

1 Poisson equation in one dimension

In this section, we will solve the one-dimensional Poisson equation

$$u_{xx} = f(x) \quad (0 < x < 1)$$

subject to a source term $f(x)$ and different boundary conditions at $x = 0$ and $x = 1$. First, we will solve it with finite difference methods of first and second order on a uniform grid. Finally, we solve it on a non-uniform grid and investigate how adaptive mesh refinement (AMR) can be used to obtain accurate solutions by distributing fewer points more cleverly along the grid. a

First, consider Dirichlet and Neumann boundary conditions at opposite ends and the source given by

$$u_{xx} = x + \cos(2\pi x) \quad (0 < x < 1), \quad u(0) = a, \quad u_x(1) = b.$$

The analytical solution is

$$u(x) = C_1 + C_2x + \frac{1}{6}x^3 - \frac{1}{4\pi^2} \cos(2\pi x),$$

where the constants C_1 and C_2 are determined from the boundary conditions. To solve the equation numerically, we impose a uniform grid of $M + 2$ points and step length h defined by

$$\begin{array}{ccccccc} x_0 = 0 & x_1 & x_2 & & x_m & & x_{M-1} & x_M & x_{M+1} = 1 \\ & h & h & & & & h & h & \end{array}$$

To generate finite difference methods of both first and second order, we approximate the second derivative at interior points using the forward difference and central difference

$$\begin{aligned} u_{xx}(x_m) &= \frac{u_m - 2u_{m+1} + u_{m+2}}{h^2} + O(h^1) & (1 \leq m \leq M) \\ u_{xx}(x_m) &= \frac{u_{m-1} - 2u_m + u_{m+1}}{h^2} + O(h^2) & (1 \leq m \leq M - 1). \end{aligned}$$

To handle the Dirichlet boundary condition $u(0) = a$ at the left edge, we insert the trivial equation

$$1 \cdot u_0 = a.$$

To handle the Neumann boundary condition $u_x(1) = b$ at the right edge to first or second order, we use

$$\begin{aligned} u_x(1) &= \frac{u_{M+1} - u_M}{h} + O(h^1) = b \\ u_x(1) &= \frac{\frac{1}{2}u_{M-1} - 2u_M + \frac{3}{2}u_{M+1}}{h} + O(h^2) = b. \end{aligned}$$

By writing all these equations in $(M + 2) \times (M + 2)$ -matrix form $AU = b$, we obtain for example to second order

$$\begin{bmatrix} 1 & & & & \\ +1/h^2 & -2/h^2 & +1/h^2 & & \\ & \ddots & \ddots & \ddots & \\ & & +1/h^2 & -2/h^2 & +1/h^2 \\ & & +1/2h & -2/h & +3/2h \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_M \\ U_{M+1} \end{bmatrix} = \begin{bmatrix} a \\ f(x_1) \\ \vdots \\ f(x_M) \\ b \end{bmatrix}$$

Some remarks:

- We could handle the Dirichlet boundary condition $u(0) = a$ differently by treating $U_0 = a$ as a known variable. The system of equations is equivalent if we remove the first row and column of A and the first entries in U and b , but simultaneously modify the entry $f(x_1) \rightarrow f(x_1) - a/h^2$. This approach is more consistent with treating U_0 as a known variable, since its precise value is defined by the Dirichlet boundary condition. However, our approach of inserting a trivial equation $1 \cdot U_0 = a$ keeps the matrix dimensions independent of boundary conditions and makes it easier to reason with how the discretized differential operator represented by A operates on the grid point U_0 in the same way it operates on all other grid points.
- To handle different combinations of Dirichlet and Neumann boundary conditions at the ends, we simply replace the first or last rows of the matrix with the same type of equation. Note that if the Neumann boundary condition is imposed at the left boundary, the last row of the matrix above would have to be both reversed and negated.
- When Neumann boundary conditions are imposed at both ends, the solution is determined only up to a constant. To see this for a general Poisson boundary value problem, note that if $u_{xx} = f(x)$, $u_x(0) = a$ and $u_x(1) = b$, then also $(u+C)_{xx} = f(x)$, $(u+C)_x(0) = a$ and $(u+C)_x(1) = b$ if C is only a constant. It can also be seen from the general solution for this particular source term that the constant C_1 is undetermined when the solution is subject to boundary conditions that involve derivatives only. In this case, an additional constraint like $u(0) = 0$ must be imposed to define a unique solution.

