# 1 Charge current from the conformal anomaly

In 2016 Chernodub [Che16] showed that the conformal anomaly of QED (quantum electrodynamics) leads to electrical currents in an inhomogeneous gravitational background. This effect was further explored by Chernodub, Cortijo, and Vozmediano [CCV18], showing through Luttinger's method that such an anomalous transport could be generated from a temperature gradient, giving additional contributions to the Nernst current. The same effect was shortly after derived more formally through the Kubo formalism, by Arjona, Chernodub, and Vozmediano [ACV19].

In this chapter we extend the Kubo calculation to tilted Weyl cones. Firstly, the result for the untilted system is rederived, where we also show several simplifications compared to previous computations. The results for the untilted cone are then generalized to tilted cones. The computation is quite lengthy, and the thesis is explicit in each step, with the goal being that a graduate level student should be able to comfortably follow the calculations.

The chapter is divided into sections, each representing a somewhat contained part of the calculation. The text is not, however, written such that a reader should expect to understand a section without reading the preceeding one. Due to the nature of the work, certain sections are rather technical. For the benefit of the reader, we have included summaries of intermediate results, enabling the reader to skip the more technical parts. In particular, the latter part of section 1.2.2 and section 1.5.1 may be skipped without much loss.

We will find the current response of a single Dirac cone, with a temperature gradient $\nabla_y T$ and a magnetic field $B_z$. The current response of interest in the given geometry is thus in the $x$-direction,

$$J^x = \chi^{xy} \frac{-\nabla_y T}{T}, \tag{1.1}$$

with $\chi^{xy}$ being the response[1]. This geometry is shown in Figure 1.1. In the derivation of Chernodub, Cortijo, and Vozmediano [CCV18] the response

$$\chi^{xy} = \frac{e^2 v_F B}{18\pi^2 \hbar} \tag{1.2}$$

[1] The sign in Eq. (1.1) depends on the choice of the response function being the response of the gravitational potential or the temperature gradient. Thus, the sign may differ in the literature.

figures/setup.png

Figure 1.1.: Sketch of the geometry used in the derivation. Note that we consider only bulk response, and the finite sample is only for illustration purposes.

was found, while the derivation of Arjona, Chernodub, and Vozmediano [ACV19] found [2]

$$\chi^{xy} = \frac{e^2 v_F B}{4\pi^2\hbar}. \tag{1.3}$$

Recall the linear response from the Kubo formalism in Eq. (**??**), found through Luttinger's approach.

$$\langle J^i \rangle (t, r) = \underbrace{\int_{-\infty}^{\infty} \mathrm{d}t'\mathrm{d}r' \int_{-\infty}^{t'} \mathrm{d}t'' \left\{ \frac{-iv_F}{\hbar} \Theta(t-t') \left\langle [J^i(t,r), T^{j0}(t'',r')] \right\rangle \right\}}_{\chi^{ij}} \frac{-\partial'_j T(t',r')}{T}. \tag{1.4}$$

Fourier transforming now to the frequency and momentum domain, will be beneficial in our calculations. As before, the non-perturbed system will be taken to be time and position invariant, such that the correlator in Eq. (1.4) can be taken to depend only on the differences $t - t''$ and $r - r'$. Starting with Fourier transforming the position part, notice that the structure of Eq.

[2] The paper is somewhat unclear on what is their final result, as there is some possible confusion related to the number of Landau levels included and whether one is including both or only one Dirac cone. The above result is what is meant, to the best of our understanding.

(1.4) is

$$\langle J^i \rangle (\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{r}' \chi(\boldsymbol{r} - \boldsymbol{r}')(-\partial_j' T(\boldsymbol{r}')/T),$$

where the temporal parts were dropped for clarity. This is a convolution, and the Fourier transform is thus simply given by the product of the two factors [Rot95].

$$\langle J^i \rangle (\boldsymbol{q}) = -\chi(\boldsymbol{q})(iq_j)T(\boldsymbol{q})/T, \tag{1.5}$$

where it was also used that the Fourier transform of a derivative gives the component of the variable. Showing explicitly how to find the form of the response $\chi$ in momentum space is often overlooked in much literature, and as it does involve some finesse, we want to show it here. This trick is courtesy of Chang [Cha18]. By definition, the Fourier transform of the response is, where the variable of integration has been chosen to be $\boldsymbol{r} - \boldsymbol{r}'$ for later convenience,

$$\begin{aligned} \chi(\boldsymbol{q}) &= \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} \chi(\boldsymbol{r} - \boldsymbol{r}') & (1.6) \\ &= \int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} C \left\langle \left[ J^i(\boldsymbol{r}), T^{j0}(\boldsymbol{r}') \right] \right\rangle, & (1.7) \\ & & (1.8) \end{aligned}$$

where $C$ denotes $t$-dependent prefactors and integrals over time are omitted, again for clarity of notation. Note that

$$\int \mathrm{d}(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \mathrm{d}\boldsymbol{r}', \tag{1.9}$$

where $\mathcal{V}$ is the volume of the system. Thus,

$$\begin{aligned} \chi(\boldsymbol{q}) &= \frac{1}{\mathcal{V}} \int \mathrm{d}\boldsymbol{r} \mathrm{d}\boldsymbol{r}' e^{-i\boldsymbol{q}(\boldsymbol{r}-\boldsymbol{r}')} C \left\langle \left[ J^i(\boldsymbol{r}), T^{j0}(\boldsymbol{r}') \right] \right\rangle \\ &= \frac{C}{\mathcal{V}} \left\langle \left[ J^i(\boldsymbol{q}), T^{j0}(-\boldsymbol{q}) \right] \right\rangle. \end{aligned} \tag{1.10}$$

Considering now the temporal part, the procedure is simpler. The linear response still has the form of a convolution, as the response function is only dependent on the difference $t - t'$ by

**Add indices to energy-momentum tensor**

$$\chi(t - t') = \int_{-\infty}^{0} \mathrm{d}t'' \Theta(t - t') \left\langle \left[ J^i(t - t'), T^{j0}(t'') \right] \right\rangle, \tag{1.11}$$

where $t''$ was shifted by $t'$, and then the translational invariance of the correlator was used. In frequency space

$$\chi(\omega) = \int \mathrm{d}t e^{i\omega t} \chi(t) \tag{1.12}$$

$$= \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t'' \Theta(t) \left\langle \left[ J^i(t), T^{j0}(t'') \right] \right\rangle . \tag{1.13}$$

In frequency and momentum space the response function is thus

$$\chi^{ij}(w, \boldsymbol{q}) = \frac{-iv_F}{\mathcal{V}\hbar} \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \left\langle \left[ J^i(t, \boldsymbol{q}), T^{j0}(t', -\boldsymbol{q}) \right] \right\rangle . \tag{1.14}$$

## 1.1. General remarks

Before beginning the computation, we here briefly mention some complications and considerations important to our result. Firstly we discuss how the charge current form a Kubo calculation relate to experimentally measurable currents. Secondly we discuss the ambiguity related to the energy-momentum tensor.

### 1.1.1. Transport and magnetization

Recall that we generally define the transport coeffieicents (Eq. (**??**))

$$J^i = -eL^{11}_{ij} \left[ E_j - T\nabla_j \frac{\mu}{T} \right] - eL^{12}_{ij} T\nabla_j \frac{1}{T},$$

where $J^i$ is the electrical current. In our work, we focus on the $L^{12}$ coefficient, however the following discussion is valid also more generally. The definition of transport currents becomes more subtle in systems with broken time-reversal symmetry[vdWS19; Che+21]. In such systems, unobservable, circulating *magnetization* currents arise. These currents do not contribute to transport, but the Kubo treatment derives the local current, which in general also includes non-transporting currents. Let

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{tr}} + \boldsymbol{J}_M, \tag{1.15}$$

where $\boldsymbol{J}$ is the total local current, $\boldsymbol{J}_{\mathrm{tr}}$ is the transport current, and $\boldsymbol{J}_M$ is the circulating magnetization current. The Kubo formalism generally gives the response to the total local current, $\chi$; we are more interested in the experimentally measurable transport response $L_{ij}^{12}$, related to our Kubo result as [Che+21]

$$L_{ij}^{12} = -\chi_{ij}/e + \epsilon^{ijl} M_l, \tag{1.16}$$

with $M_l$ the magnetization. For zero chemical potential, however, these magnetization currents have been shown to go to zero as $T \to 0$ [vdWS19]. The result from the Kubo calculation is therefore the actual transport current.

### 1.1.2. Comment on the energy-momentum tensor

There is some ambiguity with regards to the definition of the energy-momentum tensor[Kac18; Che+21; vdWS19; FR04]. The *canonical* energy-momentum tensor, derived from Lagrangian mechanics, is defined as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i} \partial^\nu \psi_i - \eta^{\mu\nu} \mathcal{L}. \tag{1.17}$$

On the other hand, from general relativity, the *dynamical* energy-momentum tensor is defined by the variation of the (matter) action with respect to the metric[Kac18]

**Signs depend on choice of g**

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{1.18}$$

Immediately, we see that the first definition is in general not symmetric, while the latter is, as the metric is always symmetric [3]. As the energy-momentum tensor is an observable, this presents a problem: how should the tensor be defined? This issue is not trivial, and has puzzled physicists for decades[FR04].

Superficially, we make the following observations. The *defining* property of the energy-momentum tensor is its conservation law

$$\partial_\mu T^{\mu\nu} = 0, \tag{1.19}$$

on a flat manifold. This, of course, only defines the tensor up to a total divergence. Denote by $\hat{T}^{\mu\nu}$ the *canonical* energy-momentum tensor. We can then define another tensor

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \partial_\alpha S^{\alpha\mu\nu}. \tag{1.20}$$

[3] something with torsion never

By letting the additional term be anti-symmetric $S^{\alpha\mu\nu} = -S^{\mu\alpha\nu}$, it is divergence free. This is easily shown as follows:

$$\begin{aligned} \partial_\mu \partial_\alpha S^{\alpha\mu\nu} &= -\partial_\mu \partial_\alpha S^{\mu\alpha\nu} && (1.21) \\ &= -\partial_\alpha \partial_\mu S^{\mu\alpha\nu} && (1.22) \\ &= -\partial_\mu \partial_\alpha S^{\alpha\mu\nu}, && (1.23) \end{aligned}$$

where we used the commutation of partial derivatives and relabelling of the dummy indices $\mu, \lambda$. By an appropriate choice of $S^{\alpha\mu\nu}$ the canonical energy-momentum tensor may be symmetrized, importantly while still abiding the conservation law. The correction that symmetrizes the energy-momentum tensor is known as the "Belifnante tensor", which for the Dirac Lagrangian is[Che+21]

$$S^{\alpha\mu\nu} = \frac{1}{8} \bar{\Psi} \left[\gamma^\alpha, \sigma^{\mu\nu}\right] \Psi, \tag{1.24}$$

which gives

$$T^{\mu\nu} = \frac{1}{4} \bar{\Psi} (\gamma^\mu D^\nu + \gamma^\nu D^\mu) \Psi. \tag{1.25}$$

Which, in the case of the Dirac Lagrangian, so happens to correspond to the naive symmetrization

$$T_s^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}. \tag{1.26}$$

It is also instructive for our work to consider a more naive line of reasoning. The energy-momentum tensor is used in this work through its conservation law Eq. (1.19), whose first component gives the conservation of energy. Writing it out explicitly

$$\partial_0 T^{00} + \partial_i T^{i0} = \partial_0 \epsilon + \partial_i j_\epsilon^i = 0, \tag{1.27}$$

with $\epsilon$ the energy density and $\boldsymbol{j}_\epsilon$ the energy density current, the question is really seen to be finding the energy density current, ignoring all formal arguments about the energy-momentum tensor in a general context. Using such a line of reasoning van der Wurff and Stoof [vdWS19] argued that the appropriate form of the energy-momentum tensor that should be used in linear response calculations of Dirac material systems, is the unsymmetrized canonical tensor. In this work, we will therefore use the canonical energy-momentum tensor, as opposed to the symmetric form used in the linear response calculation of an untitled cone done by Arjona, Chernodub, and Vozmediano [ACV19]. In the untilted case, the two definitions give the same contribution, while for a titled cone, the response from the two definitions differ.

In the case of an untilted system, the components of interest of the canonical energy-momentum tensor reads

$$T^{y0} = \frac{si}{4}\left[\phi^\dagger \sigma_y \partial_0 \phi - \partial_0 \phi^\dagger \sigma_y \phi\right], \tag{1.28a}$$

$$T^{0y} = \frac{v_F}{4}\left[\phi^\dagger p_y \phi - p_y \phi^\dagger \phi\right]. \tag{1.28b}$$

The symmetrized energy-momentum tensor used by Arjona, Chernodub, and Vozmediano [ACV19]

$$T_s^{y0} = \frac{T^{y0} + T^{0y}}{2}. \tag{1.29}$$

The response was found to be

$$\chi = [\dots] \sum_{\substack{m,n \\ N=M-1}} \int \mathrm{d}\kappa_z (F^{(1)} + F^{(2)}) \alpha^2_{\kappa_z m s}, \tag{1.30}$$

with $[\dots]$ prefactors not relevant here, and $F^{(i)}$, $i = 1, 2$ the contribution from $T^{y0}$ and $T^{y0}$, respectively. They are

$$F^{(1)} = \epsilon_{\kappa_z m s} + \epsilon_{\kappa_z n s}, \tag{1.31}$$

$$F^{(2)} = s\alpha_{\kappa_z n s}\sqrt{M-1} + \frac{s\sqrt{M}}{\alpha_{\kappa_z m s}}, \tag{1.32}$$

where we have used the dimensionless

$$\epsilon_{\kappa_z m s} = \frac{\sqrt{2eB}}{v_F} E_{k_z m s}, \quad \kappa_z = \sqrt{2eB} k_z.$$

Using the explicit form of the normalization factor

$$\alpha_{k_z m s} = -\frac{s\sqrt{M}}{\epsilon_m - s\kappa_z},$$

and energy eigenstates

$$\epsilon_m = \mathrm{sign}(m)\sqrt{M + \kappa_z^2},$$

it is not difficult to show that

$$F^{(2)} = \epsilon_{\kappa_z m s} + \epsilon_{\kappa_z n s} = F^{(1)}. \tag{1.33}$$

Tilting parallel to the magnetic field does not alter the eigenstates, it only changes the eigenvalues by a factor $t_\parallel v_F k_z$ [YYY16; TCG16], as we will show later. The results from the untilted case may thus be applied directly, with rescaled energies. As the normalization factor $\alpha_{\kappa_z ms}$ is invariant under the tilt, $F^{(2)}$ does not change. However, $F^{(1)}$ changes to

$$F^{(1)} = \epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} = \epsilon^0_{\kappa_z ms} + \epsilon^0_{\kappa_z ns} + 2\kappa_z t_{\parallel}, \tag{1.34}$$

where $\epsilon^0_{\kappa_z ms}$ is the energy levels of the untilted system. The last term in Eq. (**??**) gives a non-zero contribution to the total response, and so the results for a tilted cone is generally dependent on the choice of the energy-momentum tensor.

The calculation presented in the thesis has for completeness been carried out also for the symmetric energy-momentum tensor as well. The result presented in appendix A.

**Decide dimless or dimfull quantities**

## 1.2. Eigenvalue problem of the Landau levels of a Weyl Hamiltonian

To evaluate the correlator of the response function, the matrix elements of the current and stress-energy tensor must be found. In order to do this, we find eigenstates in the Landau basis of the system. We will first consider the untilted Hamiltonian, which we will then use to find the Landau levels of the tilted Hamiltonian.

### 1.2.1. The untilted Hamiltonian

Couple the Weyl Hamiltonian to the magnetic field through minimal coupling

$$H_s = s v_F \sigma^i \left(p_i + eA_i\right), \tag{1.35}$$

with $s$ being the chirality, $p_i$ the momentum operator, and $e = |e|$ the coupling constant to the electromagnetic field $\boldsymbol{A}$. Choose coordinates such that $\boldsymbol{B} = B_z \hat{\boldsymbol{z}}$, which in the Landau gauge gives $\boldsymbol{A} = -B_z y\, \hat{\boldsymbol{x}}$. As the Hamiltonian is invariant in $x$ and $z$, take the plane wave ansatz $\phi(\boldsymbol{r}) = e^{ik_x x + ik_z z}\phi(y)$. It then follows

$$H_s \phi(\boldsymbol{r}) = E\phi(\boldsymbol{r}) \implies \tilde{H}_s \phi(y) = E\phi(y), \tag{1.36}$$

where $\tilde{H}$ is the result of replacing $p_z \to k_z, p_x \to k_x$ in $H_s$, as the plane wave part of $\phi$ have these eigenvalues. Absorb the chirality $s$ as a sign in the velocity $v_F$, for more concise notation. Thus, writing everything explicitly, the spectrum is given by

$$-v_F \begin{pmatrix} -k_z & \partial_y + eyB_z/ - k_x \\ -\partial_y + eyB_z/ - k_x & k_z \end{pmatrix} \phi(y) = E\phi(y). \tag{1.37}$$

We will now find the spectrum $E$ of the Hamiltonian.

Inspired by the derivation for the spectrum of the 2D Dirac Hamiltonian in [WBB14], we introduce the length scale $l_B = 1/\sqrt{eB}$, and the dimensionless quantity $\chi = y/l_B - k_x l_B$. In dimensionless quantities Eq. (1.37) is

$$-\frac{v_F}{l_B} \begin{pmatrix} -k_z l_B & \partial_\chi + \chi \\ -\partial_\chi + \chi & k_z l_B \end{pmatrix} \phi(y) = E\phi(y). \tag{1.38}$$

Let the operators $a = (\chi + \partial_\chi)/\sqrt{2},\ a^\dagger = (\chi - \partial_\chi)/\sqrt{2}$. One may easily verify the commutation relation $[a, a^\dagger] = 1$; they are ladder operators of the harmonic oscillators, whose eigenstates are $|n\rangle$, with $a|n\rangle = \sqrt{n}\,|n-1\rangle, a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. In terms of these operators, the system is

$$-\frac{\sqrt{2}v_F}{l_B} \begin{pmatrix} -\frac{k_z l_B}{\sqrt{2}} & a \\ a^\dagger & \frac{k_z l_B}{\sqrt{2}} \end{pmatrix} |\phi\rangle = E\,|\phi\rangle\,. \tag{1.39}$$

Take the ansatz

$$|\phi\rangle = \begin{pmatrix} \beta\,|n-1\rangle \\ \alpha\,|n\rangle \end{pmatrix}, \tag{1.40}$$

which is the most general form of $|\phi\rangle$ with any hope of being an eigenstate. This leads to

$$-\frac{\sqrt{2}v_F}{l_B} \begin{pmatrix} (-\gamma\beta + \alpha\sqrt{n})\,|n-1\rangle \\ (\beta\sqrt{n} + \gamma\alpha)\,|n\rangle \end{pmatrix} = E\,|\phi\rangle\,, \tag{1.41}$$

with $\gamma = k_z l_B/\sqrt{2}$. For $n > 0$ this leads to the equation for $\phi$ to be an eigenfunction

$$-\gamma + \frac{\alpha}{\beta}\sqrt{n} = \frac{\beta}{\alpha}\sqrt{n} + \gamma. \tag{1.42}$$

Solving for $\alpha/\beta$ this gives

$$\frac{\alpha}{\beta} = \frac{\gamma}{\sqrt{n}} \pm \sqrt{1 + \frac{\gamma^2}{n}}, \tag{1.43}$$

and thus

$$E = \pm v_F \sqrt{\frac{2n}{l_B^2} + k_z^2} = \pm s v_F \sqrt{2neB + k_z^2}, \tag{1.44}$$

where we reintroduced the explicit $s$. For $n = 0$ the annihilation operator $a$ destroys the vacuum state $|0\rangle$, and the energy is instead $E_0 = -s k_z v_F$. The excited energy states are doubly degenerate; we choose to denote the energy levels by $m \in \mathbb{Z}$, where the sign from $\pm s$ is taken care of by the sign of this quantum number, and the harmonic oscillator levels $n$ are given by its absolute value $|m|$. The energy levels are

$$E_{k_z m s} = \operatorname{sign}(m) v_F \sqrt{2|m|eB + k_z^2} \qquad \text{for } m \neq 0, \tag{1.45a}$$

$$E_{k_z 0 s} = -s k_z v_F \qquad \text{for } m = 0. \tag{1.45b}$$

We now find the corresponding eigenvectors of the system. The solution to the one dimensional harmonic oscillator in position space is, in dimensionless coordinates $\xi$, [Olv+, Eq. 18.39.5]

$$\langle \xi | n \rangle = \phi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \pi^{-\frac{1}{4}} e^{-\frac{\xi^2}{2}} H_n(\xi), \tag{1.46}$$

where $H_n$ are the Hermite polynomials. Thus,

$$\langle \chi | \phi \rangle = \begin{pmatrix} \beta \langle \chi | n-1 \rangle \\ \alpha \langle \chi | n \rangle \end{pmatrix} = e^{-\frac{\chi^2}{2}} \begin{pmatrix} \frac{\beta}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}} H_{n-1}(\chi) \\ \frac{\alpha}{\sqrt{2^n n! \sqrt{\pi}}} H_n(\chi) \end{pmatrix}, \tag{1.47}$$

where we defined $H_{-1} = 0$ in order to get a more general expression. Choosing

$$\alpha = \sqrt{\frac{\gamma^2}{n}} \implies \beta = \frac{1}{1 \pm \sqrt{1 + \frac{n}{\gamma^2}}} = \pm \frac{\gamma^2}{n} \left( \sqrt{1 + \frac{n}{\gamma^2}} - 1 \right), \tag{1.48}$$

gives

$$\phi(\chi) = e^{-\frac{\chi^2}{2}} \sqrt{\frac{\gamma^2}{n}} \begin{pmatrix} \frac{\pm\sqrt{\frac{\gamma^2}{n}}\left(\sqrt{1+\frac{n}{\gamma^2}}-1\right)}{\sqrt{2^{n-1}(n-1)!\sqrt{\pi}}} H_{n-1}(\chi) \\ \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(\chi) \end{pmatrix}. \tag{1.49}$$

There are thus four quantum numbers related to the eigenvectors, $k_x, k_z, m, s$. Reintroducing $\chi = (y - k_x l_B^2)/l_B$ and normalizing we get:

**Summary 1**

*The Landau levels of a Weyl cone coupled to a magnetic field $B_z$ has the* *eigenvalues*

$$E_{k_z ms} = \operatorname{sign}(m) v_F \sqrt{2eBM + k_z^2} \qquad \textit{for } m \neq 0, \tag{1.50a}$$

$$E_{k_z 0s} = -s k_z v_F \qquad \textit{for } m = 0. \tag{1.50b}$$

*The eigenstates are*

$$\phi_{kms}(r) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{\left(y - k_x l_B^2\right)^2}{2 l_B^2}} \times \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y - k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M \left(\frac{y - k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{1.51}$$

*where* $\boldsymbol{k} = (k_x, k_z)$, *and with the normalization factor*

$$\alpha_{k_z ms} = \frac{-s\sqrt{2eBM}}{\frac{E_{k_z ms}}{v_F} - s k_z}. \tag{1.52}$$

*Here, capital letters indicate absolute value of corresponding quantity,* $M = |m|$, *a convention we will use throughout the chapter.*

### 1.2.2. The tilted Hamiltonian

As we have seen, the eigenvalues of a Type-II Weyl semimetal are simple to find, and are not qualitatively different from those of Type-I, other than the appearance of particle and hole pockets at the Fermi level. We will also consider the Landau levels of these materials, which importantly are very different from Type-I. In fact, erroneous treatment of the Landau spectrum of Type-II semimetals caused the original paper describing Type-II materials to mistakenly assert that the chiral anomaly would not be present for certain directions of a background magnetic field [Sol+15; SGT17].

