$$U_{xx} + U_{yy} = 0 \qquad U = x(x)Y(y)$$

$$X''(x)Y(y) + X(x)Y''(y) = 0 \Rightarrow \frac{-X''}{X} = \frac{Y''}{Y} = c$$

$$X = A\cos(cx) + B\sin(cx)$$

$$Y = C_1 e^{+cx} + D e^{-cx}$$

$$U(0,y) = 0 = A\,Y(cy) \Rightarrow A = 0 \Rightarrow X = B\sin(cx)$$

$$U(1,y) = 0 = B\sin(c)\,Y(cy) \Rightarrow c = n\pi$$

$$U(x,0) = 0 = B\sin(cx)(C_1 + D) \Rightarrow C_1 + D = 0 \Rightarrow Y = C_1(e^{cy} - e^{-cy})$$
$$= C_1 \sinh(cy)$$

$$U(x,1) = \sin(2\pi x) = B\sin(2\pi x)\,C_1\sinh(c)$$

$$= B\,C_1\sinh(c)\,\sin(cx)$$

$$\Rightarrow D\,C_1\sinh(c)\sin(\pi n x) \Rightarrow n = 2,\quad B C_1\sinh(2\pi) = 1$$

$$U = X(x)Y(y) = B C_1\sinh(2\pi y)\sin(2\pi x) = \frac{\sinh(2\pi y)}{\sinh(2\pi)}\sin(2\pi x)$$