Contents

1. Topological materials

In this chapter, we consider various concepts from physics that are relevant in the context of topological materials. Firstly, the symmetry related concepts of parity, time reversal, Kramer's degeneracy, and accidental degeneracy are explained. Then, the concept of linear dispersion in Weyl and Dirac cones is discussed, along with some useful results. Lastly follows a quick summary of spin-orbit interactions. The chapter is intended as a quick introduction to the vast field of topological materials for someone who are not familiar with these concepts.

Some topics discussed are directly applicable to the thesis, while others are included both in order to put the concepts of the thesis in a greater context, and also as it might be useful to know some of them in further continuation of this thesis.

1.1. Parity

We consider now the discrete transformation of space inversion, or *parity*. Firstly, basic properties of the transformation will be presented and discussed. Its effect on the position, momentum, and angular momentum operators will be discussed, before a more general discussion on how it transforms proper- and pseudo-tensors. This will be applied to see how the parity transformation affects electric and magnetic fields.

Let the parity operator P be a unitary operator

$$P: \quad |a\rangle \to P\,|a\rangle\,. \tag{1.1}$$

By definition, we require

$$P^\dagger x P = -x, \tag{1.2}$$

$$P^\dagger p P = -p, \tag{1.3}$$

where x, p are the position and momentum operators. By the unitarity of P, which means that $P^\dagger P = I$,

$$xP = -Px.$$

We now use this anticommutation to find an explicit form of the transformation in the position representation. By noting that, given the position eigenstate $|x_1\rangle$,

$$xP\,|x_1\rangle = -Px_1\,|x_1\rangle = -x_1P\,|x_1\rangle\,, \tag{1.4}$$

with x_1 the eigenvalue of the state, we may conclude

$$P\,|x_1\rangle = |-x_1\rangle$$

up to some arbitrary phase. We chose this phase to be unity. Then

$$P^2\,|x_1\rangle = |x_1\rangle \tag{1.5}$$

for any position eigenstate, which gives the operator relation $P^2 = 1 \implies P = \pm 1$. This also means that P is Hermitian,

$$P = P^{-1} = P^{\dagger}.$$

The treatment of angular momentum is somewhat more involved. Some sources simply state that as the orbital angular momentum

$$L = x \times p$$

is a product of two odd quantities, it must be even under parity. This, of course, is a gross over simplification, as extra care must be taken when considering the spin angular momentum S contributing to the total angular momentum

$$J = L + S.$$

The angular momentum operator is the generator of rotations

$$R = e^{-i\epsilon J\cdot n} \approx 1 - i\epsilon J \cdot n$$

where we expanded the operator under the assumption of a small angle, $\epsilon \ll 1$. As rotations are invariant during space inversion,

$$P^{\dagger}RP = R$$
$$P^{\dagger}J \cdot nP = J \cdot n$$

from which it follows that

$$P^{\dagger}JP = J,$$

as the parity operator obviously does not act on the normal vector n. Thus, the angular momentum operator, unlike the linear momentum operator, is even under parity.

For a general vector-like quantity V, we will consider how it transforms during space inversion. If the quantity "flips" during space inversion, $P^\dagger V P = -V$, we say simply that it is a vector, also sometimes known as a polar vector. Quantities that do not "flip", so that they turn into their opposites in the flipped image, we denote pseudo vectors. Thus, depending on whether the eigenvalue of an operator under space inversion is $+1$ or -1 we say that it is either a pseudo-vector or vector, respectively. Position and momentum are examples of vectors, while angular momentum and the magnetic field are examples of pseudo-vectors. An illustrative explanation of this is shown in Figure 1.1, which explains both angular momentum and magnetic fields.

Remark about dimensionality: The above discussion about parity, which is the standard way to present parity in condensed matter physics, is valid for three dimensions. In two dimensions, however, one must separate *parity* and *space inversion*. The former takes a right-handed system to a left-handed system [24], while the latter inverts space, $\boldsymbol{x} \to -\boldsymbol{x}$. In odd dimensions this is the same, while in even dimensions they differ. In even dimensions, inversion corresponds to a rotation, while a parity transform is different from any rotation. In more formal terms, inversion is part of the group of proper rotations $SO(n)$ for even dimensions, as the determinant is $+1$, the definition of a proper rotation. Parity should in general be taken to be the operation P such that the group of all rotatinos $O(n) = SO(n) \times \{E, P\}$, with E the identity transformation. This will not be of importance here, but it is an important detail to note.

1.2. Time reversal

We will now consider the time reversal operator Θ. Firstly we will show that it must be antiunitary, then we will show $\Theta^2 = \pm 1$, and find a more specific form of Θ for half-integer spin systems.

The time reversal operator by definition will invert the value of the time

$$\Theta : t \to -t$$

while leaving space unchanged. The invariance of space is summaries by the operator relation,

$$\Theta x \Theta^{-1} = x, \tag{1.6}$$

where x is understood as the position operator. The momentum operator,

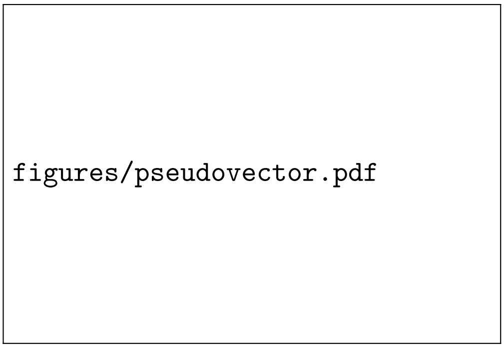


Figure 1.1.: Schematic illustration of vectors and pseudovectors. A vector field with curl, which may be taken to be either momentum or current, is shown as a rotating arrow. The curl of this field, which will respectively be the angular momentum or B-field, is shown as a straight arrow. Under inversion, shown as a mirror operation, the curl generated by the field is inverted in addition to the mirroring, i.e. rotated. This non-formal illustration gives an intuitive explanation of the concepts vector and pseudovector. Note that as the example is two-dimensional, mirror symmetry here the same as parity, and not inversion. See main text for details.

however, is flipped due to its time dependence

$$\Theta p \Theta^{-1} = -p. \tag{1.7}$$

A schematic representation of inversion symmetry and time reversal symmetry is given in Figure 1.2.

We are now in a position to show that Θ must be antiunitary by requiring the invariance of the commutation relation between momentum and position, $[x, p] = i\hbar$.

$$\Theta[x, p]\Theta^{-1} = \Theta i\hbar \Theta^{-1} = -[x, p] = -i\hbar. \tag{1.8}$$

In the first equality, the commutation relation was used directly. In the second equality, Eqs. (1.6) and (1.7) were used to gain a minus sign. This all leads to the relation

$$\Theta i \Theta^{-1} = -i. \tag{1.9}$$

From this, we gather that the time reversal operator must be antiunitary. An antiunitary transformation is a transformation

$$|a\rangle \to |\tilde{a}\rangle = \theta\,|a\rangle\,, \quad |b\rangle \to |\tilde{b}\rangle = \theta\,|b\rangle\,,$$

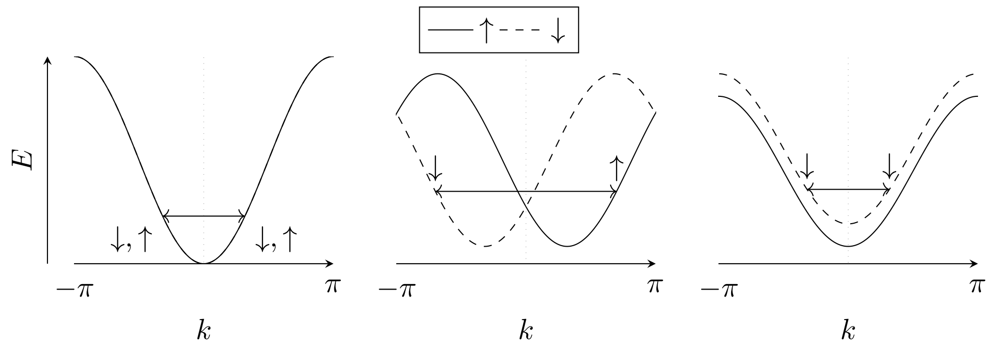


Figure 1.2.: Schematic illustration of time and inversion breaking of degenerate energy bands of a two-level system. The two levels are denoted ↑ and ↓. **(Left:)** Both time-reversal and inversion symmetry present, with the two energy bands being degenerate at all momenta. **(Center:)** Inversion symmetry is broken. Notice how at the TRIM (time reversal independent momenta) points, $-\pi, 0, \pi$, the two energy levels are degenerate, as, by definition, we have $\boldsymbol{k} = -\boldsymbol{k}$. **(Right:)** Time reversal symmetry is broken. Notice how in the time reversal symmetric case Kramer's doublet is present, as for any state at k, the state at $-k$ is degenerate in energy and has opposite spin. This is not the case in the case where time reversal is broken, as the spin at $-k$ has the same spin. Figure inspired by Ramazashvili [22].

such that

$$\langle \tilde{b} | \tilde{a} \rangle = \langle b | a \rangle^* , \tag{1.10}$$

$$\theta \left(c_1 \left| a \right\rangle + c_2 \left| b \right\rangle \right) = c_1^* \theta \left| a \right\rangle + c_2^* \theta \left| b \right\rangle . \tag{1.11}$$

A note of caution: the Dirac bra-ket notation was originally designed to handle linear operators, where it excels. For anti-linear operators, which antiunitary operators are, the bra-ket notation can be deceiving. We will always take anti-linear operators to work on kets, never on bras from the right. So, for example,

$$\langle a | O | b \rangle$$

should be understood as

$$\langle a | \left(| O | b \rangle \right)$$

and *never*

$$\left(\langle a | O | \right) | b \rangle .$$

The left operation of an anti-linear operator on a bra, $\langle a| O$, will not be defined.

We will in general write

$$\Theta = UK \tag{1.12}$$

where U is a unitary transformation and K is the complex conjugation. Now, we will show that $\Theta^2 = \pm 1$, by an elegant method inspired by Bernevig and Hughes [3]. Consider

$$\Theta^2 = UKUK = UU^* = U(U^T)^{-1} \equiv \phi, \tag{1.13}$$

where we in the second last equality used the unitarity of U. As applying the time reversal operator twice must result in the original state, up to some phase, ϕ must surely be diagonal. From Eq. (1.13) it follows

$$U = \phi U^T, \quad U^T = U\phi \tag{1.14}$$

where the fact that $\phi^T = \phi$ for any diagonal matrix was used. From this follows that

$$U = \phi U \phi \Rightarrow U\phi^{-1} = \phi U. \tag{1.15}$$

This holds in general only for $\phi = \pm 1$, and thus $\Theta^2 = \pm 1$. Furthermore, we will later show that for integer spin particles $\Theta^2 = 1$ while for half-integer spin particles $\Theta^2 = -1$.

1.2.1. Time reversal operator on spinful particles

When considering spinful particles, we must enforce yet another property on the time reversal operator. As spin is odd under time reversal one must have

$$\Theta S \Theta^{-1} = -S. \tag{1.16}$$

Consider now specifically a spin-s state, with the basis $|s, m\rangle$, being an eigenstate of $S_z, \boldsymbol{S}^2$, with eigenvalues $m\hbar, s(s+1)\hbar^2$ respectively. By Eq. (1.16) it follows that $\Theta |s, m\rangle$ is also an eigenstate of S_z, with eigenvalue $-m\hbar$, since

$$S_z \Theta |s, m\rangle = -\Theta S_z |s, m\rangle = -m\hbar \Theta |s, m\rangle . \tag{1.17}$$

Let

$$\Theta |s, m\rangle = \eta |s, -m\rangle ,$$

where η is some phase. Consider now the commutation of the ladder operators $J_\pm = S_x \pm iS_y$ with the time reversal operator.

$$\begin{aligned} \underbrace{[S_x \pm iS_y]}_{S_\pm} \Theta &= -\Theta S_x \mp i\Theta S_y \\ &= -\Theta \underbrace{[S_x \mp iS_y]}_{S_\mp}, \end{aligned} \tag{1.18}$$

where the anti-linearity of Θ is emphasized. Thus, operating with S_+ on $\Theta|s,m\rangle$ gives

$$S_+\Theta|s,m\rangle = \eta_{sm}S_+|s,-m\rangle \tag{1.19}$$
$$= \eta_{sm}\hbar\sqrt{(s+m)(s-m+1)}\,|s,-m+1\rangle\,. \tag{1.20}$$

On the other hand, commuting the two operators first gives

$$S_+\Theta|s,m\rangle = -\Theta S_-|s,m\rangle \tag{1.21}$$
$$= -\Theta\hbar\sqrt{(s+m)(s-m+1)}\,|s,m-1\rangle \tag{1.22}$$
$$= -\hbar\sqrt{(s+m)(s-m+1)}\eta_{s,m-1}\,|s,-m+1\rangle\,. \tag{1.23}$$

By comparison, $\eta_{sm} = -\eta_{s,m-1}$; η_{sm} has a flip of its sign under increments of m. The m dependence should therefore be $(-1)^m$. For later convenience, we will choose to also include an s-term in the exponent, so that the exponent is integer also for half-integer systems, resulting in

$$\eta_{sm} = (-1)^{s-m}f(s), \tag{1.24}$$

where $f(s)$ is some phase that does not depend on m. We are now in a position where we may find Θ^2, by acting on a general spin s system.

$$\Theta^2\sum_{m=-s}^{s} a_m|s,m\rangle = \Theta\sum_m a_m^* f(s)(-1)^{s-m}|s,-m\rangle \tag{1.25}$$
$$= \sum_m a_m f^*(s)(-1)^{s-m}\Theta|s,-m\rangle \tag{1.26}$$
$$= \sum_m a_m|f(s)|^2(-1)^{2s}|s,m\rangle\,. \tag{1.27}$$

Note that it was important that $(-1)^{s-m}$ was real, which is taken care of by the s-term. As $f(s)$ is only a phase, this gives

$$\Theta^2 = (-1)^{2s}, \tag{1.28}$$

for any spin s system. Thus, for half integer spin, $\frac{1}{2}, \frac{3}{2}, \ldots$, $\Theta^2 = -1$, while for integer spin $\Theta^2 = +1$.

1.3. Kramer's degeneracy

Kramer's degeneracy states that for any half-integer system that is time-reversal symmetric, energy levels are at least two-fold degenerate. The proof of this is simple, and uses the fact that for any half-integer spin system, $\Theta^2 = -1$. A heuristic way to see this is the fact that spin is odd under time-reversal, and for half-integer systems there is no zero-spin state, so reversing the spin cannot result in the same state.

Proof Proof: Assume

$$[H, \Theta] = 0$$

and that $|n\rangle$ is an eigenstate of the system

$$H|n\rangle = E_n |n\rangle .$$

Then

$$H\Theta|n\rangle = \Theta H|n\rangle = \Theta E_n |n\rangle = E_n \Theta |n\rangle$$

and so $\Theta|n\rangle$ is also an eigenstate with the eigenvalue E_n. To assert that the eigenvalue is in fact degenerate, one must also show that the two states are not the same ray. That is $\Theta|n\rangle \neq e^{i\delta}|n\rangle$, where δ is some phase. Suppose that the above is *not* true, $\Theta|n\rangle = e^{i\delta}|n\rangle$. Then,

$$\Theta^2|n\rangle = \Theta e^{i\delta}|n\rangle = e^{-i\delta}\Theta|n\rangle = +|n\rangle .$$

However, as was stated above, $\Theta^2 = -1$ for all half-integer systems. The assumption must therefore be wrong, and the eigenvalue is degenerate.

The two states, $|n\rangle$ and $\Theta|n\rangle$, are often referred to as Kramer's doublet. Note that the two states have opposite spin.

1.3.1. Generalization to the $P\Theta$ operator

Consider now a time reversal and parity symmetric system, $[H, P\Theta] = 0$. This will, similarly to the case for time reversal, make the energy levels at least two-fold degenerate.

Proof Proof: Assume

$$[H, P\Theta] = 0$$

and that $|n\rangle$ is an eigenstate of the system

$$H|n\rangle = E_n |n\rangle .$$

Then

$$HP\Theta|n\rangle = P\Theta H|n\rangle = P\Theta E_n |n\rangle = E_n P\Theta |n\rangle .$$

Assume now that $P\Theta|n\rangle = e^{i\delta}|n\rangle$, which we will prove to be false. That would lead to

$$(P\Theta)^2|n\rangle = P\Theta e^{i\delta}|n\rangle = |n\rangle .$$

However, as $[P, \Theta] = 0$, we have

$$(P\Theta)^2 = P\Theta P\Theta = P\Theta^2 P = -1$$

as $P^2 = 1$. As above, the states are thus different, and the eigenvalue is degenerate.

1.4. Accidental degeneracy

In general, for a two level system depending on some parameter the energy levels of the two levels will not cross, i.e. be degenerate, unless there are symmetries in the system forcing them to be degenerate, as is the case in for example Kramer's degeneracy. However, even without any symmetries [1] there will be so-called *accidental degeneracies* if the parameter space is sufficiently large. Consider a general two-level Hamiltonian

$$H = f_1\sigma_x + f_2\sigma_y + f_3\sigma_z, \tag{1.29}$$

which will have an energy splitting between the two levels

$$\Delta E = 2\sqrt{f_1^2 + f_2^2 + f_3^2}. \tag{1.30}$$

In general, we may solve $\Delta E = 0$ by tuning the three parameters simultaneously, and thus there must be degenerate points – accidental degeneracies. Supposing that the parameters f_i can be expressed as functions of the momentum components, $f_i = f_i(p_i)$, this will correspond to degenerate points in momentum space.

If there are in addition some symmetry constraints on the system, the space of degenerate points may increase. Suppose, for example, the system is time reversal symmetric. Recalling the time reversal operator defined in Eq. (1.12)

$$\Theta = UK,$$

with U being a unitary operator and K the complex conjugate, the imaginary Pauli matrix σ_y must be excluded. Thus, the solution to the closing of the band gap has a free parameter, and the degenerate space has dimension one.

1.5. Weyl and Dirac cones in condensed matter physics

Dirac and Weyl cones are the emergence of non-gapped linear energy bands in condensed matter physics, in effect exhibiting relativistic behavior at non-relativistic speeds. We here give a very brief introduction to these materials. Firstly, we will consider the so-called band crossing, and how the opening of a gap at the band crossing behaves differently in two and three dimensions. Then, various perturbations that do not open a gap will be considered, giving

[1] There will always, for a degenerate system, be some symmetry, although it might be a *hidden* symmetry. We here mean no a priori apparent symmetry.

interesting effects in the dispersion relations. Lastly, a consideration of these materials in light of Berry curvature and the topological quantity of Chern numbers will be given.

While the nearly free quasi-particle model performs very well for most metals, with the Hamiltonian $p^2/(2m^*)$, with m^* some effective mass, this model fails for the Dirac-materials. Instead of obeying the Schrödinger equation as most materials, they obey a Dirac equation, with the speed of light being replaced by the Fermi velocity v_F. As in the high energy case, the Dirac equation may be decomposed into chiral Weyl equations in the massless case. Setting $v_F = 1$ for simplicity one gets the Hamiltonian

$$H_D = s v_F \boldsymbol{\sigma} \boldsymbol{p}, \tag{1.31}$$

where $\boldsymbol{\sigma}$ are the Pauli matricies, v_F the Fermi velocity, $\boldsymbol{p}$ the momentum, and $s = \pm 1$ denotes the chirality. It is here important to note that the Pauli matrices represent either real spin degree of freedom or some pseudo spin degree of freedom. Examples of pseudo spin is that of bipartite lattices, such as Graphene, in which case one must be careful when for example applying time reversal, as only real spin is odd under this operation, and not pseudo spin.

The dispersion of the Hamiltonian (1.31) has a band crossing at $\boldsymbol{p} = 0$. For the two-dimensional case, a perturbation on the form $m\sigma_z$, with m some parameter, can open up a gap in the dispersion relation. This is easily verified by writing out the Hamiltonian and solving the eigenproblem

$$H_D^{(2D)} = s v_F (p_x \sigma_x + p_y \sigma_y) + m\sigma_z. \tag{1.32}$$

$$\left| H_D^{(2D)} - E \right| = 0. \tag{1.33}$$

As the Hamiltonian commutes with the momentum operator, we replace the momentum operator with its eigenvalues

$$E = \pm v_F \hbar \sqrt{k_x^2 + k_y^2 + \frac{m^2}{\hbar^2 v_F^2}}. \tag{1.34}$$

There are no solutions k_x, k_y making the energy levels degenerate. The crossing is thus only protected by symmetry considerations, and is not *topologically protected*.

In three dimensions the situation is somewhat different, with the Hamiltonian

$$H_D^{(3D)} = s v_F (p_x \sigma_x + p_y \sigma_y + p_z \sigma_z). \tag{1.35}$$

In this case, no perturbing term may open a gap at the crossing. There is no 2×2 matrix σ_4 that anticommutes with the Pauli matricies and also is linearly independent, i.e. there is no "fourth" Pauli matrix, and thus no perturbative term will open the gap. Say for example we add a term like $m\sigma_z$, where the z-direction was chosen arbitrarily. The only effect this will have on the crossing is to translate it in p_z. Tying this back to the accidental degeneracy, we see that no matter the perturbation, the three-dimensional momentum space will always have a point of degeneracy, i.e., a crossing. The crossing is *topologically protected*. A more formal approach to topological materials, is that of topological invariants – numbers related to the topology of the material. Having a non-trivial topological invariant number, is the very definition of topological materials, and we will in subsection 1.5.1 show that Dirac cones makes the Chern number of these materials non-trivial.

The Hamiltonian in Eq. (1.31) is not the most general, if we allow for anisotropy in the system. In three dimensions we have more generally the Hamiltonian

$$H(\boldsymbol{k}) = \boldsymbol{v}_0\boldsymbol{k} + (\boldsymbol{v} \odot \boldsymbol{k})\boldsymbol{\sigma}, \tag{1.36}$$

where $\boldsymbol{v}_0$ is the *tilt vector*, $\boldsymbol{v}$ is some, anisotropic velocity, $(\boldsymbol{v} \odot \boldsymbol{k})_i = v_i k_i$ is the Hadamard product of the anisotropic velocity and the momentum, and $\boldsymbol{\sigma}$ are the Pauli matrices corresponding to spin degree of freedom. Here we will consider two interesting cases. Firstly, we will consider perturbations in the tilt-less isotropic case, $\boldsymbol{v}_0 = 0, \boldsymbol{v}_i = v_F\hat{x}_i$. Then, a tilted system without perturbations is considered.

Consider an isotropic tilt-less system; introduce to the system a pseudospin degree of freedom, thus extending the system to 4×4-matricies. The Hamiltonian of the system [2]

$$H = v_F\tau_x \otimes \boldsymbol{\sigma k} + m\tau_z \otimes I_2 + bI_2 \otimes \sigma_z + b'\tau_z \otimes \sigma_x, \tag{1.37}$$

with $\boldsymbol{\tau}$ the Pauli matricies related to the pseudospin, and I_2 the identity matrix of dimension 2. The perturbing parameters m, b, b' are a mass parameter, and Zeeman fields in the z and x direction, respectively. Ignore for now b', i.e. $b' = 0$, which is related to a state known as the line node semimetal. Notice that the b term breaks time reversal symmetry in the system, as the real spin σ is odd under time reversal. The eigenvalues of this system [2]

$$E_{s\mu}(\boldsymbol{k}) = s\left[m^2 + b^2 + v_F^2k^2 + 2\mu b\sqrt{v_F^2k_z^2 + m^2}\right]^{\frac{1}{2}}, \tag{1.38}$$

with $s = \pm1, \mu = \pm1$ encoding the degeneracies related to the spin and pseudospin degrees of freedom, respectively. There are still linear dispersions for

$b > m$. For $b < m$, a gap opens, and the dispersion is non-linear. In fact, this is simply a shift in k_z of the Dirac cone, as is seen by rewriting

$$E_{s\mu}(\boldsymbol{k}) = sv_F \left[k_x^2 + k_y^2 + \left(\sqrt{k_z^2 + \frac{m^2}{v_F^2}} + \mu \frac{b}{v_F} \right)^2 \right]^{\frac{1}{2}}. \tag{1.39}$$

This still has Weyl node solutions at $k_z^2 = (b^2 - m^2)/v_F^2$, where the dispersion is linear in the vicinity of the nodal solutions. This thus separates two Dirac nodes in momentum space, giving a *Weyl* semimetal. This also illustrates that the decomposition in Eq. (1.31) is valid around either of the shifted nodes. Expanding around one of the Dirac points of the Weyl semimetal, the Hamiltonian is exactly Eq. (1.31), after decomposing the 4×4 Hamiltonian into its two chiral 2×2 Weyl constituents.

If one instead perturbs the system with a Zeeman field in the x-direction, i.e. having a $b' > 0$, the splitting is instead in energy, giving nodal loop where the two cones intersect. We will not go into any depth on these types of materials.

Possibly rewrite the following sentence

The three cases described here: unperturbed, where the two cones are superimposed; perturbed by b, where the cones are separated in momentum; and perturbed by b', where the cones are separated in energy, are shown in Figure 1.3. Notice that in the two latter cases, the Dirac points, i.e. crossings, are not superimposed. As will be discussed in section 1.5, this makes the crossings very robust, as the two nodes must merge before a gap may be opened.

The second case to consider, is a finite tilting vector $\boldsymbol{v}_0$, where we will consider only real spin, thus reducing the system back to the two-dimensional case in Eq. (1.36). For the isotropic case, $\boldsymbol{v}_i = v_F \hat{x}_i$, the energy bands are [21]

$$E_s(\boldsymbol{k}) = \boldsymbol{v}_0 \boldsymbol{k} + sv_F |k|. \tag{1.40}$$

Tehse types of systems, which are the systems of interest for this thesis, are considered in detail in section 1.6.

1.5.1. Chern number of the Weyl point

In order to more explicitly demonstrate the topological nature of the state in Eq. (1.31), we will find a non-zero topological invariant associated with that state. Thereby showing that the material is a topological material. The topological number we will calculate is the Chern number, related to the Berry curvature of the bands in some enclosed surface. In order to calculate the Chern number, we must first find an expression for the Berry curvature of our system. This figures/cones-types-col1-opaque.png

Figure 1.3.: Dispersion curves in the k_z, k_x-plane. **(Left)** Dirac material with superimposed cones. **(Center)** Time reversal symmetry broken, giving a Weyl material with the cones separated in momentum space. **(Right)** The cones shifted in energy, givng a nodal loop.

figures/cones-tilt-color1.png

Figure 1.4.: Tilted Dirac cones. From left to right the tilt increases, from no tilt in the first cone to overtilt in the last. The three first are Type-I Weyl semimetals, the last is a Type-II semimetal. See main text for details.

derivation will follow closely Berry's original derivation [4] of the Berry phase of a two-level system with the Hamiltonian

$$H(\boldsymbol{R}) = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{R}. \tag{1.41}$$

Some notation has been modernized with inspiration from the treatment of the Berry phase of the spin-1/2 particle in an external magnetic field in Holstein [13].

Suppose we have a Hamiltonian $H(t)$, and that its t-dependence can be parameterized by $\boldsymbol{R} = \boldsymbol{R}(t)$, as in $H(t) = H(\boldsymbol{R}(t))$. Any evolution of the Hamiltonian through time, may then be described as a geometric path through the $\boldsymbol{R}$-space. As the reader might be aware, Berry's most famous discovery was that a closed path through $\boldsymbol{R}$-space gives an observable phase to the system, unlike the non-physical dynamical phase, which may be removed by a suitable choice of gauge. Here we will however focus on the so-called Berry curvature, $\boldsymbol{B}$, a vector field which will be shown to be useful in the categorization of topological materials. Note that there is some variation in the literature on the naming of the various quantities, and the sign convention used. In particular, the word Berry curvature will in some literature refer to a rank two tensor, while our quantity $\boldsymbol{B}$ is referred to as the Berry field strength. In particular, if we let the rank two tensor be denoted F_{ij}, the Berry field strength $\boldsymbol{B}$ is given by

$$B_i = \epsilon_{ijk}F_{jk}. \tag{1.42}$$

Consider rewriting some of this. Look in topology book

The Berry curvature for the state n is explicitly defined as [4]

Should we add some more comments about adabiatic? See topo book

$$\boldsymbol{B_n}(\boldsymbol{R}) = -Im\sum_{m\neq n}\frac{\langle n(\boldsymbol{R})|\nabla_{\boldsymbol{R}}H|m(\boldsymbol{R})\rangle \times \langle m(\boldsymbol{R})|\nabla_{\boldsymbol{R}}H|n(\boldsymbol{R})\rangle}{(E_m(\boldsymbol{R}) - E_n(\boldsymbol{R}))^2}, \tag{1.43}$$

where $\times$ denotes the cross product. Notice that for a degeneracy $E_n = E_m$ there will be an infinity in $\boldsymbol{B_n}$. Considering the Berry curvature as a field in $\boldsymbol{R}$-space, this resembles a source, as will become relevant later. This may now be applied to for example the Weyl semimetal, both in the interest of solidifying the above theory, and as it will be useful in future consideration.

The Hamiltonian around the Weyl point is

$$H = v_F(\boldsymbol{\sigma} + \boldsymbol{t}) \cdot \boldsymbol{p}, \tag{1.44}$$

with v_F the Fermi velocity, $\boldsymbol{\sigma}$ the Pauli matrices, and $\boldsymbol{p}$ the momentum operator. By letting $\boldsymbol{R} = v_F\boldsymbol{p}$, the Berry curvature of the Hamiltonian can be found. The eigenvalues of this system are

$$E_+ - \boldsymbol{R}\cdot\boldsymbol{t} = -E_- - \boldsymbol{R}\cdot\boldsymbol{t} = |R|. \tag{1.45}$$

The aforementioned degeneracy is here of course the Weyl point, where $E_+ = E_- = 0$. Noting that

$$\nabla_{\boldsymbol{R}} H = \boldsymbol{\sigma} + v_F\boldsymbol{t}, \tag{1.46}$$

we can calculate the Berry curvature easily. Denote by $|+\rangle$ the state with the eigenvalue E_+ and $|-\rangle$ the state with the eigenvalue E_-. Take also, without loss of generality, $\boldsymbol{R}$ to be in the z-direction. This gives

$$\boldsymbol{B}_+ = -Im\frac{\langle +|\boldsymbol{\sigma} + v_F\boldsymbol{t}|-\rangle \times \langle -|\boldsymbol{\sigma} + v_F\boldsymbol{t}|+\rangle}{4R^2}. \tag{1.47}$$

As $|+\rangle$ and $|-\rangle$ are eigenstates of σ_z and orthogonal to each other, only the z-component of the cross product may contain non-zero contributions. Again, as the states are orthogonal, the $\boldsymbol{t}$ terms obviously disappears.

$$\begin{aligned}\boldsymbol{B}_+ &= -\frac{\hat{z}}{4R^2} Im\left(\langle +|\sigma_x|-\rangle\langle -|\sigma_y|+\rangle - \langle +|\sigma_y|-\rangle\langle -|\sigma_x|+\rangle\right)\\ &= -\frac{\hat{z}}{2\boldsymbol{R}^2}.\end{aligned} \tag{1.48}$$

Here, the effect of the Pauli matrices on the eigenvectors was used, according to

$$\sigma_x |\pm\rangle = |\mp\rangle \tag{1.49}$$
$$\sigma_y |\pm\rangle = \pm i |\mp\rangle \tag{1.50}$$

Returning to general axis orientations, one has

$$B_+ = -\hat{R}/2\boldsymbol{R}^2 = -\boldsymbol{R}/2\boldsymbol{R}^3, \tag{1.51}$$

independent of $\boldsymbol{t}$. For the $|+\rangle$-band, the Weyl point thus takes the form of a negative monopole in R-space; this motivates the requirement that Weyl points must always appear in pairs of opposite chirality, as the divergence of the Berry curvature must always be zero over the entire sample.

There should probably be some care taken here with the sign of v_F.

As mentioned, the Chern number is one of several numbers that is used to classify topological materials. The Chern number is defined as

$$C = \frac{1}{2\pi}\oint_{\partial C} \boldsymbol{B}_+ \cdot \mathrm{d}\boldsymbol{S}, \tag{1.52}$$

where the integral is taken over the closed surface ∂C, enclosing the volume C. Noting that the Berry curvature has the shape of a monopole source at $\boldsymbol{p} = 0$, we immediately know the value of this quantity from electromagnetism. We will, however, carry out the computation explicitly here. With the divergence theorem in mind, it behooves us to find the divergence of the Berry curvature. This divergence is zero everywhere except in the monopole source, giving

$$\nabla \cdot \boldsymbol{B}_{+} = -\frac{1}{2}\nabla \cdot \hat{R}/R^2 = -2\pi\delta(\boldsymbol{p}), \tag{1.53}$$

where δ is the Dirac delta distribution. By virtue of the divergence theorem the Chern number is then found to be

$$C = \frac{1}{2\pi}\int_C \nabla \cdot \boldsymbol{B}_{+} \mathrm{d}C = -1, \tag{1.54}$$

where the property of integrals over Dirac delta distributions was used.

Note that some literature will have a Chern number differing from (1.54) by the sign of the Fermi velocity,

$$C = -\operatorname{sign}(v_F). \tag{1.55}$$

This simply comes from the definition of the eigenstates. We have put the sign dependence in the state, making the E_{+} state always have positive eigenenergy. In literature that instead defines $E_{+} = v_F|R|$ the state's energy will depend on the sign of the Fermi velocity, and as a consequence, the sign dependence will end up in the Chern number instead.