The issue with the Landau level description is that for certain directions of the $B$-field, the Landau levels break down. For Type-I materials, the description is valid for all directions of the $B$-field, but as the cone tilts into a Type-II material, the description breaks down when the $B$-field and tilt direction are perpendicular [SGT17], and as the magnitude of the tilt increases, the Landau levels are only valid up to a certain angle between the tilt direction and magnetic field. We will in this section derive and elucidate the Landau levels and their regions of validity.

Consider again the Hamiltonian

$$H = v_F \boldsymbol{t}^s \boldsymbol{p} + s v_F \boldsymbol{p}\boldsymbol{\sigma}, \tag{1.53}$$

with the *tilt vector* as defined in Eq. (**??**)

$$\boldsymbol{t}^s = \begin{cases} \boldsymbol{t} & \text{broken inversion symmetry,} \\ s\boldsymbol{t} & \text{inversion symmetry.} \end{cases}$$

To find the Landau levels in a magnetic field $\boldsymbol{B} = B_z\hat{z}$, we will "Lorentz boost" the system to a frame where the cone is not tilted, where we may use the usual approach for finding the Landau levels.

**Make sure this is not a repetition**

Generally, consider $\boldsymbol{t}$ to consist of two components: $\boldsymbol{t}_{\parallel}$ which is parallel to the magnetic field, and $\boldsymbol{t}_{\perp}$ perpendicular to the magnetic field. Without loss of generality, we take the magnetic field to be in the $z$-direction, with the Landau gauge $\boldsymbol{A} = -B_z y\hat{x}$. By a rotation around $z$, we may also in general take $\boldsymbol{t}_{\perp} \parallel \hat{x}$. [4] Thus, let $\boldsymbol{t} = (t_{\perp}, 0, t_{\parallel})$, and introduce the magnetic field through the minimal coupling $\boldsymbol{p} \to \boldsymbol{p}^B = \boldsymbol{p} + e\boldsymbol{A}$.

The Landau level equation is

$$(H_B - E)\left|\psi\right\rangle = 0, \tag{1.54}$$

with

$$H_B = v_F\left(t^s_{\perp} p^B_x + t^s_{\parallel} p^B_z\right)\mathcal{I}_2 + \sum_i s v_F p^B_i \sigma_i, \tag{1.55}$$

where $\mathcal{I}_2$ is the identity matrix of size 2. We may again make the plane wave ansatz $\phi(\boldsymbol{r}) = e^{ik_x x + ik_z z}\phi(y)$, similar to what was done for the untilted Hamiltonian in section 1.2.1, to replace $p_{(x/z)} \to k_{(x/z)}$. In order to use the ladder operator method used for the untilted cone, we must get rid of the $k^B_x$ on the diagonal of the Hamiltonian, as we must reformulate the equation in terms of the ladder operators. [5] To achieve this, we will use a "Lorentz

[4] The setup considered in the response calculation does not have $U(1)$ symmetry around the $\boldsymbol{B}$-field, due to the temperature gradient $\nabla T$. However, the Landau levels are here computed generally, and when later introducing the symmetry breaking components like the temperature gradient, we simply rotate to an appropriate frame.

[5] It would also be possible to choose the frame such that the tilt was both in $x$ and $y$ direction, in which case we would get ladder operators also on the diagonal. This system, albeight tedious, could also have been solved directly.

**Verify this**

boost", which as we will show only leave $k_z$ and $E$ in the diagonal. Act with the hyperbolic rotation operator $R = \exp[\Theta/2\sigma_x]$ on Eq. (1.54) from the left, and insert identity on the form $RR^{-1}$ before the state vector. By introducing the state in the rotated frame $|\tilde{\psi}\rangle = R^{-1}\mathcal{N}|\psi\rangle$, with $\mathcal{N}$ a normalization factor compensating for the non-unitarity of the transformation, we get the eigenvalue equation

$$(RH_BR - ER^2)|\tilde{\psi}\rangle = 0. \tag{1.56}$$

We now make the fortunate observation that the diagonal elements of

$$R\sigma_iR$$

are zero for $i = y$ and non-zero for $i = x, z$. We may thus rotate the $x$- and $z$-components in and out of the diagonal elements, without accidentaly rotating the $y$-components into the diagonal.

We will now find the boost parameter that eliminates $k_x$ from the diagonal. Note that

$$R^2 = e^{\Theta\sigma_x} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{1.57}$$

and as $[R, \sigma_x] = 0$,

$$R\sigma_xR = R^2\sigma_x = \begin{pmatrix} \sinh\theta & \cosh\theta \\ \cosh\theta & \sinh\theta \end{pmatrix}, \tag{1.58}$$

as the effect of $\sigma_x$ is to transpose the rows. The problematic part of the Hamiltonian with regards to finding the Landau levels, are the terms containing $k_x^B$ on the diagonal, i.e.

$$v_F t_\perp^s k_x^B \mathcal{I}_2 + s v_F k_x^B \sigma_x.$$

The requirement for $k_x^B$ to be rotated out of the diagonal is thus

$$t_\perp^s \cosh\theta + s\sinh\theta = 0. \tag{1.59}$$

Solving for $\theta$ we get

$$\theta = \log(\pm\frac{\sqrt{s - t_\perp^s}}{\sqrt{s + t_\perp^s}}). \tag{1.60}$$

Alternatively, written in a slightly suggestive form,

$$\tanh\theta = -st_\perp^s, \tag{1.61}$$

which is of course on the form of the *rapiditiy* known from Lorentz transformations, with $-st_\perp^s$ taking the place of the $\beta = v/c$ factor. From this observation, we also find it instructive to introduce the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-t_\perp^2}}. \tag{1.62}$$

The required hyperbolic tilt angle to eliminate the $k_x^B$ in the diagonal elements of the Hamiltonian, originating from the tilt, is thus

$$\theta = -s \tanh^{-1} t_\perp^s. \tag{1.63}$$

The inverse of tan, of course, diverges as the argument approaches $\pm 1$, as shown in Figure 1.2. For $|t_\perp| < 1$ we are able to find an angle $\theta$ which

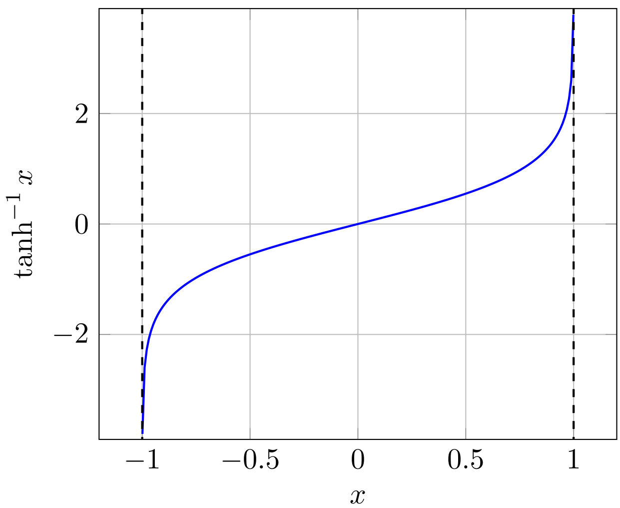



Figure 1.2.: Plot of $\tanh^{-1}$, which diverges as the argument goes to $\pm 1$.

transforms our Hamiltonian into a form which we may solve. For $|t_\perp| \geq 1$, however, no (real) solution of $\theta$ exits, and the Landau level description collapses. More concretely, as we will show later, the separation of the landau levels is reduced as the perpendicular tilt increases, and as $|t_\perp| \to 1$, the level separation $\Delta E \to 0$.

Interestingly, there are no restrictions in the perpendicual tilt, $t_\parallel$. The $t$ parametrization of the tilt is conveniently visualized by plotting the $t$-vector inside a unit sphere, shown in Figure 1.3. If the vector is outside the unit sphere, it is a Type-II, if it is inside, it is a Type-I. Also, if the projection of the vector onto the $x, y$-plane is on the unit disk, the Landau level description is valid, if not, the Landau levels collapse. When the projection is on the unit disc, the system is in the *magnetic* regime, otherwise we denote it by the *electric* regime. All Type-I materials may thus be described by Landau levels, while it for Type-II is only valid for certain directions of $\boldsymbol{t}$. As the $t$-vector gets larger, the magnetic regime is restricted to smaller angles between $\boldsymbol{t}$ and $\boldsymbol{B}$.

figures/tiltSpherewBackground.png

Figure 1.3.: Geometric visualization of the *tilt vector* $\boldsymbol{t}$. When the vector is inside the unit sphere ($t < 1$), the system is in the Type-I regime. When the vector is outside the unit sphere ($t > 1$), the system is in the Type-II regime. When the projection onto the $xy$-plane is on the unit disc, the system is in the *magnetic* regime, otherwise it is in the *electric* regime. Shown is Type-I tilt (blue), Type-II magnetic (red), and Type-II electric (green). Figure inspired by Tchoumakov, Civelli, and Goerbig [TCG16].

We now return to solving Eq. (1.56), using the solution angle we just found. By insertion, and after some clean up, we get

$$(RH_BR - ER^2)\,|\tilde{\psi}\rangle = v_F A\,|\tilde{\psi}\rangle = 0, \tag{1.64}$$

with

$$
\begin{aligned}
A_{11} &= k_z(s + t^s_{\|}\gamma) - E/v_F\gamma, \\
A_{12} &= -s(ik_y + k_z t_\perp t_\| \gamma - k_x/\gamma - E/v_F\gamma t^s_\perp), \\
A_{21} &= s(ik_y - k_z t_\perp t_\| \gamma + k_x/\gamma + E/v_F\gamma t^s_\perp), \\
A_{22} &= -k_z(s - t^s_{\|}\gamma) - E/v_F\gamma.
\end{aligned}
$$

In order to simplify this further, absorb $\gamma t^s_\perp(k_z t^s_\| - E/v_F)$ into $k_x$. Thus, let

$$
\begin{aligned}
\tilde{k}_x &= k_x/\gamma + \gamma t^s_\perp(E/v_F - k_z t^s_\|), \\
\tilde{k}_y &= k_y, \\
\tilde{k}_z &= k_z.
\end{aligned}
\tag{1.65}
$$

These expressions warrant some explanation, as the Lorentz boost is of course

$$
\tilde{k}_x = \gamma(k_x - t_\perp \frac{E}{v_F}), \tag{1.66}
$$

where we used the four momentum $p^\mu = (\frac{E}{v_F}, \boldsymbol{p})$, and the effective speed of light $v_F$. It can thus look like our expression in Eq. (1.65) is wrong. The solution to this seeming inconsitency is that the proper energy is not $\frac{E}{v_F} - k_z t_\|$, but rather $\frac{E}{v_F} - k_z t_\| - k_x t_\perp$.

**Something smart here, or remove discussion**

The eigenvalue equation is simply

$$
\left[\gamma\left(t^s_\| \tilde{k}_z - \frac{E}{v_F}\right)\mathcal{I}_2 + s\tilde{k}_i\sigma_i\right]|\tilde{\psi}\rangle = 0. \tag{1.67}
$$

If we now again introduce the magnetic field using minimal coupling, $k_x \to k_x - eyB_z$, this corresponds to an effective field $B_z/\gamma$ in the new quantities. This is because $\tilde{k}_x \to \tilde{k}_x - eyB_z/\gamma$. The Landau level equation thus reads

$$
\left[\sum_i s v_F\left(\tilde{k}_i + e\tilde{A}_i\right)\sigma_i\right]|\tilde{\psi}\rangle = (E - t^s_\| v_F \tilde{k}_z)\gamma\,|\tilde{\psi}\rangle\,, \tag{1.68}
$$

where $\tilde{\boldsymbol{A}} = -B_z/\gamma y\hat{\boldsymbol{x}}$. We may thus use directly the result for the untilted cone, Eq. (1.45), giving

$$
\left(E - t^s_\| v_F \tilde{k}_z\right)\gamma = \mathrm{sign}(m) v_F\sqrt{2|m|e\frac{B}{\gamma} + \tilde{k}_z^2}, \qquad m \neq 0, \tag{1.69a}
$$

$$
\left(E - t^s_\| v_F \tilde{k}_z\right)\gamma = -s\tilde{k}_z v_F, \qquad m = 0. \tag{1.69b}
$$

Cleaning up, we get

$$E = t_{\parallel}^{s} v_F \tilde{k}_z + \mathrm{sign}(m) v_F \sqrt{2|m|e\frac{B}{\gamma^3} + \tilde{k}_z^2/\gamma^2}, \qquad m \neq 0, \tag{1.70a}$$

$$E = \tilde{k}_z v_F \left(t_{\parallel}^{s} - s/\gamma\right), \qquad m = 0. \tag{1.70b}$$

As the perpendicular tilt is increased, $\gamma = 1/\sqrt{1-t_{\perp}^2}$ diverges to infinity. With the trivial substitution $\alpha = 1/\gamma$, which goes to zero, this gets an intuitive interpretation. As the perpendicular tilt increases, the Landau levels converge towards $t_{\parallel} v_F \tilde{k}_z$. In particular, the separation between Landau levels is reduced by a factor $\alpha^{3/2}$. The effect of the tilt on the Landau levels is to squeeze the Landau levels together, and we will call the $\alpha$ the *squeezing factor*. We note that when approaching the degree of tilt where we are no longer able to find a boost which enables us to solve for the Landau levels, i.e. when $|t_{\perp}| \to 1$, the squeezing factor goes to zero. As the tilt exceeds this limit, the squeezing factor is imaginary. The Landau level description is only valid for $|t_{\perp}| < 1$.

The energy levels of the tilted cone expressed in terms of the energy levels of the untilted cone

$$E = t_{\parallel}^{s} v_F k_z + \alpha E_{m,\alpha B}^{0},$$

where $E_{m,\alpha B}^{0}$ is the energy in the untilted case, with magnetic field $\alpha B$. Tilting of the Landau levels is induced by the parallel tilt component, $t_{\parallel}$. In fact, the Landau levels cross the Fermi level at the transition from Type-I to Type-II as well. The Landau levels are shown in Figure 1.4.

The eigenstate of

$$H = v_F \sigma^i (p_i + eA_i),$$

with $A_i = -B_z y \delta_{ix}$, given in the position basis, is

$$\phi_{\boldsymbol{k}ms}(\boldsymbol{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{ik_x x} e^{ik_z z}}{\sqrt{\alpha_{k_z ms}^2 + 1}} e^{-\frac{y-k_x l^2}{2l_B^2}} \begin{pmatrix} \frac{\alpha_{k_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ \frac{1}{\sqrt{2^M M! \sqrt{\pi} l_B}} H_M \left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}, \tag{1.71}$$

where capital letters indicate absolute value of corresponding quantity, $M = |m|$, $\boldsymbol{k} = (k_x, k_z)$, and with the normalization factor

$$\alpha_{k_z ms} = \frac{-\sqrt{2eBM}}{\frac{E_{k_z ms}}{s v_F} - k_z}. \tag{1.72}$$

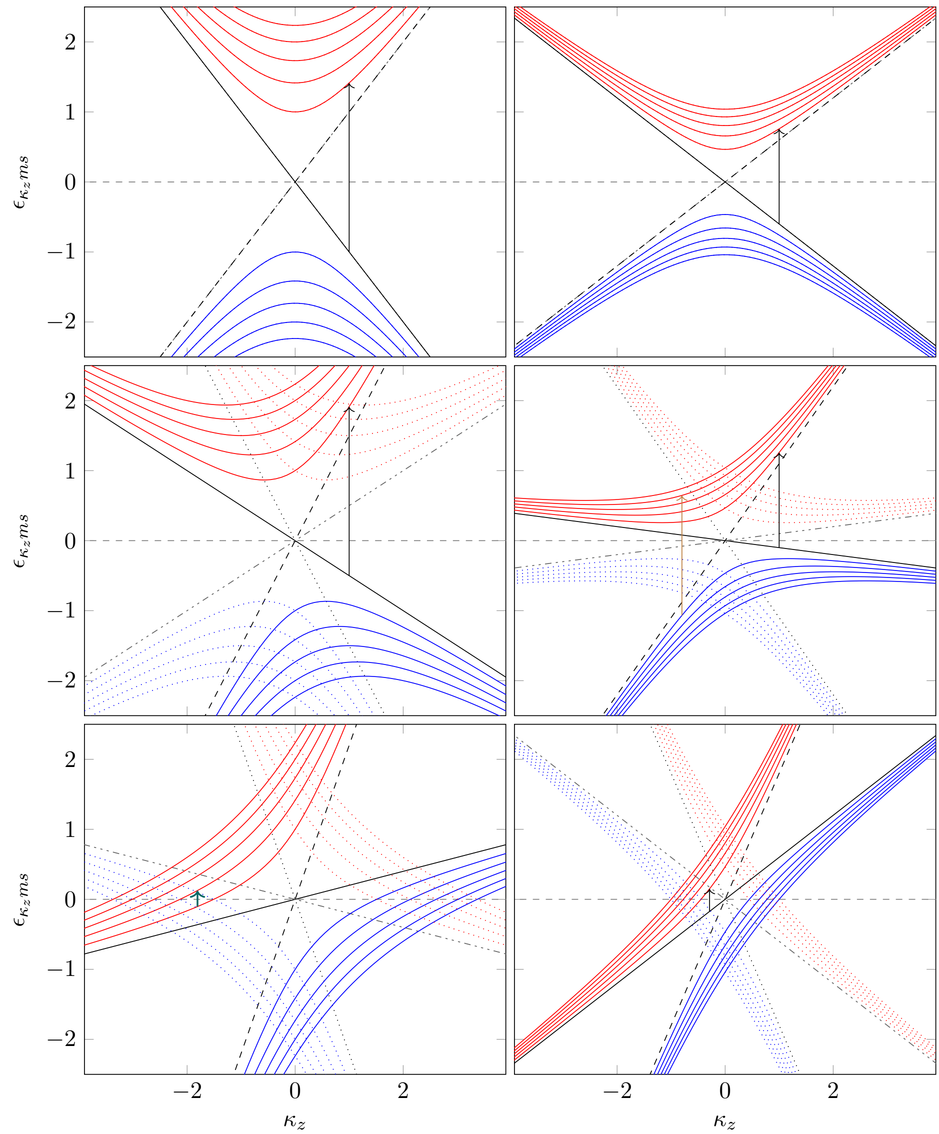



Figure 1.4.: Landau levels for different values of $t_x, t_z$. The top two rows show Type-I, while the lowest row shows Type-II. Left column shows $t_x = 0$, right column $t_x = 0.64$ ($\alpha = 0.6$). The rows shows $t_z = 0, 0.5, 1.2$, from top to bottom. The dotted lines show the Landau levels with opposite sign of $t_z$, the dashed show the opposite chirality. The arrows indicate valid “transitions”, namely the $0 \to 1$ interband in black, $-1 \to 4$ interband in brown, and $1 \to 2$ intraband in teal.

Taking care to keep track of boosted and rescaled quantites, the eigenstate in the boosted frame is

$$\tilde{\psi}(\tilde{r}) = \frac{1}{\sqrt{L_x L_z}} \frac{e^{i\tilde{k}_x \tilde{x}} e^{ik_z z}}{\sqrt{\alpha_{\tilde{k}_z ms}^2 + 1}} e^{-\frac{\left(\tilde{y}-\tilde{k}_x l_{B'}^2\right)^2}{2l_{B'}^2}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_{B'}}} H_{M-1}\left(\frac{\tilde{y}-\tilde{k}_x l_{B'}^2}{l_{B'}}\right) \\ \frac{1}{\sqrt{2^M M!\sqrt{\pi} l_{B'}}} H_M\left(\frac{\tilde{y}-\tilde{k}_x l_{B'}^2}{l_{B'}}\right) \end{pmatrix}, \tag{1.73}$$

with

$$\alpha_{\tilde{k}_z ms} = \frac{-\sqrt{2eB'M}}{\gamma \frac{E_{\tilde{k}_z ms} - t_\parallel^s v_F \tilde{k}_z}{s v_F} - \tilde{k}_z}, \tag{1.74}$$

where

$$B' = B\alpha.$$

We note that $\alpha_{k_z 0 s} = 0$, so using the explicit form of the energy we may simplify the expression some. For $m \neq 0$

$$\frac{E_{k_z ms} - t_\parallel^s v_F k_z}{s v_F} = \mathrm{sign}(m) s\alpha\sqrt{2MeB\alpha + k_z^2}$$

and thus

$$\alpha_{k_z ms} = \frac{-\sqrt{\alpha M}}{\mathrm{sign}(m) s\sqrt{\alpha M + \kappa^2} - \kappa} \tag{1.75}$$

where we defined the dimensionless $\kappa_z = \sqrt{2eB} k_z$.

