# Contents

# 1 Topological materials

## 1.1 Weyl and Dirac cones in condensed matter physics

Dirac and Weyl cones are the emergence of non-gapped linear energy bands in condensed matter physics, in effect exhibiting relativistic behavior at non-relativistic speeds. We here give a very brief introduction to these materials. Firstly, we will consider the so-called band crossing, and how the opening of a gap at the band crossing behaves differently in two and three dimensions. Then, various perturbations that do not open a gap will be considered, giving interesting effects in the dispersion relations. Lastly, a consideration of these materials in light of Berry curvature and the topological quantity of Chern numbers will be given.

While the nearly free quasi-particle model performs very well for most metals, with the Hamiltonian $p^2/(2m^*)$, with $m^*$ some effective mass, this model fails for the Dirac-materials. Instead of obeying the Schrödinger equation as most materials, they obey a Dirac equation, with the speed of light being replaced by the Fermi velocity $v_F$. As in the high energy case, the Dirac equation may be decomposed into chiral Weyl equations in the massless case. Setting $v_F = 1$ for simplicity one gets the Hamiltonian

$$H_D = s v_F \boldsymbol{\sigma} \boldsymbol{p}, \tag{1.1}$$

where $\boldsymbol{\sigma}$ are the Pauli matricies, $v_F$ the Fermi velocity, $\boldsymbol{p}$ the momentum, and $s = \pm 1$ denotes the chirality. It is here important to note that the Pauli matrices represent either real spin degree of freedom or some pseudo spin degree of freedom. Examples of pseudo spin is that of bipartite lattices, such as Graphene, in which case one must be careful when for example applying time reversal, as only real spin is odd under this operation, and not pseudo spin.

The dispersion of the Hamiltonian (1.1) has a band crossing at $\boldsymbol{p} = 0$. For the two-dimensional case, a perturbation on the form $m\sigma_z$, with $m$ some parameter, can open up a gap in the dispersion relation. This is easily verified by writing out the Hamiltonian and solving the eigenproblem

$$H_D^{(2D)} = s v_F (p_x \sigma_x + p_y \sigma_y) + m\sigma_z. \tag{1.2}$$

$$\left| H_D^{(2D)} - E \right| = 0. \tag{1.3}$$

As the Hamiltonian commutes with the momentum operator, we replace the momentum operator with its eigenvalues

$$E = \pm v_F \hbar \sqrt{k_x^2 + k_y^2 + \frac{m^2}{\hbar^2 v_F^2}}. \tag{1.4}$$

There are no solutions $k_x, k_y$ making the energy levels degenerate. The crossing is thus only protected by symmetry considerations, and is not *topologically protected.*

In three dimensions the situation is somewhat different, with the Hamiltonian

$$H_D^{(3D)} = s v_F (p_x \sigma_x + p_y \sigma_y + p_z \sigma_z). \tag{1.5}$$

In this case, no perturbing term may open a gap at the crossing. There is no $2 \times 2$ matrix $\sigma_4$ that anticommutes with the Pauli matricies and also is linearly independent, i.e. there is no "fourth" Pauli matrix, and thus no perturbative term will open the gap. Say for example we add a term like $m\sigma_z$, where the $z$-direction was chosen arbitrarily. The only effect this will have on the crossing is to translate it in $p_z$. Tying this back to the accidental degeneracy, we see that no matter the perturbation, the three-dimensional momentum space will always have a point of degeneracy, i.e., a crossing. The crossing is *topologically protected.* A more formal approach to topological materials, is that of topological invariants – numbers related to the topology of the material. Having a non-trivial topological invariant number, is the very definition of topological materials, and we will in subsection 1.1.1 show that Dirac cones makes the Chern number of these materials non-trivial.

The Hamiltonian in Eq. (1.1) is not the most general, if we allow for anisotropy in the system. In three dimensions we have more generally the Hamiltonian

$$H(\boldsymbol{k}) = \boldsymbol{v}_0\boldsymbol{k} + (\boldsymbol{v} \odot \boldsymbol{k})\boldsymbol{\sigma}, \tag{1.6}$$

where $\boldsymbol{v}_0$ is the *tilt vector*, $\boldsymbol{v}$ is some, anisotropic velocity, $(\boldsymbol{v} \odot \boldsymbol{k})_i = v_i k_i$ is the Hadamard product of the anisotropic velocity and the momentum, and $\boldsymbol{\sigma}$ are the Pauli matrices corresponding to spin degree of freedom. Here we will consider two interesting cases. Firstly, we will consider perturbations in the tilt-less isotropic case, $\boldsymbol{v}_0 = 0, \boldsymbol{v}_i = v_F \hat{x}_i$. Then, a tilted system without perturbations is considered.

Consider an isotropic tilt-less system; introduce to the system a pseudospin degree of freedom, thus extending the system to $4 \times 4$-matricies. The Hamiltonian of the system [2]

$$H = v_F \tau_x \otimes \boldsymbol{\sigma}\boldsymbol{k} + m\tau_z \otimes I_2 + bI_2 \otimes \sigma_z + b'\tau_z \otimes \sigma_x, \tag{1.7}$$

with $\boldsymbol{\tau}$ the Pauli matricies related to the pseudospin, and $I_2$ the identity matrix of dimension 2. The perturbing parameters $m, b, b'$ are a mass parameter, and Zeeman fields in the $z$ and $x$ direction, respectively. Ignore for now $b'$, i.e. $b' = 0$, which is related to a state known as the line node semimetal. Notice that the $b$ term breaks time reversal symmetry in the system, as the real spin $\sigma$ is odd under time reversal. The eigenvalues of this system [2]

$$E_{s\mu}(\boldsymbol{k}) = s\left[m^2 + b^2 + v_F^2 k^2 + 2\mu b\sqrt{v_F^2 k_z^2 + m^2}\right]^{\frac{1}{2}}, \tag{1.8}$$

with $s = \pm 1, \mu = \pm 1$ encoding the degeneracies related to the spin and pseudospin degrees of freedom, respectively. There are still linear dispersions for $b > m$. For $b < m$, a gap opens, and the dispersion is non-linear. In fact, this is simply a shift in $k_z$ of the Dirac cone, as is seen by rewriting

$$E_{s\mu}(\boldsymbol{k}) = sv_F\left[k_x^2 + k_y^2 + \left(\sqrt{k_z^2 + \frac{m^2}{v_F^2}} + \mu\frac{b}{v_F}\right)^2\right]^{\frac{1}{2}}. \tag{1.9}$$

This still has Weyl node solutions at $k_z^2 = (b^2 - m^2)/v_F^2$, where the dispersion is linear in the vicinity of the nodal solutions. This thus separates two Dirac nodes in momentum space, giving a *Weyl* semimetal. This also illustrates that the decomposition in Eq. (1.1) is valid around either of the shifted nodes. Expanding around one of the Dirac points of the Weyl semimetal, the Hamiltonian is exactly Eq. (1.1), after decomposing the $4 \times 4$ Hamiltonian into its two chiral $2 \times 2$ Weyl constituents.

If one instead perturbs the system with a Zeeman field in the $x$-direction, i.e. having a $b' > 0$, the splitting is instead in energy, giving nodal loop where the two cones intersect. We will not go into any depth on these types of materials.

**Possibly rewrite the following sentence**

The three cases described here: unperturbed, where the two cones are superimposed; perturbed by $b$, where the cones are separated in momentum; and perturbed by $b'$, where the cones are separated in energy, are shown in Figure 1.1. Notice that in the two latter cases, the Dirac points, i.e. crossings, are not superimposed. As will be discussed in section 1.1, this makes the crossings very robust, as the two nodes must merge before a gap may be opened.

The second case to consider, is a finite tilting vector $\boldsymbol{v}_0$, where we will consider only real spin, thus reducing the system back to the two-dimensional case in Eq. (1.6). For the isotropic case, $\boldsymbol{v}_i = v_F \hat{x}_i$, the energy bands are [15]

$$E_s(\boldsymbol{k}) = \boldsymbol{v}_0 \boldsymbol{k} + s v_F |k|. \tag{1.10}$$

Tehse types of systems, which are the systems of interest for this thesis, are considered in detail in section 1.2.

### 1.1.1 Chern number of the Weyl point

In order to more explicitly demonstrate the topological nature of the state in Eq. (1.1), we will find a non-zero topological invariant associated with that state. Thereby showing that the material is a topological material. The topological number we will calculate is the Chern number, related to the Berry curvature of the bands in some enclosed surface. In order to calculate the Chern number, we must first find an expression for the Berry curvature of our system. This derivation will follow closely Berry's original derivation [3] of the Berry phase of a two-level system with the Hamiltonian

$$H(\boldsymbol{R}) = \frac{1}{2} \boldsymbol{\sigma} \boldsymbol{R}. \tag{1.11}$$

Some notation has been modernized with inspiration from the treatment of the Berry phase of the spin-1/2 particle in an external magnetic field in Holstein [9].

Suppose we have a Hamiltonian $H(t)$, and that its $t$-dependence can be parameterized by $\boldsymbol{R} = \boldsymbol{R}(t)$, as in $H(t) = H(\boldsymbol{R}(t))$. Any evolution of the Hamiltonian through time, may then be described as a geometric path through the $\boldsymbol{R}$-space. As the reader might be aware, Berry's most famous discovery was that a closed path through $\boldsymbol{R}$-space gives an observable phase to the system, unlike the non-physical dynamical phase, which may be figures/cones-types-col1-opaque.png

Figure 1.1: Dispersion curves in the $k_z, k_x$-plane. **(Left)** Dirac material with superimposed cones. **(Center)** Time reversal symmetry broken, giving a Weyl material with the cones separated in momentum space. **(Right)** The cones shifted in energy, givng a nodal loop.

figures/cones-tilt-color1.png

Figure 1.2: Tilted Dirac cones. From left to right the tilt increases, from no tilt in the first cone to overtilt in the last. The three first are Type-I Weyl semimetals, the last is a Type-II semimetal. See main text for details.

removed by a suitable choice of gauge. Here we will however focus on the so-called Berry curvature, $\boldsymbol{B}$, a vector field which will be shown to be useful in the categorization of topological materials. Note that there is some variation in the literature on the naming of the various quantities, and the sign convention used. In particular, the word Berry curvature will in some literature refer to a rank two tensor, while our quantity $\boldsymbol{B}$ is referred to as the Berry field strength. In particular, if we let the rank two tensor be denoted $F_{ij}$, the Berry field strength $\boldsymbol{B}$ is given by

$$B_i = \epsilon_{ijk} F_{jk}. \tag{1.12}$$

**Consider rewriting some of this. Look in topology book**

The Berry curvature for the state $n$ is explicitly defined as [3]

**Should we add some more comments about adabiatic? See topo book**

$$\boldsymbol{B_n}(\boldsymbol{R}) = -\Im \sum_{m \neq n} \frac{\langle n(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} H | m(\boldsymbol{R}) \rangle \times \langle m(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} H | n(\boldsymbol{R}) \rangle}{(E_m(\boldsymbol{R}) - E_n(\boldsymbol{R}))^2}, \tag{1.13}$$

where $\times$ denotes the cross product. Notice that for a degeneracy $E_n = E_m$ there will be an infinity in $\boldsymbol{B_n}$. Considering the Berry curvature as a field in $\boldsymbol{R}$-space, this resembles a source, as will become relevant later. This may now be applied to for example the Weyl semimetal, both in the interest of solidifying the above theory, and as it will be useful in future consideration.

The Hamiltonian around the Weyl point is

$$H = v_F \boldsymbol{\sigma} \cdot \boldsymbol{p}, \tag{1.14}$$

with $v_F$ the Fermi velocity, $\boldsymbol{\sigma}$ the Pauli matrices, and $\boldsymbol{p}$ the momentum operator. By letting $\boldsymbol{R} = v_F \boldsymbol{p}$, the Berry curvature of the Hamiltonian can be found. The eigenvalues of this system are

$$E_+ = -E_- = |R|. \tag{1.15}$$

The aforementioned degeneracy is here of course the Weyl point, where $E_+ = E_- = 0$. Noting that

$$\nabla_{\boldsymbol{R}} H = \boldsymbol{\sigma}, \tag{1.16}$$

we can calculate the Berry curvature easily. Denote by $|+\rangle$ the state with the eigenvalue $E_+$ and $|-\rangle$ the state with the eigenvalue $E_-$. Take also, without loss of generality, $\boldsymbol{R}$ to be in the $z$-direction. This gives

$$\boldsymbol{B}_+ = -\Im \frac{\langle + | \boldsymbol{\sigma} | - \rangle \times \langle - | \boldsymbol{\sigma} | + \rangle}{4R^2}. \tag{1.17}$$

As $|+\rangle$ and $|-\rangle$ are eigenstates of $\sigma_z$ and orthogonal to each other, only the $z$-component of the cross product may contain non-zero contributions.

$$\begin{aligned} \boldsymbol{B}_+ &= -\frac{\hat{z}}{4R^2} \Im \left( \langle + | \sigma_x | - \rangle \langle - | \sigma_y | + \rangle - \langle + | \sigma_y | - \rangle \langle - | \sigma_x | + \rangle \right) \\ &= -\frac{\hat{z}}{2\boldsymbol{R}^2}. \end{aligned} \tag{1.18}$$

Here, the effect of the Pauli matrices on the eigenvectors was used, according to

$$\sigma_x |\pm\rangle = |\mp\rangle \tag{1.19}$$

$$\sigma_y |\pm\rangle = \pm i |\mp\rangle \tag{1.20}$$

Returning to general axis orientations, one has

$$B_+ = -\hat{R}/2\boldsymbol{R}^2 = -\boldsymbol{R}/2\boldsymbol{R}^3. \tag{1.21}$$

For the $|+\rangle$-band, the Weyl point thus takes the form of a negative monopole in $R$-space; this motivates the requirement that Weyl points must always appear in pairs of opposite chirality, as the divergence of the Berry curvature must always be zero over the entire sample.

**There should probably be some care taken here with the sign of $v_F$.**

As mentioned, the Chern number is one of several numbers that is used to classify topological materials. The Chern number is defined as

$$C = \frac{1}{2\pi} \oint_{\partial C} \boldsymbol{B}_+ \cdot \mathrm{d}\boldsymbol{S}, \tag{1.22}$$

where the integral is taken over the closed surface $\partial C$, enclosing the volume $C$. Noting that the Berry curvature has the shape of a monopole source at $\boldsymbol{p} = 0$, we immediately know the value of this quantity from electromagnetism. We will, however, carry out the computation explicitly here. With the divergence theorem in mind, it behooves us to find the divergence of the Berry curvature. This divergence is zero everywhere except in the monopole source, giving

$$\nabla \cdot \boldsymbol{B}_+ = -\frac{1}{2}\nabla \cdot \hat{R}/R^2 = -2\pi\delta(\boldsymbol{p}), \tag{1.23}$$

where $\delta$ is the Dirac delta distribution. By virtue of the divergence theorem the Chern number is then found to be

$$C = \frac{1}{2\pi} \int_C \nabla \cdot \boldsymbol{B}_+ \mathrm{d}C = -1, \tag{1.24}$$

where the property of integrals over Dirac delta distributions was used.

Note that some literature will have a Chern number differing from (1.24) by the sign of the Fermi velocity,

$$C = -\operatorname{sign}(v_F). \tag{1.25}$$

This simply comes from the definition of the eigenstates. We have put the sign dependence in the state, making the $E_+$ state always have positive eigenenergy. In literature that instead defines $E_+ = v_F|R|$ the state's energy will depend on the sign of the Fermi velocity, and as a consequence, the sign dependence will end up in the Chern number instead.

The overall divergence of Berry curvature must be zero, or equivalently, the sum of the Chern numbers must be zero. The Hamiltonian Eq. (1.11) chosen with the opposite chirality,

$$H(\boldsymbol{R}) = -\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{R}, \tag{1.26}$$

has the opposite Berry curvature, and also the opposite Chern number. Thus, Dirac cones must appear in pairs of opposite chirality, either superimposed as the Dirac semimetal case or separated in momentum space, as the Weyl semimetal.

**Make sure there is no discrepancy between 2D/3D materials above**

In light of the interpretation of the Dirac point as a monopole of Berry curvature, the discussion at the beginning of section 1.1 on the stability of the band crossing in two and three dimensions gets an intuitive and geometric interpretation. In Figure 1.3 the Berry curvature pole is shown in $p$-space, together with a plane parallel to the $xy$-plane, which we will denote the *state plane*. In the two-dimensional case, the state is confined to the state plane, with the $z$-position of the plane given by any mass terms $m\sigma_z$. In the three-dimensional case, the state not confined to this plane, as the parameter $p_z$ is a free variable, or alternatively it may be considered as a freedom to move the state plane freely, with its initial position simply shifted by any mass terms. It is thus obvious that one may never reach the monopole in the two-dimensional case, and thus for no $\boldsymbol{k}$ is there a band crossing. Importantly, the Berry curvature is indeed non-zero, however any closed curve of integration will give a Chern number of zero; the monopole has been moved outside the dimensionality of freedom.

`figures/berryCurvePlanewpath.png`

Figure 1.3: The state plane, transparent yellow, parallel to the $xy$-plane and a Berry curvature monopole at the origin. An integration contour is shown in blue dashed. See main text for details.

## 1.2 Type II Weyl semimetals

The conic section problem with the intersecting plane restricted to pass through the node of the cone is trivially seen to have two solutions: a point and two intersecting lines. Despite this, the possibility of a Weyl cone tilted beyond the Fermi level was never considered before Soluyanov et al. described this new class of Weyl semimetals in 2015. This now seemingly obvious possibility made an already rich field even more exciting, opening up for a wider range of novel and interesting effects.

**add some concrete examples or cites**

**Is this correct? Is a tilt at all possible in HEP?**

In the case of massless fermions, the particle physics equivalent of the Weyl semimetal, such a tilt is not possible, due to the requirement of Lorentz invariance

**add cite or explain**

. In condensed matter physics, however, this is not an issue, and it is indeed a real class of materials

**cite examples**

. We denote these types of materials Type-II Weyl semimetals, as opposed to Type-I. The transition between Type-I and Type-II is abrupt – the Fermi surface goes from a single point to two intersecting lines, in other words going from a zero dimensional to a one dimensional surface.

**Make sure this is indeed a one dimensional surface. It is kind of 1DxZ(2)**

**Make sure it is one dim also for the 3D case, quadric surface, not conic intersection**

Type-II also has electron and particle pockets at the Fermi level. While the density of states for a Type-I semimetal goes to zero as one approaches the Fermi level, this causes Type-II to have a finite density of states at the Fermi level.

**End with something like: all in all this gives type ii weyl semimetal manifestly different properties from tyep i, useful both in practical applications and as an interesting phenomena seen from a purely scientific perspective**

### 1.2.1 Hamiltonian

We will firstly consider a slightly more realistic toy model for a Weyl semimetal, with a parameter taking the system from a Type-I to a Type-II. This is instructive both in order to more intuitively see the origin of the terms causing the tilting of the Dirac cone, and also to see how two Dirac cones in the same Brillouin zone tilt in relation to each other. We will then continue by linearizing the model around the Weyl points, regaining the familiar form of a Dirac cone, with an additional anisotropy term causing the tilt.

Using the general time-reversal breaking model described by McCormick, Kimchi, and

Trivedi [13]

$$
\begin{aligned}
H(\boldsymbol{k}) = {} & \left[(\cos k_x + \cos k_z - 2)m + 2t(\cos k_x - \cos k_0)\right]\sigma_1 \\
& - 2t \sin k_y \sigma_2 - 2t \sin k_z \sigma_3 + \gamma(\cos k_x - \cos k_0).
\end{aligned} \tag{1.27}
$$

The model has Weyl nodes at $\boldsymbol{K}' = (\pm k_0, 0, 0)$, and the parameter $\gamma$ controls the tilting of the emerging cones. A value of $\gamma = 0$ gives no tilt, while for $\gamma > |2t|$ the Type-II system emerges. Figure 1.5 shows the cross section $k_y = 0$ of the eigenvalues of this system, as $\gamma$ is gradually increased from 0 to 0.15

**verify numbers**

. The $\gamma$-term "warps" the bands, and in the limit of Type-II the hole band crosses the Fermi level into positive energy, while the particle band crosses the Fermi level into negative energies. We call these hole and electron pockets, respectively.

Linearizing around the Weyl nodes reduces to the familiar expression of a Dirac cone

$$
H(\boldsymbol{K}'^{\pm} + \boldsymbol{k}) \approx \mp 2tk_x \sin k_0 \sigma_1 - 2t(k_y\sigma_2 + k_z\sigma_3) \mp \gamma k_x \sin k_0 \sigma_0, \quad k_x, k_y, k_z \ll 1. \tag{1.28}
$$

When the separation between the two nodes is $\pi$, i.e. $k_0 = \pi/2$, the linearized Hamiltonian of around the cone, is

$$
H'(\boldsymbol{k}) = \mp 2tk_x\sigma_x - 2tk_y\sigma_y - 2tk_z\sigma_z \mp \gamma k_x. \tag{1.29}
$$

However, as the two noes are brought closer together, the effective Fermi velocity in the $x$-direction is rescaled, and the system is anisotropic even for no tilt ($\gamma = 0$). The expression may be made even more clear by moving the sign $\pm$-sign into the tilt parameter $\gamma$. The Hamiltonian is invariant under a sign change of the first term, as the isotropic Dirac Hamiltonian is invariant under inversion.

**Is this not quite jalla?**

In the tilt-term, we move the sign dependence into $\gamma$, and the linearized model is

$$
H'(\boldsymbol{k}) = -2t\boldsymbol{k}\boldsymbol{\sigma} - \gamma^{\pm} k_x, \tag{1.30}
$$

where $\gamma^{\pm} = \pm\gamma$ with the upper sign corresponding to the node at $k_x = +k_0$ and the lower sign corresponds to the node at $k_x = -k_0$. As expected, we get two Dirac cones, tilting in opposite direction, but with the same amount.

**How does this affect the Berry curvature and chern number?**

**Maybe prettier/more correct to invert ky and kz, as that would also give the opposite chirality of the dirac points**

The linearized model are accurate in describing low energy interactions around the Fermi level. For higher energies their validity falls apart, and more complex models are warranted. In our calculations the linear models is sufficient, and much easier to work with, and we will thus mainly consider the linear model from here on.

For tilted Dirac cones we will consider the Hamiltonian

$$
H = s v_F \boldsymbol{k}\boldsymbol{\sigma} + v_F \boldsymbol{t}^s \boldsymbol{k}, \tag{1.31}
$$

where $s$ denotes the chirality of the Dirac cone, $v_F$ is the Fermi velocity, and $\boldsymbol{t}$ is the *tilt vector*. In general the Fermi velocity is anisotropic, as was the case in the general Dirac Hamiltonian given in Eq. (1.6). By an anisotropic scaling of the momenta $\boldsymbol{k}$, the system may always be mapped to an isotropic case, which we will consider here.

The tilt vector will in general depend on the chirality of the Dirac cone. As the Dirac cones always appear in pairs, $\boldsymbol{t}^s = s\boldsymbol{t}$ will give a system with inversion symmetry. In the case of broken inversion symmetry, we will consider the case of a tilt equal in direction and magnitude between the two cones, $\boldsymbol{t}^s = \boldsymbol{t}$. In short, we define

$$\boldsymbol{t}^s = \begin{cases} \boldsymbol{t} & \text{broken inversion symmetry,} \\ s\boldsymbol{t} & \text{inversion symmetry.} \end{cases} \tag{1.32}$$

This convention is used in most literature [21, 7].

