Separation of differently sized particles using biased Brownian motion in flashing ratchet potential

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Abstract

We have here successfully simulated a DNA transport experiment. In the original experiment, micro machined silicon-chip devices were used to generate a flashing ratchet potential, used to transport and separate differently sized DNA molecules. Here, we the same results were found numerically. We found the average drift velocity in for the DNA with only 8% relative error compared to the experiment.

1 Introduction

In bio-research one often wants to separate DNA of different size. One such technique is for example centrifugation. A rather different approach to achieving the same result is leveraging that different particles diffuse at different speeds. In one experiment[2] one successfully used a flashing ratchet potential to separate differently sized DNA molecules. Micro machined silicon chips were used to create the potential. Here we model the same experiment numerically, using the Langevin approach.

This report shows how well this computer model performs in recreating the original experiment and the results found there. We will use the simulation model to consider how well this method would perform at separating differently sized particles.

2 Theory and method

The particles are placed in a flashing ratchet potential, see figure 1. That is, the potential is a non-symmetric triangle wave. The flashing is such that for a period τ, the potential is turned on for 3/4th of the period, and then turned off. Let the potential be described as

$$U(x,t) = U_r(x) f(t), \tag{1a}$$

where

$$U_r(x) = \begin{cases} \frac{x}{\alpha L}\Delta U & , 0 \leq x < \alpha L \\ \frac{L-x}{L(1-\alpha)} & , \alpha L \leq x < L \end{cases}, \quad f(t) = \begin{cases} 0 & , 0 \leq t < \frac{3\tau}{4} \\ 1 & , \frac{3\tau}{4} \leq t < \tau \end{cases}. \tag{1b}$$

Here, α is a factor determining the asymmetry of the potential, ΔU is the strength of the potential, and L is the length of one wave or "tooth" of the potential.

Assume that each particles may be treated independently. The forces at play are those from the potential gradient, from friction, and the random walk term. From Newton's second law

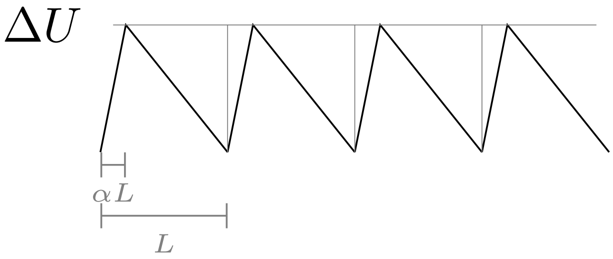


Figure 1: Ratchet potential.

this gives

$$m_i \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = -\frac{\partial U}{\partial x}(x_i, t) - \gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} + \xi(t), \tag{2}$$

where $\gamma_i = 6\pi\eta r_i$ is a friction constant and ξ is a stochastic variable modeling the random walk. The stochastic variable has a distribution such that

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = 2\gamma_i k_B T \delta(t - t'), \tag{3}$$

where k_B is the Boltzmann constant and T the temperature [1].

Removing the term with the second derivative, and rewriting

$$\gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{\partial U}{\partial x}(x_i, t) + \xi(t). \tag{4}$$

Due to the piecewise linear nature of the potential, the forward Euler scheme is sufficient. One thus gets [1]

$$x_{n+1} = x_n - \frac{1}{\gamma_i}\frac{\partial U}{\partial x}(x_n, t_n)\delta t + \sqrt{\frac{2k_B T \delta t}{\gamma_i}}\hat{\xi}_n. \tag{5}$$

The time step δt must be chosen such that

$$|x_{n+1} - x_n| \ll \alpha L.$$

Notice that αL is the shortest piece of constant force from the potential, and this condition therefore ensures that the particle does not "skip" any parts of the potential. Inserting this into (5) yields

$$\frac{1}{\gamma_i} \max \left| \frac{\partial U}{\partial x} \right| \delta t + 4\sqrt{\frac{2k_B T \delta t}{\gamma_i}} \ll \alpha L, \tag{6}$$

where the fact that $|\hat{\xi}_n| < 4$ with 99.99% probability was used.

