EE1001
Foundations of Digital Techniques

Logic – Part 1

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Outline

1. Need for Logic
2. Validity and Soundness of Argument
    - 2.1. Validity of Argument
    - 2.2. Deductive Argument
    - 2.3. Soundness of Argument
3. Propositional Logic
    - 2.1. Propositions and Truth Tables
    - 2.2. Logical Connectives (NOT/AND/OR)
    - 2.3. Tautologies and Contradictions
4. Conditionals
    - 4.1. Conditional Statement (If-then)
    - 4.2. Necessary & Sufficient Conditions

Logic – Part 1

Logic – Part 2

5. From Proposition to Predicate
    - 5.1. Nine Inference Rules to Construct Valid Arguments
6. Predicate Logic
    - 6.1. Universal Quantifier (∀) and Essential Quantifier (∃)
    - 6.2. Negation of Quantification
    - 6.3. Nested Quantification

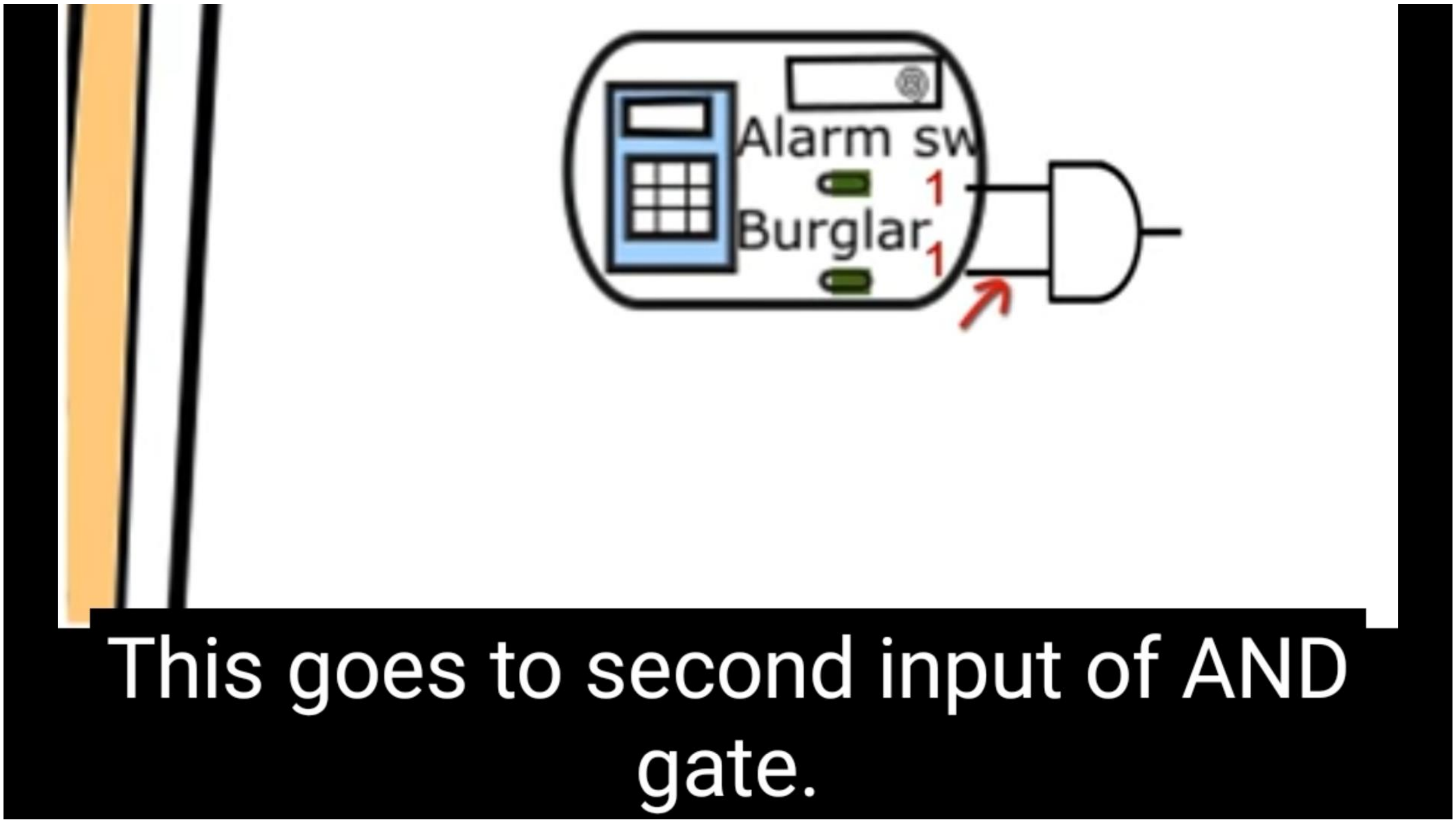
Class Intended Learning Outcomes (CILO)

- Understanding the validity and soundness of arguments in a logical way
- Identifying logic problems and solving them with logical methods.
- Designing and formulating logical arguments