The overall divergence of Berry curvature must be zero, or equivalently, the sum of the Chern numbers must be zero. The Hamiltonian Eq. (1.41) chosen with the opposite chirality,

$$H(\boldsymbol{R}) = -\frac{1}{2}\boldsymbol{\sigma}\boldsymbol{R}, \tag{1.56}$$

has the opposite Berry curvature, and also the opposite Chern number. Thus, Dirac cones must appear in pairs of opposite chirality, either superimposed as the Dirac semimetal case or separated in momentum space, as the Weyl semimetal.

Make sure there is no discrepancy between 2D/3D materials above

In light of the interpretation of the Dirac point as a monopole of Berry curvature, the discussion at the beginning of section 1.5 on the stability of the band crossing in two and three dimensions gets an intuitive and geometric interpretation. In Figure 1.5 the Berry curvature pole is shown in p-space, together with a plane parallel to the xy-plane, which we will denote the *state plane*. In the two-dimensional case, the state is confined to the state plane, with the z-position of the plane given by any mass terms $m\sigma_z$. In the three-dimensional case, the state not confined to this plane, as the parameter p_z is a free variable, or alternatively it may be considered as a freedom to move the state plane freely, with its initial position simply shifted by any mass terms. It is thus obvious that one may never reach the monopole in the two-dimensional case, and thus for no $\boldsymbol{k}$ is there a band crossing. Importantly, the Berry curvature is indeed non-zero, however any closed curve of integration will give a Chern number of zero; the monopole has been moved outside the dimensionality of freedom.

`figures/berryCurvePlanewpath.png`

Figure 1.5.: The state plane, transparent yellow, parallel to the xy-plane and a Berry curvature monopole at the origin. An integration contour is shown in blue dashed. See main text for details.

1.6. Type II Weyl semimetals

The conic section problem with the intersecting plane restricted to pass through the node of the cone is trivially seen to have two solutions: a point and two intersecting lines. Despite this, the possibility of a Weyl cone tilted beyond the Fermi level was never considered before Soluyanov et al. described this new class of Weyl semimetals in 2015. This now seemingly obvious possibility made an already rich field even more exciting, opening up for a wider range of novel and interesting effects.

add some concrete examples or cites

Is this correct? Is a tilt at all possible in HEP?

In the case of massless fermions, the particle physics equivalent of the Weyl semimetal, such a tilt is not possible, due to the requirement of Lorentz invariance

add cite or explain

. In condensed matter physics, however, this is not an issue, and it is indeed a real class of materials

cite examples

. We denote these types of materials Type-II Weyl semimetals, as opposed to Type-I. The transition between Type-I and Type-II is abrupt – the Fermi surface goes from a single point to two intersecting lines, in other words going from a zero dimensional to a one dimensional surface.

Make sure this is indeed a one dimensional surface. It is kind of 1DxZ(2)

Make sure it is one dim also for the 3D case, quadric surface, not conic intersection

Type-II also has electron and particle pockets at the Fermi level. While the density of states for a Type-I semimetal goes to zero as one approaches the Fermi level, this causes Type-II to have a finite density of states at the Fermi level.

End with something like: all in all this gives type ii weyl semimetal manifestly different properties from tyep i, useful both in practical applications and as an interesting phenomena seen from a purely scientific perspective

1.6.1. Hamiltonian

We will firstly consider a slightly more realistic toy model for a Weyl semimetal, with a parameter taking the system from a Type-I to a Type-II. This is instructive both in order to more intuitively see the origin of the terms causing the tilting of the Dirac cone, and also to see how two Dirac cones in the same Brillouin zone tilt in relation to each other. We will then continue by linearizing the model around the Weyl points, regaining the familiar form of a Dirac cone, with an additional anisotropy term causing the tilt.

Using the general time-reversal breaking model described by McCormick, Kimchi, and Trivedi [19]

$$\begin{aligned} H(\boldsymbol{k}) = {} & [(\cos k_x + \cos k_z - 2)m + 2t(\cos k_x - \cos k_0)]\,\sigma_1 \\ & - 2t \sin k_y \sigma_2 - 2t \sin k_z \sigma_3 + \gamma(\cos k_x - \cos k_0). \end{aligned} \tag{1.57}$$

The model has Weyl nodes at $\boldsymbol{K}' = (\pm k_0, 0, 0)$, and the parameter γ controls the tilting of the emerging cones. A value of $\gamma = 0$ gives no tilt, while for $\gamma > |2t|$ the Type-II system emerges. Figure 1.7 shows the cross section $k_y = 0$ of the eigenvalues of this system, as γ is gradually increased from 0 to 0.15

verify numbers

. The γ-term "warps" the bands, and in the limit of Type-II the hole band crosses the Fermi level into positive energy, while the particle band crosses the Fermi level into negative energies. We call these hole and electron pockets, respectively.

Linearizing around the Weyl nodes reduces to the familiar expression of a Dirac cone

$$H(\boldsymbol{K}'^{\pm} + \boldsymbol{k}) \approx \mp 2tk_x \sin k_0 \sigma_1 - 2t(k_y\sigma_2 + k_z\sigma_3) \mp \gamma k_x \sin k_0 \sigma_0, \quad k_x, k_y, k_z \ll 1. \tag{1.58}$$

When the separation between the two nodes is π, i.e. $k_0 = \pi/2$, the linearized Hamiltonian of around the cone, is

$$H'(\boldsymbol{k}) = \mp 2tk_x\sigma_x - 2tk_y\sigma_y - 2tk_z\sigma_z \mp \gamma k_x. \tag{1.59}$$

However, as the two noes are brought closer together, the effective Fermi velocity in the x-direction is rescaled, and the system is anisotropic even for no tilt ($\gamma = 0$). The expression may be made even more clear by moving the sign $\pm$-sign into the tilt parameter γ. The Hamiltonian is invariant under a sign change of the first term, as the isotropic Dirac Hamiltonian is invariant under inversion.

Is this not quite jalla?

In the tilt-term, we move the sign dependence into γ, and the linearized model is

$$H'(\boldsymbol{k}) = -2t\boldsymbol{k}\boldsymbol{\sigma} - \gamma^{\pm} k_x, \tag{1.60}$$

where $\gamma^{\pm} = \pm\gamma$ with the upper sign corresponding to the node at $k_x = +k_0$ and the lower sign corresponds to the node at $k_x = -k_0$. As expected, we get two Dirac cones, tilting in opposite direction, but with the same amount.

How does this affect the Berry curvature and chern number?

Maybe prettier/more correct to invert ky and kz, as that would also give the opposite chirality of the dirac points

The linearized model are accurate in describing low energy interactions around the Fermi level. For higher energies their validity falls apart, and more complex models are warranted. In our calculations the linear models is sufficient, and much easier to work with, and we will thus mainly consider the linear model from here on.

For tilted Dirac cones we will consider the Hamiltonian

$$H = s v_F \boldsymbol{k}\boldsymbol{\sigma} + v_F \boldsymbol{t}^s \boldsymbol{k}, \tag{1.61}$$

where s denotes the chirality of the Dirac cone, v_F is the Fermi velocity, and $\boldsymbol{t}$ is the *tilt vector*. In general the Fermi velocity is anisotropic, as was the case in the general Dirac Hamiltonian given in Eq. (1.36). By an anisotropic scaling of the momenta $\boldsymbol{k}$, the system may always be mapped to an isotropic case, which we will consider here.

The tilt vector will in general depend on the chirality of the Dirac cone. As the Dirac cones always appear in pairs, $\boldsymbol{t}^s = s\boldsymbol{t}$ will give a system with inversion symmetry. In the case of broken inversion symmetry, we will consider the case of a tilt equal in direction and magnitude between the two cones, $\boldsymbol{t}^s = \boldsymbol{t}$. In short, we define

$$\boldsymbol{t}^s = \begin{cases} \boldsymbol{t} & \text{broken inversion symmetry,} \\ s\boldsymbol{t} & \text{inversion symmetry.} \end{cases} \tag{1.62}$$

This convention is used in most literature [29; 9].

With no magnetic field, the eigenvalues of the system are

$$E(\boldsymbol{k}) = \boldsymbol{\omega_0}\boldsymbol{k} \pm \sqrt{(v_i k_i)^2} = \sqrt{(t_i v_i k_i)^2} \pm \sqrt{(v_i k_i)^2}, \tag{1.63}$$

where in the literature the first term is sometimes referred to as the *kinetic* term while the latter is the *potential* term. The definition for the system to be Type-II is that there exists a direction in momentum space for which the kinetic term dominates over the potential term [26]. The $\boldsymbol{t}$-vector is thus a convenient tool for categorization – if $t > 1$ we have a Type-II, else we have a Type-I.

Proof: We may always rotate our coordinate system such that, without loss of generality, $\boldsymbol{t} = t\hat{x}$. In that case, the first term obviously dominates in the x-direction, when $t > 1$. □

- gives rise to cones tilting opposite direction

- Linearized model valid for low energy interaction. For higher energy, the perfect cone model is not valid, as the cones does in fact touch.
- In this model, the hole pocket is “shared” between the two cones. There are also models with individual pockets (see [19])

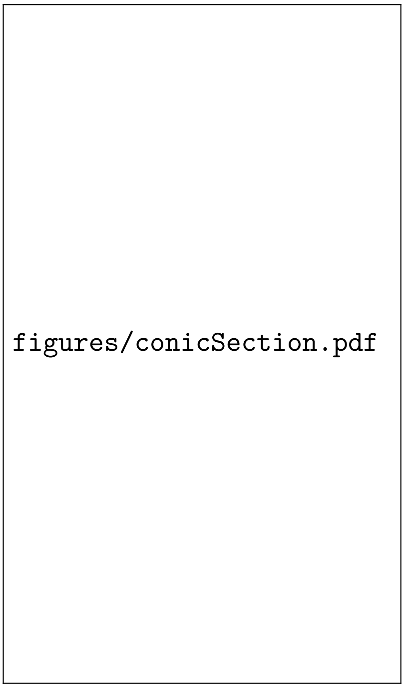


Figure 1.6.

figures/typeIIridgeline.png

Figure 1.7.:

Write this

The values of the parameters were chosen to be $m = 0.15, t = -0.05$, and $2k_0 = \pi$.

figures/movetypeiinode.png

Figure 1.8.: A Type-II Weyl semimetal with separation between the nodes $2k_0 = 0, \pi/2, \pi$. See main text for details about the model.

2. Linear response theory

We will now introduce the general theory of linear response, also referred to as the Kubo formalism. Later, the theory will be specialized to thermoelectric response. The material of this section is mostly inspired by the explanations given in Giuliani and Vignale [11]. The specialization to the electric response and Luttinger's method is also inspired by Mahan [18].

Consider making this discussion in both time and space

We are interested in expressing the response of the observable A to some field F coupling to another observable B. Let the uncoupled system be described by the Hamiltonian H_0 and the coupling term be $H_F = F(t)B$. Assume also that the coupling field F is turned on at $t = t_0$, such that $H_F(t) = 0$ for $t < t_0$. Let the unperturbed Hamiltonian be H_0, which will be assumed time independent. The total Hamiltonian describing the coupled system is

$$H(t) = H_0 + H_F = H_0 + F(t)B. \tag{2.1}$$

Linear response theory tells us then that the response δA is given by [11]

$$\delta A = -\frac{i}{\hbar} \int_{t_0}^{t} \big\langle \big[A(t), B(t')\big] \big\rangle_0 F(t') \mathrm{d}t', \tag{2.2}$$

where $[A, B]$ is the operator commutator and $\langle \ldots \rangle_0$ denotes the average in the thermal equilibrium ensemble. A non-rigorous motivation for this form of the response is the fact that

$$\dot{A} = -\frac{i}{\hbar} [A, H] + \frac{\partial A_S}{\partial t}, \tag{2.3}$$

with A_S the Schrödinger picture operator, whose derivative is from here on assumed zero. Taking $H = H_F$, the part of the Hamiltonian whose dynamics we are interested in, and integrate over the interaction time, the result is reminiscent of Eq. (2.2). For a proper derivation see for example Giuliani and Vignale [11, Chapter 3.3].

Note about time dependent vs independent operators/picture ? see Giuliani and Vaginal (3.29).

Some other formulation for what the $\langle \ldots \rangle_0$ means.

We will now try to make this expression slightly more manageable, and in the process we will highlight some important physical properties of the expression. Firstly, by taking advantage of the time translation invariance of the uncoupled Hamiltonian H_0, we may realize that the average taken in the unperturbed basis may be taken at a more convenient time, preserving the time separation of the operators

$$\big\langle \big[A(t), B(t')\big] \big\rangle_0 = \big\langle \big[A(t-t'), B(0)\big] \big\rangle_0 . \tag{2.4}$$

Inserting this back to Eq. (2.2), and performing a change of variable $\tau = t - t'$ we have

$$\delta A = -\frac{i}{\hbar} \int_0^{t-t_0} \langle [A(\tau), B(0)] \rangle_0 \, F(t-\tau) \mathrm{d}\tau. \tag{2.5}$$

In this form the retardedness of the coupling is apparent – no observable can be affected by a future perturbation, shown schematically in Figure 2.1.

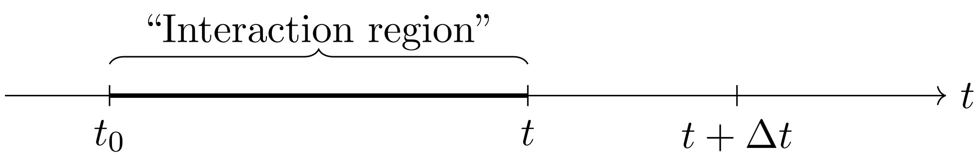


Figure 2.1.: Interacting region of a perturbation turned on at t_0. Note that the perturbation in the future, $t + \Delta t$, does not interact, as this is the retarded interaction.

For future convenience, and convention, we will in this last step introduce the *response function*

$$\chi_{AB}(\tau) = -\frac{i}{\hbar} \Theta(\tau) \langle [A(\tau), B(0)] \rangle_0 \, , \tag{2.6}$$

where the step-function Θ was introduced to make the response function explicitly *retarded*. Then our final expression for the response of A is

$$\delta A = \int_0^{t-t_0} \chi_{AB}(\tau) F(t-\tau) \mathrm{d}\tau. \tag{2.7}$$

Note of course that the limits could be altered to $\int_{-\infty}^{\infty}$ given that the coupling field is zero for times earlier than t_0 and we have chosen the retarded response function.

2.1. Linear response in charge current from electromagnetic coupling

We will now discuss the electric *conductivity* in light of the Kubo formalism, as an example to better understand and demonstrate the preceding discussion. Firstly the concept of conductivity will be presented, then it will be derived using the machinery of the Kubo formula. As mentioned above, this part follow the derivation of Mahan [18].

The charge current $\boldsymbol{J}$ that is induced from an electric field $\boldsymbol{E}$ in the linear scheme is expressed by Ohm's law

$$\boldsymbol{J}_i(\boldsymbol{r},t) = \int\limits_V \mathrm{d}\boldsymbol{x} \int\limits_{-\infty}^{t} \mathrm{dt}\ \sigma_{ij}(\boldsymbol{r},t,\boldsymbol{x},s)\boldsymbol{E}_j(\boldsymbol{x},s). \tag{2.8}$$

Above the Einstein summation convention is used, and σ is the *conductivity tensor*. We see of course that this has the familiar form of a response relation. In the case of a simple and isotropic material, meaning symmetric under SO(n) and with no transverse response, the tensor is diagonal with $\boldsymbol{\sigma} = \sigma I$ and one gets the more well-known version of Ohm's law $\boldsymbol{J} = \sigma\boldsymbol{E}$.

Again, by the principle of causality, the response of $\boldsymbol{J}$ can only depend on $\boldsymbol{E}$ in the *past*; thus $\sigma_{ij}(\boldsymbol{r},t,\boldsymbol{x},s)$ can be finite only where the time separation $t-s$ is less than the time light takes to cover the spatial separation $\boldsymbol{r}-\boldsymbol{x}$. Moreover, if we assume spatial and temporal invariance, i.e. that the response only depends on the separation $t-s$ and $\boldsymbol{r}-\boldsymbol{x}$, the expression is simplified somewhat more by transforming it to the Fourier domain. Note that this assumption is *not* valid on an atomic scale; it is here used under the assumption that currents are averaged over multiple unit cells, a common practice in electromagnetism of solids. Let $\sigma_{ij}(\boldsymbol{r}-\boldsymbol{x},t-s) \equiv \sigma_{ij}(\boldsymbol{r},t,\boldsymbol{x},s)$ and introduce the Fourier transform

$$A(\boldsymbol{q},\omega) = \iint \mathrm{d}t\mathrm{d}\boldsymbol{r} e^{i(\omega t-\boldsymbol{q}\boldsymbol{r})}A(\boldsymbol{r},t), \quad A(\boldsymbol{r},t) = \iint \frac{\mathrm{d}\omega\mathrm{d}\boldsymbol{q}}{(2\pi)^4} e^{-i(\omega t-\boldsymbol{q}\boldsymbol{r})}A(\boldsymbol{q},\omega). \tag{2.9}$$

Recognizing the right-hand side of Eq. (2.8)

Is it an issue that the t integration in (2.8) is only to t, and not infinity?

$$\int \mathrm{d}\boldsymbol{x} \int \mathrm{d}t \sigma_{ij}(\boldsymbol{r}-\boldsymbol{x}, t-s)\boldsymbol{E}_j(\boldsymbol{x}, s) \tag{2.10}$$

as a convolution, we can write Eq. (2.8) as

$$\boldsymbol{J}_i(\boldsymbol{q}, \omega) = \sigma_{ij}(\boldsymbol{q}, \omega)\boldsymbol{E}_j(\boldsymbol{q}, \omega), \tag{2.11}$$

by using the well known result that the Fourier transform of a convolution is the product of the transformed functions of the convolution [23]. Alternatively, the same result is found by simply inserting the definition Eq. (2.9) for both $\boldsymbol{E}$ and σ in Eq. (2.8), and use

$$\int \mathrm{d}x e^{-ixa} = 2\pi\delta(a).$$

We now attempt to conclude at the result (2.11) using the Kubo formalism. The current couple to the electromagnetic potential $\boldsymbol{A}$ by a Hamiltonian term

$$H_{\boldsymbol{A}} = -\int \mathrm{d}\boldsymbol{r}\boldsymbol{J}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}, t). \tag{2.12}$$

Comparing with the notation introduced earlier for general linear response, where the perturbing Hamiltonian in Eq. (2.1) was

$$F(t)B,$$

we identify the perturbing field F as $\boldsymbol{A}$ and the observable B as the current density. We thus identify the *response function*

$$\chi_{\alpha\beta}(\boldsymbol{r}, t, \boldsymbol{x}, s) = -\frac{i}{\hbar}\Theta(t-s)\left\langle[\boldsymbol{J}_\alpha(\boldsymbol{r}, t), \boldsymbol{J}_\beta(\boldsymbol{x}, s)]\right\rangle_0. \tag{2.13}$$

This gives the response

$$\delta\boldsymbol{J}(\boldsymbol{r}, t) = \int_{t_0}^{t} \mathrm{d}s \int \mathrm{d}\boldsymbol{x}\chi(\boldsymbol{r}, t, \boldsymbol{x}, s)\boldsymbol{A}(\boldsymbol{x}, s), \tag{2.14}$$

where the indices α, β has been dropped for clearer notation. Assuming spatial and temporal translational invariance,

$$\chi(\boldsymbol{r}-\boldsymbol{x}, t-s) \equiv \chi(\boldsymbol{r}, t, \boldsymbol{x}, s), \tag{2.15}$$

the expression can be simplified quite a bit. Firstly, we will make a change of variables, and then Fourier transform both the spatial and temporal argument. With $\tau = t - s$ and $\boldsymbol{x'} = \boldsymbol{r} - \boldsymbol{x}$,

$$\delta \boldsymbol{J}(\boldsymbol{r}, t) = \int_0^{t-t_0} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x'} \chi(\boldsymbol{x'}, \tau) \boldsymbol{A}(\boldsymbol{r} - \boldsymbol{x'}, t - \tau). \tag{2.16}$$

By the Fourier transformation introduced in Eq. (2.9)

$$A(\boldsymbol{q}, \omega) = \iint \mathrm{d}t \mathrm{d}\boldsymbol{r} e^{i(\omega t - \boldsymbol{q}\boldsymbol{r})} A(\boldsymbol{r}, t),$$

the time transformed version of Eq. (2.16) is

Should either implicitly or explicitly put the $t - t_0$ limit of the integral inside of A, so that the Fourier transform is simple

$$\delta \boldsymbol{J}(\boldsymbol{r}, \omega) = \int_0^{t-t_0} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x'} \chi(\boldsymbol{x'}, \tau) \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega t} \boldsymbol{A}(\boldsymbol{r} - \boldsymbol{x'}, t - \tau)}_{\equiv e^{i\omega\tau} \boldsymbol{A}(\boldsymbol{r} - \boldsymbol{x'}, \omega)}. \tag{2.17}$$

Similarly, Fourier transforming the spatial component yields

$$\delta \boldsymbol{J}(\boldsymbol{q}, \omega) = \int_0^{t-t_0} \mathrm{d}\tau \int \mathrm{d}\boldsymbol{x'} \chi(\boldsymbol{x'}, \tau) e^{i\omega\tau} \underbrace{\int \mathrm{d}\boldsymbol{r}\, e^{-i\boldsymbol{q}\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r} - \boldsymbol{x'}, \omega)}_{\equiv e^{-i\boldsymbol{q}\boldsymbol{x'}} \boldsymbol{A}(\boldsymbol{q}, \omega)}. \tag{2.18}$$

Identifying the remaining part as the Fourier transform of the response function, we finally end up with,

$$\delta \boldsymbol{J}(\boldsymbol{q}, \omega) = \chi(\boldsymbol{q}, \omega) \boldsymbol{A}(\boldsymbol{q}, \omega). \tag{2.19}$$

One could of course also have used the observation that the original expression is a convolution or the direct insertion of the Fourier transform for χ and $\boldsymbol{A}$, as shown earlier.

In the current derivation, the scalar field potential ϕ is taken to be zero, as transverse electric field is assumed, so the electric field is related to the vector potential as

$$\boldsymbol{E}(\boldsymbol{r}, t) = -\partial_t \boldsymbol{A}(\boldsymbol{r}, t) \implies \boldsymbol{E}(\boldsymbol{r}, \omega) = -i\omega \boldsymbol{A}(\boldsymbol{r}, \omega). \tag{2.20}$$

Thus, the response can be written as

$$\delta \boldsymbol{J}(\boldsymbol{q}, \omega) = \frac{i}{\omega} \chi(\boldsymbol{q}, \omega) \boldsymbol{E}(\boldsymbol{q}, \omega). \tag{2.21}$$

The expression (2.21) found using the Kubo formalism may now be compared to Ohm's equation (2.11), where we see that, re-inserting the component indices explicitly,

$$\sigma_{\alpha\beta}(\boldsymbol{q}, \omega) = \frac{i}{\omega} \chi_{\alpha\beta}(\boldsymbol{q}, \omega), \tag{2.22}$$

$$\begin{aligned} \chi_{\alpha\beta}(\boldsymbol{q}, \omega) &= \int \mathrm{d}\boldsymbol{x} \int \mathrm{d}t \; e^{i\omega t - i\boldsymbol{q}\boldsymbol{x}} \chi_{\alpha\beta}(\boldsymbol{x}, t) \\ &= -\frac{i}{\hbar} \int \mathrm{d}\boldsymbol{r} \int \mathrm{d}t \; e^{i\omega t - i\boldsymbol{q}\boldsymbol{x}} \Theta(t) \left\langle [\boldsymbol{J}_\alpha(\boldsymbol{r}, t), \boldsymbol{J}_\beta(0,0)] \right\rangle_0 . \end{aligned} \tag{2.23}$$

It is here important to remember that it was here assumed only transverse current. If that was not the case, there would be an additional contribution to the σ_{ii} components.

2.2. Luttinger approach to thermal transport

Thermal transport, i.e. response to thermal gradients, is more convoluted than the response to an electromagnetic field, as there is no well-defined Hamiltonian describing the temperature gradient, which of course is a statistical property of the system.

Consider including some of Mahan's disucssion about having a thermal gradient when we in the calculations have assumed constant temperature.

In his now illustrious paper [17] Luttinger seeks to make the theory of transport due to temperature gradients more formal and "mechanical", as he puts it. Inspired by the mechanical derivation of Kubo for the electric transport, he introduces a method where the transport may be derived mechanically from a phenomenological term in the Hamiltonian – the *Luttinger term*. Earlier calculations of the transport properties of temperature gradients was conducted from local variable theories; Luttinger [17] mentions the derivations of Green and Mori, where they respectively had assumed a Markoff process and "local equilibrium distribution". Luttinger's method attempts to put the results of those calculations on a "more solid basis".

We will here simply outline the basic idea of Luttinger, without a rigorous derivation. Introduce to the Hamiltonian a *gravitational* scalar potential field ψ

coupling to the energy density T^{00} of the (flat) system [17]

$$H_L = \int \mathrm{d}\boldsymbol{r}\psi T^{00}. \tag{2.24}$$

Note that the T^{00} component of the stress-energy tensor must not be confused with the temperature T. Luttinger showed that the system is in equilibrium, i.e. the thermal and gravitational driving forces balance out, given that the gravitational field is related to the temperature by

$$\nabla\psi + \frac{\nabla T}{T} = 0. \tag{2.25}$$

Borrowing the language of Tatara [27], this is essentially a trick to be able to calculate transport coefficients without introducing temperature gradients in the Hamiltonian. Instead, one introduces the fictitious field ψ, for which the origin is not addressed, and find the transport coefficients for this system. The situation is depicted in Figure 2.2, where the temperature field is shown, together with an accompanying gravitational field.

figures/LinearResponse_bump.pdf

Figure 2.2.: Illustration of Luttinger's solution to heat transport. To include a temperature fluctuation T, couple the system to some (fictions) gravitational potential ψ giving the same current response as the temperature fluctuation.

A temperature gradient, together with external electric potential ϕ and chemical potential μ, gives a response in the electrical current $\boldsymbol{j}$ and energy current $\boldsymbol{j}_E$ [18]

$$j_\alpha = -M^{(11)}_{\alpha\beta}\left[\frac{e}{T}\nabla_\beta\phi + \nabla_\beta\left(\frac{\mu}{T}\right)\right] + M^{(12)}_{\alpha\beta}\nabla_\beta\left(\frac{1}{T}\right), \tag{2.26}$$

$$J_{E,\alpha} = -M^{(21)}_{\alpha\beta}\left[\frac{e}{T}\nabla_\beta\phi + \nabla_\beta\left(\frac{\mu}{T}\right)\right] + M^{(22)}_{\alpha\beta}\nabla_\beta\left(\frac{1}{T}\right). \tag{2.27}$$

Or, more compactly,

$$\begin{pmatrix} j_\alpha \\ j_{E,\alpha} \end{pmatrix} = -M_{\alpha\beta}\begin{pmatrix} \frac{e}{T}\nabla_\beta\phi + \nabla_\beta\left(\frac{\mu}{T}\right) \\ \nabla_\beta\left(\frac{1}{T}\right) \end{pmatrix}. \tag{2.28}$$

The coefficients of transportation, $M_{\alpha\beta}^{(ij)}$ is a widely used convention. The success of Luttinger's method was that the transport coefficients $M_{\alpha\beta}^{(ij)}$ could now be calculated directly, and yielded the same results as had previously been found by less formal approaches.

By the introduction of the Hamiltonian perturbation H_L, the response may now be investigated in the Kubo formalism. By the response in Eq. (2.2) the electric current generated from the gravitational perturbation is

$$\langle \boldsymbol{J}^i \rangle (t, \boldsymbol{r}) = \int \mathrm{d}t' \mathrm{d}\boldsymbol{r'} \left\{ \frac{-i}{\hbar} \Theta(t - t') \left\langle \left[\boldsymbol{J}^i(t, \boldsymbol{r}), T^{00}(t', \boldsymbol{r'}) \right] \right\rangle \right\} \psi(t', \boldsymbol{r'}), \quad (2.29)$$

where the integration is taken over the entire spacetime. In order to express this as a response to the thermal gradient, we wish to get the gradient of the gravitational potential. To do this, firstly the 00-element of the stress-energy tensor will be expressed in terms derivatives of T^{0j}, and then a partial integration will swap the derivative between the stress-energy tensor and gravitational potential. Note first that in the flat system the conservation law of the energy and momentum is simply

$$\partial_0 T^{00}(t, \boldsymbol{r}) + v_F \partial_i T^{0i}(t, \boldsymbol{r}) = 0, \quad (2.30)$$

where v_F is the Fermi velocity. By the fundamental theorem of calculus this obviously gives for the zero-zero component of the stress-energy tensor

$$T^{00}(t, \boldsymbol{r}) = - \int_{-\infty}^{t} \mathrm{d}t' v_F \partial_i T^{0i}(t', \boldsymbol{r}). \quad (2.31)$$

Introduce Eq. (2.31) in the response relation (2.29), and use integration by parts

$$\int uv' = uv - \int u'v, \quad (2.32)$$

giving

$$\langle \boldsymbol{J}^i \rangle (t, \boldsymbol{r}) = \int \mathrm{d}t' \mathrm{d}\boldsymbol{r'} \int_{-\infty}^{t'} \mathrm{d}t'' \left\{ \frac{-iv_F}{\hbar} \Theta(t - t') \left\langle \left[\boldsymbol{J}^i(t, \boldsymbol{r}), T^{0j}(t'', \boldsymbol{r'}) \right] \right\rangle \right\} \partial'_j \psi(t', \boldsymbol{r'}). \quad (2.33)$$

By Luttinger's relation

$$\langle \boldsymbol{J}^i \rangle (t, \boldsymbol{r}) = \int \mathrm{d}t' \mathrm{d}\boldsymbol{r'} \int_{-\infty}^{t'} \mathrm{d}t'' \left\{ \frac{iv_F}{\hbar} \Theta(t - t') \left\langle \left[\boldsymbol{J}^i(t, \boldsymbol{r}), T^{0j}(t'', \boldsymbol{r'}) \right] \right\rangle \right\} \frac{\partial'_j T(t', \boldsymbol{r'})}{T(t', \boldsymbol{r'})}, \quad (2.34)$$

where care must be taken to distinguish the stress-energy tensor T^{0j} and the temperature T, differentiated by the indices, or lack thereof.

Check the sign here. Depending on how we understand Luttinger's method, it should be positive or negative.

3. Anomalies in quantum field theory

From Noether's theorem, described in the following section, we know that any continuous symmetry of the Lagrangian $\mathcal{L}$ in a classical consideration will lead to a conserved current. However, we know from the path integral formulation of QFT (quantum field theory) that for a system with fields ϕ and an external source J, it is the generating functional

$$Z[J] \equiv \int \mathcal{D}\phi \exp\left[i\left(S[\phi] + \int \mathrm{d}^4x J(x)\phi(x)\right)\right] \tag{3.1}$$

that must be invariant for a transformation to be a symmetry operation of the system. Quantum corrections from the second quantization can lead to the symmetry group of the generating functional to be smaller than the symmetry group of the classical action, in which case we say there is an *anomaly*. In that case, the conserved current predicted by Noether's theorem is no longer protected by symmetry, as the operation is indeed not a symmetry of the system. The terms breaking the classical conservation are called *anomalies*.

It should also be noted that the terminology *anomaly* and *breaks the classical symmetry* are somewhat misleading; there is no actual symmetry breaking – in the quantum theory there is no symmetry to begin with, and a more fitting language to describe the situation is that there is an anomalous symmetry in the classical Lagrangian, which is not there in the "real" theory. Thus, the situation must not be confused with spontaneous symmetry breaking, and there is no Goldstone boson present.

3.1. Noether's theorem

The following section is inspired by the derivation of Kachelriess [15].

Noether's theorem is one of the most central results in theoretical quantum physics. It relates continuous symmetries with conserved quantities, which for example explain fundamental principles such as conservation of momentum and conservation of energy. Given a Lagrangian $\mathcal{L}(\phi_a, \partial_\mu\phi_a)$ dependent on the fields ϕ_a, we will consider the variations $\delta\phi_a$ that leave the action, and thus equations of motion, invariant. That is, the variations that are generators for some continuous symmetry of the system. Firstly, we will restrict our consideration to the case where the Lagrangian itself is invariant

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\partial_\mu\phi_a. \tag{3.2}$$

In the last term use that the variation and derivation may be exchanged, $[\delta\partial_\mu, \partial_\delta\delta] = 0$, and in the first term use the Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\phi_a} = \delta_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\right). \tag{3.3}$$

By the product rule it follows that

$$0 = \delta\mathcal{L} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\right)\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a\right). \tag{3.4}$$

Thus, we see that the quantity in the parenthesis after the last equality must be conserved. We denote this quantity j^μ, and call it a *current.*

So far, we have the result that for any variation $\delta\phi_a$ that leave the Lagrangian invariant, there is a conserved current

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a, \quad \partial_\mu j^\mu = 0. \tag{3.5}$$

There is, however, an even stronger formulation of Noether's theorem. As the equations of motion are only dependent on the transformation being a symmetry transformation of the action, we realize that even a change in the Lagrangian of the form $\delta\mathcal{L} = \partial_\mu K^\mu$ will not change the equations of motion, as long as boundary terms of the integral over the Lagrangian may be dropped ($K \to 0, r \to \infty$). Thus, altering the starting point in Eq. (3.4) to $0 = \delta\mathcal{L} - \partial_\mu K^\mu$ we get Noether's theorem, theorem 1.

