

# Manuscript Title: with Forced Linebreak\*

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We calculate a Nernst current in tilted Weyl semimetals using the Kubo formalism.

## I. CALCULATION

### A. Setup and response

Consider the Hamiltonian

$$H^s = sv_F \boldsymbol{\sigma} \cdot \mathbf{p} + v_F \mathbf{t}^s \cdot \mathbf{p}, \quad (1)$$

with  $s = \pm 1$  the chirality of the cone,  $v_F$  the Fermi velocity,  $\boldsymbol{\sigma}$  the Pauli matrix vector,  $\mathbf{p}$  the momentum operator and  $\mathbf{t}^s$  the *tilt vector*. For inversion symmetric systems,  $\mathbf{t}^s = s\mathbf{t}$ , while broken inversion symmetry we consider  $\mathbf{t}^s = \mathbf{t}$  [1].

... The response from the Kubo formalism is thus

$$\chi^{ij}(\omega, \mathbf{q}) = \frac{-iv_F}{\mathcal{V}} \int dt e^{i\omega t} \int_{-\infty}^0 dt' \Theta(t) \times \langle [J^i(t, \mathbf{q}), T^{j0}(t', -\mathbf{q})] \rangle, \quad (2)$$

where  $\hbar = 1$ . The charge current operator

$$\mathbf{J} = ev_F (s\boldsymbol{\sigma} + \mathbf{t}^s). \quad (3)$$

For the energy-momentum tensor choose [2]

$$T^{\mu\nu} = \frac{i}{2} (\phi^\dagger \tilde{\sigma}^\mu \partial_\nu \phi - \tilde{\sigma}^\mu \phi^\dagger \partial_\nu \phi - \eta^{\mu\nu} \mathcal{L}), \quad (4)$$

where we have defined the modified Pauli matrices  $\tilde{\sigma}^\mu = \sigma^\mu + (t^s)^\mu$  with  $(t^s)^\mu = (0, \mathbf{t}^s)$ .

For concreteness, we will, without loss of generality, consider the geometry  $\mathbf{J} \parallel \hat{x}$ ,  $-\nabla T \parallel \hat{y}$ ,  $\mathbf{B} \parallel \hat{z}$ . We separate between *perpendicular* and *parallel* tilt, with the former being  $\mathbf{t} \perp \mathbf{B}$  and the latter  $\mathbf{t} \parallel \mathbf{B}$ . In the current work we have furthermore restricted the perpendicular tilt to be parallel to the current, i.e.  $\mathbf{t}_\perp = t_x \hat{x}$ .

### B. Landau levels

The Landau levels can be shown to be

$$E_{k_z m s} = \begin{cases} t_z^s v_F k_z + \text{sign}(m) v_F \alpha \sqrt{2eB\alpha M + k_z^2} & m \neq 0, \\ t_z^s v_F k_z - s\alpha v_F k_z & m = 0, \end{cases} \quad (5)$$

where we have defined the *squeezing factor*  $\alpha = \sqrt{1 + t_x^2}$ . The eigenstates in the position basis are

$$\phi(\mathbf{r}) = \sqrt{\alpha} e^{\theta/2\sigma_x} \frac{e^{ik_x x + ik_z z}}{\sqrt{L_x L_z}} e^{-\frac{1}{2}\chi^2} \begin{pmatrix} a_{k_z m s} H_{M-1}(\chi) \\ b_{k_z m s} H_M(\chi) \end{pmatrix}, \quad (6)$$

where we define the dimensionless quantity  $\chi = \sqrt{\alpha} \frac{y - k_x t_B^2}{l_B} + \frac{t_x^s l_B}{\sqrt{\alpha} v_F} E_{m, \alpha B}^0$ .