We will use the Boltzmann distribution and diffusion equation to check numeric validity. During the trapping phase, the particles should follow the Boltzmann distribution for the potential. When the potential is off, the particles should diffuse freely.

From [1] the Boltzmann distribution is given as

$$p(U) = \frac{e^{-\frac{U}{k_B T}}}{k_B T\left(1 - e^{-\frac{\Delta U}{k_B T}}\right)}. \tag{7}$$

and that the diffusion equation gives a particle distribution with a standard deviation proportional to the square root of time.

Introducing dimensionless quantities

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \omega t, \quad \omega = \frac{\Delta U}{\gamma_i L^2}, \quad \hat{U}(\hat{x}, \hat{t}) = \frac{U(x,t)}{\Delta U}, \quad \hat{D} = \frac{k_B T}{\Delta U} \tag{8}$$

the Euler scheme (5) may be written

$$\hat{x}_{n+1} = \hat{x}_n - \frac{\partial \hat{U}}{\partial \hat{x}}(\hat{x}, \hat{t})\delta\hat{t} + \sqrt{2\hat{D}\delta\hat{t}}\hat{\xi}_n. \tag{9}$$

Notice that the particles radius only appears in ω, through γ. Changing particle size thus corresponds to simply scaling the time.

3 Results and discussion

In the following, the values from the original experiment were used, with thermal energy of 26 meV, potential $\Delta U = 80\,\mathrm{eV}$, $r = 12\,\mathrm{nm}$, potential periodicity $L = 20\,\mathrm{\mu m}$, and a diffusion constant of $1.8 \times 10^{-7}\,\mathrm{cm^2\,s^{-1}}$. Note that the radius r was not given explicitly in the original experiment, but deduced from other quantities.

Firstly, to check the validity of the model a simulation was run with a constant ratchet potential. Figure 4 shows the trajectory of one particle. In figure 3 the distribution of particles that have been allowed to diffuse in the time independent potential is shown for two different values of the diffusion constant, $D = 0.1$ and $D = 0.2$. Both figures indicate that the model performs well. Note however, that the potential used in figure 4 was not sufficient to completely trap the particle, and it did sometimes escape into other wells. This was done in order to get as much movement as possible in the figure. The distribution does follow the Boltzmann distribution during the trapping phase. From the trajectory one can see that the particle is biased towards the negative side of the potential well, where the gradient is smaller. In figure 2 the standard deviation σ of an ensemble of particles is shown. As expected from theory, $\sigma \propto \sqrt{t}$.

What remains is determining an optimal flashing period, τ_{op}, that maximizes the drift velocity and use this flashing period on two particles of different size, to see if one is able to separate them. Figure 5 shows the average drift distance using different flashing periods. For each flashing period, 500 realizations were made and the particles were simulated until $\hat{t} = 4000$. With a particle of $r = 12\,\mathrm{nm}$ the optimal flashing period is found to be 0.50s, or 2.0 Hz, giving a drift velocity of $4.3\,\mathrm{\mu m\,s^{-1}}$. With a 1.4s period, the period used in the original experiment, the particle drifted $4.6L$ per 20 cycle, or $3.3\,\mathrm{\mu m\,s^{-1}}$. This is in very good correspondence with the original experiment, where the particles drifted $5L$ in 20 cycles.