With no magnetic field, the eigenvalues of the system are

$$E(\boldsymbol{k}) = \boldsymbol{\omega_0 k} \pm \sqrt{(v_i k_i)^2} = \sqrt{(t_i v_i k_i)^2} \pm \sqrt{(v_i k_i)^2}, \tag{1.33}$$

where in the literature the first term is sometimes referred to as the *kinetic* term while the latter is the *potential* term. The definition for the system to be Type-II is that there exists a direction in momentum space for which the kinetic term dominates over the potential term [19]. The $\boldsymbol{t}$-vector is thus a convenient tool for categorization – if $t > 1$ we have a Type-II, else we have a Type-I.

*Proof:* We may always rotate our coordinate system such that, without loss of generality, $\boldsymbol{t} = t\hat{x}$. In that case, the first term obviously dominates in the $x$-direction, when $t > 1$. $\square$

- gives rise to cones tilting opposite direction
- Linearized model valid for low energy interaction. For higher energy, the perfect cone model is not valid, as the cones does in fact touch.
- In this model, the hole pocket is “shared” between the two cones. There are also models with individual pockets (see [13])

figures/conicSection.pdf

Figure 1.4

figures/typeIIridgeline.png

Figure 1.5:

**Write this**

The values of the parameters were chosen to be $m = 0.15, t = -0.05$, and $2k_0 = \pi$.

figures/movetypeiinode.png

Figure 1.6: A Type-II Weyl semimetal with separation between the nodes $2k_0 = 0, \pi/2, \pi$. See main text for details about the model.

# 2 Charge current from the conformal anomaly

We will find the current response of a single Dirac cone, with a temperature gradient $\nabla_y T$ and a magnetic field $B_z$. The current response of interest in the given geometry is thus in the $x$-direction,

$$J^x = \chi^{xy} \frac{-\nabla_y T}{T}, \tag{2.1}$$

with $\chi^{xy}$ being the response[1]. This geometry is shown in Figure 2.1. In the derivation

figures/setup.png

Figure 2.1: Sketch of the geometry used in the derivation. Note that we consider only bulk response, and the finite sample is only for illustration purposes.

of Chernodub, Cortijo, and Vozmediano [5] the response

$$\chi^{xy} = \frac{e^2 v_F B}{18\pi^2 \hbar} \tag{2.2}$$

was found, while the derivation of Arjona, Chernodub, and Vozmediano [1] found [2]

$$\chi^{xy} = \frac{e^2 v_F B}{4\pi^2 \hbar}. \tag{2.3}$$

Recall the linear response from the Kubo formalism in Eq. (**??**), found through Luttinger's approach.

$$\langle J^i \rangle (t, \boldsymbol{r}) = \int_{-\infty}^{\infty} \mathrm{d}t' \mathrm{d}\boldsymbol{r}' \int_{-\infty}^{t'} \mathrm{d}t'' \left\{ \frac{-i v_F}{\hbar} \Theta(t - t') \left\langle [J^i(t, \boldsymbol{r}), T^{0j}(t'', \boldsymbol{r}')] \right\rangle \right\} \partial'_j \psi(t', \boldsymbol{r}'). \tag{2.4}$$

[1] The sign in Eq. (2.1) depends on the choice of the response function being the response of the gravitational potential or the temperature gradient. Thus, the sign may differ in the literature.

[2] The paper is somewhat unclear on what is their final result, as there is some possible confusion related to the number of Landau levels included and whether one is including both or only one Dirac cone. The above result is what is meant, to the best of our understanding.

Fourier transforming now to the frequency and momentum domain, will be beneficial in our calculations. As before, the non-perturbed system will be taken to be time and position invariant, such that the correlator in Eq. (2.4) can be taken to depend only on the differences $t - t''$ and $\boldsymbol{r} - \boldsymbol{r}'$. Starting with Fourier transforming the position part, notice that the structure of Eq. (2.4) is

$$\langle J^i \rangle (\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}' \chi(\boldsymbol{r} - \boldsymbol{r}') \partial_j' \psi(\boldsymbol{r}'),$$

where the temporal parts were dropped for clarity. This is a convolution, and the Fourier transform is thus simply given by the product of the two factors [16].

$$\langle J^i \rangle (\boldsymbol{q}) = \chi(\boldsymbol{q})(iq_j)\psi(\boldsymbol{q}), \tag{2.5}$$

where it was also used that the Fourier transform of a derivative gives the component of the variable. Showing explicitly how to find the form of the response $\chi$ in momentum space is often overlooked in much literature, and as it does involve some finesse, we want to show it here. This trick is courtesy of Chang [4]. By definition, the Fourier transform of the response is, where the variable of integration has been chosen to be $\boldsymbol{r} - \boldsymbol{r}'$ for later convenience,

$$\begin{aligned}
\chi(\boldsymbol{q}) &= \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} \chi(\boldsymbol{r} - \boldsymbol{r}') &(2.6)\\
&= \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} C \left\langle \left[ J^i(\boldsymbol{r}), T^{0j}(\boldsymbol{r}') \right] \right\rangle, &(2.7)\\
& &(2.8)
\end{aligned}$$

where $C$ denotes $t$-dependent prefactors and integrals over time are omitted, again for clarity of notation. Note that

$$\int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \mathrm{d}\boldsymbol{r}', \tag{2.9}$$

where $\mathcal{V}$ is the volume of the system. Thus,

$$\begin{aligned}
\chi(\boldsymbol{q}) &= \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \mathrm{d}\boldsymbol{r}' e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} C \left\langle \left[ J^i(\boldsymbol{r}), T^{0j}(\boldsymbol{r}') \right] \right\rangle \\
&= \frac{C}{\mathcal{V}} \left\langle \left[ J^i(\boldsymbol{q}), T^{0j}(-\boldsymbol{q}) \right] \right\rangle.
\end{aligned} \tag{2.10}$$

Considering now the temporal part, the procedure is simpler. The linear response still has the form of a convolution, as the response function is only dependent on the difference $t - t'$ by

$$\chi(t - t') = \int_{-\infty}^{0} \mathrm{d}t'' \Theta(t - t') \left\langle \left[ J(t - t'), T(t'') \right] \right\rangle, \tag{2.11}$$

where $t''$ was shifted by $t'$, and then the translational invariance of the correlator was used. In frequency space

$$\chi(\omega) = \int \mathrm{d}t e^{i\omega t} \chi(t) \tag{2.12}$$

$$= \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t'' \Theta(t) \left\langle \left[ J(t), T(t'') \right] \right\rangle . \tag{2.13}$$

In frequency and momentum space the response function is thus

$$\chi^{ij}(w, \boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}\hbar} \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \left\langle \left[ J^i(t, \boldsymbol{q}), T^{0j}(t', -\boldsymbol{q}) \right] \right\rangle . \tag{2.14}$$

### 2.0.1 Transport and magnetization

Recall that we generally define the transport coeffieicents

$$J^i = e^2 L_{11}^{ij} E_j + e L_{12}^{ij} \nabla_j T,$$

where $J^i$ is the electrical current. In our work, we focus on the $L_{12}$ coefficient, however the following discussion is valid also more generally. The definition of transport currents becomes more subtle in systems with broken time-reversal symmetry[21, 6]. In such systems, unobservable, circulating *magnetization* currents arise. These currents do not contribute to transport, but the Kubo treatment derives the local current, which in general also includes non-transporting currents. Let

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{tr}} + \boldsymbol{J}_M, \tag{2.15}$$

where $\boldsymbol{J}$ is the total local current, $\boldsymbol{J}_{\mathrm{tr}}$ is the transport current, and $\boldsymbol{J}_M$ is the circulating magnetization current. While our response $\chi$ relates to the total current, we are more interested in the experimentally measurable transport resonse $L_{12}^{ij}$, related to our Kubo result as [1]

**this might not be a first-hand source. See thermal transport...geometry chernodub eq. 62**

$$L_{12}^{ij} = \chi^{12} - \epsilon^{ijl} M_l, \tag{2.16}$$

with $M_l$ the magnetization. For zero chemical potential, however, these magnetization currents have been shown to go to zero as $T \to 0$.

## 2.1 Eigenvalue problem of the Landau levels of a Weyl Hamiltonian

To evaluate the correlator of the response function, the matrix elements of the current and stress-energy tensor must be found. In order to do this, we find eigenstates in the Landau basis of the system. We will first consider the untilted Hamiltonian, which we will then use to find the Landau levels of the tilted Hamiltonian.

### 2.1.1 The untilted Hamiltonian

The Weyl Hamiltonian

$$H_s = s v_F \sigma^i \left(p_i + eA_i\right), \tag{2.17}$$

with $s$ being the chirality, $p_i$ the momentum operator, and $e = |e|$ the coupling constant to the electromagnetic field $\boldsymbol{A}$. Choose coordinates such that $\boldsymbol{B} = B_z\hat{\boldsymbol{z}}$, which in the Landau gauge gives $\boldsymbol{A} = -B_z y\,\hat{\boldsymbol{x}}$. As the Hamiltonian is invariant in $x$ and $z$, take the plane wave ansatz $\phi(\boldsymbol{r}) = e^{ik_x x + ik_z z}\phi(y)$. It then follows

$$H_s\phi(\boldsymbol{r}) = E\phi(\boldsymbol{r}) \implies \tilde{H}_s\phi(y) = E\phi(y), \tag{2.18}$$

where $\tilde{H}$ is the result of replacing $p_z \to \hbar k_z, p_x \to \hbar k_x$ in $H_s$, as the plane wave part of $\phi$ have these eigenvalues. Absorb the chirality $s$ as a sign in the velocity $v_F$, for more concise notation. Thus, writing everything explicitly, the spectrum is given by

$$-\hbar v_F \begin{pmatrix} -k_z & \partial_y + eyB_z/\hbar - k_x \\ -\partial_y + eyB_z/\hbar - k_x & k_z \end{pmatrix}\phi(y) = E\phi(y). \tag{2.19}$$

We will now find the spectrum $E$ of the Hamiltonian.

Inspired by the derivation for the spectrum of the 2D Dirac Hamiltonian in [22], we introduce the length scale $l_B = \sqrt{\hbar/eB}$, and the dimensionless quantity $\chi = y/l_B - k_x l_B$. In dimensionless quantities Eq. (2.19) becomes

$$-\frac{\hbar v_F}{l_B}\begin{pmatrix} -k_z l_B & \partial_\chi + \chi \\ -\partial_\chi + \chi & k_z l_B \end{pmatrix}\phi(y) = E\phi(y). \tag{2.20}$$

Let the operators $a = \left(\chi + \partial_\chi\right)/\sqrt{2}$, $a^\dagger = \left(\chi - \partial_\chi\right)/\sqrt{2}$. One may easily verify the commutation relation $[a, a^\dagger] = 1$; they are ladder operators of the harmonic oscillators, whose eigenstates are $|n\rangle$, and where $a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. In terms of these operators, the system is

$$-\frac{\sqrt{2}\hbar v_F}{l_B}\begin{pmatrix} -\frac{k_z l_B}{\sqrt{2}} & a \\ a^\dagger & \frac{k_z l_B}{\sqrt{2}} \end{pmatrix}|\phi\rangle = E\,|\phi\rangle\,. \tag{2.21}$$

Take the ansatz

$$|\phi\rangle = \begin{pmatrix} \beta\,|n-1\rangle \\ \alpha\,|n\rangle \end{pmatrix}, \tag{2.22}$$

which is the most general form of $|\phi\rangle$ with any hope of being an eigenstate. This leads to

$$-\frac{\sqrt{2}\hbar v_F}{l_B}\begin{pmatrix}(-\gamma\beta+\alpha\sqrt{n})\,|n-1\rangle\\ (\beta\sqrt{n}+\gamma\alpha)\,|n\rangle\end{pmatrix}=E\,|\phi\rangle\,, \tag{2.23}$$

with $\gamma = k_z l_B/\sqrt{2}$. For $n>0$ this leads to the equation for $\phi$ to be an energy eigenfunction

$$-\gamma+\frac{\alpha}{\beta}\sqrt{n}=\frac{\beta}{\alpha}\sqrt{n}+\gamma. \tag{2.24}$$

Solving for $\alpha/\beta$ this gives

$$\frac{\alpha}{\beta}=\frac{\gamma}{\sqrt{n}}\pm\sqrt{1+\frac{\gamma^2}{n}}, \tag{2.25}$$

and thus

$$E=\pm v_F\sqrt{\frac{2n\hbar^2}{l_B^2}+k_z^2\hbar^2}=\pm s v_F\sqrt{2neB\hbar+k_z^2\hbar^2}, \tag{2.26}$$

where we reintroduced the explicit $s$. For $n=0$ the annihilation operator $a$ destroys the vacuum state $|0\rangle$, and the energy is instead $E_0=-\hbar s k_z v_F$. The excited energy states are doubly degenerate; we choose to denote the energy levels by $m\in\mathbb{Z}$, where the sign from $\pm s$ is taken care of by the sign of this quantum number, and the harmonic oscillator levels $n$ are given by its absolute value $|m|$. The energy levels are

$$E_{k_z m s}=\operatorname{sign}(m)v_F\sqrt{2|m|eB\hbar+k_z^2\hbar^2}\qquad\text{for } m\neq 0, \tag{2.27a}$$

$$E_{k_z 0 s}=-s\hbar k_z v_F\qquad\text{for } m=0. \tag{2.27b}$$

We now find the corresponding eigenvectors of the system. The solution to the one dimensional harmonic oscillator in position space is, in dimensionless coordinates $\xi$, [14, Eq. 18.39.5]

$$\langle\xi|n\rangle=\phi_n(\xi)=\frac{1}{\sqrt{2^n n!}}\pi^{-\frac{1}{4}}e^{-\frac{\xi^2}{2}}H_n\left(\xi\right), \tag{2.28}$$

where $H_n$ are the Hermite polynomials. Thus,

$$\langle\chi|\phi\rangle=\begin{pmatrix}\beta\,\langle\chi|n-1\rangle\\ \alpha\,\langle\chi|n\rangle\end{pmatrix}=e^{-\frac{\chi^2}{2}}\begin{pmatrix}\frac{\beta}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}}H_{n-1}\left(\chi\right)\\ \frac{\alpha}{\sqrt{2^n n!\sqrt{\pi}}}H_n\left(\chi\right)\end{pmatrix}, \tag{2.29}$$

where we defined $H_{-1}=0$ in order to get a more general expression. Choosing

$$\alpha=\sqrt{\frac{\gamma^2}{n}}\implies\beta=\frac{1}{1\pm\sqrt{1+\frac{n}{\gamma^2}}}=\pm\frac{\gamma^2}{n}\left(\sqrt{1+\frac{n}{\gamma^2}}-1\right), \tag{2.30}$$

gives

$$\phi(\chi)=e^{-\frac{\chi^2}{2}}\sqrt{\frac{\gamma^2}{n}}\begin{pmatrix}\frac{\pm\sqrt{\frac{\gamma^2}{n}}\left(\sqrt{1+\frac{n}{\gamma^2}}-1\right)}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}}H_{n-1}(\chi)\\ \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n\left(\chi\right)\end{pmatrix}. \tag{2.31}$$

There are thus four quantum numbers related to the eigenvectors, $k_x, k_z, m, s$. Reintroducing $\chi = (y - k_x l_B^2)/l_B$ and normalizing

$$\phi_{\boldsymbol{k}ms}(\boldsymbol{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{\left(y - k_x l^2\right)^2}{2l_B^2}} \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y - k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M\left(\frac{y - k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{2.32}$$

where capital letters indicate absolute value of corresponding quantity, $M = |m|$, $\boldsymbol{k} = (k_x, k_z)$, and with the normalization factor

$$\alpha_{k_z ms} = \frac{-\sqrt{2eB\hbar M}}{\frac{E_{k_z ms}}{s v_F} - \hbar k_z}. \tag{2.33}$$

### 2.1.2 The tilted Hamiltonian

**Consider which formalism to use for this section. Should we already here use the geometry, or keep it with parallell, perpendicular?**

**I think it is better to use perp, parallell here, and then transition to using the explicit geometry later**

The eigenvalues of a Type-II Weyl semimetal are simple to find, and are not qualitatively different from those of Type-I, other than the appearance of particle and hole pockets at the Fermi level. We will also consider the Landau levels of these materials, which importantly are very different from Type-I. In fact, erroneous treatment of the Landau spectrum of Type-II semimetals caused the original paper describing Type-II materials to mistakenly assert that the chiral anomaly would not be present for certain directions of a background magnetic field [19][18].

Eigenstates, spin, berry, etc

The issue with the Landau level description is that for certain directions of the $B$-field, the Landau levels break down. For Type-I materials, the description is valid for all directions of the $B$-field, but as the cone tip into a Type-II material, the description breaks down when the $B$-field and tilt direction are perpendicular [18], and as the magnitude of the tilt increases, the Landau levels are only valid up to a certain angle between the tilt direction and magnetic field. We will in this section derive and elucidate the Landau levels and their regions of validity.

Consider again the Hamiltonian [3]

$$H = v_F \boldsymbol{t}^s \boldsymbol{k} + s v_F \boldsymbol{k}\boldsymbol{\sigma}, \tag{2.34}$$

[3] In general, the Fermi velocity may be anisotropic, in which case the momentum enter as $v_i k_i$, instead of $v_F k_i$. By a rescaling of the momenta, we may consider any, in general anisotropic, system to be isotropic in velocity.

with the *tilt vector* as defined in Eq. (1.32)

$$\boldsymbol{t}^s = \begin{cases} \boldsymbol{t} & \text{broken inversion symmetry,} \\ s\boldsymbol{t} & \text{inversion symmetry.} \end{cases}$$

To find the Landau levels in a magnetic field $\boldsymbol{B} = B_z\hat{z}$, we will "Lorentz boost" the system to a frame where the cone is not tilted, where we may use the usual approach for finding the Landau levels.

Generally, consider $\boldsymbol{t}$ to consist of two components, $\boldsymbol{t}_{\|}$ which is parallel to the magnetic field, and $\boldsymbol{t}_{\perp}$ perpendicular to the magnetic field. In this work, we restrict ourselves to the case where the perpendicular component is parallel to the charge current, i.e. the $x$ direction in the chosen geometry. The $\boldsymbol{t}_{\perp}$ vector may of course also have a component parallel to the temperature gradient $\nabla T$, in this geometry the $y$ direction, which, although interesting, is not considered here. Thus, let $\boldsymbol{t} = (t_{\perp}, 0, t_{\|})$. Introduce the $\boldsymbol{B}$-field by the minimal coupling $\boldsymbol{k} \to \boldsymbol{k}^B = \boldsymbol{k} + e\boldsymbol{A}$. We take the field to be in the $z$-direction, and use the Landau gague $\boldsymbol{A} = -B_z y\hat{x}$.

**Before applying the temperature gradient, when we still only consider finding the LLs, we may in fact say that we have generally $t_{\perp}$ only in x, and hen later rotate into coordinates where $\nabla T$ is in y. Maybe that is interesting after all...?**

The Landau level equation is

$$(H_B - E)\,|\psi\rangle = 0, \tag{2.35}$$

with

$$H_B = v_F\left(t^s_{\perp}k^B_x + t^s_{\|}k^B_z\right)\mathcal{I}_2 + \sum_i s v_F k^B_i \sigma_i, \tag{2.36}$$

where $\mathcal{I}_2$ is the identity matrix of size 2. In order to use the ladder operator method used for the untilted cone, we must get rid of the $k^B_x$ on the diagonal of the Hamiltonian. [4] To achieve this, we will use a "Lorentz transformation", which as we will show only leave $k_z$ and $E$ in the diagonal. Act with the hyperbolic rotation operator $R = \exp[\Theta/2\sigma_x]$ on Eq. (2.35), and insert identity on the form $\exp[\Theta/2\sigma_x]\exp[-\Theta/2\sigma_x]$ before the state vector. By introducing the state in the rotated frame $|\tilde{\psi}\rangle = \exp[-\Theta/2\sigma_x]\mathcal{N}\,|\psi\rangle$, with $\mathcal{N}$ a normalization factor compensating for the non-unitarity of the transformation, we get the eigenvalue equation

$$(\exp[\Theta/2\sigma_x]H_B\exp[\Theta/2\sigma_x] - E\exp[\Theta\sigma_x])\,|\tilde{\psi}\rangle = 0. \tag{2.37}$$

We now make the fortunate observation that the diagonal elements of

$$R\sigma_i R$$

[4] It would also be possible to choose the frame such that the tilt was both in $x$ and $y$ direction, in which case we would get ladder operators also on the diagonal. This system, albeight tedious, could also have been solved directly.

**Verify this**

are zero for $i = y$ and non-zero for $i = x, z$. We may thus rotate the $x$ and $z$ in and out of the diagonal elements, without accidentaly rotating the $y$ components into the diagonal.

The problematic part of the Hamiltonian with regards to finding the Landau levels, are the terms containing $k_x^B$ on the diagonal, i.e.

$$v_F t_\perp^s k_x^B \mathcal{I}_2 + s v_F k_x^B \sigma_x.$$

We will now find the boost parameter that eliminates $k_x$ from the diagonal. We have

$$R^2 = e^{\Theta\sigma_x} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{2.38}$$

and as $[R, \sigma_x] = 0$,

$$R\sigma_x R = R^2\sigma_x = \begin{pmatrix} \sinh\theta & \cosh\theta \\ \cosh\theta & \sinh\theta \end{pmatrix}, \tag{2.39}$$

as the effect of $\sigma_x$ is to transpose the rows. The requirement for $k_x^B$ to be rotated out of the diagonal is thus

$$t_\perp^s \cosh\theta + s\sinh\theta = 0. \tag{2.40}$$

Solving for $\theta$ we get

$$\theta = \log(\pm\frac{\sqrt{s - t_\perp^s}}{\sqrt{s + t_\perp^s}}). \tag{2.41}$$

**NB: depending of choice of sign in log, we get different signs in answer**

Alternatively, written in a slightly suggestive form,

$$\tanh\theta = -st_\perp^s. \tag{2.42}$$

**For pedagogic reasons, include arctanh, which is only valid for -1 ¡ x ¡ 1, explicitly showing the collapse?**

The required hyperbolic tilt angle to eliminate the $k_x^B$ in the diagonal elements of the Hamiltonian, originating from the tilt, is thus

$$\theta = -s\tanh^{-1} t_x^s. \tag{2.43}$$

The inverse of tan, of course, diverges as the argument approaches $\pm 1$, as shown in Figure 2.2. For $|t_x| < 1$ we are able to find an angle $\theta$ which transforms our Hamiltonian into a form which we may solve. For $|t_x| \geq 1$, however, no (real) solution of $\theta$ exits, and the Landau level description collapses. More concretely, as we will show later, the separation of the landau levels is reduced as the perpendicular tilt increases, and as $|t_x| \to 1$, the level separation $\Delta E \to 0$.

