1 Poisson equation in one dimension

In this section, we will solve the one-dimensional Poisson equation

$$u_{xx} = f(x) \qquad (0 < x < 1)$$

subject to a source term $f(x)$ and different boundary conditions at $x = 0$ and $x = 1$. First, we will solve it with finite difference methods of first and second order on a uniform grid. Finally, we solve it on a non-uniform grid and investigate how adaptive mesh refinement (AMR) can be used to obtain accurate solutions by distributing fewer points more cleverly along the grid. a

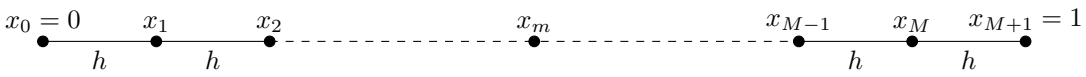
First, consider Dirichlet and Neumann boundary conditions at opposite ends and the source given by

$$u_{xx} = x + \cos(2\pi x) \qquad (0 < x < 1), \qquad u(0) = a, \qquad u_x(1) = b.$$

The analytical solution is

$$u(x) = C_1 + C_2 x + \frac{1}{6}x^3 - \frac{1}{4\pi^2}\cos(2\pi x)),$$

where the constants C_1 and C_2 are determined from the boundary conditions. To solve the equation numerically, we impose a uniform grid of $M + 2$ points and step length h defined by



To generate finite difference methods of both first and second order, we approximate the second derivative at interior points using the forward difference and central difference

$$u_{xx}(x_m) = \frac{u_m - 2u_{m+1} + u_{m+2}}{h^2} + O(h^1) \qquad (1 \le m \le M)$$
$$u_{xx}(x_m) = \frac{u_{m-1} - 2u_m + u_{m+1}}{h^2} + O(h^2) \qquad (1 \le m \le M - 1).$$

To handle the Dirichlet boundary condition $u(0) = a$ at the left edge, we insert the trivial equation

$$1 \cdot u_0 = a.$$

To handle the Neumann boundary condition $u_x(1) = b$ at the right edge to first or second order, we use

$$u_x(1) = \frac{u_{M+1} - u_M}{h} \qquad +O(h^1) \quad = b$$
$$u_x(1) = \frac{\frac{1}{2}u_{M-1} - 2u_M + \frac{3}{2}u_{M+1}}{h} \qquad +O(h^2) \quad = b.$$

By writing all these equations in $(M + 2) \times (M + 2)$-matrix form $AU = b$, we obtain for example to second order

$$\begin{bmatrix} 1 & & & & & \\ +1/h^2 & -2/h^2 & +1/h^2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & +1/h^2 & -2/h^2 & +1/h^2 & \\ & & +1/2h & -2/h & +3/2h \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_M \\ U_{M+1} \end{bmatrix} = \begin{bmatrix} a \\ f(x_1) \\ \vdots \\ f(x_M) \\ b \end{bmatrix}$$

Some remarks:

- We could handle the Dirichlet boundary condition $u(0) = a$ differently by treating $U_0 = a$ as a known variable. The system of equations is equivalent if we remove the first row and column of A and the first entries in U and b, but simultaneously modify the entry $f(x_1) \to f(x_1) - a/h^2$. This approach is more consistent with treating U_0 as a known variable, since its precise value is defined by the Dirichlet boundary condition. However, our approach of inserting a trivial equation $1 \cdot U_0 = a$ keeps the matrix dimensions independent of boundary conditions and makes it easier to reason with how the discretized differential operator represented by A operates on the grid point U_0 in the same way it operates on all other grid points.
- To handle different combinations of Dirichlet and Neumann boundary conditions at the ends, we simply replace the first or last rows of the matrix with the same type of equation. Note that if the Neumann boundary condition is imposed at the left boundary, the last row of the matrix above would have to be both reversed and negated.
- When Neumann boundary conditions are imposed at both ends, the solution is determined only up to a constant. To see this for a general Poisson boundary value problem, note that if $u_{xx} = f(x)$, $u_x(0) = a$ and $u_x(1) = b$, then also $(u+C)_{xx} = f(x)$, $(u+C)_x(0) = a$ and $(u+C)_x(1) = b$ if C is only a constant. It can also be seen from the general solution for this particular source term that the constant C_1 is undetermined when the solution is subject to boundary conditions that involve derivatives only. In this case, an additional constraint like $u(0) = 0$ must be imposed to define a unique solution.

With this in mind, it is now straightforward to solve the Poisson equation subject to any combination of Dirichlet and Neumann boundary conditions at the ends to both first and second order.