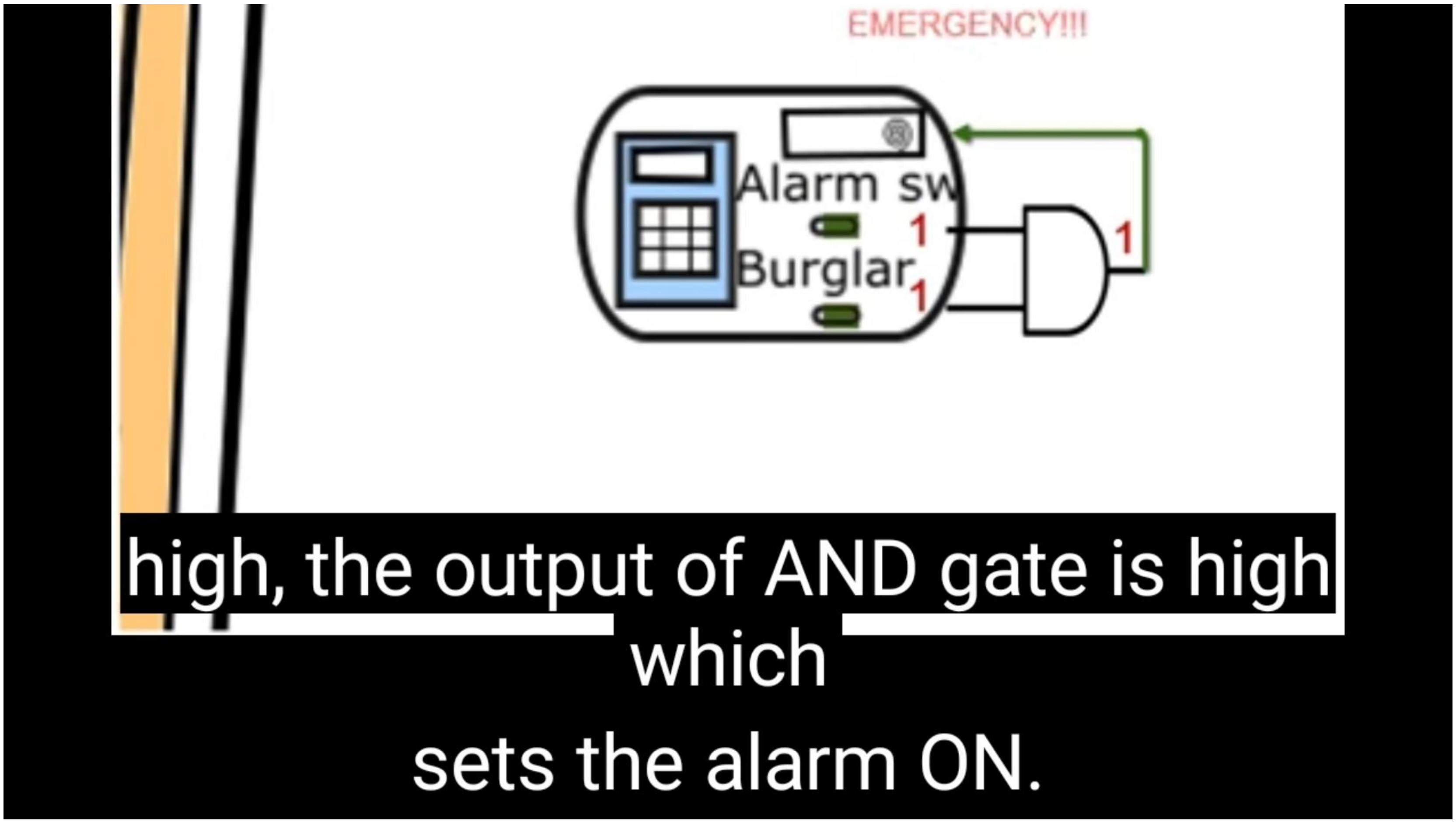
•Background

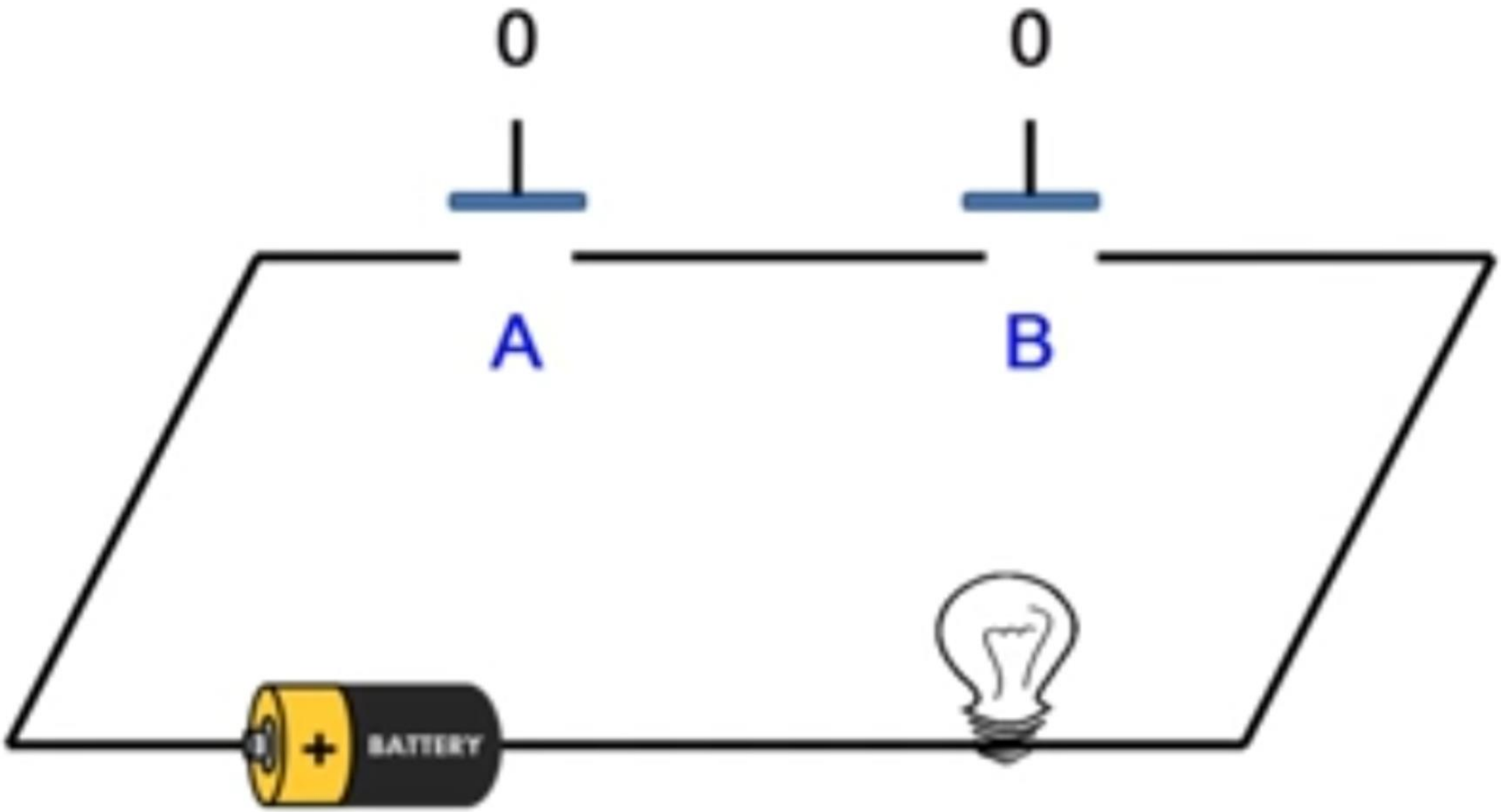
- **AND**

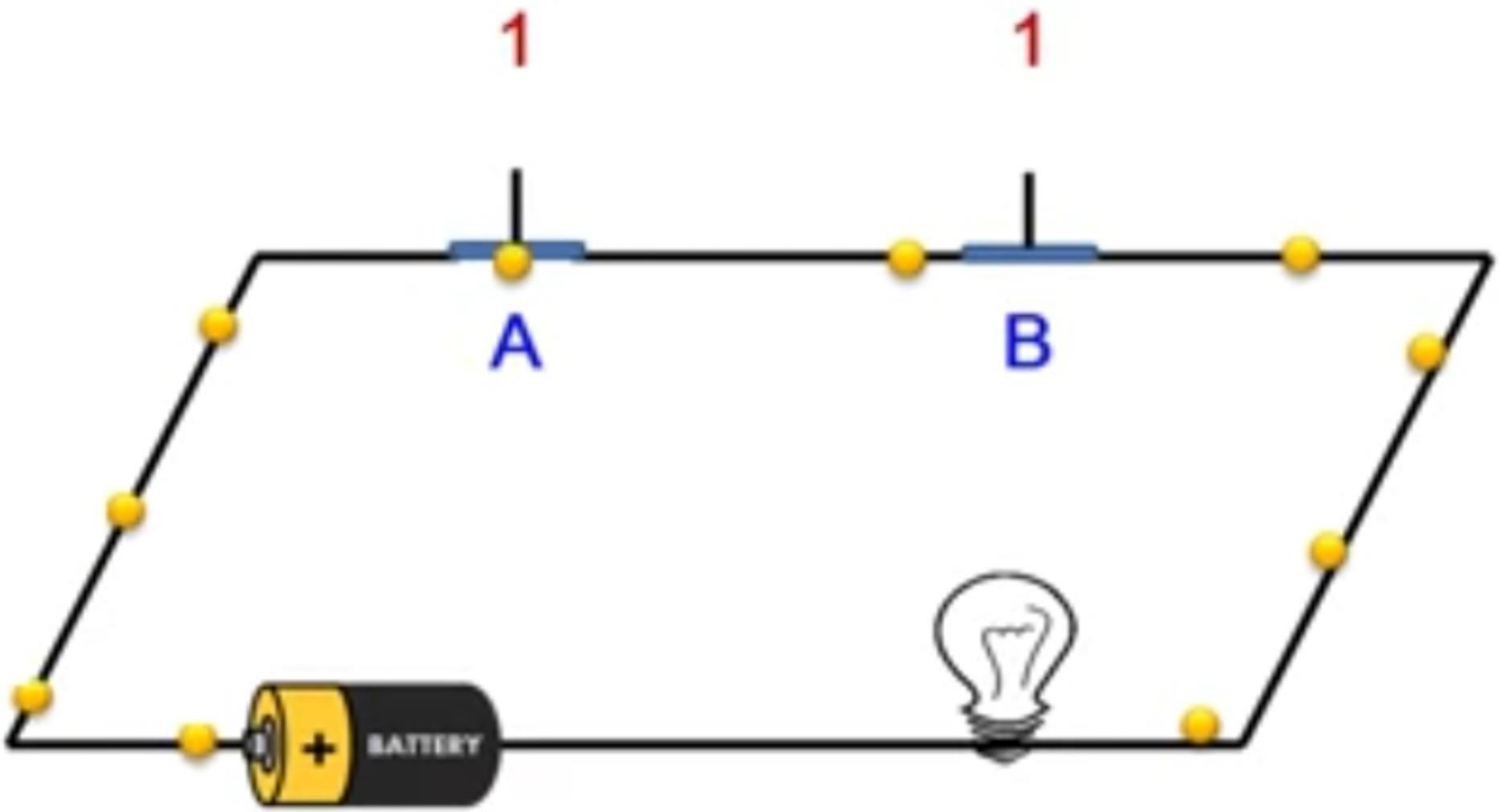

Alarm sw
Burglar

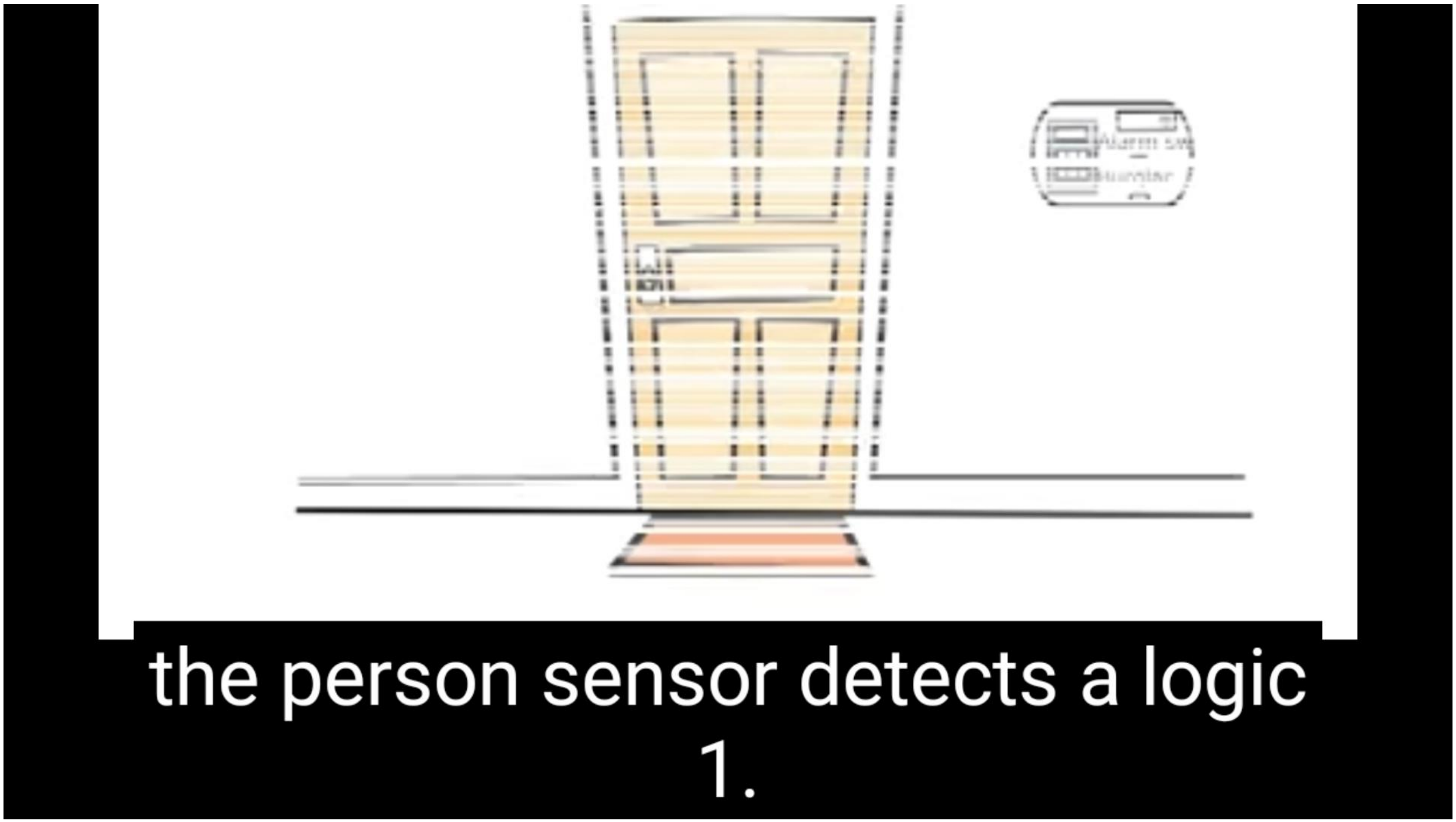

Alarm sw
1
Burglar
1
This goes to second input of AND gate.


Alarm sw
Burglar


EMERGENCY!!!
Alarm sw
1
Burglar
1
1
high, the output of AND gate is high
which
sets the alarm ON.