In the local limit  $\mathbf{q} \rightarrow 0$  we find

$$J_{k_z m n s} = \Gamma_{k_z m n s} s v_F e (\alpha_{k_z m s} \delta_{M-1, N} + m \leftrightarrow n), \quad (7)$$

$$T_{k_z m n s}^{0y} = \frac{is\Gamma_{k_z m n s}}{4} (E_{k_z m s} + E_{k_z n s} - 2\mu) \times (\alpha_{k_z m s} \delta_{M-1, N} - m \leftrightarrow n), \quad (8)$$

where  $m \leftrightarrow n$  represents the preceding term under the interchange of  $m, n$ , and we have defined  $\Gamma_{k_z m n s} = [(\alpha_{k_z m s}^2 + 1)(\alpha_{k_z n s}^2 + 1)]^{-\frac{1}{2}}$ .

## II. RESULTS

We find the response

$$\lim_{\omega \rightarrow 0} \lim_{\mathbf{q} \rightarrow 0} \chi^{xy} = \gamma_N \frac{e^2 v_F B}{(2\pi)^2}, \quad (9)$$

where  $\gamma_N$  is a term generally depending on the number of Landau levels included in the sum, and the tilting vector  $\mathbf{t}$ .

The case of no tilt was first solved by Chernodub *et al.* [24] and then using Kubo formalism by Arjona *et al.* [3]. We have simplified their computation and found an analytical expression for the prefactor, with the iterative expression

$$\gamma_N - \gamma_{N-1} = \frac{1}{4} \left[ 1 + 2N \left\{ 1 - (1 + N) \log \left( 1 + \frac{1}{N} \right) \right\} \right], \quad (10)$$

for  $N > 0$ , and  $\gamma_0 = \frac{1}{4}$ . It can be shown that

$$\begin{aligned} \gamma_N = \gamma_0 + \frac{1}{12} & \left( 6\zeta^{(1,0)}(-2, N+1) - 6\zeta^{(1,0)}(-2, N+2) \right. \\ & + 6\zeta^{(1,0)}(-1, N+1) + 6\zeta^{(1,0)}(-1, N+2) \\ & \left. + 12 \log(\xi) + 3N^2 + 6N - 1 \right), \end{aligned} \quad (11)$$

where  $\xi \approx 1.28243$  is Glaisher's constant.

\* A footnote to the article title

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The contribution from the cone with chirality  $s = -1$  can be found from the result of the positive chirality cone. In the case of perpendicular tilt, they are exactly the same. In the case of parallel tilt, it depends on the symmetry of the tilt. For systems with broken inversion symmetry, the response from the two cones are the same. On the other hand, for inversion symmetric systems, the contribution from the cone with chirality  $s = -1$  is the same as that of the  $s = +1$  cone at the opposite tilt  $t_z \rightarrow -t_z$ . Therefore, it is useful to separate the contribution into even and odd components, for finding the total contribution from the two cones combined. For some contribution  $\gamma(t_{x/z})$ , we define

$$\gamma_{\text{even}}(t_{x/z}) = \frac{\gamma(t_{x/z}) + \gamma(-t_{x/z})}{2}, \quad (12)$$

$$\gamma_{\text{odd}}(t_{x/z}) = \frac{\gamma(t_{x/z}) - \gamma(-t_{x/z})}{2}. \quad (13)$$

$$(14)$$

All results will be given in terms of these components, at  $t_{x/y} > 0$ .

### A. Perpendicular tilt

We found the response to be even in  $t_x$ , and independent of the chirality  $s$ . The response was solved numerically and is shown in figure 3.

### B. Parallel tilt

In the Type-I regime, the response has an extra term compared to the untitled case,

$$\frac{\gamma_{\text{div}, N}}{2} = \int d\kappa_z \chi(\kappa_z) \kappa_z t_z \alpha_{\kappa_z m s}^2, \quad (15)$$

where  $\kappa_z = \sqrt{2eB}k_z$ . This contribution has a UV divergence, and we introduce the momentum cutoff  $\Lambda$ . The contribution is

$$\frac{t_z}{4} \left\{ \Lambda \left( \sqrt{1 + \Lambda^2 + m} - \sqrt{\Lambda^2 + m} \right) + m \tanh^{-1} \left[ \frac{\Lambda}{\sqrt{\Lambda^2 + m}} \right] - (m + 1) \tanh^{-1} \left[ \frac{\Lambda}{\sqrt{1 + \Lambda^2 + m}} \right] \right\}. \quad (16)$$

