High Dimensional Quantum Communication with Structured Light

[1.EXERCISE]

Using the definition Hermite-Gauss Modes:

$$HG_{mn}\left(x,y\right)=\frac{a_{mn}}{w(z)}H_m\left(\frac{\sqrt{2}x}{w(z)}\right)H_n\left(\frac{\sqrt{2}y}{w(z)}\right)e^{-\frac{x^2+y^2}{w(z)}}\,e^{-iK\frac{x^2+y^2}{2R(z)}}\,e^{-i\varphi_N(z)} \quad (1)$$

$$N=m+n \quad , \quad a_{mn}=\left(\frac{2^{1-N}}{\pi n!m!}\right)^{1/2}$$

for $m=0$ and $n=1$,

$$HG_{01}\left(x,y\right)=\frac{a_{01}}{w(z)}\underbrace{H_0\left(\frac{\sqrt{2}x}{w(z)}\right)}_{1}H_1\left(\frac{\sqrt{2}y}{w(z)}\right)\underbrace{e^{-\frac{x^2+y^2}{w(z)}}}_{G}\underbrace{e^{-iK\frac{x^2+y^2}{2R(z)}}\,e^{-i\varphi_N(z)}}_{W}$$

$$HG_{01}=\frac{2\sqrt{2}\,a_{01}}{w^2}y\,G\,W \quad \Longrightarrow \quad HG_{01}=A_{01}\,y\,G\,W \quad (2)$$

for $m=1$ and $n=0$

$$HG_{10}\left(x,y\right)=\frac{a_{10}}{w(z)}H_1\left(\frac{\sqrt{2}x}{w(z)}\right)\underbrace{H_0\left(\frac{\sqrt{2}y}{w(z)}\right)}_{1}\underbrace{e^{-\frac{x^2+y^2}{w(z)}}}_{G}\underbrace{e^{-iK\frac{x^2+y^2}{2R(z)}}\,e^{-i\varphi_N(z)}}_{W}$$

$$HG_{10}=\frac{2\sqrt{2}\,a_{10}}{w^2}x\,G\,W \quad \Longrightarrow \quad HG_{10}=A_{10}\,x\,G\,W \quad (3)$$

We conclude that,

$$A_{10}=A_{01}=\frac{2\sqrt{2}}{\sqrt{\pi}\,w^2} \quad (4)$$

Now, a definition of Laguerre-Gauss modes

$$LG_{pl}=\frac{b_{pl}}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|}L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)e^{-r^2/w(z)^2}\,e^{-iKr^2/2R}\,e^{i\,l\phi}\,e^{-i\,\varphi_N} \quad (5)$$

$$b_{pl}=p!\left(\frac{2}{\pi\,p!\,(|l|+p)!}\right)^{1/2} \quad , \quad N=2p+|l|$$

for $p=0 \Rightarrow L_0^{|l|}(...)=1$.Then,

$$LG_{0l}=\frac{b_{0l}}{w}\left(\frac{\sqrt{2}r}{w}\right)^{|l|} e^{-r^2/w^2}\, e^{-iKr^2/2R}\, e^{i\,l\phi}\, e^{-i\,\varphi_{|l|}}$$

in the case $l=1$, we have:

$$LG_{01}=\left(\frac{2}{\sqrt{\pi}w^2}\right) r \underbrace{e^{-r^2/w^2}}_{G}\, \underbrace{e^{-iKr^2/2R}\, e^{-i\,\varphi}}_{W} e^{i\,\phi}$$

$$LG_{01}=B_{01}\, r\, e^{i\,\phi}\, G\, W \tag{6}$$

in the case $l=-1$ , we have :

$$LG_{0-1}=B_{0-1}\, r\, e^{-i\,\phi}\, G\, W \tag{7}$$

We conclude that,

$$B_{01}=B=_{0-1}=\frac{2}{\sqrt{\pi}w^2} \tag{8}$$

Comparing the terms $A_{01}$with $B_{01}$we obtain

$$B_{01}=A_{01}/\sqrt{2} \tag{9}$$

We have,

$$HG^*_\theta\,(x,y)\ HG_\theta\,(x,y)=\left[\cos\theta\, HG^*_{10}+\sin\theta\, HG^*_{01}\right]\,.\,\left[\cos\theta\, HG_{10}+\sin\theta\, HG_{01}\right]$$

$$HG^*_\theta\,(x,y)\ HG_\theta\,(x,y)=\left[\cos^2\theta\, HG^*_{10}\, HG_{10}+\sin\theta\,\cos\theta\, HG^*_{01}\, HG_{10}\right]\,.$$

$$.\,\left[\cos\theta\,\sin\theta\, HG^*_{10} HG_{01}+\sin^2\theta\, HG^*_{01}\, HG_{01}\right]$$