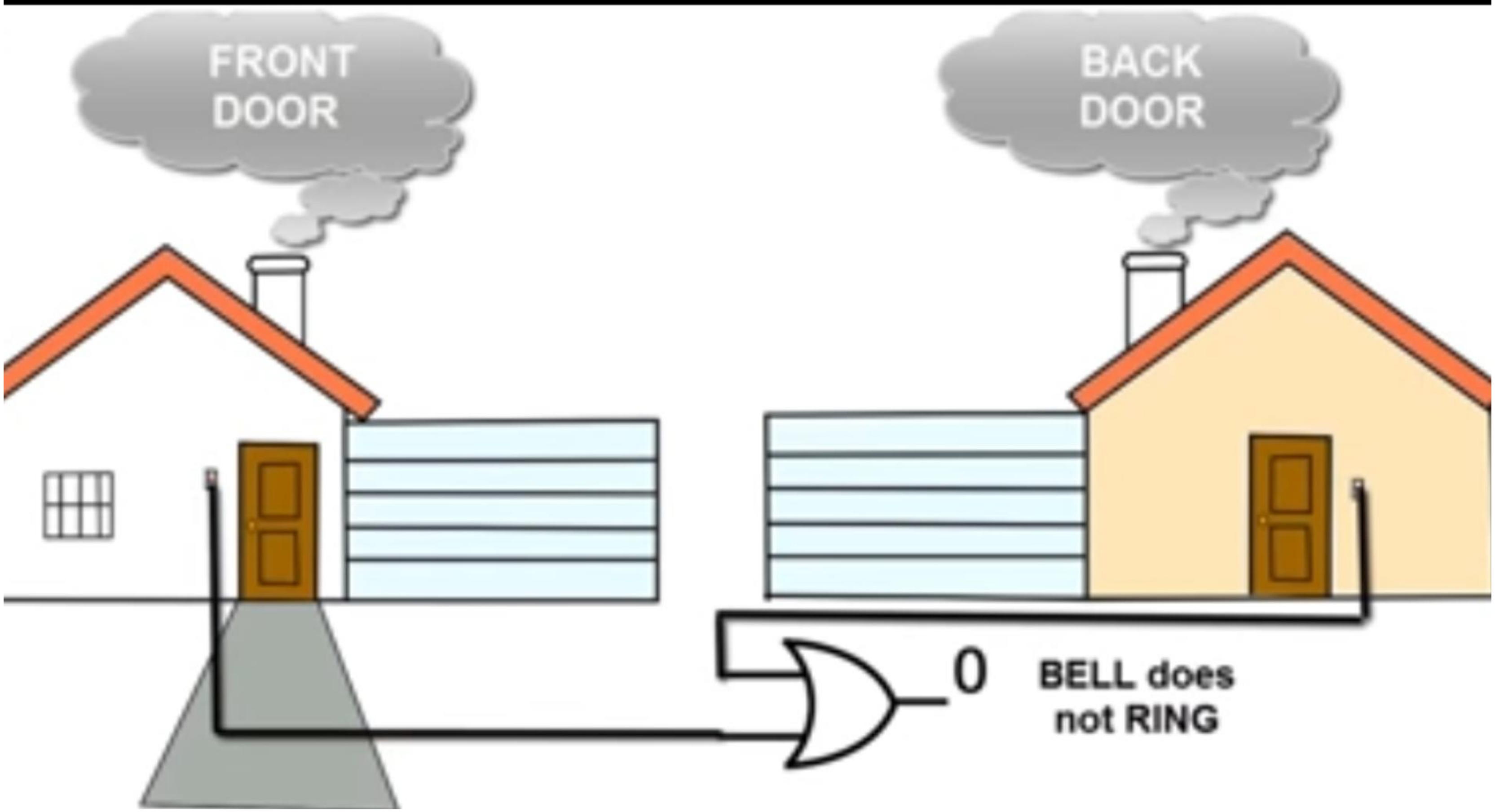

0
0
A
B
+
BATTERY

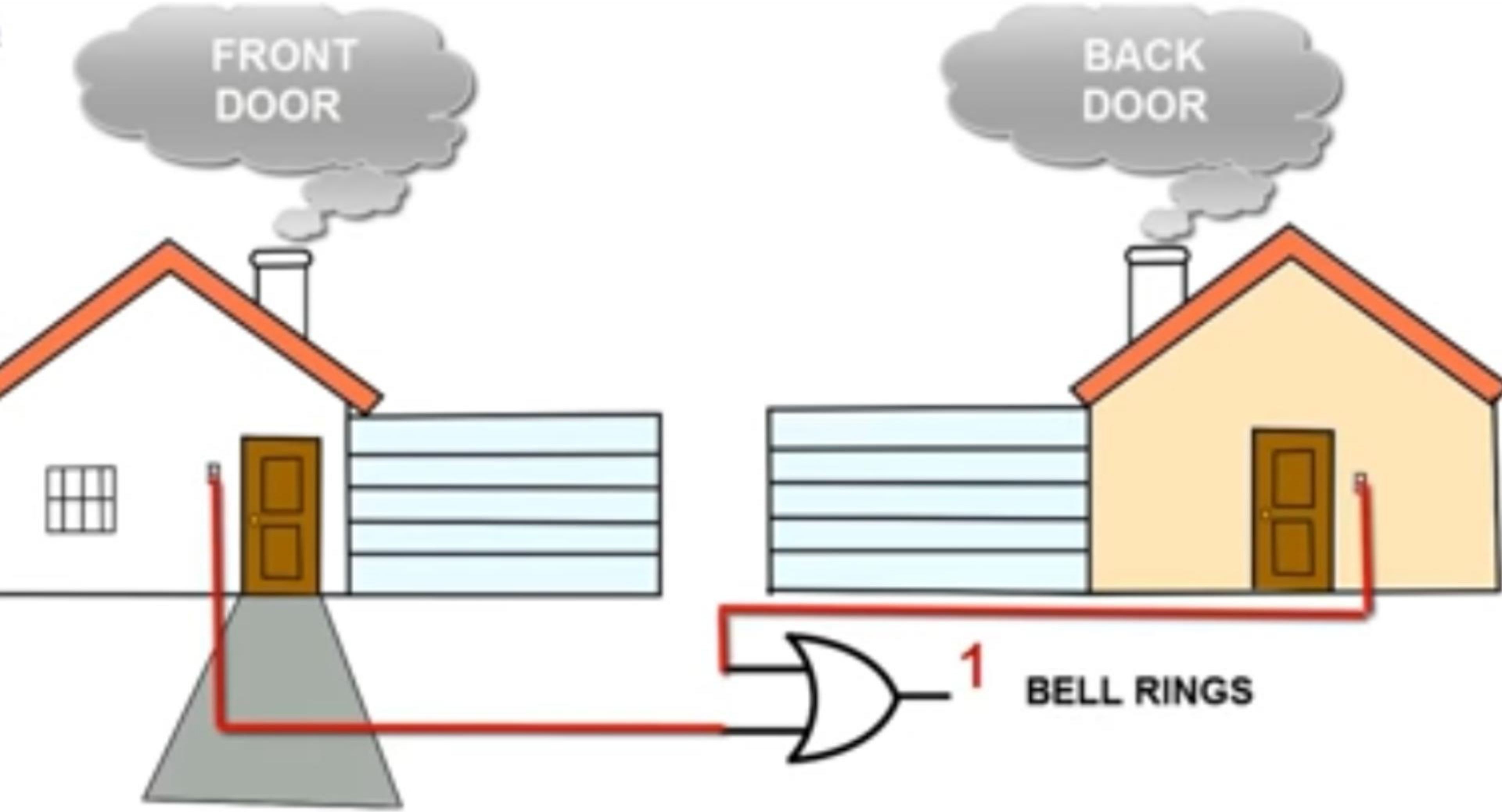