The original eigenstate $|\psi\rangle = 1/\mathcal{N} e^{\theta/2\sigma_x} |\tilde{\psi}\rangle$ of the tilted system is easily found. Reinserting explicitly, in the boosted frame, that

$$\tilde{k}_x = \alpha k_x + \frac{t_\perp^s}{\alpha}(E_{k_z ms}/v_F - k_z t_\parallel^s) = \alpha k_x + t_\perp^s \frac{E_{m,\alpha B}^0}{v_F}$$

and $l_{B'} = \frac{l_B}{\sqrt{\alpha}}$ we define

$$\chi = \frac{y - \tilde{k}_x l_{B'}^2}{l_{B'}} = \sqrt{\alpha}(y - k_x l_B^2)/l_B + \frac{t_\perp^s l_B}{\sqrt{\alpha} v_F} E_{m,\alpha B}^0, \tag{1.76}$$

which is the argument of the Hermite polynomials. For later convenience, let us explicitly define

$$\tilde{\phi}_{kms}(\tilde{r}) = \frac{e^{i\tilde{k}_x \tilde{x} + ik_z z}}{\sqrt{L_x L_z}} \underbrace{\frac{e^{-\frac{1}{2}\chi^2}\sqrt[4]{\alpha}}{\sqrt{\alpha_{\tilde{k}_z ms}^2 + 1}} \begin{pmatrix} \frac{\alpha_{\tilde{k}_z ms}}{\sqrt{2^{M-1}(M-1)!\sqrt{\pi} l_B}} H_{M-1}(\chi) \\ \frac{1}{\sqrt{2^M M!\sqrt{\pi} l_B}} H_M(\chi) \end{pmatrix}}_{\tilde{\phi}_{kms}(y)}, \tag{1.77}$$

and thus

$$\tilde{\phi}_{kms}(y) = e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{kms}H_{M-1}(\chi) \\ b_{kms}H_M(\chi)\end{pmatrix}, \tag{1.78}$$

with

$$a_{kms} = \frac{\alpha_{\tilde{k}_z ms}\sqrt[4]{\alpha}}{\sqrt{\alpha^2_{\tilde{k}_z ms}+1}\sqrt{2^{M-1}(M-1)!\sqrt{\pi}l_B}}, \tag{1.79}$$

$$b_{kms} = \frac{\sqrt[4]{\alpha}}{\sqrt{\alpha^2_{\tilde{k}_z ms}+1}\sqrt{2^M M!\sqrt{\pi}l_B}}. \tag{1.80}$$

We proceed now to find the normalization factor $\mathcal{N}$, as it will become necessary in later steps. Recall that

$$|\psi\rangle = \frac{1}{\mathcal{N}}e^{\theta/2\sigma_x}|\tilde{\psi}\rangle,$$

and

$$e^{\theta\sigma_x} = \frac{1}{\alpha}\begin{pmatrix} 1 & -st_\perp^s \\ -st_\perp^s & 1\end{pmatrix}.$$

The upper and lower part of the spinor are orthogonal, thus we have

$$\langle\psi|\psi\rangle = \frac{1}{\mathcal{N}^*\mathcal{N}}\frac{1}{\alpha}\langle\tilde{\psi}|\tilde{\psi}\rangle = 1 \implies \mathcal{N}^*\mathcal{N} = \frac{1}{\alpha}. \tag{1.81}$$

We choose $\mathcal{N} = \alpha^{-\frac{1}{2}}$.

**Summary 2**

*The tilted Hamiltonian*

$$H = v_F \boldsymbol{t}^s \boldsymbol{p} + s v_F \boldsymbol{p}\boldsymbol{\sigma}$$

*in a magnetic field $B$ has the Landau levels*

$$E = \begin{cases} t_\parallel^s v_F k_z + \operatorname{sign}(m) v_F \alpha\sqrt{2eB\alpha M + k_z^2} & m \neq 0, \\ t_\parallel^s v_F k_z - s\alpha v_F k_z & m = 0, \end{cases}$$

*with the the* squeezing factor $\alpha = \sqrt{1-t_\perp^2}$. *The associated eigenstates in the position basis are*

$$\psi(\boldsymbol{r}) = \sqrt{\alpha}e^{\theta/2\sigma_x}\frac{e^{ik_x x + ik_z z}}{\sqrt{L_x L_z}}\tilde{\psi}(y),$$

*where*

$$\tilde{\psi}(y) = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{k_z ms} H_{M-1}(\chi) \\ b_{k_z ms} H_M(\chi) \end{pmatrix},$$

*where we have defined* $\chi = \sqrt{\alpha}\frac{y-k_x l_B^2}{l_B} + \frac{t_\perp^s l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}$ *and* $a_{k_z ms}, b_{k_z ms}$ *are given in Eqs. (1.79, 1.80).*

## 1.3. Analytical expression for the response function

We will here find analytical expressions for the current operator $J^i(\omega, \boldsymbol{q})$ and stress-energy tensor $T^{j0}(\omega, \boldsymbol{q})$, needed to calculate the correlation function. The fields are given, in the position basis, by

$$\psi = \sum_{kn} \langle r|kns\rangle \, a_{kns}(t) = \sum_{kn} \phi_{kns}(r) a_{kns}(t), \tag{1.82}$$

$$\psi^\dagger = \sum_{kn} \langle kns|r\rangle \, a^\dagger_{kns}(t) = \sum_{kn} \phi^*_{kns}(r) a^\dagger_{kns}(t). \tag{1.83}$$

**Is the asterisk supposed to be dagger? $\phi^* \to \phi^\dagger$ as it must be transpose??**

Here $a^\dagger_\lambda(t) = \exp(iE_\lambda t/)a^\dagger_\lambda$ and $a^\dagger_\lambda, a_\lambda$ are the creation and annihilation operators of the state with quantum numbers $\lambda$.

### 1.3.1. Expressions for the operators

#### The current operator

The current operator $\hat{\boldsymbol{J}} = e\hat{\boldsymbol{v}}$, where $\hat{\boldsymbol{v}}$ is the velocity operator. Using the relation of Heisenberg operators $\dot{A} = -i[A, H]$ [SN17], for the operator $A$ and Hamiltonian $H$, and with the minimal coupling $\boldsymbol{p}^B = \boldsymbol{p} + e\boldsymbol{A}$,

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = -i\,[\boldsymbol{r}, H] \tag{1.84}$$

$$= -iv_F(s\sigma^i + (t^s)^i)\left[\boldsymbol{r}, p_i^B\right] \tag{1.85}$$

$$= v_F(s\boldsymbol{\sigma} + \boldsymbol{t}^s), \tag{1.86}$$

where we used the canonical commutation relation $[r_i, p_j] = i\delta_{ij}$ and the commutation between the position operator and magnetic potential $\boldsymbol{A}$. We thus get

$$J^x = \psi^\dagger \hat{J}^x \psi = s v_F e \sum_{km,ln} \phi^*_{kms}(\boldsymbol{r}) \left(\sigma^x + s t^s_x\right) \phi_{lns}(\boldsymbol{r}) a^\dagger_{kms}(t) a_{lns}(t). \tag{1.87}$$

### The energy-momentum tensor

The *canonical* energy-momentum tensor is generally defined by

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_i)} \partial_\nu \phi_i - \eta^{\mu\nu} \mathcal{L}, \tag{1.88}$$

where the index $i$ runs over the types of fields. This definition is correct for commuting fields, however, for non-commuting fields like ours, this formula is slightly wrong. This is often overlooked in many textbooks and papers, so we will here elucidate the issue to some degree. While a proper derivation requires the use of Grassman variables and defining left and right derivation, which we will not do here, some simple considerations help in understanding the issue. In the standard textbook derivation of the canonical energy-momentum tensor, one expands the total derivative of the Lagrangian $\mathcal{L}(\psi_i, \partial\psi_i)$ in terms of the fields,

$$\frac{\mathrm{d}\mathcal{L}(\psi_i, \partial\psi_i)}{\mathrm{d}x_\nu} \equiv \mathrm{d}^\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial(\partial_\mu \psi_i)}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial \psi_i} \frac{\partial \psi_i}{\partial x_\nu}. \tag{1.89}$$

This expansion, however, ignores the non-commutative nature of the fields. For concreteness, consider $\psi_i = \bar{\psi}$. Heuristically, the correct expression would be obtained by reordering the factors in the two terms. By naively employing Eq. (1.88), the resulting canonical energy-momentum tensor of the Dirac theory would be

$$T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} \partial^\nu \bar{\psi} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}, \tag{1.90}$$

while the correct form is [IZ80, Eq. 3-153]

$$T^{\mu\nu} = \partial^\nu \bar{\psi} \frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\psi})} + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L}. \tag{1.91}$$

The untilted Weyl Hamiltonian

$$H_s = s\sigma^i p_i, \tag{1.92}$$

where natural units ($c = v_F = 1$) are used, to have the expressions explicitly match those of QFT (quantum field theory) literature; the associated Lagrange density [Kac18]

$$\mathcal{L}_s = i\phi^\dagger \sigma_s^\mu \partial_\mu \phi, \tag{1.93}$$

with $\sigma_s^\mu = (I_2, s\boldsymbol{\sigma})$, i.e. $\sigma_{s=1}^\mu = \sigma^\mu, \sigma_{s=-1}^\mu = \bar{\sigma}^\mu$ known from the Dirac solutions. This is seen directly from the Dirac Lagrangian $i\bar{\psi}\partial\!\!\!/\psi$ by taking

$\psi = (\phi_L, \phi_R)^T$ and setting, for example, $\phi_R = 0$. Symmetrize in daggered and undaggered fields [6]

$$\mathcal{L}_s = \frac{i}{2}(\phi^\dagger \sigma_s^\mu \partial_\mu \phi - \partial_\mu \phi^\dagger \sigma_s^\mu \phi), \tag{1.94}$$

which will prove more convenient to work with. From the definition of the canonical energy-momentum tensor for Dirac fields Eq. (1.91), one gets

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \sigma_s^\mu \partial_\nu \phi - \partial_\nu \phi^\dagger \sigma_s^\mu \phi - \eta^{\mu\nu} \mathcal{L}). \tag{1.95}$$

Consider now the tilted Weyl Hamiltonian

$$H_s = s\sigma^i k_i + (t^s)^i p_i. \tag{1.96}$$

Exactly analogous to the treatment of van der Wurff and Stoof [vdWS19] for the full $4 \times 4$ tilted Dirac Lagrangian, absorb the tilt term into the Pauli matrices, giving the Lagrangian density

$$\mathcal{L}_s = i\phi^\dagger \tilde{\sigma}_s^\mu \partial_\mu \phi, \tag{1.97}$$

where $\tilde{\sigma}_s^\mu = \sigma_s^\mu + (t^s)^\mu$, $(t^s)^\mu = (0, \boldsymbol{t}^s)$. The corresponding energy-momentum tensor, after again symmetrizing in the fields,

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}_s^\mu \partial_\nu \phi - \partial_\nu \phi^\dagger \tilde{\sigma}_s^\mu \phi - \eta^{\mu\nu} \mathcal{L}). \tag{1.98}$$

Reintroducing the explicit effective speed of light $v_F$ and recalling $\partial_0 = \partial_t / v_F$ this gives

$$\begin{aligned} T^{y_0}(t, \boldsymbol{r}) = &\frac{1}{2} \sum_{km,ln} \phi_{kms}^*(\boldsymbol{r})(s\sigma^y + (t^s)^y)\phi_{lns}(\boldsymbol{r}) \\ &\times \left[a_{kms}^\dagger(t) i\partial_t a_{lns}(t) - i\left(\partial_t a_{kms}^\dagger(t)\right) a_{lns}(t) + 2\mu a_{kms}^\dagger(t) a_{lns}(t)\right]. \end{aligned} \tag{1.99}$$

Here, also a non-zero potential $\mu$ is included. Our final result will be given at zero potential, however it is included in the calculations as it might be of interest to consider finite potential in later work. Recalling the time dependence of $a(t), a^\dagger(t)$ we have that

$$i\partial_t a_\lambda(t) = E_\lambda a_\lambda, \quad i\partial_t a_\lambda^\dagger(t) = -E_\lambda a_\lambda^\dagger,$$

which further simplifies the expression.

[6] The Lagrangian itself is unphysical, and we may transform it in any way that leaves the action $\int \mathcal{L}$ invariant.

**Summary 3**

*The current- and energy-momentum tensor operator are*

$$J^x = sv_F e \sum_{km,ln} \phi^*_{kms}(r)\left(\sigma^x + st^s_x\right)\phi_{lns}(r) a^\dagger_{kms}(t) a_{lns}(t), \tag{1.100}$$

$$\begin{aligned} T^{y_0}(t,r) = \frac{1}{2} \sum_{km,ln} \phi^*_{kms}(r)(s\sigma^y + (t^s)^y)\phi_{lns}(r) \\ \times \left[a^\dagger_{kms}(t) i\partial_t a_{lns}(t) - i\left(\partial_t a^\dagger_{kms}(t)\right) a_{lns}(t) + 2\mu a^\dagger_{kms}(t) a_{lns}(t)\right]. \end{aligned} \tag{1.101}$$

### 1.3.2. Response function in momentum space

Fourier transforming the position gives

$$J^x(t,q) = \sum_{km,ln} J^x_{kms,lns}(q) a^\dagger_{kms}(t) a_{lns}(t), \tag{1.102}$$

$$T^{y0}(t,-q) = \sum_{km,ln} T^{y0}_{kms,lns}(q) a^\dagger_{kms}(t) a_{lns}(t), \tag{1.103}$$

where the matrix elements in momentum space are given by

$$J^x_{kms,lns}(q) = \int \mathrm{d}r e^{-iqr} sv_F e \phi^*_{kms}(r)(\sigma^x + st^s_x)\phi_{lns}(r), \tag{1.104}$$

$$T^{y0}_{kms,lns}(q) = \frac{1}{2}\int \mathrm{d}r e^{iqr} \phi^*_{kms}(r)(s\sigma^y + (t^s)^y)(E_{k_z ms} + E_{l_z ns} - 2\mu)\phi_{lns}(r). \tag{1.105}$$

Note that as $T^{y0}(t,-q)$ will be used later, we here for convenience included the sign into the definition of the matrix element $T^{y0}_{kms,lns}$, as is reflected in the sign of the exponent of Eq. (1.105).

As was noted earlier, the eigenvectors are plane waves in the $x, z$-directions, and the non-trivial part is the $y$-dependent $\phi(y)$. Thus, we want to express these matrix elements in terms of $\phi(y)$. The sum over $l$ in Eq. (1.102) can be replaced by an integration, as it is a good quantum number. As usual, the measure in the integration is given by the density of states in momentum space, the well known $L_i/2\pi$, with $L_i$ being the length of the system in the $i$-direction.

$$\begin{aligned} J^x(t,q) &= \sum_{km,n} \int \mathrm{d}l_x \mathrm{d}l_z \frac{L_x L_z}{4\pi^2} J^x_{kms,lns}(q) a^\dagger_{kms}(t) a_{lns}(t) \\ &= \int \mathrm{d}l_x \mathrm{d}l_z \int \mathrm{d}y e^{-iq_y y}\delta(l_x - k_x - q_x)\delta(l_z - k_z - q_z) sv_F e \phi^*_{kms}(y)\sigma^x\phi_{lns}(y). \end{aligned} \tag{1.106}$$

The Dirac delta functions appeared from taking the integrals from the matrix element over $x$ and $z$, as the integrand in these variables was only plane waves. The exact same procedure may be done for the stress-energy tensor in Eq. (1.103). Eliminating $l$ by doing the integrals yields

$$J^x(t,q) = \sum_{k,mn} J^x_{kms,k+\mathfrak{q}ns}(q) a^\dagger_{kms}(t) a_{k+\mathfrak{q}ns}(t), \tag{1.107}$$

$$T^{y0}(t,-q) = \sum_{\kappa,\mu\nu} T^{y0}_{\kappa\mu s,\kappa-\mathfrak{q},\nu s}(q) a^\dagger_{\kappa\mu s}(t) a_{\kappa-\mathfrak{q}\nu s}(t), \tag{1.108}$$

where $\mathfrak{q} = (q_x, q_z)$. Keeping in mind that $a^\dagger_\lambda(t) = e^{iE_\lambda t/} a^\dagger_\lambda$, and that

$$\left\langle \left[ a^\dagger_{kms} a_{k+\mathfrak{q}ns}, a^\dagger_{\kappa\mu s} a_{\kappa-\mathfrak{q}\nu s} \right] \right\rangle = \delta_{k,\kappa-\mathfrak{q}} \delta_{m,\nu} \delta_{k+\mathfrak{q},\kappa} \delta_{n,\mu} \left[ n_{kms} - n_{k+\mathfrak{q}ns} \right], \tag{1.109}$$

where $n_{kms}$ is the Fermi-Dirac distribution, the correlation function is given by

$$\left\langle \left[ J^x(t,q), T^{y0}(t',-q) \right] \right\rangle = \sum_{kmn} e^{i(E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns})t} e^{i(E_{k_z+\mathfrak{q}_z ns} - E_{k_z ms})t'} \\ \times J^x_{kms,k+\mathfrak{q}ns}(q) T^{y0}_{k+\mathfrak{q}ns,kms}(q) \left[ n_{kms} - n_{k+\mathfrak{q}ns} \right]. \tag{1.110}$$

We are now ready to find the correlation function $\chi^{xy}$ given in Eq. (1.14)

$$\chi^{xy}(\omega,q) = \frac{-iv_F}{\mathcal{V}} \int \mathrm{d}t e^{i\omega t} \int_{-\infty}^{0} \mathrm{d}t' \Theta(t) \left\langle \left[ J^x(t,q), T^{y0}(t',-q) \right] \right\rangle. \tag{1.111}$$

Introduce as usual a decay factor $e^{-\eta(t-t')}$ to ensure convergence in the time integrals, and make a change of variables $t' \to -t'$. The integral part of Eq. (1.111), ignoring everything without time dependence for clarity, is then

$$\lim_{\eta\to 0} \int_0^\infty \mathrm{d}t\mathrm{d}t' \exp\left[ i\left( E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega + i\eta \right) t \right] \exp\left[ i\left( E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta \right) t' \right] \\ = \lim_{\eta\to 0} i \left[ E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega + i\eta \right]^{-1} i \left[ E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta \right]^{-1}. \tag{1.112}$$

The response function then reads

$$\chi^{xy}(\omega,q) = \frac{iv_F}{\mathcal{V}} \lim_{\eta\to 0} \sum_{kmn} J^x_{kms,k+\mathfrak{q}ns}(q) T^{y0}_{k+\mathfrak{q}ns,kms}(q) \left[ n_{kms} - n_{k+\mathfrak{q}ns} \right] \\ \left[ E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + \omega + i\eta \right]^{-1} \left[ E_{k_z ms} - E_{k_z+\mathfrak{q}_z ns} + i\eta \right]^{-1}, \tag{1.113}$$

where the matrix elements are

$$J^x_{kms,k+qns}(q) = \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{kms}(y)(\sigma^x + s t^s_x)\phi_{k+qns}(y), \tag{1.114}$$

$$T^{y0}_{k+qns,kms}(q) = \frac{1}{2}\int \mathrm{d}y e^{iq_y y}\phi^*_{k+qns}(y)(s\sigma^y + t^s_y)\left(E_{k_z ms} + E_{k_z+q_z ns} - 2\mu\right)\phi_{kms}(y). \tag{1.115}$$

We will consider the response function in the static limit $\lim_{\omega\to 0}\lim_{q\to 0}$. We may use the property of the limit of a product of functions $\lim A\cdot B = \lim A \cdot \lim B$ to write

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy}(\omega,q) = \frac{iv_F}{\mathcal{V}}\sum_{kmn}\frac{J^x_{kms,kns}T^{y0}_{kns,kms}[n_{kms} - n_{kns}]}{(E_{k_z ms} - E_{k_z ns})(E_{k_z ms} - E_{k_z ns})}, \tag{1.116}$$

where the current and energy-momentum tensor matrix elements are the expression given in Eqs. (1.114) and (1.115) taken in the limit. Furthermore, we will take the zero temperature limit $T \to 0$, where $n_{kms} = \theta(\mu - E_{k_z ms})$.

## 1.4. Response of an untilted cone

### 1.4.1. Explicit form of the matrix elements

Compared to the procedure used by Arjona, Chernodub, and Vozmediano[ACV19], taking the limit of each matrix element by itself greatly simplifies the calculation.

Let

$$\phi_{kms}(y) = e^{-\frac{(y-k_x l_B^2)^2}{2l_B^2}}\begin{pmatrix} a_{k_z ms} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ b_{k_z ms} H_M\left(\frac{y-k_x l_B^2}{l_B}\right)\end{pmatrix}, \tag{1.117}$$

thus implicitly defining the prefactors $a_{k_z ms}, b_{k_z ms}$.