Changing the size of the particle corresponds to simply changing the friction constant γ, which in turn, according to (8), corresponds to scaling the time. Considering a particle with radius $r_2 = 3r$ gives a new $\omega_2 = \omega/3$, thus

$$\hat{t} = \omega_2 t_2 \Rightarrow t_2 = 3\frac{\hat{t}}{\omega} = 3t.$$

Thus, using a flashing period optimized for the r_2 particle will only give a drift velocity of a third of that of the smaller particle. Moreover, when using the flashing period optimized for the r particle, this will be sub optimal for the r_2 particle. The optimal drift velocity, in dimensionless units, is according to figure 5 is $\tau = 70$. The bigger particle will experience a period of three times that, $\tau = 210$, which gives a total drift of $4.2L$. Remembering that one also has to divide by three, as the bigger particle has experienced three times as much time,

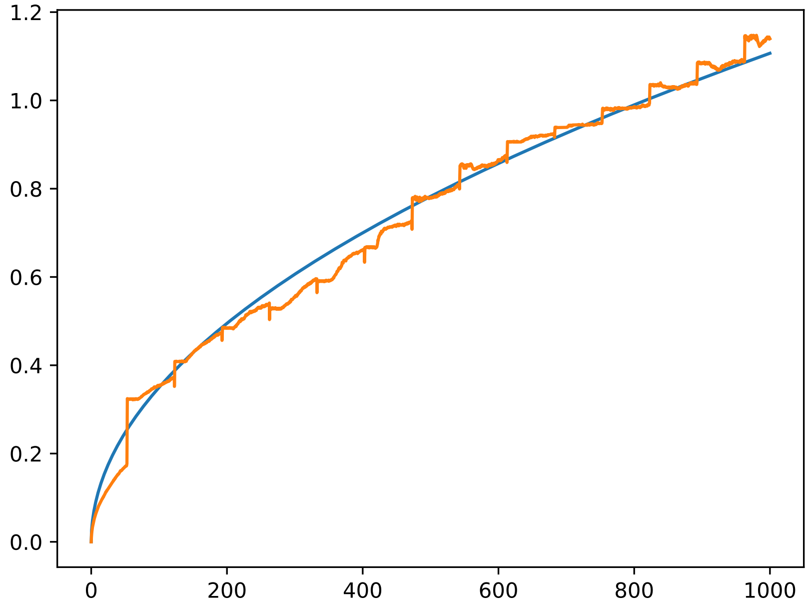
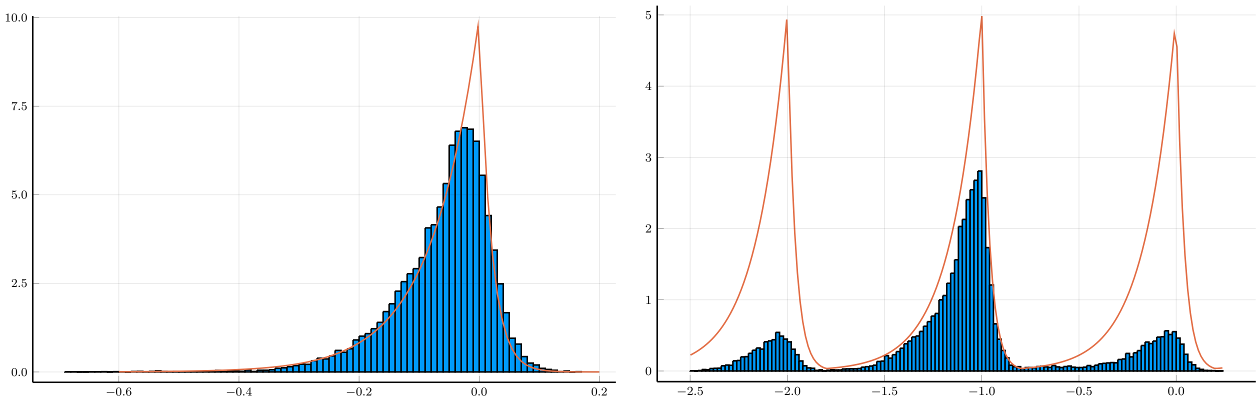


Figure 2: The standard deviation of an ensemble of particles, orange, and the square root of time, blue.

one gets $1.4L$, opposed to the smaller particle's $6.1L$. The smaller particle has a drift velocity of 4.4 that of the bigger particle. Figure 6 shows the particle density where the two types of particles have been simulated for the same length of time with the flashing period optimized for the smaller particles. From inspection, the calculated velocity difference of 4.4 is consistent with the difference in total travel for the two types of particles.