With this in mind, it is now straightforward to solve the Poisson equation subject to any combination of Dirichlet and Neumann boundary conditions at the ends to both first and second order.



Figure 1: Analytical and numerical solutions (left) and convergence plots (right) for solutions to the Poisson equation subject to three different boundary conditions.



Figure 2: Nice fig

2 Heat equation in one dimension

3 Laplace equation in two dimensions

4 Linearized Korteweg-De Vries equation in one dimension

In this section, we will study the one-dimensional linearized Korteweg-De Vries equation

$$\frac{\partial u}{\partial t} + \left(1 + \pi^2\right) \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \quad (t \geq 0) \quad (-L/2 \leq x \leq +L/2), \quad (1)$$

where the solution $u = u(x, t)$ is subject to periodic boundary conditions

$$u(x + L, t) = u(x, t).$$

4.1 Analytical solution

As the solution is periodic in space at every instant t , it can be expressed as a Fourier-series

$$u(x, t) = \sum_{n=-\infty}^{+\infty} c_n(t) \exp(ik_n x) \quad (2)$$

with wavenumbers $k_n = 2\pi n/L$ and time-dependent coefficients $c_n(t)$ that ensure spatial periodicity at all times. This can be derived formally by separation of variables. Inserting the Fourier series into the equation gives

$$\sum_n \left(\dot{c}_n(t) + i \left(\left(1 + \pi^2\right) k_n - k_n^3 \right) c_n(t) \right) \exp(ik_n x) = 0.$$

Due to orthogonality of the Fourier basis functions $\exp(ik_n x)$, this can only vanish if all coefficients separately vanish, so

$$c_n(t) = c_n(0) \exp \left(i \left(\left(1 + \pi^2\right) k_n - k_n^3 \right) t \right). \quad (3)$$

The solution has the property that the L_2 -norm $\|u(x, t)\|_2 = \left(\frac{1}{2} \int_{-L/2}^{+L/2} |u(x, t)|^2 dx \right)^{1/2}$ is preserved over time. To understand this, first note carefully how the first order time derivative combined with spatial derivatives of odd order produced coefficients of variable phase, but constant magnitude $|c_n(t)|^2 = |c_n(0)|^2$. This means that each Fourier mode is undamped in time, so

$$\begin{aligned} \int_{-1}^{+1} dx |u(x, t)|^2 &= \int_{-L/2}^{+L/2} \sum_{m,n} c_m(t) c_n^*(t) \exp(ik_m x) \exp(-ik_n x) dx \\ &= \sum_{m,n} c_m(t) c_n^*(t) \underbrace{\int_{-L/2}^{+L/2} \exp(i(k_m - k_n)x) dx}_{L\delta_{mn}} \\ &= L \sum_n |c_n(t)|^2 \\ &= L \sum_n |c_n(0)|^2 \\ &= \int_{-L/2}^{+L/2} |u(x, 0)|^2 dx. \end{aligned} \quad (4)$$

4.2 Numerical solution method

GRID FIGURE?? x_0 to x_{M-1} makes modulo arithmetic intuitive.

We will solve the Korteweg-De Vries equation with the central finite differences

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\delta}{2h} + O(h^2) = \frac{u_{m+1}^n - u_{m-1}^n}{2h} \\ \frac{\partial^3 u}{\partial x^3} &= \frac{\delta^3}{(2h)^3} + O(h^2) = \frac{u_{m+3}^n - 3u_{m+1}^n + 3u_{m-1}^n - u_{m-3}^n}{8h^3}\end{aligned}$$

in space, and doing the integration of $u_t = F(u_x, u_{xxx})$ in time using both the Euler method and the Crank-Nicholson method. They can be written simultaneously by inserting $\theta = 0$ and $\theta = 1/2$, respectively, in the Theta method

$$\begin{aligned}\frac{u(x, t+k) - u(x, t)}{k} &= (1-\theta)F(u_x(x, t), u_{xxx}(x, t)) + \theta F(u_x(x, t+k), u_{xxx}(x, t+k)) \\ &\quad + O\left(k\left(\frac{1}{2} - \theta\right) + k^2\left(\frac{1}{6} - \frac{\theta}{2}\right)\right).\end{aligned}$$