**These are already explicitly defined**

### The current operator

The matrix element

$$
\begin{aligned}
&J_{kms;k+\mathfrak{q}ns}(q) && (1.118)\\
&= \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{kms}(y)\sigma^x \phi_{k+\mathfrak{q}ns}(y) \\
&= s v_F e \int \mathrm{d}y \exp\left\{-iq_y y - \frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} && (1.119)\\
&\quad \left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.\\
&\left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right]\\
&= s v_F e \int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] && (1.120)\\
&\quad \exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x+q_x)q_y\right\}\right]\\
&\quad \left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.\\
&\left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right],
\end{aligned}
$$

where we completed the square in the exponent, to get the form $e^{-a(y+b)^2}$. Also, $\mathfrak{q}_y = (q_x, q_y)$, was introduced, not to be confused with $\mathfrak{q} = (q_x, q_z)$. By introducing $\tilde{y} = \frac{y}{l_B} + l_B(iq_y - q_x - 2k_x)/2$ the matrix element may be rewritten

$$
\begin{aligned}
J_{kms;k+\mathfrak{q}ns}(q) = s v_F e \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{1}{4} l_B^2 \left\{\mathfrak{q}_y^2 + 2i(2k_x+q_x)q_y\right\}\right]&\\
e^{-\tilde{y}^2}\left[a_{k_z ms} b_{k_z+\mathfrak{q}_z ns} H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_N\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right.&\\
\left. + b_{k_z ms} a_{k_z+\mathfrak{q}_z ns} H_M\left(\tilde{y} + \frac{l_B}{2}(q_x - iq_y)\right) H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(-q_x - iq_y)\right)\right].&
\end{aligned}
\tag{1.121}
$$

Taking the limit we find the simple form

$$
J_{kms;kns} = J_{k_z mns} = s v_F e l_B \int \mathrm{d}\tilde{y} e^{-\tilde{y}} \left[a_{k_z ms} b_{k_z ns} H_{M-1}(\tilde{y}) H_N(\tilde{y}) + m \leftrightarrow n\right], \tag{1.122}
$$

where $m \leftrightarrow n$ are the repetition of the previous term under the interchange of $m, n$. We employ now the orthogonality relation of the Hermite polynomials [Olv+, Table 18.3.1]

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_n(x) H_m(x) = \sqrt{\pi} 2^n n! \delta_{n,m} \tag{1.123}$$

to write

$$J_{kms,kns} = J_{k_z mns} = s v_F e l_B \sqrt{\pi} (a_{k_z ms} b_{k_z ns} \delta_{M-1,N} 2^N N! + m \leftrightarrow n). \tag{1.124}$$

With

$$a_{kms} b_{kns} = \frac{\alpha_{k_z ms}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}} \left[2^{N+M-1}(M-1)! N! \pi l_B^2\right]^{-\frac{1}{2}}, \tag{1.125}$$

$$b_{kms} a_{kns} = \frac{\alpha_{k_z ns}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}} \left[2^{N+M-1}(N-1)! M! \pi l_B^2\right]^{-\frac{1}{2}}. \tag{1.126}$$

we find explicitly

$$J_{kms,kns} = J_{k_z mns} = s v_F e \frac{\alpha_{k_z ms}\delta_{M-1,N} + \alpha_{k_z ns}\delta_{M,N-1}}{\sqrt{\alpha_{k_z ms}^2 + 1}\sqrt{\alpha_{k_z ns}^2 + 1}}. \tag{1.127}$$

#### The stress-energy tensor operator

Consider now the matrix element of the energy-momentum tesnor

$$T^{y0}_{k+\mathrm{q}ns,kms}(q) = \frac{1}{2}\int \mathrm{d}y e^{iq_y y} \phi^*_{k+\mathrm{q}ns}(y) s\sigma^y (E_{k_z ms} + E_{k_z+\mathrm{q}_z ns} - 2\mu)\phi_{kms}(y). \tag{1.128}$$

Recall that

$$\phi_{kms}(y) = e^{-\frac{(y-k_x l_B^2)^2}{2l_B^2}} \begin{pmatrix} a_{k_z ms} H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) \\ b_{k_z ms} H_M\left(\frac{y-k_x l_B^2}{l_B}\right) \end{pmatrix}. \tag{1.129}$$

The form of the integrand is very similar to the current matrix case, with the exchange of the Pauli matrix $\sigma^x \to \sigma^y$, thus giving an additional $i$ and a negative sign to the first term.

$$\begin{aligned} T^{y0}_{k+\mathrm{q}ns,kms}(q) &= \frac{is}{2}(E_{k_z ms} + E_{k_z+\mathrm{q}_z ns} - 2\mu) \int \mathrm{d}y e^{iq_y y} e^{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x) l_B^2)^2}{2l_B^2}} \\ &\times \left[-a_{k_z+\mathrm{q}_z ns} b_{k_z ms} H_{N-1}(\dots) H_M(\dots) + b_{k_z+\mathrm{q}_z ns} a_{k_z ms} H_N(\dots) H_{M-1}(\dots)\right]. \end{aligned} \tag{1.130}$$

Taking care to note that the factor from the Fourier transform, that was $e^{-iq_y y}$ in the current matrix element is here $e^{+iq_y y}$, a similar completion of the square is done

$$
\begin{aligned}
&T^{y0}_{k+\mathrm{q}ns,kms}(q) \\
&= \frac{is}{2}(E_{k_z ms} + E_{k_z+\mathrm{q}_z ns} - 2\mu)\exp\left[-\frac{l_B^2}{4}\left\{\mathrm{q}_y^2 - 2iq_y(2k_x+q_x)\right\}\right] \\
&\int \mathrm{d}y \exp\left[-\left\{y + \frac{l_B^2}{2}(-iq_y - 2k_x - q_x)\right\}^2 / l_B^2\right] \\
&[-a_{k_z+\mathrm{q}_z ns} b_{k_z ms} H_{N-1}(\dots)H_M(\dots) + b_{k_z+\mathrm{q}_z ns} a_{k_z ms} H_N(\dots)H_{M-1}(\dots)]\,.
\end{aligned}
\tag{1.131}
$$

The arguments of the Hermite polynomials have been dropped for brevity of notation. As before make a change of variables to get the integral on the form of the shifted orthogonality relation for the Hermite polynomials Eq. (1.159). Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$ the shifted orthogonality relation is used on the expression

$$
\begin{aligned}
T^{y0}_{k+\mathrm{q}ns,kms}(q) = \frac{is}{2}(E_{k\mu s} + E_{\lambda\nu s} - 2\mu)\exp\left[-\frac{l_B^2}{4}\left\{\mathrm{q}_y^2 - 2iq_y(2k_x+q_x)\right\}\right]\int \mathrm{d}\tilde{y}\; l_B e^{-\tilde{y}^2} \\
\left[-a_{k+\mathrm{q}ns} b_{kms} H_{N-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right) H_M\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right. \\
\left. + b_{k+\mathrm{q}ns} a_{kms} H_N\left(\tilde{y} + \frac{l_B}{2}(iq_y - q_x)\right) H_{M-1}\left(\tilde{y} + \frac{l_B}{2}(iq_y + q_x)\right)\right].
\end{aligned}
\tag{1.132}
$$

The terms in the integrand are exactly the same as in the current matrix element case, just in the reverse order and with $q_y \to -q_y$.

$$
T^{y0}_{kns,kms}(q) = \frac{is}{2}\frac{(E_{k_z ms} + E_{k_z ns} - 2\mu)}{\sqrt{\alpha^2_{k_z ms}+1}\sqrt{\alpha^2_{k_z ns}+1}}\left(\alpha_{k_z ms}\delta_{M-1,N} - \alpha_{k_z ns}\delta_{M,N-1}\right).
\tag{1.133}
$$

**Summary 4**

*For a untilted case, in the local limit $q \to 0$, we have the matrix elements*

$$
J_{kms;kns} = \Gamma_{k_z mns} s v_F e\left(\alpha_{k_z ms}\delta_{M-1,N} + m \leftrightarrow n\right), \tag{1.134}
$$

$$
T^{y0}_{kns,kms} = \frac{is\Gamma_{k_z mns}}{2}\left(E_{k_z ms} + E_{k_z ns} - 2\mu\right)\left(\alpha_{k_z ms}\delta_{M-1,N} - m \leftrightarrow n\right), \tag{1.135}
$$

*where $m \leftrightarrow n$ represent the preceding term under the interchange of $m, n$ and where we have defined* $\Gamma_{k_z mns} = \left[(\alpha^2_{k_z ms} + 1)(\alpha^2_{k_z ns} + 1)\right]^{-\frac{1}{2}}$.

### 1.4.2. Computing the reponse function

It is now finally possible to write out the entire response function. We begin by replacing the sum over $\boldsymbol{k}$ by an integral. Firstly, we will show that the sum over $k_x$ is restricted; recall that the eigenfunctions are exponentially centered around $y_0 = k_x l_B^2$, which for a finite sample we expect to be restricted to $0 \leq y_0 \leq L_y$. This restricts the $k_x$ sum to $0 \leq k_x \leq L_y/l_B^2 = L_y eB/$, resulting in the $k_x$ summation giving a finite degeneracy contribution [Ton, Ch. 1.4.1; Lin17], as the integrand is independent of $k_x$.

$$\sum_{\boldsymbol{k}} = \sum_{k_x=0}^{L_y eB/} \sum_{k_z} \to \frac{L_x L_z}{(2\pi)^2} \int_0^{L_y eB/} \mathrm{d}k_x \int \mathrm{d}k_z \tag{1.136}$$

$$= \frac{\mathcal{V}eB}{(2\pi)^2} \int \mathrm{d}k_z. \tag{1.137}$$

Recall the response function

$$\chi^{xy}(\omega, \boldsymbol{q}) = \lim_{\eta \to 0} \sum_{\boldsymbol{k}, mn} \frac{1}{\mathcal{V}} \frac{i v_F J^x_{\boldsymbol{k}ms, \boldsymbol{k}+\mathrm{q}ns}(\boldsymbol{q}) T^{y0}_{\boldsymbol{k}+\mathrm{q}ns, \boldsymbol{k}ms}(\boldsymbol{q}) \left[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}+\mathrm{q}ns}\right]}{(E_{k_z ms} - E_{k_z+\mathrm{q}_z ns} + i\eta)(E_{k_z ms} - E_{k_z+\mathrm{q}_z ns} + \omega + i\eta)}. \tag{1.138}$$

Firstly, introduce the dimensionless quantities $\kappa_z \sqrt{2eB} = k_z, \epsilon_{k_z ms} v_F \sqrt{2eB} = E_{k_z ms}$, in order to facilitate solving the integral over $k_z$. Collecting dimensionfull quantites, the response function reads

$$\lim_{\omega \to 0} \lim_{q \to 0} \chi^{xy} = -\frac{e^2 v_F B}{2(2\pi)^2} \sum_{mn} \int \mathrm{d}\kappa_z [n_{\kappa_z ms} - n_{\kappa_z ns}][(\alpha^2_{\kappa_z ms} + 1)(\alpha^2_{\kappa_z ns} + 1)]^{-1} \times \frac{(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms} \delta_{M-1,N} - \alpha^2_{\kappa_z ns} \delta_{N-1,M})}{(\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns} + i\eta)^2}. \tag{1.139}$$

Let us now define

$$\xi(\kappa_z) = \frac{\left[n_{\kappa ms} - n_{\kappa+\mathrm{q}ns}\right] \left[(\alpha^2_{\kappa ms} + 1)(\alpha^2_{\kappa+\mathrm{q}ns} + 1)\right]^{-1}}{(\epsilon_{\kappa ms} - \epsilon_{\kappa+\mathrm{q}ns} + i\frac{\eta}{v_F\sqrt{2eB}})(\epsilon_{\kappa ms} - \epsilon_{\kappa+\mathrm{q}ns} + \frac{\omega}{v_F\sqrt{2eB}} + i\frac{\eta}{v_F\sqrt{2eB}})}. \tag{1.140}$$

As is shown in table 1.1, $\xi(\kappa_z)$ is odd under interchagne of $m, n$ and inversion of $\kappa_z$.

**Clean up. Is it inversion or sign flip or what?**

Using this, we may simplify our expressions some. In the last term of Eq. (1.139), relabel the summation indices $m \leftrightarrow n$, and then use that $\xi$ is odd under interchange of $m, n$. This renders the two terms equal, and we may consider

$$\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M} \to 2\alpha^2_{\kappa_z ms}\delta_{M-1,N}.$$

The final expression is then

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy} = -\frac{e^2 v_F B}{(2\pi)^2}\sum_{\substack{mn \\ N=M-1}}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} - 2\mu)\alpha^2_{\kappa_z ms}. \quad (1.141)$$

| Transformation | $\xi(\kappa_z)$ | $\epsilon_{\kappa_z ms}$ | $\alpha_{\kappa_z ms}$ |
|---|---|---|---|
| $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$ | -1 | -1 | -1 |
| $(\kappa_z, s) \mapsto (-\kappa_z, -s)$ | +1 | +1 | -1 |
| $(m, n) \mapsto (n, m)$ | -1 | | |

Table 1.1.: Sign change of factors under various transformations. Note that $\xi(\kappa_z)$ is taken in the limit $\omega \to 0, q \to 0, \eta \to 0$.

Before solving the integral, we note that in addition to the

**say the word diatomic?**

$N = M - 1$ selection rule of the sum, the factor with the distributions $n_{\kappa_z ms} - n_{\kappa_z ns}$ impose further restrictions on which transitions are energetically allowed. We consider the limit $T \to 0$

**something about the Luttinger in this limit? I.e. the fact we get finite result in T-> 0 is the interesting thing about this result**

, where the distributions take the form of step functions, $n_{\kappa_z ms} \to \theta(-\epsilon_{\kappa_z ms})$. As the sign of energy level $m$, for $m \neq 0$, is given by the sign of $m$ itself, this gives a rather simple restriction on the sum. For the zeroth energy level, the sign of the energy is given by $\mathrm{sign}(-s\kappa_z)$. The distribution factor is

$$n_{kms} - n_{kns} = \begin{cases} 0 & mn > 0 \text{ or } m, n = 0, \\ -\,\mathrm{sign}(m) & m, n \neq 0, \\ -\,\mathrm{sign}(m)\theta\left[\mathrm{sign}(m)s\kappa_z\right] & n = 0. \end{cases} \quad (1.142)$$

Combining this with the selection rule $N = M - 1$, we see that the only allowed transitions are

$$M \to -N = -(M-1), \; -M \to N = (M-1).$$

The last simplification we will make, is to note that the step function is odd under $(m, n, \kappa_z) \to (-m, -n, -\kappa_z)$, and likewise with $\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns}$.

**is it supposed to be $\epsilon + \epsilon$?**

In the case of zero chemical potential, the expression may be simplified further, by considering only $-N \to M = N+1$ transitions, adding a factor 2.

Lastly, we now show that the contributions from cones of opposite chirality $s$ are the same. Under the transformation $(\kappa_z, s) \mapsto (-\kappa_z, -s)$, the product $\kappa_z s$ is obviously invariant. Note that $\epsilon_{\kappa_z ms}$ only depends on $s$ and $\kappa_z$ through the product $\kappa_z s$. While it is not the case for $\alpha_{\kappa_z ms}$, it is the case for its square. Consequently, the integrand is invariant under $(\kappa_z, s) \mapsto (-\kappa_z, -s)$. Similarly to the argumentation used above, as the integral goes over all $\kappa_z$, the integral is invariant under $s \mapsto -s$.

**Proposition 1**

*We have shown the following simplifications of Eq. (1.139):*

- *The contributions from the terms $\alpha^2_{\kappa_z ms}\delta_{M-1,N}$ and $-\alpha^2_{\kappa_z ns}\delta_{N-1,M}$ are equal, and we consider therefore only one of them, adding a degeneracy factor 2.*
- *The difference of the step functions takes the form Eq. (1.142), which limits the transitions to states with energies of opposite sign. For each value of $M, N$, this means the only valid transitions are $m = M, n = -N$ and $m = -M, n = N$.*
- *As the integrand is invariant under $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, we may consider only one of the transitions mentioned in the previous point, adding once again a degeneracy factor of 2.*
- *We lastly showed that the contribution is independent of the chirality $s$.*

For zero chemical potential, the response function is

$$\lim_{\omega \to 0} \lim_{q \to 0} \chi^{xy} = -\frac{2e^2 v_F B}{(2\pi)^2} \sum_{N=0} \int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns})\alpha^2_{\kappa_z ms}\Big|_{\substack{m=N+1 \\ n=-N}}, \tag{1.143}$$

**Consider change sum to sum over i to N**

where the integration limits are $(-\infty, \infty)$ for $N \neq 0$, $(-\infty, 0)$ for $N = 0, s = -1$, and $(0, \infty)$ for $N = 0, s = 1$.

Including only the first term of the sum, we find

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy} = \frac{e^2 v_F B}{2(2\pi)^2\hbar}, \tag{1.144}$$

where we have reinserted the explicit $\hbar$. Including contributions from the $N$ lowest landau levels, one acquire additional numerical prefactors,

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy} = \gamma_N \frac{e^2 v_F B}{2(2\pi)^2\hbar}, \tag{1.145}$$

where the factor by analytical integration was found to be $\gamma_0 = 1, \gamma_{20} \approx 2$. Furthermore, $\gamma_N$ goes like $\log N$. The first 300 contributions are shown in Figure 1.5.

Solving the integral analytically, we obtained the contribution from each term

$$\gamma_N - \gamma_{N-1} = 1 + 2N\left\{1 - (1+N)\log(1+\frac{1}{N})\right\}, \quad N > 0. \tag{1.146}$$

The sum can be shown to equal the rather nasty expression

$$\begin{aligned}\gamma_N = \gamma_0 + \frac{1}{3}\Big(&6\zeta^{(1,0)}(-2, N+1) - 6\zeta^{(1,0)}(-2, N+2) + 6\zeta^{(1,0)}(-1, N+1)\\ &+6\zeta^{(1,0)}(-1, N+2) + 12\log(\xi) + 3N^2 + 6N - 1\Big),\end{aligned} \tag{1.147}$$

where $\xi \approx 1.28243$ is Glaisher's constant. This expression goes like $\log N$.

**Series expand in $x = \frac{1}{N}$**

## 1.5. The response of a tilted cone

**Write this better**
**Repeating the calculation of the response function is now straightforward, but rather tedious. Due to the boost transformation, the elements of the spinor in the untilted system, Eq. (1.73), mix. We thus have twice as many terms to keep track of.**

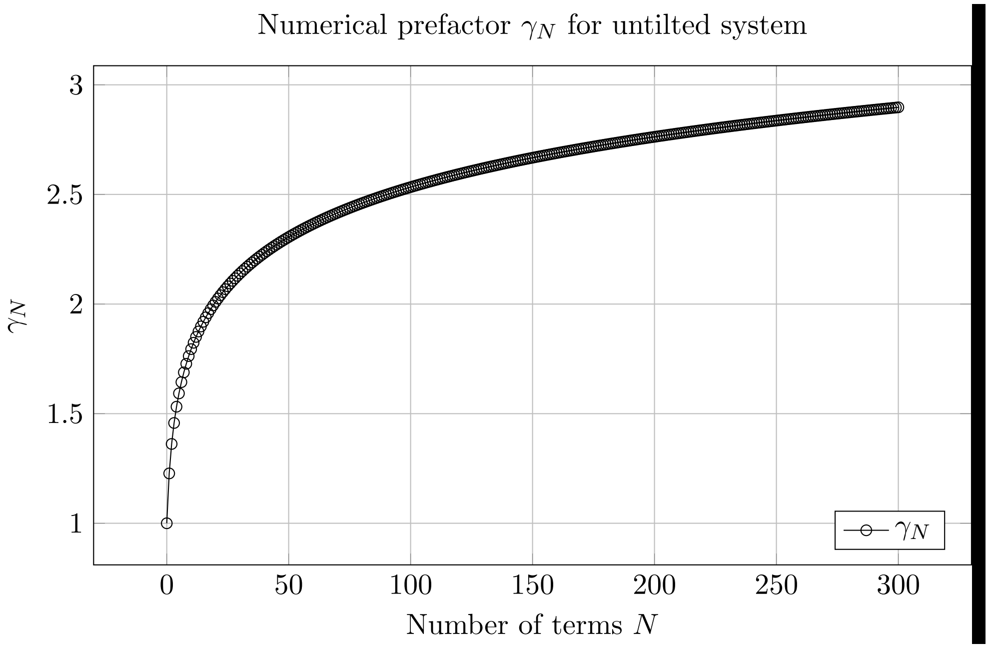



Figure 1.5.: Prefactor $\gamma_N$ as a function of the number of inlcuded Landau levels $N$.

### 1.5.1. Explicit form of the matrix elements

We will here find an explicit form of the matrix elements, starting with the charge current

$$J_{kms;k+qns}(\boldsymbol{q}) = \int \mathrm{d}y e^{-iq_y y} s v_F e \phi^*_{kms}(y)(\sigma^x + s t^s_x)\phi_{k+qns}(y), \tag{1.148}$$

which we will split into two parts, $J^{(1)}, J^{(2)}$, corresponding to the terms $\sigma_x$ and $st^s_x$. For the first part, we must find the matrix product $\phi\sigma_x\phi$. Recall from summary 2 that $\phi = \sqrt{\alpha} e^{\theta/2\sigma_x}\tilde{\phi}$, and thus we must find

$$\phi^* \sigma_x \phi = \alpha \tilde{\phi}^* e^{\theta/2\sigma_x} \sigma_x e^{\theta/2\sigma_x} \tilde{\phi} = \alpha \tilde{\phi}^* \sigma_x e^{\theta\sigma_x} \tilde{\phi}.$$

As defined in summary 2

$$\tilde{\phi} = e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{kms} H_{M-1}(\chi) \\ b_{kms} H_M(\chi) \end{pmatrix}, \quad \chi = \sqrt{\alpha}\frac{y - k_x l_B^2}{l_B} + \frac{t^s_\perp l_B}{\sqrt{\alpha} v_F} E^0_{m,\alpha B}.$$

With the previously found solution $\theta = -\tanh^{-1} t_x^s$, we get the rather simple form

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -st_x^s \\ -st_x^s & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x}}.$$

Where we in the untilted case only have off-diagonal contributions from $\sigma_x$, the hyperbolic rotation gives contributions on the diagonal as well.