1
1
A
B
+
BATTERY

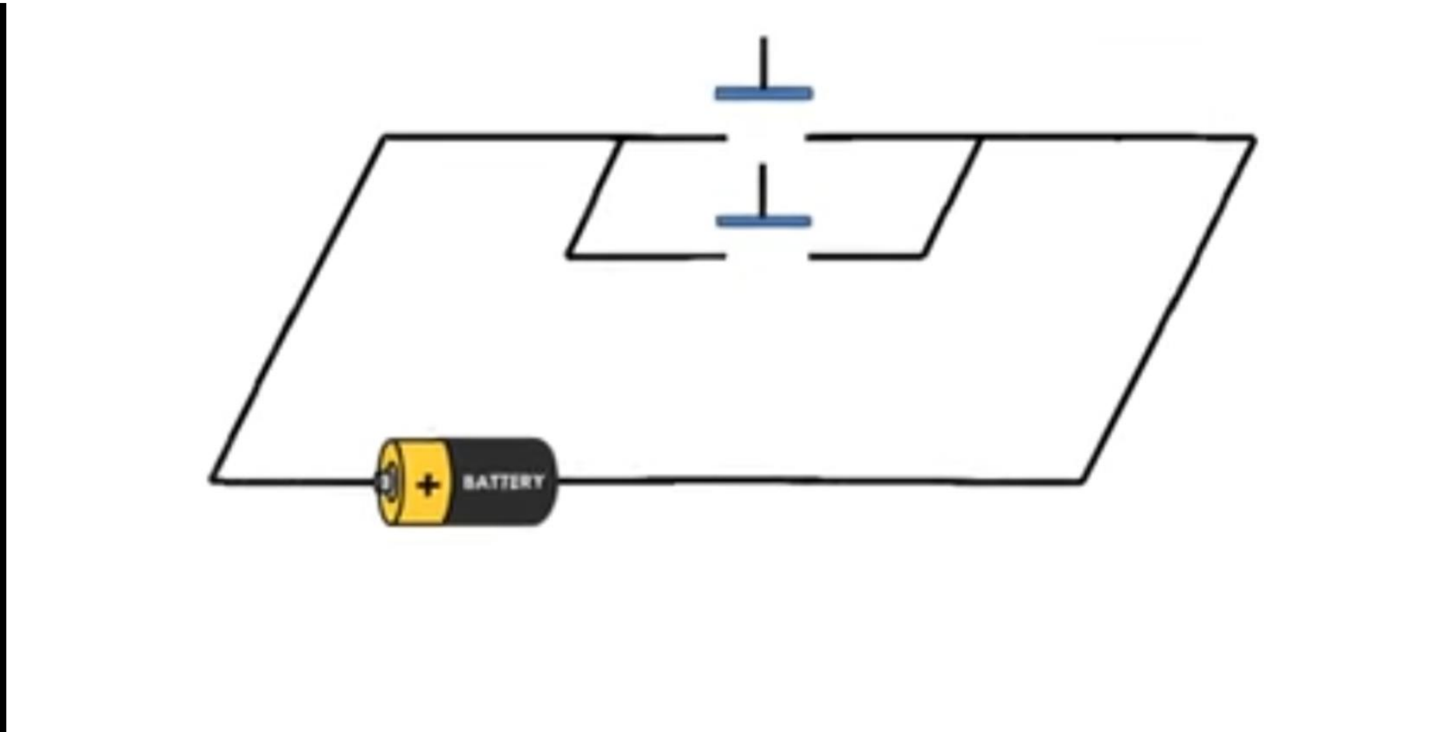

the person sensor detects a logic
1.

