Exercice 2:

a) Deduce the expression for the expentation value of the Hamiltonian:

$$H=\sum_{i=1}^{N}\left[\frac{p_i^2}{2m}+V\left(r_i\right)\right]+U_0\sum_{i<j}\delta\left(r_i-r_j\right)$$

and,

$$E=\langle\Psi|H|\Psi\rangle=\sum_{i=1}^{N}\int d^3r\frac{\hbar^2}{2m}|\nabla_{r_i}\Psi\left(r_1,...,r_N\right)|^2+\sum_{i=1}^{N}\int d^3rV\left(r_i\right)|\Psi\left(r_1,...,r_N\right)|^2+$$

$$U_0\sum_{i<j}\int d^3r|\Psi\left(r_1,...,r_N\right)|^2\delta\left(r_i-r_j\right)$$

where,

$$\Psi\left(r_1,...,r_N\right)=\prod_{i=1}^{N}\phi(r_i)$$

$$\int d^3r_i|\phi(r_i)|^2=1$$

so,

$$E=\frac{\hbar^2}{2m}\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_j|\nabla_{r_i}\phi\left(r_j\right)|^2+\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_jV\left(r_i\right)|\phi\left(r_j\right)|^2+$$

$$U_0\sum_{i<j}\int\prod_{j=1}^{N}d^3r_j|\phi\left(r_j\right)|^2\delta\left(r_i-r_j\right)$$

we make,

$$E=\frac{\hbar^2}{2m}\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_j|\nabla_{r_i}\phi\left(r_j\right)|^2+\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_jV\left(r_i\right)|\phi\left(r_j\right)|^2+$$

$$U_0\int\prod_{i=1}^{N}d^3r_i|\phi\left(r_i\right)|^2\left\{\sum_{i<j}\int\prod_{j=1}^{N}d^3r_j|\phi\left(r_j\right)|^2\delta\left(r_i-r_j\right)\right\}$$

where,

$$\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_j|\nabla_{r_i}\phi\left(r_j\right)|^2=\sum_{i=1}^{N}\int d^3r_i|\nabla_{r_i}\phi\left(r_i\right)|^2=N\int d^3r|\nabla_r\phi\left(r\right)|^2$$

$$\sum_{i=1}^{N}\int\prod_{j=1}^{N}d^3r_jV\left(r_i\right)|\phi\left(r_j\right)|^2=\sum_{i=1}^{N}\int d^3r_jV\left(r_j\right)|\phi\left(r_j\right)|^2=N\int d^3rV\left(r\right)|\phi\left(r\right)|^2$$

and,

$$\int \prod_{i=1}^{N} d^3r_i|\phi(r_i)|^2 \left\{ \sum_{i<j} \int \prod_{j=1}^{N} d^3r_j|\phi(r_j)|^2 \delta(r_i - r_j) \right\} =$$

$$\frac{1}{2} \sum_{i} \int \prod_{i=1}^{N} d^3r_i|\phi(r_i)|^2 \left\{ \sum_{j} \int \prod_{j=1}^{N} d^3r_j|\phi(r_j)|^2 \delta(r_i - r_j) \right\}$$

so,

$$\sum_{j} \int \prod_{j=1}^{N} d^3r_j|\phi(r_j)|^2 \delta(r_i - r_j) = N|\phi(r_i)|^2$$

replacing in the expectation value of the hamiltonian,

$$E = N \int d^3r \frac{\hbar^2}{2m} |\nabla_r \phi(r)|^2 + N \int d^3r V(r)|\phi(r)|^2 +$$

$$U_0 \frac{N}{2} \sum_{i=1}^{N} \int d^3r_i|\phi(r_i)|^2|\phi(r_i)|^2$$

finally,

$$E = N \int d^3r \frac{\hbar^2}{2m} |\nabla_r \phi(r)|^2 + N \int d^3r V(r)|\phi(r)|^2 +$$

$$U_0 \frac{N^2}{2} \int d^3r|\phi(r)|^4$$

$$E = N \int d^3r \left[\frac{\hbar^2}{2m} |\nabla \phi(r)|^2 + V(r)|\phi(r)|^2 + U_0 \frac{N}{2} |\phi(r)|^4 \right]$$

b) obtain time-independent GPE. from:

$$E(\psi) = \int d^3r \left[\frac{\hbar^2}{2m} \mid \nabla \psi(r) \mid^2 + V(r) \mid \psi(r) \mid^2 + \frac{1}{2} U_0 \mid \psi(r) \mid^4 \right]$$

and,

$$N = \int d^3r \mid \psi(r) \mid^2$$

by using Lagrange multipliers,

$$\delta E - \mu \delta N = 0$$

and,

$$\frac{\partial E}{\partial \psi^*} \delta \psi^* + \frac{\partial E}{\partial(\nabla \psi^*)} \delta(\nabla \psi^*) - \mu \frac{\partial N}{\partial \psi^*} \delta \psi^* = 0$$

$$\frac{\partial E}{\partial \psi^*} \delta \psi^* + \frac{\partial E}{\partial(\nabla \psi^*)} \nabla(\delta \psi^*) - \mu \frac{\partial N}{\partial \psi^*} \delta \psi^* = 0$$

and,

$$\left\{\frac{\partial E}{\partial\psi^*}-\nabla\left(\frac{\partial E}{\partial\left(\nabla\psi^*\right)}\right)-\mu\frac{\partial N}{\partial\psi^*}\right\}\delta\psi^*+\nabla\left(\frac{\partial E}{\partial\left(\nabla\psi^*\right)}\delta\psi^*\right)=0$$

$$\frac{\partial E}{\partial\psi^*}=\int d^3r\left[V\left(r\right)\psi\left(r\right)+U_0\mid\psi\left(r\right)\mid^2\psi\left(r\right)\right]$$

$$\frac{\partial E}{\partial\left(\nabla\psi^*\right)}=\int d^3r\frac{\hbar^2}{2m}\nabla\psi\left(r\right)$$

$$\frac{\partial N}{\partial\psi^*}=\int d^3\psi\left(r\right)$$

replacing,

$$\int d^3r\left\{V\left(r\right)\psi\left(r\right)+U_0\mid\psi\left(r\right)\mid^2\psi\left(r\right)-\nabla\left(\frac{\hbar^2}{2m}\nabla\psi\left(r\right)\right)-\mu\psi\left(r\right)\right\}\delta\psi^*+$$

$$\int d^3r\nabla\left(\frac{\partial E}{\partial\left(\nabla\psi^*\right)}\delta\psi^*\right)=0$$

and,

$$\int d^3r\left\{-\frac{\hbar^2}{2m}\nabla^2\psi\left(r\right)+V\left(r\right)\psi\left(r\right)+U_0\mid\psi\left(r\right)\mid^2\psi\left(r\right)-\mu\psi\left(r\right)\right\}\delta\psi^*=0$$

a GPE is,

$$-\frac{\hbar^2}{2m}\nabla^2\psi\left(r\right)+V\left(r\right)\psi\left(r\right)+U_0\mid\psi\left(r\right)\mid^2\psi\left(r\right)=\mu\psi\left(r\right)$$

c) with for uniform gas $V\left(r\right)=0$, $\psi\left(r\right)\approx cte$, defining $g=U_0$, $g\mid\psi\mid^2=\mu$. We can consider the time-dependende GPE.

$$i\hbar\frac{\partial\psi\left(r,t\right)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi\left(r\right)+g\mid\psi\left(r\right)\mid^2\psi\left(r\right)$$