This results in the system of equations

$$\left(1 - \theta k \left(-\left(1 + \pi^2\right) \frac{\delta}{2h} - \frac{\delta^3}{(2h)^3}\right)\right) U_m^{n+1} = \left(1 - (1-\theta)k \left(-\left(1 + \pi^2\right) \frac{\delta}{2h} - \frac{\delta^3}{(2h)^3}\right)\right) U_m^n$$

for the unknown values $U_0^{n+1}, \dots, U_{M-1}^{n+1}$ at the next time step in terms of the known values U_0^n, \dots, U_{M-1}^n at the current time step. In matrix form, the system can be written

$$(I - \theta k A) U^{n+1} = (I - (1-\theta)k A) U^n, \quad (5)$$

where $U^n = [U_0^n \ \dots \ U_{M-1}^n]^T$ and $A =$

$$-\frac{1}{2h} \begin{bmatrix} 0 & +1 & & & & & -1 \\ -1 & 0 & +1 & & & & \\ & -1 & 0 & +1 & & & \\ & & -1 & 0 & +1 & & \\ & & & -1 & 0 & +1 & \\ & & & & \ddots & \ddots & \ddots \\ & & & & & -1 & 0 & +1 \\ & & & & & & -1 & 0 & +1 \\ & & & & & & & -1 & 0 \\ +1 & & & & & & & & -1 & 0 \end{bmatrix} - \frac{1 + \pi^2}{h^3} \begin{bmatrix} 0 & -3 & 0 & +1 & & & -1 & 0 & +3 \\ +3 & 0 & -3 & 0 & +1 & & & -1 & 0 \\ 0 & +3 & 0 & -3 & 0 & +1 & & & -1 \\ -1 & 0 & +3 & 0 & -3 & 0 & +1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 0 & +3 & 0 & -3 & 0 & +1 \\ +1 & & & -1 & 0 & +3 & 0 & -3 & 0 \\ 0 & +1 & & & -1 & 0 & +3 & 0 & -3 \\ -3 & 0 & +1 & & & -1 & 0 & +3 & 0 \end{bmatrix}.$$

where we have imposed periodic boundary conditions $U_m^n = U_{m+M}^n$ by simply wrapping the spatial derivative stencils around the matrix.