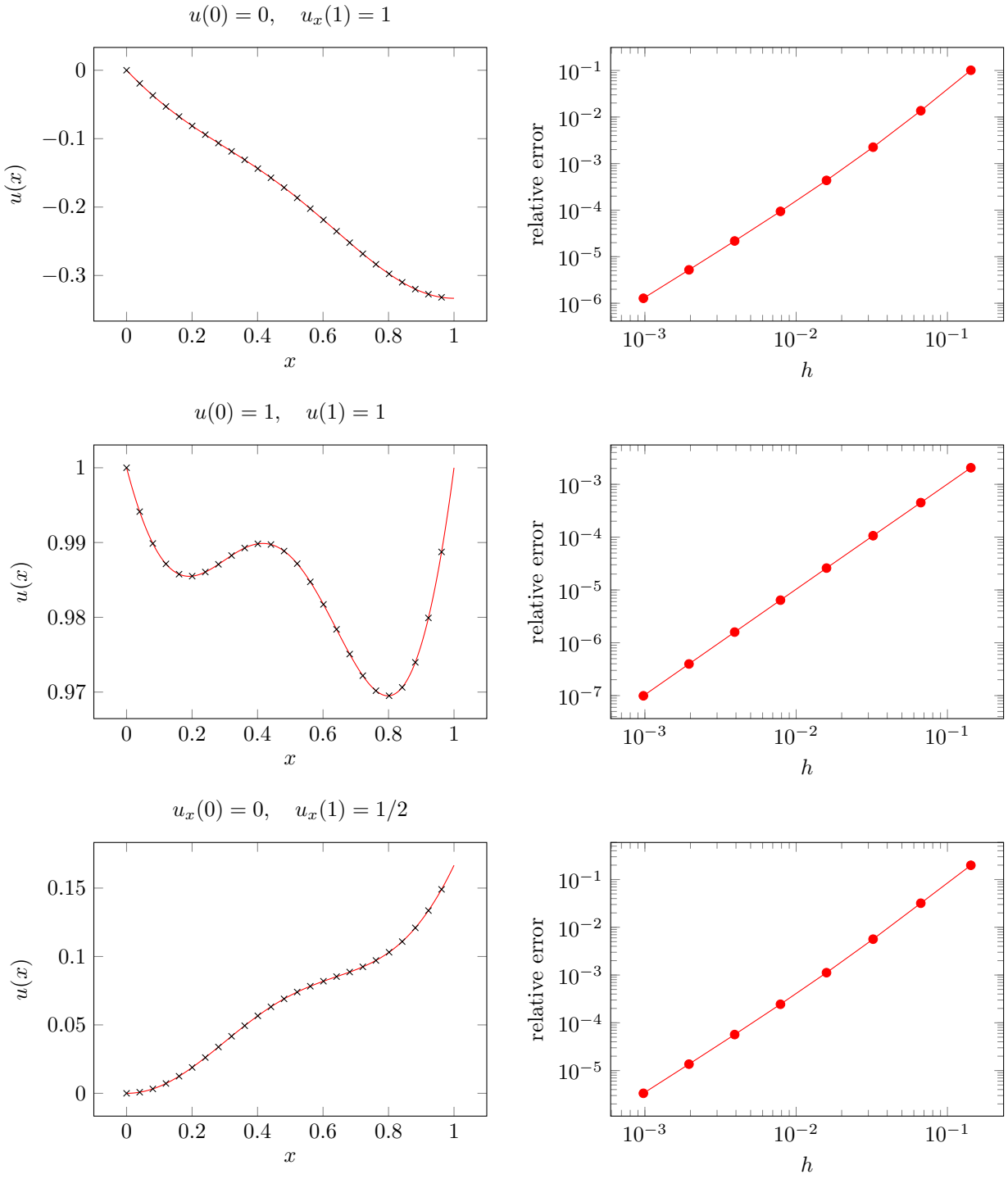


Figure 1: Analytical and numerical solutions (left) and convergence plots (right) for solutions to the Poisson equation subject to three different boundary conditions.

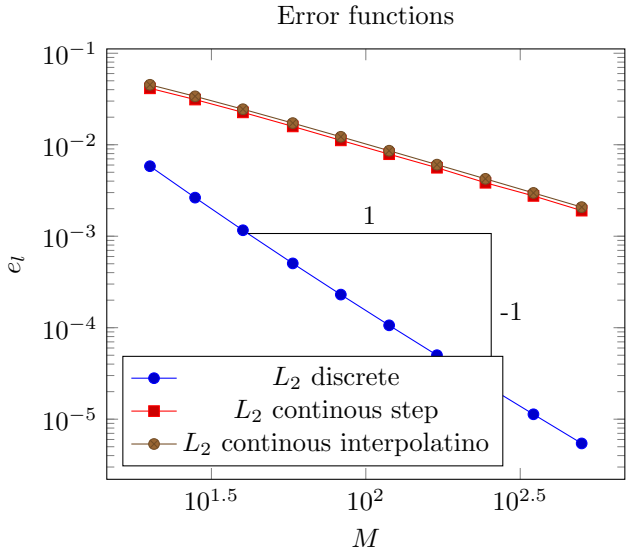


Figure 2: Nice fig

2 Heat equation in one dimension

In this seection, we will solve one-dimensional differential equations with time evolution. Firstly, we will solve the one-dimensional heat equation with Neumann boundary conditions, using both the Backward Euler method and Crank-Nicolson's method. We will then analyse the error of our methods using a manufactured solution to which the analytical solution is known. For this, we will use both uniform and adaptive mesh refinement, and the different methods will be discussed. Finally, we will solve the inviscid Burgers' equation and evaluate the breaking which this equation exhibits.

