Nernst effect from conformal anomaly in tilted Weyl semimetals

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We calculate a Nernst current in tilted Weyl semimetals using the Kubo formalism.

I. CALCULATION

A. Setup and response

Consider the Hamiltonian

$$H^s = sv_F \boldsymbol{\sigma} \cdot \boldsymbol{p} + v_F \boldsymbol{t}^s \boldsymbol{p}, \tag{1}$$

with $s = \pm 1$ the chirality of the cone, v_F the Fermi velocity, $\boldsymbol{\sigma}$ the Pauli matrix vector, $\boldsymbol{p}$ the momentum operator and $\boldsymbol{t}^s$ the *tilt vector*. For inversion symmetric systems $\boldsymbol{t}^s = s\boldsymbol{t}$, while broken inversion symmetry we consider $\boldsymbol{t}^s = \boldsymbol{t}$ [1].

... The response from the Kubo formalism is thus

$$\begin{aligned}\chi^{ij}(\omega, \boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}} \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \\ \times \left\langle \left[J^i(t, \boldsymbol{q}), T^{j0}(t', -\boldsymbol{q}) \right] \right\rangle,\end{aligned} \tag{2}$$

where $\hbar = 1$. The charge current operator

$$\boldsymbol{J}_s = ev_F(s\boldsymbol{\sigma} + \boldsymbol{t}^s). \tag{3}$$

For the energy-momentum tensor choose [2]

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}^\mu \partial_\nu \phi - \tilde{\sigma}^\mu \phi^\dagger \partial_\nu \phi - \eta^{\mu\nu} \mathcal{L}), \tag{4}$$

where we have defined the modified Pauli matrices[3] $\tilde{\sigma}^\mu = \sigma^\mu + (t^s)^\mu$ with $(t^s)^\mu = (0, \boldsymbol{t}^s)$.

For concreteness, we will, without loss of generality, consider the geometry $\boldsymbol{J} \parallel \hat{x}, -\nabla T \parallel \hat{y}, \boldsymbol{B} \parallel \hat{z}$. We separate between *perpendicular* and *parallel* tilt, with the former being $\boldsymbol{t} \perp \boldsymbol{B}$ and the latter $\boldsymbol{t} \parallel \boldsymbol{B}$. In the current work we have furthermore restricted the perpendicular tilt to be parallel to the current, i.e. $\boldsymbol{t}_\perp = t_x \hat{x}$.

B. Landau levels

The Landau levels (LLs) can be shown to be

$$E_{k_z ms} = \begin{cases} t_z^s v_F k_z + \mathrm{sign}(m) v_F \alpha \sqrt{2eB\alpha M + k_z^2} & m \neq 0, \\ t_z^s v_F k_z - s\alpha v_F k_z & m = 0, \end{cases} \tag{5}$$

where we have defined the *squeezing factor* $\alpha = \sqrt{1 + t_x^2}$. The eigenstates in the position basis are

$$\phi(\boldsymbol{r}) = \sqrt{\alpha} e^{\theta/2\sigma_x} \frac{e^{ik_x x + ik_z z}}{\sqrt{L_x L_z}} e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{k_z ms} H_{M-1}(\chi) \\ b_{k_z ms} H_M(\chi) \end{pmatrix}, \tag{6}$$

where we defiend the dimesionless quantity $\chi = \sqrt{\alpha} \frac{y - k_x l_B^2}{l_B} + \frac{t_x^s l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}$.

In the local limit $\boldsymbol{q} \to 0$ we find

$$J_{k_z mns} = \Gamma_{k_z mns} s v_F e(\alpha_{k_z ms} \delta_{M-1,N} + m \leftrightarrow n), \tag{7}$$

$$\begin{aligned}T^{0y}_{k_z mns} = \frac{is\Gamma_{k_z mns}}{4}(E_{k_z ms} + E_{k_z ns} - 2\mu) \\ \times (\alpha_{k_z ms} \delta_{M-1,N} - m \leftrightarrow n),\end{aligned} \tag{8}$$

where $m \leftrightarrow n$ represents the preceding term under the interchange of m, n, and we have defined $\Gamma_{\kappa_z mns} = \left[(\alpha^2_{k_z ms} + 1)(\alpha k_z ns^2 + 1)\right]^{-\frac{1}{2}}$.