Add discussion about linear model and cut-off

In the case of Type-II systems, the calculation is more involved, because of the Landau levels crossing the Fermi surface. This gives both intraband and interband transitions. The zeroth transition  $0 \rightarrow 1$  was found to be

$$\frac{\text{sign}(t_z)}{2} \left( |t_z| \sinh^{-1} \left( \frac{1}{\sqrt{t_z^2 - 1}} \right) - 1 \right). \quad (17)$$

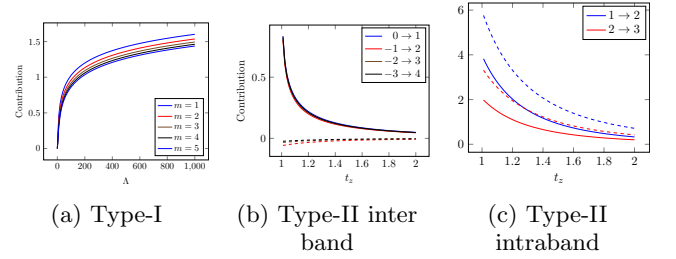


FIG. 1: Contributions in for parallel tilt, for Type-I and Type-II. Components even in  $t_z$  are given in solid line, while odd components are given in dashed line.

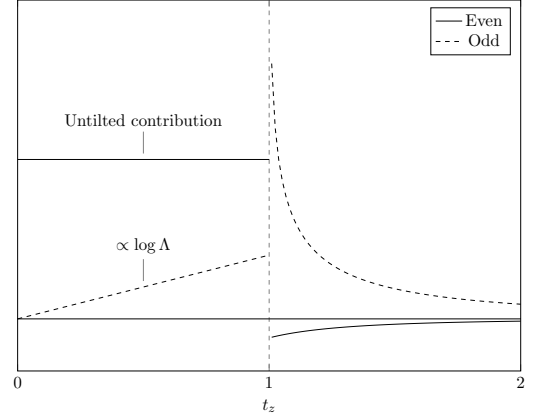


FIG. 2: Schematic summary of the contributions for perpendicular tilt  $t_z$ . Shown is the even (solid) and odd (dashed) parts as functions of  $t_z$ .

The higher order contributions were also computed analytically, with very lengthy expressions. A schematic summary of all the contributions are given in figure 2.

[1] We here chose the “antisymmetric” version for the broken inversion symmetry, where the cones tilt in the same direction with the same magnitude. Of course, many other

choices also break inversion symmetry.

[2] TODO: add some discussion here or in appendix.

[3] V. Arjona, M. N. Chernodub, and M. A. H. Vozmediano,

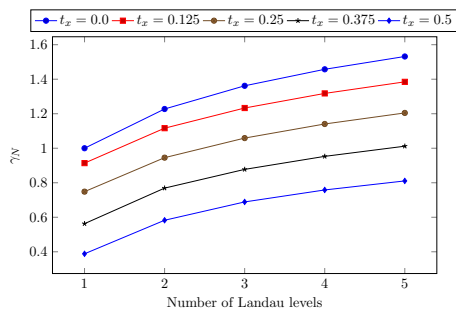


FIG. 3: Contribution as a function of number of Landau levels  $N$  for various values of  $t_x$ .

Fingerprints of the conformal anomaly on the thermoelectric transport in Dirac and Weyl semimetals: Result from a Kubo formula, **99**, 235123 (), arXiv:1902.02358.