Firstly, we consider the one-dimensional heat equation with Neumann boundary conditions

$$u_t = u_{xx}, \quad u_x(0,t) = u_x(1,t) = 0, \quad u(x,0) = 2\pi x - \sin(2\pi x),$$

valid for $x \in [0,1] := \Omega$ and $t > 0$. The equation can be solved numeriacally by dividing the interval Ω in M parts, as described in section 1. We also divide the time interval $t \in [0, t_{end}$ into parts, so that we approximate the solution at N finite times $t_n = nk, \quad n = 1, \dots N$. For $t = 0$, the solution is the given initial condition.

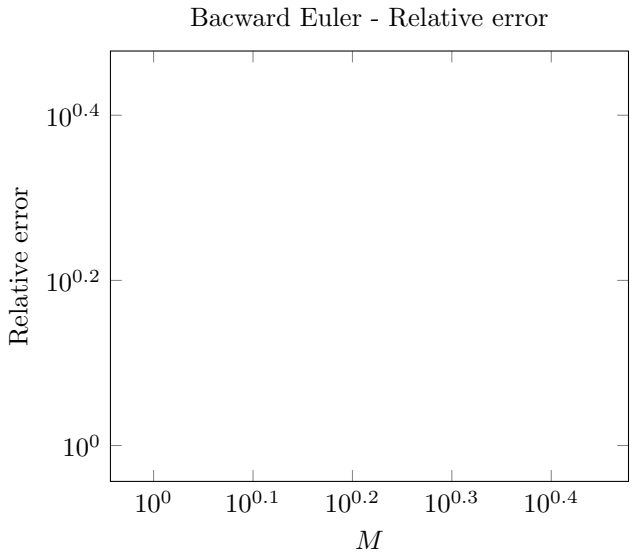


Figure 3: Convergence plot, Relative err. of BE, same time t, different M

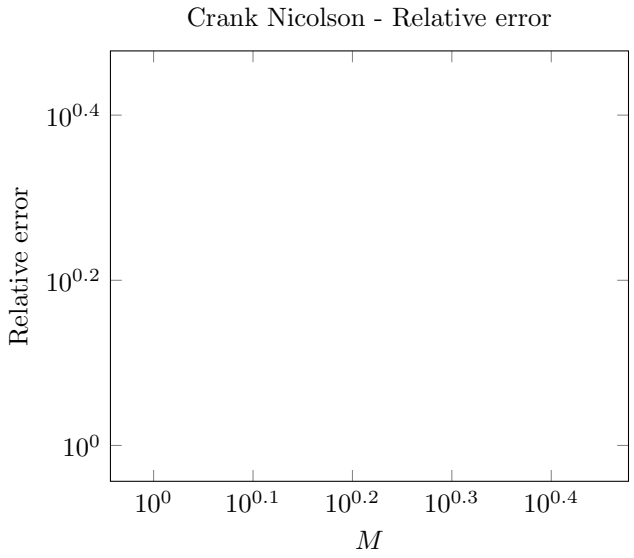


Figure 4: Convergence plot, Relative err. of CN, same time t, different M

3 Laplace equation in two dimensions

4 Linearized Korteweg-De Vries equation in one dimension

In this section, we will study the one-dimensional linearized Korteweg-De Vries equation

$$\frac{\partial u}{\partial t} + \left(1+\pi^2\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad (t \geq 0) \quad (-L/2 \leq x \leq +L/2), \tag{1}$$

where the solution $u = u(x,t)$ is subject to periodic boundary conditions

$$u(x+L,t) = u(x,t).$$

4.1 Analytical solution

As the solution is periodic in space at every instant t, it can be expressed as a Fourier-series

$$u(x,t) = \sum_{n=-\infty}^{+\infty} c_n(t)\exp\left(ik_n x\right) \tag{2}$$

with wavenumbers $k_n = 2\pi n/L$ and time-dependent coefficients $c_n(t)$ that ensure spatial periodicity at all times. This can be derived formally by separation of variables. Inserting the Fourier series into the equation gives

$$\sum_n \left(\dot{c}_n(t) + i\left(\left(1+\pi^2\right)k_n - k_n^3\right)c_n(t)\right)\exp\left(ik_n x\right) = 0.$$

Due to orthogonality of the Fourier basis functions $\exp(ik_n x)$, this can only vanish if all coefficients separately vanish, so

$$c_n(t) = c_n(0)\exp\left(-i\left(\left(1+\pi^2\right)k_n - k_n^3\right)t\right). \tag{3}$$

