Prospects and challenges for quantum machine learning

October 14-16

1 [Exercise]

Since $\mathbb{1} \cdot \mathbb{1} = \mathbb{1}$, $X \cdot X = \mathbb{1}$ and $\mathbb{1} \cdot X = X \cdot \mathbb{1} = X$, we see that the product of any two elements from $\{\mathbb{1}, X\}$ is also in $\{\mathbb{1}, X\}$, and that the Caley table for this set is the same as the one for the cyclic group $\mathbb{Z}_2$. We also see that each element is their own inverse, and that $\mathbb{1}$ is the identity. Also, the associativity follows from the fact that matrix multiplication is associative.

Therefore, from the existence of the inverse, the identity and from the associativity, we can conclude that the set $\{\mathbb{1}, X\}$ is indeed a group.

2 [Exercise]

From

$$\frac{d}{d\phi_1} e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}\bigg|_{(\phi_1,\phi_2,\phi_3)=(0,0,0)} = -iY \tag{1}$$

and

$$\frac{d}{d\phi_2} e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y}\bigg|_{(\phi_1,\phi_2,\phi_3)=(0,0,0)} = -iX \tag{2}$$

we see that one generator is proportional to iX and another to iY.

Next, we expand the three exponentials around small values of (ϕ_1, ϕ_2, ϕ_3):

$$\begin{aligned} e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y} &\approx (\mathbb{1} - i\phi_3 Y)(\mathbb{1} - i\phi_2 X)(\mathbb{1} - i\phi_1 Y) && (3)\\ &\approx \mathbb{1} - i(\phi_3 + \phi_1)Y - i\phi_2 X - \phi_3\phi_2 YX - \phi_2\phi_1 XY && (4)\\ &\approx \mathbb{1} - i(\phi_3 + \phi_1)Y - i\phi_2 X - i(\phi_1 - \phi_3)\phi_2 Z\ , && (5)\end{aligned}$$

where $XY = iZ$ was used. From the above equation, we get that it is possible to write the product of the exponentials in the form

$$e^{-i\phi_3 Y} e^{-i\phi_2 X} e^{-i\phi_1 Y} = e^{-iaX - ibY - icZ}\ , \tag{6}$$

and so we can show, by differentiating with respect to c and setting all the parameters to zero, that one generator is also proportional to iZ. Since the Pauli matrices obey the commutation relations,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \ , \tag{7}$$

we can define the generators $T_i = \frac{1}{2}i\sigma_i$, that obey

$$[T_i, T_j] = \epsilon_{ijk}T_k \ , \tag{8}$$

which is the $su(2)$ Lie algebra. Therefore, $e^{-i\phi_3 Y}e^{-i\phi_2 X}e^{-i\phi_1 Y}$ constitutes a representation of the SU(2) Lie group.

3 [Challenge]

The elements of the representation can be written as a direct sum of the irreps in the following way

$$R(g) = \bigoplus_\lambda \mathbb{1}_{m_\lambda} \otimes R_\lambda(g) \tag{9}$$

for all g in the group. Since, for an element A of the commutant, $[A, R(g)] = 0$ for all g, we have that A can be written as

$$A = \bigoplus_\lambda A_\lambda \otimes \mathbb{1}_{d_\lambda} \ , \tag{10}$$

where A_λ is an $m_\lambda \times m_\lambda$ arbitrary matrix.

Since each A_λ is an arbitrary matrix, and its dimension is $m_\lambda \times m_\lambda$, then there are $m_\lambda \times m_\lambda = m_\lambda^2$ possible A_λ matrices for each lambda. Then summing for all λ, we have that the total number of elements in the basis of the commutant, that is, the dimension of the commutant, is

$$D = \sum_\lambda m_\lambda^2 \ , \tag{11}$$

which is what we wanted to prove.