**Discuss magnetic vs electric regime**

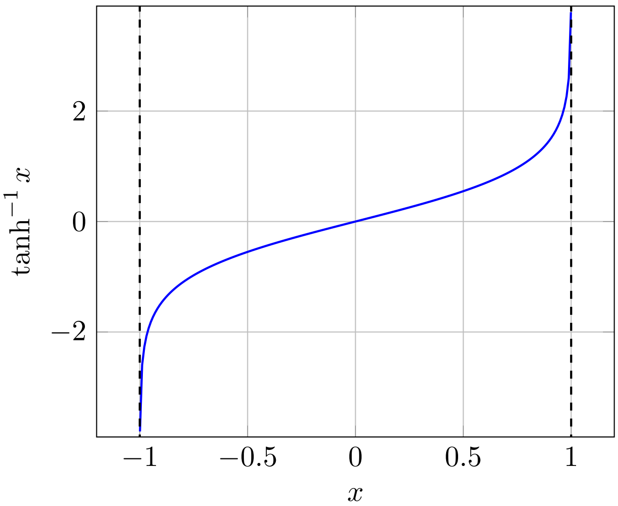



Figure 2.2: Plot of $\tanh^{-1}$, which diverges as the argument goes to $\pm 1$.

Interestingly, there are no restrictions in the perpendicual tilt, $t_z$. The $\boldsymbol{t}$ parametrization of the tilt is conveniently visualized by plottin the $t$-vector inside a unit sphere, shown in Figure 2.3. If the vector is outside the unit sphere, it is a Type-II, if it is inside, it is a Type-I. Also, if the projection of the vector onto the $x, y$-plane is on the unit disk, the Landau level description is valid, if not, the Landau levels collapse. When the projection is on the unit disc, the system is in the *magnetic* regime, otherwise we denote it by the *electric* regime. All Type-I materials may thus be described by Landau levels, while it for Type-II is only valid for certain directions of the $t$-vector. As the $t$-vector gets larger, the magnetic regime is restricted to smaller angles between $\boldsymbol{t}$ and $\boldsymbol{B}$.

We now return to solving Eq. (2.37), using the solution angle we just found. By insertion, and after some clean up, we get

$$\begin{aligned}&(\exp[\Theta/2\sigma_x] H_B \exp[\Theta/2\sigma_x] - E\exp[\Theta\sigma_x])\,|\tilde{\psi}\rangle = 0 = v_F\\ &\times\begin{pmatrix} k_z(s + t_z^s\gamma) - E/v_F\gamma & -s(ik_y + k_z t_x^s t_z^s\gamma - k_x/\gamma - E/v_F\gamma t_x^s) \\ s(ik_y - k_z t_x^s t_z^s\gamma + k_x/\gamma + E/v_F\gamma t_x^s) & -k_z(s - t_z^s\gamma) - E\gamma \end{pmatrix}|\tilde{\psi}\rangle\,.\end{aligned} \tag{2.44}$$

In order to simplify this further, absorb $\gamma t_x^s(k_z t_\parallel^s - E/v_F)$ into $k_x$. Thus, let

$$\begin{aligned}\tilde{k}_x &= k_x/\gamma + \gamma t_x^s(E/v_F - k_z t_\parallel^s),\\ \tilde{k}_y &= k_y,\\ \tilde{k}_z &= k_z.\end{aligned} \tag{2.45}$$

These expressions warrant some explanation, as the Lorentz boost is of course

$$\tilde{k}_x = \gamma(k_x - \beta\frac{E}{v_F}), \tag{2.46}$$

figures/tiltSpherewBackground.png

Figure 2.3: TODO

**comment on beta = tx, or change to tx**

where we used the four momentum $p^\mu = (\frac{E}{v_F}, \boldsymbol{p})$, and the effective speed of light $v_F$. It can thus look like our expression in Eq. (2.45) is wrong. The solution to this seeming inconsitency is that the proper energy is not $\frac{E}{v_F} - k_z t_\parallel$, but rather $\frac{E}{v_F} - k_z t_\parallel - k_x t_\perp$.

**Something smart here**

The eigenvalue equation is simply

$$\left[\gamma\left(t_\parallel^s \tilde{k}_z - \frac{E}{v_F}\right)\mathcal{I}_2 + s\tilde{k}_i\sigma_i\right]|\tilde{\psi}\rangle = 0. \tag{2.47}$$

If we now again introduce the magnetic field using minimal coupling, $k_x \to k_x - eyB_z$, this corresponds to an effective field $B_z\gamma$ in the new quantities. This is because $\tilde{k}_x \to \tilde{k}_x - eyB_z/\gamma$.

The Landau level equation thus reads

$$\left[\sum_i s v_F\left(\tilde{k}_i + e\tilde{A}_i\right)\sigma_i\right]|\tilde{\psi}\rangle = (E - t_\parallel^s v_F \tilde{k}_z)\gamma\,|\tilde{\psi}\rangle\,, \tag{2.48}$$

where $\tilde{\boldsymbol{A}} = -B_z/\gamma y\hat{x}$. We may thus use directly the result for the untilted cone, Eq. (2.27), giving

$$\left(E - t_{\parallel}^s v_F \tilde{k}_z\right)\gamma = \operatorname{sign}(m) v_F \sqrt{2|m|e\frac{B}{\gamma}\hbar + \tilde{k}_z^2\hbar^2}, \qquad m \neq 0, \tag{2.49a}$$

$$\left(E - t_{\parallel}^s v_F \tilde{k}_z\right)\gamma = -s\hbar\tilde{k}_z v_F, \qquad m = 0. \tag{2.49b}$$

Cleaning up, we get

$$E = t_{\parallel}^s v_F \tilde{k}_z + \operatorname{sign}(m) v_F \sqrt{2|m|e\frac{B}{\gamma^3}\hbar + \tilde{k}_z^2\hbar^2/\gamma^2}, \qquad m \neq 0, \tag{2.50a}$$

$$E = \tilde{k}_z v_F \left(t_{\parallel}^s - s\hbar/\gamma\right), \qquad m = 0. \tag{2.50b}$$

As the perpendicular tilt is increased, $\gamma = 1/\sqrt{1-\beta^2}$ diverges to infinity. With the trivial substitution $\alpha = \frac{1}{\gamma}$, which goes to zero, this gets an intuitive interpretation.

$$E = t_{\parallel}^s v_F \tilde{k}_z + \operatorname{sign}(m) v_F \alpha \sqrt{2|m|eB\alpha\hbar + \tilde{k}_z^2\hbar^2}. \tag{2.51}$$

As the perpendicular tilt increases, the Landau levels converge towards $t_{\parallel} v_F \tilde{k}_z$. In particular, the separation between Landau levels $m$

**maybe use the word cyclotron frequency**

is reduced by a factor $\alpha^{\frac{3}{2}}$. The effect of the tilt on the Landau levels is to squeeze the Landau levels together, and we will call the $\alpha$ the *squeezing factor*. We note that when approaching the degree of tilt where we are no longer able to find a boost which enables us to solve for the Landau levels, i.e. when $\beta \to 1$, the squeezing factor goes to zero. As the tilt exceeds this limit, the squeezing factor is imaginary. Note also that the energy levels

$$E = t_{\parallel}^s v_F k_z + \alpha E^0_{m,\alpha B},$$

where $E^0_{m,\alpha B}$ is the energy in the untilted case, with magnetic field $\alpha B$. Tilting of the Landau levels is induced by the parallel tilt component, $t_{\parallel}$. In fact, the Landau levels cross the Fermi surface at the transition from Type-I to Type-II as well. The Landau levels are shown in Figure 2.4.

The eigenstate of

$$H = v_F \sigma^i (p_i + eA_i),$$

with $A_i = -B_z y \delta_{ix}$, given in the position basis, is

$$\phi_{\boldsymbol{k}ms}(\boldsymbol{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{y - k_x l^2}{2l_B^2}} \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M!\sqrt{\pi} l_B}} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{2.52}$$

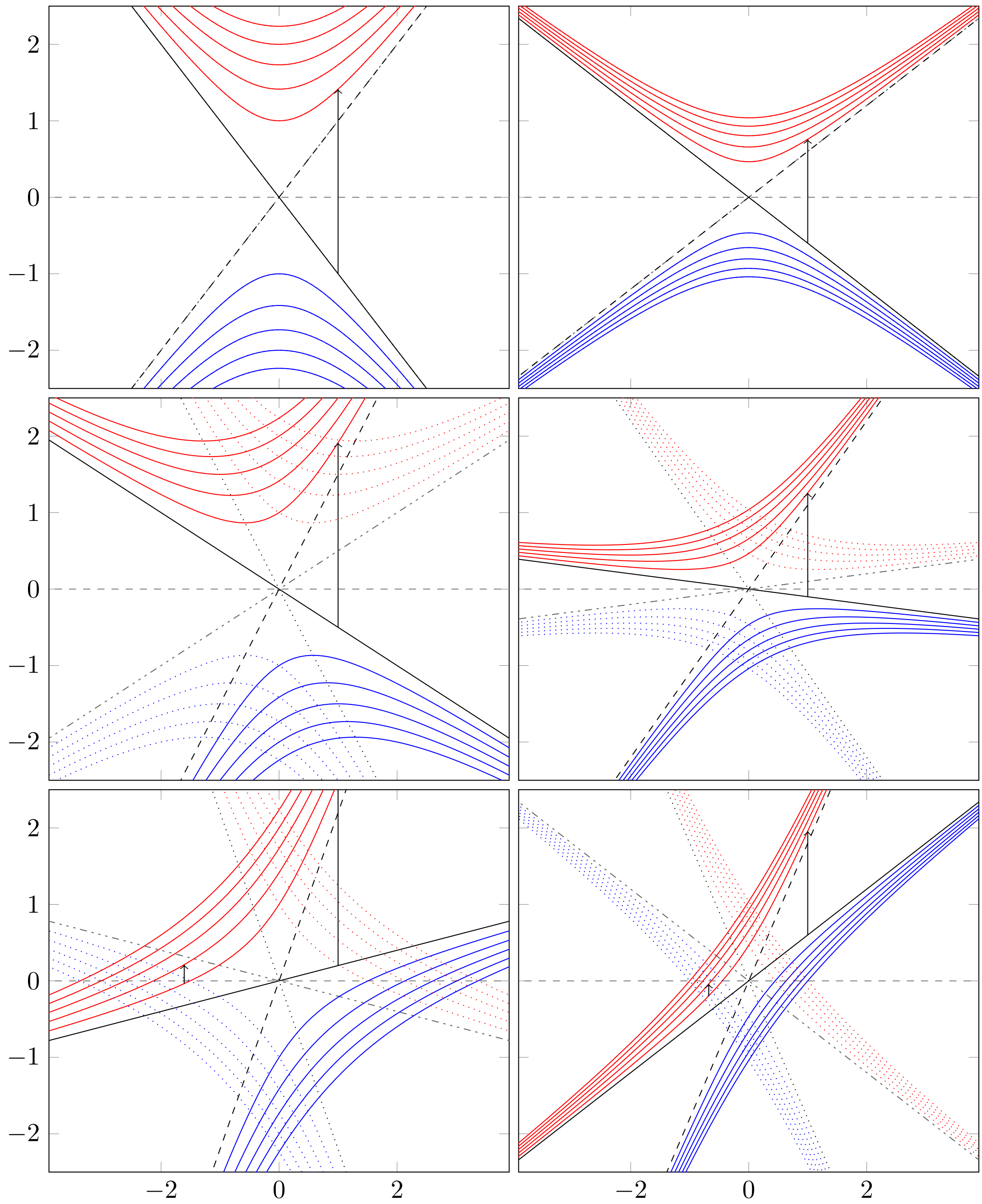



Figure 2.4: Landau levels for different values of $t_x, t_z$. The top two rows show Type-I, while the lowest row shows Type-II. Left column shows $t_x = 0$, right column $t_x = 0.64$ ($\alpha = 0.6$). The rows shows $t_z = 0, 0.5, 1.2$, from top to bottom. The dotted lines show the Landau levels with opposite sign of $t_z$.

where capital letters indicate absolute value of corresponding quantity, $M = |m|, \boldsymbol{k} = (k_x, k_z)$, and with the normalization factor

$$\alpha_{k_z ms} = \frac{-\sqrt{2eB\hbar M}}{\frac{E_{k_z ms}}{sv_F} - \hbar k_z}. \tag{2.53}$$

Taking care to keep track of boosted and rescaled quantites, the eigenstate in the boosted frame is

$$\tilde{\psi}(\tilde{\boldsymbol{r}}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{i\tilde{k}_x \tilde{x}} e^{ik_z z}}{\sqrt{\alpha^2_{\tilde{k}_z ms} + 1}} e^{-\frac{\left(\tilde{y} - \tilde{k}_x l^2_{B'}\right)^2}{2l^2_{B'}}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_{B'}}} H_{M-1}\left(\frac{\tilde{y} - \tilde{k}_x l^2_{B'}}{l_{B'}}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_{B'}}} H_M\left(\frac{\tilde{y} - \tilde{k}_x l^2_{B'}}{l_{B'}}\right) \end{pmatrix}, \tag{2.54}$$

with

$$\alpha_{\tilde{k}_z ms} = \frac{-\sqrt{2eB'\hbar M}}{\gamma \frac{E_{\tilde{k}_z ms} - t^s_{\parallel} v_F \tilde{k}_z}{sv_F} - \hbar \tilde{k}_z}, \tag{2.55}$$

where

$$B' = B\alpha.$$

We note that $\alpha_{k_z 0s} = 0$, so using the explicit form of the energy we may simplify the expression some. For $m \neq 0$

$$\frac{E_{k_z ms} - t^s_{\parallel} v_F k_z}{sv_F} = \operatorname{sign}(m) s\alpha \sqrt{2MeB\alpha + k_z^2}$$

and thus

$$\alpha_{k_z ms} = \frac{-\sqrt{\alpha M}}{\operatorname{sign}(m) s \sqrt{\alpha M + \kappa^2} - \kappa} \tag{2.56}$$

where we defined the dimensionless $\kappa_z = \sqrt{2eB} k_z$.

The original eigenstate $|\psi\rangle = 1/\mathcal{N} e^{\theta/2\sigma_x} |\tilde{\psi}\rangle$ of the tilted system is easily found. Reinserting explicitly, in the boosted frame, that

$$\tilde{k}_x = \alpha k_x + \frac{t^s_x}{\alpha}(E_{k_z ms}/v_F - k_z t^s_{\parallel}) = \alpha k_x + t^s_x \frac{E^0_{m,\alpha B}}{v_F}$$

and $l_{B'} = \frac{l_B}{\sqrt{\alpha}}$ we define

$$\chi = \frac{y - \tilde{k}_x l^2_{B'}}{l_{B'}} = \sqrt{\alpha}(y - k_x l_B^2)/l_B + \frac{t^s_x l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}, \tag{2.57}$$

which is the argument of the Hermite polynomials. For later convenience, let us explicitly define

$$\tilde{\phi}_{\boldsymbol{k}ms}(\tilde{\boldsymbol{r}}) = \frac{e^{i\tilde{k}_x \tilde{x} + ik_z z}}{\sqrt{L_x L_z}} \underbrace{\frac{e^{-\frac{1}{2}\chi^2} \sqrt[4]{\alpha}}{\sqrt{\alpha^2_{\tilde{k}_z ms} + 1}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}(\chi) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M(\chi) \end{pmatrix}}_{\tilde{\phi}_{\boldsymbol{k}ms}(y)}, \tag{2.58}$$

and thus

$$\tilde{\phi}_{\boldsymbol{k}ms}(y) = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix}, \tag{2.59}$$

with

$$a_{\boldsymbol{k}ms} = \frac{\alpha_{\tilde{k}_z ms} \sqrt[4]{\alpha}}{\sqrt{\alpha^2_{\tilde{k}_z ms} + 1} \sqrt{2^{M-1}(M-1)! \sqrt{\pi} l_B}}, \tag{2.60}$$

$$b_{\boldsymbol{k}ms} = \frac{\sqrt[4]{\alpha}}{\sqrt{\alpha^2_{\tilde{k}_z ms} + 1} \sqrt{2^M M! \sqrt{\pi} l_B}}. \tag{2.61}$$

We proceed now to find the normalization factor $\mathcal{N}$, as it will become necessary in later steps. Recall that

$$|\psi\rangle = \frac{1}{\mathcal{N}} e^{\theta/2\sigma_x} |\tilde{\psi}\rangle,$$

and

$$e^{\theta\sigma_x} = \frac{1}{\alpha} \begin{pmatrix} 1 & -st^s_x \\ -st^s_x & 1 \end{pmatrix}.$$

The upper and lower part of the spinor are orthogonal, thus we have

$$\langle\psi|\psi\rangle = \frac{1}{\mathcal{N}^*\mathcal{N}} \frac{1}{\alpha} \langle\tilde{\psi}|\tilde{\psi}\rangle = 1 \implies \mathcal{N}^*\mathcal{N} = \frac{1}{\alpha}. \tag{2.62}$$

We choose $\mathcal{N} = \alpha^{-\frac{1}{2}}$.

**Summary 1**

*The tilted Hamiltonian*

$$H = v_F \boldsymbol{t}^s \boldsymbol{k} + s v_F \boldsymbol{k}\boldsymbol{\sigma}$$

*in a magnetic field $\boldsymbol{B}$ has the Landau levels*

$$E = \begin{cases} t^s_\parallel v_F k_z + \mathrm{sign}(m) v_F \alpha \sqrt{2eB\alpha M + k_z^2} & m \neq 0, \\ t^s_\parallel v_F k_z - s\alpha v_F k_z & m = 0. \end{cases}$$

*The associated eigenstates in the position basis are*

$$\psi(\boldsymbol{r}) = \sqrt{\alpha} e^{\theta/2\sigma_x} \frac{e^{ik_x x + ik_z z}}{\sqrt{L_x L_z}} \psi(y),$$

*where*

$$\psi(y) = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{k_z ms} H_{M-1}(\chi) \\ b_{k_z ms} H_M(\chi) \end{pmatrix},$$

*where we have defined $\chi = \sqrt{\alpha}\frac{y - k_x l_B^2}{l_B} + \frac{t^s_x l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}$ and $a_{k_z ms}, b_{k_z ms}$ are given in Eqs. (2.60, 2.61).*

## 2.2 Analytical expressions for the operators

We will here find analytical expressions for the current operator $J^i(\omega, \boldsymbol{q})$ and stress-energy tensor $T^{0j}(\omega, \boldsymbol{q})$, needed to calculate the correlation function. The fields are given, in the position basis, by

$$\psi = \sum_{\boldsymbol{k}n} \langle \boldsymbol{r}|\boldsymbol{k}ns\rangle\, a_{\boldsymbol{k}ns}(t) = \sum_{\boldsymbol{k}n} \phi_{\boldsymbol{k}ns}(\boldsymbol{r}) a_{\boldsymbol{k}ns}(t), \tag{2.63}$$

$$\psi^\dagger = \sum_{\boldsymbol{k}n} \langle \boldsymbol{k}ns|\boldsymbol{r}\rangle\, a^\dagger_{\boldsymbol{k}ns}(t) = \sum_{\boldsymbol{k}n} \phi^*_{\boldsymbol{k}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ns}(t). \tag{2.64}$$

Here $a^\dagger_\lambda(t) = \exp(iE_\lambda t/\hbar) a^\dagger_\lambda$ and $a^\dagger_\lambda, a_\lambda$ are the creation and annihilation operators of the state with quantum numbers $\lambda$. The current operator $\hat{\boldsymbol{J}} = e\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is the velocity operator. Using the relation of Heisenberg operators $\dot{A} = [A, H]/i\hbar$ [17], for the operator $A$ and Hamiltonian $H$, the operator

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{1}{i\hbar}\,[\boldsymbol{r}, H] \tag{2.65}$$

$$= \frac{sv_F\sigma^i}{i\hbar}\,[\boldsymbol{r}, p_i + eA_i] + \frac{v_F}{i\hbar}[\boldsymbol{r}, \boldsymbol{t}^s\boldsymbol{k}] \tag{2.66}$$

$$= \frac{sv_F\sigma^i}{i\hbar}\,(i\hbar\hat{\boldsymbol{x}}_i + e[\boldsymbol{r}, A_i]) + v_F\boldsymbol{t}^s \tag{2.67}$$

$$= sv_F\sigma^i\hat{\boldsymbol{x}}_i + v_F\boldsymbol{t}^s, \tag{2.68}$$

and thus

$$J^x = \psi^\dagger \hat{J}^x\psi = sv_F e \sum_{\boldsymbol{k}m,\boldsymbol{l}n} \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r})\,(\sigma^x + st^s_x)\,\phi_{\boldsymbol{l}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t). \tag{2.69}$$

### 2.2.1 The energy momentum tensor

The *canonical* energy-momentum tensor is generally defined by

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\partial_\nu\phi_i - \eta^{\mu\nu}\mathcal{L}, \tag{2.70}$$

where the index $i$ runs over the types of fields. This definition is correct for commuting fields, however, for non-commuting fields like ours, this formula is slightly wrong. This is often overlooked in many textbooks and papers, so we will here elucidate the issue to some degree. While a proper derivation requires the use of Grassman variables and defining left and right derivation, which we will not do here, some simple considerations help in understanding the issue. In the standard text book derivation of then canonical energy-momentum tensor, one expands the total derivative of the Lagrangian $\mathcal{L}(\psi_i, \partial\psi_i)$ in terms of the fields

$$\frac{\mathrm{d}\mathcal{L}(\psi_i, \partial\psi_i)}{\mathrm{d}x_\nu} \equiv \mathrm{d}^\nu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\frac{\partial(\partial_\mu\psi_i)}{\partial x_\nu} + \frac{\partial\mathcal{L}}{\partial\psi_i}\frac{\partial\psi_i}{\partial x_\nu}. \tag{2.71}$$

This expansion, however, ignores the non-commutative nature of the fields. For concreteness, consider $\psi_i = \bar{\psi}$. Heuristically, the correct expression would be obtained by reordering the factors in the two terms. By naively employing Eq. (2.70), the resulting canonical energy-momentum tensor of the Dirac theory would be

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} \partial^\nu \bar{\psi} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}, \tag{2.72}$$

while the correct form is [10, Eq. 3-153]

$$T^{\mu\nu} = \partial^\nu \bar{\psi} \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}. \tag{2.73}$$

Our Hamiltonian

$$H_s = s\sigma^i k_i$$

may of course be considered as a Weyl decomposition of a full massless Dirac equation.