Integrating in space,

$$\int\!\!\int HG^*_\theta\, HG_\theta\, dx\, dy=\cos^2\theta\underbrace{\int\!\!\int HG^*_{10}\, HG_{10}\, dx\, dy}_{I}+\sin\theta\,\cos\theta\underbrace{\int\!\!\int HG^*_{01}\, HG_{10}\, dx\, dy}_{II}$$

$$+\cos\theta\,\sin\theta\underbrace{\int\!\!\int HG^*_{10}\, HG_{01}\, dx\, dy}_{III}+\sin^2\theta\underbrace{\int\!\!\int HG^*_{01}\, HG_{01}\, dx\, dy}_{IV}$$

for $I$,

$$I=\int\!\!\int A_{10}\, x\, G\, W^*\, A_{10}\, x\, G\, W dx\, dy$$

$$I = A_{10}^2 \int x^2\, e^{-2x^2/w^2} dx \int e^{-2y^2/w^2} dy$$

Using the table of Gaussian integrals,

$$I = 1$$

for $II$,

$$II = \int\int A_{01}\, y\, G\, W^*\, A_{10}\, x\, G\, W dx\, dy = A_{10}^2 \int x\, e^{-2x^2/w^2} dx \int y\, e^{-2y^2/w^2} dy \quad \Rightarrow \quad II = 0$$

for$III$,

$$III = \int\int A_{10}\, x\, G\, W^*\, A_{01}\, y\, G\, W dx\, dy = A_{10}^2 \int x\, e^{-2x^2/w^2} dx \int y\, e^{-2y^2/w^2} dy \quad \Rightarrow \quad III = 0$$

for $IV$,

$$IV = \int\int A_{01}\, y\, G\, W^*\, A_{01}\, y\, G\, W dx\, dy$$

$$IV = A_{01}^2 \int y^2\, e^{-2y^2/w^2} dy \int e^{-2x^2/w^2} dx$$

Using the table of Gaussian integrals,

$$IV = 1 \tag{10}$$

Finally,

$$\int\int HG_\theta^*\, HG_\theta\, dx\, dy = \cos^2\theta\, I + \sin\theta\, \cos\theta\, II + \cos\theta\, \sin\theta\, III + \sin^2\theta\, IV = \cos^2\theta + \sin^2\theta$$

$$\boxed{\int\int HG_\theta^*\, HG_\theta\, dx\, dy = 1} \tag{11}$$

The integrals do not depend on $\vartheta$. Thus, using the same integrals from the previous section, we obtain:

$$HG_\theta^*(x,y)\ HG_{\theta+\frac{\pi}{2}}(x,y) = [\cos\theta\, HG_{10}^* + \sin\theta\, HG_{01}^*]\ .\ [\cos(\theta+\pi/2)\, HG_{10} + \sin(\theta+\pi/2)\, HG_{01}]$$

$$\int\int HG_\theta^*\, HG_{\theta+\frac{\pi}{2}}\, dx\, dy = \cos\theta\, \cos(\theta+\pi/2)\, I + \sin\theta\, \cos(\theta+\pi/2)\, II$$

$$+ \cos\theta\, \sin(\theta+\pi/2)\, III + \sin\theta\, \sin(\theta+\pi/2)\, IV$$

we know $\cos(\theta + \pi/2) = -\sin\theta$ and $\sin(\theta + \pi/2) = \cos\theta$. Then,

$$\boxed{\int\int HG_{\theta}^{*}\, HG_{\theta+\frac{\pi}{2}}\, dx\, dy = -\cos\theta\, \sin\theta + \sin\theta\, \cos\theta = 0 \qquad (12)}$$

Using Euler's identity, we can show the relation for $LG_{01}$,

$$LG_{01} = B_{01}\, r\, e^{i\phi}\, G\, W = B_{01}\, (x + iy)\, G\, W$$

$$LG_{01} = B_{01}\, x\, G\, W + iB_{01}\, y\, G\, W$$

$$LG_{01} = \frac{1}{\sqrt{2}}(A_{01}\, x\, G\, W + iA_{01}\, y\, G\, W)$$

$$LG_{01} = \frac{1}{\sqrt{2}}(HG_{10} + iHG_{01}) \tag{13}$$

for $LG_{0-1}$,

$$LG_{0-1} = B_{01}\, r\, e^{-i\phi}\, G\, W = B_{01}\, (x - iy)\, G\, W$$

$$LG_{0-1} = B_{01}\, x\, G\, W - iB_{01}\, y\, G\, W$$

$$LG_{0-1} = \frac{1}{\sqrt{2}}(A_{01}\, x\, G\, W - iA_{01}\, y\, G\, W)$$

$$LG_{0-1} = \frac{1}{\sqrt{2}}(HG_{10} - iHG_{01}) \tag{14}$$

Combining the above equations, we show that

$$\boxed{LG_{0\pm 1}(r, \phi) = \frac{1}{\sqrt{2}}(HG_{10} \pm iHG_{01}) \qquad (15)}$$

[2.EXERCISE]

We have,

$$\textstyle\sum_{n,m=0}^{\infty} u_{nm}(x,y)\,.\,u_{nm}^{*}(x',y') = \delta(x-x')\delta(y-y')$$

$$\sum_{n,m=0}^{\infty} < x,y|u_{nm} > \;.\; < u_{nm}|x',y' >= \delta(x-x')\delta(y-y')$$

$$< x,y| \underbrace{\sum_{n,m=0}^{\infty} [u_{nm} > \;.\; < u_{nm}}_{completeness} |x',y' >= \delta(x-x')\delta(y-y')$$

we know that $\sum_{n,m=0}^{\infty}[u_{nm} > \;.\; < u_{nm} = 1$.Then,

$$< x,y|x',y' >= \delta(x-x')\delta(y-y')$$

$$\underbrace{< x|x' >}_{\delta(x-x')} . \underbrace{< y|y'}_{\delta(y-y')} = \delta(x-x')\delta(y-y')$$

Finally, we show

$$\boxed{\delta(x-x')\delta(y-y') = \delta(x-x')\delta(y-y') = \sum_{n,m=0}^{\infty} u_{nm}(x,y)\,.\,u_{nm}^{*}(x',y')}$$