II. RESULTS

We find the response in the limit $T \to 0$ and with zero chemical potential $\mu = 0$,

$$\lim_{\omega \to 0} \lim_{\boldsymbol{q} \to 0} \chi^{xy} = \gamma_N \frac{e^2 v_F B}{(2\pi)^2}, \tag{9}$$

where γ_N is a term generally depending on the number of LLs included in the sum, and the tilting vector $\boldsymbol{t}$.

The case of no tilt was first solved by Chernodub *et al.* [4] and then using Kubo formalism by Arjona *et al.* [5]. We have simplified their computation and found an analytical expression for the prefactor, with the iterative expression

$$\gamma_{\bar{N}} - \gamma_{\bar{N}-1} = \frac{1}{4} \left[1 + 2\bar{N} \left\{ 1 - (1 + \bar{N}) \log \left(1 + \frac{1}{\bar{N}} \right) \right\} \right], \tag{10}$$

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for $\bar{N} > 0$, and $\gamma_0 = \frac{1}{4}$. It can be shown that

$$\gamma_{\bar{N}} = \gamma_0 + \frac{1}{12}\Big(6\zeta^{(1,0)}(-2, \bar{N}+1) - 6\zeta^{(1,0)}(-2, \bar{N}+2) + 6\zeta^{(1,0)}(-1, \bar{N}+1) + 6\zeta^{(1,0)}(-1, \bar{N}+2) + 12\log(\xi) + 3\bar{N}^2 + 6\bar{N} - 1\Big), \quad (11)$$

where $\xi \approx 1.28243$ is Glaisher's constant.

The contribution from the cone with chirality $s = -1$ can be found from the result of the positive chirality cone. In the case of perpendicular tilt, they are exactly the same. In the case of parallel tilt, it depends on the symmetry of the tilt. For systems with broken inversion symmetry, the response from the two cones are the same. On the other hand, for inversion symmeric systems, the contribution form the cone with chirality $s = -1$ is the same as that of the $s = +1$ cone at the opposite tilt $t_z \to -t_z$. Therefore, it is useful to separate the contribution into even and odd components, for finding the total contribution from the two cones combined. For some contribution $\gamma(t_{x/z})$, we define

$$\gamma_{\text{even}}(t_{x/z}) = \frac{\gamma(t_{x/z}) + \gamma(-t_{x/z})}{2}, \quad (12)$$

$$\gamma_{\text{odd}}(t_{x/z}) = \frac{\gamma(t_{x/z}) - \gamma(-t_{x/z})}{2}. \quad (13)$$

All results will be given in terms of these components, at $t_{x/y} > 0$.

A. Perpendicular tilt

We found the response to be even in t_x, and independent of the chirality s. The response was solved numerically and is shown in fig. 4. As the tilt is increased, the response is gradually reduced.

1. *Including only 0 to n transitions*

In these calculations, there is always the need to cap the LLs included. In the results above, the LLs were capped by considering all transitions between LLs below some cutoff level $\bar{N}$. In the untilted case, in the deep quantum limit were only the lowest LL is filled, one considers the only the $0 \to \pm 1$ transition. In the case of perpendicular tilt, however, where the dipolar selection rule $M = N+1$ is broken, it might be reasonable to consider all transitions $0 \to n$ $|n| < \bar{N}$. Doing this, yields a very interesting result. Instead of the response begin strictly decreasing as a function of the tilt $t_\perp$, there is a maximum at non-zero tilt, as shown in fig. 1.