The solution has the property that the L_2-norm $\|u(x,t)\|_2 = (\frac{1}{2}\int_{-L/2}^{+L/2}|u(x,t)|^2\,\mathrm{d}x)^{1/2}$ is preserved over time. To understand this, first note carefully how the first order time derivative combined with spatial derivatives of odd order produced coefficients of variable phase, but constant magnitude $|c_n(t)|^2 = |c_n(0)|^2$. This means that each Fourier mode is undamped in time, so

$$\begin{aligned}
\int_{-1}^{+1}\mathrm{d}x\left|u(x,t)\right|^2 &= \int_{-L/2}^{+L/2}\sum_{m,n} c_m(t)c_n^*(t)\exp\left(ik_m x\right)\exp\left(-ik_n x\right)\mathrm{d}x \\
&= \sum_{m,n} c_m(t)c_n^*(t)\underbrace{\int_{-L/2}^{+L/2}\exp\left(i(k_m-k_n)x\right)}_{L\delta_{mn}\,\mathrm{d}x} \\
&= L\sum_n \left|c_n(t)\right|^2 \\
&= L\sum_n \left|c_n(0)\right|^2 \\
&= \int_{-L/2}^{+L/2}\left|u(x,0)\right|^2\mathrm{d}x.
\end{aligned} \tag{4}$$

4.2 Numerical solution method

GRID FIGURE?? x_0 to x_{M-1} makes modulo arithmetic intuitive.

We will solve the Korteweg-De Vries equation with the central finite differences

$$\frac{\partial u}{\partial x} = \frac{\delta}{2h} + O(h^2) = \frac{u^n_{m+1} - u^n_{m-1}}{2h} + O(h^2)$$
$$\frac{\partial^3 u}{\partial x^3} = \frac{\delta^3}{(2h)^3} + O(h^2) = \frac{u^n_{m+3} - 3u^n_{m+1} + 3u^n_{m-1} - u^n_{m-3}}{8h^3} + O(h^2)$$

in space, and doing the integration of $u_t = F(u_x, u_{xxx})$ in time using both the Euler method and the Crank-Nicholson method. They can be written simultaneously by inserting $\theta = 0$ and $\theta = 1/2$, respectively, in the Theta method

$$\frac{U^{n+1}_m - U^n_m}{\Delta t} = (1-\theta)F(U^n_m) + \theta F(U^{n+1}_m),$$

where all spatial discretization is contained in

$$\begin{aligned} F(U^n_m) &= -\left(1+\pi^2\right)\frac{\delta}{2h}U^n_m - \frac{\delta^3}{(2h)^3}U^n_m \\ &= -\left(1+\pi^2\right)\frac{u^n_{m+1} - u^n_{m-1}}{2h} - \frac{u^n_{m+3} - 3u^n_{m+1} + 3u^n_{m-1} - u^n_{m-3}}{8h^3} \end{aligned}$$

This results in the system of equations

$$\left(1 - \theta k\left(-\left(1+\pi^2\right)\frac{\delta}{2h} - \frac{\delta^3}{(2h)^3}\right)\right)U^{n+1}_m = \left(1 - (1-\theta)\,k\left(-\left(1+\pi^2\right)\frac{\delta}{2h} - \frac{\delta^3}{(2h)^3}\right)\right)U^n_m \tag{5}$$

for the unknown values $U^{n+1}_0, \ldots, U^{n+1}_{M-1}$ at the next time step in terms of the known values $U^n_0, \ldots, U^n_{M-1}$ at the current time step. In matrix form, the system can be written

$$(I - \theta k A)\,U^{n+1} = \left(I - (1-\theta)\,kA\right)U^n, \tag{6}$$

where $U^n = \begin{bmatrix} U^n_0 & \ldots & U^n_{M-1} \end{bmatrix}^T$ and $A =$

$$\frac{-1}{2h}\begin{bmatrix} 0 & +1 & & & & & & & -1 \\ -1 & 0 & +1 & & & & & & \\ & -1 & 0 & +1 & & & & & \\ & & -1 & 0 & +1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -1 & 0 & +1 & & \\ & & & & & -1 & 0 & +1 & \\ & & & & & & -1 & 0 & +1 \\ +1 & & & & & & & -1 & 0 \end{bmatrix} - \frac{1+\pi^2}{h^3}\begin{bmatrix} 0 & -3 & 0 & +1 & & & -1 & 0 & +3 \\ +3 & 0 & -3 & 0 & +1 & & & -1 & 0 \\ 0 & +3 & 0 & -3 & 0 & +1 & & & -1 \\ -1 & 0 & +3 & 0 & -3 & 0 & +1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 0 & +3 & 0 & -3 & 0 & +1 \\ +1 & & & -1 & 0 & +3 & 0 & -3 & 0 \\ 0 & +1 & & & -1 & 0 & +3 & 0 & -3 \\ -3 & 0 & +1 & & & -1 & 0 & +3 & 0 \end{bmatrix}.$$

where we have imposed periodic boundary conditions $U^n_m = U^n_{m+M}$ by simply wrapping the spatial derivative stencils around the matrix.