**Why do we have to consider 4x4? Is the definitions not also valid for 2x2?**

**Regarding the non-symmetry of the stress tensor, see keichelriess eq 5.16 with discussion**

The Hamiltonian

$$H_s = s\sigma^i k_i$$

can be considered the Hamiltonian of one part of a Weyl decomposition of a Dirac system. The Weyl field has the Lagrangian density [11]

$$\mathcal{L} = i\phi^\dagger \sigma^\mu \partial_\mu \phi, \tag{2.74}$$

which may be seen directly from the Dirac Lagrangian $i\bar{\psi}\partial\!\!\!/\psi$ by taking $\psi = (\phi_L, \phi_R)^T$ and set, for example, $\phi_R = 0$. Symmetrizing in daggered and undaggered fields [5]

**Alternatively argue by directly showning that this does not affect the action by doing an integration by parts**

$$\mathcal{L} = \frac{i}{2}\left(\phi^\dagger \sigma^\mu \partial_\mu \phi - \partial_\mu \phi^\dagger \sigma^\mu \phi\right),$$

which will prove to be more convenient to work with. Adapting the definition Eq. (2.73) the energy-momentum tensor for the untilted Dirac cone is thus

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \sigma^\mu \partial_\nu \phi - \sigma^\mu \phi \partial_\nu \phi^\dagger - \eta^{\mu\nu}\mathcal{L}). \tag{2.75}$$

Moving now to the tilted case, the 4x4 Lagrangian becomes [21]

**check sign compared to action in stoof**

$$\mathcal{L}_{\text{tilt}} = i\bar{\psi}\Gamma^\mu \partial_\mu \psi, \tag{2.76}$$

[5]The Lagrangian itself is unphysical, and we may transform it in any way that leaves the action $\int \mathcal{L}$ invariant.

where we have introduced modified gamma matrices

$$\Gamma^\mu = \begin{cases} \gamma^\mu + t^\mu \gamma^0 & \text{inversion symmetry broken,} \\ \gamma^\mu + t^\mu \gamma^0 \gamma^5 & \text{inversion symmetric,} \end{cases} \tag{2.77}$$

with $t^\mu = (0, \boldsymbol{t})$. Decomposing to $2 \times 2$, this yields

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}^\mu \partial_\nu \phi - \tilde{\sigma}^\mu \phi \partial_\nu \phi^\dagger - \eta^{\mu\nu}\mathcal{L}), \tag{2.78}$$

where we defined the modified Pauli matrices

$$\tilde{\sigma}^\mu = \begin{cases} \sigma^\mu + t^\mu & \text{inversion symmetry broken,} \\ \sigma^\mu + st^\mu & \text{inversion symmetric.} \end{cases} \tag{2.79}$$

Similarly, the $T^{0y}$ component of the stress-energy tensor of the theory is given by [1]

$$\begin{aligned} T^{0y}(t,\boldsymbol{r}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} \frac{1}{4}\Big\{ & \left[v_F \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) p_y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) - v_F \left(p_y \phi^*_{\boldsymbol{k}ms}\right) \phi_{\boldsymbol{l}ns}\right] a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t) \\ & + \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s\sigma^y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) \left[a^\dagger_{\boldsymbol{k}ms}(t) i\hbar\partial_0 a_{\boldsymbol{l}ns}(t) - i\hbar\left(\partial_0 a^\dagger_{\boldsymbol{k}ms}(t)\right) a_{\boldsymbol{l}ns}(t)\right] \\ & + \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s\sigma^y (2\mu) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t) \Big\}. \end{aligned} \tag{2.80}$$

Here, also a non-zero potential $\mu$ is included. Our final result will be given at zero potential, however it is included in the calculations as it might be of interest to consider finite potential in later work. Recalling the time dependence of $a(t), a^\dagger(t)$ we have that

$$i\hbar\partial_0 a_\lambda(t) = E_\lambda a_\lambda, \quad i\hbar\partial_0 a^\dagger_\lambda(t) = -E_\lambda a^\dagger_\lambda,$$

which further simplifies the expression.

Fourier transforming the position gives

$$J^x(t,\boldsymbol{q}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t), \tag{2.81}$$

$$T^{0y}(t,-\boldsymbol{q}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t), \tag{2.82}$$

where the matrix elements in momentum space are given by

$$J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{r} e^{-i\boldsymbol{q}\boldsymbol{r}} s v_F e \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) \sigma^x \phi_{\boldsymbol{l}ns}(\boldsymbol{r}), \tag{2.83}$$

$$\begin{aligned} T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) = & \frac{1}{4}\int \mathrm{d}\boldsymbol{r} e^{i\boldsymbol{q}\boldsymbol{r}} \left[v_F \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) p_y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) - v_F (p_y \phi^*_{\boldsymbol{k}ms}) \phi_{\boldsymbol{l}ns}(\boldsymbol{r})\right] \\ & + \frac{1}{4}\int \mathrm{d}\boldsymbol{r} e^{i\boldsymbol{q}\boldsymbol{r}} \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s\sigma^y (E_{\boldsymbol{k}_z ms} + E_{\boldsymbol{l}_z ns} - 2\mu) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}). \end{aligned} \tag{2.84}$$

Note that as $T^{0y}(t,-\boldsymbol{q})$ will be used later, we here for convenience included the sign into the definition of the matrix element $T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}$, as is reflected in the sign of the exponent of Eq. (2.84).

As was noted earlier, the eigenvectors are plane waves in the $x,z$-directions, and the non-trivial part is the $y$-dependent $\phi(y)$. Thus, we want to express these matrix elements in terms of $\phi(y)$. The sum over $\boldsymbol{l}$ in Eq. (2.81) can be replaced by an integration, as it is a good quantum number. As usual, the measure in the integration is given by the density of states in momentum space, the well known $L_i/2\pi$, with $L_i$ being the length of the system in the $i$-direction.

$$\begin{aligned} J^x(t,\boldsymbol{q}) &= \sum_{\boldsymbol{k}m,n}\int \mathrm{d}l_x\mathrm{d}l_z\frac{L_xL_z}{4\pi^2}J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q})a^\dagger_{\boldsymbol{k}ms}(t)a_{\boldsymbol{l}ns}(t) \\ &= \int \mathrm{d}l_x\mathrm{d}l_z\int \mathrm{d}ye^{-iq_yy}\delta(l_x-k_x-q_x)\delta(l_z-k_z-q_z)sv_Fe\phi^*_{\boldsymbol{k}ms}(y)\sigma^x\phi_{\boldsymbol{l}ns}(y). \end{aligned} \tag{2.85}$$

The Dirac delta functions appeared from taking the integrals from the matrix element over $x$ and $z$, as the integrand in these variables was only plane waves. The exact same procedure may be done for the stress-energy tensor in Eq. (2.82). Eliminating $l$ by doing the integrals yields

$$J^x(t,\boldsymbol{q}) = \sum_{\boldsymbol{k},mn}J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q})a^\dagger_{\boldsymbol{k}ms}(t)a_{\boldsymbol{k}+\mathfrak{q}ns}(t), \tag{2.86}$$

$$T^{0y}(t,-\boldsymbol{q}) = \sum_{\boldsymbol{\kappa},\mu\nu}T^{0y}_{\boldsymbol{\kappa}\mu s,\boldsymbol{\kappa}-\mathfrak{q},\nu s}(\boldsymbol{q})a^\dagger_{\boldsymbol{\kappa}\mu s}(t)a_{\boldsymbol{\kappa}-\mathfrak{q}\nu s}(t), \tag{2.87}$$

where $\mathfrak{q}=(q_x,q_z)$. Keeping in mind that $a^\dagger_\lambda(t)=e^{iE_\lambda t/\hbar}a^\dagger_\lambda$, and that

$$\left\langle\left[a^\dagger_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns},a^\dagger_{\boldsymbol{\kappa}\mu s}a_{\boldsymbol{\kappa}-\mathfrak{q}\nu s}\right]\right\rangle = \delta_{\boldsymbol{k},\boldsymbol{\kappa}-\mathfrak{q}}\delta_{m,\nu}\delta_{\boldsymbol{k}+\mathfrak{q},\boldsymbol{\kappa}}\delta_{n,\mu}\left[n_{\boldsymbol{k}ms}-n_{\boldsymbol{k}+\mathfrak{q}ns}\right], \tag{2.88}$$

the correlation function is given by

$$\begin{aligned} \left\langle\left[J^x(t,\boldsymbol{q}),T^{0y}(t',-\boldsymbol{q})\right]\right\rangle = \sum_{\boldsymbol{k}mn} & e^{\frac{i}{\hbar}(E_{k_zms}-E_{k_z+\mathfrak{q}_zns})t}e^{\frac{i}{\hbar}(E_{k_z+\mathfrak{q}_zns}-E_{k_zms})t'} \\ & \times J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})\left[n_{\boldsymbol{k}ms}-n_{\boldsymbol{k}+\mathfrak{q}ns}\right]. \end{aligned} \tag{2.89}$$

We are now ready to find the correlation function $\chi^{xy}$ given in Eq. (2.14)

$$\chi^{xy}(\omega,\boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}\hbar}\int \mathrm{d}te^{i\omega t}\int_{-\infty}^{0}\mathrm{d}t'\Theta(t)\left\langle\left[J^x(t,\boldsymbol{q}),T^{0y}(t',-\boldsymbol{q})\right]\right\rangle. \tag{2.90}$$

Introduce as usual a decay factor $e^{-\eta(t-t')}$ to ensure convergence in the time integrals, and make a change of variables $t'\to -t'$. The integral part of Eq. (2.90), ignoring everything without time dependence for clarity, is then

$$\lim_{\eta\to 0}\int_0^\infty \mathrm{d}t\mathrm{d}t' \exp\left[\frac{i}{\hbar}\left(E_{k_zms}-E_{k_z+\mathsf{q}_zns}+\omega\hbar+i\eta\hbar\right)t\right]\exp\left[\frac{i}{\hbar}\left(E_{k_zms}-E_{k_z+\mathsf{q}_zns}+i\eta\hbar\right)t'\right]$$
$$=\lim_{\eta\to 0}\frac{\hbar}{i}\left[E_{k_zms}-E_{k_z+\mathsf{q}_zns}+\omega\hbar+i\eta\hbar\right]^{-1}\frac{\hbar}{i}\left[E_{k_zms}-E_{k_z+\mathsf{q}_zns}+i\eta\hbar\right]^{-1}. \quad (2.91)$$

The response function then reads

$$\chi^{xy}(\omega,\boldsymbol{q})=\frac{iv_F\hbar}{\mathcal{V}}\lim_{\eta\to 0}\sum_{\boldsymbol{k}mn}J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathsf{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})\left[n_{\boldsymbol{k}ms}-n_{\boldsymbol{k}+\mathsf{q}ns}\right]$$
$$\left[E_{k_zms}-E_{k_z+\mathsf{q}_zns}+\omega\hbar+i\eta\hbar\right]^{-1}\left[E_{k_zms}-E_{k_z+\mathsf{q}_zns}+i\eta\hbar\right]^{-1}, \quad (2.92)$$

where the matrix elements are

$$J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathsf{q}ns}(\boldsymbol{q})=\int \mathrm{d}y e^{-iq_yy}sv_Fe\phi^*_{\boldsymbol{k}ms}(y)\sigma^x\phi_{\boldsymbol{k}+\mathsf{q}ns}(y), \quad (2.93)$$

$$T^{0y}_{\boldsymbol{k}ms,\boldsymbol{k}-\mathsf{q}ns}(\boldsymbol{q})=\frac{1}{4}\int \mathrm{d}ye^{iq_yy}\left[v_F\phi^*_{\boldsymbol{k}ms}(y)p_y\phi_{\boldsymbol{k}-\mathsf{q}ns}(y)-v_Fp_y\phi^*_{\boldsymbol{k}ms}(y)\phi_{\boldsymbol{k}-\mathsf{q}ns}(y)\right] \quad (2.94)$$
$$+\frac{1}{4}\int \mathrm{d}ye^{iq_yy}\phi^*_{\boldsymbol{k}ms}(y)s\sigma^y\left(E_{k_zms}+E_{k_z+\mathsf{q}_zns}-2\mu\right)\phi_{\boldsymbol{k}-\mathsf{q}ns}(y).$$

We will consider the response function in the static limit $\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}$. We may use the property of the limit of a product of functions $\lim A\cdot B=\lim A\cdot\lim B$ to write

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}(\omega,\boldsymbol{q})=\frac{iv_F\hbar}{\mathcal{V}}\sum_{\boldsymbol{k}mn}\frac{J^x_{\boldsymbol{k}ms,\boldsymbol{k}ns}T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms}[n_{\boldsymbol{k}ms}-n_{\boldsymbol{k}ns}]}{(E_{k_zms}-E_{k_zns})(E_{k_zms}-E_{k_zns})}, \quad (2.95)$$

where the current and energy-momentum tensor matrix elements are the expression given in Eqs. (2.93) and (2.94) taken in the limit.

## 2.3 Response of an untilted cone

### 2.3.1 Explicit form of the matrix elements

Compared to the procedure used by Arjona, Chernodub, and Vozmediano[1], taking the limit of each matrix element by itself greatly simplifies the calculation.

Let

$$\phi_{\boldsymbol{k}ms}(y)=e^{-\frac{(y-k_xl_B^2)^2}{2l_B^2}}\begin{pmatrix}a_{k_zms}H_{M-1}\left(\frac{y-k_xl_B^2}{l_B}\right)\\ b_{k_zms}H_M\left(\frac{y-k_xl_B^2}{l_B}\right)\end{pmatrix}, \quad (2.96)$$

thus implicitly defining the prefactors $a_{k_zms}, b_{k_zms}$.

### The current operator

The matrix element

$$
\begin{aligned}
&J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) && (2.97)\\
&= \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{\boldsymbol{k}ms}(y) \sigma^x \phi_{\boldsymbol{k}+\mathfrak{q}ns}(y) \\
&= s v_F e \int \mathrm{d}y \exp\left\{-iq_y y - \frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} && (2.98)\\
&\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.\\
&\left.+ b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right]\\
&= s v_F e \int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] && (2.99)\\
&\exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x+q_x)q_y\right\}\right]\\
&\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.\\
&\left.+ b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right],
\end{aligned}
$$

where we completed the square in the exponent, to get the form $e^{-a(y+b)^2}$. Also, $\mathfrak{q}_y = (q_x, q_y)$, was introduced, not to be confused with $\mathfrak{q} = (q_x, q_z)$. By introducing $\tilde{y} = \frac{y}{l_B} + l_B(iq_y - q_x - 2k_x)/2$ the matrix element may be rewritten

$$
\begin{aligned}
J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = s v_F e \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x+q_x)q_y\right\}\right]\\
e^{-\tilde{y}^2}\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_N\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right.\\
\left.+ b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right].
\end{aligned}
\tag{2.100}
$$

Taking the limit we find the simple form

$$
J_{\boldsymbol{k}ms;\boldsymbol{k}ns} = J_{k_z mns} = s v_F e l_B \int \mathrm{d}\tilde{y} e^{-\tilde{y}} \left[a_{k_z ms} b_{k_z ns} H_{M-1}(\tilde{y}) H_N(\tilde{y}) + m \leftrightarrow n\right], \tag{2.101}
$$

where $m \leftrightarrow n$ are the repetition of the previous term under the interchange of $m, n$. We employ now the orthogonality relation of the Hermite polynomials [14, Table 18.3.1]

$$
\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi} 2^n n! \delta_{n,m} \tag{2.102}
$$

to write

$$J_{\boldsymbol{k}ms,\boldsymbol{k}ns} = J_{k_zmns} = sv_F e l_B \sqrt{\pi}(a_{k_zms} b_{k_zns} \delta_{M-1,N} 2^N N! + m \leftrightarrow n). \tag{2.103}$$

With

$$a_{\boldsymbol{k}ms} b_{\boldsymbol{k}ns} = \frac{\alpha_{k_zms}}{\sqrt{\alpha_{k_zms}^2+1}\sqrt{\alpha_{k_zns}^2+1}} \left[2^{N+M-1}(M-1)!N!\pi l_B^2\right]^{-\frac{1}{2}}, \tag{2.104}$$

$$b_{\boldsymbol{k}ms} a_{\boldsymbol{k}ns} = \frac{\alpha_{k_zns}}{\sqrt{\alpha_{k_zms}^2+1}\sqrt{\alpha_{k_zns}^2+1}} \left[2^{N+M-1}(N-1)!M!\pi l_B^2\right]^{-\frac{1}{2}}. \tag{2.105}$$

we find explicitly

$$J_{\boldsymbol{k}ms,\boldsymbol{k}ns} = J_{k_zmns} = sv_F e \frac{\alpha_{k_zms}\delta_{M-1,N} + \alpha_{k_zns}\delta_{M,N-1}}{\sqrt{\alpha_{k_zms}^2+1}\sqrt{\alpha_{k_zns}^2+1}}. \tag{2.106}$$

#### The stress-energy tensor operator

Consider the first part of the stress-energy matrix element

$$T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{1}{4}\int \mathrm{d}y e^{iq_y y} \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) s\sigma^y (E_{k_zms} + E_{k_z+\mathfrak{q}_zns} - 2\mu)\phi_{\boldsymbol{k}ms}(y). \tag{2.107}$$

Recall that

$$\phi_{\boldsymbol{k}ms}(y) = e^{-\frac{(y-k_x l_B^2)^2}{2l_B^2}} \begin{pmatrix} a_{k_zms} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ b_{k_zms} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}. \tag{2.108}$$

The form of the integrand is very similar to the current matrix case, with the exchange of the Pauli matrix $\sigma^x \to \sigma^y$, thus giving an additional $i$ and a negative sign to the first term.

$$\begin{aligned} &T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \\ &= \frac{is}{4}(E_{k_zms} + E_{k_z+\mathfrak{q}_zns} - 2\mu) \int \mathrm{d}y e^{iq_y y} e^{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}} \\ &\left[-a_{k_z+\mathfrak{q}_zns} b_{k_zms} H_{N-1}(\dots)H_M(\dots) + b_{k_z+\mathfrak{q}_zns} a_{k_zms} H_N(\dots) H_{M-1}(\dots)\right]. \end{aligned} \tag{2.109}$$

Taking care to note that the factor from the Fourier transform, that was $e^{-iq_y y}$ in the current matrix element is here $e^{+iq_y y}$, a similar completion of the square is done

$$\begin{aligned} &T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \\ &= \frac{is}{4}(E_{k_zms} + E_{k_z+\mathfrak{q}_zns} - 2\mu) \exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2 - 2iq_y(2k_x+q_x)\right\}\right] \\ &\int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(-iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] \\ &\left[-a_{k_z+\mathfrak{q}_zns} b_{k_zms} H_{N-1}(\dots)H_M(\dots) + b_{k_z+\mathfrak{q}_zns} a_{k_zms} H_N(\dots) H_{M-1}(\dots)\right]. \end{aligned} \tag{2.110}$$

The arguments of the Hermite polynomials have been dropped for brevity of notation. As before make a change of variables to get the integral on the form of the shifted orthogonality relation for the Hermite polynomials Eq. (2.146). Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$ the shifted orthogonality relation is used on the expression

$$T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is}{4}(E_{k\mu s} + E_{\lambda\nu s} - 2\mu)\exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2 - 2iq_y(2k_x + q_x)\right\}\right]\int \mathrm{d}\tilde{y}\, l_B e^{-\tilde{y}^2}$$
$$\left[-a_{\boldsymbol{k}+\mathfrak{q}ns}b_{\boldsymbol{k}ms}H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right)H_M\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right.$$
$$\left. + b_{\boldsymbol{k}+\mathfrak{q}ns}a_{\boldsymbol{k}ms}H_N\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right)H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right]. \tag{2.111}$$

The terms in the integrand are exactly the same as in the current matrix element case, just in the reverse order and with $q_y \to -q_y$.

$$T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is}{4}\frac{(E_{k_zms} + E_{k_zns} - 2\mu)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_zns}+1}}\left(\alpha_{k_zms}\delta_{M-1,N} - \alpha_{k_zns}\delta_{M,N-1}\right). \tag{2.112}$$

**Summary 2**

*For a untilted case, in the local limit $q \to 0$, we have the matrix elements*

$$J_{\boldsymbol{k}ms;\boldsymbol{k}ns} = \Gamma_{k_zmns}sv_Fe\left(\alpha_{k_zms}\delta_{M-1,N} + m \leftrightarrow n\right), \tag{2.113}$$
$$T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms} = \frac{is\Gamma_{k_zmns}}{4}\left(E_{k_zms} + E_{k_zns} - 2\mu\right)\left(\alpha_{k_zms}\delta_{M-1,N} - m \leftrightarrow n\right). \tag{2.114}$$

*where $m \leftrightarrow n$ represent the preceding term under the interchange of $m, n$ and where we have defined $\Gamma_{k_zmns} = \left[(\alpha^2_{k_zms}+1)(\alpha^2_{k_zns}+1)\right]^{-\frac{1}{2}}$.*

### 2.3.2 Comment on the energy-momentum tensor

There is some ambiguity with regards to the definition of th energy-momentum tensor

**cite**

. The *canonical* energy-momentum tensor, derived from Lagrangian mechanics, is defined as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial\partial_\mu\psi_i}\partial^\nu\psi_i - \eta^{\mu\nu}\mathcal{L}. \tag{2.115}$$

On the other hand, from general relativity, the (*dynamical*?) energy-momentum tensor is defined by the variation of the action with respsect to the metric

$$T^{\mu\nu} = something something\frac{\delta S}{\delta g_{\mu\nu}}. \tag{2.116}$$

Immediately, we see that the first definition is in general not symmetric, while the latter is, as the metric is always symmetric [6]. ... Something about the non-defitness of the tensor, we may add some total derivative or something.

We may of course symmetrize the energy-momentum tensor. Denote by $T^{\mu\nu}$ the *canonical* tensor, and let the symmetrized tensor

$$T_S^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}. \tag{2.117}$$

In the case of our untilted system, the Weyl Lagrangian, the components of interest are

$$T^{0y} = \frac{v_F}{4}\left[\phi^\dagger p_y \phi - p_y \phi^\dagger \phi\right], \tag{2.118}$$
$$T^{y0} = \frac{si}{4}\left[\phi^\dagger \sigma_y \partial_0 \phi - \partial_0 \phi^\dagger \sigma_y \phi\right]. \tag{2.119}$$
$$\tag{2.120}$$

The symmetric form of the energy-momentum tensor, used by Arjona, Chernodub, and Vozmediano, gives additional contributions to the energy-momentum matrix element. We will here show that in the case of no tilt, these contributions are identical to those of the non-symmetric tensor. In the tilted case, however, the contributions differ.

The first other contribution is

**take care of prefactors**

$$\left(\frac{\sqrt{M}}{\alpha_{k_z ms}} + \sqrt{(M-1)}\alpha_{k_z ns}\right)\alpha_{k_z ms}\delta_{M-1,N}. \tag{2.121}$$

The normalization factor, given in dimensionless quantities is,

$$\alpha_{k_z ms} = -\frac{s\sqrt{M}}{\epsilon_m - s\kappa}.$$

Inserting this, and using the explicit form of the energy for $m \neq 0$

$$\epsilon_n = \text{sign}(m)\sqrt{M + \kappa^2},$$

for the case $N > 0$ the contribution can be shown to be

$$-s(\epsilon_m + \epsilon_n)\alpha_{k_z ms}\delta_{M-1,N}. \tag{2.122}$$

For $n = 0$, the second term of Eq. (2.121) is zero, and we have

**mising s?**

$$-(\epsilon_m - s\kappa)\alpha_{kms}\delta_{M-1,N}, \tag{2.123}$$

and by identifying $\epsilon_0 = -s\kappa$ this has the same form as Eq. (2.122).

[6] something with torsion never

In the case of tilt, however, the contribution can be shown to be

$$-\frac{s}{\sqrt{\alpha}}(\epsilon^0_{m,\alpha B}+\epsilon^0_{n,\alpha B}), \tag{2.124}$$

where $\epsilon^0_{m,\alpha B}=\mathrm{sign}(m)\sqrt{\alpha M+\kappa^2}$ and we used

$$\alpha_{k_z ms}=-\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B}-\kappa}$$

in the tilted case. Thus, we see that in the case of tilt perpendicular to the $B$-field, the contribution is scaled compared to the non-symmetric term. In the case of tilt parallel to the $B$-field, one gets an additional term proportional to $t_{\parallel}\kappa$.