Theorem 1 (Noether's theorem)**.** *For any continuous transformation that leave the Lagrangian $\mathcal{L}$ invariant up to a total derivative $\partial_\mu K^\mu$, there must be an associated conserved current*

$$j^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a - K^\mu, \quad \partial_\mu j^\mu = 0. \tag{3.6}$$

3.2. The axial/chiral anomaly

We will first give a quick and somewhat superficial introduction to the axial anomaly[1], and why it matters in condensed matter physics. That discussion will

[1] Also known as the chiral anomaly.

be based on the discussion given in Wehling, Black-Schaffer, and Balatsky [30] and Tong [28, Ch. 3]. Then we will present a more thorough derivation of the anomaly, based on the derivation of Zee [31] and Kachelriess [15].

In the massless case the Dirac equation reduce to the Weyl equation, whose solutions are right and left moving fermions. In 1+1 dimensions they have the energy dispersion

$$\epsilon_{\pm} = \pm|p|,$$

where the $\pm$ indicate positive and negative energy solutions. Consider the case now in the Dirac sea picture. The negative energy solutions, antiparticles in high energy physics and holes in condensed matter physics, are all filled, with the energy band going to $\pm\infty$ momentum. The particles with energy $\epsilon = +p$ are right moving solutions, while $\epsilon = -p$ represent left moving solutions. Note that in this language, an antiparticle with negative momentum, is right moving, and of course a particle with positive momentum is right moving. The situation is shown in the left pane of Figure 3.1. Introduce now an electric field E. This will cause the states to shift, according to $\dot{p} = eE$, with e being the electric coupling, which is here taken to be the fundamental charge; note that this shift does not discriminate against left and right movers, they are both shifted the same. For a field $E > 0$ the right movers are shifted towards higher energies and the left movers are shifted towards lower energies, shown in the right pane of Figure 3.1. This also shifts the densities of left and right movers! Denote by n_+ the right movers and n_- the left movers. The total density $n = n_+ + n_-$ is constant, however, the difference $n_+ - n_-$ is not conserved. Identifying $J = n_+ + n_-$ as the vector current and $J_A = n_+ - n_-$ as the axial current, we see that the vector current is conserved, but the *axial* current is not! Notice how the origin of the anomaly in this context, is the infinite depth of the Dirac sea.

We will now give a purely field theoretical derivation of the axial anomaly, under a gauge transformation

$$\psi \to e^{i\theta + i\theta\gamma^5}\psi, \tag{3.7}$$

corresponding to a gauge transformation of the coupling fields.

remove or write more about these coupling fields, as we do not have them in our Lagrangian. See Zee and Kachelriess

As is often the case, there are many ways to do this. For example, one could show directly that the measure of the path integral is not invariant under a transformation. We will, however, show it in a somewhat crude way, but where there are no complicated formal considerations, only brute force calculation which is hopefully more readily appreciated by those not familiar to the concept. The calculation also has some historical importance, as the problem we will

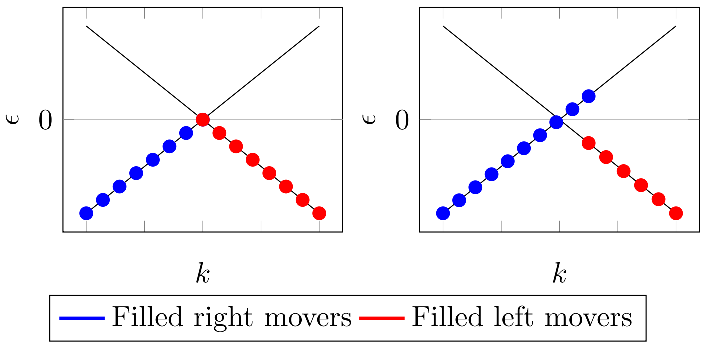


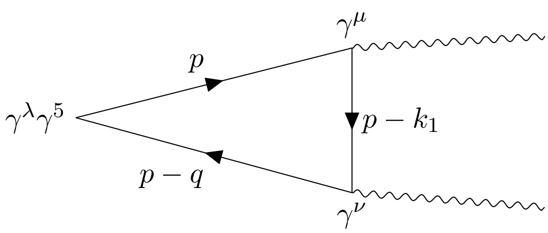
Figure 3.1.: Dispersion of Weyl fermions, black showing unfilled states, blue filled right movers, and red filled left movers. **(Left)** No electric field applied, Fermi level at the crossing. **(Right)** Electric field in the positive direction applied, shifting the filled states. See main text for details.

Consider changing for solid lines (not marks). In that case, ha thiccc lines.

solve is in fact exactly the same as the problem that led to the discovery of anomalies!

Consider to spice up this historic reference

We will calculate the triangle diagram



and show that this leads to the conclusion that either the vector current or the axial current is non-conserved. The amplitude of the diagram is

$$\left\langle 0 \middle| T J_5^\lambda J^\mu J^\nu \middle| 0 \right\rangle, \tag{3.8}$$

with the vector current $J^\mu = \bar{\psi}\gamma^\mu\psi$ and the axial current $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. Written out explicitly in momentum space

$$\mathcal{A}^{\lambda\mu\nu}(k_1,k_2) =$$
$$(-1)i^3 \int \frac{\mathrm{d}^4p}{(2\pi)^4} \operatorname{Tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu \frac{1}{\not{p}-\not{k}_1}\gamma^\mu \frac{1}{\not{p}} + \gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\mu \frac{1}{\not{p}-\not{k}_2}\gamma^\nu \frac{1}{\not{p}}\right), \tag{3.9}$$

where $q = k_1 + k_2$. For the vector current to be conserved the requirement $k_{1\mu}\mathcal{A}^{\lambda\mu\nu} = k_{2\nu}\Delta^{\lambda\mu\nu} = 0$ must hold. For the axial current to be conserved, the requirement is $q_\lambda\mathcal{A}^{\lambda\mu\nu} = 0$.

show or cite the requirements. I think maybe it is quite simple to see from the wick contractions

One must be careful when carrying out this calculation, as is also stressed in many textbooks dealing with this issue, for example [15] and [31]. Consider the criterion for the vector current to be conserved

$$k_{1\mu}\mathcal{A}^{\lambda\mu\nu}(k_1,k_2) = i\int \frac{\mathrm{d}^4p}{(2\pi)^4} \operatorname{Tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{q}}\gamma^\nu \frac{1}{\not{p}-\not{k}_1} - \gamma^\lambda\gamma^5 \frac{1}{\not{p}-\not{k}_2}\gamma^\nu \frac{1}{\not{p}}\right) = 0. \tag{3.10}$$

When calculating the integral it might be tempting to simply perform a change of variables, rendering the two terms equal and thus concluding that the criterion is met. However, we must notice that the integrand goes like $1/p^2$ while the boundary surface of a 3-sphere is proportional to p^3. The boundary terms does therefore not vanish, and there is an extra term associated with performing such a change of variables.

Consider that we want to integrate over the function f

$$\int \mathrm{d}^dp f(p). \tag{3.11}$$

If we perform the change of variables $p \to p+a$, one could in theory get an extra contribution from boundary terms, which we will now find. We will calculate

$$\int \mathrm{d}^dp \left[f(p+a) - f(p)\right], \tag{3.12}$$

where we in the first term has "naively" performed a change of variables, without considering the boundary terms. Thus, the result of this integral is indeed the boundary terms. Firstly, we will perform a Wick rotation into Euclidean space

some note about why this is allowd, i.e. that there exists some Feynamn parametrization for the

$$\int \mathrm{d}_E^dp \left[f(p+a) - f(p)\right] = \int \mathrm{d}_E^dp \left[a^\mu\partial_\mu f(p) + \ldots\right]. \tag{3.13}$$

Ignoring the higher order terms, the RHS may be rewritten as a surface integral by Gauss's theorem. Taking the average over the surface integral, and denoting by $S_d(r)$ the surface of a d-sphere with radius r, we write the integral as

$$\lim_{P\to\infty} a^\mu \left(\frac{P_\mu}{P}\right) f(P) S_{d-1}(P). \tag{3.14}$$

Rotating back to Minkowski space we gain an additional i, with

$$\int \mathrm{d}^d p \left[f(p+a) - f(p)\right] = \lim_{P\to\infty} i a^\mu \left(\frac{P_\mu}{P}\right) f(P) S_{d-1}(P). \tag{3.15}$$

We will now perform such a shift of variables in the second term of the trace in Eq. (3.10), as we notice that shifting $p \to p - k_1$ makes the two terms cancel, leaving only the boundary term. Let

$$f(p) = \mathrm{Tr}\left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{k}_2} \gamma^\nu \frac{1}{\not{p}}\right) = \frac{\mathrm{Tr}\left(\gamma^5 (\not{p} - \not{k}_2)\gamma^\nu \not{p} \gamma^\lambda\right)}{(p-k_2)^2 p^2} = \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} P_\sigma}{(p-k_2)p^2}. \tag{3.16}$$

Here we used in the first equality the property $1/\not{a} = \not{a}/a^2$ twice, and the cyclic permutation invariance of the trace, $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$. In the second equality, we first wrote the Feynman slash operator by its definition $\not{a} = \gamma^\mu a_\mu$, and then used the property

$$\mathrm{Tr}(\gamma^5 \gamma^\tau \gamma^\nu \gamma^\sigma g^\lambda) = -4i\epsilon^{\tau\nu\sigma\lambda}, \tag{3.17}$$

where ϵ is the totally antisymmetric tensor. The trace can be split into two terms, where the first vanishes as it is proportional to $\epsilon^{\tau\nu\sigma\lambda} p_\tau p_\sigma$, and one is left with the expression in Eq. (3.16). Thus, Eq. (3.10) becomes

$$k_{1\mu} \mathcal{A}^{\lambda\mu\nu} = \frac{i}{(2\pi)^4} \lim_{P\to\infty} i(-k_1)^\mu \frac{P_\mu}{P} \frac{4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} P_\sigma}{P^4} 2\pi^2 P^3 = \frac{i}{8\pi^2} \epsilon^{\lambda\nu\tau\sigma} k_{1\tau} k_{2\sigma}. \tag{3.18}$$

Consider now, however, what happens if we shift $p \to p + k_2$ in the first term of Eq. (3.10) instead. Surely, if our answer above is correct, any arbitrary shift must yield the same answer. Similarly to before, let

$$\begin{aligned} f(p) &= \mathrm{Tr}\left(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1}\right) \\ &= \frac{\mathrm{Tr}\left(\gamma^5 (\not{p} - \not{q})\gamma^\nu (\not{p} - \not{k}_1) \gamma^\lambda\right)}{(p-q)^2 (p-k_1)^2} = \frac{-4i\epsilon^{\tau\nu\sigma\lambda} k_{2\tau} (k_{1\sigma} - p_\sigma)}{(p-q)^2 (p-k_1)^2}, \end{aligned} \tag{3.19}$$

where we as above removed all terms symmetric under $\sigma \leftrightarrow \tau$. Now, Eq. (3.10) becomes

$$k_{1\mu}\mathcal{A}^{\lambda\mu\nu} = \frac{i}{(2\pi)^4}\lim_{P\to\infty} ik_2^\mu \frac{P_\mu}{P}\frac{-4i\epsilon^{\tau\nu\sigma\lambda}k_{2\tau}(k_{1\sigma}-p_\sigma)}{P^4}2\pi^2P^3 = \frac{i}{8\pi^2}\epsilon^{\lambda\nu\tau\sigma}k_{2\tau}k_{2\sigma}. \tag{3.20}$$

Check there is not misssing a -1 in last espression

Where we used that the only term contributing is the p_σ, as the term with $k_{1\sigma}$ goes like P^{-1}. Our results differ depending on the non-physical shift of variables! As is shown by several textbooks, [31] [15], this comes from the fact that the integral we started with is in fact linearly divergent – its value is not well-defined. What we will have to do, is consider an arbitrary shift a in the integration variable of the amplitude Eq. (3.9), which we will show changes the amplitude by a quantity dependent on a. To cancel this, a counter term must be inserted; however, as we will see, this counter term can only make either the axial current or the vector current conserved! Consider now a shift in the integration variable $p \to p - a$ in the amplitude (3.9), where we denote the amplitude with shifted integration variable

$$\begin{aligned}\mathcal{A}^{\lambda\mu\nu}(a,k_1,k_2) = (-1)i^3\int\frac{\mathrm{d}^4p}{(2\pi)^4}\operatorname{Tr}\bigg(&\gamma^\lambda\gamma^5\frac{1}{\not p-\not a-\not q}\gamma^\nu\frac{1}{\not p-\not a-\not k_1}\gamma^\mu\frac{1}{\not p-\not a}\\ &+\gamma^\lambda\gamma^5\frac{1}{\not p-\not a-\not q}\gamma^\mu\frac{1}{\not p-\not a-\not k_2}\gamma^\nu\frac{1}{\not p-\not a}\bigg).\end{aligned} \tag{3.21}$$

From Eq. (3.15) we already have a formula for the difference

$$\mathcal{A}^{\lambda\mu\nu}(a,k_1,k_2) - \mathcal{A}^{\lambda\mu\nu}(k_1,k_2), \tag{3.22}$$

by choosing

$$f(p) = \frac{i}{(2\pi)^4}\operatorname{Tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k_1}\gamma^\mu\frac{1}{\not p}\right).$$

Ignore for now the prefactor, and note that in the limit

$$\begin{aligned}\lim_{p\to\infty} f(p) &= \frac{\operatorname{Tr}(\gamma^\lambda\gamma^5\not p\gamma^\nu\not p\gamma^\mu\not p)}{p^6}\\ &= \frac{2\operatorname{Tr}(\gamma^\lambda\gamma^5\not p\gamma^\nu\not p) - p^2\operatorname{Tr}(\gamma^\lambda\gamma^5\not p\gamma^\nu\gamma^\mu)}{p^6}\\ &= \frac{4ip_\sigma\epsilon^{\sigma\nu\mu\lambda}}{p^4}.\end{aligned} \tag{3.23}$$

In the second equality we used the anti-commutation relation of gamma matrices in $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$ and $\not{a}^2 = a^2$. In the last equality, we used again Eq. (3.17), and the vanishing of all terms symmetric under interchanging indices when contracted with the fully antisymmetric tensor. We now find the amplitude difference (3.22). Firstly, we simplify the expression slightly as

$$\Delta\mathcal{A}^{\lambda\mu\nu}(a, k_1, k_2) \equiv \int \mathrm{d}^4 p f(p-a) - f(p) + \{(k_1, \mu) \leftrightarrow (k_2, \nu)\}, \tag{3.24}$$

where the last term indicates to repeat the preceding expression with interchange of $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Thus, by Eq. (3.15),

$$\begin{aligned}\Delta\mathcal{A}^{\lambda\mu\nu}(a, k_1, k_2) &= \lim_{p\to\infty} ia^\mu \left(\frac{p_\mu}{p}\right) \frac{i}{(2\pi)^4} \frac{4ip_\sigma\epsilon^{\sigma\nu\mu\lambda}}{p^4} 2\pi^2 p^3 \\ &\quad + \{(k_1, \mu) \leftrightarrow (k_2, \nu)\} \\ &= \lim_{p\to\infty} \frac{-ia^\mu}{2\pi^2} \frac{p_\mu p_\sigma}{p^2} \epsilon^{\sigma\nu\mu\lambda} + \{(k_1, \mu) \leftrightarrow (k_2, \nu)\} \\ &= -\frac{ia_\sigma}{8\pi^2}\epsilon^{\sigma\nu\mu\lambda} + \{(k_1, \mu) \leftrightarrow (k_2, \nu)\}.\end{aligned} \tag{3.25}$$

Figure out if there is missing an overall minus sign

Now is the time to take a break from the calculations and consider in some detail what this result means, before we will finally carry out the derivation to its end and show the anomaly. A priori $\mathcal{A}^{\lambda\mu\nu}(a, k_1, k_2)$ should be just as valid as $\mathcal{A}^{\lambda\mu\nu}(k_1, k_2)$, i.e. setting $a = 0$. In fact, that formulation is quite the misnomer, as $a = 0$ is no less arbitrary than any $a \neq 0$ in this setting; p is simply a name by which we denote the moment transfer in our diagram. *However*, using Eq. (3.25), leading to

$$\begin{aligned}k_{1\mu}\mathcal{A}^{\lambda\mu\nu}(a, k_1, k_2) - k_{1\mu}\mathcal{A}^{\lambda\mu\nu}(a = 0, k_1, k_2) = \\ -\frac{i}{8\pi^2}\left[\epsilon^{\sigma\nu\mu\lambda}a_\sigma + \{(k_1, \mu) \leftrightarrow (k_2, \nu)\}\right] k_{1\mu},\end{aligned} \tag{3.26}$$

we see that the criterion for vector current conservation (3.10) may or may not be met depending on our choice of a!

Should this be related to counter terms as Kachelriess does? What does it really mean that we have to choose some shift

Owning to a trick from Zee [31], we will show that the resolve of this is to chose one particular a, and the choice will be that a which preserves the consistency of our theory. Now, this may indeed seem both strange and ad-hoc, how can we justify *choosing* some parameter to get the result we want? This is, in fact, common in QFT. Recall that both the UV-cutoff and dimensional regularization schemes introduce a parameter, which must be determined "outside" of our theory.

Let $a = \alpha(k_1 + k_2) + \beta(k_1 - k_2)$. This is allowed as k_1, k_2 are independent, and the only parameters of our equations. The α term is obviously symmetric under interchange of k_1, k_2, while the β term is antisymmetric. Thus, we see that in Eq. (3.26) only the β part survives when adding the pair with interchanged indices and momenta. Thus,

$$
\begin{aligned}
k_{1\mu}\mathcal{A}^{\lambda\mu\nu}(a, k_1, k_2) &= -\frac{i}{4\pi^2}\epsilon^{\sigma\nu\mu\lambda}\beta(k_{1\sigma} - k_{2\sigma})k_{1\mu} + k_{1\mu}\mathcal{A}^{\lambda\mu\nu}(k_1, k_2) && (3.27)\\
&= \frac{i}{8\pi^2}\left(\epsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma} - 2\epsilon^{\sigma\nu\mu\lambda}\beta(k_{1\sigma} - k_{2\sigma})k_{1\mu}\right) && (3.28)\\
&= \frac{i}{8\pi^2}\epsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}(1 + 2\beta). && (3.29)
\end{aligned}
$$

Here we inserted our previous result for $k_{1\mu}\mathcal{A}^{\lambda\mu\nu}$ given in Eq. (3.18). In the last equation we used that $k_{1\sigma}k_{1\mu}$ vanishes when contracted with the Levi Cevita symbol, and relabeled the dummy indices. It is now apparent that choosing $\beta = -1/2$ makes the criterion for conservation of vector current hold!

By choosing the shift appropriately, the vector current is preserved. However, it does come at a price. The requirement for the axial current to be conserved, as mentioned earlier, is

$$q_\lambda \mathcal{A}^{\lambda\mu\nu} = 0.$$

This amplitude is in fact also set by the parameter β, as also here α drops out; we have no free parameter to tune after fixing β. With the choice $\beta = -1/2$, required to conserve the vector current, the axial current will not be conserved! This is the chiral anomaly.

We could have, of course, instead chosen β such that the axial current is conserved, at the expense of the conservation of the vector current. However, as Zee [31] describes, this would have catastrophic consequences, rendering the entire theory useless. A non-conserved vector current, would make the fermion number not conserved, clearly non-acceptable. We therefore chose to sacrifice the axial current instead of the vector current.

3.3. The conformal anomaly

Consider massless QED (quantum electrodynamics)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi, \tag{3.30}$$

with ψ the Dirac field, $\bar{\psi} = \psi^\dagger\gamma^0$, $\not{D} = \gamma^\mu D_\mu$, D the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$, γ^μ the Dirac matrices, and F the electromagnetic field. A is the electromagnetic potential, $\partial_\mu A_\nu - \partial_\nu A_\mu$, and e is the coupling, here the fundamental charge. The theory is classically scale invariant. That is, under the transformation

$$x \to \lambda^{-1}x, \quad A_\mu \to \lambda A_\mu, \quad \psi \to \lambda^{\frac{3}{2}}\psi, \tag{3.31}$$

the Lagrangian transforms as

$$\mathcal{L} \to \lambda^4\mathcal{L}, \tag{3.32}$$

which is canceled by the transformation of measure $\mathrm{d}^4x \to \mathrm{d}^4x\lambda^{-4}$ in the action. As the action is invariant, thus so are the equations of motion.

By Noether's theorem there must be some conserved current corresponding to this symmetry transformation, which we will now show is the dilation current $j^\mu_D = T^{\mu\nu}x_\nu$. Consider a conformal transformation of the type $g_{\mu\nu} = e^{2\tau}\eta_{\mu\nu}$, also known as a Weyl transformation of the metric. The variation of the metric is obviously $\delta g_{\mu\nu} = 2\tau\eta_{\mu\nu}$. Recall also that the stress-energy tensor is defined as the response of the action to a variation of the metric

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{3.33}$$

where g is the determinant of the metric. Now, using this we see that

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \\ &= \int \mathrm{d}^4x \frac{\sqrt{|g|}}{2}T_{\mu\nu}\delta g^{\mu\nu} \\ &= \int \mathrm{d}^4x T_{\mu\nu}\sqrt{|g|}\tau(x)\eta^{\mu\nu}(x) \\ &= \int \mathrm{d}^4x \sqrt{|g|}\tau(x)T^\mu_\mu.\end{aligned} \tag{3.34}$$

As the scaling is a symmetry operation, Eq. 3.34 must be zero. As the scaling factor τ is an arbitrary function, we conclude that the trace T^μ_μ must vanish.

The vanishing trace ensures the conservation of the dilation current as

$$\begin{aligned} \partial_\mu J^\mu_D &= T^{\mu\nu}\partial_\mu x_\nu + \underbrace{(\partial_\mu T^{\mu\nu})}_{=0} x_\nu \\ &= T^{\mu\nu}\delta_{\mu\nu} = T^\mu_\mu, \end{aligned} \tag{3.35}$$

where we used the property of the stress-energy tensor that $\partial_\mu T^{\mu\nu} = 0$. As the trace is zero, the dilation current is conserved in the classical picture.

However, this symmetry does not hold when quantum corrections are taken into account. Loop effects give non-vanishing contributions to the trace, and by Eq. (3.35) this makes the dilation current non-conserved. Due to this, the conformal anomaly is also often referred to as the trace anomaly. Recall that when calculating the propagators of the QED theory, we end up with infinities. These, we regularize and renormalize, for example with dimensional regularization or UV-cutoff. In any case, this introduces some dimensionfull scale, μ, the renormalization scale and the cutoff energy respectively for the regulators mentioned. This scale dependence is encoded in the beta function of the theory, encoding the dependence of the coupling e on the scale,

$$\beta(e) = \frac{\partial e}{\partial \log \mu}; \tag{3.36}$$

if the beta function does not vanish, our theory now has a scale dependence, rendering our theory no longer scale invariant!

When taking into account the loop effects, the trace of the stress-energy tensor is [15]

$$T^\mu_\mu = \frac{\beta(e)}{2e} F_{\mu\nu}F^{\mu\nu}, \tag{3.37}$$

where $\beta(e)$ is the beta function of the theory. This beta function makes the anomaly not exact in one loop, as opposed to the axial anomaly. In one loop, the massless fermion beta function is [6]

$$\beta^{(1)} = \frac{e^3}{12\pi^2}. \tag{3.38}$$

4. Charge current from the conformal anomaly

In 2016 Chernodub [6] showed that the conformal anomaly of QED leads to electrical currents in an inhomogeneous gravitational background. This effect was further explored by Chernodub, Cortijo, and Vozmediano [7], showing through Luttinger's method that such an anomalous transport could be generated from a temperature gradient, giving additional contributions to the Nernst current. The same effect was shortly after derived more formally through the Kubo formalism, by Arjona, Chernodub, and Vozmediano [1].

In this chapter we extend the Kubo calculation to tilted Weyl cones. Firstly, the result for the untilted system is rederived, where we also show several simplifications which makes the computation much easier and shorter. The results for the untilted cone are then generalized to tilted cones. The computation is quite lengthy, and the thesis is explicit in each step, with the goal being that a graduate level student should be able to comfortably follow the calculations.

The chapter is divided into sections, each representing a somewhat contained part of the calculation. The text is not, however, written such that a reader should expect to understand a section without reading the preceeding one.

We will find the current response of a single Dirac cone, with a temperature gradient $\nabla_y T$ and a magnetic field B_z. The current response of interest in the given geometry is thus in the x-direction,

$$J^x = \chi^{xy} \frac{-\nabla_y T}{T}, \tag{4.1}$$

with χ^{xy} being the response[1]. This geometry is shown in Figure 4.1. In the derivation of Chernodub, Cortijo, and Vozmediano [7] the response

$$\chi^{xy} = \frac{e^2 v_F B}{18\pi^2 \hbar} \tag{4.2}$$

[1]The sign in Eq. (4.1) depends on the choice of the response function being the response of the gravitational potential or the temperature gradient. Thus, the sign may differ in the literature.

figures/setup.png

Figure 4.1.: Sketch of the geometry used in the derivation. Note that we consider only bulk response, and the finite sample is only for illustration purposes.

was found, while the derivation of Arjona, Chernodub, and Vozmediano [1] found [2]

$$\chi^{xy} = \frac{e^2 v_F B}{4\pi^2 \hbar}. \tag{4.3}$$

Recall the linear response from the Kubo formalism in Eq. (2.33), found through Luttinger's approach.

$$\langle J^i \rangle (t, \boldsymbol{r}) = \int_{-\infty}^{\infty} \mathrm{d}t' \mathrm{d}\boldsymbol{r}' \int_{-\infty}^{t'} \mathrm{d}t'' \left\{ \frac{-iv_F}{\hbar} \Theta(t - t') \left\langle [J^i(t, \boldsymbol{r}), T^{0j}(t'', \boldsymbol{r}')] \right\rangle \right\} \partial_j' \psi(t', \boldsymbol{r}'). \tag{4.4}$$

Fourier transforming now to the frequency and momentum domain, will be beneficial in our calculations. As before, the non-perturbed system will be taken to be time and position invariant, such that the correlator in Eq. (4.4) can be taken to depend only on the differences $t - t''$ and $\boldsymbol{r} - \boldsymbol{r}'$. Starting with Fourier transforming the position part, notice that the structure of Eq. (4.4) is

$$\langle J^i \rangle (\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}' \chi(\boldsymbol{r} - \boldsymbol{r}') \partial_j' \psi(\boldsymbol{r}'),$$

[2]The paper is somewhat unclear on what is their final result, as there is some possible confusion related to the number of Landau levels included and whether one is including both or only one Dirac cone. The above result is what is meant, to the best of our understanding.

where the temporal parts were dropped for clarity. This is a convolution, and the Fourier transform is thus simply given by the product of the two factors [23].

$$\langle J^i \rangle (\boldsymbol{q}) = \chi(\boldsymbol{q})(iq_j)\psi(\boldsymbol{q}), \tag{4.5}$$

where it was also used that the Fourier transform of a derivative gives the component of the variable. Showing explicitly how to find the form of the response χ in momentum space is often overlooked in much literature, and as it does involve some finesse, we want to show it here. This trick is courtesy of Chang [5]. By definition, the Fourier transform of the response is, where the variable of integration has been chosen to be $\boldsymbol{r} - \boldsymbol{r}'$ for later convenience,

$$\chi(\boldsymbol{q}) = \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}')e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}\chi(\boldsymbol{r} - \boldsymbol{r}') \tag{4.6}$$

$$= \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}')e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}C\left\langle \left[J^i(\boldsymbol{r}), T^{0j}(\boldsymbol{r}')\right]\right\rangle, \tag{4.7}$$

$$\tag{4.8}$$

where C denotes t-dependent prefactors and integrals over time are omitted, again for clarity of notation. Note that

$$\int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{\mathcal{V}}\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}', \tag{4.9}$$

where $\mathcal{V}$ is the volume of the system. Thus,

$$\begin{aligned}\chi(\boldsymbol{q}) &= \frac{1}{\mathcal{V}}\int \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')}C\left\langle \left[J^i(\boldsymbol{r}), T^{0j}(\boldsymbol{r}')\right]\right\rangle \\ &= \frac{C}{\mathcal{V}}\left\langle \left[J^i(\boldsymbol{q}), T^{0j}(-\boldsymbol{q})\right]\right\rangle.\end{aligned} \tag{4.10}$$

Considering now the temporal part, the procedure is simpler. The linear response still has the form of a convolution, as the response function is only dependent on the difference $t - t'$ by

$$\chi(t - t') = \int_{-\infty}^{0} \mathrm{d}t''\Theta(t - t')\left\langle \left[J(t - t'), T(t'')\right]\right\rangle, \tag{4.11}$$

where t'' was shifted by t', and then the translational invariance of the correlator was used. In frequency space

$$\chi(\omega) = \int \mathrm{d}te^{i\omega t}\chi(t) \tag{4.12}$$

$$= \int \mathrm{d}te^{i\omega t}\int_{-\infty}^{0} \mathrm{d}t''\Theta(t)\left\langle \left[J(t), T(t'')\right]\right\rangle. \tag{4.13}$$

In frequency and momentum space the response function is thus

$$\chi^{ij}(w, \boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}\hbar} \int \mathrm{d}te^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \left\langle \left[J^i(t, \boldsymbol{q}), T^{0j}(t', -\boldsymbol{q})\right] \right\rangle . \tag{4.14}$$

4.0.1. Transport and magnetization

Recall that we generally define the transport coeffieicents

$$J^i = e^2 L_{11}^{ij} E_j + e L_{12}^{ij} \nabla_j T,$$

where J^i is the electrical current. In our work, we focus on the L_{12} coefficient, however the following discussion is valid also more generally. The definition of transport currents becomes more subtle in systems with broken time-reversal symmetry[29; 8]. In such systems, unobservable, circulating *magnetization* currents arise. These currents do not contribute to transport, but the Kubo treatment derives the local current, which in general also includes non-transporting currents. Let

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{tr}} + \boldsymbol{J}_M, \tag{4.15}$$

where $\boldsymbol{J}$ is the total local current, $\boldsymbol{J}_{\mathrm{tr}}$ is the transport current, and $\boldsymbol{J}_M$ is the circulating magnetization current. While our response χ relates to the total current, we are more interested in the experimentally measurable transport resonse L_{12}^{ij}, related to our Kubo result as [1]

this might not be a first-hand source. See thermal transport...geometry chernodub eq. 62

$$L_{12}^{ij} = \chi^{12} - \epsilon^{ijl} M_l, \tag{4.16}$$

with M_l the magnetization. For zero chemical potential, however, these magnetization currents have been shown to go to zero as $T \to 0$.

4.1. Eigenvalue problem of the Landau levels of a Weyl Hamiltonian

To evaluate the correlator of the response function, the matrix elements of the current and stress-energy tensor must be found. In order to do this, we find eigenstates in the Landau basis of the system. We will first consider the untilted Hamiltonian, which we will then use to find the Landau levels of the tilted Hamiltonian.

4.1.1. The untilted Hamiltonian

The Weyl Hamiltonian

$$H_s = s v_F \sigma^i \left(p_i + eA_i\right), \tag{4.17}$$

with s being the chirality, p_i the momentum operator, and $e = |e|$ the coupling constant to the electromagnetic field $\boldsymbol{A}$. Choose coordinates such that $\boldsymbol{B} = B_z\hat{\boldsymbol{z}}$, which in the Landau gauge gives $\boldsymbol{A} = -B_z y\,\hat{\boldsymbol{x}}$. As the Hamiltonian is invariant in x and z, take the plane wave ansatz $\phi(\boldsymbol{r}) = e^{ik_x x + ik_z z}\phi(y)$. It then follows

$$H_s\phi(\boldsymbol{r}) = E\phi(\boldsymbol{r}) \implies \tilde{H}_s\phi(y) = E\phi(y), \tag{4.18}$$

where $\tilde{H}$ is the result of replacing $p_z \to \hbar k_z, p_x \to \hbar k_x$ in H_s, as the plane wave part of ϕ have these eigenvalues. Absorb the chirality s as a sign in the velocity v_F, for more concise notation. Thus, writing everything explicitly, the spectrum is given by

$$-\hbar v_F \begin{pmatrix} -k_z & \partial_y + eyB_z/\hbar - k_x \\ -\partial_y + eyB_z/\hbar - k_x & k_z \end{pmatrix} \phi(y) = E\phi(y). \tag{4.19}$$

We will now find the spectrum E of the Hamiltonian.