- **OR**

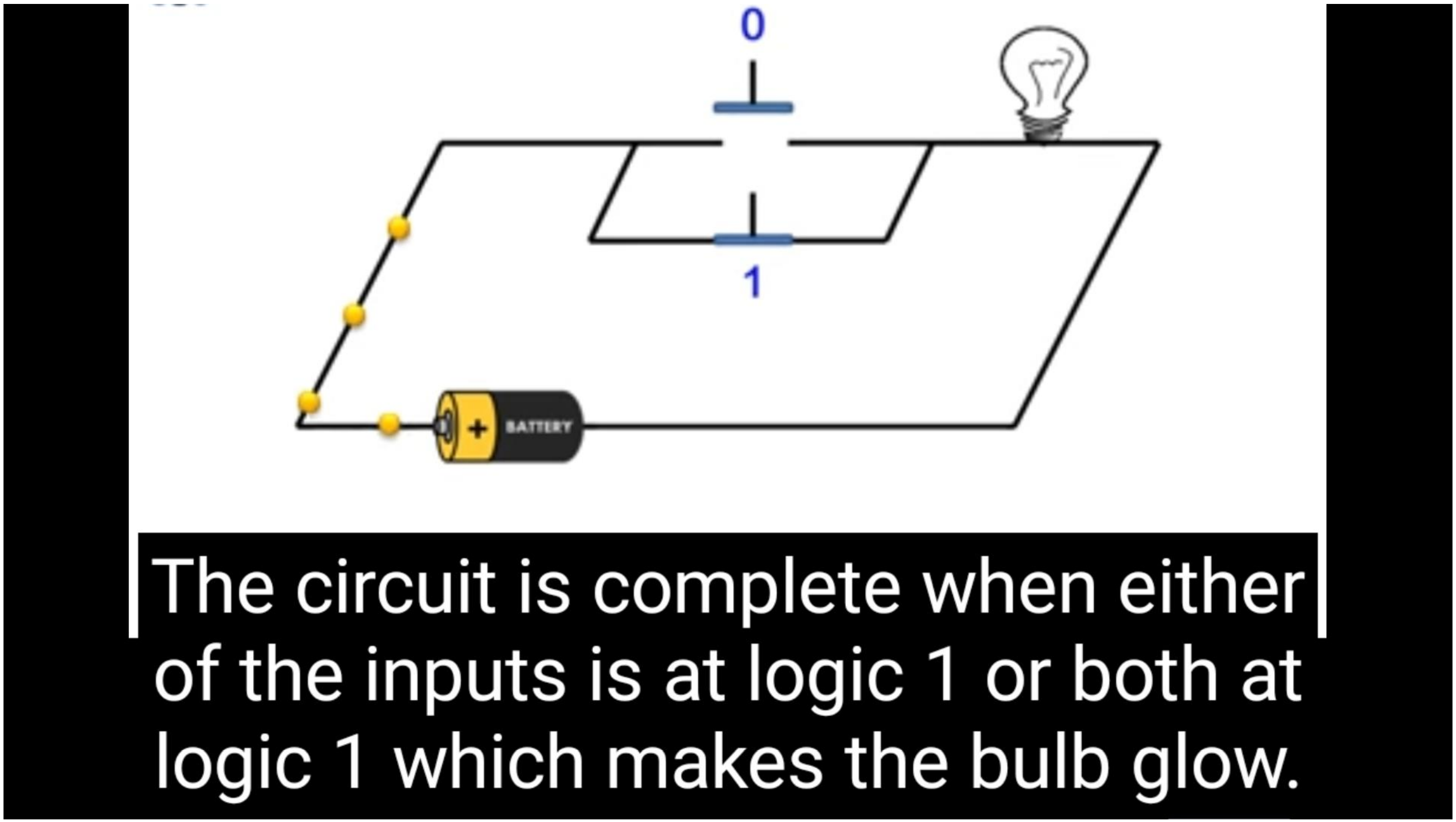

When a house has a front door and a back door, guests can arrive at any door

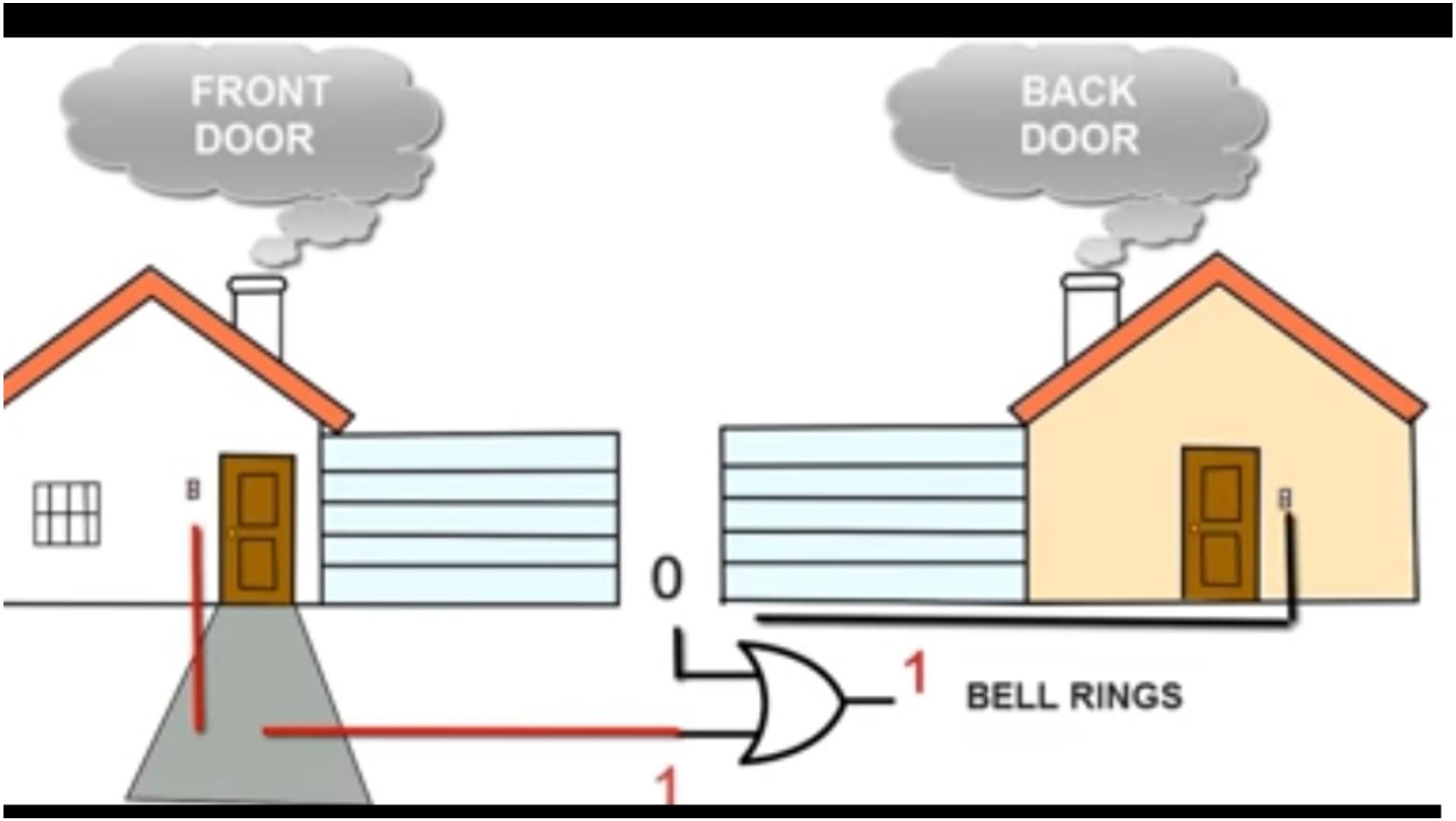

FRONT DOOR
BACK DOOR
0
BELL does not RING


FRONT DOOR
BACK DOOR
1
BELL RINGS



It is similar to 2 switches in parallel.


0
1
BATTERY
+
The circuit is complete when either of the inputs is at logic 1 or both at logic 1 which makes the bulb glow.

