1 Exercise 1

Consider the one-dimensional Laplace equation

$$u_{xx} = f(x) \qquad (0 < x < 1)$$

with the source term $f(x) = x + \cos(2\pi x)$, subject to different kinds of boundary conditions

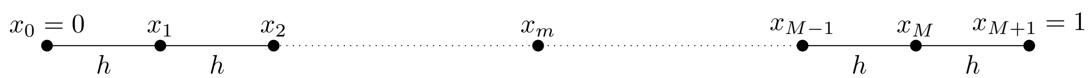
$$\begin{aligned}
u(0) &= a, & u(1) &= b & &\text{(Dirichlet-Dirichlet)}\\
u_x(0) &= a, & u_x(1) &= b & &\text{(Neumann-Neumann)}\\
u(0) &= a, & u_x(1) &= b & &\text{(Dirichlet-Neumann).}
\end{aligned}$$

An analytical solution is

$$u(x) = C_1 + C_2 x + \frac{1}{6}x^3 - \frac{1}{4\pi^2}\cos(2\pi x)),$$

where the constants C_1 and C_2 are determined from two boundary conditions. Note that when Neumann-Neumann boundary conditions are imposed, C_1 is undetermined and so the solution is determined only up to a constant.

Grid:



Inner points:

$$\frac{U_{m+1} - 2U_m + U_{m-1}}{h^2} = f(x_m)$$

Dirichlet, left:

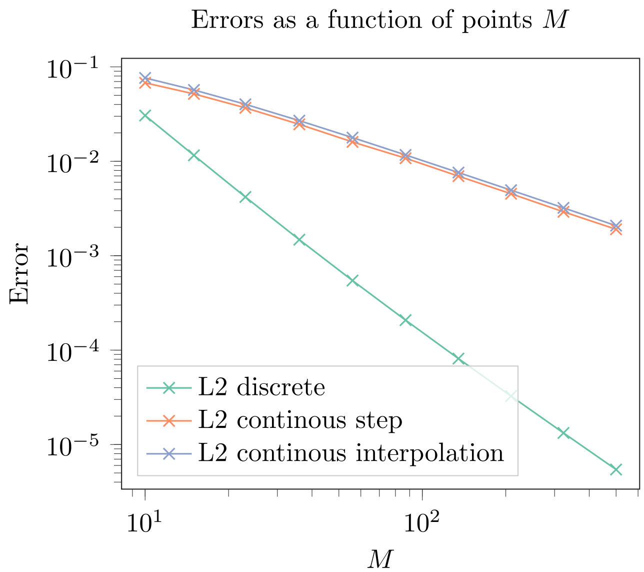
$$u(0) = U_0$$

Von-Neumann, left, 2nd order:

$$u_x(0) \approx -\frac{(3/2)U_0 - 2U_1 + (1/2)U_2}{h}$$

Discretized equation:

$$\begin{bmatrix}
-2/h^2 & +1/h^2 & 0 & \cdots & 0\\
+1/h^2 & -2/h^2 & +1/h^2 & \ddots & \vdots\\
0 & \ddots & \ddots & \ddots & 0\\
\vdots & \ddots & +1/h^2 & -2/h^2 & +1/h^2\\
0 & \cdots & -1/2h & +2/h & -3/2h
\end{bmatrix}
\begin{bmatrix}
U_1\\ U_2\\ \vdots\\ U_M\\ U_{M+1}
\end{bmatrix}
=
\begin{bmatrix}
f(x_1) - \alpha/h^2\\ f(x_2)\\ \vdots\\ f(x_M)\\ \sigma
\end{bmatrix}$$


Errors as a function of points M
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
Error
L2 discrete
L2 continous step
L2 continous interpolation
10^{1}
10^{2}
M

2 Exercise 2