- [4] Automatically placing footnotes into the bibliography requires using BibTeX to compile the bibliography.
- [5] V. Arjona, E. V. Castro, and M. A. H. Vozmediano, Collapse of Landau levels in Weyl semimetals, **96**, 081110 (), arXiv:1703.05399.
- [6] V. Arjona Romano, Novel thermoelectric and elastic responses in Dirac matter.
- [7] V. Arjona, J. Borge, and M. A. H. Vozmediano, Thermoelectric relations in the conformal limit in Dirac and Weyl semimetals, arXiv:1903.00019 [cond-mat] ().
- [8] V. Arjona, J. Borge, and M. A. H. Vozmediano, Thermoelectric Relations in the Conformal Limit in Dirac and Weyl Semimetals, **12**, 814 ().
- [9] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, **90**, 015001 ().
- [10] N. P. Armitage, E. J. Mele, and A. Vishwanath, Weyl and Dirac semimetals in three-dimensional solids, **90**, 015001 ().
- [11] G. Basar and G. V. Dunne, The Chiral Magnetic Effect and Axial Anomalies, .
- [12] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press).
- [13] M. Berry, Quantal phase factors accompanying adiabatic changes 10.1098/rspa.1984.0023.
- [14] R. A. Bertlmann, *Anomalies in Quantum Field Theory*, Oxford Science Publications No. 91 (Clarendon Press).
- [15] A. Bilal, Lectures on Anomalies, arXiv:0802.0634 [hep-th].
- [16] A. A. Burkov, Chiral anomaly and transport in Weyl metals, **27**, 113201 ().
- [17] A. A. Burkov, M. D. Hook, and L. Balents, Topological nodal semimetals, **84**, 235126, arXiv:1110.1089.
- [18] A. A. Burkov, Topological semimetals, **15**, 1145 ().
- [19] A. A. Burkov, Topological semimetals, **15**, 1145 ().
- [20] M.-C. Chang, Lecture notes for Manybody Physics I.
- [21] M. N. Chernodub, C. Corianò, and M. M. Maglio, Anomalous gravitational TTT vertex, temperature inhomogeneity, and pressure anisotropy, **802**, 135236 ().
- [22] M. N. Chernodub, Anomalous Transport Due to the Conformal Anomaly, **117**, 141601.
- [23] M. N. Chernodub, V. A. Goy, and A. V. Molochkov, Conformal magnetic effect at the edge: A numerical study in scalar QED, **789**, 556 (), arXiv:1811.05411.
- [24] M. N. Chernodub, A. Cortijo, and M. A. H. Vozmediano, Generation of a Nernst Current from the Conformal Anomaly in Dirac and Weyl Semimetals, **120**, 206601 ().
- [25] M. N. Chernodub and M. A. Zubkov, Scale Magnetic Effect in Quantum Electrodynamics and the Wigner-Weyl Formalism, **96**, 056006, arXiv:1703.06516.
- [26] M. N. Chernodub, Y. Ferreira, A. G. Grushin, K. Landsteiner, and M. A. H. Vozmediano, Thermal transport, geometry, and anomalies, arXiv:2110.05471 [cond-mat, physics:hep-th] ().
- [27] Y. W. Chow, R. Pietranico, and A. Mukerji, Studies of oxygen binding energy to hemoglobin molecule, **66**, 1424, 6.
- [28] R. D'Auria and M. Trigiante, *From Special Relativity to Feynman Diagrams*, UNITEXT (Springer Milan).
- [29] H.-A. Engel, E. I. Rashba, and B. I. Halperin, Theory of Spin Hall Effects in Semiconductors, arXiv:cond-mat/0603306.
- [30] Y. Ferreira, A. A. Zyuzin, and J. H. Bardarson, Anomalous Nernst and thermal Hall effects in tilted Weyl semimetals, **96**, 115202.
- [31] M. Forger and H. Römer, Currents and the Energy-Momentum Tensor in Classical Field Theory: A fresh look at an Old Problem, **309**, 306, arXiv:hep-th/0307199.
- [32] M. Fruchart and D. Carpentier, An Introduction to Topological Insulators, **14**, 779, arXiv:1310.0255.
- [33] K. Fujikawa, Comment on Chiral and Conformal Anomalies, **44**, 1733.
- [34] G. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press).
- [35] M. O. Goerbig, Electronic properties of graphene in a strong magnetic field, **83**, 1193.
- [36] M. Gotay and J. Marsden, Stress-Energy-Momentum Tensors and the Belinfante-Rosenfeld Formula **132**, 10.1090/conm/132/1188448.
- [37] I. S. Gradshteyn and D. Zwillinger, *Table of Integrals, Series, and Products*, eighth edition ed. (Elsevier, Academic Press is an imprint of Elsevier).
- [38] V. P. Gusynin, S. G. Sharapov, and J. P. Carbotte, Anomalous Absorption Line in the Magneto-Optical Response of Graphene, **98**, 157402.
- [39] K. Hagiwara, P. Rüßmann, X. L. Tan, Y.-J. Chen, K. Ueno, V. Feyer, G. Zamborlini, M. Jugovac, S. Suga, S. Blügel, C. M. Schneider, and C. Tuschke, Link between Weyl-fermion chirality and spin texture, arXiv:2205.15252 [cond-mat].
- [40] C. Herring, Accidental Degeneracy in the Energy Bands of Crystals, **52**, 365.
- [41] B. R. Holstein, The adiabatic theorem and Berry's phase, **57**, 1079.
- [42] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, International Series in Pure and Applied Physics (McGraw-Hill International Book Co).
- [43] A. Janik, [complaints of the mentally disordered (author's transl)], **73**, 1, 852047.
- [44] M. Kachelriess, *Quantum Fields: From the Hubble to the Planck Scale*, first edition ed., Oxford Graduate Texts (Oxford University Press).
- [45] D. B. Kaplan, Chiral Symmetry and Lattice Fermions, arXiv:0912.2560 [hep-lat, physics:hep-ph, physics:hep-th].