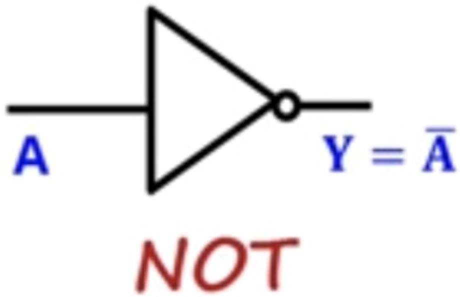

FRONT DOOR
BACK DOOR
0
1
BELL RINGS
1

OR

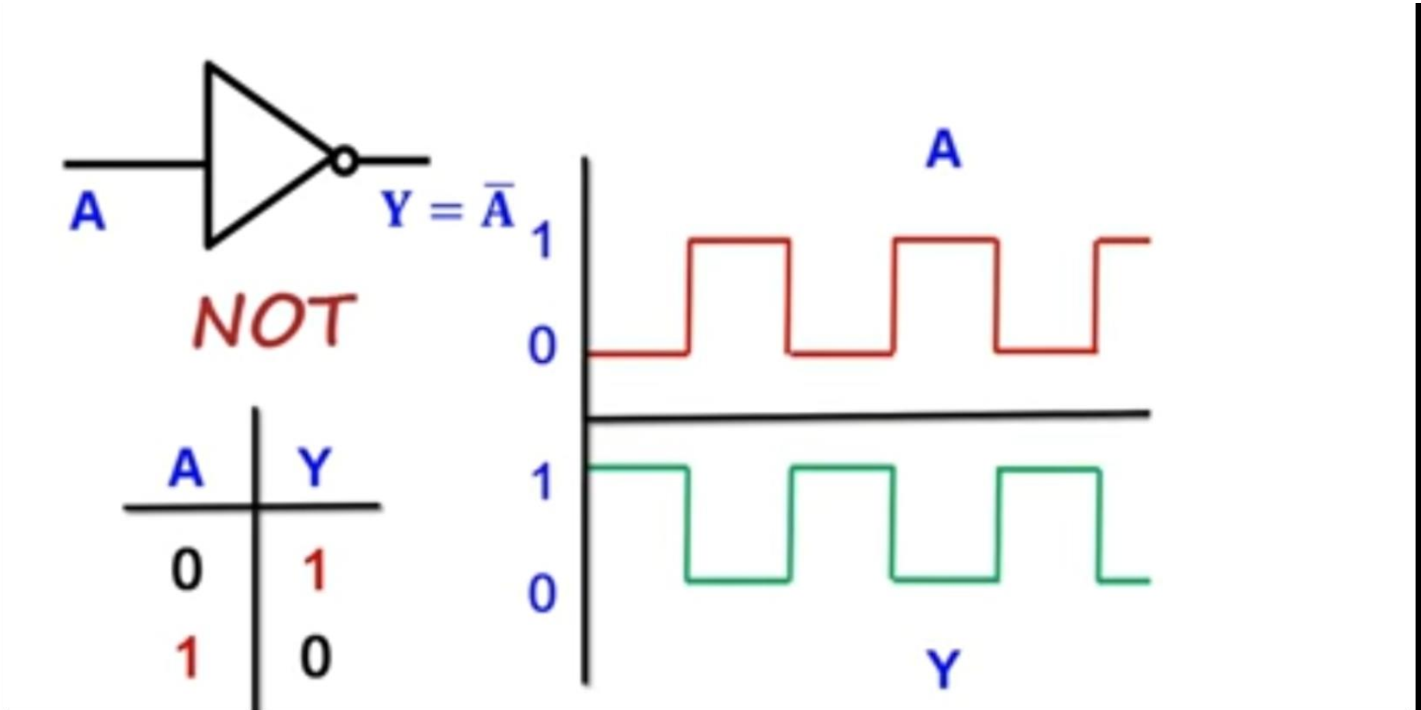
$$Y = A + B$$

[redacted]

- **NOT**

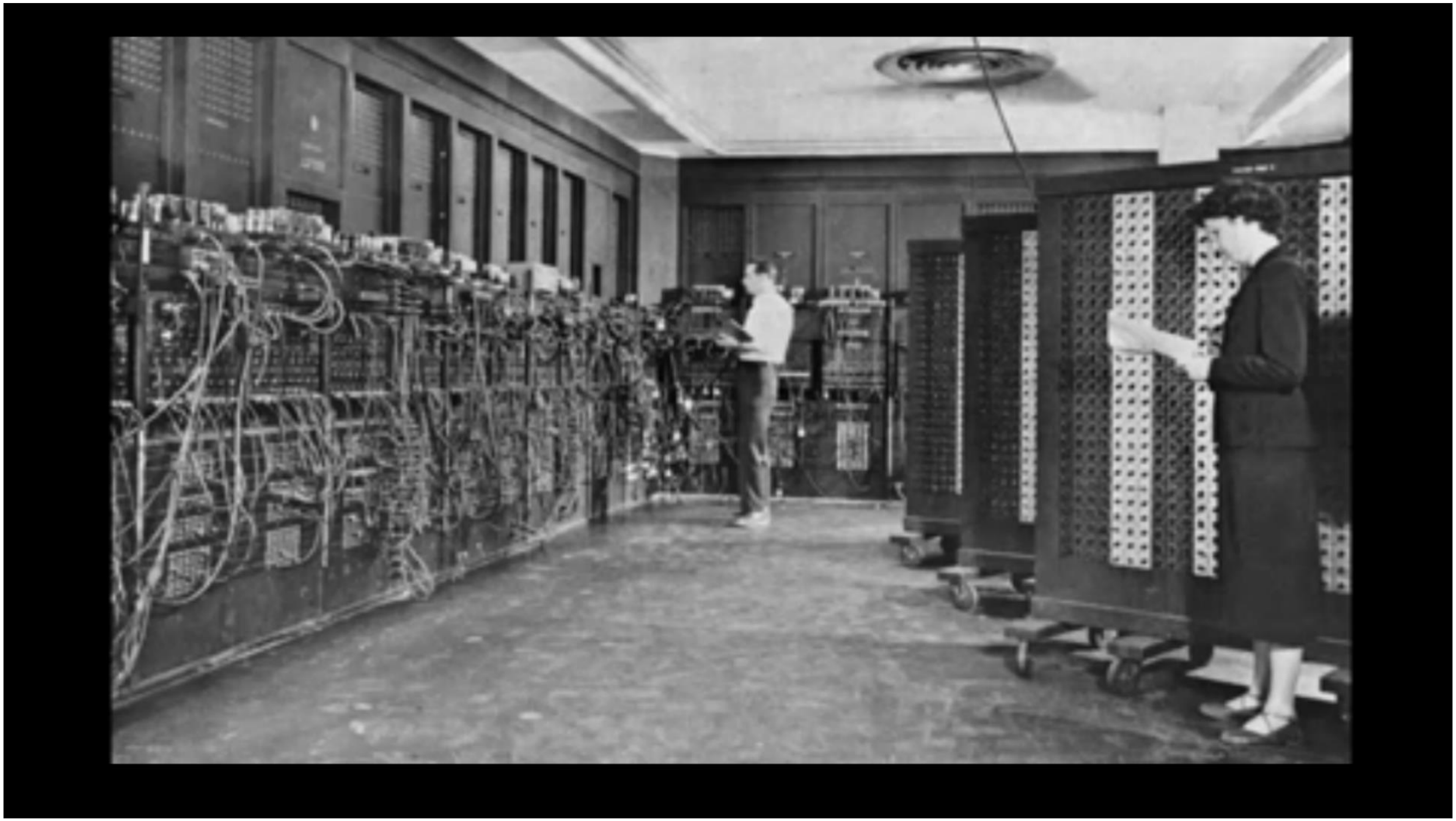


A complement is denoted by a bar over the variable. The boolean expression for input, A is A bar.