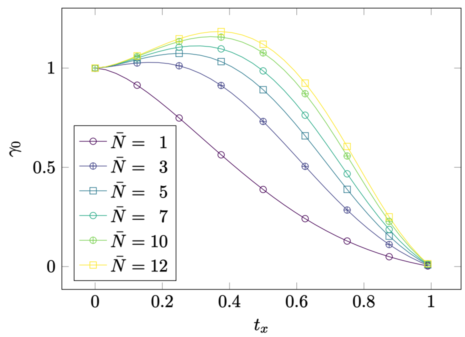

FIG. 1. Numerically computed values of the prefactor $\gamma_{\bar{N}}$ with only the $\bar{N}$ lowest, $0 \to n$ transitions included for perpendicular tilt t_x. The contribution is even in t_x, and vanish as $|t_x| \to 1$. For clarity, only every 4th mark is drawn.

B. Parallel tilt

In the Type-I regime, the response has an extra term compared to the untilted case, $\gamma_N = \gamma_N^0 + \gamma_{\text{div},N}$, with γ_N^0 the untitled response and

$$\gamma_{\text{div},N} = -2\sum_{i=0}^{N}\int \mathrm{d}\kappa_z \xi(\kappa_z)\kappa_z t_z \alpha^2_{\kappa_z m s}\Big|_{\substack{m=i+1,\\ n=-i}} \quad (14)$$

where $\kappa_z = \sqrt{2eB}k_z$ and

$$\xi(\kappa_z) = \frac{[n_{\kappa m s} - n_{\kappa n s}]}{[(\alpha^2_{\kappa m s}+1)(\alpha^2_{\kappa n s}+1)]\,(\epsilon_{\kappa m s} - \epsilon_{\kappa n s})^2}. \quad (15)$$

This contribution has a UV divergence, and we introduce the momentum cutoff Λ.

Add discussion about linear model and cutoff

The integral was solved analytically, and the contribution from each term found to be

$$\gamma_{\text{div},N} - \gamma_{\text{div},N-1} = \frac{t_z}{4}\left\{\Lambda\left(\sqrt{1+\Lambda^2+m} - \sqrt{\Lambda^2+m}\right) + m\tanh^{-1}\left[\frac{\Lambda}{\sqrt{\Lambda^2+m}}\right] - (m+1)\tanh^{-1}\left[\frac{\Lambda}{\sqrt{1+\Lambda^2+m}}\right]\right\}. \quad (16)$$

In the case of Type-II systems, the calculation is more involved, because of the Landau levels crossing the Fermi surface. This gives both intraband and interband transitions, and the integration limits are more involved. The zeroth transition $0 \to 1$ was found to be

$$\frac{\operatorname{sign}(t_z)}{2}\left(|t_z|\sinh^{-1}\left(\frac{1}{\sqrt{t_z^2-1}}\right)-1\right). \tag{17}$$

The higher order contributions were also computed analytically, with very lengthy expressions. A schematic summary of all the contributions are given in fig. 3.

Appendix A: Formulation of tilt as non-flat metric

In the main text, we included the tilt explicitly in the Lagrangian, through *modified* Pauli matrices, similar to what van der Wurff and Stoof [6] did. However, we may instead incldue the tilt in the metric, leaving to explicit tilt in the Lagrangian itself. Let $g^{\mu\nu} = \eta^{\mu\nu} + t^\mu \delta_0^\nu$, where η is the Minkowski-metric. Thus,

$$\mathcal{L} = \phi^\dagger g^{\mu\lambda}\sigma_\lambda \partial_\mu \phi = \phi^\dagger(\sigma^\mu + t^\mu)\partial_\mu \phi. \tag{A1}$$

Note, importantly, that the metric is *not* symmetric.

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[1] We here chose the "antisymmetric" version for the broken inversion symmetry, where the cones tilt in the same direction with the same magnitude. Of course, many other choices also break inversion symmetry.