We then solve the system by preparing U^0 from the initial condition and solve equation (6) repeatedly to step forward in time. Note that with the constant time step k, all matrices in equation (6) are constant in time, and the process of solving the system many times can be accelerated by for example LU-factorizing the matrix on the left side.

Next, we test our numerical solution on the problem defined by the initial condition $u(x,0) = \sin(\pi x)$ on $x \in [-1, +1]$ with $L = 2$. The Fourier series of the analytical solution then has nonzero coefficients

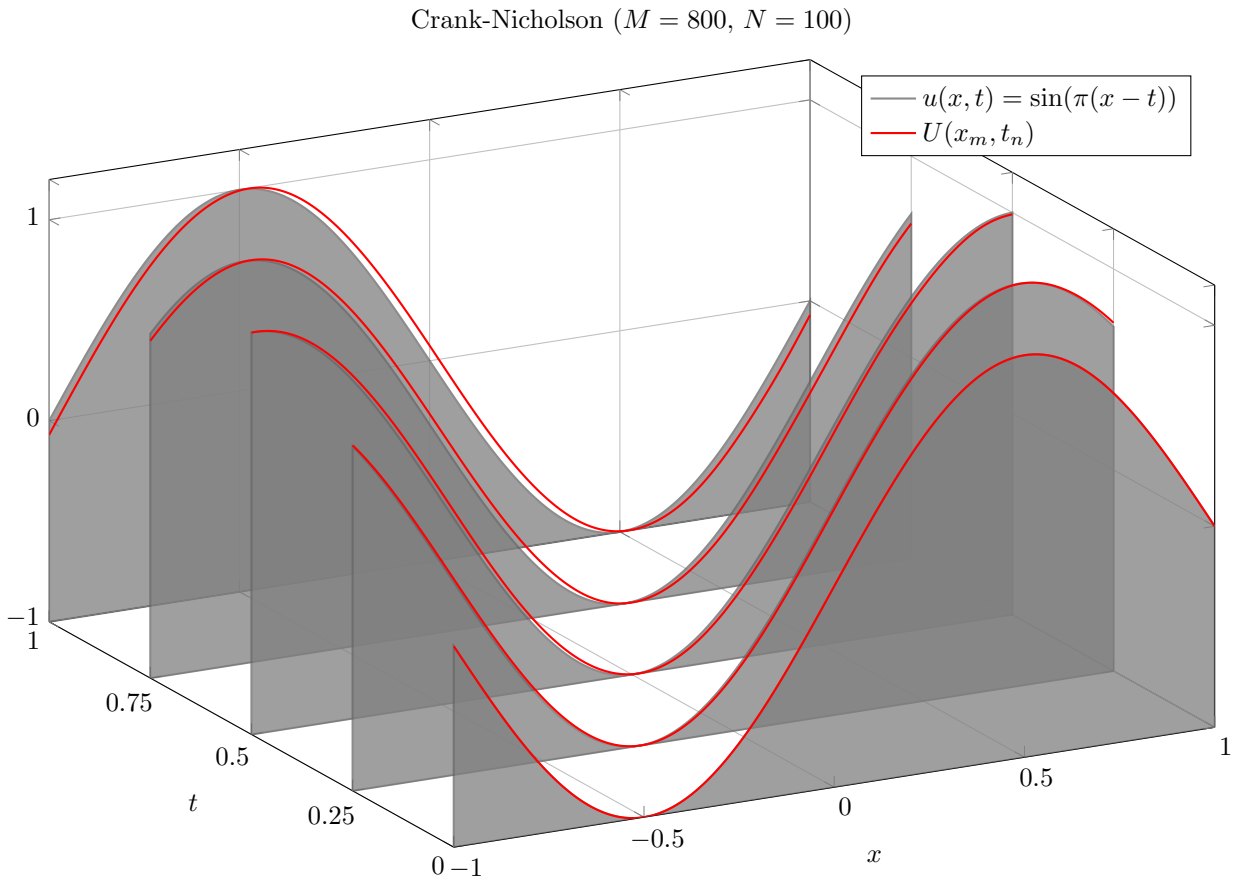

Crank-Nicholson ($M = 800$, $N = 100$)
$u(x,t) = \sin(\pi(x-t))$
$U(x_m, t_n)$
1
0
−1
1
0.75
0.5
0.25
0
t
−1
−0.5
0
0.5
1
x