Inspired by the derivation for the spectrum of the 2D Dirac Hamiltonian in [30], we introduce the length scale $l_B = \sqrt{\hbar/eB}$, and the dimensionless quantity $\chi = y/l_B - k_x l_B$. In dimensionless quantities Eq. (4.19) becomes

$$-\frac{\hbar v_F}{l_B} \begin{pmatrix} -k_z l_B & \partial_\chi + \chi \\ -\partial_\chi + \chi & k_z l_B \end{pmatrix} \phi(y) = E\phi(y). \tag{4.20}$$

Let the operators $a = \left(\chi + \partial_\chi\right)/\sqrt{2},\ a^\dagger = \left(\chi - \partial_\chi\right)/\sqrt{2}$. One may easily verify the commutation relation $[a, a^\dagger] = 1$; they are ladder operators of the harmonic oscillators, whose eigenstates are $|n\rangle$, and where $a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$. In terms of these operators, the system is

$$-\frac{\sqrt{2}\hbar v_F}{l_B} \begin{pmatrix} -\frac{k_z l_B}{\sqrt{2}} & a \\ a^\dagger & \frac{k_z l_B}{\sqrt{2}} \end{pmatrix} |\phi\rangle = E\,|\phi\rangle\,. \tag{4.21}$$

Take the ansatz

$$|\phi\rangle = \begin{pmatrix} \beta\,|n-1\rangle \\ \alpha\,|n\rangle \end{pmatrix}, \tag{4.22}$$

which is the most general form of $|\phi\rangle$ with any hope of being an eigenstate. This leads to

$$-\frac{\sqrt{2}\hbar v_F}{l_B} \begin{pmatrix} \left(-\gamma\beta + \alpha\sqrt{n}\right)|n-1\rangle \\ \left(\beta\sqrt{n} + \gamma\alpha\right)|n\rangle \end{pmatrix} = E\,|\phi\rangle\,, \tag{4.23}$$

with $\gamma = k_z l_B/\sqrt{2}$. For $n > 0$ this leads to the equation for ϕ to be an energy eigenfunction

$$-\gamma + \frac{\alpha}{\beta}\sqrt{n} = \frac{\beta}{\alpha}\sqrt{n} + \gamma. \tag{4.24}$$

Solving for α/β this gives

$$\frac{\alpha}{\beta} = \frac{\gamma}{\sqrt{n}} \pm \sqrt{1 + \frac{\gamma^2}{n}}, \tag{4.25}$$

and thus

$$E = \pm v_F \sqrt{\frac{2n\hbar^2}{l_B^2} + k_z^2\hbar^2} = \pm s v_F \sqrt{2neB\hbar + k_z^2\hbar^2}, \tag{4.26}$$

where we reintroduced the explicit s. For $n = 0$ the annihilation operator a destroys the vacuum state $|0\rangle$, and the energy is instead $E_0 = -\hbar s k_z v_F$. The excited energy states are doubly degenerate; we choose to denote the energy levels by $m \in \mathbb{Z}$, where the sign from $\pm s$ is taken care of by the sign of this quantum number, and the harmonic oscillator levels n are given by its absolute value $|m|$. The energy levels are

$$E_{k_z m s} = \operatorname{sign}(m) v_F \sqrt{2|m|eB\hbar + k_z^2\hbar^2} \qquad \text{for } m \neq 0, \tag{4.27a}$$

$$E_{k_z 0 s} = -s\hbar k_z v_F \qquad \text{for } m = 0. \tag{4.27b}$$

We now find the corresponding eigenvectors of the system. The solution to the one dimensional harmonic oscillator in position space is, in dimensionless coordinates ξ, [20, Eq. 18.39.5]

$$\langle \xi | n \rangle = \phi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \pi^{-\frac{1}{4}} e^{-\frac{\xi^2}{2}} H_n(\xi), \tag{4.28}$$

where H_n are the Hermite polynomials. Thus,

$$\langle \chi | \phi \rangle = \begin{pmatrix} \beta \langle \chi | n-1 \rangle \\ \alpha \langle \chi | n \rangle \end{pmatrix} = e^{-\frac{\chi^2}{2}} \begin{pmatrix} \frac{\beta}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}} H_{n-1}(\chi) \\ \frac{\alpha}{\sqrt{2^n n! \sqrt{\pi}}} H_n(\chi) \end{pmatrix}, \tag{4.29}$$

where we defined $H_{-1} = 0$ in order to get a more general expression. Choosing

$$\alpha = \sqrt{\frac{\gamma^2}{n}} \implies \beta = \frac{1}{1 \pm \sqrt{1 + \frac{n}{\gamma^2}}} = \pm \frac{\gamma^2}{n}\left(\sqrt{1 + \frac{n}{\gamma^2}} - 1\right), \tag{4.30}$$

gives

$$\phi(\chi) = e^{-\frac{\chi^2}{2}} \sqrt{\frac{\gamma^2}{n}} \begin{pmatrix} \frac{\pm\sqrt{\frac{\gamma^2}{n}}\left(\sqrt{1+\frac{n}{\gamma^2}}-1\right)}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}} H_{n-1}(\chi) \\ \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} H_n(\chi) \end{pmatrix}. \tag{4.31}$$

There are thus four quantum numbers related to the eigenvectors, k_x, k_z, m, s. Reintroducing $\chi = (y - k_x l_B^2)/l_B$ and normalizing

$$\phi_{\boldsymbol{k}ms}(\boldsymbol{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{\left(y - k_x l^2\right)^2}{2 l_B^2}} \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y - k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M\left(\frac{y - k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{4.32}$$

where capital letters indicate absolute value of corresponding quantity, $M = |m|, \boldsymbol{k} = (k_x, k_z)$, and with the normalization factor

$$\alpha_{k_z ms} = \frac{-\sqrt{2eB\hbar M}}{\frac{E_{k_z ms}}{s v_F} - \hbar k_z}. \tag{4.33}$$

4.1.2. The tilted Hamiltonian

Consider which formalism to use for this section. Should we already here use the geometry, or keep it with parallell, perpendicular?

I think it is better to use perp, parallell here, and then transition to using the explicit geometry later

The eigenvalues of a Type-II Weyl semimetal are simple to find, and are not qualitatively different from those of Type-I, other than the appearance of particle and hole pockets at the Fermi level. We will also consider the Landau levels of these materials, which importantly are very different from Type-I. In fact, erroneous treatment of the Landau spectrum of Type-II semimetals caused the original paper describing Type-II materials to mistakenly assert that the chiral anomaly would not be present for certain directions of a background magnetic field [26][25].

Eigenstates, spin, berry, etc

The issue with the Landau level description is that for certain directions of the B-field, the Landau levels break down. For Type-I materials, the description is valid for all directions of the B-field, but as the cone tip into a Type-II material, the description breaks down when the B-field and tilt direction are perpendicular [25], and as the magnitude of the tilt increases, the Landau levels are only valid up to a certain angle between the tilt direction and magnetic field. We will in this section derive and elucidate the Landau levels and their regions of validity.

Consider again the Hamiltonian [3]

$$H = v_F \boldsymbol{t}^s \boldsymbol{k} + s v_F \boldsymbol{k}\boldsymbol{\sigma}, \tag{4.34}$$

with the *tilt vector* as defined in Eq. (1.62)

$$\boldsymbol{t}^s = \begin{cases} \boldsymbol{t} & \text{broken inversion symmetry,} \\ s\boldsymbol{t} & \text{inversion symmetry.} \end{cases}$$

To find the Landau levels in a magnetic field $\boldsymbol{B} = B_z \hat{z}$, we will "Lorentz boost" the system to a frame where the cone is not tilted, where we may use the usual approach for finding the Landau levels.

Generally, consider $\boldsymbol{t}$ to consist of two components, $\boldsymbol{t}_{\|}$ which is parallel to the magnetic field, and $\boldsymbol{t}_{\perp}$ perpendicular to the magnetic field. In this work, we restrict ourselves to the case where the perpendicular component is parallel to the charge current, i.e. the x direction in the chosen geometry. The $\boldsymbol{t}_{\perp}$ vector may of course also have a component parallel to the temperature gradient ∇T, in this geometry the y direction, which, although interesting, is not considered here. Thus, let $\boldsymbol{t} = (t_{\perp}, 0, t_{\|})$. Introduce the $\boldsymbol{B}$-field by the minimal coupling $\boldsymbol{k} \to \boldsymbol{k}^B = \boldsymbol{k} + e\boldsymbol{A}$. We take the field to be in the z-direction, and use the Landau gague $\boldsymbol{A} = -B_z y \hat{x}$.

Before applying the temperature gradient, when we still only consider finding the LLs, we may in fact say that we have generally $t_{\perp}$ only in x, and hen later rotate into coordinates where ∇T is in y. Maybe that is interesting after all...?

The Landau level equation is

$$(H_B - E)\,|\psi\rangle = 0, \tag{4.35}$$

with

$$H_B = v_F \left(t^s_{\perp} k^B_x + t^s_{\|} k^B_z \right) \mathcal{I}_2 + \sum_i s v_F k^B_i \sigma_i, \tag{4.36}$$

where $\mathcal{I}_2$ is the identity matrix of size 2. In order to use the ladder operator method used for the untilted cone, we must get rid of the k^B_x on the diagonal

[3] In general, the Fermi velocity may be anisotropic, in which case the momentum enter as $v_i k_i$, instead of $v_F k_i$. By a rescaling of the momenta, we may consider any, in general anisotropic, system to be isotropic in velocity.

of the Hamiltonian. [4] To achieve this, we will use a "Lorentz transformation", which as we will show only leave k_z and E in the diagonal. Act with the hyperbolic rotation operator $R = \exp[\Theta/2\sigma_x]$ on Eq. (4.35), and insert identity on the form $\exp[\Theta/2\sigma_x]\exp[-\Theta/2\sigma_x]$ before the state vector. By introducing the state in the rotated frame $|\tilde{\psi}\rangle = \exp[-\Theta/2\sigma_x]\mathcal{N}\,|\psi\rangle$, with $\mathcal{N}$ a normalization factor compensating for the non-unitarity of the transformation, we get the eigenvalue equation

$$(\exp[\Theta/2\sigma_x]H_B\exp[\Theta/2\sigma_x] - E\exp[\Theta\sigma_x])\,|\tilde{\psi}\rangle = 0. \tag{4.37}$$

We now make the fortunate observation that the diagonal elements of

$$R\sigma_i R$$

are zero for $i = y$ and non-zero for $i = x, z$. We may thus rotate the x and z in and out of the diagonal elements, without accidentaly rotating the y components into the diagonal.

The problematic part of the Hamiltonian with regards to finding the Landau levels, are the terms containing k_x^B on the diagonal, i.e.

$$v_F t_\perp^s k_x^B \mathcal{I}_2 + s v_F k_x^B \sigma_x.$$

We will now find the boost parameter that eliminates k_x from the diagonal. We have

$$R^2 = e^{\Theta\sigma_x} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{4.38}$$

and as $[R, \sigma_x] = 0$,

$$R\sigma_x R = R^2\sigma_x = \begin{pmatrix} \sinh\theta & \cosh\theta \\ \cosh\theta & \sinh\theta \end{pmatrix}, \tag{4.39}$$

as the effect of σ_x is to transpose the rows. The requirement for k_x^B to be rotated out of the diagonal is thus

$$t_\perp^s \cosh\theta + s\sinh\theta = 0. \tag{4.40}$$

Solving for θ we get

$$\theta = \log(\pm\frac{\sqrt{s - t_\perp^s}}{\sqrt{s + t_\perp^s}}). \tag{4.41}$$

[4] It would also be possible to choose the frame such that the tilt was both in x and y direction, in which case we would get ladder operators also on the diagonal. This system, albeight tedious, could also have been solved directly.

Verify this

NB: depending of choice of sign in log, we get different signs in answer

Alternatively, written in a slightly suggestive form,

$$\tanh\theta = -st_{\perp}^{s}. \tag{4.42}$$

For pedagogic reasons, include arctanh, which is only valid for -1 ¡ x ¡ 1, explicitly showing the collapse?

The required hyperbolic tilt angle to eliminate the k_x^B in the diagonal elements of the Hamiltonian, originating from the tilt, is thus

$$\theta = -s\tanh^{-1} t_x^s. \tag{4.43}$$

The inverse of tan, of course, diverges as the argument approaches ± 1, as shown in Figure 4.2. For $|t_x| < 1$ we are able to find an angle θ which transforms our

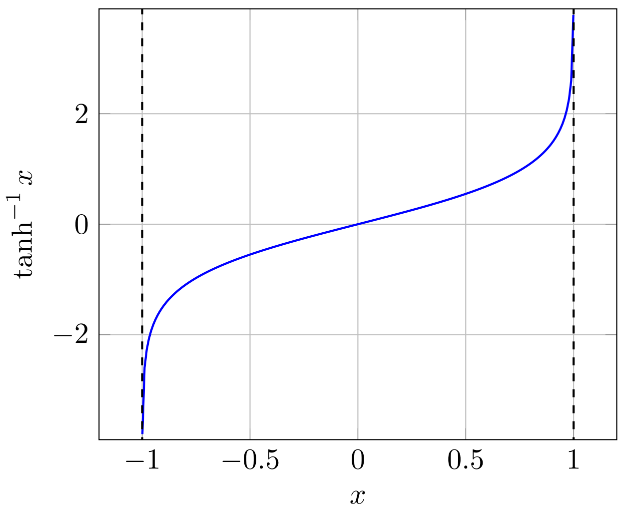


Figure 4.2.: Plot of $\tanh^{-1}$, which diverges as the argument goes to ± 1.

Hamiltonian into a form which we may solve. For $|t_x| \geq 1$, however, no (real) solution of θ exits, and the Landau level description collapses. More concretely, as we will show later, the separation of the landau levels is reduced as the perpendicular tilt increases, and as $|t_x| \to 1$, the level separation $\Delta E \to 0$.

Discuss magnetic vs electric regime

Interestingly, there are no restrictions in the perpendicual tilt, t_z. The $\boldsymbol{t}$ parametrization of the tilt is conveniently visualized by plottin the t-vector inside a unit sphere, shown in Figure 4.3. If the vector is outside the unit sphere, it is a Type-II, if it is inside, it is a Type-I. Also, if the projection of the vector onto the x, y-plane is on the unit disk, the Landau level description is valid, if not, the Landau levels collapse. When the projection is on the unit disc, the system is in the *magnetic* regime, otherwise we denote it by the *electric* regime. All Type-I materials may thus be described by Landau levels, while it for Type-II is only valid for certain directions of the t-vector. As the t-vector gets larger, the magnetic regime is restricted to smaller angles between $\boldsymbol{t}$ and $\boldsymbol{B}$.

`figures/tiltSpherewBackground.png`

Figure 4.3.: TODO

We now return to solving Eq. (4.37), using the solution angle we just found.

By insertion, and after some clean up, we get

$$(\exp[\Theta/2\sigma_x]H_B\exp[\Theta/2\sigma_x]-E\exp[\Theta\sigma_x])\,|\tilde{\psi}\rangle=0=v_F \times\begin{pmatrix} k_z(s+t_z^s\gamma)-E/v_F\gamma & -s(ik_y+k_zt_x^st_z^s\gamma-k_x/\gamma-E/v_F\gamma t_x^s) \\ s(ik_y-k_zt_x^st_z^s\gamma+k_x/\gamma+E/v_F\gamma t_x^s) & -k_z(s-t_z^s\gamma)-E\gamma \end{pmatrix}|\tilde{\psi}\rangle\,. \tag{4.44}$$

In order to simplify this further, absorb $\gamma t_x^s(k_zt_\parallel^s-E/v_F)$ into k_x. Thus, let

$$\begin{aligned}\tilde{k}_x&=k_x/\gamma+\gamma t_x^s(E/v_F-k_zt_\parallel^s),\\ \tilde{k}_y&=k_y,\\ \tilde{k}_z&=k_z.\end{aligned} \tag{4.45}$$

These expressions warrant some explanation, as the Lorentz boost is of course

$$\tilde{k}_x=\gamma(k_x-\beta\frac{E}{v_F}), \tag{4.46}$$

comment on beta = tx, or change to tx

where we used the four momentum $p^\mu=(\frac{E}{v_F},\boldsymbol{p})$, and the effective speed of light v_F. It can thus look like our expression in Eq. (4.45) is wrong. The solution to this seeming inconsistency is that the proper energy is not $\frac{E}{v_F}-k_zt_\parallel$, but rather $\frac{E}{v_F}-k_zt_\parallel-k_xt_\perp$.

Something smart here

The eigenvalue equation is simply

$$\left[\gamma\left(t_\parallel^s\tilde{k}_z-\frac{E}{v_F}\right)\mathcal{I}_2+s\tilde{k}_i\sigma_i\right]|\tilde{\psi}\rangle=0. \tag{4.47}$$

If we now again introduce the magnetic field using minimal coupling, $k_x\to k_x-eyB_z$, this corresponds to an effective field $B_z\gamma$ in the new quantities. This is because $\tilde{k}_x\to\tilde{k}_x-eyB_z/\gamma$.

The Landau level equation thus reads

$$\left[\sum_i sv_F\left(\tilde{k}_i+e\tilde{A}_i\right)\sigma_i\right]|\tilde{\psi}\rangle=(E-t_\parallel^sv_F\tilde{k}_z)\gamma\,|\tilde{\psi}\rangle\,, \tag{4.48}$$

where $\tilde{\boldsymbol{A}} = -B_z/\gamma y\hat{x}$. We may thus use directly the result for the untilted cone, Eq. (4.27), giving

$$\left(E - t^s_{\parallel} v_F \tilde{k}_z\right)\gamma = \operatorname{sign}(m) v_F \sqrt{2|m|e\frac{B}{\gamma}\hbar + \tilde{k}_z^2\hbar^2}, \qquad m \neq 0, \tag{4.49a}$$

$$\left(E - t^s_{\parallel} v_F \tilde{k}_z\right)\gamma = -s\hbar\tilde{k}_z v_F, \qquad m = 0. \tag{4.49b}$$

Cleaning up, we get

$$E = t^s_{\parallel} v_F \tilde{k}_z + \operatorname{sign}(m) v_F \sqrt{2|m|e\frac{B}{\gamma^3}\hbar + \tilde{k}_z^2\hbar^2/\gamma^2}, \qquad m \neq 0, \tag{4.50a}$$

$$E = \tilde{k}_z v_F \left(t^s_{\parallel} - s\hbar/\gamma\right), \qquad m = 0. \tag{4.50b}$$

As the perpendicular tilt is increased, $\gamma = 1/\sqrt{1-\beta^2}$ diverges to infinity. With the trivial substitution $\alpha = \frac{1}{\gamma}$, which goes to zero, this gets an intuitive interpretation.

$$E = t^s_{\parallel} v_F \tilde{k}_z + \operatorname{sign}(m) v_F \alpha \sqrt{2|m|eB\alpha\hbar + \tilde{k}_z^2\hbar^2}. \tag{4.51}$$

As the perpendicular tilt increases, the Landau levels converge towards $t_{\parallel} v_F \tilde{k}_z$. In particular, the separation between Landau levels m

maybe use the word cyclotron frequency

is reduced by a factor $\alpha^{\frac{3}{2}}$. The effect of the tilt on the Landau levels is to squeeze the Landau levels together, and we will call the α the *squeezing factor*. We note that when approaching the degree of tilt where we are no longer able to find a boost which enables us to solve for the Landau levels, i.e. when $\beta \to 1$, the squeezing factor goes to zero. As the tilt exceeds this limit, the squeezing factor is imaginary. Note also that the energy levels

$$E = t^s_{\parallel} v_F k_z + \alpha E^0_{m,\alpha B},$$

where $E^0_{m,\alpha B}$ is the energy in the untilted case, with magnetic field αB. Tilting of the Landau levels is induced by the parallel tilt component, $t_{\parallel}$. In fact, the Landau levels cross the Fermi surface at the transition from Type-I to Type-II as well. The Landau levels are shown in Figure 4.4.

The eigenstate of

$$H = v_F \sigma^i (p_i + eA_i),$$

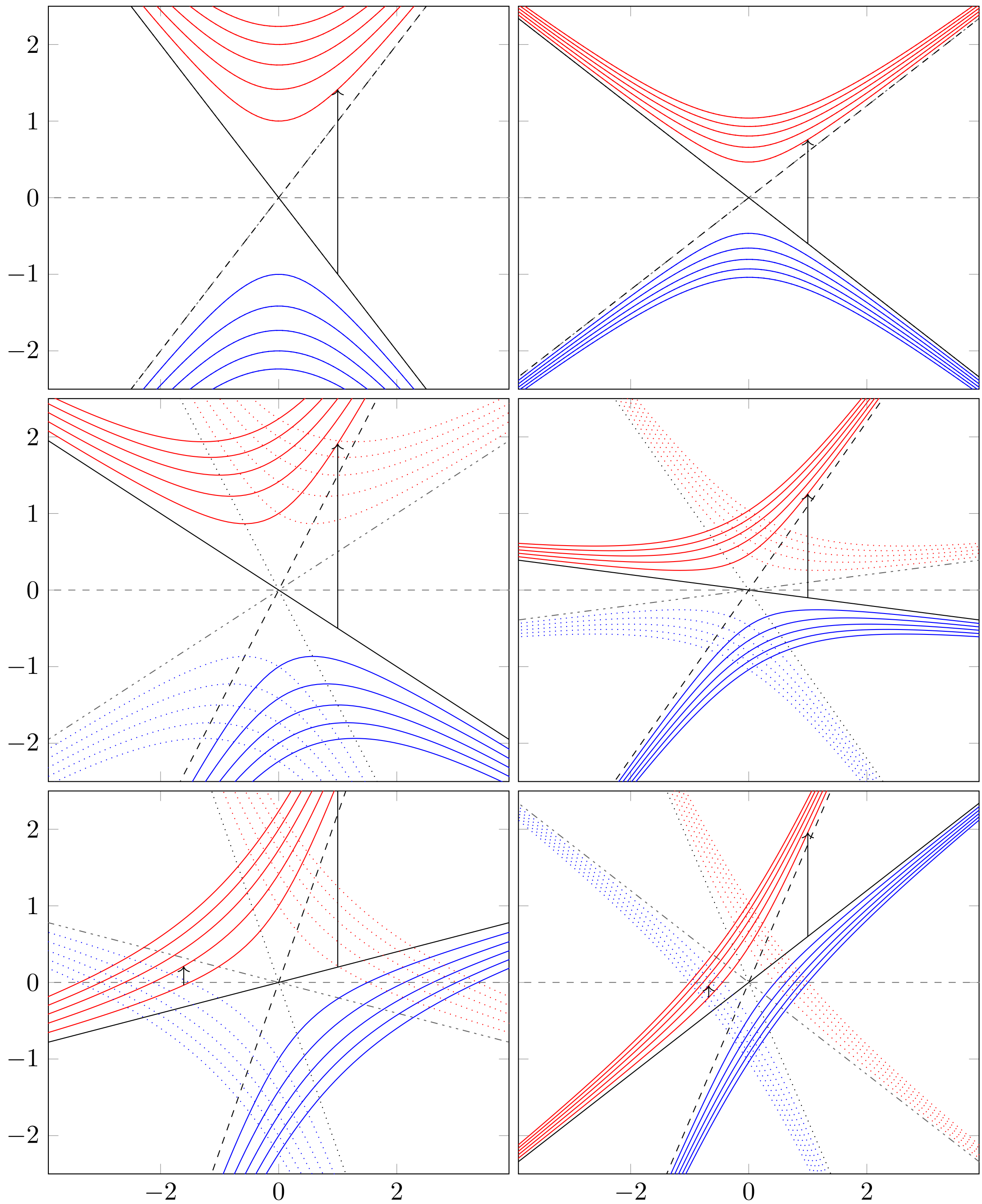


Figure 4.4.: Landau levels for different values of t_x, t_z. The top two rows show Type-I, while the lowest row shows Type-II. Left column shows $t_x = 0$, right column $t_x = 0.64$ ($\alpha = 0.6$). The rows shows $t_z = 0, 0.5, 1.2$, from top to bottom. The dotted lines show the Landau levels with opposite sign of t_z.

with $A_i = -B_z y \delta_{ix}$, given in the position basis, is

$$\phi_{\boldsymbol{k}ms}(\boldsymbol{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{y - k_x l^2}{2l_B^2}} \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M\left(\frac{y - k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{4.52}$$

where capital letters indicate absolute value of corresponding quantity, $M = |m|, \boldsymbol{k} = (k_x, k_z)$, and with the normalization factor

$$\alpha_{k_z ms} = \frac{-\sqrt{2eB\hbar M}}{\frac{E_{k_z ms}}{s v_F} - \hbar k_z}. \tag{4.53}$$

Taking care to keep track of boosted and rescaled quantites, the eigenstate in the boosted frame is

$$\tilde{\psi}(\tilde{\boldsymbol{r}}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{i\tilde{k}_x \tilde{x}} e^{ik_z z}}{\sqrt{\alpha_{\tilde{k}_z ms}^2 + 1}} e^{-\frac{\left(\tilde{y} - \tilde{k}_x l_{B'}^2\right)^2}{2l_{B'}^2}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_{B'}}} H_{M-1}\left(\frac{\tilde{y}-\tilde{k}_x l_{B'}^2}{l_{B'}}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_{B'}}} H_M\left(\frac{\tilde{y} - \tilde{k}_x l_{B'}^2}{l_{B'}}\right) \end{pmatrix}, \tag{4.54}$$

with

$$\alpha_{\tilde{k}_z ms} = \frac{-\sqrt{2eB'\hbar M}}{\gamma \frac{E_{\tilde{k}_z ms} - t_\parallel^s v_F \tilde{k}_z}{s v_F} - \hbar \tilde{k}_z}, \tag{4.55}$$

where

$$B' = B\alpha.$$

We note that $\alpha_{k_z 0 s} = 0$, so using the explicit form of the energy we may simplify the expression some. For $m \neq 0$

$$\frac{E_{k_z ms} - t_\parallel^s v_F k_z}{s v_F} = \operatorname{sign}(m) s \alpha \sqrt{2MeB\alpha + k_z^2}$$

and thus

$$\alpha_{k_z ms} = \frac{-\sqrt{\alpha M}}{\operatorname{sign}(m) s \sqrt{\alpha M + \kappa^2} - \kappa} \tag{4.56}$$

where we defined the dimensionless $\kappa_z = \sqrt{2eB} k_z$.

The original eigenstate $|\psi\rangle = 1/\mathcal{N} e^{\theta/2\sigma_x} |\tilde{\psi}\rangle$ of the tilted system is easily found. Reinserting explicitly, in the boosted frame, that

$$\tilde{k}_x = \alpha k_x + \frac{t_x^s}{\alpha}(E_{k_z ms}/v_F - k_z t_\parallel^s) = \alpha k_x + t_x^s \frac{E_{m,\alpha B}^0}{v_F}$$

and $l_{B'} = \frac{l_B}{\sqrt{\alpha}}$ we define

$$\chi = \frac{y - \tilde{k}_x l_{B'}^2}{l_{B'}} = \sqrt{\alpha}(y - k_x l_B^2)/l_B + \frac{t_x^s l_B}{\sqrt{\alpha} v_F} E_{m,\alpha B}^0, \tag{4.57}$$

which is the argument of the Hermite polynomials. For later convenience, let us explicitly define

$$\tilde{\phi}_{\boldsymbol{k}ms}(\tilde{\boldsymbol{r}}) = \frac{e^{i\tilde{k}_x\tilde{x}+ik_z z}}{\sqrt{L_x L_z}} \underbrace{\frac{e^{-\frac{1}{2}\chi^2}\sqrt[4]{\alpha}}{\sqrt{\alpha_{\tilde{k}_z ms}^2+1}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}(\chi) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M(\chi) \end{pmatrix}}_{\tilde{\phi}_{\boldsymbol{k}ms}(y)}, \tag{4.58}$$

and thus

$$\tilde{\phi}_{\boldsymbol{k}ms}(y) = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix}, \tag{4.59}$$

with

$$a_{\boldsymbol{k}ms} = \frac{\alpha_{\tilde{k}_z ms}\sqrt[4]{\alpha}}{\sqrt{\alpha_{\tilde{k}_z ms}^2+1}\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}}, \tag{4.60}$$

$$b_{\boldsymbol{k}ms} = \frac{\sqrt[4]{\alpha}}{\sqrt{\alpha_{\tilde{k}_z ms}^2+1}\sqrt{2^M M!\sqrt{\pi} l_B}}. \tag{4.61}$$

We proceed now to find the normalization factor $\mathcal{N}$, as it will become necessary in later steps. Recall that

$$|\psi\rangle = \frac{1}{\mathcal{N}} e^{\theta/2\sigma_x} |\tilde{\psi}\rangle,$$

and

$$e^{\theta\sigma_x} = \frac{1}{\alpha} \begin{pmatrix} 1 & -st_x^s \\ -st_x^s & 1 \end{pmatrix}.$$

The upper and lower part of the spinor are orthogonal, thus we have

$$\langle\psi|\psi\rangle = \frac{1}{\mathcal{N}^*\mathcal{N}} \frac{1}{\alpha} \langle\tilde{\psi}|\tilde{\psi}\rangle = 1 \implies \mathcal{N}^*\mathcal{N} = \frac{1}{\alpha}. \tag{4.62}$$

We choose $\mathcal{N} = \alpha^{-\frac{1}{2}}$.

Summary 1

The tilted Hamiltonian

$$H = v_F \boldsymbol{t}^s \boldsymbol{k} + s v_F \boldsymbol{k}\boldsymbol{\sigma}$$

in a magnetic field $\boldsymbol{B}$ has the Landau levels

$$E = \begin{cases} t_{\parallel}^{s} v_F k_z + \operatorname{sign}(m) v_F \alpha \sqrt{2eB\alpha M + k_z^2} & m \neq 0, \\ t_{\parallel}^{s} v_F k_z - s\alpha v_F k_z & m = 0. \end{cases}$$

The associated eigenstates in the position basis are

$$\psi(\boldsymbol{r}) = \sqrt{\alpha} e^{\theta/2\sigma_x} \frac{e^{ik_x x + ik_z z}}{\sqrt{L_x L_z}} \psi(y),$$

where

$$\psi(y) = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{k_z m s} H_{M-1}(\chi) \\ b_{k_z m s} H_M(\chi) \end{pmatrix},$$

where we have defined $\chi = \sqrt{\alpha}\frac{y - k_x l_B^2}{l_B} + \frac{t_x^s l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}$ and $a_{k_z m s}, b_{k_z m s}$ are given in Eqs. (4.60, 4.61).

4.2. Analytical expressions for the operators

We will here find analytical expressions for the current operator $J^i(\omega, \boldsymbol{q})$ and stress-energy tensor $T^{0j}(\omega, \boldsymbol{q})$, needed to calculate the correlation function. The fields are given, in the position basis, by

$$\psi = \sum_{\boldsymbol{k}n} \langle \boldsymbol{r} | \boldsymbol{k}ns \rangle \, a_{\boldsymbol{k}ns}(t) = \sum_{\boldsymbol{k}n} \phi_{\boldsymbol{k}ns}(\boldsymbol{r}) a_{\boldsymbol{k}ns}(t), \tag{4.63}$$

$$\psi^\dagger = \sum_{\boldsymbol{k}n} \langle \boldsymbol{k}ns | \boldsymbol{r} \rangle \, a^\dagger_{\boldsymbol{k}ns}(t) = \sum_{\boldsymbol{k}n} \phi^*_{\boldsymbol{k}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ns}(t). \tag{4.64}$$

Here $a^\dagger_\lambda(t) = \exp(iE_\lambda t/\hbar) a^\dagger_\lambda$ and $a^\dagger_\lambda, a_\lambda$ are the creation and annihilation operators of the state with quantum numbers λ. The current operator $\hat{\boldsymbol{J}} = e\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is the velocity operator. Using the relation of Heisenberg operators $\dot{A} = [A, H]/i\hbar$ [24], for the operator A and Hamiltonian H, the operator

$$\begin{aligned} \boldsymbol{v} = \dot{\boldsymbol{r}} &= \frac{1}{i\hbar} [\boldsymbol{r}, H] && (4.65) \\ &= \frac{s v_F \sigma^i}{i\hbar} [\boldsymbol{r}, p_i + eA_i] + \frac{v_F}{i\hbar} [\boldsymbol{r}, \boldsymbol{t}^s \boldsymbol{k}] && (4.66) \\ &= \frac{s v_F \sigma^i}{i\hbar} (i\hbar \hat{\boldsymbol{x}}_i + e[\boldsymbol{r}, A_i]) + v_F \boldsymbol{t}^s && (4.67) \\ &= s v_F \sigma^i \hat{\boldsymbol{x}}_i + v_F \boldsymbol{t}^s, && (4.68) \end{aligned}$$

and thus

$$J^x = \psi^\dagger \hat{J}^x \psi = s v_F e \sum_{\boldsymbol{k}m,\boldsymbol{l}n} \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) \left(\sigma^x + s t^s_x\right) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t). \tag{4.69}$$

4.2.1. The energy momentum tensor

The *canonical* energy-momentum tensor is generally defined by

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \partial_\nu \phi_i - \eta^{\mu\nu} \mathcal{L}, \tag{4.70}$$

where the index i runs over the types of fields. This definition is correct for commuting fields, however, for non-commuting fields like ours, this formula is slightly wrong. This is often overlooked in many textbooks and papers, so we will here elucidate the issue to some degree. While a proper derivation requires the use of Grassman variables and defining left and right derivation, which we will not do here, some simple considerations help in understanding the issue. In the standard textbook derivation of then canonical energy-momentum tensor, one expands the total derivative of the Lagrangian $\mathcal{L}(\psi_i, \partial\psi_i)$ in terms of the fields

$$\frac{\mathrm{d}\mathcal{L}(\psi_i, \partial\psi_i)}{\mathrm{d}x_\nu} \equiv \mathrm{d}^\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial(\partial_\mu \psi_i)}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_\nu}. \tag{4.71}$$

This expansion, however, ignores the non-commutative nature of the fields. For concreteness, consider $\psi_i = \bar{\psi}$. Heuristically, the correct expression would be obtained by reordering the factors in the two terms. By naively employing Eq. (4.70), the resulting canonical energy-momentum tensor of the Dirac theory would be

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} \partial^\nu \bar{\psi} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}, \tag{4.72}$$

while the correct form is [14, Eq. 3-153]

$$T^{\mu\nu} = \partial^\nu \bar{\psi} \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}. \tag{4.73}$$

Our Hamiltonian

$$H_s = s\sigma^i k_i$$

may of course be considered as a Weyl decomposition of a full massless Dirac equation.