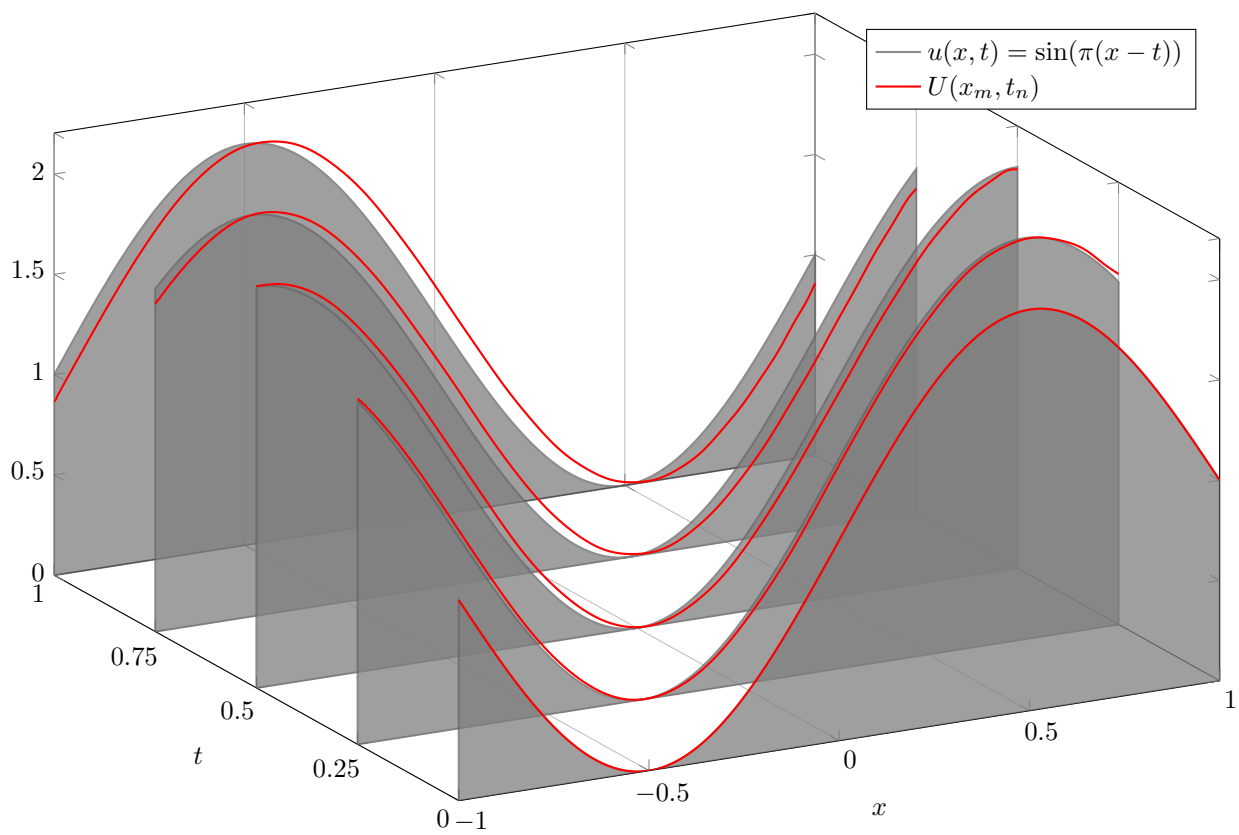
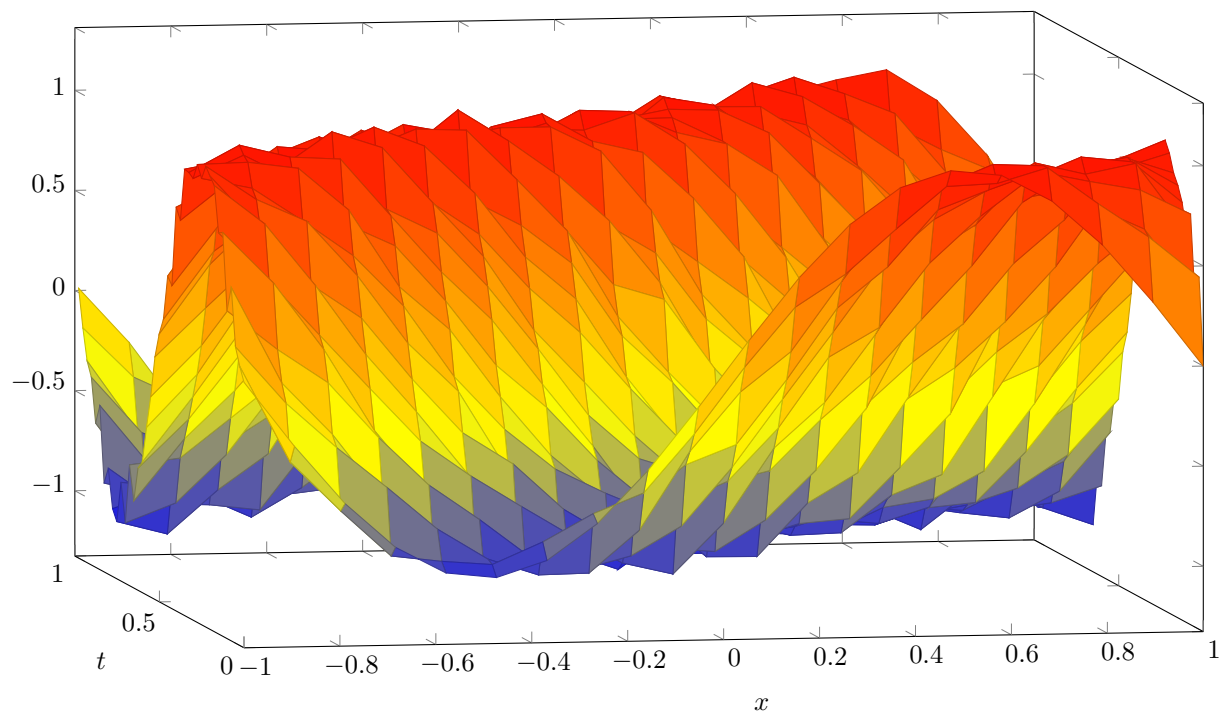
We then solve the system by preparing U^0 from the initial condition and solve equation (5) repeatedly to step forward in time. Note that with the constant time step k , all matrices in equation (5) are constant in time, and the process of solving the system many times can be accelerated by for example LU-factorizing the matrix on the left side.