4 Conclusion

Using a simple Langevin approach, solved with Forward Euler, one is able to recreate the same findings numerically as was found in the original experiment. For the type of particle used in the experiment, the numerical approach found the optimal flashing period to be 0.50 s, giving



(a) $D = \frac{1}{10}$

(b) $D = \frac{1}{5}$

Figure 3: The simulated distribution in a time independent potential for $D = 0.1$ and $D = 0.2$. The orange lines show the theoretical Boltzmann distribution.

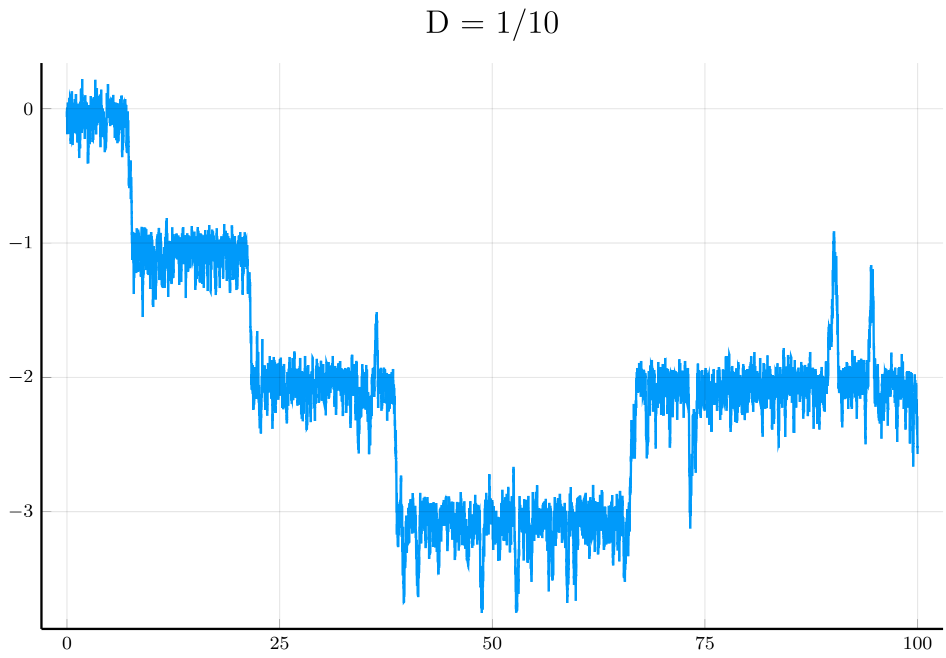


Figure 4: Particle trajectory in time independent potential. Notice that the position is not symmetric when trapped, but biased towards the negative end, due to the lower slope of the potential there.

a drift velocity of $4.3\,\mu\mathrm{m\,s^{-1}}$. With the flashing period used in the original experiment, we here found the drift velocity to be $4.6L$ per 20 cycle, in good correspondence with the original experiment's $5L$ per 20 cycle. We also showed that one is able to get good separation of particles of different sizes. With two particles, with one having three times the radius of the other, the smaller particle was found to have a drift velocity of 4.4 that of the bigger particle.

References

[1] J.P. Banon and I. Simonsen, *Assignment 2: Biased Brownian Motion: An Application to Particle Separation*, Department of Physics, NTNU, Norway, 2020.

[2] J. S. Bader, R. W. Hammond, S. A Henck, M. W. Deem, G. A. McDermott, J. M. Bustillo, J. W. Simpson, G. T. Mulhern, and J. M. Rothberg. *DNA transport by a micromachined Brownian ratchet device*. Proceedings of the National Academy of Sciences of the United States of America, 96 (23):13165–13169, Nov. 9, 1999.