Why do we have to consider 4x4? Is the definitions not also valid for 2x2?

Regarding the non-symmetry of the stress tensor, see keichelriess eq 5.16 with discussion

The Hamiltonian

$$H_s = s\sigma^i k_i$$

can be considered the Hamiltonian of one part of a Weyl decomposition of a Dirac system. The Weyl field has the Lagrangian density [15]

$$\mathcal{L} = i\phi^\dagger \sigma^\mu \partial_\mu \phi, \tag{4.74}$$

which may be seen directly from the Dirac Lagrangian $i\bar{\psi}\partial\!\!\!/\psi$ by taking $\psi = (\phi_L, \phi_R)^T$ and set, for example, $\phi_R = 0$. Symmetrizing in daggered and undaggered fields [5]

Alternatively argue by directly shownning that this does not affect the action by doing an integration by parts

$$\mathcal{L} = \frac{i}{2}\left(\phi^\dagger \sigma^\mu \partial_\mu \phi - \partial_\mu \phi^\dagger \sigma^\mu \phi\right),$$

which will prove to be more convenient to work with. Adapting the definition Eq. (4.73) the energy-momentum tensor for the untilted Dirac cone is thus

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \sigma^\mu \partial_\nu \phi - \sigma^\mu \phi \partial_\nu \phi^\dagger - \eta^{\mu\nu}\mathcal{L}). \tag{4.75}$$

Moving now to the tilted case, the 4x4 Lagrangian becomes [29]

check sign compared to action in stoof

$$\mathcal{L}_{\text{tilt}} = i\bar{\psi}\Gamma^\mu \partial_\mu \psi, \tag{4.76}$$

where we have introduced modified gamma matrices

$$\Gamma^\mu = \begin{cases} \gamma^\mu + t^\mu \gamma^0 & \text{inversion symmetry broken,} \\ \gamma^\mu + t^\mu \gamma^0 \gamma^5 & \text{inversion symmetric,} \end{cases} \tag{4.77}$$

with $t^\mu = (0, \boldsymbol{t})$. Decomposing to 2×2, this yields

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}^\mu \partial_\nu \phi - \tilde{\sigma}^\mu \phi \partial_\nu \phi^\dagger - \eta^{\mu\nu}\mathcal{L}), \tag{4.78}$$

[5] The Lagrangian itself is unphysical, and we may transform it in any way that leaves the action $\int \mathcal{L}$ invariant.

where we defined the modified Pauli matrices

$$\tilde{\sigma}^\mu = \begin{cases} \sigma^\mu + t^\mu & \text{inversion symmetry broken,} \\ \sigma^\mu + s t^\mu & \text{inversion symmetric.} \end{cases} \tag{4.79}$$

Similarly, the T^{0y} component of the stress-energy tensor of the theory is given by [1]

$$\begin{aligned} T^{0y}(t, \boldsymbol{r}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} \frac{1}{4} \Big\{ & \left[v_F \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) p_y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) - v_F \left(p_y \phi^*_{\boldsymbol{k}ms} \right) \phi_{\boldsymbol{l}ns} \right] a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t) \\ & + \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s \sigma^y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) \left[a^\dagger_{\boldsymbol{k}ms}(t) i\hbar\partial_0 a_{\boldsymbol{l}ns}(t) - i\hbar \left(\partial_0 a^\dagger_{\boldsymbol{k}ms}(t) \right) a_{\boldsymbol{l}ns}(t) \right] \\ & + \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s \sigma^y (2\mu) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t) \Big\}. \end{aligned} \tag{4.80}$$

Here, also a non-zero potential μ is included. Our final result will be given at zero potential, however it is included in the calculations as it might be of interest to consider finite potential in later work. Recalling the time dependence of $a(t), a^\dagger(t)$ we have that

$$i\hbar\partial_0 a_\lambda(t) = E_\lambda a_\lambda, \quad i\hbar\partial_0 a^\dagger_\lambda(t) = -E_\lambda a^\dagger_\lambda,$$

which further simplifies the expression.

Fourier transforming the position gives

$$J^x(t, \boldsymbol{q}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t), \tag{4.81}$$

$$T^{0y}(t, -\boldsymbol{q}) = \sum_{\boldsymbol{k}m,\boldsymbol{l}n} T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t), \tag{4.82}$$

where the matrix elements in momentum space are given by

$$J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) = \int \mathrm{d}\boldsymbol{r} e^{-i\boldsymbol{q}\boldsymbol{r}} s v_F e \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) \sigma^x \phi_{\boldsymbol{l}ns}(\boldsymbol{r}), \tag{4.83}$$

$$\begin{aligned} T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) = & \frac{1}{4} \int \mathrm{d}\boldsymbol{r} e^{i\boldsymbol{q}\boldsymbol{r}} \left[v_F \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) p_y \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) - v_F (p_y \phi^*_{\boldsymbol{k}ms}) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}) \right] \\ & + \frac{1}{4} \int \mathrm{d}\boldsymbol{r} e^{i\boldsymbol{q}\boldsymbol{r}} \phi^*_{\boldsymbol{k}ms}(\boldsymbol{r}) s \sigma^y (E_{\boldsymbol{k}_z ms} + E_{\boldsymbol{l}_z ns} - 2\mu) \phi_{\boldsymbol{l}ns}(\boldsymbol{r}). \end{aligned} \tag{4.84}$$

Note that as $T^{0y}(t, -\boldsymbol{q})$ will be used later, we here for convenience included the sign into the definition of the matrix element $T^{0y}_{\boldsymbol{k}ms,\boldsymbol{l}ns}$, as is reflected in the sign of the exponent of Eq. (4.84).

As was noted earlier, the eigenvectors are plane waves in the x, z-directions, and the non-trivial part is the y-dependent $\phi(y)$. Thus, we want to express these matrix elements in terms of $\phi(y)$. The sum over $\boldsymbol{l}$ in Eq. (4.81) can be replaced by an integration, as it is a good quantum number. As usual, the measure in the integration is given by the density of states in momentum space, the well known $L_i/2\pi$, with L_i being the length of the system in the i-direction.

$$\begin{aligned} J^x(t, \boldsymbol{q}) &= \sum_{\boldsymbol{k}m,n} \int \mathrm{d}l_x \mathrm{d}l_z \frac{L_x L_z}{4\pi^2} J^x_{\boldsymbol{k}ms,\boldsymbol{l}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{l}ns}(t) \qquad (4.85) \\ &= \int \mathrm{d}l_x \mathrm{d}l_z \int \mathrm{d}y e^{-iq_y y} \delta(l_x - k_x - q_x)\delta(l_z - k_z - q_z) s v_F e \phi^*_{\boldsymbol{k}ms}(y) \sigma^x \phi_{\boldsymbol{l}ns}(y). \end{aligned}$$

The Dirac delta functions appeared from taking the integrals from the matrix element over x and z, as the integrand in these variables was only plane waves. The exact same procedure may be done for the stress-energy tensor in Eq. (4.82). Eliminating l by doing the integrals yields

$$J^x(t, \boldsymbol{q}) = \sum_{\boldsymbol{k},mn} J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) a^\dagger_{\boldsymbol{k}ms}(t) a_{\boldsymbol{k}+\mathfrak{q}ns}(t), \qquad (4.86)$$

$$T^{0y}(t, -\boldsymbol{q}) = \sum_{\boldsymbol{\kappa},\mu\nu} T^{0y}_{\boldsymbol{\kappa}\mu s,\boldsymbol{\kappa}-\mathfrak{q},\nu s}(\boldsymbol{q}) a^\dagger_{\boldsymbol{\kappa}\mu s}(t) a_{\boldsymbol{\kappa}-\mathfrak{q}\nu s}(t), \qquad (4.87)$$

where $\mathfrak{q} = (q_x, q_z)$. Keeping in mind that $a^\dagger_\lambda(t) = e^{iE_\lambda t/\hbar} a^\dagger_\lambda$, and that

$$\left\langle \left[a^\dagger_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathfrak{q}ns}, a^\dagger_{\boldsymbol{\kappa}\mu s} a_{\boldsymbol{\kappa}-\mathfrak{q}\nu s} \right] \right\rangle = \delta_{\boldsymbol{k},\boldsymbol{\kappa}-\mathfrak{q}} \delta_{m,\nu} \delta_{\boldsymbol{k}+\mathfrak{q},\boldsymbol{\kappa}} \delta_{n,\mu} \left[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns} \right], \qquad (4.88)$$

the correlation function is given by

$$\begin{aligned} \left\langle \left[J^x(t, \boldsymbol{q}), T^{0y}(t', -\boldsymbol{q}) \right] \right\rangle = \sum_{\boldsymbol{k}mn} & e^{\frac{i}{\hbar}(E_{k_z ms} - E_{k_z + \mathfrak{q}_z ns})t} e^{\frac{i}{\hbar}(E_{k_z+\mathfrak{q}_z ns} - E_{k_z ms})t'} \\ & \times J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \left[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns} \right]. \qquad (4.89) \end{aligned}$$

We are now ready to find the correlation function χ^{xy} given in Eq. (4.14)

$$\chi^{xy}(\omega, \boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}\hbar} \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \left\langle \left[J^x(t, \boldsymbol{q}), T^{0y}(t', -\boldsymbol{q}) \right] \right\rangle. \qquad (4.90)$$

Introduce as usual a decay factor $e^{-\eta(t-t')}$ to ensure convergence in the time integrals, and make a change of variables $t' \to -t'$. The integral part of Eq.

(4.90), ignoring everything without time dependence for clarity, is then

$$\lim_{\eta\to 0}\int_0^\infty \mathrm{d}t\mathrm{d}t' \exp\left[\frac{i}{\hbar}\left(E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega\hbar + i\eta\hbar\right)t\right]\exp\left[\frac{i}{\hbar}\left(E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta\hbar\right)\right.$$
$$= \lim_{\eta\to 0}\frac{\hbar}{i}\left[E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega\hbar + i\eta\hbar\right]^{-1}\frac{\hbar}{i}\left[E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta\hbar\right]^{-1}. \tag{4.91}$$

The response function then reads

$$\chi^{xy}(\omega,\boldsymbol{q}) = \frac{iv_F\hbar}{\mathcal{V}}\lim_{\eta\to 0}\sum_{\boldsymbol{k}mn} J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})\left[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns}\right]$$
$$\left[E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega\hbar + i\eta\hbar\right]^{-1}\left[E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta\hbar\right]^{-1}, \tag{4.92}$$

where the matrix elements are

$$J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{\boldsymbol{k}ms}(y)\sigma^x\phi_{\boldsymbol{k}+\mathfrak{q}ns}(y), \tag{4.93}$$

$$T^{0y}_{\boldsymbol{k}ms,\boldsymbol{k}-\mathfrak{q}ns}(\boldsymbol{q}) = \frac{1}{4}\int \mathrm{d}y e^{iq_y y}\left[v_F\phi^*_{\boldsymbol{k}ms}(y)p_y\phi_{\boldsymbol{k}-\mathfrak{q}ns}(y) - v_F p_y\phi^*_{\boldsymbol{k}ms}(y)\phi_{\boldsymbol{k}-\mathfrak{q}ns}(y)\right] \tag{4.94}$$
$$+\frac{1}{4}\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}ms}(y)s\sigma^y\left(E_{k_z ms} + E_{k_z+\mathfrak{q}_z ns} - 2\mu\right)\phi_{\boldsymbol{k}-\mathfrak{q}ns}(y).$$

We will consider the response function in the static limit $\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}$. We may use the property of the limit of a product of functions $\lim A\cdot B = \lim A\cdot\lim B$ to write

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \frac{iv_F\hbar}{\mathcal{V}}\sum_{\boldsymbol{k}mn}\frac{J^x_{\boldsymbol{k}ms,\boldsymbol{k}ns}T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms}[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}ns}]}{(E_{k_z ms} - E_{k_z ns})(E_{k_z ms} - E_{k_z ns})}, \tag{4.95}$$

where the current and energy-momentum tensor matrix elements are the expression given in Eqs. (4.93) and (4.94) taken in the limit.

4.3. Response of an untilted cone

4.3.1. Explicit form of the matrix elements

Compared to the procedure used by Arjona, Chernodub, and Vozmediano[1], taking the limit of each matrix element by itself greatly simplifies the calculation.

Let

$$\phi_{\boldsymbol{k}ms}(y) = e^{-\frac{(y-k_x l_B^2)^2}{2l_B^2}} \begin{pmatrix} a_{k_z ms} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ b_{k_z ms} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{4.96}$$

thus implicitly defining the prefactors $a_{k_z ms}, b_{k_z ms}$.

These are already explicitly defined

The current operator

The matrix element

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) \tag{4.97}$$
$$= \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{\boldsymbol{k}ms}(y) \sigma^x \phi_{\boldsymbol{k}+\mathfrak{q}ns}(y)$$
$$= s v_F e \int \mathrm{d}y \exp\left\{-iq_y y - \frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} \tag{4.98}$$
$$\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.$$
$$\left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right]$$
$$= s v_F e \int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] \tag{4.99}$$
$$\exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x+q_x)q_y\right\}\right]$$
$$\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.$$
$$\left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right],$$

where we completed the square in the exponent, to get the form $e^{-a(y+b)^2}$. Also, $\mathfrak{q}_y = (q_x, q_y)$, was introduced, not to be confused with $\mathfrak{q} = (q_x, q_z)$. By introducing $\tilde{y} = \frac{y}{l_B} + l_B(iq_y - q_x - 2k_x)/2$ the matrix element may be rewritten

$$\begin{aligned} J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = sv_F e \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x + q_x)q_y\right\}\right] \\ e^{-\tilde{y}^2}\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_N\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right. \\ \left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right]. \end{aligned} \tag{4.100}$$

Taking the limit we find the simple form

$$J_{\boldsymbol{k}ms;\boldsymbol{k}ns} = J_{k_z mns} = sv_F e l_B \int \mathrm{d}\tilde{y} e^{-\tilde{y}} \left[a_{k_z ms} b_{k_z ns} H_{M-1}(\tilde{y}) H_N(\tilde{y}) + m \leftrightarrow n\right], \tag{4.101}$$

where $m \leftrightarrow n$ are the repetition of the previous term under the interchange of m, n. We employ now the orthogonality relation of the Hermite polynomials [20, Table 18.3.1]

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi} 2^n n! \delta_{n,m} \tag{4.102}$$

to write

$$J_{\boldsymbol{k}ms,\boldsymbol{k}ns} = J_{k_z mns} = sv_F e l_B \sqrt{\pi} (a_{k_z ms} b_{k_z ns} \delta_{M-1,N} 2^N N! + m \leftrightarrow n). \tag{4.103}$$

With

$$a_{\boldsymbol{k}ms} b_{\boldsymbol{k}ns} = \frac{\alpha_{k_z ms}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}} \left[2^{N+M-1}(M-1)! N! \pi l_B^2\right]^{-\frac{1}{2}}, \tag{4.104}$$

$$b_{\boldsymbol{k}ms} a_{\boldsymbol{k}ns} = \frac{\alpha_{k_z ns}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}} \left[2^{N+M-1}(N-1)! M! \pi l_B^2\right]^{-\frac{1}{2}}. \tag{4.105}$$

we find explicitly

$$J_{\boldsymbol{k}ms,\boldsymbol{k}ns} = J_{k_z mns} = sv_F e \frac{\alpha_{k_z ms}\delta_{M-1,N} + \alpha_{k_z ns}\delta_{M,N-1}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}}. \tag{4.106}$$

The stress-energy tensor operator

Consider the first part of the stress-energy matrix element

$$T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{1}{4}\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) s\sigma^y (E_{k_z ms} + E_{k_z+\mathfrak{q}_z ns} - 2\mu)\phi_{\boldsymbol{k}ms}(y). \tag{4.107}$$

Recall that

$$\phi_{\boldsymbol{k}ms}(y) = e^{-\frac{(y-k_x l_B^2)^2}{2l_B^2}} \begin{pmatrix} a_{k_z ms} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ b_{k_z ms} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}. \tag{4.108}$$

The form of the integrand is very similar to the current matrix case, with the exchange of the Pauli matrix $\sigma^x \to \sigma^y$, thus giving an additional i and a negative sign to the first term.

$$\begin{aligned} &T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \\ &= \frac{is}{4}(E_{k_z ms} + E_{k_z+\mathfrak{q}_z ns} - 2\mu)\int \mathrm{d}y e^{iq_y y} e^{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}} \\ &\quad \left[-a_{k_z+\mathfrak{q}_z ns} b_{k_z ms} H_{N-1}(\dots)H_M(\dots) + b_{k_z+\mathfrak{q}_z ns} a_{k_z ms} H_N(\dots)H_{M-1}(\dots)\right]. \end{aligned} \tag{4.109}$$

Taking care to note that the factor from the Fourier transform, that was $e^{-iq_y y}$ in the current matrix element is here $e^{+iq_y y}$, a similar completion of the square is done

$$\begin{aligned} &T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \\ &= \frac{is}{4}(E_{k_z ms} + E_{k_z+\mathfrak{q}_z ns} - 2\mu)\exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2 - 2iq_y(2k_x + q_x)\right\}\right] \\ &\quad \int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(-iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] \\ &\quad \left[-a_{k_z+\mathfrak{q}_z ns} b_{k_z ms} H_{N-1}(\dots)H_M(\dots) + b_{k_z+\mathfrak{q}_z ns} a_{k_z ms} H_N(\dots)H_{M-1}(\dots)\right]. \end{aligned} \tag{4.110}$$

The arguments of the Hermite polynomials have been dropped for brevity of notation. As before make a change of variables to get the integral on the form of the shifted orthogonality relation for the Hermite polynomials Eq. (4.154). Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$ the shifted orthogonality relation is used on the expression

$$T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is}{4}(E_{k\mu s} + E_{\lambda\nu s} - 2\mu)\exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2 - 2iq_y(2k_x + q_x)\right\}\right]\int \mathrm{d}\tilde{y}\; l_B e^{-\tilde{y}^2}$$
$$\left[-a_{\boldsymbol{k}+\mathfrak{q}ns}b_{\boldsymbol{k}ms}H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right)H_M\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right.$$
$$\left.+b_{\boldsymbol{k}+\mathfrak{q}ns}a_{\boldsymbol{k}ms}H_N\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right)H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right]. \tag{4.111}$$

The terms in the integrand are exactly the same as in the current matrix element case, just in the reverse order and with $q_y \to -q_y$.

$$T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is}{4}\frac{(E_{k_zms} + E_{k_zns} - 2\mu)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_zns}+1}}\left(\alpha_{k_zms}\delta_{M-1,N} - \alpha_{k_zns}\delta_{M,N-1}\right). \tag{4.112}$$

Summary 2

For a untilted case, in the local limit $q \to 0$, we have the matrix elements

$$J_{\boldsymbol{k}ms;\boldsymbol{k}ns} = \Gamma_{k_zmns}sv_Fe\left(\alpha_{k_zms}\delta_{M-1,N} + m \leftrightarrow n\right), \tag{4.113}$$
$$T^{0y}_{\boldsymbol{k}ns,\boldsymbol{k}ms} = \frac{is\Gamma_{k_zmns}}{4}\left(E_{k_zms} + E_{k_zns} - 2\mu\right)\left(\alpha_{k_zms}\delta_{M-1,N} - m \leftrightarrow n\right), \tag{4.114}$$

where $m \leftrightarrow n$ represent the preceding term under the interchange of m, n and where we have defined $\Gamma_{k_zmns} = \left[(\alpha^2_{k_zms}+1)(\alpha^2_{k_zns}+1)\right]^{-\frac{1}{2}}$.

4.3.2. Comment on the energy-momentum tensor

There is some ambiguity with regards to the definition of the energy-momentum tensor[15; 8; 29; 10]. The *canonical* energy-momentum tensor, derived from Lagrangian mechanics, is defined as

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_i}\partial^\nu\psi_i - \eta^{\mu\nu}\mathcal{L}. \tag{4.115}$$

On the other hand, from general relativity, the *dynamical* energy-momentum tensor is defined by the variation of the (matter) action with respect to the metric[15]

Signs depend on choice of g

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}. \tag{4.116}$$

Immediately, we see that the first definition is in general not symmetric, while the latter is, as the metric is always symmetric [6]. As the energy-momentum tensor is an observable, this presents a problem: how should the tensor be defined? This issue is not trivial, and has puzzled physicists for decades[10].

Superficially, we make the following observations. The *defining* property of the energy-momentum tensor is its conservation law

$$\partial_\mu T^{\mu\nu} = 0, \tag{4.117}$$

on a flat manifold. This, of course, only defines the tensor up to a total divergence. Denote by $\hat{T}^{\mu\nu}$ the *canonical* energy-momentum tensor. We can then define another tensor

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \partial_\alpha S^{\alpha\mu\nu}. \tag{4.118}$$

By letting the additional term be anti-symmetric $S^{\alpha\mu\nu} = -S^{\mu\alpha\nu}$, it is divergence free. This is easily shown as follows:

$$\begin{aligned}\partial_\mu\partial_\alpha S^{\alpha\mu\nu} &= -\partial_\mu\partial_\alpha S^{\mu\alpha\nu} &(4.119)\\ &= -\partial_\alpha\partial_\mu S^{\mu\alpha\nu} &(4.120)\\ &= -\partial_\mu\partial_\alpha S^{\alpha\mu\nu}, &(4.121)\end{aligned}$$

where we used the commutation of partial derivatives and relabelling of the dummy indices μ, λ. By an appropriate choice of $S^{\alpha\mu\nu}$ the canonical energy-momentum tensor may be symmetrized, importantly while still abiding the conservation law. The correction that symmetrizes the energy-momentum tensor is known as the “Belifnante tensor”, which for the Dirac Lagrangian is[8]

$$S^{\alpha\mu\nu} = \frac{1}{8}\bar{\Psi}\left[\gamma^\alpha, \sigma^{\mu\nu}\right]\Psi, \tag{4.122}$$

which gives

$$T^{\mu\nu} = \frac{1}{4}\bar{\Psi}(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\Psi. \tag{4.123}$$

Which, in the case of the Dirac Lagrangian, so happens to correspond to the naive symmetrization

$$T_s^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}. \tag{4.124}$$

[6] something with torsion never

It is also instructive for our work to consider a more naive line of reasoning. The energy-momentum tensor is used in this work through its conservation law Eq. (4.117), whose first component gives the conservation of energy. Writing it out explicitly

$$\partial_0 T^{00} + \partial_i T^{i0} = \partial_0 \epsilon + \partial_i j^i_\epsilon = 0, \tag{4.125}$$

with ϵ the energy density and $\boldsymbol{j}_\epsilon$ the energy density current, the question is really seen to be finding the energy density current, ignoring all formal arguments about the energy-momentum tensor in a general context. Using such a line of reasoning van der Wurff and Stoof [29] argued that the appropriate form of the energy-momentum tensor that should be used in linear response calculations of Dirac material systems, is the unsymmetrized canonical tensor. In this work, we will therefore use the canonical energy-momentum tensor, as opposed to the symmetric form used in the linear response calculation of an untitled cone done by Arjona, Chernodub, and Vozmediano [1]. In the untilted case, the two definitions give the same contribution, while for a titled cone, the response from the two definitions differ.

In the case of our untilted system, the Weyl Lagrangian, the components of interest are

$$T^{0y} = \frac{v_F}{4}\left[\phi^\dagger p_y \phi - p_y \phi^\dagger \phi\right], \tag{4.126}$$

$$T^{y0} = \frac{si}{4}\left[\phi^\dagger \sigma_y \partial_0 \phi - \partial_0 \phi^\dagger \sigma_y \phi\right]. \tag{4.127}$$

$$\tag{4.128}$$

The symmetric form of the energy-momentum tensor, used by Arjona, Chernodub, and Vozmediano [1], gives additional contributions to the energy-momentum matrix element. We will here show that in the case of no tilt, these contributions are identical to those of the non-symmetric tensor. In the tilted case, however, the contributions differ.

The first other contribution is

take care of prefactors

$$\left(\frac{\sqrt{M}}{\alpha_{k_z m s}} + \sqrt{(M-1)}\alpha_{k_z n s}\right)\alpha_{k_z m s}\delta_{M-1,N}. \tag{4.129}$$

The normalization factor, given in dimensionless quantities is,

$$\alpha_{k_z m s} = -\frac{s\sqrt{M}}{\epsilon_m - s\kappa}.$$

Inserting this, and using the explicit form of the energy for $m \neq 0$

$$\epsilon_n = \operatorname{sign}(m)\sqrt{M + \kappa^2},$$

for the case $N > 0$ the contribution can be shown to be

$$-s(\epsilon_m + \epsilon_n)\alpha_{k_z m s}\delta_{M-1,N}. \tag{4.130}$$

For $n = 0$, the second term of Eq. (4.129) is zero, and we have

mising s?

$$-(\epsilon_m - s\kappa)\alpha_{kms}\delta_{M-1,N}, \tag{4.131}$$

and by identifying $\epsilon_0 = -s\kappa$ this has the same form as Eq. (4.130).

In the case of tilt, however, the contribution can be shown to be

$$-\frac{s}{\sqrt{\alpha}}(\epsilon^0_{m,\alpha B} + \epsilon^0_{n,\alpha B}), \tag{4.132}$$

where $\epsilon^0_{m,\alpha B} = \operatorname{sign}(m)\sqrt{\alpha M + \kappa^2}$ and we used

$$\alpha_{k_z m s} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B} - \kappa}$$

in the tilted case. Thus, we see that in the case of tilt perpendicular to the B-field, the contribution is scaled compared to the non-symmetric term. In the case of tilt parallel to the B-field, one gets an additional term proportional to $t_\parallel \kappa$.

4.3.3. Computing the reponse function

It is now finally possible to write out the entire response function. We begin by replacing the sum over $\boldsymbol{k}$ by an integral. Firstly, we will show that the sum over k_x is restricted; recall that the eigenfunctions are exponentially centered around $y_0 = k_x l_B^2$, which for a finite sample we expect to be restricted to $0 \leq y_0 \leq L_y$. This restricts the k_x sum to $0 \leq k_x \leq L_y/l_B^2 = L_y eB/\hbar$, resulting in the k_x summation giving a finite degeneracy contribution [28, Ch. 1.4.1; 16], as the integrand is independent of k_x.

$$\sum_{\boldsymbol{k}} = \sum_{k_x=0}^{L_y eB/\hbar} \sum_{k_z} \to \frac{L_x L_z}{(2\pi)^2} \int_0^{L_y eB/\hbar} \mathrm{d}k_x \int \mathrm{d}k_z \tag{4.133}$$

$$= \frac{\mathcal{V} eB}{(2\pi)^2 \hbar} \int \mathrm{d}k_z. \tag{4.134}$$

Recall the response function

$$\chi^{xy}(\omega, \boldsymbol{q}) = \lim_{\eta\to 0} \sum_{\boldsymbol{k},mn} \frac{1}{\mathcal{V}} \frac{i v_F \hbar J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) T^{0y}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) \left[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathrm{q}ns}\right]}{(E_{k_z ms} - E_{k_z+\mathrm{q}_z ns} + i\hbar\eta)(E_{k_z ms} - E_{k_z+\mathrm{q}_z ns} + \hbar\omega + i\hbar\eta)}. \tag{4.135}$$

Firstly, introduce the dimensionless quantities $\kappa_z\sqrt{2eB} = k_z$, $\epsilon_{k_z ms} v_F \sqrt{2eB} = E_{k_z ms}$, in order to facilitate solving the integral over k_z. Collecting dimensionfull quantites, the response function reads

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0} \chi^{xy} = -\frac{e^2 v_F B}{4(2\pi)^2} \sum_{mn} \int \mathrm{d}\kappa_z [n_{\kappa_z ms} - n_{\kappa_z ns}][(\alpha^2_{\kappa_z ms}+1)(\alpha^2_{\kappa_z ns}+1)]^{-1} \times \frac{(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M})}{(\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns} + i\eta)^2}. \tag{4.136}$$

Let us now define

$$\xi(\kappa_z) = \frac{\left[n_{\kappa ms} - n_{\kappa+\mathrm{q}ns}\right]\left[(\alpha^2_{\kappa ms}+1)(\alpha^2_{\kappa+\mathrm{q}ns}+1)\right]^{-1}}{(\epsilon_{\kappa ms} - \epsilon_{\kappa+\mathrm{q}ns} + i\frac{\hbar\eta}{v_F\sqrt{2eB\hbar}})(\epsilon_{\kappa ms} - \epsilon_{\kappa+\mathrm{q}ns} + \frac{\hbar\omega}{v_F\sqrt{2eB\hbar}} + i\frac{\hbar\eta}{v_F\sqrt{2eB\hbar}})}. \tag{4.137}$$

As is shown in table 4.1, $\xi(\kappa_z)$ is odd under interchagne of m, n and inversion of κ_z.

Clean up. Is it inversion or sign flip or what?

Using this, we may simplify our expressions some. In the last term of Eq. (4.136), relabel the summation indices $m \leftrightarrow n$, and then use that ξ is odd under interchange of m, n. This renders the two terms equal, and we may consider

$$\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M} \to 2\alpha^2_{\kappa_z ms}\delta_{M-1,N}.$$

The final expression is then

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0} \chi^{xy} = -\frac{e^2 v_F B}{2(2\pi)^2} \sum_{\substack{mn \\ N=M-1}} \int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} - 2\mu)\alpha^2_{\kappa_z ms}. \tag{4.138}$$

Before solving the integral, we note that in addition to the

say the word diatomic?

$N = M - 1$ selection rule of the sum, the factor with the distributions $n_{\kappa_z ms} - n_{\kappa_z ns}$ impose further restrictions on which transitions are energetically allowed. We consider the limit $T \to 0$

something about the Luttinger in this limit? I.e. the fact we get finite result in T-¿ 0 is the interesting thing about this result

Transformation	$\xi(\kappa_z)$	$\epsilon_{\kappa_z ms}$	$\alpha_{\kappa_z ms}$
$(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$	-1	-1	-1
$(\kappa_z, s) \mapsto (-\kappa_z, -s)$	+1	+1	-1
$(m, n) \mapsto (n, m)$	-1		

Table 4.1.: Sign change of factors under various transformations. Note that $\xi(\kappa_z)$ is taken in the limit $\omega \to 0, \boldsymbol{q} \to 0, \eta \to 0$.

, where the distributions take the form of step functions, $n_{\kappa_z ms} \to \theta(-\epsilon_{\kappa_z ms})$. As the sign of energy level m, for $m \neq 0$, is given by the sign of m itself, this gives a rather simple restriction on the sum. For the zeroth energy level, the sign of the energy is given by $\text{sign}(-s\kappa_z)$. The distribution factor is

$$n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}ns} = \begin{cases} 0 & mn > 0 \text{ or } m, n = 0, \\ -\,\text{sign}(m) & m, n \neq 0, \\ -\,\text{sign}(m)\theta\left[\text{sign}(m) s\kappa_z\right] & n = 0. \end{cases} \tag{4.139}$$

Combining this with the selection rule $N = M - 1$, we see that the only allowed transitions are

$$M \to -N = -(M-1), \; -M \to N = (M-1).$$

The last simplification we will make, is to note that the step function is odd under $(m, n, \kappa_z) \to (-m, -n, -\kappa_z)$, and likewise with $\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns}$.

is it supposed to be $\epsilon + \epsilon$?

In the case of zero chemical potential, the expression may be simplified further, by considering only $-N \to M = N + 1$ transitions, adding a factor 2.

Lastly, we now show that the contributions from cones of opposite chirality s are the same. Under the transformation $(\kappa_z, s) \mapsto (-\kappa_z, -s)$, the product $\kappa_z s$ is obviously invariant. Note that $\epsilon_{\kappa_z ms}$ only depends on s and κ_z through the product $\kappa_z s$. While it is not the case for $\alpha_{\kappa_z ms}$, it is the case for its square. Consequently, the integrand is invariant under $(\kappa_z, s) \mapsto (-\kappa_z, -s)$. Similarly to the argumentation used above, as the integral goes over all κ_z, the integral is invariant under $s \mapsto -s$.