First of all, let us consider the exponent of the product. We want to complete the square similarly to what was done for the untilted cone in section 1.4.1. Due to the extra term in $\chi$, this becomes more elaborate. The exponent in the current matrix element Eq. (1.148) is of course

$$\exp\{-iq_y y - \frac{1}{2}\chi_k^2 - \frac{1}{2}\chi_{k+\mathrm{q}}^2\}. \tag{1.149}$$

A straightforward but tedious calculation shows that the argument of the exponent can be written as

$$-\frac{\alpha}{l_B^2}\left(y + \frac{l_B^2}{2\alpha}(iq_y - (q_x' + 2k_x'))\right)^2 - \frac{l_B^2}{4\alpha}(q_y^2 + 2i(q_x' + 2k_x')q_y + (q_x')^2), \tag{1.150}$$

where we have defined

$$q_x' = q_x\alpha - \frac{\beta}{v_F}(E_{n,\alpha B}^0 - E_{m,\alpha B}^0), \tag{1.151}$$

$$k_x' = k_x\alpha - \frac{\beta}{v_F}E_{m,\alpha B}^0. \tag{1.152}$$

**check sign of E above**

These must not be confused with the transformed momenta $\tilde{k}$, which are similar in form. Eq. (1.150) is on the same for as in the untilted cone case, and we may thus proceed with the same method. Make a change of variable

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y + \frac{l_B^2}{2\alpha}(iq_y - 2k_x' - q_x')\right),$$

**Follow up the substitution of the root in the integral. Consider moving the root into $\Xi$**

to get the exponent on the form $e^{-\tilde{y}^2}$. With this substitution,

$$\chi_k = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q_x' - iq_y\right), \tag{1.153}$$

$$\chi_{k+\mathrm{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q_x' - iq_y\right). \tag{1.154}$$

The first part of the current matrix element, Eq. (1.148), is thus

$$
\begin{aligned}
J^{(1)}_{kms;k+\mathrm{q}ns}(q) = \frac{sv_F e}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y}\, l_B \exp\left[-\frac{l_B^2}{4\alpha}\left(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2\right)\right] \\
e^{-\tilde{y}^2}\, [a_{kms} b_{k+\mathrm{q}ns} H_{M-1}\left(\chi_k\right) H_N\left(\chi_{k+\mathrm{q}}\right) \\
- st_x a_{kms} a_{k+\mathrm{q}ns} H_{M-1}\left(\chi_k\right) H_{N-1}\left(\chi_{k+\mathrm{q}}\right) \\
+ b_{kms} a_{k+\mathrm{q}ns} H_M\left(\chi_k\right) H_{N-1}\left(\chi_{k+\mathrm{q}}\right) \\
- st_x b_{kms} b_{k+\mathrm{q}ns} H_M\left(\chi_k\right) H_N\left(\chi_{k+\mathrm{q}}\right)] .
\end{aligned}
\tag{1.155}
$$

Next consider the second term of the current operator,

$$
J^{(2)}_{kms;k+\mathrm{q}ns}(q) = ev_F t_x^s \int \mathrm{d}y e^{-iq_y y} \phi^*_{kms}(y) \phi_{k+\mathrm{q}ns}(y). \tag{1.156}
$$

With a procedure similar to above, with the same substituion and completion of the square

$$
\begin{aligned}
J^{(2)}_{kms;k+\mathrm{q}ns}(q) = \frac{sv_F e t_x^s}{\sqrt{\alpha}} \int \mathrm{d}\tilde{y} l_B \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2)\right] \\
e^{-\tilde{y}^2}\, [a_{kms} H_{M-1}(\chi_k)\left(s a_{k+\mathrm{q}ns} H_{N-1}(\chi_{k+\mathrm{q}}) - t_x^s b_{k+\mathrm{q}ns} H_N(\chi_{k+\mathrm{q}})\right) \\
+ b_{kms} H_M(\chi_k)\left(-t_x^s a_{k+\mathrm{q}ns} H_{N-1}(\chi_{k+\mathrm{q}}) + s b_{k+\mathrm{q}ns} H_N(\chi_{k+\mathrm{q}})\right)].
\end{aligned}
\tag{1.157}
$$

By inspection, recalling $\sqrt{1 - t_x^2} = \alpha$, we see

$$
\begin{aligned}
J_{kms;k+\mathrm{q}ns}(q) = sv_F e\sqrt{\alpha} \int \mathrm{d}\tilde{y} l_B \exp\left[-\frac{l_B^2}{4\alpha}\left(q_y^2 + 2i(2k_x' + q_x')q_y + (q_x')^2\right)\right] e^{-\tilde{y}^2} \\
\times \left[a_{kms} b_{k+\mathrm{q}ns} H_{M-1}\left(\chi_k\right) H_N\left(\chi_{k+\mathrm{q}}\right) + b_{kms} a_{k+\mathrm{q}ns} H_M\left(\chi_k\right) H_{N-1}\left(\chi_{k+\mathrm{q}}\right)\right].
\end{aligned}
\tag{1.158}
$$

To perform the integration, we use the *shifted orthogonality* relation for Hermite polynomials [GZ15, Eq. (7.377)]

$$
\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_m(x+y) H_n(x+z) = 2^n \pi^{\frac{1}{2}} m! y^{n-m} L_m^{n-m}(-2yz), \quad m \le n,
\tag{1.159}
$$

where $L_b^a$ are the *generalized Laguerre polynomial* of order $b$ and type $a$. Define the functions $\Xi_1, \Xi_2$ by

$$\frac{\sqrt{\alpha}\alpha_{k_z ms}\Xi_1(\boldsymbol{q}, m, n, s)}{\sqrt{\alpha^2_{k_z ms}+1}\sqrt{\alpha^2_{k_z+\mathrm{q}_z ns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B a_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathrm{q}ns} H_{M-1}(\chi_{\boldsymbol{k}}) H_N(\chi_{\boldsymbol{k}+\mathrm{q}}), \tag{1.160}$$

$$\frac{\sqrt{\alpha}\alpha_{k_z+qns}\Xi_2(\boldsymbol{q}, m, n, s)}{\sqrt{\alpha^2_{k_z ms}+1}\sqrt{\alpha^2_{k_z+\mathrm{q}_z ns}+1}} = \int \mathrm{d}\tilde{y}\; e^{-\tilde{y}^2} l_B b_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathrm{q}ns} H_M(\chi_{\boldsymbol{k}}) H_{N-1}(\chi_{\boldsymbol{k}+\mathrm{q}}). \tag{1.161}$$

Using that

$$a_{\boldsymbol{k}ms} b_{\boldsymbol{k}+\mathrm{q}ns} = \frac{\sqrt{\alpha}\alpha_{k_z ms}}{\sqrt{\alpha^2_{k_z ms}+1}\sqrt{\alpha^2_{k_z+\mathrm{q}_z ns}+1}} \left[2^{N+M-1}(M-1)!N!\pi l_B^2\right]^{-\frac{1}{2}}, \tag{1.162}$$

$$b_{\boldsymbol{k}ms} a_{\boldsymbol{k}+\mathrm{q}ns} = \frac{\sqrt{\alpha}\alpha_{k_z+\mathrm{q}_z ns}}{\sqrt{\alpha^2_{k_z ms}+1}\sqrt{\alpha^2_{k_z+\mathrm{q}_z ns}+1}} \left[2^{N+M-1}(N-1)!M!\pi l_B^2\right]^{-\frac{1}{2}}, \tag{1.163}$$

we use Eq. (1.159) to find explicit expressions

$$\Xi_1^{(1)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}} \left(\frac{q'_x - iq_y}{2\sqrt{\alpha}} l_B\right)^{N-M+1} L_{M-1}^{N-M+1}\left(\frac{\mathrm{q}_y^2 l_B^2}{2\alpha}\right), \tag{1.164a}$$

$$\Xi_1^{(2)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}} \left(\frac{-q'_x - iq_y}{2\sqrt{\alpha}} l_B\right)^{M-N-1} L_N^{M-N-1}\left(\frac{\mathrm{q}_y^2 l_B^2}{2\alpha}\right), \tag{1.164b}$$

$$\Xi_1(\boldsymbol{q}, m, n, s) = \begin{cases} \Xi_1^{(1)} & \text{if } N \geq M-1 \\ \Xi_1^{(2)} & \text{if } N \leq M-1 \end{cases} \quad \text{for } M > 0, N \geq 0, \tag{1.164c}$$

$$\Xi_2^{(1)}(q,m,n,s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}} \left(\frac{q'_x - iq_y}{2\sqrt{\alpha}} l_B\right)^{N-1-M} L_M^{N-1-M}\left(\frac{\mathrm{q}_y^2 l_B^2}{2\alpha}\right), \tag{1.165a}$$

$$\Xi_2^{(2)}(q,m,n,s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}} \left(\frac{-q'_x - iq_y}{2\sqrt{\alpha}} l_B\right)^{M-N+1} L_{N-1}^{M-N+1}\left(\frac{\mathrm{q}_y^2 l_B^2}{2\alpha}\right), \tag{1.165b}$$

$$\Xi_2(q,m,n,s) = \begin{cases} \Xi_2^{(1)} & \text{if } N-1 \geq M \\ \Xi_2^{(2)} & \text{if } N-1 \leq M \end{cases} \quad \text{for } M \geq 0, N > 0, \tag{1.165c}$$

Here, $\mathrm{q}_y = (q'_x, q_y)$.

Thus, the current matrix element in terms of the functions $\Xi_i$ is

$$J_{kms;k+\mathrm{q}ns}(q) = ev_F s\alpha^2 \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]}{\sqrt{\alpha_{k_zms}^2+1}\sqrt{\alpha_{k_z+\mathrm{q}_zns}^2+1}} \times \left[\alpha_{k_zms}\Xi_1(q,m,n,s) + \alpha_{k_z+\mathrm{q}_zns}\Xi_2(q,m,n,s)\right]. \tag{1.166}$$

### Energy-momentum tensor

Consider now the energy-momentum tensor matrix element

$$T^{0y}_{k+\mathrm{q}ns,kms}(q) = \frac{1}{2}\int \mathrm{d}y e^{iq_y y}\phi^*_{k+\mathrm{q}ns}(y)s\sigma^y(E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu)\phi_{kms}(y). \tag{1.167}$$

As

$$\sigma_y e^{\theta/2\sigma_x} = e^{-\theta/2\sigma_x}\sigma_y \tag{1.168}$$

we get the very fortunate result

$$\phi^*\sigma_y\phi = \frac{1}{\mathcal{N}^*\mathcal{N}}\tilde{\phi}^*\sigma_y\tilde{\phi} = \alpha\tilde{\phi}^*\sigma_y\tilde{\phi}. \tag{1.169}$$

The first term of the stress-energy tensor thus has the exact same form as the untilted case, however with a prefactor $\alpha$ and using the transformed coordinates $\chi$. We thus get

$$T^{0y(1)}_{k+\mathrm{q}ns,kms}(q) = \frac{is\alpha}{2}(E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu)\int \mathrm{d}y e^{iq_y y} e^{-\frac{1}{2}(\chi^2_{k+\mathrm{q}} + \chi^2_k)} \left[-a_{k+\mathrm{q}ns}b_{kms}H_{N-1}(\chi_{k+\mathrm{q}})H_M(\chi_k) + b_{k+\mathrm{q}ns}a_{kms}H_N(\chi_{k+\mathrm{q}})H_{M-1}(\chi_k)\right]. \tag{1.170}$$

We will perform once again the completion of the square and substituion of $y$. The exponent is the same as that which we found for the current operator case, Eq. (1.150), with the change $q_y \to -q_y$. We thus make the change of variables

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left(y - \frac{l_B^2}{2\alpha}(iq_y + (2k_x' + q_x'))\right), \tag{1.171}$$

giving

$$\chi_k = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(q_x' + iq_y\right), \tag{1.172}$$

$$\chi_{k+\mathrm{q}} = \tilde{y} + \frac{l_B}{2\sqrt{\alpha}}\left(-q_x' + iq_y\right). \tag{1.173}$$

Thus, after inserting and employing the defining relations for the $\Xi_i$ functions, the matrix element reads

$$T^{0y}_{k+\mathrm{q}ns,kms}(q) = \frac{is\alpha}{2}\frac{E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu}{\sqrt{\alpha^2_{k_zms}+1}\sqrt{\alpha^2_{k_z+\mathrm{q}_zns}+1}} \tag{1.174}$$

$$\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k_x' + q_x')q_y + (q_x')^2)\right] \tag{1.175}$$

$$(-\alpha_{k_z+\mathrm{q}_zns}\Xi_2(\bar{q},m,n,s) + \alpha_{k_zms}\Xi_1(\bar{q},m,n,s)), \tag{1.176}$$

where $\bar{q} = (q_x, -q_y, q_z)$.

**Summary 5**

*In summary we have*

$$J_{kms;k+\mathrm{q}ns}(q) = v_F es\alpha^2\Gamma^-_{k\mathrm{q}mns}\left[\alpha_{k_zms}\Xi_1(q,m,n,s) + \alpha_{k_z+\mathrm{q}_zns}\Xi_2(q,m,n,s)\right], \tag{1.177}$$

$$T^{0y}_{k+\mathrm{q}ns,kms}(q) = \frac{is\alpha}{2}(E_{k_zms} + E_{k_z+\mathrm{q}_zns} - 2\mu)\Gamma^+_{k\mathrm{q}mns} \tag{1.178}$$
$$\times\left[-\alpha_{k_z+\mathrm{q}_zns}\Xi_2(\bar{q},m,n,s) + \alpha_{k_zms}\Xi_1(\bar{q},m,n,s)\right],$$

*with* $\bar{q} = (q_x, -q_y, q_z)$ *and*

$$\Gamma^{\pm}_{k\mathrm{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm iq_y l_B^2(k_x' + \frac{q_x'}{2})\right]}{\left[(\alpha^2_{k_zms}+1)(\alpha^2_{k_z+\mathrm{q}_zns}+1)\right]^{\frac{1}{2}}}.$$

### 1.5.2. Static limit and dimensionless form of the matrix elements

We are interested in the response in the static limit $\boldsymbol{q} \to 0$. We may use the property of limits that

$$\lim_{n \to a} A \cdot B = \lim_{n \to a} A \cdot \lim_{n \to a} B.$$

We may thus consider the limits of the current and energy-momentum matrix elements separately, which we will do here. Furthermore, to facilitate for more easily solving the integration later, we will introduce dimensionless quantities.

Let the dimensionless energy and momentum $\epsilon_{\kappa_z ms} = v_F\sqrt{2eB}E_{k_z ms}$, $\kappa_z = \sqrt{2eB}k_z$. Consider firstly the exponent in the $\Gamma^{\pm}$ factor from summary 5,

$$\Gamma^{\pm}_{k\mathrm{q}mns} \propto \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm iq_y l_B^2(k_x' + \frac{q_x'}{2})\right].$$

Define

$$P = \lim_{q \to 0} \frac{l_B q_x'}{\sqrt{2\alpha}} = \frac{\beta}{\sqrt{\alpha}}(\epsilon^0_{n,\alpha B} - \epsilon^0_{m,\alpha B}), \tag{1.179}$$

where $q_x'$ was defined in Eq. (1.151),

$$q_x' = q_x\alpha - \frac{\beta}{v_F}(E^0_{n,\alpha B} - E^0_{m,\alpha B}).$$

In the limit, the exponent is thus

$$\lim_{q \to 0} \Gamma_{k\mathrm{q}mns} \propto \exp\left[-\frac{\beta^2}{2\alpha}(\epsilon^0_{n,\alpha B} - \epsilon^0_{m,\alpha B})^2\right]. \tag{1.180}$$

The normalization factor $\alpha_{k_z ms}$ is independent on $\boldsymbol{q}$, and already dimensionless. Explicity, it is given in dimensionless quantities as

$$\alpha_{k_z ms} = -\frac{\sqrt{2e\alpha BM}}{\frac{E_{k_z ms} - t_\parallel v_F k_z}{v_F s\alpha} - k_z} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B} - \kappa}. \tag{1.181}$$

In the tilted case, the $\Xi_i$ functions, defined in Eqs. (1.164, 1.165), do not have a trivial form in the static limit, as was the case in the untilted case. Expressed in the quantities introduced here, they simplify to

$$\Xi_1^{(1)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^N(M-1)!}{2^{M-1}N!}}\left(\frac{P}{\sqrt{2}}\right)^{N-M+1} L_{M-1}^{N-M+1}\left(P^2\right), \tag{1.182a}$$

$$\Xi_1^{(2)}(\boldsymbol{q}, m, n, s) = \sqrt{\frac{2^{M-1}N!}{2^N(M-1)!}}\left(-\frac{P}{\sqrt{2}}\right)^{M-N-1} L_N^{M-N-1}\left(P^2\right), \tag{1.182b}$$

$$\Xi_2^{(1)}(q,m,n,s) = \sqrt{\frac{2^{N-1}M!}{2^M(N-1)!}}\left(\frac{P}{\sqrt{2}}\right)^{N-1-M} L_M^{N-1-M}\left(P^2\right), \tag{1.183a}$$

$$\Xi_2^{(2)}(q,m,n,s) = \sqrt{\frac{2^M(N-1)!}{2^{N-1}M!}}\left(-\frac{P}{\sqrt{2}}\right)^{M-N+1} L_{N-1}^{M-N+1}\left(P^2\right), \tag{1.183b}$$

Lastly, notice that in the static limit, the entire expression of the response function is independen of $k_x$, and so the same procedure as was done for the untilted cone in section 1.4.2 is valid for the tilted cone, replacing the $\boldsymbol{k}$ sum with an integral over $k_z$ and a degeneracy factor

$$\sum_{\boldsymbol{k}} \rightarrow \frac{\mathcal{V}eB}{(2\pi)^2}\int \mathrm{d}k_z. \tag{1.184}$$

Importantly, the degeneracy factor does *not* depend on the renormalized magnetic field $\alpha B$, but rather $B$ itself.

### 1.5.3. Perpendicular tilt

We consider here the specialized situation where $\boldsymbol{t} = t_x\hat{x}$, i.e. only tilt perpendicular to the magnetic field. The response function

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0}\frac{eBiv_F}{(2\pi)^2}\sum_{mn}\int \mathrm{d}k_z[n_{\boldsymbol{k}ms} - n_{\boldsymbol{k}ns}] \times \frac{J^x_{\boldsymbol{k}ms,\boldsymbol{k}ns}(\boldsymbol{q}\to 0)T^{y0}_{\boldsymbol{k}ns,\boldsymbol{k}ms}(\boldsymbol{q}\to 0)}{(E_{k_zms} - E_{k_zns} + i\eta)(E_{k_zms} - E_{k_zns} + i\eta)}.$$

Writing out the matrix products we have

$$J^x_{\boldsymbol{k}ms,\boldsymbol{k}ns}(\boldsymbol{q}\to 0)T^{y0}_{\boldsymbol{k}ns,\boldsymbol{k}ms}(\boldsymbol{q}\to 0) = \frac{v_Fei\alpha^3}{2}e^{-P^2} \frac{(E_{k_zms} + E_{k_zns})(\alpha^2_{k_zms}\Xi_1(0,m,n,s)^2 - \alpha^2_{k_zns}\Xi_2(0,m,n,s)^2)}{(\alpha^2_{k_zms}+1)(\alpha^2_{k_zns}+1)}. \tag{1.185}$$

And so, inserting into the response function

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy}(\omega,\boldsymbol{q}) = \lim_{\eta\to 0}\frac{-e^2\alpha^3 v_F B}{2(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z e^{-P^2} \frac{[n_{\kappa_zms} - n_{\kappa_zns}](\epsilon_{\kappa_zms} + \epsilon_{\kappa_zns})(\alpha^2_{\kappa_zms}\Xi_1(0,m,n,s)^2 - \alpha^2_{\kappa_zns}\Xi_2(0,m,n,s)^2)}{(\alpha^2_{\kappa_zms}+1)(\alpha^2_{\kappa_zns}+1)(\epsilon_{\kappa_zms} - \epsilon_{\kappa_zns} + i\eta)^2}, \tag{1.186}$$

where we also made a change of variables $k_z = \sqrt{2eB}\kappa_z$.