### 2.3.3 Computing the reponse function

It is now finally possible to write out the entire response function. We begin by replacing the sum over $\boldsymbol{k}$ by an integral. Firstly, we will show that the sum over $k_x$ is restricted; recall that the eigenfunctions are exponentially centered around $y_0=k_x l_B^2$, which for a finite sample we expect to be restricted to $0\leq y_0\leq L_y$. This restricts the $k_x$ sum to $0\leq k_x\leq L_y/l_B^2=L_y eB/\hbar$, resulting in the $k_x$ summation giving a finite degeneracy contribution [20, Ch. 1.4.1, 12], as the integrand is independent of $k_x$.

$$\sum_{\boldsymbol{k}}=\sum_{k_x=0}^{L_y eB/\hbar}\sum_{k_z}\rightarrow\frac{L_xL_z}{(2\pi)^2}\int_0^{L_yeB/\hbar}\mathrm{d}k_x\int\mathrm{d}k_z \tag{2.125}$$

$$=\frac{\mathcal{V}eB}{(2\pi)^2\hbar}\int\mathrm{d}k_z. \tag{2.126}$$

Recall the response function

$$\chi^{xy}(\omega,\boldsymbol{q})=\lim_{\eta\to 0}\sum_{\boldsymbol{k},mn}\frac{1}{\mathcal{V}}\frac{iv_F\hbar J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})\;[n_{\boldsymbol{k}ms}-n_{\boldsymbol{k}+\mathrm{q}ns}]}{(E_{k_zms}-E_{k_z+\mathrm{q}_zns}+i\hbar\eta)(E_{k_zms}-E_{k_z+\mathrm{q}_zns}+\hbar\omega+i\hbar\eta)}. \tag{2.127}$$

Firstly, introduce the dimensionless quantities $\kappa_z\sqrt{2eB}=k_z$, $\epsilon_{k_zms}v_F\sqrt{2eB}=E_{k_zms}$, in order to facilitate solving the integral over $k_z$. Collecting dimensionfull quantites, the response function reads

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}=-\frac{e^2v_FB}{4(2\pi)^2}\sum_{mn}\int\mathrm{d}\kappa_z[n_{\kappa_zms}-n_{\kappa_zns}][(\alpha^2_{\kappa_zms}+1)(\alpha^2_{\kappa_zns}+1)]^{-1}$$
$$\times\frac{(\epsilon_{\kappa_zms}+\epsilon_{\kappa_zns})(\alpha^2_{\kappa_zms}\delta_{M-1,N}-\alpha^2_{\kappa_zns}\delta_{N-1,M})}{(\epsilon_{\kappa_zms}-\epsilon_{\kappa_zns}+i\eta)^2}. \tag{2.128}$$

Let us now define

$$\xi(\kappa_z)=\frac{[n_{\kappa ms}-n_{\kappa+\mathrm{q}ns}]\left[(\alpha^2_{\kappa ms}+1)(\alpha^2_{\kappa+\mathrm{q}ns}+1)\right]^{-1}}{(\epsilon_{\kappa ms}-\epsilon_{\kappa+\mathrm{q}ns}+i\frac{\hbar\eta}{v_F\sqrt{2eB\hbar}})(\epsilon_{\kappa ms}-\epsilon_{\kappa+\mathrm{q}ns}+\frac{\hbar\omega}{v_F\sqrt{2eB\hbar}}+i\frac{\hbar\eta}{v_F\sqrt{2eB\hbar}})}. \tag{2.129}$$

As is shown in table 2.1, $\xi(\kappa_z)$ is odd under interchagne of $m, n$ and inversion of $\kappa_z$.

**Clean up. Is it inversion or sign flip or what?**

Using this, we may simplify our expressions some. In the last term of Eq. (2.128), relabel the summation indices $m \leftrightarrow n$, and then use that $\xi$ is odd under interchange of $m, n$. This renders the two terms equal, and we may consider

$$\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M} \to 2\alpha^2_{\kappa_z ms}\delta_{M-1,N}.$$

The final expression is then

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = -\frac{e^2 v_F B}{2(2\pi)^2}\sum_{\substack{mn\\ N=M-1}}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} - 2\mu)\alpha^2_{\kappa_z ms}. \tag{2.130}$$

| Transformation | $\xi(\kappa_z)$ | $\epsilon_{\kappa_z ms}$ | $\alpha_{\kappa_z ms}$ |
|---|---|---|---|
| $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$ | -1 | -1 | -1 |
| $(\kappa_z, s) \mapsto (-\kappa_z, -s)$ | +1 | +1 | -1 |
| $(m, n) \mapsto (n, m)$ | -1 | | |

Table 2.1: Sign change of factors under various transformations. Note that $\xi(\kappa_z)$ is taken in the limit $\omega \to 0, \boldsymbol{q} \to 0, \eta \to 0$.

Before solving the integral, we note that in addition to the

**say the word diatomic?**

$N = M - 1$ selection rule of the sum, the factor with the distributions $n_{\kappa_z ms} - n_{\kappa_z ns}$ impose further restrictions on which transitions are energetically allowed. We consider the limit $T \to 0$

**something about the Luttinger in this limit? I.e. the fact we get finite result in T-¿ 0 is the interesting thing about this result**

, where the distributions take the form of step functions, $n_{\kappa_z ms} \to \theta(-\epsilon_{\kappa_z ms})$. As the sign of energy level $m$, for $m \neq 0$, is given by the sign of $m$ itself, this gives a rather simple restriction on the sum. For the zeroth energy level, the sign of the energy is given by $\mathrm{sign}(-s\kappa_z)$. The distribution factor is

$$n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}ns} = \begin{cases} 0 & mn > 0 \text{ or } m, n = 0, \\ -\,\mathrm{sign}(m) & m, n \neq 0, \\ -\,\mathrm{sign}(m)\theta\left[\mathrm{sign}(m)s\kappa_z\right] & n = 0. \end{cases} \tag{2.131}$$

Combining this with the selection rule $N = M - 1$, we see that the only allowed transitions are

$$M \to -N = -(M-1), \; -M \to N = (M-1).$$

The last simplification we will make, is to note that the step function is odd under $(m, n, \kappa_z) \to (-m, -n, -\kappa_z)$, and likewise with $\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns}$.

**is it supposed to be $\epsilon + \epsilon$?**

In the case of zero chemical potential, the expression may be simplified further, by considering only $-N \to M = N+1$ transitions, adding a factor 2.

Lastly, we now show that the contributions from cones of opposite chirality $s$ are the same. Under the transformation $(\kappa_z, s) \mapsto (-\kappa_z, -s)$, the product $\kappa_z s$ is obviously invariant. Note that $\epsilon_{\kappa_z m s}$ only depends on $s$ and $\kappa_z$ through the product $\kappa_z s$. While it is not the case for $\alpha_{\kappa_z m s}$, it is the case for its square. Consequently, the integrand is invariant under $(\kappa_z, s) \mapsto (-\kappa_z, -s)$. Similarly to the argumentation used above, as the integral goes over all $\kappa_z$, the integral is invariant under $s \mapsto -s$.

**Proposition 1**

*We have shown the following simplifications of Eq.* (2.128)*:*

- *The contributions from the terms $\alpha^2_{\kappa_z m s}\delta_{M-1,N}$ and $-\alpha^2_{\kappa_z n s}\delta_{N-1,M}$ are equal, and we consider therefore only one of them, adding a degeneracy factor 2.*
- *The difference of the step functions takes the form Eq.* (2.131)*, which limits the transitions to states with energies of opposite sign. For each value of $M, N$, this means the only valid transitions are $m = M, n = -N$ and $m = -M, n = N$.*
- *As the integrand is invariant under $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, we may consider only one of the transitions mentioned in the previous point, adding once again a degeneracy factor of 2.*
- *We lastly showed that the contribution is independent of the chirality $s$.*

For zero chemical potential, the response function is

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = -\frac{e^2 v_F B}{(2\pi)^2}\sum_{N=0}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z m s} + \epsilon_{\kappa_z n s})\alpha^2_{\kappa_z m s}\Big|_{\substack{m=N+1\\ n=-N}}, \tag{2.132}$$

where the integration limits are $(-\infty, \infty)$ for $N \neq 0$, $(-\infty, 0)$ for $N = 0, s = -1$, and $(0, \infty)$ for $N = 0, s = 1$.

Including only the first term of the sum, we find

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \frac{e^2 v_F B}{(2\pi)^2}. \tag{2.133}$$

Including contributions from the $N$ lowest landau levels, one acquire additional numerical prefactors,

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \gamma_N \frac{e^2 v_F B}{(2\pi)^2}, \tag{2.134}$$

where the factor by analytical integration was found to be $\gamma_0 = 1, \gamma_{20} \approx 2$. Furthermore, $\gamma_N$ goes like $\log N$. The first 300 contributions are shown in Figure 2.5.

Solving the integral analytically, we obtained the contribution from each term

$$\gamma_N - \gamma_{N-1} = \frac{1}{4}\left[1 + 2N\left\{1 - (1+N)\log(1 + \frac{1}{N})\right\}\right], \quad N > 0.$$

The sum can be shown to equal the rather nasty expression

$$\begin{aligned}\gamma_N = \gamma_0 + \frac{1}{12}\Big(&6\zeta^{(1,0)}(-2, N+1) - 6\zeta^{(1,0)}(-2, N+2) + 6\zeta^{(1,0)}(-1, N+1) \\ &+6\zeta^{(1,0)}(-1, N+2) + 12\log(\xi) + 3N^2 + 6N - 1\Big),\end{aligned} \tag{2.135}$$

where $\xi \approx 1.28243$ is Glaisher's constant. This expression goes like $\log N$.

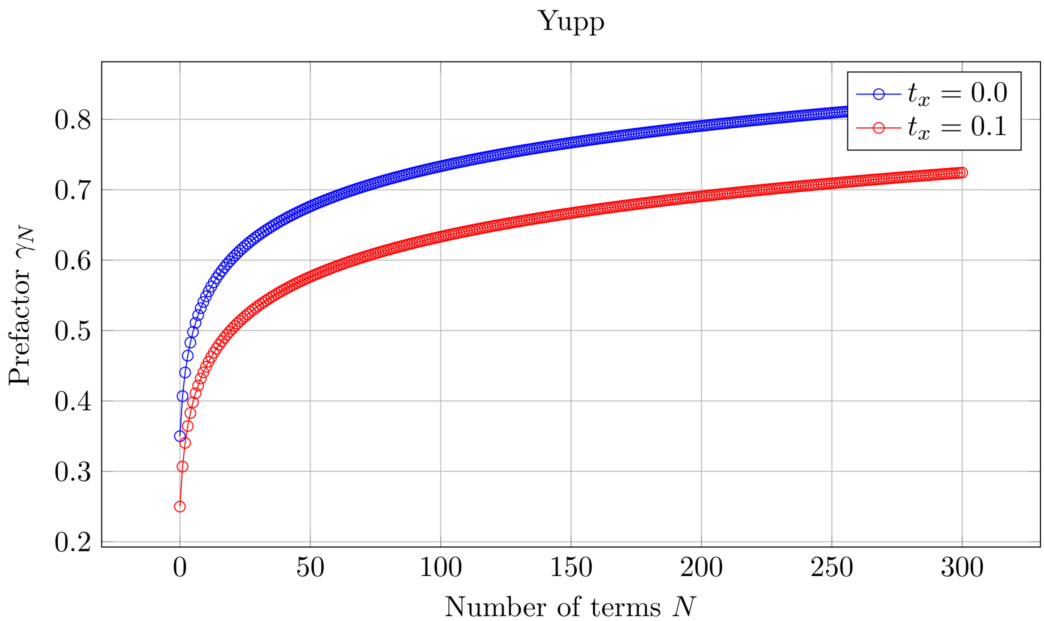



Figure 2.5: Prefactor $\gamma_N$ as a function of the number of inlcuded terms $N$. TODO remove the tx other test function

## 2.4 The response of a tilted cone

Repeating the calculation of the response function is now straightforward, but rather tedious. Due to the boost transformation, the elements of the spinor in the untilted system, Eq. (2.54), mix. We thus have twice as many terms to keep track of.

### 2.4.1 Explicit form of the matrix elements

We will here find an explicit form of the matrix elements, starting with the charge current

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{\boldsymbol{k}ms}(y) \sigma^x \phi_{\boldsymbol{k}+\mathrm{q}ns}(y).$$

We must find the matrix product $\phi\sigma_x\phi$. Recall that $\phi = \frac{1}{\mathcal{N}} e^{\theta/2\sigma_x}\tilde{\phi}$, and thus we must find

$$\phi^*\sigma_x\phi = \frac{1}{\mathcal{N}^*\mathcal{N}}\tilde{\phi}^* e^{\theta/2\sigma_x}\sigma_x e^{\theta/2\sigma_x}\tilde{\phi} = \alpha\tilde{\phi}^*\sigma_x e^{\theta\sigma_x}\tilde{\phi}.$$

With the previously found solution $\theta = -\tanh^{-1} t^s_x$, we get the rather simple form

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -st^s_x \\ -st^s_x & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x}}.$$

With

$$\tilde{\phi} = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix} \tag{2.136}$$

we see how the expressions change when $t^s_x$ become non-zero. Where we previoulsy had

$$\tilde{\phi}^*_{\boldsymbol{k}ms}\sigma_x\tilde{\phi}_{\boldsymbol{k}+\mathrm{q}ns} = a_{\boldsymbol{k}ms}H_{M-1}(\dots)\left[b_{\boldsymbol{k}+\mathrm{q}ns}H_N(\dots)\right] + \dots \tag{2.137}$$

the contents of the square brackets must now include also the other element of the spinor:

$$\alpha\tilde{\phi}^*_{\boldsymbol{k}ms}\sigma_x e^{\theta\sigma_x}\tilde{\phi}_{\boldsymbol{k}+\mathrm{q}ns} = a_{\boldsymbol{k}ms}H_{M-1}(\dots)\left[b_{\boldsymbol{k}+\mathrm{q}ns}H_N(\dots) - st^s_x a_{\boldsymbol{k}+\mathrm{q}ns}H_{N-1}(\dots)\right] + \dots. \tag{2.138}$$

First of all, let us consider the exponent of the product. Due to the extra term in $\chi$, this becomes more elaborate. The exponent is of course

$$\exp\{-iq_y y - \frac{1}{2}\chi^2_{\boldsymbol{k}} - \frac{1}{2}\chi^2_{\boldsymbol{k}+\mathrm{q}}\} \tag{2.139}$$

A straightforward but tedious calculation shows that the argument of the exponent can be written as

$$-\frac{\alpha}{l_B^2}\left(y + \frac{l_B^2}{2\alpha}(iq_y - (q'_x + 2k'_x))\right)^2 - \frac{l_B^2}{4\alpha}(q_y^2 + 2i(q'_x + 2k'_x)q_y + (q'_x)^2), \tag{2.140}$$

where we have defined

$$q'_x = q_x\alpha - \frac{\beta}{v_F}(E^0_{n,\alpha B} - E^0_{m,\alpha B}), \tag{2.141}$$

$$k'_x = k_x\alpha - \frac{\beta}{v_F}E^0_{m,\alpha B}. \tag{2.142}$$

**check sign of E above**

These must not be confused with the transformed momenta $\tilde{k}$, which are similar in form. Eq. (2.140) is on the same for as in the untilted cone case, and we may thus proceed with the same method. Make a change of variable

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y + \frac{l_B^2}{2\alpha}(iq_y - 2k'_x - q'_x)\right),$$

**Follow up the substitution of the root in the integral. Consider moving the root into $\Xi$**

to get the exponent on the form $e^{-\tilde{y}^2}$. With this substitution,

$$\chi_{\boldsymbol{k}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q'_x - iq_y\right), \tag{2.143}$$

$$\chi_{\boldsymbol{k}+\mathfrak{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q'_x - iq_y\right). \tag{2.144}$$

Doing this, Eq. (4.59)

**fix ref**

in the project thesis, becomes

$$\begin{aligned} J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = \frac{sv_F e}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{l_B^2}{4\alpha}\left(q_y^2 + 2i(2k'_x + q'_x)q_y + (q'_x)^2\right)\right] \\ e^{-\tilde{y}^2}\,[a_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathfrak{q}ns} H_{M-1}\left(\chi_{\boldsymbol{k}}\right) H_N\left(\chi_{\boldsymbol{k}+\mathfrak{q}}\right) \\ - st_x a_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathfrak{q}ns} H_{M-1}\left(\chi_{\boldsymbol{k}}\right) H_{N-1}\left(\chi_{\boldsymbol{k}+\mathfrak{q}}\right) \\ + b_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathfrak{q}ns} H_M\left(\chi_{\boldsymbol{k}}\right) H_{N-1}\left(\chi_{\boldsymbol{k}+\mathfrak{q}}\right) \\ - st_x b_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathfrak{q}ns} H_M\left(\chi_{\boldsymbol{k}}\right) H_N\left(\chi_{\boldsymbol{k}+\mathfrak{q}}\right)]. \end{aligned} \tag{2.145}$$

To perform the integration, we use the *shifted orthogonality* relation for Hermite polynomials [8, Eq. (7.377)]

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_m(x+y) H_n(x+z) = 2^n \pi^{\frac{1}{2}} m! y^{n-m} L_m^{n-m}(-2yz), \quad m \leq n, \tag{2.146}$$

where $L_b^a$ is the *generalized Laguerre polynomial* of order $b$ and type $a$. Using that

$$a_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_zms}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-1}(M-1)!N!\pi l_B^2\right]^{-\frac{1}{2}} \tag{2.147}$$

$$b_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_z+\mathfrak{q}_zns}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-1}(N-1)!M!\pi l_B^2\right]^{-\frac{1}{2}} \tag{2.148}$$

$$a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_zms}\alpha_{k_z+\mathfrak{q}_zns}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-2}(M-1)!(N-1)!\pi l_B^2\right]^{-\frac{1}{2}} \tag{2.149}$$

$$b_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{1}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M}M!N!\pi l_B^2\right]^{-\frac{1}{2}} \tag{2.150}$$

we define $\Xi_1, \Xi_2$ by

$$\frac{\sqrt{\alpha}\alpha_{k_zms}\Xi_1(\boldsymbol{q},m,n,s)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B a_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}(\chi_{\boldsymbol{k}})H_N(\chi_{\boldsymbol{k}+\mathfrak{q}}), \tag{2.151}$$

$$\frac{\sqrt{\alpha}\alpha_{k_z+\boldsymbol{q}ns}\Xi_2(\boldsymbol{q},m,n,s)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B b_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M}(\chi_{\boldsymbol{k}})H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}}). \tag{2.152}$$

Evaluating, we find

$$\Xi_1^{(1)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}}\left(\frac{q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{N-M+1} L_{M-1}^{N-M+1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{2.153}$$

$$\Xi_1^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}}\left(\frac{-q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{M-N-1} L_{N}^{M-N-1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{2.154}$$

$$\Xi_1(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_1^{(1)} & \text{if } N \geq M-1\\ \Xi_1^{(2)} & \text{if } N \leq M-1\end{cases} \quad \text{for } M>0, N\geq 0, \tag{2.155}$$

$$\Xi_2^{(1)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}}\left(\frac{q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{N-1-M} L_{M}^{N-1-M}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{2.156}$$

$$\Xi_2^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}}\left(\frac{-q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{M-N+1} L_{N-1}^{M-N+1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{2.157}$$

$$\Xi_2(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_2^{(1)} & \text{if } N-1 \geq M\\ \Xi_2^{(2)} & \text{if } N-1 \leq M\end{cases} \quad \text{for } M\geq 0, N> 0, \tag{2.158}$$

Here, $\mathfrak{q}_y = (q'_x, q_y)$.

This gives the first part of the current matrix element

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = sv_F e \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z+\mathrm{q}_z ns}^2 + 1}}$$
$$\Big[\alpha_{k_z ms}\Xi_1(\boldsymbol{q}, m, n, s) + \alpha_{k_z+\mathrm{q}_z ns}\Xi_2(\boldsymbol{q}, m, n, s)$$
$$- st_x \alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}\Xi_1(\boldsymbol{q}, m, n \mp 1, s) - st_x \Xi_2(\boldsymbol{q}, m, n \pm 1, s)\Big]. \quad (2.159)$$

**In the future, might be nice to go over to having only one function, Xi1, and simply mix around the arguments**

Now we will consider the second term of the current operator.

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = ev_F t_x^s \int \mathrm{d}y e^{-iq_y y}\phi^*_{\boldsymbol{k}ms}(y)\phi_{\boldsymbol{k}+\mathrm{q}ns}(y). \quad (2.160)$$

Using the results of Summary 1 we find

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = \frac{ev_F t_x}{\mathcal{N}^*\mathcal{N}} \int \mathrm{d}y e^{-iq_y y - \frac{1}{2}\chi_{\boldsymbol{k}}^2 - \frac{1}{2}\chi_{\boldsymbol{k}+\mathrm{q}}^2}\tilde{\phi}^*_{\boldsymbol{k}ms}(y)e^{\theta\sigma_x}\tilde{\phi}_{\boldsymbol{k}+\mathrm{q}ns}(y). \quad (2.161)$$

Using the same substitution and completion of the square as above, this is

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = \frac{ev_F t_x^s l_B}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y} \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]$$
$$e^{-\tilde{y}^2}\left[a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}})\left(a_{\boldsymbol{k}+\mathrm{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}}) - st_x^s b_{\boldsymbol{k}+\mathrm{q}ns}H_N(\chi_{\boldsymbol{k}+\mathrm{q}})\right)\right.$$
$$\left. + b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\left(-st_x^s a_{\boldsymbol{k}+\mathrm{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}}) + b_{\boldsymbol{k}+\mathrm{q}ns}H_N(\chi_{\boldsymbol{k}+\mathrm{q}})\right)\right]. \quad (2.162)$$

Thus [7]

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = ev_F t_x^s \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z+\mathrm{q}_z ns}^2 + 1}}$$
$$\Big[- st_x^s[\alpha_{k_z ms}\Xi_1(\boldsymbol{q}, m, n) + \alpha_{k_z+\mathrm{q}_z ns}\Xi_2(\boldsymbol{q}, m, n)]$$
$$+ \alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}\Xi_1(\boldsymbol{q}, m, n \mp 1) + \Xi_2(\boldsymbol{q}, m, n \pm 1)\Big]. \quad (2.163)$$

We notice that this part has the same form as $J^{(1)}$, with only a change of the prefactors of the $\Xi$-functions.

[7] Note to self: note that we dropped the $\frac{1}{\sqrt{\alpha}}$ for the $\sqrt{\alpha}$ coming from the $\Xi$ definition.

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = ev_F s\alpha^2 \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]}{\sqrt{\alpha_{k_zms}^2+1}\sqrt{\alpha_{k_z+\mathrm{q}_zns}^2+1}}$$
$$[\alpha_{k_zms}\Xi_1(\boldsymbol{q},m,n) + \alpha_{k_z+\mathrm{q}_zns}\Xi_2(\boldsymbol{q},m,n)] . \quad (2.164)$$

We used here that $1 - t_x^2 = \alpha^2$.