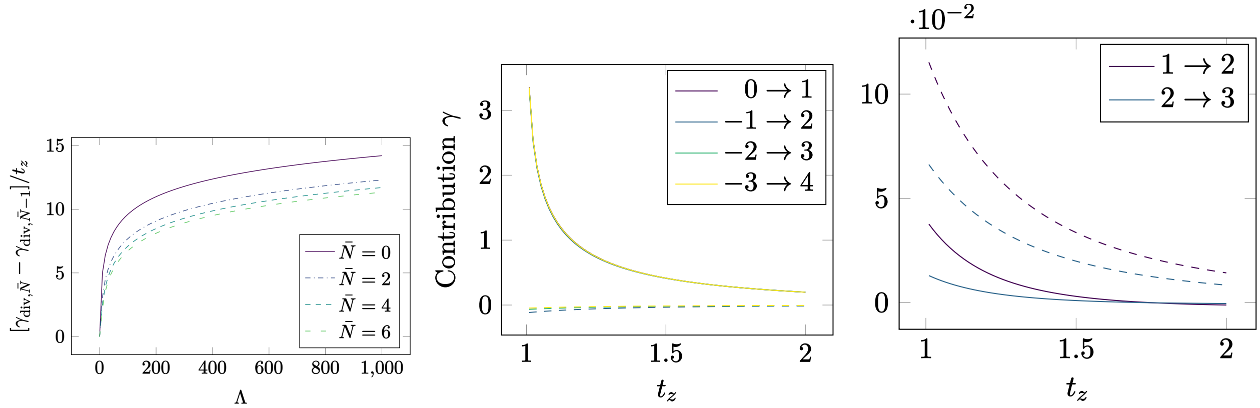
[2] TODO: add some discussion here or in appendix.

[3] See appendix A.

[4] M. N. Chernodub, A. Cortijo, and M. A. H. Vozmediano, Generation of a Nernst Current from the Conformal Anomaly in Dirac and Weyl Semimetals, Physical Review Letters **120**, 206601 (2018).

[5] V. Arjona, M. N. Chernodub, and M. A. H. Vozmediano, Fingerprints of the conformal anomaly on the thermoelectric transport in Dirac and Weyl semimetals: Result from a Kubo formula, Physical Review B **99**, 235123 (2019), arXiv:1902.02358.

[6] E. C. I. van der Wurff and H. T. C. Stoof, Magnetovortical and thermoelectric transport in tilted Weyl metals, Physical Review B **100**, 045114 (2019).



(a) Type-I (b) Type-II inter band (c) Type-II intraband

FIG. 2. Contributions in for parallel tilt, for Type-I and Type-II. Components even in t_z are given in solid line, while odd components are given in dashed line.

