$

$$f(x) := \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that $g'(0) > 0$. But there is no neigh. of 0 on which g is increasing. (Every interval containing zero has a subinterval where g is decreasing)

↳ By sum rule of derivative $g'(x) = f'(x) + \frac{1}{2}$ and at $x = 0$.

$$g'(0) = 0 + \frac{1}{2} > 0.$$

For g to be ↗ (increasing) at any interval I containing zero. we should have $g'(x) > 0$ for any $x \in I$. $\quad g'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2}$

(Hint: arc property)

$\frac{1}{2\pi}$ 'nin katlarını seç

By Archimedean property in any I, we can choose n large enough so that $\frac{1}{2\pi n} \in I$, and $g'\left(\frac{1}{2\pi n}\right) = -\frac{1}{2} < 0$. Hence there is no such interval I around zero where g is increasing.

4) $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ Show that f is diffable at $x = 0$ eventhough it is discontinuous at every other point.

To show differentiability at $x = 0$, we want to find $m \in \mathbb{R}$ (a candidate for the derivative) and $E(h)$ s.t $\lim_{h \to 0} \frac{E(h)}{h} = 0$ s.t $f(h) = f(0+h) = f(0) + mh + E(h) = f(0) + 0 \cdot h + f(h)$

Observe that $\lim_{h \to 0} \frac{f(h)}{h} = \begin{cases} \lim_{h \to 0} h & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} = 0$

$f(h)$ is $o(h)$ and taking $E(h) = f(h)$ and $m = 0$, we see that f is diffable at zero.

Show that $f(x) = x^2$ and at $x = 1$ $f'(1) = 2$

$f(1+h) = f(1) + \underset{\sim}{m} \cdot h + E(h)$

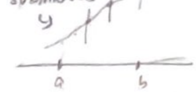
$(1+h)^2 = 1^2 + 2h + h^2$

for $m = 2$, $E(h) = h^2$, f is diffable at $x = 1$, $f'(1) = m = 2$.

5) Folland (2.1.1)

Suppose f is diffable on I and $f'(x) > 0$ for all $x \in I$ except finitely many points at which $f'(x) = 0$. Show that f is strictly $\nearrow$.

look at the subintervals



↳ WTS : for any $a < b \in I$, $f(b) - f(a) > 0$ (Hint: Use MVT)

↳ If f is diff (a,b) $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

Consider finitely many points $x_1 < x_2 < \dots < x_n \in (a,b)$ for which $f'(x_i) = 0$. And $f(b) - f(a) = \underbrace{f(b) - f(x_n)}_{>0} + \underbrace{f(x_n) - f(x_{n-1})}_{>0} + \dots + \underbrace{f(x_1) - f(a)}_{>0}$

For any $c_i \in (x_i, \dots x_{i+1})$, $i = 0, \dots n$ $(x_0 = a,\ x_{n+1} = b)$

$f'(c_i) > 0$ by assumption. By MVT

$$\frac{f(x_{i+1}) - f(x_i)}{\underbrace{x_{i+1} - x_i}_{>0}} = f'(c_i) > 0 \Rightarrow f(x_{i+1}) - f(x_i) > 0 \quad \forall i = 1, \dots n$$

for some c_i

$$\Rightarrow f(b) - f(a) > 0$$

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6) Folland (2.1.10)

↳ Darboux Thm : IVT applying for f' → we don't know f' is cts or not.

Darboux Thm : f is diff on $[a,b]$. If v is any number between $f'(a)$ and $f'(b)$ then $\exists\, c \in (a,b)$ s.t. $f'(c) = v$. Prove it.

Proof: WTS : $f'(c) - v = 0$ for some $c \in (a,b)$.

$g'(x) = f'(x) - v$ has a zero in (a,b)

$\Leftrightarrow$ $g'(x) = f'(x) - v$

we don't know whether it is cts or not!

Consider the function $g: [a,b] \longrightarrow \mathbb{R}$ $g(x) = f(x) + v \cdot x$, f is diffable so f is cts $\Rightarrow g$ is also cts.

observe that $g'(x) = f'(x) - v$

By EVT g attains its maximum and minimum in $[a,b]$. If we can show that at least one of them occurs at the interior points of $[a,b]$ then we are done!

observe that either g attains its max & min on the boundary or it attains at least one of them inside the interval.

Therefore at least one of the max or min occurs at some $c \in (a,b)$

i.e. $g'(c) = 0$

$\Rightarrow f'(c) = v$

① Assume $g(a)$ is maximum and $g(b)$ is min. then for any $t \in (a,b)$ $\frac{g(t) - g(a)}{\underbrace{t-a}_{>0}} < 0 \Rightarrow g'(a) < 0$. (as $t \to 0$)

$\Rightarrow f'(a) < v$ Moreover $\frac{g(t) - g(b)}{\underbrace{t-b}_{<0}} < 0 \Rightarrow g'(b) < 0$ (as $t \to 0$)

$\Rightarrow f'(b) < v$. But we assumed v to be in between $f'(a)$ & $f'(b)$. Contradiction. ※

② Similarly if $g(a)$ is min $g(b)$ is max.

$\Rightarrow f'(a) > v$, $f'(b) > v$ ※