We make the observation that $\Xi_1(m,n) = \Xi_2(n,m)$, where it is important to note that $P$ changes sign under interchange of $m, n$. The rest of the factors are invariant under the interchange $m \leftrightarrow n$, except for the step functions, which gives an overall sign change. Thus, using $\Xi_1(m,n) = \Xi_2(n,m)$ and relabelling the summation indices we may consider

$$\alpha^2_{\kappa_z ms}\Xi_1^2 - \alpha^2_{\kappa_z ns} \to 2\alpha^2_{\kappa_z ms}\Xi_1^2.$$

We may also simplify the step function expression. Physically, the step function term corresponds to only considering transitions between states with energies of opposite sign. For Type-I systems, which we are restricted to here as we consider currently only perpendicular tilt, the energy of the state with quantum number $n$ has the same sign as $n$ itself, excluding of course the zeroth state. For the zeroth state, the sign of the energy is $\text{sign}(-s\kappa_z)$. Using these considerations, we may make certain seletion rules for the sum. In the $(m,n)$-plane, the first and third quadrant give no contribution, as there $mn > 0$, i.e. they have the same sign. Our sum is thus restricted to the second and fourth quadrant. It is easy to show that

$$n_{kms} - n_{k+qns} = \begin{cases} 0 & mn > 0 \text{ or } m,n = 0, \\ -\text{sign}(m) & m,n \neq 0, \\ \ \ \text{sign}(n)\theta(\text{sign}(n)s\kappa) & m = 0, \\ -\text{sign}(m)\theta(\text{sign}(m)s\kappa) & n = 0. \end{cases} \tag{1.187}$$

Furthermore, the contributions from the second and fourth quadrant are equal, which we will now show. The mapping $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, i.e. a $\pi$ rotation, transforms points from the $m < 0$ half plane to the $m > 0$ half plane, including mapping the second quadrant to the fourth quadrant. We want to consider how the integrand in question transforms under such a mapping. Recall

$$\alpha_{\kappa_z ms} = -\frac{\sqrt{\alpha M}}{s\epsilon^0_{m,\alpha B} - \kappa_z},$$
$$\epsilon^0_{m,\alpha B} = \text{sign}(m)\sqrt{\alpha M + \kappa_z^2}, \quad m \neq 0.$$

Under the above mapping, we have the following relations

$$\epsilon^0_{m,\alpha B} \mapsto -\epsilon^0_{m,\alpha B}, \tag{1.188}$$
$$\alpha_{\kappa_z ms} \mapsto -\alpha_{\kappa_z ms}, \tag{1.189}$$
$$P \mapsto -P. \tag{1.190}$$

The $\Xi$ functions also acquries a sign for some values of $m, n$, however, we only consdier $\Xi^2$. The integrand in Eq. (1.186) is thus invariant under the transformation from the second to the fourth quadrant, and so we may consider only the fourth quadrant, adding a degeneracy factor 2.

Lastly, completely analogous to the untilted case, the integrand only depend on $s$ and $\kappa_z$ through their product $s\kappa_z$, and thus is invariant under $(s, \kappa_z) \mapsto (-s, -\kappa_z)$. As the integral spans all of $\kappa_z$, the contribution is independent of the chirality $s$, and may be calcluated for a specific choice, which is here taken to be $s = +1$.

**Make a note about $M = N$ always giving zero contributinos. Maybe also show in figure. This is important wrt. saying that $\gamma_0$ is all contributison withtin square etc.**

**Summary 6**

*The response of a perpendicularly tilted cone is given by*

$$\lim_{\omega \to 0} \lim_{q \to 0} \chi^{xy}(\omega, q) = \frac{e^2 v_F B}{2(2\pi)^2} \gamma_N^{t_x}, \tag{1.191}$$

*with*

$$\gamma_N^{t_x} = 2\alpha^3 \sum_{mn}^{N} \int \mathrm{d}\kappa_z e^{-P^2} \frac{(\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns}) \alpha^2_{\kappa_z ms} \Xi_1(0, m, n, s)^2}{(\alpha^2_{\kappa_z ms} + 1)(\alpha^2_{\kappa_z ns} + 1)(\epsilon_{\kappa_z ms} - \epsilon_{\kappa_z ns})^2}, \tag{1.192}$$

[7] *where the summation goes over $m > 0, n \leq 0$, capped at the Landau level $N$. The integration limits are $(-\infty, \infty)$, except for $n = 0$, where they are $[0, \infty)$*[8].

***Make sure numerical prefactors are correct. In particular, have we included the 2 from restricting to half plane?***

### 1.5.4. Tilt parallell to the magnetic field

Even though the treatment above for a general tilt is valid for parallel tilt, the response can be found more directly from the untilted case. For $\boldsymbol{t} = t_z \hat{z}$, the energy momentum tensor $T^{y0}$, charge current $J^x$, and wave functions $\phi(\boldsymbol{r})$ are all independent of $t_z$, and the only difference compared to the untilted system is a change in the energies of the Landau levels. We may thus immediately

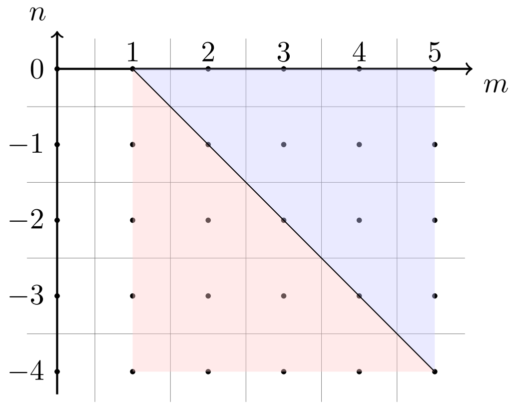



Figure 1.6.: The region of $(m, n)$ to sum over for a Type-I perpendicularly tilted cone. The black line represents the combinations that give a finite contribution also in the untilted case. As the cone is tilted, this sharp line “diffuse” into the red and blue regions as well. Note that, as $\Xi_1$ defined only for $M > 0$, the region with $m = 0$ gives no contribution.

use the result from the untilted case

$$\lim_{\omega\to0}\lim_{q\to0}\chi^{xy} = -\frac{e^2 v_F B}{2(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms}+\epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\delta_{M-1,N}-\alpha^2_{\kappa_z ns}\delta_{N-1,M}), \tag{1.193}$$

with

$$\epsilon_{\kappa_z ms} = \begin{cases} t_z^s\kappa_z + \operatorname{sign} m\sqrt{M+\kappa_z^2} & m \neq 0, \\ (t_z^s - s)\kappa_z & m = 0, \end{cases} \tag{1.194}$$

$$\alpha_{\kappa_z ms} = -s\frac{\sqrt{M}}{\epsilon^0_{\kappa_z ms} - s\kappa_z}, \tag{1.195}$$

$$\lim_{\omega\to0}\lim_{q\to0}\xi(\kappa_z) = \frac{[n_{\kappa ms} - n_{\kappa ns}]\left[(\alpha^2_{\kappa ms}+1)(\alpha^2_{\kappa ns}+1)\right]^{-1}}{(\epsilon_{\kappa ms} - \epsilon_{\kappa ns})^2}. \tag{1.196}$$

In the untilted case we made several simplifications to this expression, especially with regards to limiting the summation domain. We will here consider which of those simplifications apply also in the case of tilt $t_z$.

Under the transformation $(m, n, \kappa_z) \mapsto (-m, -n, -\kappa_z)$, $\xi(\kappa_z), \epsilon_{\kappa_z ms}, \alpha_{\kappa_z ms}$ are all still odd, and so the integrand is invariant under such a transformation. As the integral is over all $\kappa_z$, we may therefore consider only half the $m, n$

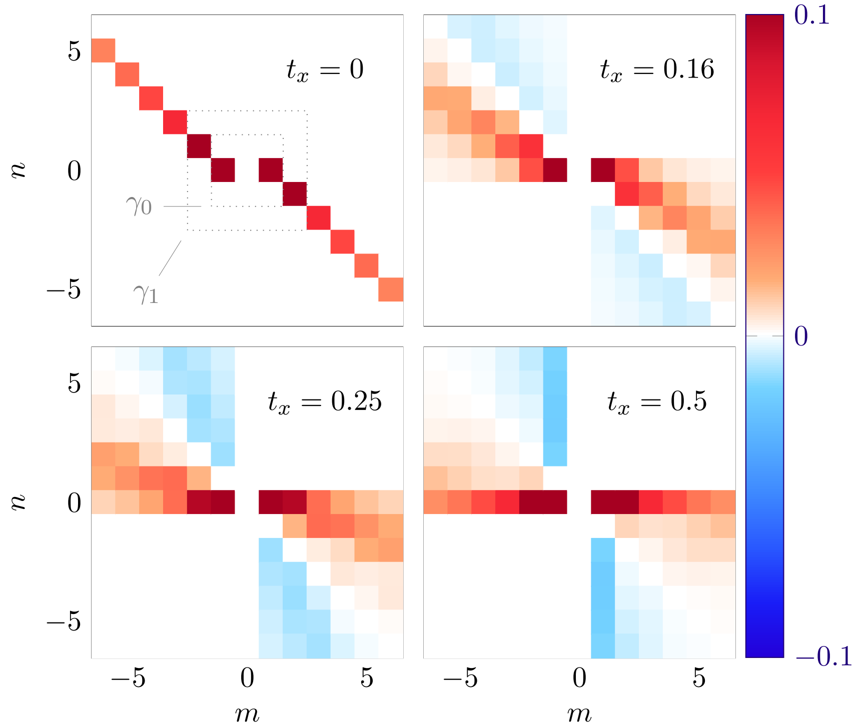



Figure 1.7.: Contributions to $\gamma_N$ from $m \to n$ transitions for different values of $t_x$. In order to retain contrast, the color values are capped at 0.1, meaning that the $\gamma_0$ contributions are clipped. Note that all quadrants of the $m, n$-plane is shown, although as proven in the main text, only the fourth quadrant needs to be computed, as the second quadrant contributions are equal.

plane, as was the case in the untilted case. However, in the untilted case the sum was in fact restricted to only one quadrant, as at $T \to 0$ the transitions must be between states with energy of opposite sign. In the case of Type-II systems, this requirement does not restrict the sum to one quadrant. It is thus convenient to consider Type-I and Type-II separately.

In the untitled system, the contributions from the two chiralities where the same, as $\kappa_z$ and $s$ always appeared in conjunction, $\kappa_z s$. In the case of $t_z$ tilt, this is not the case. The proof for the response from the two chiralities being the same in the untilted case was that $s$ and $\kappa_z$ appeared only through the product $s\kappa_z$, and so the expression was invariant under $(s, \kappa_z) \mapsto (-s, -\kappa_z)$. As our integration spans all $\kappa_z$, the total response is invariant under $s \to -s$. The tilt parameter enters the expression only through $\epsilon_{\kappa_z ms} = \epsilon^0_{\kappa_z ms} + \kappa_z t^s_z$, and in the inversion symmetric case, $t^s_z = st_z$, the argument still holds. In the case of broken inversion symmetry, however, where $t^s_z = t_z$, the argument fails. A similar argument may, however, be made for the transformation $(s, \kappa_z, t_z) \mapsto (-s, -\kappa_z, -t_z)$, for which the (inversion broken) system is invariant. The response of a cone with chirality $s = -1$ is thus equal the response with $s = +1$ and $t_z \to -t_z$. We therefore compute all responses for $s = +1$; for symmetric systems the response is equal for $s = -1$, while for broken inversion symmetry, the response is given at $t_z \to -t_z$.

### Type-I

In Type-I systems, the selection rules from the step functions are independent of $t_z$, and the only difference from the untilted case is the term $\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns} = \epsilon^0_{\kappa_z ms} + \epsilon^0_{\kappa_z ns} + 2\kappa_z t^s_z$. The response is therefore

$$\lim_{\omega \to 0} \lim_{q \to 0} \chi^{xy} = \frac{e^2 v_F B}{2(2\pi)^2} (\gamma^0_N + \gamma_{\text{div},N}), \tag{1.197}$$

where $\gamma^0_N$ is the prefactor of the untilted case, and according to Eq. (1.143)

$$\gamma_{\text{div},N} = -4 \sum_{i=0}^{N} \int \mathrm{d}\kappa_z \xi(\kappa_z) 2\kappa_z t^s_z \alpha^2_{\kappa_z ms} \Big|_{\substack{m=i+1 \\ n=-i}}, \tag{1.198}$$

which has an UV divergence. Introduce the momentum cutoff $\Lambda$, in which case the integral can be solved analytically, with the result[9]

$$\gamma_{\text{div},0} = 2t_z \left( \Lambda \left( \Lambda - \sqrt{\Lambda^2 + 1} \right) + \sinh^{-1}(\Lambda) \right) \tag{1.199}$$

and the contribution from each term of the sum

$$\begin{aligned} \gamma_{\text{div},N} - \gamma_{\text{div},N-1} = 2t_z \Bigg\{ & \Lambda \left( \sqrt{\Lambda^2 + N} - \sqrt{\Lambda^2 + N + 1} \right) \\ & + (N+1) \tanh^{-1} \left[ \frac{\Lambda}{\sqrt{\Lambda^2 + N + 1}} \right] - N \tanh^{-1} \left[ \frac{\Lambda}{\sqrt{\Lambda^2 + N}} \right] \Bigg\}, \end{aligned} \tag{1.200}$$

where we used the selection rule of the sum $N = M - 1$ and $m > 0, n < 0$. This contribution is shown in figure 1.9.

[9] Note the minus sign introduced by the step function in $\xi$.

**is it ok to write 'the contribution (1.200)', or must it always be 'the contribution Eq. (1.200)'?**

The contribution (1.200) is odd in $t_z$, and so for systems with broken inversion symmetry, the total contribution from two cones cancel.

Assuming $\Lambda \gg 1$ the expression is approximated by

$$t_z\left(\left[-1+N\log\left(\frac{N}{N+1}\right)-\log\frac{N+1}{4}\right]+2\log\Lambda\right)+\mathcal{O}\left(\frac{1}{\Lambda^2}\right). \tag{1.201}$$

The contribution is shown in figure 1.9 for the first Landau levels.

### Type-II

For Type-I semimetals, the sign of energy state $m \neq 0$ is given by the sign of $m$ itself. For $m = 0$ the sign of the energy is given by $-s\,\mathrm{sign}\,\kappa$. Due to this, the sum is restricted to $n = M+1, m = -M$ and $n = -M-1, m = M$. In the case of Type-II, however, the situation is not so simple. The energy bands cross the Fermi surface, and we must also include in our sum overlap between states of the same sign, i.e. $n = M+1, m = M$ and $n = -M-1, m = -M$, which is non-zero for certain intervals of $\kappa$. See plot of the tilted Landau levels in figure 1.4.

In order to find explicitly the limits of integration for the Type-II case, we must find the roots of the energy levels. The zeroth Landau level always has only one root, which is in the origin. For the higher order Landau levels, we solve

$$\epsilon_{\kappa_z m s} = t_z^s\kappa_z + \mathrm{sign}(m)\sqrt{M+\kappa_z^2} = 0, \tag{1.202}$$

whose solution is

$$\kappa_z^2 = \frac{M}{t_z^2-1}.$$

The actual roots of the energies are

$$\kappa_z = -\,\mathrm{sign}(mt_x^s)\sqrt{\frac{M}{t_z^2-1}}. \tag{1.203}$$

The integration limit for the $0 \to 1$ transition is thus, for $t_z > 1$, $[-\sqrt{t_z^2-1}^{\,-1}, 0]$ The $1 \to 2$ transition is $[-\sqrt{2}/\sqrt{t_z^2-1}, -\sqrt{t_z^2-1}^{\,-1}]$, and so forth. The general $n \to m$ transition has the integration limits

$$\left[-\,\mathrm{sign}(t_z)\sqrt{\frac{m}{t_z^2-1}}, -\,\mathrm{sign}(t_z n)\sqrt{\frac{-n}{t_z^2-1}}\right].$$

The $0 \to 1$ transitions was computed analytically, and found to be

$$\gamma_0 = 2\,\mathrm{sign}(t_z)\left(|t_z|\sinh^{-1}\left(\frac{1}{\sqrt{t_z^2-1}}\right)-1\right). \tag{1.204}$$

For a general $n \to m$, $N > 0, M = N+1$ transition, the contribution $\gamma_N - \gamma_{N-1}$ was found to have very lengthy expressions. Consult Table 1.2 to find the appropriate expressions for positive and negative tilt, and interband and intraband transitions.

| **Band type** | **Tilt direction** $t_x > 1$ | **Tilt direction** $t_x < -1$ |
|---|---|---|
| $n < 0$ | Lst. ?? | Lst. ?? |
| $n > 0$ | Lst. ?? | Lst. ?? |

Table 1.2.: Decision matrix for the expression of the $m \to n; N > 0, M = N+1$ transition over different regions. Expressions given in Mathematica code format. Code listings starting on page ?? in ??. See main text for details.

**Make sure I am not hanged for having vertical lines**

## 1.6. Results

In the static and local limit $\lim_{\omega\to 0}\lim_{q\to 0}$ the transverse response function $\chi^{xy}$ of the charge current to a temperature perturbation

$$J^x = \chi^{xy}\frac{-\nabla^y T}{T} \tag{1.205}$$

from a single Weyl cone was found to be

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy} = \gamma_N\frac{e^2 B v_F}{2(2\pi)^2\hbar}, \tag{1.206}$$

with $\gamma_N$ a prefactor dependent on the chirality $s$, the tilt $t$, and how many landau levels are included in the final evaluation of the response function.

In general, the prefactor $\gamma_N$ diverges as $N \to \infty$. However, not all Landau levels are filled, and thus the sum should not be taken to all levels. Similarly to a quantum Hall effect, the number of filled bands, the filling factor $\nu$, is inverse proportional to the $B$-field strength

$$\nu \propto \frac{1}{B}. \tag{1.207}$$

Thus, we expect that the $N$-sum should be truncated at a Landau level, given by the filling factor $\nu$. A detailed derivation of the exact truncation of the $N$-sum has not been done. If a precise result for the numerical prefactor is found to be of importance, this should be straightforward.

As described above, the contribution from the cone with chirality $s = -1$ can be found from the result of the positive chirality cone. In the case of perpendicular tilt, they are exactly the same. In the case of parallel tilt, it depends on the symmetry of the tilt. For systems with broken inversion symmetry, the response from the two cones are the same. On the other hand, for inversion symmetric systems, the contribution from the cone with chirality $s = -1$ is the same as that of the $s = +1$ cone at the opposite tilt $t_z \to -t_z$. Therefore, it is useful to separate the contribution into even and odd components, for finding the total contribution from the two cones combined. For some contribution $\gamma(t_{x/z})$, we define

$$\gamma_{\text{even}}(t_{x/z}) = \frac{\gamma(t_{x/z}) + \gamma(-t_{x/z})}{2}, \tag{1.208}$$

$$\gamma_{\text{odd}}(t_{x/z}) = \frac{\gamma(t_{x/z}) - \gamma(-t_{x/z})}{2}. \tag{1.209}$$

All results will be given in terms of these components, at $t_{x/y} > 0$.

### 1.6.1. Perpendicular tilt

In the case of a tilt perpendicular to the magnetic field, we are, as previously explained, restricted to Type-I materials, as the Landau level description breaks down for Type-II perpendicular tilt. Importantly, this does not generally mean that the effect is not present for Type-II systems, but simply that the linear model Landau level description is not a good basis for the system. The collapse of the Landau levels caused Soluyanov et al. [Sol+15] to erroneously predict the collapse of the chiral anomaly in their now famous paper first describing Type-II Weyl semimetals.

As explained in section 1.5.3, the $m, n$ summation is restricted to the fourth quadrant in the $m, n$ plane. In the case of no tilt, only contributions from $M = N + 1$ were non-zero; we named the contribution from the $0 \to 1$ transition $\gamma_0$, the $-1 \to 2$ transition $\gamma_1$ and so forth. For perpendicular tilt, as there are contributions also away from the $M = N + 1$ line, we denote by $\gamma_0$ the contributions from inside the square of length 2 centered at the origin. The $\gamma_1$ contributions are those inside the square of length 4, and in general $\gamma_n$ the square with length $2n$. This is indicated in figure 1.7. This definition effectively sets a ceiling on which Landau levels we consider.

**Correct which values**

The integral was computed numerically for $M, N \leq 6$ over different values of $t_x$ with $t_z = 0$, with the individual contributions shown in figure 1.7. Note that the figure shows contributions for the entire $m, n$-plane, not only the fourth quadrant as discussed above. This is purely for illustration purposes, and only the fourth quadrant needs to be computed. The total contribution $\gamma_N$ as a function of $N$ is shown in figure 1.8. The contribution is even in $t_x$, and the two cones have the same contribution, as shown analytically in section 1.5.3. Also shown in figure 1.8 is $\gamma_0$ as a function of $t_x$, which is seen to be strictly decreasing to zero as $t_x \to 1$. This last observation is discussed further in section 1.6.3, under "Perpendicular tilt with only zeroth level transitions".

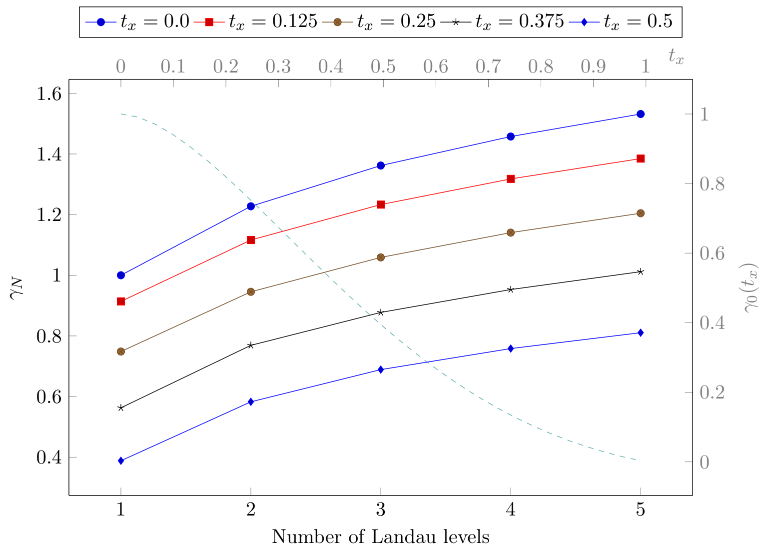



Figure 1.8.: Total contribution $\gamma_N$ for a perpendicular tilt $t_x$, which only has an even component. See main text for details on how $\gamma_N$ is defined. Shown in dashed teal on secondary axis (gray labels) is $\gamma_0$ as a function of $t_x$, which is strictly decreasing from 1 at $t_x = 0$ to 0 at $t_x = 1$.

### 1.6.2. Parallel tilt

#### Type-I

**Should we also compute the momentum cutoff for nontilted terms?**

In the Type-I regime, the contributions differ from that of the untitled system by $\gamma_{\text{div},N}$, Eq. (1.198), dependent on a momentum cutoff $\Lambda = k^{\text{cutoff}}/\sqrt{2eB}$, where $k^{\text{cutoff}}$ is the physical cutoff. The contribution is proportional to $t_z$, i.e. there is no even component, so for systems with broken inversion symmetry, the two chiralities cancel, and the response is equal to the untilted case. In case of inversion symmetry, the contributions from the two chiralities are equal and add up.