Proposition 1

We have shown the following simplifications of Eq. (4.136)*:*

- *The contributions from the terms $\alpha^2_{\kappa_z ms}\delta_{M-1,N}$ and $-\alpha^2_{\kappa_z ns}\delta_{N-1,M}$ are equal, and we consider therefore only one of them, adding a degeneracy* *factor 2.*

- *The difference of the step functions takes the form Eq.* (4.139)*, which limits the transitions to states with energies of opposite sign. For each value of* M, N*, this means the only valid transitions are* $m = M, n = -N$ *and* $m = -M, n = N$*.*
- *As the integrand is invariant under* $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$*, we may consider only one of the transitions mentioned in the previous point, adding once again a degeneracy factor of 2.*
- *We lastly showed that the contribution is independent of the chirality* s*.*

For zero chemical potential, the response function is

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = -\frac{e^2 v_F B}{(2\pi)^2}\sum_{N=0}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z m s} + \epsilon_{\kappa_z n s})\alpha^2_{\kappa_z m s}\Big|_{\substack{m=N+1\\ n=-N}}, \quad (4.140)$$

where the integration limits are $(-\infty, \infty)$ for $N \neq 0$, $(-\infty, 0)$ for $N = 0, s = -1$, and $(0, \infty)$ for $N = 0, s = 1$.

Including only the first term of the sum, we find

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \frac{e^2 v_F B}{(2\pi)^2}. \quad (4.141)$$

Including contributions from the N lowest landau levels, one acquire additional numerical prefactors,

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \gamma_N\frac{e^2 v_F B}{(2\pi)^2}, \quad (4.142)$$

where the factor by analytical integration was found to be $\gamma_0 = 1, \gamma_{20} \approx 2$. Furthermore, γ_N goes like $\log N$. The first 300 contributions are shown in Figure 4.5.

Solving the integral analytically, we obtained the contribution from each term

$$\gamma_N - \gamma_{N-1} = \frac{1}{4}\left[1 + 2N\left\{1 - (1+N)\log(1+\frac{1}{N})\right\}\right], \quad N > 0.$$

The sum can be shown to equal the rather nasty expression

$$\begin{aligned}\gamma_N = \gamma_0 + \frac{1}{12}\Big(&6\zeta^{(1,0)}(-2, N+1) - 6\zeta^{(1,0)}(-2, N+2) + 6\zeta^{(1,0)}(-1, N+1)\\ &+6\zeta^{(1,0)}(-1, N+2) + 12\log(\xi) + 3N^2 + 6N - 1\Big),\end{aligned} \quad (4.143)$$

where $\xi \approx 1.28243$ is Glaisher's constant. This expression goes like $\log N$.

Series expand in $x = \frac{1}{N}$

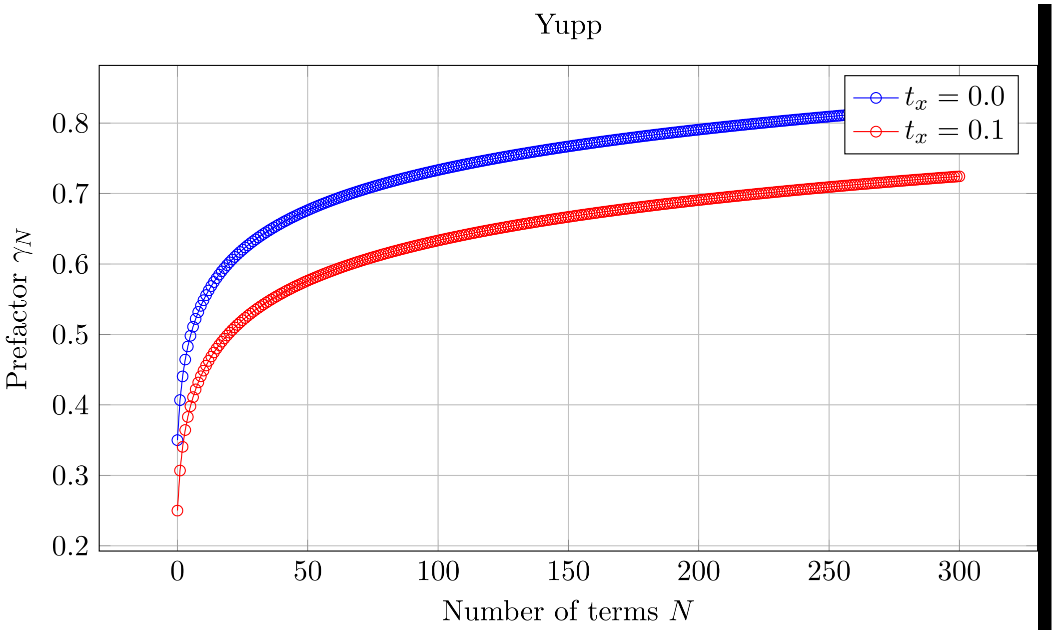


Figure 4.5.: Prefactor γ_N as a function of the number of inlcuded terms N. TODO remove the tx other test function

4.4. The response of a tilted cone

Repeating the calculation of the response function is now straightforward, but rather tedious. Due to the boost transformation, the elements of the spinor in the untilted system, Eq. (4.54), mix. We thus have twice as many terms to keep track of.

4.4.1. Explicit form of the matrix elements

We will here find an explicit form of the matrix elements, starting with the charge current

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{\boldsymbol{k}ms}(y) \sigma^x \phi_{\boldsymbol{k}+\mathfrak{q}ns}(y).$$

We must find the matrix product $\phi\sigma_x\phi$. Recall that $\phi = \frac{1}{\mathcal{N}} e^{\theta/2\sigma_x}\tilde{\phi}$, and thus we must find

$$\phi^*\sigma_x\phi = \frac{1}{\mathcal{N}^*\mathcal{N}}\tilde{\phi}^* e^{\theta/2\sigma_x}\sigma_x e^{\theta/2\sigma_x}\tilde{\phi} = \alpha\tilde{\phi}^*\sigma_x e^{\theta\sigma_x}\tilde{\phi}.$$

With the previously found solution $\theta = -\tanh^{-1} t_x^s$, we get the rather simple form

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -st_x^s \\ -st_x^s & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x}}.$$

With

$$\tilde{\phi} = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix} \tag{4.144}$$

we see how the expressions change when t_x^s become non-zero. Where we previoulsy had

$$\tilde{\phi}^*_{\boldsymbol{k}ms} \sigma_x \tilde{\phi}_{\boldsymbol{k}+\mathfrak{q}ns} = a_{\boldsymbol{k}ms} H_{M-1}(\dots)\,[b_{\boldsymbol{k}+\mathfrak{q}ns} H_N(\dots)] + \dots \tag{4.145}$$

the contents of the square brackets must now include also the other element of the spinor:

$$\alpha\tilde{\phi}^*_{\boldsymbol{k}ms} \sigma_x e^{\theta\sigma_x} \tilde{\phi}_{\boldsymbol{k}+\mathfrak{q}ns} = a_{\boldsymbol{k}ms} H_{M-1}(\dots)\,[b_{\boldsymbol{k}+\mathfrak{q}ns} H_N(\dots) - st_x^s a_{\boldsymbol{k}+\mathfrak{q}ns} H_{N-1}(\dots)] + \dots \tag{4.146}$$

First of all, let us consider the exponent of the product. Due to the extra term in χ, this becomes more elaborate. The exponent is of course

$$\exp\{-iq_y y - \frac{1}{2}\chi^2_{\boldsymbol{k}} - \frac{1}{2}\chi^2_{\boldsymbol{k}+\mathfrak{q}}\} \tag{4.147}$$

A straightforward but tedious calculation shows that the argument of the exponent can be written as

$$-\frac{\alpha}{l_B^2}\left(y + \frac{l_B^2}{2\alpha}(iq_y - (q_x' + 2k_x'))\right)^2 - \frac{l_B^2}{4\alpha}(q_y^2 + 2i(q_x' + 2k_x')q_y + (q_x')^2), \tag{4.148}$$

where we have defined

$$q_x' = q_x\alpha - \frac{\beta}{v_F}(E^0_{n,\alpha B} - E^0_{m,\alpha B}), \tag{4.149}$$

$$k_x' = k_x\alpha - \frac{\beta}{v_F}E^0_{m,\alpha B}. \tag{4.150}$$

check sign of E above

These must not be confused with the transformed momenta $\tilde{k}$, which are similar in form. Eq. (4.148) is on the same for as in the untilted cone case, and we may thus proceed with the same method. Make a change of variable

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y + \frac{l_B^2}{2\alpha}(iq_y - 2k_x' - q_x')\right),$$

Follow up the substitution of the root in the integral. Consider moving the root into Ξ

to get the exponent on the form $e^{-\tilde{y}^2}$. With this substitution,

$$\chi_{\boldsymbol{k}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q'_x - iq_y\right), \tag{4.151}$$

$$\chi_{\boldsymbol{k}+\mathsf{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q'_x - iq_y\right). \tag{4.152}$$

Doing this, Eq. (4.59)

fix ref

in the project thesis, becomes

$$\begin{aligned} J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathsf{q}ns}(\boldsymbol{q}) = \frac{sv_F e}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{l_B^2}{4\alpha}\left(q_y^2 + 2i(2k'_x + q'_x)q_y + (q'_x)^2\right)\right] \\ e^{-\tilde{y}^2}\left[a_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathsf{q}ns} H_{M-1}\left(\chi_{\boldsymbol{k}}\right) H_N\left(\chi_{\boldsymbol{k}+\mathsf{q}}\right)\right. \\ - st_x a_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathsf{q}ns} H_{M-1}\left(\chi_{\boldsymbol{k}}\right) H_{N-1}\left(\chi_{\boldsymbol{k}+\mathsf{q}}\right) \\ + b_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathsf{q}ns} H_M\left(\chi_{\boldsymbol{k}}\right) H_{N-1}\left(\chi_{\boldsymbol{k}+\mathsf{q}}\right) \\ \left. - st_x b_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathsf{q}ns} H_M\left(\chi_{\boldsymbol{k}}\right) H_N\left(\chi_{\boldsymbol{k}+\mathsf{q}}\right)\right]. \end{aligned} \tag{4.153}$$

To perform the integration, we use the *shifted orthogonality* relation for Hermite polynomials [12, Eq. (7.377)]

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_m(x+y) H_n(x+z) = 2^n \pi^{\frac{1}{2}} m! y^{n-m} L_m^{n-m}(-2yz), \quad m \leq n, \tag{4.154}$$

where L_b^a is the *generalized Laguerre polynomial* of order b and type a. Using that

$$a_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_zms}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-1}(M-1)!N!\pi l_B^2\right]^{-\frac{1}{2}} \tag{4.155}$$

$$b_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_z+\mathfrak{q}_zns}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-1}(N-1)!M!\pi l_B^2\right]^{-\frac{1}{2}} \tag{4.156}$$

$$a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{\alpha_{k_zms}\alpha_{k_z+\mathfrak{q}_zns}}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M-2}(M-1)!(N-1)!\pi l_B^2\right]^{-\frac{1}{2}} \tag{4.157}$$

$$b_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns} = \sqrt{\alpha}\frac{1}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}\left[2^{N+M}M!N!\pi l_B^2\right]^{-\frac{1}{2}} \tag{4.158}$$

we define Ξ_1, Ξ_2 by

$$\frac{\sqrt{\alpha}\alpha_{k_zms}\Xi_1(\boldsymbol{q},m,n,s)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B a_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}(\chi_{\boldsymbol{k}})H_N(\chi_{\boldsymbol{k}+\mathfrak{q}}), \tag{4.159}$$

$$\frac{\sqrt{\alpha}\alpha_{k_z+\boldsymbol{q}ns}\Xi_2(\boldsymbol{q},m,n,s)}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B b_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M}(\chi_{\boldsymbol{k}})H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}}). \tag{4.160}$$

Evaluating, we find

$$\Xi_1^{(1)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}}\left(\frac{q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{N-M+1} L_{M-1}^{N-M+1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{4.161}$$

$$\Xi_1^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}}\left(\frac{-q'_x-iq_y}{2\sqrt{\alpha}}l_B\right)^{M-N-1} L_{N}^{M-N-1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{4.162}$$

$$\Xi_1(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_1^{(1)} & \text{if } N \geq M-1 \\ \Xi_1^{(2)} & \text{if } N \leq M-1\end{cases} \quad \text{for } M>0, N\geq 0, \tag{4.163}$$

$$\Xi_2^{(1)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}} \left(\frac{q_x' - iq_y}{2\sqrt{\alpha}} l_B\right)^{N-1-M} L_M^{N-1-M}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{4.164}$$

$$\Xi_2^{(2)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}} \left(\frac{-q_x' - iq_y}{2\sqrt{\alpha}} l_B\right)^{M-N+1} L_{N-1}^{M-N+1}\left(\frac{\mathfrak{q}_y^2 l_B^2}{2\alpha}\right), \tag{4.165}$$

$$\Xi_2(\boldsymbol{q}, m, n, s) = \begin{cases} \Xi_2^{(1)} & \text{if } N-1 \geq M \\ \Xi_2^{(2)} & \text{if } N-1 \leq M \end{cases} \text{for } M \geq 0, N > 0, \tag{4.166}$$

Here, $\mathfrak{q}_y = (q_x', q_y)$.

This gives the first part of the current matrix element

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = sv_F e \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z+\mathfrak{q}_z ns}^2 + 1}}$$
$$\Big[\alpha_{k_z ms}\Xi_1(\boldsymbol{q}, m, n, s) + \alpha_{k_z+\mathfrak{q}_z ns}\Xi_2(\boldsymbol{q}, m, n, s)$$
$$- st_x\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}\Xi_1(\boldsymbol{q}, m, n \mp 1, s) - st_x\Xi_2(\boldsymbol{q}, m, n \pm 1, s)\Big]. \tag{4.167}$$

In the future, might be nice to go over to having only one function, Xi1, and simply mix around the arguments

Now we will consider the second term of the current operator.

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = ev_F t_x^s \int \mathrm{d}y e^{-iq_y y}\phi^*_{\boldsymbol{k}ms}(y)\phi_{\boldsymbol{k}+\mathfrak{q}ns}(y). \tag{4.168}$$

Using the results of Summary 1 we find

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = \frac{ev_F t_x}{\mathcal{N}^*\mathcal{N}} \int \mathrm{d}y e^{-iq_y y - \frac{1}{2}\chi_{\boldsymbol{k}}^2 - \frac{1}{2}\chi_{\boldsymbol{k}+\mathfrak{q}}^2}\tilde{\phi}^*_{\boldsymbol{k}ms}(y)e^{\theta\sigma_x}\tilde{\phi}_{\boldsymbol{k}+\mathfrak{q}ns}(y). \tag{4.169}$$

Using the same substitution and completion of the square as above, this is

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = \frac{ev_F t_x^s l_B}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y} \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right]$$
$$e^{-\tilde{y}^2}\big[a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}})\left(a_{\boldsymbol{k}+\mathfrak{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}}) - st_x^s b_{\boldsymbol{k}+\mathfrak{q}ns}H_N(\chi_{\boldsymbol{k}+\mathfrak{q}})\right)$$
$$+ b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\left(-st_x^s a_{\boldsymbol{k}+\mathfrak{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}}) + b_{\boldsymbol{k}+\mathfrak{q}ns}H_N(\chi_{\boldsymbol{k}+\mathfrak{q}})\right)\big]. \tag{4.170}$$

Thus [7]

$$J^{(2)}_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = ev_F t^s_x \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}$$
$$\Big[- st^s_x[\alpha_{k_zms}\Xi_1(\boldsymbol{q},m,n) + \alpha_{k_z+\mathfrak{q}_zns}\Xi_2(\boldsymbol{q},m,n)]$$
$$+ \alpha_{k_zms}\alpha_{k_z+\mathfrak{q}_zns}\Xi_1(\boldsymbol{q},m,n\mp 1) + \Xi_2(\boldsymbol{q},m,n\pm 1)\Big]. \quad (4.171)$$

We notice that this part has the same form as $J^{(1)}$, with only a change of the prefactors of the Ξ-functions.

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q}) = ev_F s\alpha^2 \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathfrak{q}_zns}+1}}$$
$$[\alpha_{k_zms}\Xi_1(\boldsymbol{q},m,n) + \alpha_{k_z+\mathfrak{q}_zns}\Xi_2(\boldsymbol{q},m,n)]\,. \quad (4.172)$$

We used here that $1 - t_x^2 = \alpha^2$.

Stress-energy tensor

Consider now

$$T^{0y(1)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{1}{4}\int \mathrm{d}y e^{iq_y y}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) s\sigma^y(E_{k_zms} + E_{k_z+\mathfrak{q}_zns} - 2\mu)\phi_{\boldsymbol{k}ms}(y). \quad (4.173)$$

As

$$\sigma_y e^{\theta/2\sigma_x} = e^{-\theta/2\sigma_x}\sigma_y \quad (4.174)$$

we get the very fortunate result

$$\phi^*\sigma_y\phi = \frac{1}{\mathcal{N}^*\mathcal{N}}\tilde{\phi}^*\sigma_y\tilde{\phi}. \quad (4.175)$$

The first term of the stress-energy tensor thus has the exact same form as the untilted case. Recalling the expression for $\tilde{\phi}$ from Eq. (4.59),

$$\tilde{\phi} = e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{\boldsymbol{k}ms}H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms}H_M(\chi)\end{pmatrix},$$

[7] Note to self: note that we dropped the $\frac{1}{\sqrt{\alpha}}$ for the $\sqrt{\alpha}$ coming from the Ξ definition.

where

$$\chi = \sqrt{\alpha}(y - q_x l_B^2)/l_B + \mathrm{sign}(m)\beta\sqrt{2|m| + \frac{k_z^2 l_B^2}{\alpha}}.$$

We thus get

$$T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}(E_{k_z ms} + E_{k_z+\mathsf{q}_z ns} - 2\mu)\int \mathrm{d}y e^{iq_y y} e^{-\frac{1}{2}(\chi^2_{\boldsymbol{k}+\mathsf{q}} + \chi^2_{\boldsymbol{k}})}$$
$$[-a_{\boldsymbol{k}+\mathsf{q}ns} b_{\boldsymbol{k}ms} H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}}) H_M(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}+\mathsf{q}ns} a_{\boldsymbol{k}ms} H_N(\chi_{\boldsymbol{k}+\mathsf{q}}) H_{M-1}(\chi_{\boldsymbol{k}})]. \tag{4.176}$$

We will perform once again the completion of the square and substituion of y. The exponent is the same as that which we found for the current operator case, Eq. (4.148), with the change $q_y \to -q_y$. We thus make the change of variables

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y - \frac{l_B^2}{2\alpha}(iq_y + (2k'_x + q'_x))\right), \tag{4.177}$$

giving

$$\chi_{\boldsymbol{k}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q'_x + iq_y\right), \tag{4.178}$$

$$\chi_{\boldsymbol{k}+\mathsf{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q'_x + iq_y\right). \tag{4.179}$$

Thus, analogous to Eq. (4.79), we get

$$T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) =$$
$$\frac{is\sqrt{\alpha}}{4}(E_{k_z ms} + E_{k_z+\mathsf{q}_z ns} - 2\mu)\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]\int \mathrm{d}\tilde{y} l_B e^{-\tilde{y}^2}$$
$$\Big[-a_{\boldsymbol{k}+\mathsf{q}ns} b_{\boldsymbol{k}ms} H_{N-1}(\chi_{\boldsymbol{k}}) H_M(\chi_{\boldsymbol{k}+\mathsf{q}}) + b_{\boldsymbol{k}+\mathsf{q}ns} a_{\boldsymbol{k}ms} H_N(\chi_{\boldsymbol{k}}) H_{M-1}(\chi_{\boldsymbol{k}+\mathsf{q}})\Big] \tag{4.180}$$

And thus we have

$$T^{0y(1)}_{\boldsymbol{k}+\mathsf{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}\frac{E_{k_z ms} + E_{k_z+\mathsf{q}_z ns} - 2\mu}{\sqrt{\alpha^2_{k_z ms} + 1}\sqrt{\alpha^2_{k_z+\mathsf{q}_z ns} + 1}} \tag{4.181}$$

$$\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right] \tag{4.182}$$

$$(-\alpha_{k_z+\mathsf{q}_z ns}\Xi_2(\bar{\boldsymbol{q}}, m, n, s) + \alpha_{k_z ms}\Xi_1(\bar{\boldsymbol{q}}, m, n, s)), \tag{4.183}$$

where $\bar{\boldsymbol{q}} = (q_x, -q_y, q_z)$.

Summary 3

In summary we have

$$J_{\boldsymbol{k}ms;\boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) = v_F e s \alpha^2 \Gamma^-_{\boldsymbol{k}\mathrm{q}mns} \left[\alpha_{k_z ms}\Xi_1(\boldsymbol{q},m,n,s) + \alpha_{k_z+\mathrm{q}_z ns}\Xi_2(\boldsymbol{q},m,n,s)\right], \tag{4.184}$$

$$T^{0y(1)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{is\alpha}{4}(E_{k_z ms} + E_{k_z+\mathrm{q}_z ns} - 2\mu)\Gamma^+_{\boldsymbol{k}\mathrm{q}mns} \tag{4.185}$$

$$(-\alpha_{k_z+\mathrm{q}_z ns}\Xi_2(\bar{\boldsymbol{q}},m,n,s) + \alpha_{k_z ms}\Xi_1(\bar{\boldsymbol{q}},m,n,s)), \tag{4.186}$$

with

$$\Gamma^{\pm}_{\boldsymbol{k}\mathrm{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm i q_y l_B^2 (k_x' + \frac{q_x'}{2})\right]}{\left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathrm{q}_z ns}+1)\right]^{\frac{1}{2}}}$$

4.4.2. Static limit and dimensionless form

We are interested in the response in the static limit $\boldsymbol{q} \to 0$. We may use the property of limits that

$$\lim_{n\to a} A \cdot B = \lim_{n\to a} A \cdot \lim_{n\to a} B.$$

We are thus interested in the current and energy-momentum matrix elements in the static limit. Furthermore, we will write them in dimensionless quantities.

Consider firstly the exponent in the $\Gamma^{\pm}$ factor,

$$\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm i q_y l_B^2 (k_x' + \frac{q_x'}{2})\right].$$

Recall the definition Eq. (4.149),

$$q_x' = q_x\alpha - \frac{\beta}{v_F}(E^0_{n,\alpha B} - E^0_{m,\alpha B}).$$

In the static limit

$$\lim_{\boldsymbol{q}\to 0} q_x' = -\frac{\beta}{v_F}(E^0_{n,\alpha B} - E^0_{m,\alpha B}),$$

and thus the for the exponent one has in the limit

$$\exp\left[-\frac{l_B^2\beta^2}{4\alpha v_F^2}(E^0_{n,\alpha B} - E^0_{m,\alpha B})^2\right].$$

Expressed in the dimensionless energies $\epsilon = \frac{E}{v_F\sqrt{2eB}}$

$$\exp\left[-\frac{\beta^2}{2\alpha}(\epsilon^0_{n,\alpha B}-\epsilon^0_{m,\alpha B})^2\right].$$

The normalization factor $\alpha_{k_z ms}$ is independent on $\boldsymbol{q}$, and already dimensionless. Explicity, it is given in dimensionless quantities as

$$\alpha_{k_z ms} = -\frac{\sqrt{2e\alpha BM}}{\frac{E_{k_z ms}-t_\parallel v_F k_z}{v_F s\alpha}-k_z} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B}-\kappa}. \tag{4.187}$$

In the tilted case, the Ξ functions do not have a trivial form in the static limit, as was the case in the untilted case. Define

$$P = \lim_{\boldsymbol{q}\to 0}\frac{l_B q'_x}{\sqrt{2\alpha}} = \frac{\beta}{\sqrt{\alpha}}(\epsilon^0_{n,\alpha B}-\epsilon^0_{m,\alpha B}).$$

In the static limit, the Ξ functions thus take the form

$$\Xi_1^{(1)}(m,n,s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}}\left(\frac{P}{\sqrt{2}}\right)^{N-M+1} L_{M-1}^{N-M+1}\left(P^2\right), \tag{4.188}$$

$$\Xi_1^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}}\left(-\frac{P}{\sqrt{2}}\right)^{M-N-1} L_{N}^{M-N-1}\left(P^2\right), \tag{4.189}$$

$$\Xi_1(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_1^{(1)} & \text{if } N\geq M-1\\ \Xi_1^{(2)} & \text{if } N\leq M-1\end{cases} \quad \text{for } M>0, N\geq 0, \tag{4.190}$$

$$\Xi_2^{(1)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}}\left(\frac{P}{\sqrt{2}}\right)^{N-1-M} L_{M}^{N-1-M}\left(P^2\right), \tag{4.191}$$

$$\Xi_2^{(2)}(\boldsymbol{q},m,n,s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}}\left(-\frac{P}{\sqrt{2}}\right)^{M-N+1} L_{N-1}^{M-N+1}\left(P^2\right), \tag{4.192}$$

$$\Xi_2(\boldsymbol{q},m,n,s) = \begin{cases}\Xi_2^{(1)} & \text{if } N-1\geq M\\ \Xi_2^{(2)} & \text{if } N-1\leq M\end{cases} \quad \text{for } M\geq 0, N>0. \tag{4.193}$$

Lastly, notice that in the static limit, the dependence on k_z disappears, and so the same procedure as was done for the untilted cone in section 4.3.3 is valid for the tilted cone, replacing the $\boldsymbol{k}$ sum with an integral over k_z and a degeneracy factor

$$\sum_{\boldsymbol{k}} \to \frac{\mathcal{V}eB}{(2\pi)^2\hbar}\int \mathrm{d}k_z. \tag{4.194}$$

Importantly, the degeneracy factor does *not* depend on the renormalized magnetic field αB, but rather B itself.

4.4.3. Perpendicular tilt

We consider here the specialized situation where $\boldsymbol{t} = t_x\hat{x}$, i.e. only tilt perpendicular to the magnetic field. The response function

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0}\frac{eBiv_F}{(2\pi)^2}\sum_{mn}\int \mathrm{d}k_z[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns}] \times \frac{J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})}{(E_{\boldsymbol{k}ms} - E_{\boldsymbol{k}+\mathfrak{q}ns} + i\eta)(E_{\boldsymbol{k}ms} - E_{\boldsymbol{k}+\mathfrak{q}ns} + i\eta)}.$$

Writing out the matrix products we have

$$J^x_{\boldsymbol{k}ms,\boldsymbol{k}+\mathfrak{q}ns}(\boldsymbol{q})T^{0y}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{v_F e i\alpha^3}{4}e^{-P^2} \frac{(E_{\boldsymbol{k}ms} + E_{\boldsymbol{k}+\mathfrak{q}ns})(\alpha^2_{k_z ms}\Xi_1(m,n)^2 - \alpha^2_{k_z+\mathfrak{q}_z ns}\Xi_2(m,n)^2)}{(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathfrak{q}_z ns}+1)}. \tag{4.195}$$

And so, inserting into the response function

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0}\frac{-e^2\alpha^3 v_F B}{4(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z e^{-P^2} \frac{[n_{\kappa_z ms} - n_{\kappa_z ns}](\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\Xi_1(m,n)^2 - \alpha^2_{\kappa_z ns}\Xi_2(m,n)^2)}{(\alpha^2_{\kappa_z ms}+1)(\alpha^2_{\kappa_z ns}+1)(\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns} + i\eta)^2}, \tag{4.196}$$

where we also made a change of variables $k_z = \sqrt{2eB}\kappa_z$.

We make the observation that $\Xi_1(m,n) = \Xi_2(n,m)$, where it is important to note that P changes sign under interchange of m, n. The rest of the factors are invariant under the interchange $m \leftrightarrow n$, except for the step functions, which gives an overall sign change. Thus, using $\Xi_1(m,n) = \Xi_2(n,m)$ and relabelling the summation indices we may consider

$$\alpha^2_{\kappa_z ms}\Xi_1^2 - \alpha^2_{\kappa_z ns} \to 2\alpha^2_{\kappa_z ms}\Xi_1^2.$$

We may also simplify the step function expression. Physically, the step function term corresponds to only considering transitions between states with energies of opposite sign. For Type-I systems, which we are restricted to here as we consider currently only perpendicular tilt, the energy of the state with quantum number n has the same sign as n itself, excluding of course the zeroth state. For the zeroth state, the sign of the energy is $\mathrm{sign}(-s\kappa_z)$. Using these considerations, we may make certain seletion rules for the sum. In the (m,n)-plane, the first and third quadrant give no contribution, as there $mn > 0$, i.e.

they have the same sign. Our sum is thus restricted to the second and fourth quadrant. It is easy to show that

$$
n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathfrak{q}ns} = \begin{cases} 0 & mn > 0 \text{ or } m, n = 0, \\ -\operatorname{sign}(m) & m, n \neq 0, \\ \operatorname{sign}(n)\theta(\operatorname{sign}(n)s\kappa) & m = 0, \\ -\operatorname{sign}(m)\theta(\operatorname{sign}(m)s\kappa) & n = 0. \end{cases} \tag{4.197}
$$

Furthermore, the contributions from the second and fourth quadrant are equal, which we will now show. The mapping $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, i.e. a π rotation, transforms points from the $m < 0$ half plane to the $m > 0$ half plane, including mapping the second quadrant to the fourth quadrant. We want to consider how the integrand in question transforms under such a mapping. Recall

$$
\begin{aligned}
\alpha_{\kappa_z ms} &= -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B} - \kappa_z}, \\
\epsilon^0_{m,\alpha B} &= \operatorname{sign}(m)\sqrt{\alpha M + \kappa_z^2}, \quad m \neq 0.
\end{aligned}
$$

Under the above mapping, we have the following relations

$$
\epsilon^0_{m,\alpha B} \mapsto -\epsilon^0_{m,\alpha B}, \tag{4.198}
$$

$$
\alpha_{\kappa_z ms} \mapsto -\alpha_{\kappa_z ms}, \tag{4.199}
$$

$$
P \mapsto -P. \tag{4.200}
$$

The Ξ functions also acquries a sign for some values of m, n, however, we only consdier Ξ^2. The integrand in Eq. (4.196) is thus invariant under the transformation from the second to the fourth quadrant, and so we may consider only the fourth quadrant, adding a degeneracy factor 2.

Lastly, completely analogous to the untilted case, the integrand only depend on s and κ_z through their product $s\kappa_z$, and thus is invariant under $(s, \kappa_z) \mapsto (-s, -\kappa_z)$. As the integral spans all of κ_z, the contribution is independent of the chirality s, and may be calcluated for a specific choice, which is here taken to be $s = +1$.

Make a note about $M = N$ always giving zero contributinos. Maybe also show in figure. This is important wrt. saying that γ_0 is all contributison withtin square etc.

Summary 4

The response of a perpendicularly tilted cone is given by

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \frac{e^2 v_F B}{(2\pi)^2}\gamma_N^{tx}, \tag{4.201}$$

with

$$\gamma_N^{tx} = \frac{\alpha^3}{2}\sum_{mn}^{N}\int \mathrm{d}\kappa_z e^{-P^2}\frac{(\epsilon_{\kappa_z ms}+\epsilon_{\kappa_z ns})\alpha^2_{\kappa_z ms}\Xi_1(m,n)^2}{(\alpha^2_{\kappa_z ms}+1)(\alpha^2_{\kappa_z ns}+1)(\epsilon_{\kappa_z ms}-\epsilon_{\kappa_z ns})^2}, \tag{4.202}$$

[8] *where the summation goes over $m>0, n\leq 0$, capped at the Landau level N. The integration limits are $(-\infty,\infty)$, except for $n=0$, where they are $[0,\infty)$*[9].

Make sure numerical prefactors are correct. In particular, have we included the 2 from restricting to half plane?

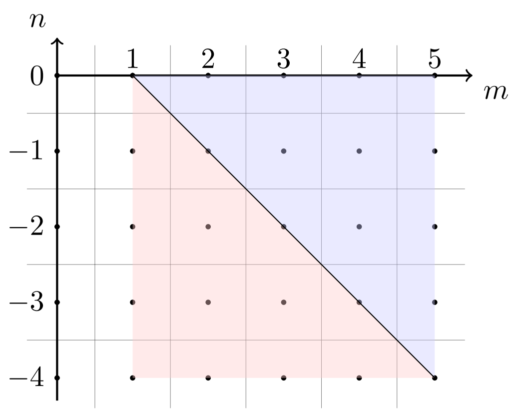


Figure 4.6.: The region of (m,n) to sum over for a Type-I perpendicularly tilted cone. The black line represents the combinations that give a finite contribution also in the untilted case. As the cone is tilted, this sharp line "diffuse" into the red and blue regions as well. Note that, as Ξ_1 defined only for $M>0$, the region with $m=0$ gives no contribution.

4.4.4. Tilt parallell to the magnetic field

Even though the treatment above for a general tilt is valid for parallel tilt, the response can be found more directly from the untilted case. For $\boldsymbol{t} = t_z\hat{z}$, the energy momentum tensor T^{0y}, charge current J^x, and wave functions $\phi(\boldsymbol{r})$ are all independent of t_z, and the only difference compared to the untilted system is a change in the energies of the Landau levels. We may thus immediately use the result from the untilted case

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = -\frac{e^2 v_F B}{4(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms}+\epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\delta_{M-1,N}-\alpha^2_{\kappa_z ns}\delta_{N-1,M}), \tag{4.203}$$

with

$$\epsilon_{\kappa_z ms} = \begin{cases} t_z^s\kappa_z + \operatorname{sign} m\sqrt{M+\kappa_z^2} & m \neq 0, \\ (t_z^s - s)\kappa_z & m = 0, \end{cases} \tag{4.204}$$

$$\alpha_{\kappa_z ms} = -s\frac{\sqrt{M}}{\epsilon^0_{\kappa_z ms} - s\kappa_z}, \tag{4.205}$$

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\xi(\kappa_z) = \frac{[n_{\kappa ms} - n_{\kappa ns}]\left[(\alpha^2_{\kappa ms}+1)(\alpha^2_{\kappa ns}+1)\right]^{-1}}{(\epsilon_{\kappa ms}-\epsilon_{\kappa ns})^2}. \tag{4.206}$$

In the untilted case we made several simplifications to this expression, especially with regards to limiting the summation domain. We will here consider which of those simplifications apply also in the case of tilt t_z.