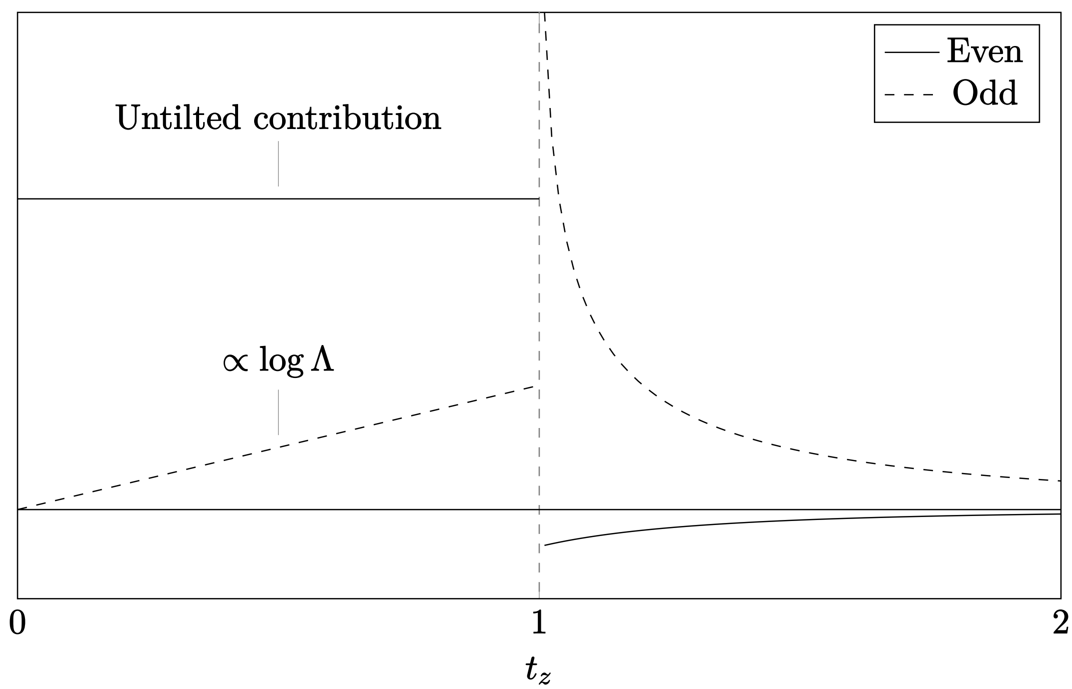


FIG. 3. Schematic summary of the contributions for perpendicular tilt t_z. Shown is the even (solid) and odd (dashed) parts as functions of t_z.

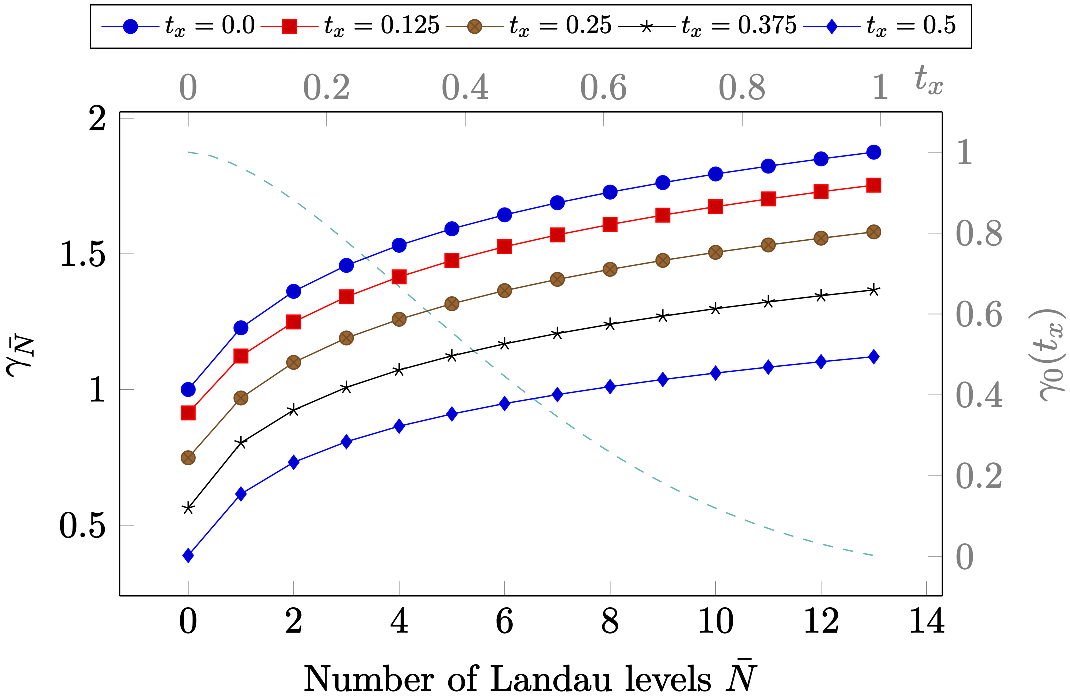


FIG. 4. Contribution as a function of number of Landau levels N for various values of t_x.