### Stress-energy tensor

Consider now

$$T^{0y(1)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{1}{4}\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)s\sigma^y(E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu)\phi_{\boldsymbol{k}ms}(y). \quad (2.165)$$

As

$$\sigma_y e^{\theta/2\sigma_x} = e^{-\theta/2\sigma_x}\sigma_y \quad (2.166)$$

we get the very fortunate result

$$\phi^*\sigma_y\phi = \frac{1}{\mathcal{N}^*\mathcal{N}}\tilde{\phi}^*\sigma_y\tilde{\phi}. \quad (2.167)$$

The first term of the stress-energy tensor thus has the exact same form as the untilted case. Recalling the expression for $\tilde{\phi}$ from Eq. (2.59),

$$\tilde{\phi} = e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{\boldsymbol{k}ms}H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms}H_M(\chi)\end{pmatrix},$$

where

$$\chi = \sqrt{\alpha}(y - q_x l_B^2)/l_B + \mathrm{sign}(m)\beta\sqrt{2|m| + \frac{k_z^2 l_B^2}{\alpha}}.$$

We thus get

$$T^{0y(1)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}(E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu)\int \mathrm{d}y e^{iq_y y}e^{-\frac{1}{2}(\chi^2_{\boldsymbol{k}+\mathrm{q}} + \chi^2_{\boldsymbol{k}})}$$
$$[-a_{\boldsymbol{k}+\mathrm{q}ns}b_{\boldsymbol{k}ms}H_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}})H_M(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}+\mathrm{q}ns}a_{\boldsymbol{k}ms}H_N(\chi_{\boldsymbol{k}+\mathrm{q}})H_{M-1}(\chi_{\boldsymbol{k}})]. \quad (2.168)$$

We will perform once again the completion of the square and substituion of $y$. The exponent is the same as that which we found for the current operator case, Eq. (2.140), with the change $q_y \to -q_y$. We thus make the change of variables

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y - \frac{l_B^2}{2\alpha}(iq_y + (2k_x' + q_x'))\right), \quad (2.169)$$

giving

$$\chi_{\boldsymbol{k}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q'_x + iq_y\right), \tag{2.170}$$

$$\chi_{\boldsymbol{k}+\mathsf{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q'_x + iq_y\right). \tag{2.171}$$

Thus, analogous to Eq. (4.79), we get

$$\begin{aligned}
&T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \\
&\frac{is\sqrt{\alpha}}{4}(E_{k_zms} + E_{k_z+\mathsf{q}_zns} - 2\mu)\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]\int \mathrm{d}\tilde{y} l_B e^{-\tilde{y}^2} \\
&\qquad\left[-a_{\boldsymbol{k}+\mathsf{q}ns}b_{\boldsymbol{k}ms}H_{N-1}(\chi_{\boldsymbol{k}})H_M(\chi_{\boldsymbol{k}+\mathsf{q}}) + b_{\boldsymbol{k}+\mathsf{q}ns}a_{\boldsymbol{k}ms}H_N(\chi_{\boldsymbol{k}})H_{M-1}(\chi_{\boldsymbol{k}+\mathsf{q}})\right]
\end{aligned} \tag{2.172}$$

And thus we have

$$T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}\frac{E_{k_zms} + E_{k_z+\mathsf{q}_zns} - 2\mu}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathsf{q}_zns}+1}} \tag{2.173}$$

$$\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right] \tag{2.174}$$

$$(-\alpha_{k_z+\mathsf{q}_zns}\Xi_2(\bar{\boldsymbol{q}}, m, n, s) + \alpha_{k_zms}\Xi_1(\bar{\boldsymbol{q}}, m, n, s)), \tag{2.175}$$

where $\bar{\boldsymbol{q}} = (q_x, -q_y, q_z)$.

**Summary 3**

*In summary we have*

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathsf{q}ns}(\boldsymbol{q}) = v_F e s\alpha^2\Gamma^-_{\boldsymbol{k}\mathsf{q}mns}\left[\alpha_{k_zms}\Xi_1(\boldsymbol{q}, m, n, s) + \alpha_{k_z+\mathsf{q}_zns}\Xi_2(\boldsymbol{q}, m, n, s)\right], \tag{2.176}$$

$$T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}(E_{k_zms} + E_{k_z+\mathsf{q}_zns} - 2\mu)\Gamma^+_{\boldsymbol{k}\mathsf{q}mns} \tag{2.177}$$

$$(-\alpha_{k_z+\mathsf{q}_zns}\Xi_2(\bar{\boldsymbol{q}}, m, n, s) + \alpha_{k_zms}\Xi_1(\bar{\boldsymbol{q}}, m, n, s)), \tag{2.178}$$

*with*

$$\Gamma^{\pm}_{\boldsymbol{k}\mathsf{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q'_x)^2) \pm iq_y l_B^2(k'_x + \frac{q'_x}{2})\right]}{\left[(\alpha^2_{k_zms}+1)(\alpha^2_{k_z+\mathsf{q}_zns}+1)\right]^{\frac{1}{2}}}$$

### 2.4.2 Static limit and dimensionless form

We are interested in the response in the static limit $\boldsymbol{q} \to 0$. We may use the property of limits that

$$\lim_{n\to a} A \cdot B = \lim_{n\to a} A \cdot \lim_{n\to a} B.$$

We are thus interested in the current and energy-momentum matrix elements in the static limit. Furthermore, we will write them in dimensionless quantities.

Consider firstly the exponent in the $\Gamma^{\pm}$ factor,

$$\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2+(q_x')^2)\pm iq_y l_B^2(k_x'+\frac{q_x'}{2})\right].$$

Recall the definition Eq. (2.141),

$$q_x' = q_x\alpha - \frac{\beta}{v_F}(E^0_{n,\alpha B}-E^0_{m,\alpha B}).$$

In the static limit

$$\lim_{\boldsymbol{q}\to 0} q_x' = -\frac{\beta}{v_F}(E^0_{n,\alpha B}-E^0_{m,\alpha B}),$$

and thus the for the exponent one has in the limit

$$\exp\left[-\frac{l_B^2\beta^2}{4\alpha v_F^2}(E^0_{n,\alpha B}-E^0_{m,\alpha B})^2\right].$$

Expressed in the dimensionless energies $\epsilon = \frac{E}{v_F\sqrt{2eB}}$

$$\exp\left[-\frac{\beta^2}{2\alpha}(\epsilon^0_{n,\alpha B}-\epsilon^0_{m,\alpha B})^2\right].$$

The normalization factor $\alpha_{k_zms}$ is independent on $\boldsymbol{q}$, and already dimensionless. Explicity, it is given in dimensionless quantities as

$$\alpha_{k_zms} = -\frac{\sqrt{2e\alpha BM}}{\frac{E_{k_zms}-t_\parallel v_F k_z}{v_F s\alpha}-k_z} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B}-\kappa}. \tag{2.179}$$

In the tilted case, the $\Xi$ functions do not have a trivial form in the static limit, as was the case in the untilted case. Define

$$P = \lim_{\boldsymbol{q}\to 0}\frac{l_B q_x'}{\sqrt{2\alpha}} = \frac{\beta}{\sqrt{\alpha}}(\epsilon^0_{n,\alpha B}-\epsilon^0_{m,\alpha B}).$$

In the static limit, the $\Xi$ functions thus take the form

$$\Xi_1^{(1)}(m,n,s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}}\left(\frac{P}{\sqrt{2}}\right)^{N-M+1} L_{M-1}^{N-M+1}\left(P^2\right), \tag{2.180}$$

$$\Xi_1^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}}\left(-\frac{P}{\sqrt{2}}\right)^{M-N-1} L_{N}^{M-N-1}\left(P^2\right), \tag{2.181}$$

$$\Xi_1(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_1^{(1)} & \text{if } N\geq M-1\\ \Xi_1^{(2)} & \text{if } N\leq M-1\end{cases} \quad \text{for } M>0, N\geq 0, \tag{2.182}$$

$$\Xi_2^{(1)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}} \left(\frac{P}{\sqrt{2}}\right)^{N-1-M} L_M^{N-1-M}\left(P^2\right), \tag{2.183}$$

$$\Xi_2^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}} \left(-\frac{P}{\sqrt{2}}\right)^{M-N+1} L_{N-1}^{M-N+1}\left(P^2\right), \tag{2.184}$$

$$\Xi_2(\boldsymbol{q},m,n,s) = \begin{cases} \Xi_2^{(1)} & \text{if } N-1 \geq M \\ \Xi_2^{(2)} & \text{if } N-1 \leq M \end{cases} \text{for } M \geq 0, N > 0. \tag{2.185}$$

Lastly, notice that in the static limit, the dependence on $k_z$ disappears, and so the same procedure as was done for the untilted cone in section 2.3.3 is valid for the tilted cone, replacing the $\boldsymbol{k}$ sum with an integral over $k_z$ and a degeneracy factor

$$\sum_{\boldsymbol{k}} \to \frac{\mathcal{V}eB}{(2\pi)^2\hbar} \int \mathrm{d}k_z. \tag{2.186}$$

Importantly, the degeneracy factor does *not* depend on the renormalized magnetic field $\alpha B$, but rather $B$ itself.

### 2.4.3 Perpendicular tilt

We consider here the specialized situation where $\boldsymbol{t} = t_x\hat{x}$, i.e. only tilt perpendicular to the magnetic field. The response function

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0} \chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0} \frac{eBiv_F}{(2\pi)^2} \sum_{mn} \int \mathrm{d}k_z [n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathrm{q}ns}] \\ \times \frac{J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) T^{0y}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})}{(E_{\boldsymbol{k}ms} - E_{\boldsymbol{k}+\mathrm{q}ns} + i\eta)(E_{\boldsymbol{k}ms} - E_{\boldsymbol{k}+\mathrm{q}ns} + i\eta)}.$$

Writing out the matrix products we have

$$J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) T^{0y}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{v_F e i\alpha^3}{4} e^{-P^2} \\ \frac{(E_{\boldsymbol{k}ms} + E_{\boldsymbol{k}+\mathrm{q}ns})(\alpha^2_{k_zms}\Xi_1(m,n)^2 - \alpha^2_{k_z+\mathrm{q}_zns}\Xi_2(m,n)^2)}{(\alpha^2_{k_zms}+1)(\alpha^2_{k_z+\mathrm{q}_zns}+1)}. \tag{2.187}$$

And so, inserting into the response function

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0} \chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0} \frac{-e\alpha^3 v_F\sqrt{eB}}{4(2\pi)^2\sqrt{2}} \sum_{mn} \int \mathrm{d}k_z e^{-P^2} \\ \frac{[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathrm{q}ns}](\epsilon_{\boldsymbol{k}ms} + \epsilon_{\boldsymbol{k}+\mathrm{q}ns})(\alpha^2_{k_zms}\Xi_1(m,n)^2 - \alpha^2_{k_z+\mathrm{q}_zns}\Xi_2(m,n)^2)}{(\alpha^2_{k_zms}+1)(\alpha^2_{k_z+\mathrm{q}_zns}+1)(\epsilon_{\boldsymbol{k}ms} - \epsilon_{\boldsymbol{k}+\mathrm{q}ns} + i\eta)^2} \tag{2.188}$$

We make the observation that $\Xi_1(m,n) = \Xi_2(n,m)$, where it is important to note that $P$ changes sign under interchange of $m,n$. The rest of the factors are invariant under the interchange $m \leftrightarrow n$, except for the step functions, which gives an overall sign change. Thus, using $\Xi_1(m,n) = \Xi_2(n,m)$ and relabelling the summation indices we may consider

$$\alpha^2_{\kappa_z ms}\Xi_1^2 - \alpha^2_{\kappa_z ns} \to 2\alpha^2_{\kappa_z ms}\Xi_1^2.$$

We may also simplify the step function expression. Physically, the step function term corresponds to only considering transitions between states with energies of opposite sign. For Type-I systems, which we are restricted to here as we consider currently only perpendicular tilt, the energy of the state with quantum number $n$ has the same sign as $n$ itself, excluding of course the zeroth state. For the zeroth state, the sign of the energy is $\mathrm{sign}(-s\kappa_z)$. Using these considerations, we may make certain seletion rules for the sum. In the $(m,n)$-plane, the first and third quadrant give no contribution, as there $mn > 0$, i.e. they have the same sign. Our sum is thus restricted to the second and fourth quadrant. It is easy to show that

$$n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns} = \begin{cases} 0 & mn > 0 \text{ or } m,n = 0, \\ -\mathrm{sign}(m) & m,n \neq 0, \\ \ \mathrm{sign}(n)\theta(\mathrm{sign}(n)s\kappa) & m = 0, \\ -\mathrm{sign}(m)\theta(\mathrm{sign}(m)s\kappa) & n = 0. \end{cases} \tag{2.189}$$

Furthermore, the contributions from the second and fourth quadrant are equal, which we will now show. The mapping $(m,n,\kappa_z) \mapsto (-m,-n,-\kappa_z)$, i.e. a $\pi$ rotation, transforms points from the $m < 0$ half plane to the $m > 0$ half plane, including mapping the second quadrant to the fourth quadrant. We want to consider how the integrand in question transforms under such a mapping. Recall

$$\alpha_{\kappa_z ms} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B} - \kappa_z},$$
$$\epsilon^0_{m,\alpha B} = \mathrm{sign}(m)\sqrt{\alpha M + \kappa_z^2}, \quad m \neq 0.$$

Under the above mapping, we have the following relations

$$\epsilon^0_{m,\alpha B} \mapsto -\epsilon^0_{m,\alpha B}, \tag{2.190}$$
$$\alpha_{\kappa_z ms} \mapsto -\alpha_{\kappa_z ms}, \tag{2.191}$$
$$P \mapsto -P. \tag{2.192}$$

The $\Xi$ functions also acquries a sign for some values of $m,n$, however, we only consdier $\Xi^2$. The integrand in Eq. (2.188) is thus invariant under the transformation from the second to the fourth quadrant, and so we may consider only the fourth quadrant, adding a degeneracy factor 2.

Lastly, completely analogous to the untilted case, the integrand only depend on $s$ and $\kappa_z$ through their product $s\kappa_z$, and thus is invariant under $(s,\kappa_z) \mapsto (-s,-\kappa_z)$. As the integral spans all of $\kappa_z$, the contribution is independent of the chirality $s$, and may be calcluated for a specific choice, which is here taken to be $s = +1$.

**Make a note about $M = N$ always giving zero contributinos. Maybe also show in figure. This is important wrt. saying that $\gamma_0$ is all contributison withtin square etc.**

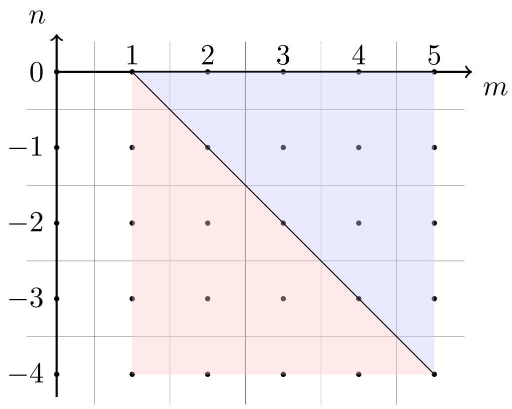



Figure 2.6: The region of $(m, n)$ to sum over for a Type-I perpendicularly tilted cone. The black line represents the combinations that give a finite contribution also in the untilted case. As the cone is tilted, this sharp line "diffuse" into the red and blue regions as well. Note that, as $\Xi_1$ defined only for $M > 0$, the region with $m = 0$ gives no contribution.

### 2.4.4 Tilt parallell to the magnetic field

Even though the treatment above for a general tilt is valid for parallel tilt, the response can be found more directly from the untilted case. For $\boldsymbol{t} = t_z\hat{z}$, the energy momentum tensor $T^{0y}$, charge current $J^x$, and wave functions $\phi(\boldsymbol{r})$ are all independent of $t_z$, and the only difference compared to the untilted system is a change in the energies of the Landau levels. We may thus immediately use the result from the untilted case

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = -\frac{e^2 v_F B}{4(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M}), \tag{2.193}$$

with

$$\epsilon_{\kappa_z ms} = \begin{cases} t_z^s\kappa_z + \operatorname{sign} m\sqrt{M+\kappa_z^2} & m \neq 0 \\ (t_z^s - s)\kappa_z & m = 0 \end{cases}, \tag{2.194}$$

$$\alpha_{\kappa_z ms} = -s\frac{\sqrt{M}}{\epsilon^0_{\kappa_z ms} - s\kappa_z}, \tag{2.195}$$

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\xi(\kappa_z) = \frac{[n_{\kappa ms} - n_{\kappa ns}]\left[(\alpha^2_{\kappa ms}+1)(\alpha^2_{\kappa ns}+1)\right]^{-1}}{(\epsilon_{\kappa ms} - \epsilon_{\kappa ns})^2}. \tag{2.196}$$

In the untilted case we made several simplifications to this expression, especially with regards to limiting the summation domain. We will here consider which of those simplifications apply also in the case of tilt $t_z$.

Under the transformation $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, $\xi(\kappa_z), \epsilon_{\kappa_z ms}, \alpha_{\kappa_z ms}$ are all still odd, and so the integrand is invariant under such a transformation. As the integral is over all $\kappa_z$, we may therefore consider only half the $m, n$ plane, as was the case in the untilted case. However, in the untilted case the sum was in fact restricted to only one quadrant, as at $T \to 0$ the transitions must be between states with energy of opposite sign. In the case of Type-II systems, this requirement does not restrict the sum to one quadrant. It is thus convenient to consider Type-I and Type-II separately.

In the untitled system, the contributions from the two chiralities where the same, as $\kappa_z$ and $s$ always appeared in conjunction, $\kappa_z s$. In the case of $t_z$ tilt, this is not the case. The proof for the response from the two chiralities being the same in the untilted case was that $s$ and $\kappa_z$ appeared only through the product $s\kappa_z$, and so the expression was invariant under $(s, \kappa_z) \mapsto (-s, -\kappa_z)$. As our integration spans all $\kappa_z$, the total response is invariant under $s \to -s$. The tilt parameter enters the expression only through $\epsilon_{\kappa_z ms} = \epsilon^0_{\kappa_z ms} + \kappa_z t^s_z$, and in the inversion symmetric case, $t^s_z = st_z$, the argument still holds. In the case of broken inversion symmetry, however, where $t^s_z = t_z$, the argument fails. A similar argument may, however, be made for the transformation $(s, \kappa_z, t_z) \mapsto (-s, -\kappa_z, -t_z)$, for which the (inversion broken) system is invariant. The response of a cone with chirality $s = -1$ is thus equal the response with $s = +1$ and $t_z \to -t_z$. We therefore compute all responses for $s = +1$; for symmetric systems the response is equal for $s = -1$, while for broken inversion symmetry, the response is given at $t_z \to -t_z$.

### Type-I

In Type-I systems, the selection rules from the step functions are independent of $t_z$, and the only difference from the untilted case is the term $\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} = \epsilon^0_{\kappa_z ms} + \epsilon^0_{\kappa_z ns} + 2\kappa_z t^s_z$. We compute therefore only the term that differ, for $s = +1$. The integral

$$\frac{\gamma_{\mathrm{div},N}}{2} = \int \mathrm{d}\kappa_z \xi(\kappa_z)\kappa_z t_z \alpha^2_{\kappa_z ms} \tag{2.197}$$

diverges. Introduce the momentum cutoff $\Lambda$, in which case the integral can be solved analytically, with the result

$$\frac{t_z}{4}\Bigg\{\Lambda\left(\sqrt{1+\Lambda^2+m} - \sqrt{\Lambda^2+m}\right) + m\tanh^{-1}\left[\frac{\Lambda}{\sqrt{\Lambda^2+m}}\right] - (m+1)\tanh^{-1}\left[\frac{\Lambda}{\sqrt{1+\Lambda^2+m}}\right]\Bigg\}, \tag{2.198}$$

where we used the selection rule of the sum $N = M + 1$ and $m > 0, n < 0$. This contribution is shown in figure 2.8.

**is it ok to write 'the contribution (2.198)', or must it always be 'the contribution Eq. (2.198)'?**

The contribution (2.198) is odd in $t_z$, and so for systems with broken inversion symmetry, the total contribution from two cones cancel.

### Type-II

**Recheck the order and specify clearly if we consider Type-I or Type-II for the different arguments**

**There should be some argument along the lines of the result being the same for s -¿ -s and tz -¿ -tz, so we may consider s=1, tz¿0 for defiteness (if that is indeed the case)**

For Type-I semimetals, the sign of energy state $m \neq 0$ is given by the sign of $m$ itself. For $m = 0$ the sign of the energy is given by $-s \operatorname{sign} \kappa$. Due to this, the sum is restricted to $n = M+1, m = -M$ and $n = -M-1, m = M$. In the case of Type-II, however, the situation is not so simple. The energy bands cross the Fermi surface, and we must also include in our sum overlap between states of the same sign, i.e. $n = M+1, m = M$ and $n = -M-1, m = -M$, which is non-zero for certain intervals of $\kappa$. See figure 2.4.

In order to find explicitly the limits of integration for the Type-II case, we must find the roots of the energy levels. The zeroth Landau level always has only one root, which is in the origin. For the higher order Landau levels, we solve

$$\epsilon_{\kappa_z m s} = t_z^s \kappa_z + \operatorname{sign}(m)\sqrt{M + \kappa_z^2} = 0, \tag{2.199}$$

whose solution is

$$\kappa_z^2 = \frac{M}{t_z^2 - 1}.$$

The actual roots of the energies are

$$\kappa_z = -\operatorname{sign}(m t_x^s)\sqrt{\frac{M}{t_z^2 - 1}}. \tag{2.200}$$

The integration limit for the $0 \to 1$ transition is thus, for $t_z^s > 1$, $[-\sqrt{t_z^2-1}^{-1}, 0]$. The $1 \to 2$ transition is $[-\sqrt{2}/\sqrt{t_z^2-1}, -\sqrt{t_z^2-1}^{-1}]$, and so forth.