Under the transformation $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, $\xi(\kappa_z), \epsilon_{\kappa_z ms}, \alpha_{\kappa_z ms}$ are all still odd, and so the integrand is invariant under such a transformation. As the integral is over all κ_z, we may therefore consider only half the m, n plane, as was the case in the untilted case. However, in the untilted case the sum was in fact restricted to only one quadrant, as at $T \to 0$ the transitions must be between states with energy of opposite sign. In the case of Type-II systems, this requirement does not restrict the sum to one quadrant. It is thus convenient to consider Type-I and Type-II separately.

In the untitled system, the contributions from the two chiralities where the same, as κ_z and s always appeared in conjunction, $\kappa_z s$. In the case of t_z tilt, this is not the case. The proof for the response from the two chiralities being the same in the untilted case was that s and κ_z appeared only through the product $s\kappa_z$, and so the expression was invariant under $(s, \kappa_z) \mapsto (-s, -\kappa_z)$. As our integration spans all κ_z, the total response is invariant under $s \to -s$. The tilt parameter enters the expression only through $\epsilon_{\kappa_z ms} = \epsilon^0_{\kappa_z ms} + \kappa_z t_z^s$, and in the inversion symmetric case, $t_z^s = st_z$, the argument still holds. In the case of broken inversion symmetry, however, where $t_z^s = t_z$, the argument fails. A similar argument may, however, be made for the transformation $(s, \kappa_z, t_z) \mapsto (-s, -\kappa_z, -t_z)$, for which the (inversion broken) system is invariant. The response of a cone with chirality $s = -1$ is thus equal the response with $s = +1$ and $t_z \to -t_z$. We therefore compute all responses for $s = +1$; for symmetric systems the response is equal for $s = -1$, while for broken inversion symmetry, the response is given at $t_z \to -t_z$.

Type-I

In Type-I systems, the selection rules from the step functions are independent of t_z, and the only difference from the untilted case is the term $\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} = \epsilon^0_{\kappa_z ms} + \epsilon^0_{\kappa_z ns} + 2\kappa_z t_z^s$. The response is therefore

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \frac{e^2 v_F B}{(2\pi)^2}(\gamma_N^0 + \gamma_{\mathrm{div},N}), \tag{4.207}$$

where γ_N^0 is the prefactor of the untilted case, and

$$\gamma_{\mathrm{div},N} = -2\sum_{i=0}^{N}\int \mathrm{d}\kappa_z \xi(\kappa_z)\kappa_z t_z^s \alpha^2_{\kappa_z ms}\Big|_{\substack{m=i+1\\ n=-i}}, \tag{4.208}$$

which has an UV divergence. Introduce the momentum cutoff Λ, in which case the integral can be solved analytically, with the result[10]

$$\gamma_{\mathrm{div},0} = \frac{t_z}{2}\left(\Lambda\left(\Lambda - \sqrt{\Lambda^2+1}\right) + \sinh^{-1}(\Lambda)\right) \tag{4.209}$$

and the contribution from each term of the sum

$$\begin{aligned}\gamma_{\mathrm{div},N} - \gamma_{\mathrm{div},N-1} = \frac{t_z}{2}\Bigg\{&\Lambda\left(\sqrt{\Lambda^2+m-1} - \sqrt{\Lambda^2+m}\right)\\ &+ m\tanh^{-1}\left[\frac{\Lambda}{\sqrt{\Lambda^2+m}}\right] - (m-1)\tanh^{-1}\left[\frac{\Lambda}{\sqrt{\Lambda^2+m-1}}\right]\Bigg\},\end{aligned} \tag{4.210}$$

what i m here?

where we used the selection rule of the sum $N = M - 1$ and $m > 0, n < 0$. This contribution is shown in figure 4.8.

[10]Note the minus sign introduced by the step function in ξ.

is it ok to write 'the contribution (4.210)**', or must it always be 'the contribution Eq.** (4.210)**'?**

The contribution (4.210) is odd in t_z, and so for systems with broken inversion symmetry, the total contribution from two cones cancel.

Assuming $\Lambda \gg 1$ the expression is approximated by

$$t_z \left(\frac{1}{4} \left[-1 - m \log\left(1 - \frac{1}{m}\right) + \log\left(\frac{m-1}{4}\right) \right] + \frac{\log(\Lambda)}{2} \right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right). \quad (4.211)$$

Type-II

For Type-I semimetals, the sign of energy state $m \neq 0$ is given by the sign of m itself. For $m = 0$ the sign of the energy is given by $-s \operatorname{sign} \kappa$. Due to this, the sum is restricted to $n = M + 1, m = -M$ and $n = -M - 1, m = M$. In the case of Type-II, however, the situation is not so simple. The energy bands cross the Fermi surface, and we must also include in our sum overlap between states of the same sign, i.e. $n = M + 1, m = M$ and $n = -M - 1, m = -M$, which is non-zero for certain intervals of κ. See plot of the tilted Landau levels in figure 4.4.

In order to find explicitly the limits of integration for the Type-II case, we must find the roots of the energy levels. The zeroth Landau level always has only one root, which is in the origin. For the higher order Landau levels, we solve

$$\epsilon_{\kappa_z m s} = t_z^s \kappa_z + \operatorname{sign}(m)\sqrt{M + \kappa_z^2} = 0, \quad (4.212)$$

whose solution is

$$\kappa_z^2 = \frac{M}{t_z^2 - 1}.$$

The actual roots of the energies are

$$\kappa_z = -\operatorname{sign}(m t_x^s)\sqrt{\frac{M}{t_z^2 - 1}}. \quad (4.213)$$

The integration limit for the $0 \to 1$ transition is thus, for $t_z > 1$, $[-\sqrt{t_z^2 - 1}^{-1}, 0]$. The $1 \to 2$ transition is $[-\sqrt{2}/\sqrt{t_z^2 - 1}, -\sqrt{t_z^2 - 1}^{-1}]$, and so forth. The general $n \to m$ transition has the integration limits

$$\left[-\operatorname{sign}(t_z)\sqrt{\frac{m}{t_z^2 - 1}}, -\operatorname{sign}(t_z n)\sqrt{\frac{-n}{t_z^2 - 1}} \right].$$

The $0 \to 1$ transitions was computed analytically, and found to be

$$\gamma_0 = \frac{\operatorname{sign}(t_z)}{2}\left(|t_z| \sinh^{-1}\left(\frac{1}{\sqrt{t_z^2 - 1}}\right) - 1\right). \tag{4.214}$$

For a general $n \to m,\ N > 0, M = N + 1$ transition, the contribution $\gamma_N - \gamma_{N-1}$ was found to have very lengthy expressions. Consult Table 4.2 to find the expressions for positive and negative tilt, and interband and intraband transitions.

		Tilt direction	
		$t_x > 1$	$t_x < -1$
Band type	$n < 0$	Lst. A.1	Lst. A.2
	$n > 0$	Lst. A.3	Lst. A.4

	$t_x > 1$	$t_x < -1$
$n < 0$	Lst. A.1	Lst. A.2
$n > 0$	Lst. A.3	Lst. A.4

Table 4.2.: Decision matrix over different regions for the expression of the $m \to n; N > 0, M = N + 1$ transition, given in code listing of the corresponding number. Listings starting on page 107 in appendix A. See main text for details.

Make sure I am not hanged for having vertical lines

4.5. Results

As described above, the contribution from the cone with chirality $s = -1$ can be found from the result of the positive chirality cone. In the case of perpendicular tilt, they are exactly the same. In the case of parallel tilt, it depends on the symmetry of the tilt. For systems with broken inversion symmetry, the response from the two cones are the same. On the other hand, for inversion symmeric systems, the contribution form the cone with chirality $s = -1$ is the same as that of the $s = +1$ cone at the opposite tilt $t_z \to -t_z$. Therefore, it is useful to separate the contribution into even and odd components, for finding the total contribution from the two cones combined. For some contribution $\gamma(t_{x/z})$, we define

$$\gamma_{\text{even}}(t_{x/z}) = \frac{\gamma(t_{x/z}) + \gamma(-t_{x/z})}{2}, \tag{4.215}$$

$$\gamma_{\text{odd}}(t_{x/z}) = \frac{\gamma(t_{x/z}) - \gamma(-t_{x/z})}{2}. \tag{4.216}$$

All results will be given in terms of these components, at $t_{x/y} > 0$.

We will here consider parallel and perpendicular tilt separately.

4.5.1. Perpendicular tilt

In the case of a tilt perpendicular to the magnetic field, we are, as previously explained, restricted to Type-I materials, as the Landau level description breaks down for Type-II perpendicular tilt. Importantly, this does not generally mean that the effect is not present for Type-II systems, but simply that the Linear model Landau level description is not a good basis for the system. The collapse of the Landau levels caused Soluyanov et al. [26] to errenously predict the collapse of the chiral anomaly in their now famous paper first describing Type-II Weyl semimetals.

As explained in section 4.4.3, the m, n summation is restricted to the fourth quadrant in the m, n plane. In the case of no tilt, only contributions from $M = N + 1$ were non-zero; we named the contribution from the $0 \to 1$ transition γ_0, the $-1 \to 2$ transition γ_1 and so fourth. Here, as there are contributions also away from the $M = N + 1$ line, we denote by γ_0 the contributions from inside the square of length 1 centered at the origin. The γ_1 contributions are those inside the square of length 2, and so fourth. This definition effectively sets a roof to which Landau levels we consider. This is indicated in figure 4.7.

Correct which values

The integral was computed numerically for $M, N \leq 6$ over different values of t_x with $t_z = 0$, shown in figure 4.7. The total contribution γ_N as a function of N is shown in figure 4.9. The contribution is even in t_x, and the two cones have the same contribution, as shown analytically in section 4.4.3.

4.5.2. Parallel tilt

Should we also compute the momentum cutoff for nontilted terms?

In the Type-I regime, the contributions differ from that of the untitled system by $\gamma_{\mathrm{div},N}$, Eq. (4.210), dependent on a momentum cutoff Λ. The contribution is odd in t_z, so for systems with broken inversion symmetry, the two chiralities cancel, and the response is equal to the untilted case. In case of inversion symmetry, the contributions from the two chiralities are equal and add up. In the large cutoff limit, the divergence goes like $\log\Lambda$, where the dimensionfull cutoff $k_z^{\mathrm{cutoff}} = \sqrt{2eB}\Lambda$.

In the Type-II regime, the contributions have more complicated form. Considering firstly only the lowest Landau level contribution, Eq. (4.214),, which is odd in t_z, the total contribution cancel between the chiralitites for broken inversion symmetry, while it adds up for inversion symmetric systems. As $|t_z| \to 1$ from above, the contribution blows up. This is to be expected as we move towards the Lifshitz transition, where we expect the linear model to perform poorly. [11]

put this on more solid footing. Discuss how the Fermi surface is massively wrong there

The contribution goes to zero as $t_z \to \infty$, shown in figure 4.10.

Considering also higher Landau level contributions, both interband and intraband transitions must be included, [12] meaning the summation is no longer restricted to a quadrant in the m, n plane, but rather to half the plane. The contributions are shown in figure 4.10. These contributions are not odd in t_z – they have a finite even component. Due to this, the contribution does not cancel for inversion broken systems, however the contribution is small in magnitude compared to the other contributions.

A schematic plot of all the contributions of a parallel tilt is shown in figure 4.11.

4.6. Notes

4.6.1. Spin states for Dirac cone

See mathematica file.

Consider a simple Dirac cone Hamiltonian $H_D = sv_F\boldsymbol{\sigma}\boldsymbol{p}$, with s denoting the chirality of the cone. The eigenvalues of the system is of course $E = \pm v_F k, \quad k =$

[11]As the Fermi surface of the linear model is vastly different from the Fermi surface of the tight binding model. See van der Wurff and Stoof [29]

[12]By band we here refer to the “conduction” band and “valence” band

$|\boldsymbol{k}|$. We want to find the eigenstates of this system. Assume plane wave state, and some arbitrary linear combination of spin up and spin down,

$$\psi_{\pm} = e^{i\boldsymbol{k}\boldsymbol{r}} \alpha \begin{pmatrix} 1 \\ b \end{pmatrix},$$

where α is some normalization. Solving the time independent Schrodinger equation

$$H\psi = E\psi,$$

we may solve for b, which gives

$$b = -\frac{k_z \pm k}{k_x - ik_y}. \tag{4.217}$$

Requiring normalization of the state $\langle \psi | \psi \rangle = 1$ gives the normalization

$$|\alpha|^2 = \frac{1}{1 + |b|^2}.$$

Having found the states, we find the spin expectation value

$$\boldsymbol{S} = \langle \psi | \hat{S} | \psi \rangle, \tag{4.218}$$

where $\boldsymbol{S}$ is the spin expectation value and $\hat{S} = \frac{\boldsymbol{\sigma}}{2}$ is the spin operator, where $\hbar$ was set to 1. Simply evaluating Eq. (4.218), yields

$$\boldsymbol{S} = \pm \frac{\boldsymbol{k}}{2k}. \tag{4.219}$$

The spin structure is that of a hedgehog.

4.6.2. Symmetries

In order to separate weyl cones in momentum, we introduce a pseuod spin degree of freedom, making the system 4x4. We may then get solutions with the cones separated in momentum (or energy). We may also ask what heppens if we try to separate tilted cones?

Firstly, in the most intuitive way to extend the 2x2 tilted cones to 4x4, we get that the cones tilt opposite direction, thus not superimposed even before separating in momentum. They are after that simple to separate in momentum. We might wonder if it makes sense to do it in this way.

The lattice model of the energy dispersion to explain tilted cones gives two cones separated in momentum, and tilting corresponds to "bending" the dispersion curves between them. Maybe we therefore always have cones separated in momentum, and thus tilting superimposed does not make sense? All depends on the origin of the tilt I believe. Also, we must not confuse the global dispersion relation, to the Dirac cones which are expansions around the nodes.

Key to understand how spin behaves in all of this, and also maybe the symmetries.

To properly investigate the symmetry properties of the system, we must consider the 4x4, not 2x2 Hamiltonians. While the 2x2 system does a goood job at describing a single cone, much important phsycis is lost when reducing the 4x4 Hamiltonian. For example, the requirement that the total Berry curvature over the entire Briolluine zone is zero is not met for the 2x2 Hamiltonian, as it describes only one cone of a certain chirality. The 4x4, however, includes two cones, which may in general be superimposed, thus conserving the total zero-divergence of the Berry curvature. As a matter of fact, the inclusion of both cones is important also for symmetry considerations.

Let

$$H = v_F \tau_x \otimes \boldsymbol{\sigma k},$$

where τ is some pseudo spin degree of freedom, transforming like $\boldsymbol{r}$ under parity in time reversal. This system describes two superimposed cones at the origin, with opposite chirality. The effect of parity $\mathcal{P}$ and time reversal $\mathcal{T}$ is

	$\mathcal{P}$	$\mathcal{T}$
τ	-	+
σ	+	-
k	-	-

$$\begin{aligned} \mathcal{P}\tau\mathcal{P}^\dagger &= -\tau, \quad \mathcal{T}\tau\mathcal{T}^\dagger = +\tau \\ \mathcal{P}\sigma\mathcal{P}^\dagger &= +\sigma, \quad \mathcal{T}\sigma\mathcal{T}^\dagger = -\sigma \\ \mathcal{P}k\mathcal{P}^\dagger &= -k, \quad \mathcal{T}k\mathcal{T}^\dagger = -k \end{aligned} \tag{4.220}$$

Obviously then, the Hamiltonian is both time reversal and parity invariant, as $\mathcal{P}\mathcal{P}^\dagger = \mathcal{T}\mathcal{T}^\dagger = 1$.

A tilt term $\tau_x \otimes \mathcal{I}\boldsymbol{\omega}_0\boldsymbol{k}$ breaks time reversal invariance, while maintaining parity invariance. This is due to the two cones of opposite chirality tilting in opposite directions.

The unperturbed Dirac Hamiltonian is Lorentz invariant, given that we consider an “effective speed of light”, namely the Fermi velocity, instead of the actual speed of light c. Specifically, Lorentz invariance means invariance under the *Lorentz group*. The Lorentz group is the $O(1,3)$ Lie group that conserves

$$x_\mu x^\mu = t^2 - x^2 - y^2 - z^2,$$

i.e. all isometries of Minkowski space. More specifically, the group consists of all 3D rotations, $O(3)$, and all *boosts*. A boost is a hyperbolic rotation from a spactial dimension to the temporal dimension. If we now direct our focus at the Hamiltonian of the Dirac cone

$$H = \pm v_F \boldsymbol{\sigma} \boldsymbol{p},$$

we may easily show the Lorentz invariance of the system. The time independent Schrodinger equation is

$$H |\psi\rangle = E |\psi\rangle \implies (H^2 - E^2) |\psi\rangle = 0. \tag{4.221}$$

As

$$p^\mu = \left(\frac{E}{c}, \boldsymbol{p} \right),$$

the operator in Eq. (4.221) is nothing more than

Make clear the matrix strucute here. There is an implicit identity matrix of size 2

$$H^2 - E^2 = v_F^2 \boldsymbol{p}^2 - c^2 \left(p^0\right)^2, \tag{4.222}$$

where we used the anticommutation relation

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

of the Pauli matrices. Using now the effective speed of light $c = v_F$, Eq. (4.222) is

$$-v_F^2 p_\mu p^\mu. \tag{4.223}$$

The invariance of $x^\mu x_\nu$ is the very definition of the Lorentz group, and so is obviously Lorentz invariant.

Consider now a *tilted* Dirac cone

$$H = \pm v_F \boldsymbol{\sigma} \boldsymbol{p} + \omega_x k_x, \tag{4.224}$$

where we, without loss of generality, chose the tilt to be in the x-direction. By the same argumentation as above, the eigenequation

$$H |\psi\rangle = E |\psi\rangle \implies (H^2 - E^2) |\psi\rangle = 0$$

leads to the equation

$$-v_F^2 p^\mu p_\mu + \omega_x k_x (2E - \omega_x k_x) = 0. \tag{4.225}$$

This is *not* invariant under a Lorentz transformation, as can be seen by, for example, a rotation around the z-axis.

Clean up p vs k

In literature, however, it is sometimes written that Type-II break Lorentz symmetry,

cite

however we showed that any finite tilt breaks Lorentz invariance. What is meant is the following: the energy of an untilted cone is

$$E = \pm v_F |k|.$$

Under a Lorentz boost, which we will take to be in the x direction without loss of generality, the energy transforms as

$$\tilde{E} = \gamma(E - \beta v_F k_x) = \gamma v_F(\pm|k| - \beta k_x).$$

Rescaling the energy by γ, this is a tilted cone. Importantly, the Lorentz boost is only properly defined up to $|\beta| < 1$, exactly restricting us to a Type-I system.

4.7. Discussion of results

Old, from specialization project

In the static and local limit $\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}$ the transverse response function χ^{xy} of the charge current to a temperature perturbation

$$J^x = \chi^{xy}\frac{-\nabla^y T}{T} \tag{4.226}$$

from a single Dirac point was found to be

$$\lim_{\omega\to 0}\lim_{\boldsymbol{q}\to 0}\chi^{xy} = \gamma_N \frac{e^2 B v_F}{4(2\pi)^2\hbar}, \tag{4.227}$$

with γ_N a prefactor dependent on how many landau levels are included in the final evaluation of the response function. The response function is independent of the chirality s of the Dirac point. It was found that $\gamma_0 = 1, \gamma_{20} \approx 2$ and that the prefactor goes like $\log N$.

Secondly, the sum will diverge as $N \to \infty$. However, not all Landau levels are filled, and thus the sum should not be taken to all levels. Similarly to a Quantum Hall effect, the number of filled bands, the filling factor ν, is inverse proportional to the B-field strength

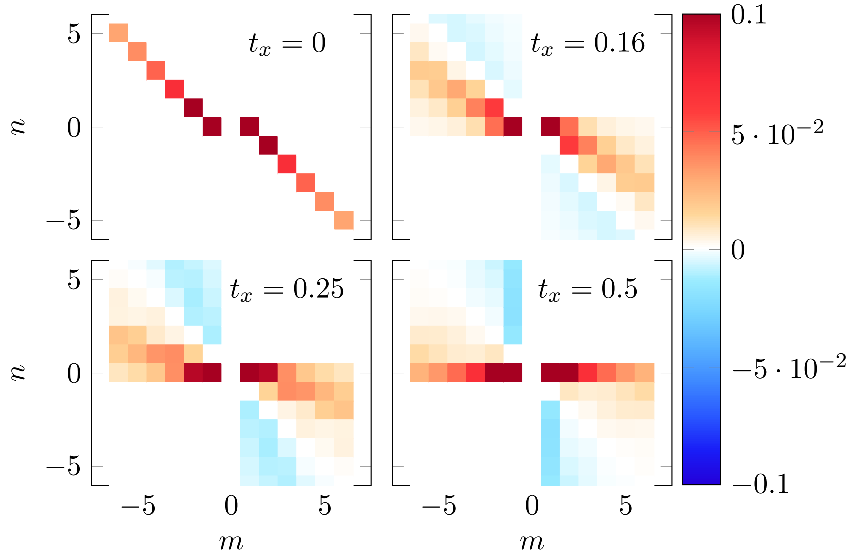
$$\nu \propto \frac{1}{B}. \tag{4.228}$$

Thus, we expect that the N-sum should be truncated at a Landau level, given by the filling factor ν. A detailed derivation of the exact truncation of the N-sum has not been done. If a precise result for the numerical prefactor is found to be of importance, this should be straightforward.

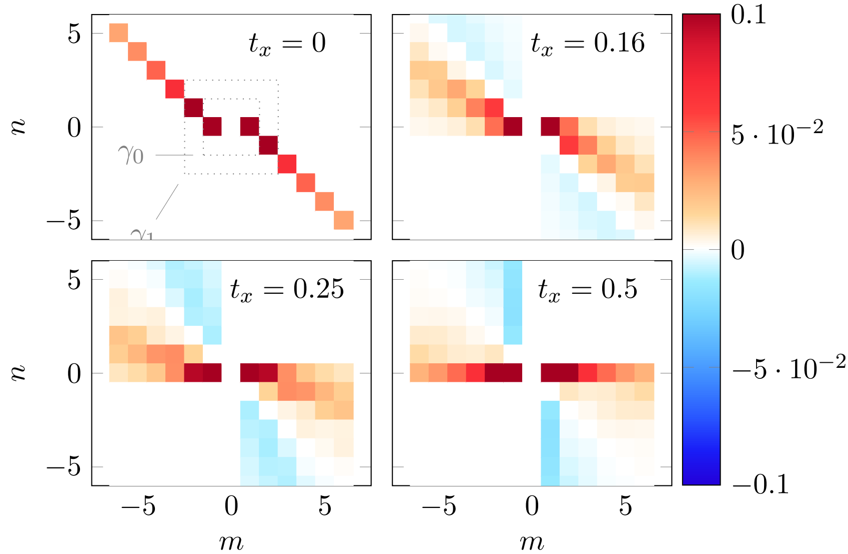
The divergence is not discussed by Arjona, Chernodub, and Vozmediano, where only the values of $N = 0$ and $N = 20$ are given, and the final result is that of $N = 20$. Furthermore, they state that the contributions from higher values of N decrease very rapidly. However, we found the contributions to go like $1/x$, which is not decreasing rapidly enough to give a finite total contribution, thus giving the total contribution diverging logarithmically.

Say that we are communicating with them to better understand their choice of truncation?

Comparing our result with the different procedure done by Chernodub, Cortijo, and Vozmediano [7], the numerical prefactor found in our calculation including only the first term ($M = 0$) coincides very well with the numerical prefactor found there, with a ratio of 16/18.



(a) Inversion symmetric case.



(b) Inversion symmetry broken case.

Figure 4.7.: Contributions to γ_N from $m \to n$ transitions for different values of t_x. TODO: Update caption

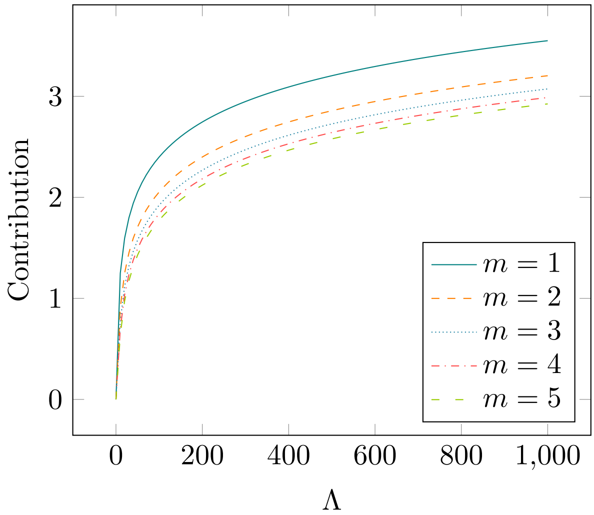


Figure 4.8.: The divergent factor $\gamma_{\text{div},N}/t_z$ for the first Landau levels, as a function of the momentum cutoff Λ.

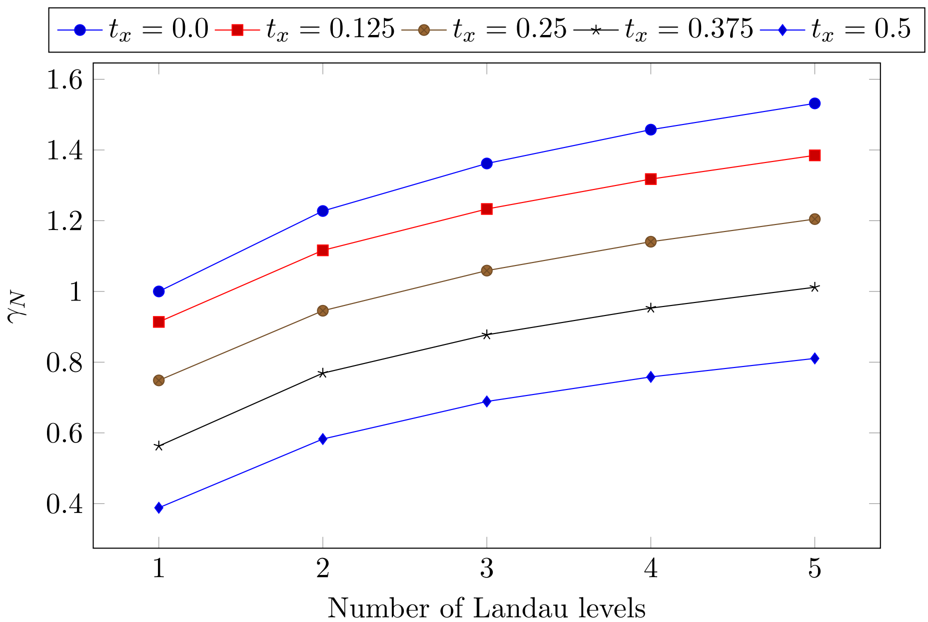
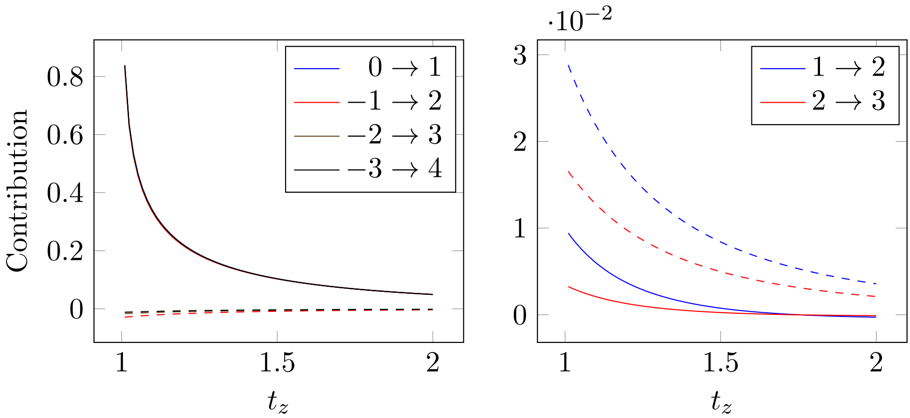


Figure 4.9.



(a) Intraband contributions, $-N \to N+1$. (b) Interband contributions, $N \to N+1$.

Figure 4.10.: The contribution from $n \to m$ transitions in a Type-II t_z tilted system. Shown in dashed line of corresponding color, is the even component of the contribution, i.e. $[\text{contrib}(|t_z|) + \text{contrib}(-|t_z|)]/2$.

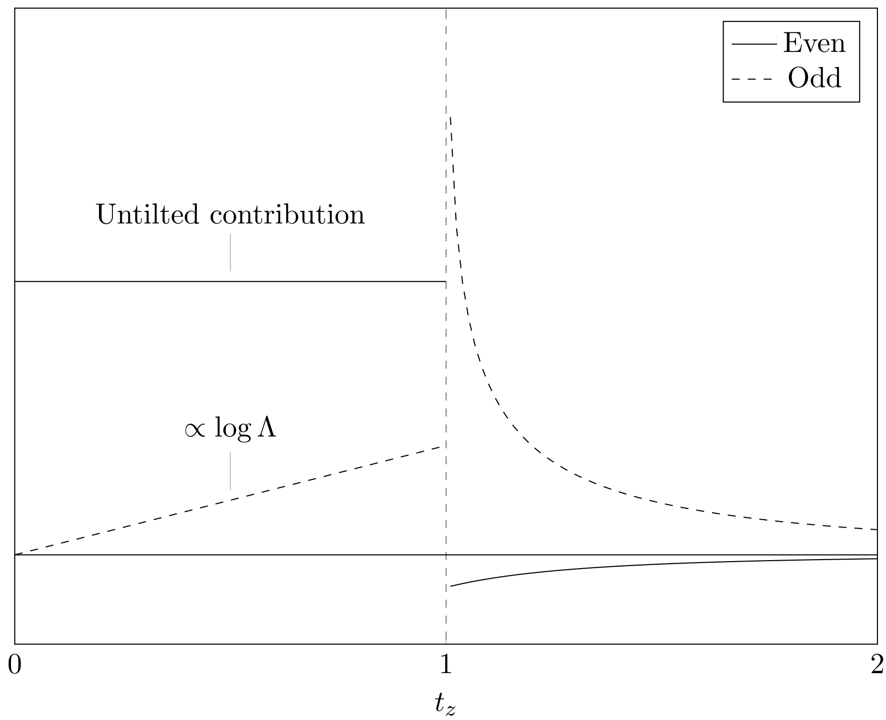


Figure 4.11.: Schematic summary of the contribution for perpendicular tilt t_z. Shown is the even (solid line) and odd (dashed line) parts as a function of t_z. As explained in the main text, the total contribution for a pair of cones is given by the sum of the even and even part in inversion symmetric systems, and by the odd part for broken inversion symmetry.

figures/spinStructureWeyl.pdf

Figure 4.12.

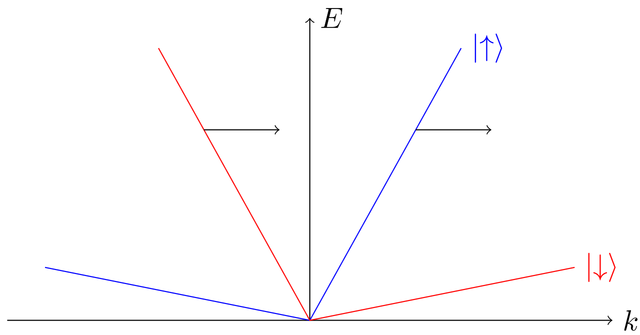


Figure 4.13.: Time reversal breaking in tilted system. Cross section in the tilt direction shown, with blue showing one cone and red the other. Black arrows indicate spin direction, which for $|\uparrow\rangle$ is proporitional to k while for $|\downarrow\rangle$ is proportional to $-k$.