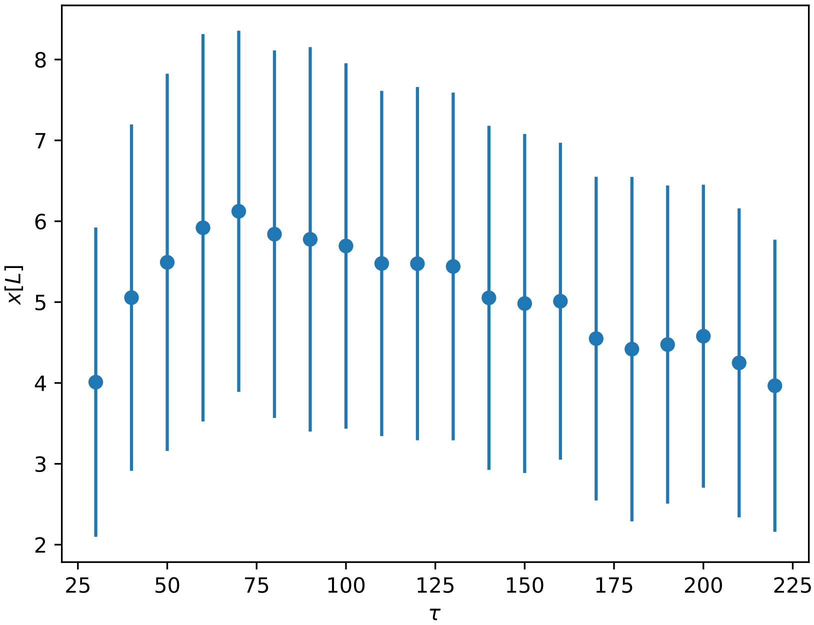


Figure 5: Total drifted distance x in units of periodicity length L for different flashing periods τ. Particles allowed to drift for 4000 time units. Quantities presented in dimensionless units in order to apply the results to different values of r.

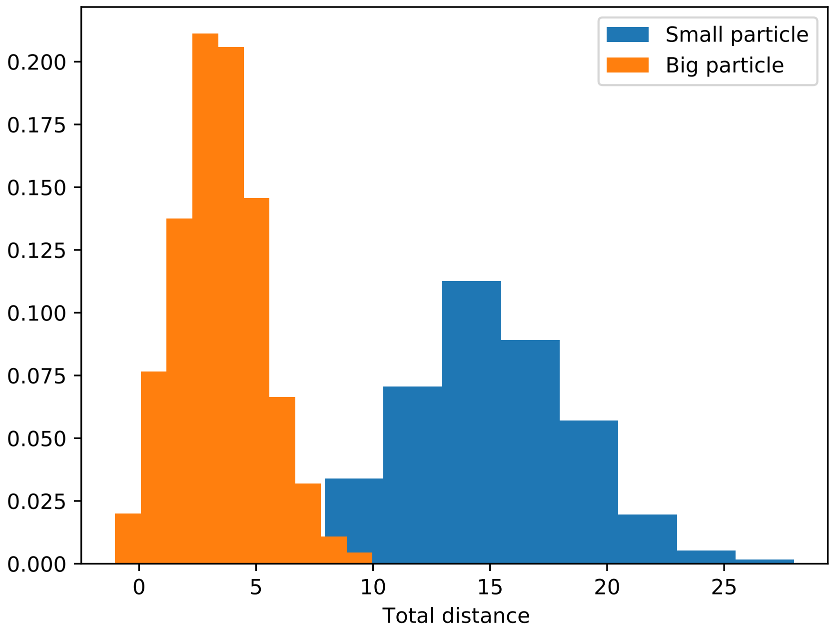


Figure 6: Separation of particles with different size. The big particles have three times the radius of the small particles.

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