In the large cutoff limit, $\Lambda \gg 1$, expanding and dropping terms $\mathcal{O}(1/\Lambda^2)$, we found Eq. (1.201),

$$\gamma_{\text{div},N} - \gamma_{\text{div},N-1} = t_z \left( \left[ -1 + N \log \left( \frac{N}{N+1} \right) - \log \frac{N+1}{4} \right] + 2 \log \Lambda \right).$$

The first term, independent of $\Lambda$, is a negative factor that decreases as $N$

increases, and goes like $-\log N$ for large $N$.

**Some discussion: we get a positive addition to the prefactor.**

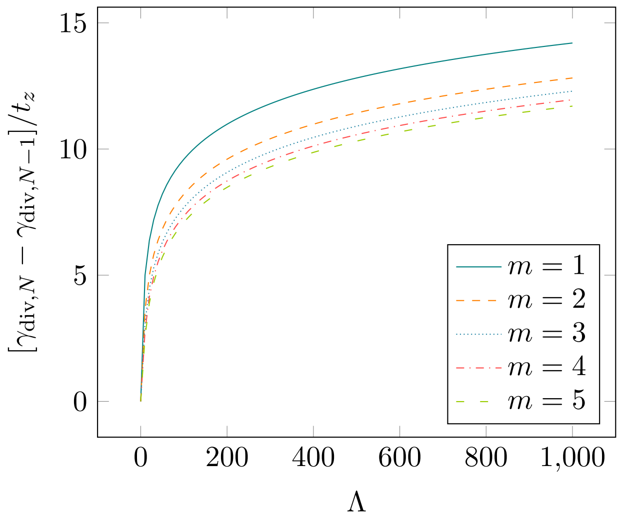



Figure 1.9.: The divergent factor $\gamma_{\text{div},N}/t_z$ for the first Landau levels, as a function of the momentum cutoff $\Lambda$.

**What is $m$ here?**

#### Type-II

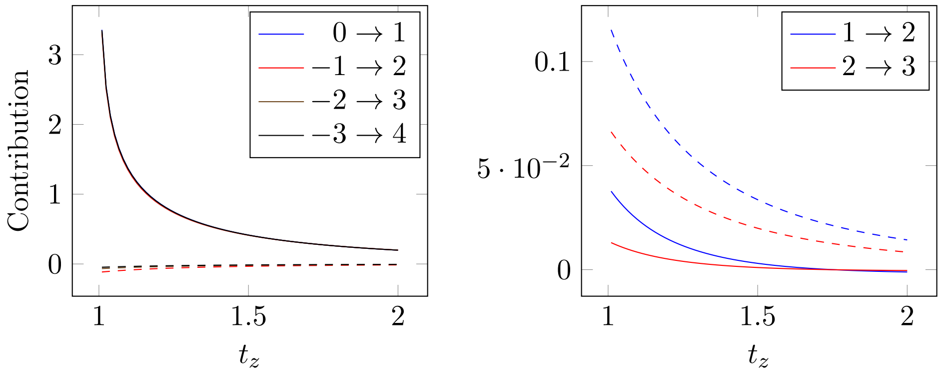
In the Type-II regime, the contributions have more complicated form. Considering firstly only the lowest Landau level contribution, Eq. (1.204), which is odd in $t_z$, the total contribution cancel between the chiralitites for broken inversion symmetry, while it adds up for inversion symmetric systems. As $|t_z| \to 1$ from above, the contribution blows up. This is to be expected as we move towards the Lifshitz transition, where we expect the linear model to perform poorly.[10] The contribution goes to zero as $t_z \to \infty$, shown in figure 1.10.

Considering also higher Landau level contributions, both interband and

[10] As the Fermi surface of the linear model is vastly different from the Fermi surface of the tight binding model. See discussion on page **??** in section **??** and van der Wurff and Stoof [vdWS19].

intraband transitions must be included, [11] meaning the summation is no longer restricted to a quadrant in the $m, n$ plane, but rather to half the plane. The contributions are shown in figure 1.10. These contributions are not odd in $t_z$ – they have a finite even component. Due to this, the contribution does not cancel for inversion broken systems, however the contribution is small in magnitude compared to the other contributions.

A schematic plot of all the contributions of a parallel tilt is shown in figure 1.11.




(a) Intraband contributions, $-N \to N+1$.

(b) Interband contributions, $N \to N+1$.

Figure 1.10.: The contribution from $n \to m$ transitions in a Type-II $t_z$ tilted system. Shown in dashed line of corresponding color, is the even component of the contribution, i.e. $[\text{contrib}(|t_z|) + \text{contrib}(-|t_z|)]/2$.

[11] By band we here refer to the "conduction" band and "valence" band.

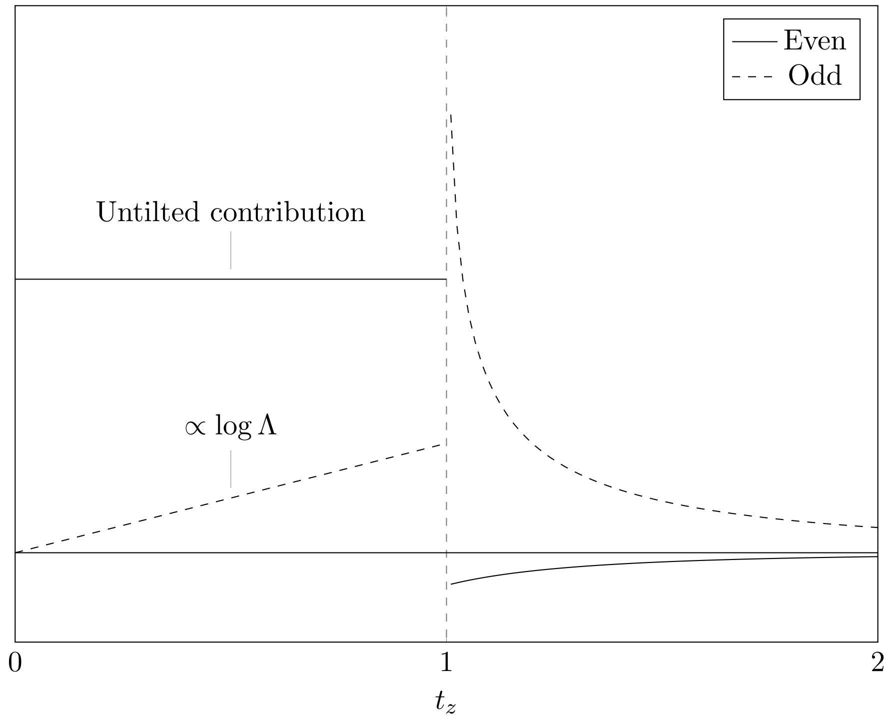



Figure 1.11.: Schematic summary of the contribution for perpendicular tilt $t_z$. Shown is the even (solid line) and odd (dashed line) parts as a function of $t_z$. As explained in the main text, the total contribution for a pair of cones is given by the sum of the even and even part in inversion symmetric systems, and by the odd part for broken inversion symmetry.

### 1.6.3. Other observations

We here present some further observations that are of interest, which we have been unable to investigate further due to time constraints. We therefore do not have conclusive result, and do not want to present them together with the main results. However, they are of great importance, and will be investigated further in future work.

#### Perpendicular tilt with only zeroth level transitions

Above, we defined the prefactor $\gamma_N$ for perpendicular tilt as all transitions $m \to n$, $|m|, |n| < N$; in other words, all transitions between Landau levels up to some cutoff level $N$. However, there are also other possible ways one may consider including the Landau level cutoff. In the untilted case, the *dipolar* selection rule $M = N + 1$ makes the choice obvious. With no such selection rule, the choice is however less obvious, and one other natural choice would be the following. Assuming a large magnetic field, only the lowest Landau level is occupied [Che+21], and so it would be natural to only consider $0 \to n, |n| < N$ transitions. Doing this, the resulting response is very interesting! If we define by $\gamma_N$ the sum of all $0 \to n, |n| < N$ transitions, and compute $\gamma_N$ as a function of the tilt $t_x$ for various $N$, we get the result shown in figure 1.12. When including transitions to higher Landau levels, $\gamma_N$ as a function of $t_x$ is no longer strictly decreasing – it has a maximum at $0 < t_x < 1$! This would be a very interesting experimental signature.

The procedure, however, is not rigorous. Due to time limitations, we have not been able to investigate this effect further in time for this print, however, we plan to do a more rigorous treatment in the future. In particular, this shows the importance of the choice of how one truncates the Landau level sum when there is no dipolar selection rule; with a dipolar selection rule, as is the case for no tilt and parallel tilt, the truncation yields only additional numerical factors, but does not change the behavior as a function of the tilt magnitude. Here, however, it qualitatively changes the response as a function of the tilt!

#### Experimental signature at finite potential and temperature

In real materials, the Fermi level is close to, but not exactly at, the Dirac point. Arjona, Chernodub, and Vozmediano [ACV19] investigated this, which is of great interest with regards to experimental observations, by extending the computation to finite chemical potential and temperature. For sufficiently large magnetic field, only the zeroth Landau level is filled [ACV19; Voz21], and the

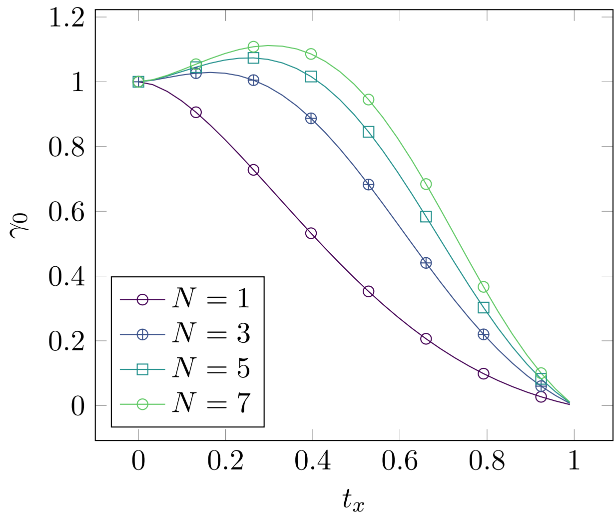



Figure 1.12.: Numerically computed values of the prefactor $\gamma_N$ with only the first Landau level included for perpendicular tilt $t_x$. The contribution is even in $t_x$, and vanish as $|t_x| \to 1$. For clarity, only every 4th mark is drawn.

only transitions are the $0 \to \pm 1$ transitions. For a chemical potential $\mu$ small enough to be contained between the $\pm 1$ Landau levels, i.e. $|\mu|/(v_F\sqrt{2eB}) < 1$, the response function was found to be invariant. Furthermore, for a finite temperature, it was found that thermally activated carriers increased the magnitude of the effect, with a stable platau around $\mu = 0$. The width of the plateau is inversely proportional to the temperature. See figure 1.13.

As tilt is introduced, the energy interval in which one only has the zeroth Landau level is reduced, and as $t \to 1$ the interval vanishes. So as the system is tilted, the width of the platau is reduced. We reproduced the calculation for finite potential and temperature [12] in the untilted situation, but have not yet extended the computation to the tilted case. This should, however, not be very difficult. However, both the issue of how to do a cutoff in the case of perpendicular tilt and the momentum cutoff in the case of parallell tilt, has to be given extra care.

**Comment on we found this also for non-symmetric energy-momentum tensor?**

[12] Using the non-symmetric choice of the energy-momentum tensor, as opposed to the symmetric one used in te original calculation [ACV19].

**Do the computation for tz? In that case, compute separately the even and odd component**

**See also [Che+21] FIG. 9 and discussion**

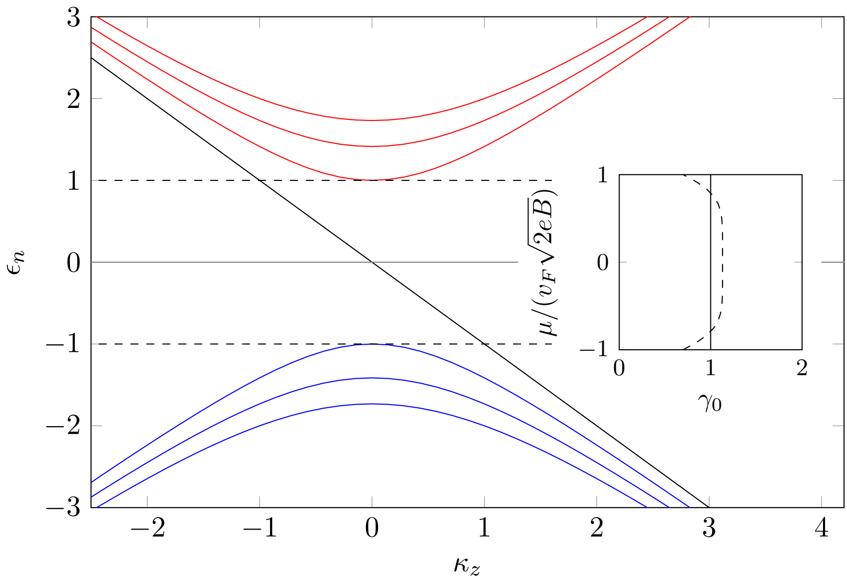



Figure 1.13.: The Landau level of an untilted Weyl cone. The inset shows the prefactor $\gamma_0$ of the response function for a small finite potential $\mu$, within the energy interval indicated with dashed lines. In the inset, the solid line is computed at zero temperature, while the dashed line is computed at a small finite temperature. Figure inspired by Arjona, Chernodub, and Vozmediano [ACV19].

## 1.7. Conclusion and outlook

**Tilt parallel to temperature gradient** -Time constraints did not compute for tilt parallel to $\nabla T$.

**The energy-momentum tensor** - Still open question

**Fully covariant calculation** - Move the explicit tilt parameter in the lagrangian into the metric

**Finite chemical potential and temperature** Similar to doen ine sec

# A Contributions from symmetric energy-momentum tensor

As noted in the main text, there are some subtlety in the definition of the energy-momentum tensor. The *canonical* definition, which we have used in the main text, is in general not symmetric. The tensor enter our calculation from the conservation law

$$\partial_\mu T^{\mu\nu} = 0,$$

which for $\nu = 0$ is nothing more than the conservation law of energy: $\partial_t \epsilon - \boldsymbol{\nabla} \boldsymbol{J}_\epsilon = 0$, where $\epsilon$ is energy density and $\boldsymbol{J}_\epsilon$ is the energy current. In the calculation by Arjona, Chernodub, and Vozmediano[ACV19], the symmetrized[1] energy-momentum tensor

$$T_S^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}$$

was used. In this appendix we show the contributions of the symmetric tensor. The contributions from $T^{\mu\nu}$ and $T^{\nu\mu}$ is shown to be equal in the non-tilted case, while they differ in the tilted case.

In the main text we have already found the contributions from the canonical tensor, and so we focus here on the contributions from $T_F^{\mu\nu} = T^{\nu\mu}$. The relevant element is $T_F^{y0} = T^{0y}$.

The tilted canonical energy-momentum tensor, eq. (1.98),

$$T^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}_s^\mu \partial_\nu \phi - \partial_\nu \phi^\dagger \tilde{\sigma}_s^\mu \phi - \eta^{\mu\nu}\mathcal{L}),$$

and so the symmetric tensor is

$$T_S^{\mu\nu} = \frac{i}{2}(\phi^\dagger \tilde{\sigma}_s^\mu \overleftrightarrow{\partial}_\nu \phi + \phi^\dagger \tilde{\sigma}_s^\nu \overleftrightarrow{\partial}_\mu \phi - \eta^{\mu\nu}\mathcal{L}), \tag{A.1}$$

where we used the notation $\phi^\dagger \overleftrightarrow{\partial} \phi = (\phi^\dagger \partial \phi - (\partial \phi^\dagger)\phi)/2$. We split $T_S^{y0}$ into three parts; the first part corresponds to the canonical energy-momentum tensor, while the two latter correspond to the two terms of $T_F^{y0}$. Explicitly

$$T_S^{y0} = \underbrace{\frac{i}{2}\phi^\dagger \tilde{\sigma}_s^y \overleftrightarrow{\partial}_0 \phi}_{T^{(1)}} + \underbrace{\frac{i}{4}\phi^\dagger \partial_y \phi}_{T^{(2)}} + \underbrace{\frac{i}{4}\phi^\dagger \partial_y \phi}_{T^{(3)}}. \tag{A.2}$$

[1] See section 1.1.2 for a more precise discussion on the symmetrization of the energy-momentum tensor.

In other words, the first part is half that found in the main text, while the two latter are unique to the symmetric tensor. For convenience, we will for the rest of the appendix rename $T^{\mu\nu} = T_S^{\mu\nu}$.

## A.1. No tilt

Begin by considering the matrix elements

$$T^{0y\ (2)}_{k+\mathrm{q}ns,kms}(q) = +\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \phi^*_{k+\mathrm{q}ns}(y) p_y \phi_{kms}(y), \tag{A.3}$$

$$T^{0y\ (3)}_{k+\mathrm{q}ns,kms}(q) = -\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \left(p_y \phi^*_{k+\mathrm{q}ns}(y)\right) \phi_{kms}(y). \tag{A.4}$$

Recall that $\phi_{kms}(y)$, defined in Eq. (1.117), consists of two $y$-dependent factors: $\exp\left[-\frac{(y-k_x l_B^2)^2}{2l_B^2}\right]$ and the Hermite polynomials. The operator $p_y$ thus produces two terms when operating on $\phi$. The first term, coming from the exponent, is proportional to $y - k_x l_B^2$. The operator in Eqs. (A.3) and (A.4) acts on $\phi$ with the quantum number $k$ and $k + \mathrm{q}$, respectively; when summing the two contributions, everything thus cancels except for a term proportional to $q_x$, which vanishes in the local limit.

It remains to consider the result of $p_y$ operating on the Hermite polynomials. Let $\tilde{p}_y$ indicate the $p_y$ operator acting only on the Hermite polynomial part of $\phi$, and use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$ [Olv+, Eq. 18.9.25].

$$\begin{aligned}\phi^*_{k+\mathrm{q}ns}(y)\tilde{p}_y\phi_{kms} = -i\hbar \exp&\left\{-\frac{(y-k_x l_B^2)^2 + (y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\} \\ \frac{2}{l_B}\Bigg\{(M-1)a_{kms}a_{k+\mathrm{q}ns}&H_{M-2}\left(\frac{y-k_x l_B^2}{l_B}\right) H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right) \\ &+ M b_{kms}b_{k+\mathrm{q}ns}H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right) H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\Bigg\}.\end{aligned} \tag{A.5}$$

Completing the square, we get

$$\int \mathrm{d}y e^{iq_y y}\phi^*_{k+\mathrm{q}ns}(y)\tilde{p}_y\phi_{kms}(y) = -i\hbar \exp\left[-\frac{l_B^2}{4}\left\{\mathrm{q}_y^2 - 2iq_y(2k_x+q_x)\right\}\right]$$
$$\int \mathrm{d}y \exp\left[-\left\{y+\frac{l_B^2}{2}\left(-iq_y-2k_x-q_x\right)\right\}^2/l_B^2\right]$$
$$\frac{2}{l_B}\Bigg\{(M-1)a_{kms}a_{k+\mathrm{q}ns}H_{M-2}\left(\frac{y-k_x l_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)$$
$$+ M b_{kms}b_{k+\mathrm{q}ns}H_{M-1}\left(\frac{y-k_x l_B^2}{l_B}\right)H_N\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\Bigg\}. \quad \text{(A.6)}$$

Upon introducing $\tilde{y} = \frac{y}{l_B} + l_B(-iq_y - q_x - 2k_x)/2$, as was also done in the main text, the expression reduces to

$$\int \mathrm{d}y e^{iq_y y}\phi^*_{k+\mathrm{q}ns}(y)\tilde{p}_y\phi_{kms}(y) = -i\hbar \exp\left[-\frac{l_B^2}{4}\left\{q_x^2 + q_y^2 - 2iq_y(2k_x+q_x)\right\}\right]$$
$$\int \mathrm{d}\tilde{y} l_B \exp\left[-\tilde{y}^2\right]$$
$$\frac{2}{l_B}\Bigg\{(M-1)a_{kms}a_{k+\mathrm{q}ns}H_{M-2}\left(\tilde{y}+\frac{l_B}{2}(iq_y+q_x)\right)H_{N-1}\left(\tilde{y}+\frac{l_B}{2}(iq_y-q_x)\right)$$
$$+ M b_{kms}b_{k+\mathrm{q}ns}H_{M-1}\left(\tilde{y}+\frac{l_B}{2}(iq_y+q_x)\right)H_N\left(\tilde{y}+\frac{l_B}{2}(iq_y-q_x)\right)\Bigg\}. \quad \text{(A.7)}$$

Considering now the local limit $\boldsymbol{q} \to 0$, the expression greatly simplifies, and we may use the orthogonality relation for the Hermite polynomials Eq. (1.123)

$$\int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} H_n(x)H_m(x) = \sqrt{\pi}2^n n!\delta_{n,m}$$

to evaulate the integral.

$$\lim_{q\to 0}\int \mathrm{d}y e^{iq_y y}\phi^*_{k+\mathrm{q}ns}(y)\tilde{p}_y\phi_{kms}(y) = -i\hbar\sqrt{2}\frac{\alpha_{kms}\alpha_{kns}\sqrt{M-1}+\sqrt{M}}{l_B\sqrt{\alpha_{kms}^2+1}\sqrt{\alpha_{kns}^2+1}}\delta_{N,M-1}. \quad \text{(A.8)}$$