The $0 \to 1$ transitions where computed analytically, and found to be

$$\frac{\operatorname{sign}(t_z)}{2}\left(|t_z| \sinh^{-1}\left(\frac{1}{\sqrt{t_z^2-1}}\right) - 1\right). \tag{2.201}$$

For a general $-N \to N+1,\ N > 0$ transition, the contribution was found to be the very lengthy expression

**Must also do $t_z < -1$**

**Be careful with tz ¡ -1. This does not only change sing of tz, but also the integration limits**

$$
\begin{aligned}
&\frac{(n-1)n}{8((n-1)n)^{3/2}(t_z^2-1)}\Bigg[nt_z \\
&\quad\left(F_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1}{1-t_z^2},-\frac{n}{(n-1)(t_z^2-1)}\right)-F_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1-n}{n(t_z^2-1)},\frac{1}{1-t_z^2}\right)\right) \\
&\qquad+t_zF_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1-n}{n(t_z^2-1)},\frac{1}{1-t_z^2}\right)+n^2(4-4t_z^2)\sqrt{(n-1)((n-1)t_z^2+1)} \\
&\qquad\qquad-4\sqrt{(n-1)n}n^2(t_z^2-1)\log\left(\frac{\sqrt{\frac{n-1}{n}}\left(\sqrt{1-nt_z^2}+\sqrt{1-n}\right)}{\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}}\right) \\
&\quad-n(2-2t_z)\sqrt{(1-n)\left(-((n-1)t_z^2)-1\right)}+2n(t_z-1)t_z\sqrt{(n-1)\left((n-1)t_z^2+1\right)} \\
&\qquad+(1-n)n\left(4-4t_z^2\right)\sqrt{-n\left(1-nt_z^2\right)}+2(1-n)\left(t_z^2-1\right)\sqrt{-n\left(1-nt_z^2\right)} \\
&\qquad\qquad-2\sqrt{(n-1)n}(t_z^2-1)\Bigg[-t_z\log\left(-\left((t_z-1)\sqrt{\frac{1-n}{t_z^2-1}}\right)\right) \\
&\qquad\qquad-t_z\log\left(\frac{\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}}{\sqrt{t_z^2-1}}\right)+t_z\log(1-n)-1\Bigg] \\
&+2\sqrt{(n-1)n}n\left(-t_z^3-2t_z^2+t_z+2\right)\log\left(\frac{\sqrt{\frac{n}{n-1}}\left(\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}\right)}{\sqrt{1-nt_z^2}+\sqrt{1-n}}\right)\Bigg],\quad\text{for } t_z<-1
\end{aligned}
\tag{2.202}
$$

and

$$
\begin{aligned}
&\frac{1}{8\sqrt{(n-1)n}\,(t_z^2-1)}\Bigg\{nt_z\Bigg(\\
&\qquad F_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1-n}{n\,(t_z^2-1)},\frac{1}{1-t_z^2}\right)\\
&\qquad -F_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1}{1-t_z^2},-\frac{n}{(n-1)\,(t_z^2-1)}\right)\Bigg)\\
&\qquad -t_zF_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1-n}{n\,(t_z^2-1)},\frac{1}{1-t_z^2}\right)\\
&\qquad +n^2(4-4t_z^2)\sqrt{(n-1)\,((n-1)t_z^2+1)}\\
&\qquad +4n^2(t_z^2-1)\sqrt{n\,(nt_z^2-1)}\\
&\qquad -4\sqrt{(n-1)n}n^2\left(t_z^2-1\right)\log\left(\frac{-\sqrt{(n-1)\,(nt_z^2-1)}+n-1}{n\sqrt{\frac{(n-1)t_z^2+1}{n}}+n}\right)\\
&\qquad +2n\left(t_z^2-1\right)\sqrt{(n-1)\,((n-1)t_z^2+1)}\\
&\qquad +n\left(6-6t_z^2\right)\sqrt{n\,(nt_z^2-1)}\\
&\qquad +2\left(t_z^2-1\right)\sqrt{n\,(nt_z^2-1)}\\
&\qquad -2\sqrt{(n-1)n}\left(t_z^2-1\right)\Bigg(-t_z\log\left((t_z+1)\sqrt{\frac{1-n}{t_z^2-1}}\right)\\
&\qquad -t_z\log\left(\frac{\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}}{\sqrt{t_z^2-1}}\right)\\
&\qquad +t_z\log(1-n)+1\Bigg)\\
&+2\sqrt{(n-1)n}n\left(-t_z^3-2t_z^2+t_z+2\right)\log\left(\frac{\sqrt{\frac{n}{n-1}}\left(\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}\right)}{\sqrt{1-nt_z^2}+\sqrt{1-n}}\right)\Bigg\},\quad\text{for } t_z>1.
\end{aligned}
\tag{2.203}
$$

where $F_1$ is the Appell hypergeometric function of two variables.

As described above, for broken inversion symmetry, the result of the chirality $s=-1$ is that of $s=+1$ at $t_z\to -t_z$. The total contribution will therefore be the sum of the contribution for $t_z > 1$ an d$t_z < -1$, which is

$$\frac{1}{4\sqrt{(n-1)n}\left(t_z^2-1\right)}\Bigg\{|t_z|\Bigg((n-1)F_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1-n}{n\left(t_z^2-1\right)},\frac{1}{1-t_z^2}\right) - nF_1\left(1;\frac{1}{2},\frac{1}{2};2;\frac{1}{1-t_z^2},-\frac{n}{(n-1)\left(t_z^2-1\right)}\right)\Bigg) + 2\left(t_z^2-1\right)\Bigg(-2n^2\sqrt{(n-1)\left((n-1)t_z^2+1\right)}+2n^2\sqrt{n\left(nt_z^2-1\right)} - \sqrt{(n-1)n}n^2\log\left(\frac{\sqrt{\frac{n-1}{n}}\left(\sqrt{1-nt_z^2}+\sqrt{1-n}\right)}{\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}}\right) - \sqrt{(n-1)n}n^2\log\left(\frac{-\sqrt{(n-1)\left(nt_z^2-1\right)}+n-1}{n\sqrt{\frac{(n-1)t_z^2+1}{n}}+n}\right) + n\sqrt{(n-1)\left((n-1)t_z^2+1\right)}-3n\sqrt{n\left(nt_z^2-1\right)}+\sqrt{n\left(nt_z^2-1\right)} - 2\sqrt{(n-1)n}n\log\left(\frac{\sqrt{\frac{n}{n-1}}\left(\sqrt{-nt_z^2+t_z^2-1}+\sqrt{-n}\right)}{\sqrt{1-nt_z^2}+\sqrt{1-n}}\right)\Bigg)\Bigg\} \tag{2.204}$$

## 2.5 Results

We will here consider parallel and perpendicular tilt separately.

**Make sure the argumentation and computations are valid also for $t_z < 0$**

### 2.5.1 Perpendicular tilt

In the case of a tilt perpendicular to the magnetic field, we are, as previously explained, restricted to Type-I materials, as the Landau level description breaks down for Type-II perpendicular tilt. Importantly, this does not generally mean that the effect is not present for Type-II systems, but simply that the Linear model Landau level description is not a good basis for the system. The collapse of the Landau levels caused Soluyanov et al. [19] to errenously predict the collapse of the chiral anomaly in their now famous paper first describing Type-II Weyl semimetals.

As explained in section 2.4.3, the $m, n$ summation is restricted to the fourth quadrant in the $m, n$ plane. In the case of no tilt, only contributions from $M = N+1$ were non-zero; we named the contribution from the $0 \to 1$ transition $\gamma_0$, the $-1 \to 2$ transition $\gamma_1$ and so fourth. Here, as there are contributions also away from the $M = N+1$ line, we denote by $\gamma_0$ the contributions from inside the square of length 1 centered at the origin. The

$\gamma_1$ contributions are those inside the square of length 2, and so fourth. This definition effectively sets a roof to which Landau levels we consider. This is indicated in figure 2.7.

**Correct which values**

The integral was computed numerically for $M, N \leq 6$ over different values of $t_x$ with $t_z = 0$, shown in figure 2.7. The total contribution $\gamma_N$ as a function of $N$ is shown in figure 2.9.

### 2.5.2 Parallel tilt

**Should we also compute the momentum cutoff for nontilted terms?**

In the Type-I regime, the contributions differ from that of the untitled system by Eq. (2.198), dependent on a momentum cutoff $\Lambda$. The contribution is odd in $t_z$, so for systems with broken inversion symmetry, the two chiralities cancel, and the response is equal to the untilted case. In case of inversion symmetry, the contributions from the two chiralities are equal and add up.

In the Type-II regime, the contributions have more complicated form. Considering firstly only the lowest Landau level contribution, Eq. (2.201). Also this contribution is odd in $t_z$, so the total contribution cancel between the chiralitites for broken inversion symmetry, while it adds up for inversion symmetric systems. As $|t_z| \to 1$ from above, the contribution blows up. This is to be expected as we move towards the Lifshitz transition, where we expect the linear model to perform poorly. [8]

**put this on more solid footing**

The contribution goes to zero as $t_z \to \infty$, shown in figure 2.10.

Considering also higher Landau level contributions, both interband and intraband transitions must be included, [9] meaning the summation is no longer restricted to a quadrant in the $m, n$ plane, but rather to half the plane. These contributions are not odd in $t_z$ – they have a finite even component. Due to this, the contribution does not cancel for inversion broken systems, however the contribution is small in magnitude compared to the other contributions.

## 2.6 Notes

### 2.6.1 Spin states for Dirac cone

See mathematica file.

Consider a simple Dirac cone Hamiltonian $H_D = s v_F \boldsymbol{\sigma p}$, with $s$ denoting the chirality of the cone. The eigenvalues of the system is of course $E = \pm v_F k, \quad k = |\boldsymbol{k}|$. We want to find the eigenstates of this system. Assume plane wave state, and some arbitrary linear

[8] As the Fermi surface of the linear model is vastly different from the Fermi surface of the tight binding model. See van der Wurff and Stoof [21]

[9] By band we here refer to the “conduction” band and “valence” band

combination of spin up and spin down,

$$\psi_{\pm} = e^{i\boldsymbol{k}\boldsymbol{r}}\alpha \begin{pmatrix} 1 \\ b \end{pmatrix},$$

where $\alpha$ is some normalization. Solving the time independent Schrodinger equation

$$H\psi = E\psi,$$

we may solve for $b$, which gives

$$b = -\frac{k_z \pm k}{k_x - ik_y}. \tag{2.205}$$

Requiring normalization of the state $\langle\psi|\psi\rangle = 1$ gives the normalization

$$|\alpha|^2 = \frac{1}{1 + |b|^2}.$$

Having found the states, we find the spin expectation value

$$\boldsymbol{S} = \langle\psi|\hat{S}|\psi\rangle, \tag{2.206}$$

where $\boldsymbol{S}$ is the spin expectation value and $\hat{S} = \frac{\boldsymbol{\sigma}}{2}$ is the spin operator, where $\hbar$ was set to 1. Simply evaluating Eq. (2.206), yields

$$\boldsymbol{S} = \pm\frac{\boldsymbol{k}}{2k}. \tag{2.207}$$

The spin structure is that of a hedgehog.

### 2.6.2 Symmetries

In order to separate weyl cones in momentum, we introduce a pseuod spin degree of freedom, making the system 4x4. We may then get solutions with the cones separated in momentum (or energy). We may also ask what heppens if we try to separate tilted cones?

Firstly, in the most intuitive way to extend the 2x2 tilted cones to 4x4, we get that the cones tilt opposite direction, thus not superimposed even before separating in momentum. They are after that simple to separate in momentum. We might wonder if it makes sense to do it in this way.

The lattice model of the energy dispersion to explain tilted cones gives two cones separated in momentum, and tilting corresponds to “bending” the dispersion curves between them. Maybe we therefore always have cones separated in momentum, and thus tilting superimposed does not make sense? All depends on the origin of the tilt I believe. Also, we must not confuse the global dispersion relation, to the Dirac cones which are expansions around the nodes.

Key to understand how spin behaves in all of this, and also maybe the symmetries.

To properly investigate the symmetry properties of the system, we must consider the 4x4, not 2x2 Hamiltonians. While the 2x2 system does a goood job at describing a single cone, much important phsycis is lost when reducing the 4x4 Hamiltonian. For example, the requirement that the total Berry curvature over the entire Briolluine zone is zero is not met for the 2x2 Hamiltonian, as it describes only one cone of a certain chirality. The 4x4, however, includes two cones, which may in general be superimposed, thus conserving the total zero-divergence of the Berry curvature. As a matter of fact, the inclusion of both cones is important also for symmetry considerations.

Let

$$H = v_F \tau_x \otimes \boldsymbol{\sigma k},$$

where $\tau$ is some pseudo spin degree of freedom, transforming like $\boldsymbol{r}$ under parity in time reversal. This system describes two superimposed cones at the origin, with opposite chirality. The effect of parity $\mathcal{P}$ and time reversal $\mathcal{T}$ is

| | $\mathcal{P}$ | $\mathcal{T}$ |
|---|---|---|
| $\tau$ | - | + |
| $\sigma$ | + | - |
| $k$ | - | - |

$$\begin{aligned} \mathcal{P}\tau\mathcal{P}^\dagger &= -\tau, \quad \mathcal{T}\tau\mathcal{T}^\dagger = +\tau \\ \mathcal{P}\sigma\mathcal{P}^\dagger &= +\sigma, \quad \mathcal{T}\sigma\mathcal{T}^\dagger = -\sigma \\ \mathcal{P}k\mathcal{P}^\dagger &= -k, \quad \mathcal{T}k\mathcal{T}^\dagger = -k \end{aligned} \tag{2.208}$$

Obviously then, the Hamiltonian is both time reversal and parity invariant, as $\mathcal{P}\mathcal{P}^\dagger = \mathcal{T}\mathcal{T}^\dagger = 1$.

A tilt term $\tau_x \otimes \mathcal{I}\boldsymbol{\omega}_0 \boldsymbol{k}$ breaks time reversal invariance, while maintaining parity invariance. This is due to the two cones of opposite chirality tilting in opposite directions.

The unperturbed Dirac Hamiltonian is Lorentz invariant, given that we consider an “effective speed of light”, namely the Fermi velocity, instead of the actual speed of light $c$. Specifically, Lorentz invariance means invariance under the *Lorentz group*. The Lorentz group is the $O(1,3)$ Lie group that conserves

$$x_\mu x^\mu = t^2 - x^2 - y^2 - z^2,$$

i.e. all isometries of Minkowski space. More specifically, the group consists of all 3D rotations, $O(3)$, and all *boosts*. A boost is a hyperbolic rotation from a spactial dimension to the temporal dimension. If we now direct our focus at the Hamiltonian of the Dirac cone

$$H = \pm v_F \boldsymbol{\sigma p},$$

we may easily show the Lorentz invariance of the system. The time independent Schrodinger equation is

$$H|\psi\rangle = E|\psi\rangle \implies (H^2 - E^2)|\psi\rangle = 0. \tag{2.209}$$

As

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right),$$

the operator in Eq. (2.209) is nothing more than

**Make clear the matrix strucute here. There is an implicit identity matrix of size 2**

$$H^2 - E^2 = v_F^2 \boldsymbol{p}^2 - c^2 \left(p^0\right)^2, \tag{2.210}$$

where we used the anticommutation relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

of the Pauli matrices. Using now the effective speed of light $c = v_F$, Eq. (2.210) is

$$-v_F^2 p_\mu p^\mu. \tag{2.211}$$

The invariance of $x^\mu x_\nu$ is the very definition of the Lorentz group, and so is obviously Lorentz invariant.

Consider now a *tilted* Dirac cone

$$H = \pm v_F \boldsymbol{\sigma p} + \omega_x k_x, \tag{2.212}$$

where we, without loss of generality, chose the tilt to be in the $x$-direction. By the same argumentation as above, the eigenequation

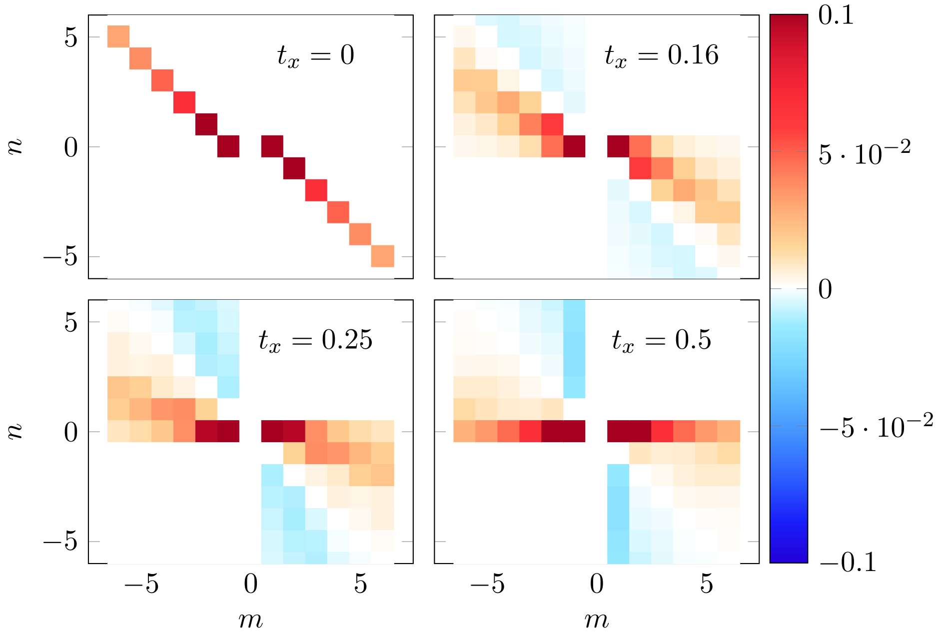
$$H\left|\psi\right\rangle = E\left|\psi\right\rangle \implies (H^2 - E^2)\left|\psi\right\rangle = 0$$

leads to the equation

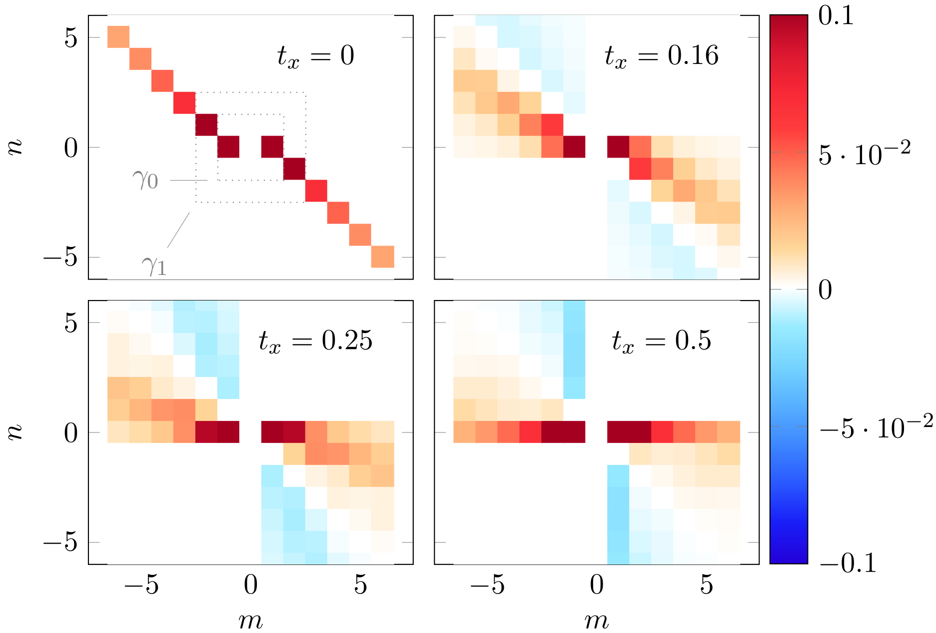
$$-v_F^2 p^\mu p_\mu + \omega_x k_x (2E - \omega_x k_x) = 0. \tag{2.213}$$

This is *not* invariant under a Lorentz transformation, as can be seen by, for example, a rotation around the $z$-axis.

**Clean up p vs k**




(a) Inversion symmetric case.




(b) Inversion symmetry broken case.

Figure 2.7: Contributions to $\gamma_N$ from $m \to n$ transitions for different values of $t_x$. TODO: Update caption

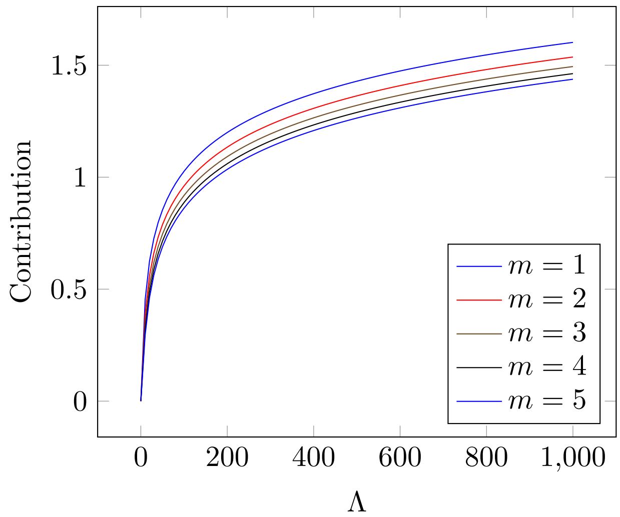



Figure 2.8: The divergent factor $\gamma_{\mathrm{div},N}/t_z$ for the first Landau levels, as a function of the momentum cutoff $\Lambda$. TODO: fix m to be m-1. Alternatively use N

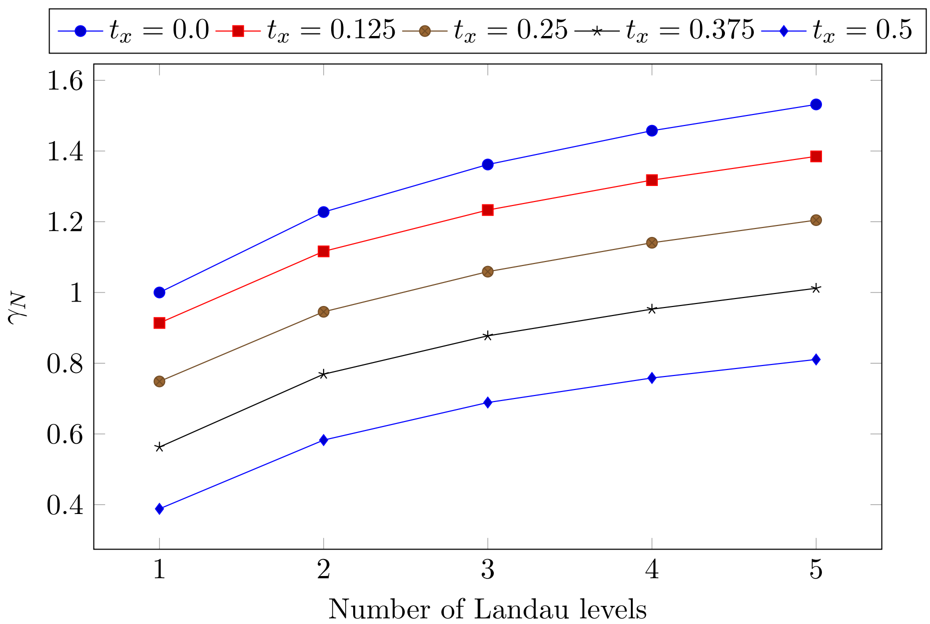
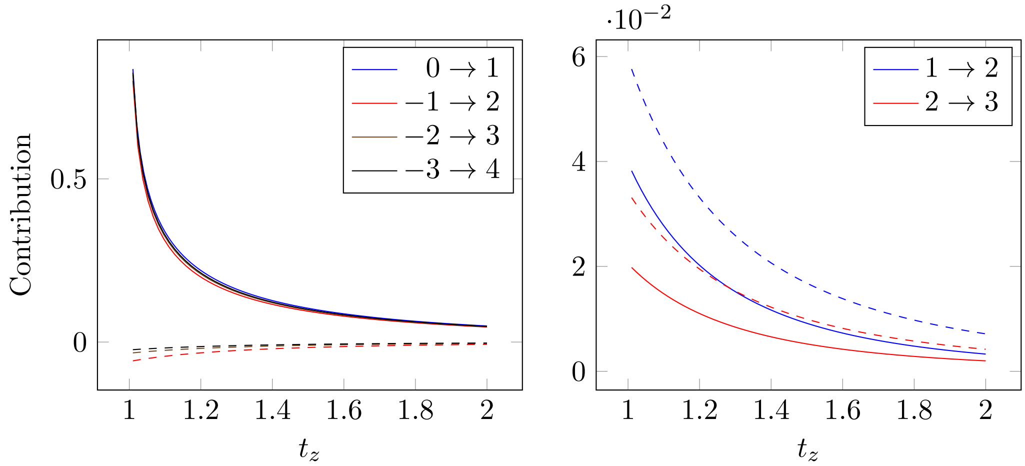



Figure 2.9




(a) Intraband contributions, $-N \to N+1$. (b) Interband contributions, $N \to N+1$.

Figure 2.10: The contribution from $n \to m$ transitions in a Type-II $t_z$ tilted system. Shown in dashed line of corresponding color, is the even component of the contribution, i.e. $\text{contrib}(|t_z|) + \text{contrib}(-|t_z|)$.

figures/spinStructureWeyl.pdf

Figure 2.11

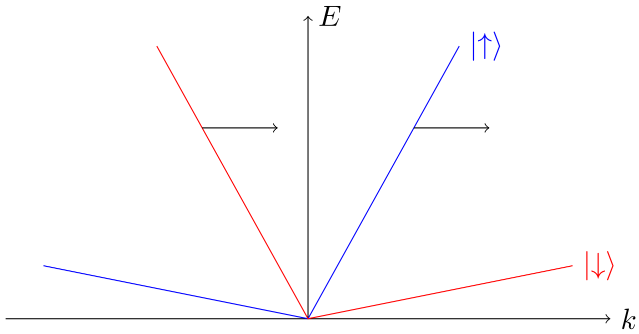



Figure 2.12: Time reversal breaking in tilted system. Cross section in the tilt direction shown, with blue showing one cone and red the other. Black arrows indicate spin direction, which for $|\uparrow\,\rangle$ is proporitional to $k$ while for $|\downarrow\,\rangle$ is proportional to $-k$.