Figure 5

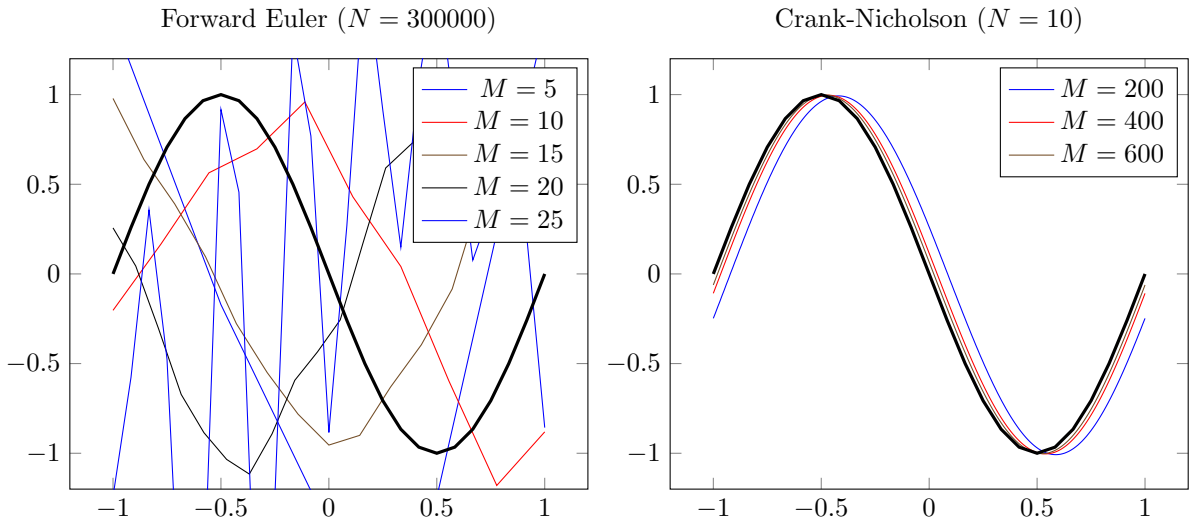


Figure 6

$c_{\pm 1}(0) = \pm 1/2i$ and wavenumbers $k_{\pm 1} = \pm\pi$, which give rise to the analytical solution $u(x,t) = \sin(\pi(x-t))$. As shown in figure 5, the solution represents a sine wave traveling with velocity 1 to the right.

In figure 6, we compare snapshots of the numerical solution at $t = 1$ from the Euler method and the Crank-Nicholson method. Note that the Crank-Nicholson method approaches the exact solution with as little as $N = 10$ time steps and a few hundred spatial grid points M, while the Euler method produces garbage with as many as $N = 300000$ time steps and seems to become unstable for very few spatial grid points. Figure 7 supports our suspicions, showing the stable second order (?) nature of the Crank-Nicholson method and the unstability of the Euler method.

4.3 Stability analysis

Motivated by the above examples of the Euler method and the Crank-Nicholson method, we perform a Von Neumann analysis of their stability. Just like the exact solution, the numerical solution is subject to periodic boundary conditions in space and can therefore be expanded in a Fourier series

$$U_m^n = U(x_m, t_n) = \sum_l C_l^n \exp\left(ik_l x_m\right). \tag{7}$$

Consider now only a single Fourier mode $C_l^n \exp(ik_l x_m)$ in the series. Inserting it into equation (5), dividing by $\exp(ik_l x_m)$ and expanding exponentials using Euler's identity gives

$$\frac{C_l^{n+1} - C_l^n}{\Delta t} = i\left(\left(1-\theta\right)C_l^n + \theta C_l^{n+1}\right), \quad \text{where } f(k_l) = \left(-\left(1+\pi^2\right)\frac{\sin(k_l h)}{h} - \frac{\sin^3(k_l h)}{h^3}\right).$$

Now look at the amplification factor $G_l = C_l^{n+1}/C_l^n$ of Fourier mode l over one time step. With $\theta = 1/2$, the Crank-Nicholson method gives

$$G_l = \frac{1 + i\Delta t f(x)/2}{1 - i\Delta t f(x)/2} \quad \Longrightarrow \quad |G_l| = 1.$$

The amplitude of all Fourier modes is thus preserved over time independently of Δt and Δx, and we say the Crank-Nicholson method is **unconditionally stable**.