Similarly, for $T^{0y\,(3)}_{k+\mathrm{q}ns,kms}(q)$, one has

$$\left(\tilde{p}_y\phi^*_{k+\mathrm{q}ns}(y)\right)\phi_{kms}(y) = -i\hbar\exp\left\{-\frac{(y-k_xl_B^2)^2+(y-(k_x+q_x)l_B^2)^2}{2l_B^2}\right\}$$
$$\frac{2}{l_B}\left\{(N-1)a_{kms}a_{k+\mathrm{q}ns}H_{M-1}\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-2}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right.$$
$$\left.+Nb_{kms}b_{k+\mathrm{q}ns}H_M\left(\frac{y-k_xl_B^2}{l_B}\right)H_{N-1}\left(\frac{y-(k_x+q_x)l_B^2}{l_B}\right)\right\} \tag{A.9}$$

which with the same procedure as above gives

$$\lim_{q\to 0}\int \mathrm{d}y e^{iq_yy}\left(\tilde{p}_y\phi^*_{k+\mathrm{q}ns}(y)\right)\phi_{kms}(y) = -i\hbar\sqrt{2}\frac{\alpha_{kms}\alpha_{kns}\sqrt{N-1}+\sqrt{N}}{l_B\sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}}\delta_{M,N-1}. \tag{A.10}$$

**Summary 7**

*In the untilted case, we have*

$$\lim_{q\to 0}T^{y0\,(2)}_{kns,kms} = -\frac{i\hbar\sqrt{2}}{4}\frac{\alpha_{kms}\alpha_{kns}\sqrt{M-1}+\sqrt{M}}{l_B\sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}}\delta_{N,M-1}, \tag{A.11}$$
$$\lim_{q\to 0}T^{y0\,(3)}_{kns,kms} = \frac{i\hbar\sqrt{2}}{4}\frac{\alpha_{kms}\alpha_{kns}\sqrt{N-1}+\sqrt{N}}{l_B\sqrt{\alpha^2_{kms}+1}\sqrt{\alpha^2_{kns}+1}}\delta_{M,N-1}. \tag{A.12}$$

## A.2. With tilt

In the tilted case, we have shown in the main text that

**insert ref**

$$T^{\mu 0} = \frac{i}{2}\left[\partial_i\bar{\psi}\Gamma^j\gamma^0\Gamma^\mu\psi - \bar{\psi}\Gamma^\mu\gamma^0\Gamma^j\partial_j\psi\right].$$

Swapping the indices, we have for $\mu \neq 0$ [vdWS19]

$$T^{0i} = \frac{i}{2}[\bar{\psi}\gamma^0\partial^\mu\psi - \partial^\mu\bar{\psi}\gamma^0\psi].$$

In our work, we have considered only tilt perpendiculat to the thermal gradient, so the component of the energy-momentum tensor of interest are not affected by the tilt.

or

$$T^{0y(2)}_{k+\mathrm{q}ns,kms}(q) = +\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F \phi^*_{k+\mathrm{q}ns}(y) p_y \phi_{kms}(y), \tag{A.13}$$

$$T^{0y(3)}_{k+\mathrm{q}ns,kms}(q) = -\frac{1}{4}\int \mathrm{d}y e^{iq_y y} v_F (p_y \phi^*_{k+\mathrm{q}ns}(y)) \phi_{kms}(y). \tag{A.14}$$

Firstly, we note that

$$[p_y, e^{\theta/2\sigma_x}] = 0.$$

Furthermore, exactly as for the untilted case, the momentum operator acting on the exponential prefactor of $\phi$ gives contributions proportional to $q_x$. In the local limit $q \to 0$ this term vanishes, and we need only consider the effect of the momentum operator acting on the Hermite polynomials.

Denote by $\tilde{p}_y$ the momentum operator $p_y$ acting only on the Hermite polynomial part of $\phi$. Furthermore, we will use the property of Hermite polynomials $\partial_x H_n(x) = 2nH_{n-1}(x)$ [Olv+, Eq. 18.9.25].

$$\tilde{p}_y \phi_{kms} = -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \partial_y \begin{pmatrix} a_{kms} H_{M-1}(\chi) \\ b_{kms} H_M(\chi) \end{pmatrix} \tag{A.15}$$

$$= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} 2\frac{\partial \chi}{\partial y} \begin{pmatrix} a_{kms}(M-1) H_{M-2}(\chi) \\ b_{kms}(M) H_{M-1}(\chi) \end{pmatrix} \tag{A.16}$$

$$= -i\hbar e^{\theta/2\sigma_x} e^{-\frac{1}{2}\chi^2} \frac{2\sqrt{\alpha}}{l_B} \begin{pmatrix} a_{kms}(M-1) H_{M-2}(\chi) \\ b_{kms}(M) H_{M-1}(\chi) \end{pmatrix}. \tag{A.17}$$

And thus, recalling that

$$e^{\theta\sigma_x} = \begin{pmatrix} 1 & -t_x \\ -t_x & 1 \end{pmatrix} \frac{1}{\sqrt{1-t_x^2}},$$

we find the product

$$\begin{aligned} \phi^*_{k+\mathrm{q}ns}(y) \tilde{p}_y \phi_{kms} = {} & -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} e^{-\frac{1}{2}\chi_k^2 - \frac{1}{2}\chi_{k+\mathrm{q}}^2} \\ & \Big[ a_{k+\mathrm{q}ns} H_{N-1}(\chi_{k+\mathrm{q}}) \left\{ a_{kms}(M-1) H_{M-2}(\chi_k) - t_x b_{kms} M H_{M-1}(\chi_k) \right\} \\ & + b_{k+\mathrm{q}ns} H_N(\chi_{k+\mathrm{q}}) \left\{ -t_x a_{kms}(M-1) H_{M-2}(\chi_k) + b_{kms} M H_{M-1}(\chi_k) \right\} \Big]. \end{aligned} \tag{A.18}$$

Completing the square and substituting

$$\tilde{y} = \frac{\sqrt{\alpha}}{l_B}\left( y - \frac{l_B^2}{2\alpha}(iq_y + (2k'_x + q'_x)) \right)$$

gives

$$
\begin{aligned}
\int \mathrm{d}y e^{iq_y} \phi^*_{k+\mathrm{q}ns}(y) \tilde{p}_y \phi_{kms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right] \\
\int \mathrm{d}\tilde{y} \frac{l_B}{\sqrt{\alpha}} \\
\Big[a_{k+\mathrm{q}ns} H_{N-1}(\chi_{k+\mathrm{q}}) \left\{a_{kms}(M-1)H_{M-2}(\chi_k) - t_x b_{kms} M H_{M-1}(\chi_k)\right\} \\
+ b_{k+\mathrm{q}ns} H_N(\chi_{k+\mathrm{q}}) \left\{-t_x a_{kms}(M-1)H_{M-2}(\chi_k) + b_{kms} M H_{M-1}(\chi_k)\right\}\Big].
\end{aligned} \tag{A.19}
$$

We must now evaluate the integral, and express the result in the $\Xi$-functions, defined in eqs. (1.164) and (1.165) of the main text.

$$
\begin{pmatrix} a_{k+\mathrm{q}ns} H_{N-1}(\chi_{k+\mathrm{q}}) \\ b_{k+\mathrm{q}ns} H_N(\chi_{k+\mathrm{q}}) \end{pmatrix}^T \underbrace{\begin{pmatrix} 1 & -t_x \\ -t_x & 1 \end{pmatrix}}_{T} \begin{pmatrix} a_{kms}(M-1)H_{M-2}(\chi_k) \\ b_{kms} M H_{M-1}(\chi_k) \end{pmatrix}
$$

For each of the entries in $T$, we get a product of Hermite polynomials. Where the untilted cone had two such terms, the tilt parameter $t_x$ now gives two extra products, which we must evaluate. Let $M^{(2)}_{ij}$ be the product corresponding to $T_{ij}$, i.e.

$$M^{(2)}_{11} = \quad a_{k+\mathrm{q}ns} a_{kms} (M-1) H_{N-1}(\chi_{k+\mathrm{q}}) H_{M-2}(\chi_k), \tag{A.20}$$

$$M^{(2)}_{12} = -t_x a_{k+\mathrm{q}ns} b_{kms} M H_{N-1}(\chi_{k+\mathrm{q}}) H_{M-1}(\chi_k), \tag{A.21}$$

$$M^{(2)}_{21} = -t_x b_{k+\mathrm{q}ns} a_{kms} (M-1) H_N(\chi_{k+\mathrm{q}}) H_{M-2}(\chi_k), \tag{A.22}$$

$$M^{(2)}_{22} = \quad b_{k+\mathrm{q}ns} b_{kms} M H_N(\chi_{k+\mathrm{q}}) H_{M-1}(\chi_k). \tag{A.23}$$

We want to evaluate

$$F^{(2)}_{ij} = \left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathrm{q}_z ns}+1)\right]^{\frac{1}{2}} \int \mathrm{d}\tilde{y} e^{-\tilde{y}^2} M^{(2)}_{ij}, \tag{A.24}$$

with the prefactor introduced for later convenience.

Notice that

**Verify $l_B$ in this section**

$$F^{(2)}_{12} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{q}, m \mp 1, n). \tag{A.25}$$

and

$$F_{21}^{(2)} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a_{kms}^2}{l_B a_{km\mp 1s}}\Xi_1(\bar{q}, m\mp 1, n, s). \tag{A.26}$$

$F_{11}^{(2)}$ and $F_{22}^{(2)}$ are the same as for the untilted case:

$$F_{11}^{(2)} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{q}, m\mp 1, n\mp 1, s), \tag{A.27}$$

and

$$F_{22}^{(2)} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{q}, m, n, s). \tag{A.28}$$

In summary we have

$$T^{0y\ (2)}_{k+\mathrm{q}ns,kms}(\boldsymbol{q}) = +\frac{v_F}{4}\int \mathrm{d}y e^{iq_y q}\phi^*_{k+\mathrm{q}ns}(y)p_y\phi_{kms}(y) \tag{A.29}$$

$$= -\frac{i\hbar v_F}{2}\Gamma^+_{k\mathrm{q}mns}\sum_{i,j}F_{ij}^{(2)}, \tag{A.30}$$

where

$$\Gamma^+_{k\mathrm{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k_x' + q_x')q_y + (q_x')^2)\right]}{\left[(\alpha^2_{k_z ms}+1)(\alpha^2_{k_z+\mathrm{q}_z ns}+1)\right]^{\frac{1}{2}}\sqrt{1-t_x^2}}$$

In a similar procedure, we find $T^{0y\ (2)}_{k+\mathrm{q}ns,kms}(\boldsymbol{q})$.

$$\tilde{p}_y\phi^*_{k+\mathrm{q}ms} = \frac{-i\hbar\sqrt{\alpha}}{l_B}e^{-\frac{1}{2}\chi^2}\begin{pmatrix} a_{k+\mathrm{q}ms}(M-1)H_{M-2}(\chi) \\ b_{k+\mathrm{q}ms}(M)H_{M-1}(\chi)\end{pmatrix}. \tag{A.31}$$

And thus,

$$\begin{aligned}\left(\tilde{p}_y\phi^*_{k+\mathrm{q}ns}(y)\right)\phi_{kms} = -&\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}}e^{-\frac{1}{2}\chi_k^2-\frac{1}{2}\chi_{k+\mathrm{q}}^2} \\ &\Big[a_{k+\mathrm{q}ns}(N-1)H_{N-2}(\chi_{k+\mathrm{q}})\left\{a_{kms}H_{M-1}(\chi_k) - t_x b_{kms}H_M(\chi_k)\right\} \\ &+ b_{k+\mathrm{q}ns}NH_{N-1}(\chi_{k+\mathrm{q}})\left\{-t_x a_{kms}H_{M-1}(\chi_k) + b_{kms}H_M(\chi_k)\right\}\Big].\end{aligned} \tag{A.32}$$

With the now well-known completion of the square and substitution, we have

$$\int \mathrm{d}y e^{iq_y} \left[\tilde{p}_y \phi^*_{k+\mathrm{q}ns}(y)\right] \phi_{kms}(y) = -\frac{i\hbar 2\sqrt{\alpha}}{l_B\sqrt{1-t_x^2}} \exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 - 2i(2k'_x + q'_x)q_y + (q'_x)^2)\right]$$
$$\int \mathrm{d}\tilde{y} \frac{l_B}{\sqrt{\alpha}}$$
$$\Big[a_{k+\mathrm{q}ns}(N-1)H_{N-2}(\chi_{k+\mathrm{q}})\left\{a_{kms}H_{M-1}(\chi_k) - t_x b_{kms}H_M(\chi_k)\right\}$$
$$+ b_{k+\mathrm{q}ns}NH_{N-1}(\chi_{k+\mathrm{q}})\left\{-t_x a_{kms}H_{M-1}(\chi_k) + b_{kms}H_M(\chi_k)\right\}\Big]. \quad \text{(A.33)}$$

Denote the terms of the integrand by

$$M_{11}^{(3)} = \quad a_{k+\mathrm{q}ns}a_{kms}(N-1)H_{N-2}(\chi_{k+\mathrm{q}})H_{M-1}(\chi_k), \quad \text{(A.34)}$$
$$M_{12}^{(3)} = -t_x a_{k+\mathrm{q}ns}b_{kms}(N-1)H_{N-2}(\chi_{k+\mathrm{q}})H_M(\chi_k), \quad \text{(A.35)}$$
$$M_{21}^{(3)} = -t_x b_{k+\mathrm{q}ns}a_{kms}NH_{N-1}(\chi_{k+\mathrm{q}})H_{M-1}(\chi_k), \quad \text{(A.36)}$$
$$M_{22}^{(3)} = \quad b_{k+\mathrm{q}ns}b_{kms}NH_{N-1}(\chi_{k+\mathrm{q}})H_M(\chi_k). \quad \text{(A.37)}$$

We must evaluate

$$F_{ij}^{(3)} = \left[(\alpha^2_{k_zms}+1)(\alpha^2_{k_z+\mathrm{q}_zns}+1)\right]^{\frac{1}{2}} \int \mathrm{d}\tilde{y} e^{-\tilde{y}^2} M_{ij}^{(3)}. \quad \text{(A.38)}$$

From the untilted case we know

$$F_{11}^{(3)} = \sqrt{\frac{N-1}{2}} \frac{\alpha_{k_zms}\alpha_{k_z+\mathrm{q}_zns}}{l_B \alpha_{k_z+\mathrm{q}_zn\mp 1s}} \Xi_2(\bar{q}, m\mp 1, n\mp 1, s), \quad \text{(A.39)}$$

$$F_{22}^{(3)} = \sqrt{\frac{N}{2}} \frac{1}{l_B \alpha_{k_z+\mathrm{q}_zns}} \Xi_2(\bar{q}, m, n, s). \quad \text{(A.40)}$$

Furthermore,

$$F_{12}^{(3)} = -t_x \frac{\alpha_{k_z+\mathrm{q}_zn}}{\alpha_{k_z+\mathrm{q}_zn\mp 1}l_B}\sqrt{\frac{N-1}{2}} \Xi_2(\bar{q}, m, n\mp 1, s), \quad \text{(A.41)}$$

$$F_{21}^{(3)} = -\frac{t_x}{l_B}\sqrt{\frac{N}{2}} \frac{\alpha_{k_zm}}{\alpha_{k_z+\mathrm{q}_zn}} \Xi_2(\bar{q}, m\mp 1, n, s). \quad \text{(A.42)}$$

We thus have

$$T^{0y\ (3)}_{k+\mathrm{q}ns,kms}(q) = -\frac{v_F}{4}\int \mathrm{d}y e^{iq_y y}\left(p_y \phi^*_{k+\mathrm{q}ns}(y)\right)\phi_{kms}(y) \quad \text{(A.43)}$$
$$= \frac{i\hbar v_F}{2}\Gamma^+_{k\mathrm{q}mns}\sum_{ij} F_{ij}^{(3)}. \quad \text{(A.44)}$$

**Summary 8**

*The non-canonical part of the energy-momentum tensor $T_F^{\mu\nu} = T^{\nu\mu}$ in a tilted system have the matrix elements*

$$T^{0y\ (2)}_{k+\mathrm{q}ns,kms}(\boldsymbol{q}) = -\frac{i\hbar v_F}{2}\Gamma^{+}_{k\mathrm{q}mns}\sum_{i,j}F^{(2)}_{ij}, \tag{A.45}$$

$$T^{0y\ (3)}_{k+\mathrm{q}ns,kms}(\boldsymbol{q}) = \frac{i\hbar v_F}{2}\Gamma^{+}_{k\mathrm{q}mns}\sum_{ij}F^{(3)}_{ij}. \tag{A.46}$$

*with*

$$\Gamma^{\pm}_{k\mathrm{q}mns} = \frac{\exp\left[-\frac{l_B^2}{4\alpha}(q_y^2 + (q_x')^2) \pm iq_y l_B^2(k_x' + \frac{q_x'}{2}))\right]}{\left[(\alpha^2_{k_z ms} + 1)(\alpha^2_{k_z+\mathrm{q}_z ns} + 1)\right]^{\frac{1}{2}}}$$

*and where the factors $F^{(n)}_{ij}$ where found to be*

$$F^{(2)}_{12} = -t_x\sqrt{\alpha}\sqrt{\frac{M}{2}}\alpha_{k+q,n}\Xi_2(\bar{q}, m \mp 1, n), \tag{A.47}$$

$$F^{(2)}_{21} = -t_x\sqrt{\alpha}\sqrt{\frac{M-1}{2}}\frac{a^2_{kms}}{l_B a_{km\mp 1s}}\Xi_1(\bar{q}, m \mp 1, n, s), \tag{A.48}$$

$$F^{(2)}_{11} = \sqrt{\alpha}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}\sqrt{M-1}}{l_B\sqrt{2}}\Xi_1(\bar{q}, m \mp 1, n \mp 1, s), \tag{A.49}$$

$$F^{(2)}_{22} = \sqrt{\alpha}\frac{\sqrt{M}}{l_B\sqrt{2}}\Xi_1(\bar{q}, m, n, s), \tag{A.50}$$

$$F^{(3)}_{11} = \sqrt{\frac{N-1}{2}}\frac{\alpha_{k_z ms}\alpha_{k_z+\mathrm{q}_z ns}}{l_B\alpha_{k_z+\mathrm{q}_z n\mp 1s}}\Xi_2(\bar{q}, m \mp 1, n \mp 1, s), \tag{A.51}$$

$$F^{(3)}_{22} = \sqrt{\frac{N}{2}}\frac{1}{l_B\alpha_{k_z+\mathrm{q}_z ns}}\Xi_2(\bar{q}, m, n, s), \tag{A.52}$$

$$F^{(3)}_{12} = -t_x\frac{\alpha_{k_z+\mathrm{q}_z n}}{\alpha_{k_z+\mathrm{q}_z n\mp 1}l_B}\sqrt{\frac{N-1}{2}}\Xi_2(\bar{q}, m, n \mp 1, s), \tag{A.53}$$

$$F^{(3)}_{21} = -\frac{t_x}{l_B}\sqrt{\frac{N}{2}}\frac{\alpha_{k_z m}}{\alpha_{k_z+\mathrm{q}_z n}}\Xi_2(\bar{q}, m \mp 1, n, s). \tag{A.54}$$

### A.2.1. Parallel tilt

The procedure greatly simplifies in the case of parallel tilt. As noted in the main text, parallel tilt only rescales the energies Landau levels, while the wave functions and operators stay invariant. The procedure for the untitled cone, done in appendix A.1, is thus relevant here as well, with an interchange of the energy levels where relevant.

The $T^{(2)}$ and $T^{(3)}$ parts of the energy-momentum tensor for parallel tilt is therefore the same as the result without tilt, found in summary 7. In the main text we showed a simplification procedure for terms of the form

$$\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M} \tag{A.55}$$

in the total response function. The outline of the idea was to note that we sum over all $m, n$, and by certain symmetries of the terms under interchange of $m \leftrightarrow n$, we could rename summation indices and replace

$$\alpha^2_{\kappa_z ms}\delta_{M-1,N} - \alpha^2_{\kappa_z ns}\delta_{N-1,M} \to 2\alpha^2_{\kappa_z ms}\delta_{M-1,N}. \tag{A.56}$$

For details on the procedure see section 1.4.2 of the main text. By simply inserting $T^{(2)}, T^{(3)}$ in the response function, one may easily show that the resulting term is on the form eq. (A.55), with the first term corresponding to $T^{(3)}$ and the second to $T^{(2)}$. The response from $T^{(2)}$ and $T^{(3)}$ is thus equal.

By the procedure explained in section 1.1.2, the response of $T^{(2)} + T^{(3)}$ may be rewritten as the response of $T^{(1)}$, which contains the factor $E_{k_z ms} + E_{k_z ns}$, with the energies replaced with the untilted energies. In other words, using the energy momentum tensor $T_F^{\mu\nu}$, the response is the same as the response found for parallel tilt in the main text, eq. (1.193),

$$\lim_{\omega\to 0}\lim_{q\to 0}\chi^{xy} = -\frac{e^2 v_F B}{2(2\pi)^2}\sum_{mn}\int \mathrm{d}\kappa_z \xi(\kappa_z)(\epsilon_{\kappa_z ms}+\epsilon_{\kappa_z ns})(\alpha^2_{\kappa_z ms}\delta_{M-1,N}-\alpha^2_{\kappa_z ns}\delta_{N-1,M}),$$

with the term $\epsilon_{\kappa_z ms} + \epsilon_{\kappa_z ns}$ replaced with the untilted energies $\epsilon^0_{\kappa_z ms} + \epsilon^0_{\kappa_z ms}$. The response from the $T_F^{\mu\nu}$ tensor is therefore the exact same as that of the untilted cone, as long as one stays in Type-I. It differs from the response found in the main text by the divergent prefactor $\gamma_{\mathrm{div},N}$.

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