## 2.7 Discussion of results

In the static and local limit $\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}$ the transverse response function $\chi^{xy}$ of the charge current to a temperature perturbation

$$J^x = \chi^{xy}\frac{-\nabla^y T}{T} \tag{2.214}$$

from a single Dirac point was found to be

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \gamma_N\frac{e^2 B v_F}{4(2\pi)^2\hbar}, \tag{2.215}$$

with $\gamma_N$ a prefactor dependent on how many landau levels are included in the final evaluation of the response function. The response function is independent of the chirality $s$ of the Dirac point. It was found that $\gamma_0 = 1, \gamma_{20} \approx 2$ and that the prefactor goes like $\log N$.

Firstly, the result differ slightly from that found by Arjona, Chernodub, and Vozmediano [1]

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = 2\gamma_N\frac{e^2 B v_F}{4(2\pi)^2\hbar}, \tag{2.216}$$

which differ by a factor of two.

Secondly, the sum will diverge as $N\to\infty$. However, not all Landau levels are filled, and thus the sum should not be taken to all levels. Similarly to a Quantum Hall effect, the number of filled bands, the filling factor $\nu$, is inverse proportional to the $B$-field strength

$$\nu \propto \frac{1}{B}. \tag{2.217}$$

Thus, we expect that the $N$-sum should be truncated at a Landau level, given by the filling factor $\nu$. A detailed derivation of the exact truncation of the $N$-sum has not been done. If a precise result for the numerical prefactor is found to be of importance, this should be straightforward.

The divergence is not discussed by Arjona, Chernodub, and Vozmediano, where only the values of $N = 0$ and $N = 20$ are given, and the final result is that of $N = 20$. Furthermore, they state that the contributions from higher values of $N$ decrease very rapidly. However, we found the contributions to go like $1/x$, which is not decreasing rapidly enough to give a finite total contribution, thus giving the total contribution diverging logarithmically.

**Say that we are communicating with them to better understand their choice of truncation?**

Comparing our result with the different procedure done by Chernodub, Cortijo, and Vozmediano [5], the numerical prefactor found in our calculation including only the first term ($M = 0$) coincides very well with the numerical prefactor found there, with a ratio of 16/18.

# Contributions from symmetric energy-momentum tensor

As noted in the main text, there are some subtlety in the definition of the energy-momentum tensor. The *canonical* definition, which we have used in the main text, is in general not symmetric. The tensor enter our calculation from the conservation law

$$\partial_\mu T^{\mu\nu} = 0,$$

which for $\nu = 0$ is nothing more than the conservation law of energy: $\partial_t \epsilon - \boldsymbol{\nabla} \boldsymbol{J}_\epsilon = 0$, where $\epsilon$ is energy density and $\boldsymbol{J}_\epsilon$ is the energy current. In the calculation by Arjona, Chernodub, and Vozmediano[1], the symmetrized energy-momentum tensor

$$T_S^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}$$

was used. In this appendix we show the contributions of the symmetric tensor. The contributions from $T^{\mu\nu}$ and $T^{\nu\mu}$ is shown to be equal in the non-tilted case.

## 1 No tilt

In the main text we have already found the contributions from the canonical tensor, and so we focus here on the contributions from $T_F^{\mu\nu} = T^{\nu\mu}$. The relevant element is $T_F^{y0} = \frac{v_F}{4}\left(\phi^\dagger p_y \phi - p_y \phi^\dagger \phi\right)$

Consider now the latter part of the stress-energy tensor, which is split into two parts

$$T^{0y\ (2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = +\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) p_y \phi_{\boldsymbol{k}ms}(y), \tag{1}$$

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \left(p_y \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\right) \phi_{\boldsymbol{k}ms}(y). \tag{2}$$

Recall that $\phi_{\boldsymbol{k}ms}(y)$, defined in Eq. (2.96), consists of two $y$-dependent factors: $\exp\left[-\frac{(y-k_x l_B^2)^2}{2l_B^2}\right]$ and the Hermite polynomials. The operator $p_y$ thus produces two terms when operating on $\phi$. The first term, coming from the exponent, is proportional to $y - k_x l_B^2$. The operator in Eqs. (1) and (2) acts on $\phi$ with the quantum number $\boldsymbol{k}$ and $\boldsymbol{k} + \mathfrak{q}$, respectively; when summing the two contributions, everything thus cancels except for a term proportional to $q_x$, which vanishes in the local limit.

It remains to consider the result of $p_y$ operating on the Hermite polynomials. Let $\tilde{p}_y$ indicate the $p_y$ operator acting only on the Hermite polynomial part of $\phi$, and use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$ [14, Eq. 18.9.25].

$$\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms} = -i\hbar \exp\left\{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\}$$
$$\frac{2}{l_B}\Bigg\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\frac{y-k_x l_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)$$
$$+ Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right)H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\Bigg\}. \quad (3)$$

Completing the square, we get

$$\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) = -i\hbar \exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2 - 2iq_y(2k_x+q_x)\right\}\right]$$
$$\int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}\left(-iq_y - 2k_x - q_x\right)\right\}^2 / l_B^2\right]$$
$$\frac{2}{l_B}\Bigg\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\frac{y-k_x l_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)$$
$$+ Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right)H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\Bigg\}. \quad (4)$$

Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$, as was also done in the main text, the expression reduces to

$$\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) = -i\hbar \exp\left[-\frac{l_B^2}{4}\left\{q_x^2 + q_y^2 - 2iq_y(2k_x+q_x)\right\}\right]$$
$$\int \mathrm{d}\tilde{y} l_B \exp\left[-\tilde{y}^2\right]$$
$$\frac{2}{l_B}\Bigg\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\tilde{y} + \frac{l_B}{2}(iq_y+q_x)\right)H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y-q_x)\right)$$
$$+ Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y+q_x)\right)H_N\left(\tilde{y} + \frac{l_B}{2}(iq_y-q_x)\right)\Bigg\}. \quad (5)$$

Considering now the local limit $\boldsymbol{q} \to 0$, the expression greatly simplifies, and we may use the orthogonality relation for the Hermite polynomials Eq. (2.102)

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2}H_n(x)H_m(x) = \sqrt{\pi}2^n n!\delta_{n,m}$$

to evaulate the integral.

$$\lim_{\boldsymbol{q}\to 0}\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) = -i\hbar\sqrt{2}\frac{\alpha_{kms}\alpha_{kns}\sqrt{M-1}+\sqrt{M}}{l_B\sqrt{\alpha_{kms}^2+1}\sqrt{\alpha_{kns}^2+1}}\delta_{N,M-1}. \quad (6)$$

Similarly, for $T^{0y\,(3)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})$, one has

$$\begin{aligned}\left(\tilde{p}_y\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right)\phi_{\boldsymbol{k}ms}(y) &= -i\hbar\exp\left\{-\frac{(y-k_xl_B^2)^2+(y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\}\\ &\frac{2}{l_B}\left\{(N-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathrm{q}ns}H_{M-1}\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-2}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.\\ &\left.+Nb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathrm{q}ns}H_M\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right\} \end{aligned}\tag{7}$$

which with the same procedure as above gives

$$\lim_{\boldsymbol{q}\to 0}\int \mathrm{d}y e^{iq_yy}\left(\tilde{p}_y\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right)\phi_{\boldsymbol{k}ms}(y) = -i\hbar\sqrt{2}\frac{\alpha_{kms}\alpha_{kns}\sqrt{N-1}+\sqrt{N}}{l_B\sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}}\delta_{M,N-1}. \tag{8}$$

## 2 With tilt

In the tilted case, we have shown in the main text that

**insert ref**

$$T^{\mu 0} = \frac{i}{2}\left[\partial_i\bar{\psi}\Gamma^j\gamma^0\Gamma^\mu\psi - \bar{\psi}\Gamma^\mu\gamma^0\Gamma^j\partial_j\psi\right].$$

Swapping the indices, we have for $\mu \neq 0$ [21]

$$T^{0i} = \frac{i}{2}[\bar{\psi}\gamma^0\partial^\mu\psi - \partial^\mu\bar{\psi}\gamma^0\psi].$$

In our work, we have considered only tilt perpendiculat to the thermal gradient, so the component of the energy-momentum tensor of interest are not affected by the tilt.

or

$$T^{0y(2)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = +\frac{1}{4}\int \mathrm{d}y e^{iq_yy}v_F\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)p_y\phi_{\boldsymbol{k}ms}(y), \tag{9}$$

$$T^{0y(3)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{1}{4}\int \mathrm{d}y e^{iq_yy}v_F(p_y\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y))\phi_{\boldsymbol{k}ms}(y). \tag{10}$$

Firstly, we note that

$$[p_y, e^{\theta/2\sigma_x}] = 0.$$

Furthermore, exactly as for the untilted case, the momentum operator acting on the exponential prefactor of $\phi$ gives contributions proportional to $q_x$. In the local limit $q \to 0$ this term vanishes, and we need only consider the effect of the momentum operator acting on the Hermite polynomials.

Denote by $\tilde{p}_y$ the momentum operator $p_y$ acting only on the Hermite polynomial part of $\phi$. Furthermore, we will use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$

[14, Eq. 18.9.25].

$$\tilde{p}_y \phi_{\boldsymbol{k}ms} = -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \partial_y \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix} \tag{11}$$

$$= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} 2 \frac{\partial \chi}{\partial y} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi) \\ b_{\boldsymbol{k}ms}(M) H_{M-1}(\chi) \end{pmatrix} \tag{12}$$

$$= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \frac{2\sqrt{\alpha}}{l_B} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi) \\ b_{\boldsymbol{k}ms}(M) H_{M-1}(\chi) \end{pmatrix}. \tag{13}$$

And thus, recalling that

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -t_x \\ -t_x & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x^2}},$$

we find the product

$$\phi^*_{\boldsymbol{k}+\mathsf{q}ns}(y) \tilde{p}_y \phi_{\boldsymbol{k}ms} = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} e^{-\frac{1}{2}\chi^2_{\boldsymbol{k}} - \frac{1}{2}\chi^2_{\boldsymbol{k}+\mathsf{q}}}$$
$$\Big[ a_{\boldsymbol{k}+\mathsf{q}ns} H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}}) \left\{ a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}}) \right\}$$
$$+ b_{\boldsymbol{k}+\mathsf{q}ns} H_N(\chi_{\boldsymbol{k}+\mathsf{q}}) \left\{ -t_x a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}}) \right\} \Big]. \tag{14}$$

Completing the square and substituting

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B} \left( y - \frac{l_B^2}{2\alpha}(iq_y + (2k'_x + q'_x)) \right)$$

gives

$$\int \mathrm{d}y e^{iq_y} \phi^*_{\boldsymbol{k}+\mathsf{q}ns}(y) \tilde{p}_y \phi_{\boldsymbol{k}ms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} \exp\left[ -\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x) q_y + (q'_x)^2) \right]$$
$$\int \mathrm{d}\tilde{y} \frac{l_B}{\sqrt{\alpha}}$$
$$\Big[ a_{\boldsymbol{k}+\mathsf{q}ns} H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}}) \left\{ a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}}) \right\}$$
$$+ b_{\boldsymbol{k}+\mathsf{q}ns} H_N(\chi_{\boldsymbol{k}+\mathsf{q}}) \left\{ -t_x a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}}) \right\} \Big]. \tag{15}$$

We must now evaluate the integral, and express the result in the Ξ-functions.

$$\begin{pmatrix} a_{\boldsymbol{k}+\mathsf{q}ns} H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}}) \\ b_{\boldsymbol{k}+\mathsf{q}ns} H_N(\chi_{\boldsymbol{k}+\mathsf{q}}) \end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & -t_x \\ -t_x & 1 \end{pmatrix}}_{T} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) \\ b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}}) \end{pmatrix}$$

For each of the entries in $T$, we get a product on of Hermite polynomials. Where the untilted cone had two such terms, the tilt parameter $t_x$ now gives two extra products, which we must evaluate. Let $M_{ij}^{(2)}$ be the product corresponding to $T_{ij}$, i.e.

$$M_{11}^{(2)} = \quad a_{\boldsymbol{k}+\mathfrak{q}ns}a_{\boldsymbol{k}ms}(M-1)H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}})H_{M-2}(\chi_{\boldsymbol{k}}), \tag{16}$$

$$M_{12}^{(2)} = -t_x a_{\boldsymbol{k}+\mathfrak{q}ns}b_{\boldsymbol{k}ms}MH_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \tag{17}$$

$$M_{21}^{(2)} = -t_x b_{\boldsymbol{k}+\mathfrak{q}ns}a_{\boldsymbol{k}ms}(M-1)H_{N}(\chi_{\boldsymbol{k}+\mathfrak{q}})H_{M-2}(\chi_{\boldsymbol{k}}), \tag{18}$$

$$M_{22}^{(2)} = \quad b_{\boldsymbol{k}+\mathfrak{q}ns}b_{\boldsymbol{k}ms}MH_{N}(\chi_{\boldsymbol{k}+\mathfrak{q}})H_{M-1}(\chi_{\boldsymbol{k}}). \tag{19}$$

We want to evaluate

$$F_{ij}^{(2)} = \left[(\alpha_{k_z ms}^2+1)(\alpha_{k_z+\mathfrak{q}_z ns}^2+1)\right]^{\frac{1}{2}} \int \mathrm{d}\tilde{y} e^{-\tilde{y}^2} M_{ij}^{(2)}, \tag{20}$$

with the prefactor introduced for later convenience.

Notice that

**Verify $l_B$ in this section**

$$F_{12}^{(2)} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n). \tag{21}$$

and

$$F_{21}^{(2)} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a_{\boldsymbol{k}ms}^2}{l_B a_{\boldsymbol{k}m\mp 1s}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n, s). \tag{22}$$

$F_{11}^{(2)}$ and $F_{22}^{(2)}$ are the same as for the untilted case:

$$F_{11}^{(2)} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \tag{23}$$

and

$$F_{22}^{(2)} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m, n, s). \tag{24}$$

In summary we have

$$T^{0y\ (2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = +\frac{v_F}{4}\int \mathrm{d}y e^{iq_y q}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)p_y\phi_{\boldsymbol{k}ms}(y) \tag{25}$$

$$= -\frac{i\hbar v_F}{2}\Gamma^+_{\boldsymbol{k}\mathfrak{q}mns}\sum_{i,j}F_{ij}^{(2)}, \tag{26}$$

where

$$\Gamma^+_{\boldsymbol{k}\mathfrak{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x+q'_x)q_y + (q'_x)^2)\right]}{\left[(\alpha_{k_z ms}^2+1)(\alpha_{k_z+\mathfrak{q}_z ns}^2+1)\right]^{\frac{1}{2}}\sqrt{1-t_x^2}}$$

In a similar procedure, we find $T^{0y\ (2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})$.

$$\tilde{p}_y\phi^*_{\boldsymbol{k}+\mathfrak{q}ms} = \frac{-i\hbar\sqrt{\alpha}}{l_B}e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{\boldsymbol{k}+\mathfrak{q}ms}(M-1)H_{M-2}(\chi) \\ b_{\boldsymbol{k}+\mathfrak{q}ms}(M)H_{M-1}(\chi)\end{pmatrix}. \tag{27}$$

And thus,

$$\left(\tilde{p}_y \phi^*_{\boldsymbol{k}+\mathsf{q}ns}(y)\right)\phi_{\boldsymbol{k}ms} = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} e^{-\frac{1}{2}\chi^2_{\boldsymbol{k}}-\frac{1}{2}\chi^2_{\boldsymbol{k}+\mathsf{q}}}$$
$$\Big[a_{\boldsymbol{k}+\mathsf{q}ns}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}$$
$$+ b_{\boldsymbol{k}+\mathsf{q}ns}NH_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{-t_x a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}\Big]. \quad (28)$$

With the now well-known completion of the square and substitution, we have

$$\int \mathrm{d}y e^{iq_y}\left[\tilde{p}_y \phi^*_{\boldsymbol{k}+\mathsf{q}ns}(y)\right]\phi_{\boldsymbol{k}ms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}}\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]$$
$$\int \mathrm{d}\tilde{y}\frac{l_B}{\sqrt{\alpha}}$$
$$\Big[a_{\boldsymbol{k}+\mathsf{q}ns}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}$$
$$+ b_{\boldsymbol{k}+\mathsf{q}ns}NH_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{-t_x a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}\Big]. \quad (29)$$

Denote the terms of the integrand by

$$M^{(3)}_{11} = a_{\boldsymbol{k}+\mathsf{q}ns}a_{\boldsymbol{k}ms}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \quad (30)$$
$$M^{(3)}_{12} = -t_x a_{\boldsymbol{k}+\mathsf{q}ns}b_{\boldsymbol{k}ms}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathsf{q}})H_M(\chi_{\boldsymbol{k}}), \quad (31)$$
$$M^{(3)}_{21} = -t_x b_{\boldsymbol{k}+\mathsf{q}ns}a_{\boldsymbol{k}ms}NH_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \quad (32)$$
$$M^{(3)}_{22} = b_{\boldsymbol{k}+\mathsf{q}ns}b_{\boldsymbol{k}ms}NH_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})H_M(\chi_{\boldsymbol{k}}). \quad (33)$$

We must evaluate

$$F^{(3)}_{ij} = \left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathsf{q}_z ns}+1)\right]^{\frac{1}{2}}\int \mathrm{d}\tilde{y}e^{-\tilde{y}^2}M^{(3)}_{ij}. \quad (34)$$

From the untilted case we know

$$F^{(3)}_{11} = \sqrt{\frac{N-1}{2}}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathsf{q}_z ns}}{l_B\alpha_{k_z+\mathsf{q}_z n\mp 1 s}}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \quad (35)$$
$$F^{(3)}_{22} = \sqrt{\frac{N}{2}}\frac{1}{l_B\alpha_{k_z+\mathsf{q}_z ns}}\Xi_2(\bar{\boldsymbol{q}}, m, n, s). \quad (36)$$

Furthermore,

$$F^{(3)}_{12} = -t_x\frac{\alpha_{k_z+\mathsf{q}_z n}}{\alpha_{k_z+\mathsf{q}_z n\mp 1}l_B}\sqrt{\frac{N-1}{2}}\Xi_2(\bar{\boldsymbol{q}}, m, n\mp 1, s), \quad (37)$$
$$F^{(3)}_{21} = -\frac{t_x}{l_B}\sqrt{\frac{N}{2}}\frac{\alpha_{k_z m}}{\alpha_{k_z+\mathsf{q}_z n}}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n, s). \quad (38)$$

We thus have

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{v_F}{4}\int \mathrm{d}y e^{iq_y y}\left(p_y \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\right)\phi_{\boldsymbol{k}ms}(y) \tag{39}$$

$$= \frac{i\hbar v_F}{2}\Gamma^+_{\boldsymbol{k}\mathfrak{q}mns}\sum_{ij} F^{(3)}_{ij}. \tag{40}$$

**Summary 4**

*The non-canonical part of the energy-momentum tensor $T^{\mu\nu}_F = T^{\nu\mu}$ in a tilted system have the matrix elements*

$$T^{0y\ (2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{i\hbar v_F}{2}\Gamma^+_{\boldsymbol{k}\mathfrak{q}mns}\sum_{i,j} F^{(2)}_{ij}, \tag{41}$$

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{i\hbar v_F}{2}\Gamma^+_{\boldsymbol{k}\mathfrak{q}mns}\sum_{ij} F^{(3)}_{ij}. \tag{42}$$

*with*

$$\Gamma^{\pm}_{\boldsymbol{k}\mathfrak{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm iq_y l_B^2(k_x' + \frac{q_x'}{2})\right)\right]}{\left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathfrak{q}_z ns}+1)\right]^{\frac{1}{2}}}$$

*and where the factors $F^{(n)}_{ij}$ where found to be*

$$F^{(2)}_{12} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{\boldsymbol{q}}, m \mp 1, n), \tag{43}$$

$$F^{(2)}_{21} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a^2_{\boldsymbol{k}ms}}{l_B a_{\boldsymbol{k}m\mp 1s}}\Xi_1(\bar{\boldsymbol{q}}, m \mp 1, n, s), \tag{44}$$

$$F^{(2)}_{11} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m \mp 1, n \mp 1, s), \tag{45}$$

$$F^{(2)}_{22} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m, n, s), \tag{46}$$

$$F^{(3)}_{11} = \sqrt{\frac{N-1}{2}}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}}{l_B\alpha_{k_z+\mathfrak{q}_z n\mp 1s}}\Xi_2(\bar{\boldsymbol{q}}, m \mp 1, n \mp 1, s), \tag{47}$$

$$F^{(3)}_{22} = \sqrt{\frac{N}{2}}\frac{1}{l_B\alpha_{k_z+\mathfrak{q}_z ns}}\Xi_2(\bar{\boldsymbol{q}}, m, n, s), \tag{48}$$

$$F^{(3)}_{12} = -t_x\frac{\alpha_{k_z+\mathfrak{q}_z n}}{\alpha_{k_z+\mathfrak{q}_z n\mp 1}l_B}\sqrt{\frac{N-1}{2}}\Xi_2(\bar{\boldsymbol{q}}, m, n \mp 1, s), \tag{49}$$

$$F^{(3)}_{21} = -\frac{t_x}{l_B}\sqrt{\frac{N}{2}}\frac{\alpha_{k_z m}}{\alpha_{k_z+\mathfrak{q}_z n}}\Xi_2(\bar{\boldsymbol{q}}, m \mp 1, n, s). \tag{50}$$

# Conformal symmetry of a tilted system

The origin of the term *conformal anomaly* is the *conformal symmetry.* Under the conformal transformation, the massless QED (quantum electrodynamics) Lagrangian is invariant, as shown in the main text. Specifically, the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi,$$

with the usual $\bar{\psi} = \psi^\dagger\gamma^0, \not{D} = \gamma^\mu D_\mu, D_\mu = \partial_{\mu - ieA_\mu}$ transforms under the scaling

$$x \to \lambda^{-1},\ A_\mu \to \lambda A_\mu,\ \psi \to \lambda^{\frac{3}{2}}\psi,$$

as

$$\mathcal{L} \to \lambda^4\mathcal{L}.$$

The action $S = \int \mathrm{d}^4x\mathcal{L}$ is thus invariant (as $\mathrm{d}^4x \to \lambda^{-4}\mathrm{d}^4x$), and the theory is classically manifestly scale invariant.

Consider now the tilted Dirac Lagrangian considered in our work,

$$\mathcal{L}ki\bar{\psi}\Gamma^\mu\partial_\mu\psi, \tag{1}$$

with $\Gamma^\mu = \gamma^\mu + t^\mu\gamma_P\gamma^0$, where $\gamma_P = I_4$ when inversion summery is broken and $\gamma_P = \gamma^5$ for inversion symmetric systems. The tilt parameter $t^\mu = (0, \boldsymbol{t})$ is invariant under scaling, and thus also this theory is classically scale invariant.

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