- [46] K. v. Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, **45**, 494.
- [47] X. Kou, Y. Fan, and K. L. Wang, Review of Quantum Hall Trio, Spin-Orbit Coupled Materials, **128**, 2.
- [48] K. Landsteiner, E. Megías, and F. Pena-Benitez, Gravitational Anomaly and Transport Phenomena, **107**, 021601.
- [49] J. Linder, *Intermediate Quantum Mechanics*, 1st ed. (Bookboon).
- [50] V. Lukose, R. Shankar, and G. Baskaran, Novel Electric Field Effects on Landau Levels in Graphene, **98**, 116802.
- [51] R. Lundgren, P. Laurell, and G. A. Fiete, Thermoelectric properties of Weyl and Dirac semimetals, **90**, 165115.
- [52] Luttinger's theorem.
- [53] J. M. Luttinger, Theory of Thermal Transport Coefficients, **135**, A1505.
- [54] G. D. Mahan, *Many-Particle Physics*, 3rd ed., Physics of Solids and Liquids (Kluwer Academic/Plenum Publishers).
- [55] A. Manchon, H. C. Koo, J. Nitta, S. M. Frolov, and R. A. Duine, New perspectives for Rashba spin-orbit coupling, **14**, 871.
- [56] T. M. McCormick, I. Kimchi, and N. Trivedi, Minimal models for topological Weyl semimetals, **95**, 075133.
- [57] H. B. Nielsen and M. Ninomiya, The Adler-Bell-Jackiw anomaly and Weyl fermions in a crystal, **130**, 389.
- [58] The Nobel Prize in Physics 2016.
- [59] T. Okuda and A. Kimura, Spin- and Angle-Resolved Photoemission of Strongly Spin-Orbit Coupled Systems, **82**, 021002.
- [60] F. W. J. Olver, C. W. C. B. R. M. B. V. S. H. S. C. A. B. O. D. D. W. B. I. S. Lozier, R. F. Boisvert, and e. M. A. McClain, NIST digital library of mathematical functions.
- [61] H. L. Ottesen, Optical Conductivity of Dirac Fermions in Antiferromagnetic Semimetals.
- [62] J. Park, O. Golan, Y. Vinkler-Aviv, and A. Rosch, Thermal Hall response: Violation of gravitational analogues and Einstein relations, arXiv:2108.06162 [cond-mat, physics:hep-th] ().
- [63] J. Park, O. Golan, Y. Vinkler-Aviv, and A. Rosch, Thermal Hall response: Violation of gravitational analogues and Einstein relations, arXiv:2108.06162 [cond-mat, physics:hep-th] ().
- [64] PhD Research Fellow in Physics – Error mitigation of quantum information (219817) — University of Oslo.
- [65] R. Ramazashvili, Zeeman spin-orbit coupling in antiferromagnetic conductors, Spin-Orbit Coupled Materials, **128**, 65.
- [66] R. Rodgers, E. Mauri, U. Gürsoy, and H. T. Stoof, Thermodynamics and transport of holographic nodal line semimetals, **2021**, 191.
- [67] K. Rottmann, *Matematisk formelsamling* (Bracan forl.).
- [68] S. Saha and S. Tewari, Anomalous Nernst effect in type-II Weyl semimetals, **91**, 4, arXiv:1707.04117.
- [69] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Cambridge university press).
- [70] A. P. Schnyder, Accidental and symmetry-enforced band crossings in topological semimetals, , 47.
- [71] G. Sharma, P. Goswami, and S. Tewari, Chiral anomaly and longitudinal magnetotransport in type-II Weyl semimetals, **96**, 045112.