A. Long expressions not included in the main text

Listing A.1: Expression for Type-II interband transition with $t_z > 1$, given in Mathematica format.

```
(m/(-1 + tz^2) - (2*m*tz)/(-1 + tz^2) - (4*(-1 + m)*m*tz)
   /(-1 + tz^2) + (m*tz^2)/(-1 + tz^2) +
  tz*Sqrt[(m*tz^2*(1 + (-1 + m)*tz^2))/(-1 + tz^2)^2] +
     2*(-1 + m)*tz*Sqrt[(m*tz^2*(1 + (-1 + m)*tz^2))/(-1
     + tz^2)^2] -
  Sqrt[m + (-1 + m)*m*tz^2]/(-1 + tz^2) - (2*(-1 + m)*
     Sqrt[m + (-1 + m)*m*tz^2])/(-1 + tz^2) +
  (2*tz*Sqrt[m + (-1 + m)*m*tz^2])/(-1 + tz^2) + 2*(-1 +
     m)*Log[(1 - Sqrt[(1 + (-1 + m)*tz^2)/m])/(1 + tz)] +
  (-1 + m)*tz*Log[(1 - Sqrt[(1 + (-1 + m)*tz^2)/m])/(1 +
     tz)] -
  2*(-1 + m)^2*Log[((1 + tz)*Sqrt[m/(-1 + tz^2)])/(Sqrt[m
     /(-1 + tz^2)] - Sqrt[(1 + (-1 + m)*tz^2)/(-1 + tz^2)
     ])] -
  (-2*(-1 + m)^(3/2)*Sqrt[-1 + m*tz^2] + (-1 + tz)*Sqrt
     [(-1 + m)*(-1 + m*tz^2)] -
    tz*(2*m^2 - (1 + tz)*(-1 + Sqrt[(-1 + m)*(-1 + m*tz
       ^2)]) + m*(-3 + tz*(-1 + 2*Sqrt[(-1 + m)*(-1 + m*
       tz^2)]))) -
    (1 - m)*(-1 + tz)*(-1 + (1 + tz)*(2 + tz)*Log[-((-1 +
        tz)*Sqrt[(-1 + m)/(-1 + tz^2)])]) -
    (-1 + tz^2)*(-2 + m*(2 + tz))*Log[(Sqrt[-1 + m] +
       Sqrt[-1 + m*tz^2])/Sqrt[-1 + tz^2]] +
    tz*Log[(-(Sqrt[-1 + m]*tz) + Sqrt[-1 + m*tz^2])/Sqrt
       [-1 + tz^2]] -
    tz^3*Log[(-(Sqrt[-1 + m]*tz) + Sqrt[-1 + m*tz^2])/
       Sqrt[-1 + tz^2]] -
    2*(-1 + m)^2*(tz + (-1 + tz^2)*Log[(-1 + m + Sqrt[(-1
        + m)*(-1 + m*tz^2)])/(-1 + m + tz - m*tz)]))/(-1
```

```
      + tz^2) -
  tz*Log[(m*tz - Sqrt[m + (-1 + m)*m*tz^2])/(-1 + tz)])/4
```

Listing A.2: Expression for Type-II interband transition with $t_z < -1$, given in Mathematica format.

```
(1 + Sqrt[(-1 + m)*(-1 + m*tz^2)] - 2*m*Sqrt[(-1 + m)*(-1
    + m*tz^2)] - Sqrt[m + (-1 + m)*m*tz^2] +
  2*m*Sqrt[m + (-1 + m)*m*tz^2] + (-1 + m)*(2 + tz)*Log
     [-((1 + tz)*Sqrt[(-1 + m)/(-1 + tz^2)])] +
  (-2 + m*(2 + tz))*Log[-((-1 + tz)*Sqrt[m/(-1 + tz^2)])]
     + 2*Log[(Sqrt[m]*(1 - tz))/(Sqrt[m] + Sqrt[1 + (-1
     + m)*tz^2])] -
  4*m*Log[(Sqrt[m]*(1 - tz))/(Sqrt[m] + Sqrt[1 + (-1 + m)
     *tz^2])] +
  2*m^2*Log[(Sqrt[m]*(1 - tz))/(Sqrt[m] + Sqrt[1 + (-1 +
     m)*tz^2])] +
  2*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt[-1 + tz
     ^2]] - 2*m*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/
     Sqrt[-1 + tz^2]] +
  tz*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt[-1 + tz
     ^2]] - m*tz*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/
     Sqrt[-1 + tz^2]] +
  tz*Log[(-(Sqrt[m]*tz) + Sqrt[1 + (-1 + m)*tz^2])/Sqrt
     [-1 + tz^2]] -
  tz*Log[(-(Sqrt[-1 + m]*tz) + Sqrt[-1 + m*tz^2])/Sqrt[-1
     + tz^2]] - 2*Log[(1 - Sqrt[(-1 + m*tz^2)/(-1 + m)])
     /(1 + tz)] +
  4*m*Log[(1 - Sqrt[(-1 + m*tz^2)/(-1 + m)])/(1 + tz)] -
     2*m^2*Log[(1 - Sqrt[(-1 + m*tz^2)/(-1 + m)])/(1 + tz
     )] +
  2*Log[-Sqrt[(-1 + m)/(-1 + tz^2)] + Sqrt[(-1 + m*tz^2)
     /(-1 + tz^2)]] -
  2*m*Log[-Sqrt[(-1 + m)/(-1 + tz^2)] + Sqrt[(-1 + m*tz
     ^2)/(-1 + tz^2)]] -
  m*tz*Log[-Sqrt[(-1 + m)/(-1 + tz^2)] + Sqrt[(-1 + m*tz
     ^2)/(-1 + tz^2)]])/4
```

Listing A.3: Expression for Type-II intraband transition with $t_z > 1$, given in Mathematica format.

```
(-((-1 + m)*m*tz*AppellF1[1, 1/2, 1/2, 2, (1 - tz^2)^(-1)
    , (1 - m)/(m*(-1 + tz^2))]) +
  m^2*tz*AppellF1[1, 1/2, 1/2, 2, -(m/((-1 + m)*(-1 + tz
      ^2))), (1 - tz^2)^(-1)] +
   (-1 + tz^2)*(-2*Sqrt[(-1 + m)*m^3] + 2*m^2*Sqrt[(-1 + m
      )*(1 + (-1 + m)*tz^2)] - 4*m^3*Sqrt[(-1 + m)*(1 +
      (-1 + m)*tz^2)] +
     2*Sqrt[m^3*(-1 + m*tz^2)] - 6*Sqrt[m^5*(-1 + m*tz^2)]
       + 4*Sqrt[m^7*(-1 + m*tz^2)] -
     (-2*Sqrt[(-1 + m)*m^5] + 2*Sqrt[(-1 + m)*m^7] + (-
       Sqrt[(-1 + m)*m^3] + Sqrt[(-1 + m)*m^5])*tz)*Log
       [-1 + m] -
     2*Sqrt[(-1 + m)*m^5]*Log[m] + 2*Sqrt[(-1 + m)*m^7]*
       Log[m] + Sqrt[(-1 + m)*m^5]*tz*Log[m] +
     2*Sqrt[(-1 + m)*m^3]*tz*Log[1 + tz] - Sqrt[(-1 + m)*m
       ^3]*tz*Log[-1 + tz^2] -
     4*Sqrt[(-1 + m)*m^5]*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)
       *tz^2])/Sqrt[-1 + tz^2]] +
     4*Sqrt[(-1 + m)*m^7]*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)
       *tz^2])/Sqrt[-1 + tz^2]] -
     2*Sqrt[(-1 + m)*m^3]*tz*Log[(Sqrt[m] + Sqrt[1 + (-1 +
        m)*tz^2])/Sqrt[-1 + tz^2]] +
     2*Sqrt[(-1 + m)*m^5]*tz*Log[(Sqrt[m] + Sqrt[1 + (-1 +
        m)*tz^2])/Sqrt[-1 + tz^2]] +
     4*Sqrt[(-1 + m)*m^5]*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*
       tz^2])/Sqrt[-1 + tz^2]] -
     4*Sqrt[(-1 + m)*m^7]*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*
       tz^2])/Sqrt[-1 + tz^2]] -
     2*Sqrt[(-1 + m)*m^5]*tz*Log[(Sqrt[-1 + m] + Sqrt[-1 +
        m*tz^2])/Sqrt[-1 + tz^2]]))/(8*Sqrt[-1 + m]*m
       ^(3/2)*(-1 + tz^2))
```

Listing A.4: Expression for Type-II intraband transition with $t_z < -1$, given in Mathematica format.

```
(4*m*(-1 + tz)*Sqrt[(-1 + m)*(1 + (-1 + m)*tz^2)]*(-1 +
    Abs[tz]) - 4*(-1 + m)*(-1 + tz)*Sqrt[m*(-1 + m*tz^2)
    ]*(-1 + Abs[tz]) +
  8*(-1 + m)^(3/2)*m*Sqrt[1 + (-1 + m)*tz^2]*(1 + tz*Abs[
      tz]) - 8*(-1 + m)^2*Sqrt[m*(-1 + m*tz^2)]*(1 + tz*
```

```
  Abs[tz]) +
2*(-1 + m)*tz*(AppellF1[1, 1/2, 1/2, 2, (1 - tz^2)^(-1)
   , (1 - m)/(m*(-1 + tz^2))] -
  AppellF1[1, 1/2, 1/2, 2, m/(-1 + m - (-1 + m)*tz^2),
     (1 - tz^2)^(-1)]) -
2*tz*AppellF1[1, 1/2, 1/2, 2, m/(-1 + m - (-1 + m)*tz
   ^2), (1 - tz^2)^(-1)] -
4*(-1 + m)^(5/2)*Sqrt[m]*(-1 + tz^2)*(Log[(-1 + m)/m] -
    2*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt[-1 +
    tz^2]] +
  2*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1 + tz
     ^2]]) - Sqrt[(-1 + m)*m]*(-1 + tz)*
 (-4 + 4*Abs[tz] + tz*Log[m/(-1 + tz^2)] + 4*tz*Log[(
    Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1 + tz^2]]
    +
  4*tz^2*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1
     + tz^2]] + 2*tz*Log[1 + Abs[tz]] -
  6*tz*Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])] - 4*tz^2*
     Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])]) +
(-1 + m)^(3/2)*Sqrt[m]*(8*tz + 8*Abs[tz] + Log[(-1 + m)
   /m] - tz^2*Log[(-1 + m)/(-1 + tz^2)] + tz^2*Log[m
   /(-1 + tz^2)] -
  8*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt[-1 +
     tz^2]] - 4*tz*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz
     ^2])/Sqrt[-1 + tz^2]] +
  8*tz^2*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt
     [-1 + tz^2]] +
  4*tz^3*Log[(Sqrt[m] + Sqrt[1 + (-1 + m)*tz^2])/Sqrt
     [-1 + tz^2]] +
  8*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1 + tz
     ^2]] + 4*tz*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])
     /Sqrt[-1 + tz^2]] -
  8*tz^2*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1
     + tz^2]] -
  4*tz^3*Log[(Sqrt[-1 + m] + Sqrt[-1 + m*tz^2])/Sqrt[-1
     + tz^2]] + 6*Log[Sqrt[(-1 + m)/(-1 + tz^2)]*(1 +
     Abs[tz])] +
  4*tz*Log[Sqrt[(-1 + m)/(-1 + tz^2)]*(1 + Abs[tz])] -
     6*tz^2*Log[Sqrt[(-1 + m)/(-1 + tz^2)]*(1 + Abs[tz
     ])] -
```

```
4*tz^3*Log[Sqrt[(-1 + m)/(-1 + tz^2)]*(1 + Abs[tz])]
    - 6*Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])] -
4*tz*Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])] + 6*tz^2*
    Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])] +
4*tz^3*Log[Sqrt[m/(-1 + tz^2)]*(1 + Abs[tz])]))/(16*
    Sqrt[(-1 + m)*m]*(-1 + tz^2))
```

(A.1)

(A.2)

(A.3)

B. Contributions from symmetric energy-momentum tensor

As noted in the main text, there are some subtlety in the definition of the energy-momentum tensor. The *canonical* definition, which we have used in the main text, is in general not symmetric. The tensor enter our calculation from the conservation law

$$\partial_\mu T^{\mu\nu} = 0,$$

which for $\nu = 0$ is nothing more than the conservation law of energy: $\partial_t \epsilon - \boldsymbol{\nabla} \boldsymbol{J}_\epsilon = 0$, where ϵ is energy density and $\boldsymbol{J}_\epsilon$ is the energy current. In the calculation by Arjona, Chernodub, and Vozmediano[1], the symmetrized energy-momentum tensor

$$T_S^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}$$

was used. In this appendix we show the contributions of the symmetric tensor. The contributions from $T^{\mu\nu}$ and $T^{\nu\mu}$ is shown to be equal in the non-tilted case.

B.1. No tilt

In the main text we have already found the contributions from the canonical tensor, and so we focus here on the contributions from $T_F^{\mu\nu} = T^{\nu\mu}$. The relevant element is $T_F^{y0} = \frac{v_F}{4}\left(\phi^\dagger p_y \phi - p_y \phi^\dagger \phi\right)$

Consider now the latter part of the stress-energy tensor, which is split into two parts

$$T^{0y\ (2)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = +\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y) p_y \phi_{\boldsymbol{k}ms}(y), \tag{B.1}$$

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \left(p_y \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right) \phi_{\boldsymbol{k}ms}(y). \tag{B.2}$$

Recall that $\phi_{\boldsymbol{k}ms}(y)$, defined in Eq. (4.96), consists of two y-dependent factors: $\exp\left[-\frac{(y-k_x l_B^2)^2}{2l_B^2}\right]$ and the Hermite polynomials. The operator p_y thus produces two terms when operating on ϕ. The first term, coming from the exponent, is proportional to $y - k_x l_B^2$. The operator in Eqs. (B.1) and (B.2) acts on ϕ

with the quantum number $\boldsymbol{k}$ and $\boldsymbol{k}+\mathfrak{q}$, respectively; when summing the two contributions, everything thus cancels except for a term proportional to q_x, which vanishes in the local limit.

It remains to consider the result of p_y operating on the Hermite polynomials. Let $\tilde{p}_y$ indicate the p_y operator acting only on the Hermite polynomial part of ϕ, and use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$ [20, Eq. 18.9.25].

$$\begin{aligned}
\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms} &= -i\hbar\exp\left\{-\frac{(y-k_xl_B^2)^2+(y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} \\
&\frac{2}{l_B}\left\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right. \\
&\left.+Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\frac{y-k_xl_B^2}{l_B}\right)H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right\}. \qquad \text{(B.3)}
\end{aligned}$$

Completing the square, we get

$$\begin{aligned}
\int \mathrm{d}y e^{iq_yy}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) &= -i\hbar\exp\left[-\frac{l_B^2}{4}\left\{\mathfrak{q}_y^2-2iq_y(2k_x+q_x)\right\}\right] \\
&\int \mathrm{d}y\exp\left[-\left\{y+\frac{l_B^2}{2}\left(-iq_y-2k_x-q_x\right)\right\}^2/l_B^2\right] \\
&\frac{2}{l_B}\left\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right. \\
&\left.+Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\frac{y-k_xl_B^2}{l_B}\right)H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right\}. \qquad \text{(B.4)}
\end{aligned}$$

Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$, as was also done in the main text, the expression reduces to

$$\begin{aligned}
\int \mathrm{d}y e^{iq_yy}\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) &= -i\hbar\exp\left[-\frac{l_B^2}{4}\left\{q_x^2+q_y^2-2iq_y(2k_x+q_x)\right\}\right] \\
&\int \mathrm{d}\tilde{y}l_B\exp\left[-\tilde{y}^2\right] \\
\frac{2}{l_B}\left\{(M-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-2}\left(\tilde{y}+\frac{l_B}{2}(iq_y+q_x)\right)\right. & H_{N-1}\left(\tilde{y}+\frac{l_B}{2}(iq_y-q_x)\right) \\
\left.+Mb_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathfrak{q}ns}H_{M-1}\left(\tilde{y}+\frac{l_B}{2}(iq_y+q_x)\right)\right. & \left.H_N\left(\tilde{y}+\frac{l_B}{2}(iq_y-q_x)\right)\right\}. \qquad \text{(B.5)}
\end{aligned}$$

Considering now the local limit $\boldsymbol{q} \to 0$, the expression greatly simplifies, and we may use the orthogonality relation for the Hermite polynomials Eq. (4.102)

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi} 2^n n! \delta_{n,m}$$

to evaulate the integral.

$$\lim_{\boldsymbol{q}\to 0} \int \mathrm{d}y e^{iq_y y} \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y) \tilde{p}_y \phi_{\boldsymbol{k}ms}(y) = -i\hbar\sqrt{2} \frac{\alpha_{kms}\alpha_{kns}\sqrt{M-1} + \sqrt{M}}{l_B \sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}} \delta_{N,M-1}. \tag{B.6}$$

Similarly, for $T^{0y\,(3)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})$, one has

$$\begin{aligned}\left(\tilde{p}_y \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right) \phi_{\boldsymbol{k}ms}(y) = -i\hbar \exp&\left\{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} \\ \frac{2}{l_B}\Bigg\{(N-1)a_{\boldsymbol{k}ms}a_{\boldsymbol{k}+\mathrm{q}ns} H_{M-1}&\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-2}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right) \\ +N b_{\boldsymbol{k}ms}b_{\boldsymbol{k}+\mathrm{q}ns} H_M&\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\Bigg\}\end{aligned} \tag{B.7}$$

which with the same procedure as above gives

$$\lim_{\boldsymbol{q}\to 0} \int \mathrm{d}y e^{iq_y y} \left(\tilde{p}_y \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right) \phi_{\boldsymbol{k}ms}(y) = -i\hbar\sqrt{2} \frac{\alpha_{kms}\alpha_{kns}\sqrt{N-1} + \sqrt{N}}{l_B \sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}} \delta_{M,N-1}. \tag{B.8}$$

B.2. With tilt

In the tilted case, we have shown in the main text that

insert ref

$$T^{\mu 0} = \frac{i}{2}\left[\partial_i \bar{\psi}\Gamma^j \gamma^0 \Gamma^\mu \psi - \bar{\psi}\Gamma^\mu \gamma^0 \Gamma^j \partial_j \psi\right].$$

Swapping the indices, we have for $\mu \neq 0$ [29]

$$T^{0i} = \frac{i}{2}[\bar{\psi}\gamma^0 \partial^\mu \psi - \partial^\mu \bar{\psi}\gamma^0 \psi].$$

In our work, we have considered only tilt perpendiculat to the thermal gradient, so the component of the energy-momentum tensor of interest are not affected by the tilt.

or

$$T^{0y(2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = +\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) p_y \phi_{\boldsymbol{k}ms}(y), \tag{B.9}$$

$$T^{0y(3)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F (p_y \phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y)) \phi_{\boldsymbol{k}ms}(y). \tag{B.10}$$

Firstly, we note that

$$[p_y, e^{\theta/2\sigma_x}] = 0.$$

Furthermore, exactly as for the untilted case, the momentum operator acting on the exponential prefactor of ϕ gives contributions proportional to q_x. In the local limit $q \to 0$ this term vanishes, and we need only consider the effect of the momentum operator acting on the Hermite polynomials.

Denote by $\tilde{p}_y$ the momentum operator p_y acting only on the Hermite polynomial part of ϕ. Furthermore, we will use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$ [20, Eq. 18.9.25].

$$\begin{aligned}
\tilde{p}_y \phi_{\boldsymbol{k}ms} &= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \partial_y \begin{pmatrix} a_{\boldsymbol{k}ms} H_{M-1}(\chi) \\ b_{\boldsymbol{k}ms} H_M(\chi) \end{pmatrix} && \text{(B.11)} \\
&= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} 2\frac{\partial \chi}{\partial y} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi) \\ b_{\boldsymbol{k}ms}(M) H_{M-1}(\chi) \end{pmatrix} && \text{(B.12)} \\
&= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \frac{2\sqrt{\alpha}}{l_B} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi) \\ b_{\boldsymbol{k}ms}(M) H_{M-1}(\chi) \end{pmatrix}. && \text{(B.13)}
\end{aligned}$$

And thus, recalling that

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -t_x \\ -t_x & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x^2}},$$

we find the product

$$\begin{aligned}
\phi^*_{\boldsymbol{k}+\mathfrak{q}ns}(y) \tilde{p}_y \phi_{\boldsymbol{k}ms} = &-\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} e^{-\frac{1}{2}\chi^2_{\boldsymbol{k}} - \frac{1}{2}\chi^2_{\boldsymbol{k}+\mathfrak{q}}} \\
&\Big[a_{\boldsymbol{k}+\mathfrak{q}ns} H_{N-1}(\chi_{\boldsymbol{k}+\mathfrak{q}}) \left\{a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}})\right\} \\
&+ b_{\boldsymbol{k}+\mathfrak{q}ns} H_N(\chi_{\boldsymbol{k}+\mathfrak{q}}) \left\{-t_x a_{\boldsymbol{k}ms}(M-1) H_{M-2}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms} M H_{M-1}(\chi_{\boldsymbol{k}})\right\}\Big].
\end{aligned} \tag{B.14}$$

Completing the square and substituting

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y - \frac{l_B^2}{2\alpha}(iq_y + (2k'_x + q'_x))\right)$$

gives

$$\begin{aligned}\int \mathrm{d}y e^{iq_y}\phi^*_{\boldsymbol{k}+\mathsf{q}ns}(y)\tilde{p}_y\phi_{\boldsymbol{k}ms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}}\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right] \\ \int \mathrm{d}\tilde{y}\frac{l_B}{\sqrt{\alpha}} \\ \Big[a_{\boldsymbol{k}+\mathsf{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{a_{\boldsymbol{k}ms}(M-1)H_{M-2}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms}MH_{M-1}(\chi_{\boldsymbol{k}})\right\} \\ + b_{\boldsymbol{k}+\mathsf{q}ns}H_N(\chi_{\boldsymbol{k}+\mathsf{q}})\left\{-t_x a_{\boldsymbol{k}ms}(M-1)H_{M-2}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms}MH_{M-1}(\chi_{\boldsymbol{k}})\right\}\Big].\end{aligned} \tag{B.15}$$

We must now evaluate the integral, and express the result in the Ξ-functions.

$$\begin{pmatrix} a_{\boldsymbol{k}+\mathsf{q}ns}H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}}) \\ b_{\boldsymbol{k}+\mathsf{q}ns}H_N(\chi_{\boldsymbol{k}+\mathsf{q}})\end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & -t_x \\ -t_x & 1\end{pmatrix}}_{T} \begin{pmatrix} a_{\boldsymbol{k}ms}(M-1)H_{M-2}(\chi_{\boldsymbol{k}}) \\ b_{\boldsymbol{k}ms}MH_{M-1}(\chi_{\boldsymbol{k}})\end{pmatrix}$$

For each of the entries in T, we get a product on of Hermite polynomials. Where the untilted cone had two such terms, the tilt parameter t_x now gives two extra products, which we must evaluate. Let $M^{(2)}_{ij}$ be the product corresponding to T_{ij}, i.e.

$$M^{(2)}_{11} = \quad a_{\boldsymbol{k}+\mathsf{q}ns}a_{\boldsymbol{k}ms}(M-1)H_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-2}(\chi_{\boldsymbol{k}}), \tag{B.16}$$
$$M^{(2)}_{12} = -t_x a_{\boldsymbol{k}+\mathsf{q}ns}b_{\boldsymbol{k}ms}MH_{N-1}(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \tag{B.17}$$
$$M^{(2)}_{21} = -t_x b_{\boldsymbol{k}+\mathsf{q}ns}a_{\boldsymbol{k}ms}(M-1)H_N(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-2}(\chi_{\boldsymbol{k}}), \tag{B.18}$$
$$M^{(2)}_{22} = \quad b_{\boldsymbol{k}+\mathsf{q}ns}b_{\boldsymbol{k}ms}MH_N(\chi_{\boldsymbol{k}+\mathsf{q}})H_{M-1}(\chi_{\boldsymbol{k}}). \tag{B.19}$$

We want to evaluate

$$F^{(2)}_{ij} = \left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathsf{q}_z ns}+1)\right]^{\frac{1}{2}}\int \mathrm{d}\tilde{y}e^{-\tilde{y}^2}M^{(2)}_{ij}, \tag{B.20}$$

with the prefactor introduced for later convenience.

Notice that

Verify l_B in this section

$$F^{(2)}_{12} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{\boldsymbol{q}}, m \mp 1, n). \tag{B.21}$$

and

$$F_{21}^{(2)} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a^2_{\boldsymbol{k}ms}}{l_B a_{\boldsymbol{k}m\mp 1s}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n, s). \tag{B.22}$$

$F_{11}^{(2)}$ and $F_{22}^{(2)}$ are the same as for the untilted case:

$$F_{11}^{(2)} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \tag{B.23}$$

and

$$F_{22}^{(2)} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m, n, s). \tag{B.24}$$

In summary we have

$$\begin{aligned} T^{0y\ (2)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) &= +\frac{v_F}{4}\int \mathrm{d}y e^{iq_y q}\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)p_y\phi_{\boldsymbol{k}ms}(y) &\text{(B.25)}\\ &= -\frac{i\hbar v_F}{2}\Gamma^+_{\boldsymbol{k}\mathrm{q}mns}\sum_{i,j}F_{ij}^{(2)}, &\text{(B.26)}\end{aligned}$$

where

$$\Gamma^+_{\boldsymbol{k}\mathrm{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]}{\left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathrm{q}_z ns}+1)\right]^{\frac{1}{2}}\sqrt{1-t_x^2}}$$

In a similar procedure, we find $T^{0y\ (2)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q})$.

$$\tilde{p}_y\phi^*_{\boldsymbol{k}+\mathrm{q}ms} = \frac{-i\hbar\sqrt{\alpha}}{l_B}e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{\boldsymbol{k}+\mathrm{q}ms}(M-1)H_{M-2}(\chi) \\ b_{\boldsymbol{k}+\mathrm{q}ms}(M)H_{M-1}(\chi)\end{pmatrix}. \tag{B.27}$$

And thus,

$$\begin{aligned} \left(\tilde{p}_y\phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right)\phi_{\boldsymbol{k}ms} &= -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}}e^{-\frac{1}{2}\chi^2_{\boldsymbol{k}}-\frac{1}{2}\chi^2_{\boldsymbol{k}+\mathrm{q}}} \\ &\Big[a_{\boldsymbol{k}+\mathrm{q}ns}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathrm{q}})\left\{a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\} \\ &+ b_{\boldsymbol{k}+\mathrm{q}ns}NH_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}})\left\{-t_x a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}\Big]. \end{aligned} \tag{B.28}$$

With the now well-known completion of the square and substitution, we have

$$\int \mathrm{d}y e^{iq_y} \left[\tilde{p}_y \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right] \phi_{\boldsymbol{k}ms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k_x' + q_x')q_y + (q_x')^2)\right]$$
$$\int \mathrm{d}\tilde{y} \frac{l_B}{\sqrt{\alpha}}$$
$$\Big[a_{\boldsymbol{k}+\mathrm{q}ns}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathrm{q}})\left\{a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) - t_x b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}$$
$$+ b_{\boldsymbol{k}+\mathrm{q}ns}NH_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}})\left\{-t_x a_{\boldsymbol{k}ms}H_{M-1}(\chi_{\boldsymbol{k}}) + b_{\boldsymbol{k}ms}H_M(\chi_{\boldsymbol{k}})\right\}\Big]. \tag{B.29}$$

Denote the terms of the integrand by

$$M_{11}^{(3)} = a_{\boldsymbol{k}+\mathrm{q}ns}a_{\boldsymbol{k}ms}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathrm{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \tag{B.30}$$
$$M_{12}^{(3)} = -t_x a_{\boldsymbol{k}+\mathrm{q}ns}b_{\boldsymbol{k}ms}(N-1)H_{N-2}(\chi_{\boldsymbol{k}+\mathrm{q}})H_M(\chi_{\boldsymbol{k}}), \tag{B.31}$$
$$M_{21}^{(3)} = -t_x b_{\boldsymbol{k}+\mathrm{q}ns}a_{\boldsymbol{k}ms}NH_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}})H_{M-1}(\chi_{\boldsymbol{k}}), \tag{B.32}$$
$$M_{22}^{(3)} = b_{\boldsymbol{k}+\mathrm{q}ns}b_{\boldsymbol{k}ms}NH_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}})H_M(\chi_{\boldsymbol{k}}). \tag{B.33}$$

We must evaluate

$$F_{ij}^{(3)} = \left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathrm{q}_z ns}+1)\right]^{\frac{1}{2}} \int \mathrm{d}\tilde{y} e^{-\tilde{y}^2} M_{ij}^{(3)}. \tag{B.34}$$

From the untilted case we know

$$F_{11}^{(3)} = \sqrt{\frac{N-1}{2}} \frac{\alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}}{l_B \alpha_{k_z+\mathrm{q}_z n\mp 1 s}} \Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \tag{B.35}$$
$$F_{22}^{(3)} = \sqrt{\frac{N}{2}} \frac{1}{l_B \alpha_{k_z+\mathrm{q}_z ns}} \Xi_2(\bar{\boldsymbol{q}}, m, n, s). \tag{B.36}$$

Furthermore,

$$F_{12}^{(3)} = -t_x \frac{\alpha_{k_z+\mathrm{q}_z n}}{\alpha_{k_z+\mathrm{q}_z n\mp 1} l_B} \sqrt{\frac{N-1}{2}} \Xi_2(\bar{\boldsymbol{q}}, m, n\mp 1, s), \tag{B.37}$$
$$F_{21}^{(3)} = -\frac{t_x}{l_B} \sqrt{\frac{N}{2}} \frac{\alpha_{k_z m}}{\alpha_{k_z+\mathrm{q}_z n}} \Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n, s). \tag{B.38}$$

We thus have

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathrm{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{v_F}{4} \int \mathrm{d}y e^{iq_y y} \left(p_y \phi^*_{\boldsymbol{k}+\mathrm{q}ns}(y)\right) \phi_{\boldsymbol{k}ms}(y) \tag{B.39}$$
$$= \frac{i\hbar v_F}{2} \Gamma^+_{\boldsymbol{k}\mathrm{q}mns} \sum_{ij} F_{ij}^{(3)}. \tag{B.40}$$

Summary 5

The non-canonical part of the energy-momentum tensor $T_F^{\mu\nu} = T^{\nu\mu}$ in a tilted system have the matrix elements

$$T^{0y\ (2)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = -\frac{i\hbar v_F}{2}\Gamma^{+}_{\boldsymbol{k}\mathfrak{q}mns}\sum_{i,j} F^{(2)}_{ij}, \tag{B.41}$$

$$T^{0y\ (3)}_{\boldsymbol{k}+\mathfrak{q}ns,\boldsymbol{k}ms}(\boldsymbol{q}) = \frac{i\hbar v_F}{2}\Gamma^{+}_{\boldsymbol{k}\mathfrak{q}mns}\sum_{ij} F^{(3)}_{ij}. \tag{B.42}$$

with

$$\Gamma^{\pm}_{\boldsymbol{k}\mathfrak{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm iq_y l_B^2(k_x' + \frac{q_x'}{2}))\right]}{\left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathfrak{q}_z ns}+1)\right]^{\frac{1}{2}}}$$

and where the factors $F^{(n)}_{ij}$ where found to be

$$F^{(2)}_{12} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n), \tag{B.43}$$

$$F^{(2)}_{21} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a^2_{\boldsymbol{k}ms}}{l_B a_{\boldsymbol{k}m\mp 1s}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n, s), \tag{B.44}$$

$$F^{(2)}_{11} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \tag{B.45}$$

$$F^{(2)}_{22} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{\boldsymbol{q}}, m, n, s), \tag{B.46}$$

$$F^{(3)}_{11} = \sqrt{\frac{N-1}{2}}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathfrak{q}_z ns}}{l_B\alpha_{k_z+\mathfrak{q}_z n\mp 1s}}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n\mp 1, s), \tag{B.47}$$

$$F^{(3)}_{22} = \sqrt{\frac{N}{2}}\frac{1}{l_B\alpha_{k_z+\mathfrak{q}_z ns}}\Xi_2(\bar{\boldsymbol{q}}, m, n, s), \tag{B.48}$$

$$F^{(3)}_{12} = -t_x\frac{\alpha_{k_z+\mathfrak{q}_z n}}{\alpha_{k_z+\mathfrak{q}_z n\mp 1}l_B}\sqrt{\frac{N-1}{2}}\Xi_2(\bar{\boldsymbol{q}}, m, n\mp 1, s), \tag{B.49}$$

$$F^{(3)}_{21} = -\frac{t_x}{l_B}\sqrt{\frac{N}{2}}\frac{\alpha_{k_z m}}{\alpha_{k_z+\mathfrak{q}_z n}}\Xi_2(\bar{\boldsymbol{q}}, m\mp 1, n, s). \tag{B.50}$$

B.2.1. Parallel tilt

The procedure greatly simplifies in the case of parallel tilt. As noted in the main text, parallel tilt only rescales the energies Landau levels, while the wave functions and operators stay invariant

C. Conformal symmetry of a tilted system

The origin of the term *conformal anomaly* is the *conformal symmetry*. Under the conformal transformation, the massless QED Lagrangian is invariant, as shown in the main text. Specifically, the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar{\psi}\not{D}\psi,$$

with the usual $\bar{\psi} = \psi^\dagger\gamma^0, \not{D} = \gamma^\mu D_\mu, D_\mu = \partial_{\mu - ieA_\mu}$ transforms under the scaling

$$x \to \lambda^{-1},\ A_\mu \to \lambda A_\mu,\ \psi \to \lambda^{\frac{3}{2}}\psi,$$

as

$$\mathcal{L} \to \lambda^4\mathcal{L}.$$

The action $S = \int \mathrm{d}^4x\mathcal{L}$ is thus invariant (as $\mathrm{d}^4x \to \lambda^{-4}\mathrm{d}^4x$), and the theory is classically manifestly scale invariant.

Consider now the tilted Dirac Lagrangian considered in our work,

$$\mathcal{L}ki\bar{\psi}\Gamma^\mu\partial_\mu\psi, \tag{C.1}$$

with $\Gamma^\mu = \gamma^\mu + t^\mu\gamma_P\gamma^0$, where $\gamma_P = I_4$ when inversion summery is broken and $\gamma_P = \gamma^5$ for inversion symmetric systems. The tilt parameter $t^\mu = (0, \boldsymbol{t})$ is invariant under scaling, and thus also this theory is classically scale invariant.

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