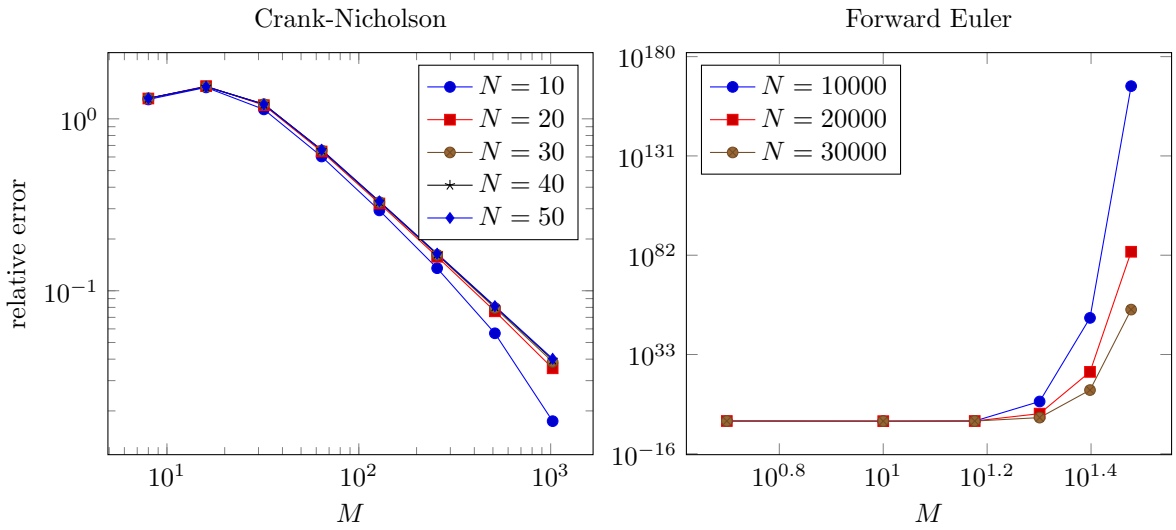


Figure 7: Convergence plot at $t = 1$, comparison between Crank-Nicholson and Forward Euler

The Euler method has $\theta = 0$ and gives

$$G_l = 1 + i\Delta t f(x) \quad \Longrightarrow \quad |G_l| = \sqrt{1 + \Delta t^2 f(k_l)^2}.$$

Since $\left|\sin(k_l h)\right| \leq 1$ for all k_l, we can bound $f(k_l)$ by

$$\left|f(k_l)\right| \leq \frac{(1+\pi^2)}{\Delta x} + \frac{1}{\Delta x^3} = \frac{1}{\Delta x^3}\left((1+\pi^2)\Delta x^2 + 1\right) \leq \frac{1}{\Delta x^3}\left((1+\pi^2)L^2 + 1\right).$$

Then $|G_l| = \sqrt{1 + O(\Delta t^2/\Delta x^6)}$. The Von Neumann stability criterion $|G_l| \leq 1 + O(\Delta t)$ [1] is attained only with $\Delta t \geq O(\Delta x^6)$, corresponding to *extremely small* time steps. Thus, while the Euler method in theory is **conditionally stable**, it is unstable for all practical time steps. Other methods are far superior, as they allow for both a much greater resolution of the spatial grid in addition to larger time steps.

Theta method:

$$\frac{u_m^{n+1} - u_m^n}{k} = (1-\theta)F(u^n) + \theta F(u^{n+1}) \tag{8}$$

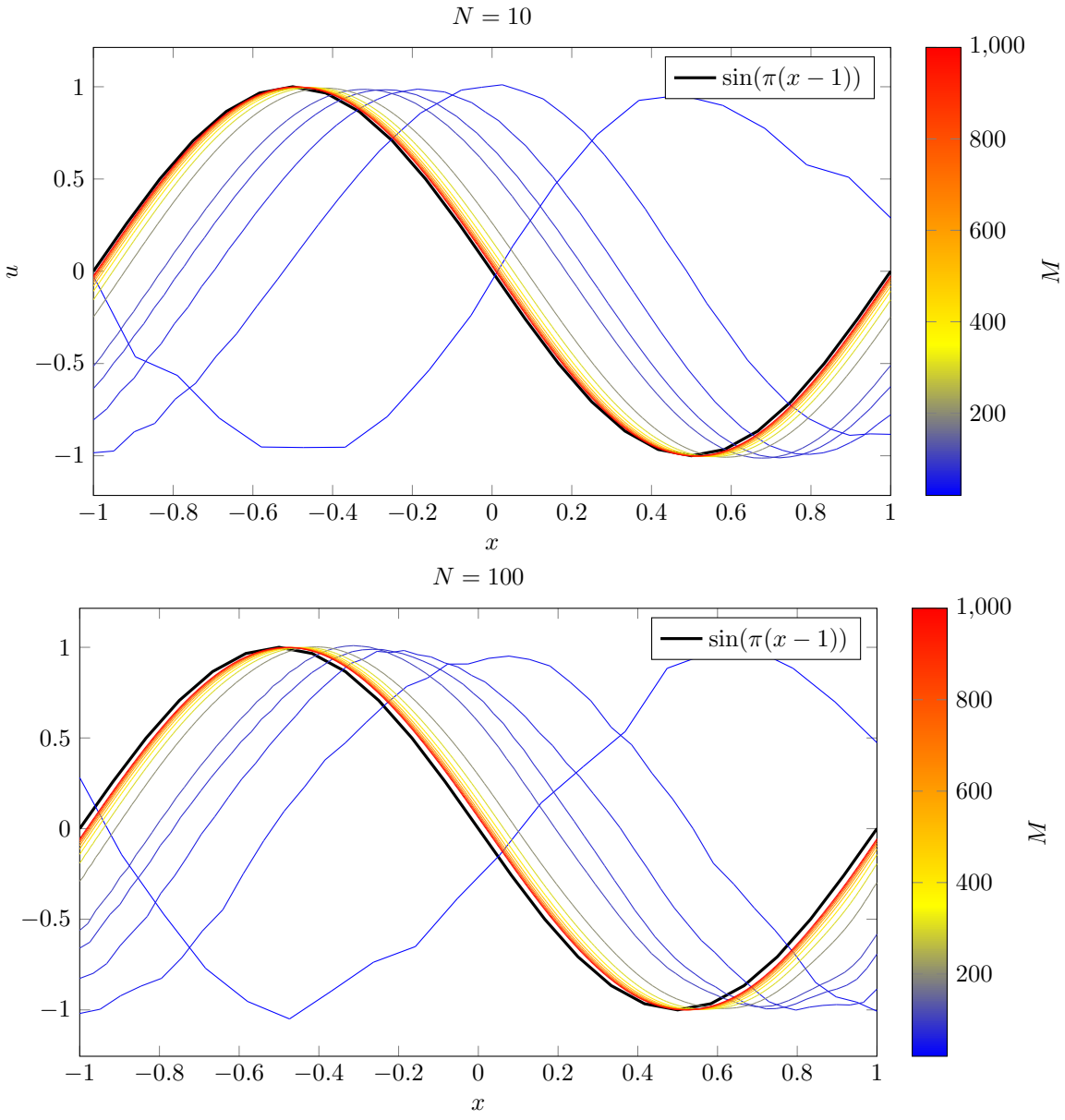
where

$$F(u^n) = -(1+\pi^2)\frac{u_{m+1}^n - u_{m-1}^n}{2h} - \frac{u_{m+3}^n - 3u_{m+1}^n + 3u_{m-1}^n - u_{m-3}^n}{8h^3} \tag{9}$$

$\theta = 0$ is forward Euler, $\theta = 1/2$ is Crank-Nicholson, $\theta = 1$ is backward Euler.

References

[1] Brynjulf Owren: *TMA4212 Numerical solution of partial differential equations with finite difference methods* (2017) [`http://www.math.ntnu.no/emner/TMA4212/2020v/notes/master.pdf`]


$N = 10$
$\sin(\pi(x-1))$
u
x
M
1,000
800
600
400
200
$N = 100$
$\sin(\pi(x-1))$
u
x
M
1,000
800
600
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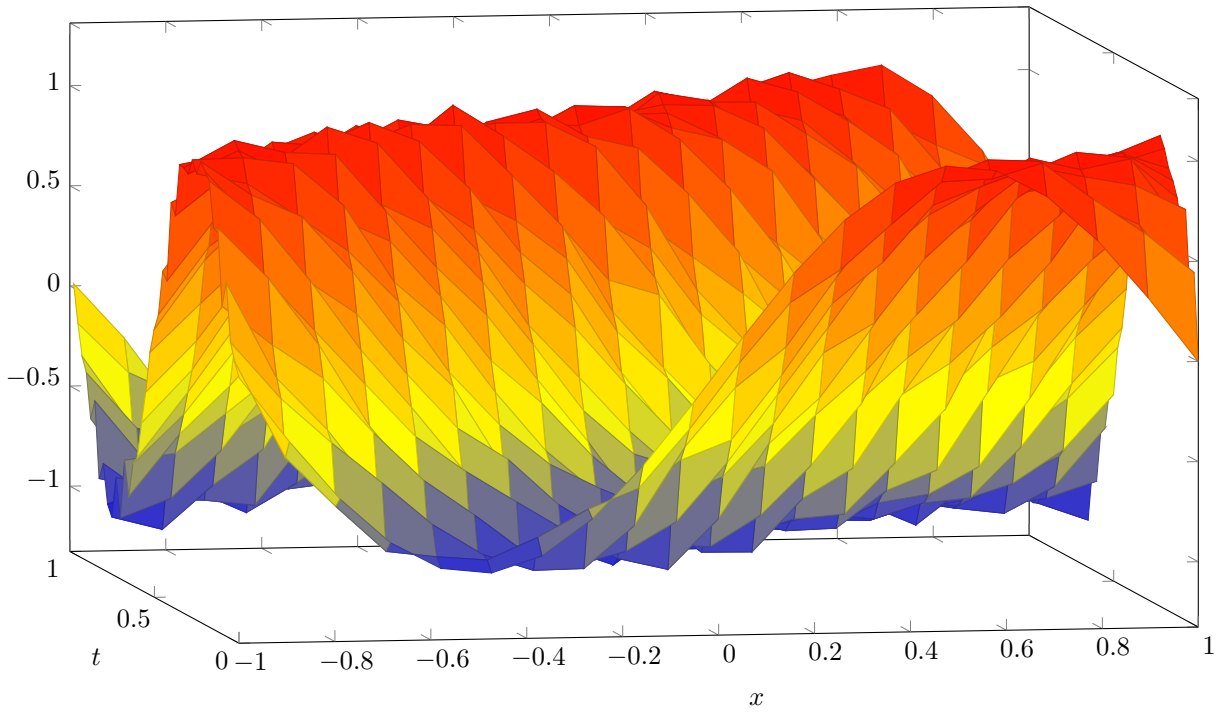
200


1
0.5
0
−0.5
−1
1
0.5
t
0
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x