- [72] B. S. Shastry, Electro Thermal Transport Coefficients at Finite Frequencies, arXiv:0806.4629 [cond-mat].
- [73] M. A. Shifman, Anomalies in gauge theories, **209**, 341.
- [74] J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Universal Intrinsic Spin Hall Effect, **92**, 126603.
- [75] A. A. Soluyanov, D. Gresch, Z. Wang, Q. Wu, M. Troyer, X. Dai, and B. A. Bernevig, Type-II Weyl semimetals, **527**, 495.
- [76] M. Stone, Gravitational anomalies and thermal Hall effect in topological insulators, **85**, 184503.
- [77] P. Tang, Q. Zhou, G. Xu, and S.-C. Zhang, Dirac fermions in an antiferromagnetic semimetal, **12**, 1100.
- [78] G. Tatara, Thermal Vector Potential Theory of Transport Induced by a Temperature Gradient, **114**, 196601 ().
- [79] G. Tatara, Thermal Vector Potential Theory of Transport Induced by a Temperature Gradient, **114**, 196601 ().
- [80] S. Tchoumakov, M. Civelli, and M. O. Goerbig, Magnetic description of the Fermi arc in type-I and type-II Weyl semimetals, , 17 ().
- [81] S. Tchoumakov, M. Civelli, and M. O. Goerbig, Magnetic-Field-Induced Relativistic Properties in Type-I and Type-II Weyl Semimetals, **117**, 086402 ().
- [82] D. Tong, Gauge Theory lecture notes.
- [83] W.-K. Tse and A. H. MacDonald, Magneto-optical Faraday and Kerr effects in topological insulator films and in other layered quantized Hall systems, **84**, 205327.
- [84] A. M. Turner and A. Vishwanath, Chapter 11 - Beyond Band Insulators: Topology of Semimetals and Interacting Phases, in *Contemporary Concepts of Condensed Matter Science*, Topological Insulators, Vol. 6, edited by M. Franz and L. Molenkamp (Elsevier) pp. 293–324.
- [85] M. Udagawa and E. J. Bergholtz, Field-Selective Anomaly and Chiral Mode Reversal in Type-II Weyl Materials, **117**, 086401 (), arXiv:1604.08457.
- [86] M. Udagawa and E. J. Bergholtz, Field-Selective Anomaly and Chiral Mode Reversal in Type-II Weyl Materials, **117**, 086401 ().
- [87] E. C. I. van der Wurff and H. T. C. Stoof, Magnetovortical and thermoelectric transport in tilted Weyl metals, **100**, 045114.
- [88] T. Wehling, A. Black-Schaffer, and A. Balatsky, Dirac materials, **63**, 1.
- [89] A. Wirzba, Lecture on Anomalies and the Infinite Hotel, , 75.
- [90] K. Wu, J. Chen, H. Ma, L. Wan, W. Hu, and J. Yang, Two-Dimensional Giant Tunable Rashba Semiconductors with Two-Atom-Thick Buckled Honeycomb Structure **21**, 10.1021/acs.nanolett.0c04429.
- [91] Z.-M. Yu, Y. Yao, and S. A. Yang, Predicted Unusual Magnetoresponse in Type-II Weyl Semimetals, **117**, 077202.
- [92] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd ed., In a Nutshell (Princeton University Press).
- [93] L. Zhang, Y. Jiang, D. Smirnov, and Z. Jiang, Landau quantization in tilted Weyl semimetals with broken symmetry, **129**, 105107, arXiv:2012.14846.