Theta method:

$$\frac{u_m^{n+1} - u_m^n}{k} = (1-\theta)F(u^n) + \theta F(u^{n+1}) \quad (6)$$

where

$$F(u^n) = -(1 + \pi^2) \frac{u_{m+1}^n - u_{m-1}^n}{2h} - \frac{u_{m+3}^n - 3u_{m+1}^n + 3u_{m-1}^n - u_{m-3}^n}{8h^3} \quad (7)$$



$\theta = 0$ is forward Euler, $\theta = 1/2$ is Crank-Nicholson, $\theta = 1$ is backward Euler.

Separation of variables:

$$u(x, t) = X(x)T(t) \quad (8)$$

Insert into KdV equation to get $T = Ae^{zt}$. $X(x)$ is periodic, so expand it in a Fourier series:

$$X(x) = \sum_k C_k e^{ikx} \quad (9)$$

The general solution is then

$$u(x, t) = \sum_k C_k e^{kt} e^{ikx} \quad (10)$$

so

$$u_m^n = \sum_k C_k e^{kt_n} e^{ikx_m} \quad (11)$$

With constant time steps, $t_n = nk$, we can write

$$u_m^n = \sum_k C_k G^n e^{ikx_m} \quad (12)$$

Consider a general term in the series $u_m^n = G^n e^{ikx_m}$. Insert into discretized KdV equation to get

$$\frac{G-1}{k} = [(1-\theta) + \theta G] \left[-(1+\pi^2) \frac{e^{iqh} - e^{-iqh}}{2h} - \frac{e^{3iqh} - e^{-3iqh} - 3(e^{iqh} - e^{-iqh})}{8h^3} \right] \quad (13)$$

Expand using Euler's identity to get

$$\frac{G-1}{k} = i[(1-\theta) + \theta G] \left[-(1+\pi^2) \frac{\sin(qh)}{h} - \frac{\sin^3(qh)}{h^3} \right] \quad (14)$$

$\theta = 0$ gives $|G| > 1$ (unstable), but $\theta = 1/2$ gives $|G| = 1$ (stable)! (unconditionally)

Proof of conserved norm: Due to periodicity (if one is very pedantic, one can obtain this by separation of variables)

$$u(x, t) = \sum_n c_n(t) e^{ik_n x} \quad (15)$$

Insert into the KdV equation to get

$$\sum_n \left\{ \dot{c}_n(t) + i[(1+\pi^2)k_n - k_n^3]c_n(t) \right\} e^{ik_n x} = 0 \quad (16)$$

For the sum to be 0, each term inside the curly brackets must vanish, so

$$c_n(t) = c_n(0) \exp \left\{ i[(1+\pi^2)k_n - k_n^3]t \right\} \quad (17)$$

Thus,

$$\begin{aligned}
\int_{-1}^{+1} dx |u(x, t)|^2 &= \int_{-L/2}^{+L/2} \sum_{m, n} c_m(t) c_n^*(t) \exp(i k_m x) \exp(-i k_n x) \\
&= \sum_{m, n} c_m(t) c_n^*(t) \underbrace{\int_{-L/2}^{+L/2} \exp(i(k_m - k_n)x)}_{L \delta_{mn}} \\
&= L \sum_n |c_m(t)|^2 \\
&= L \sum_n |c_m(0)|^2 \\
&= \int_{-L/2}^{+L/2} dx |u(x, 0)|^2,
\end{aligned} \tag{18}$$

so the spatial L_2 -norm is preserved over time.