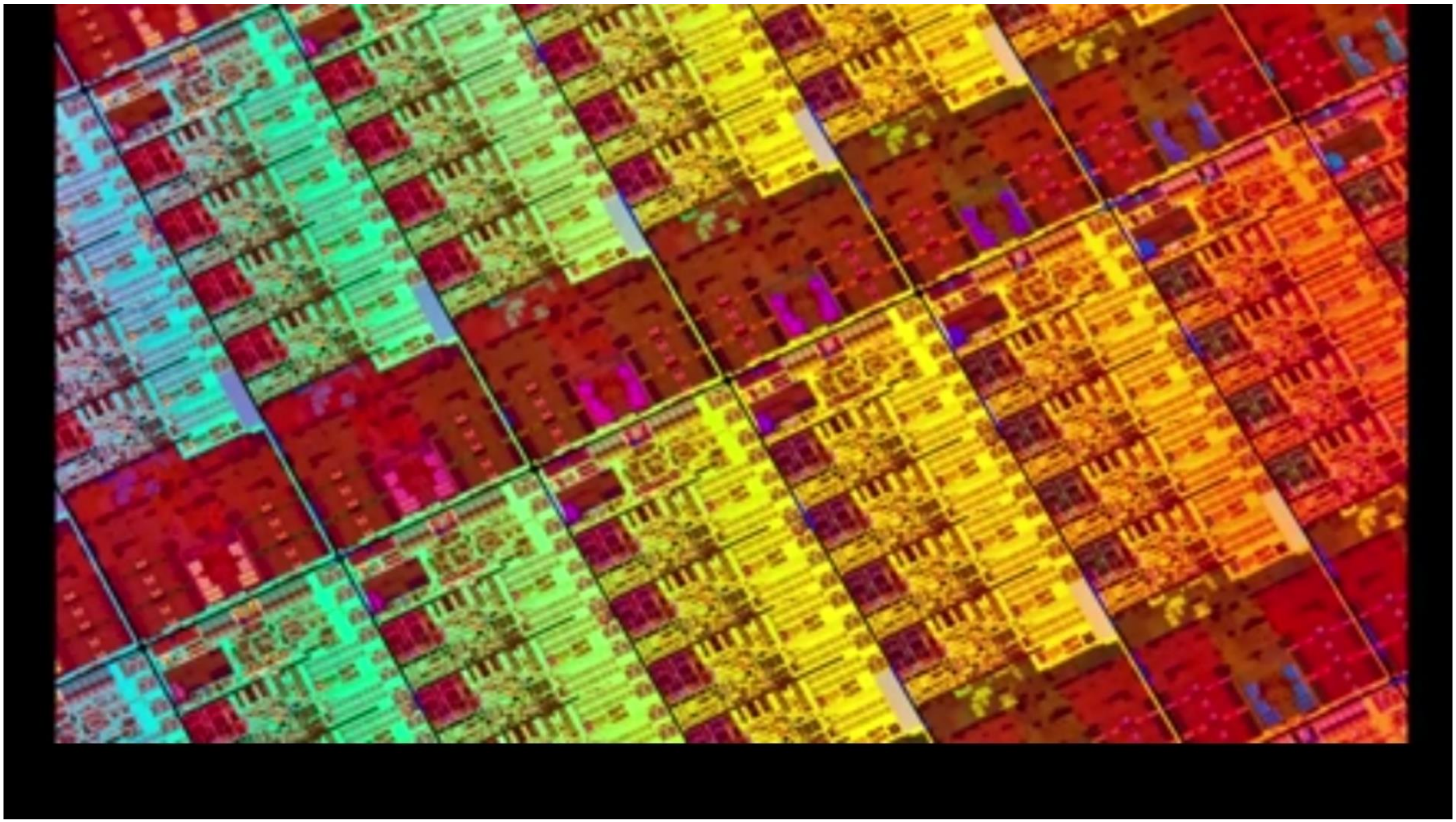
When the input is 0, output is 1 and vice versa

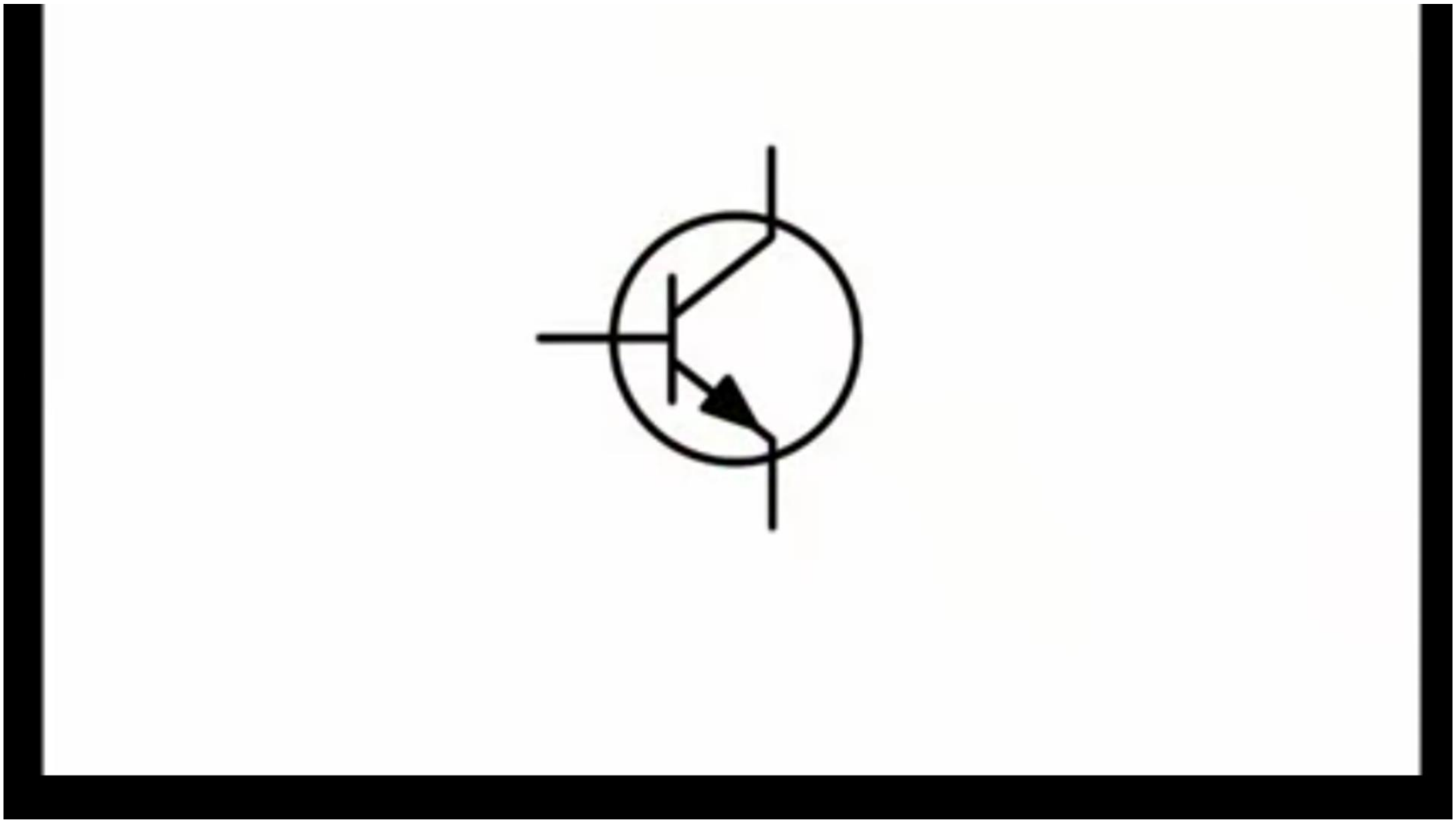
- **Logic Gates**

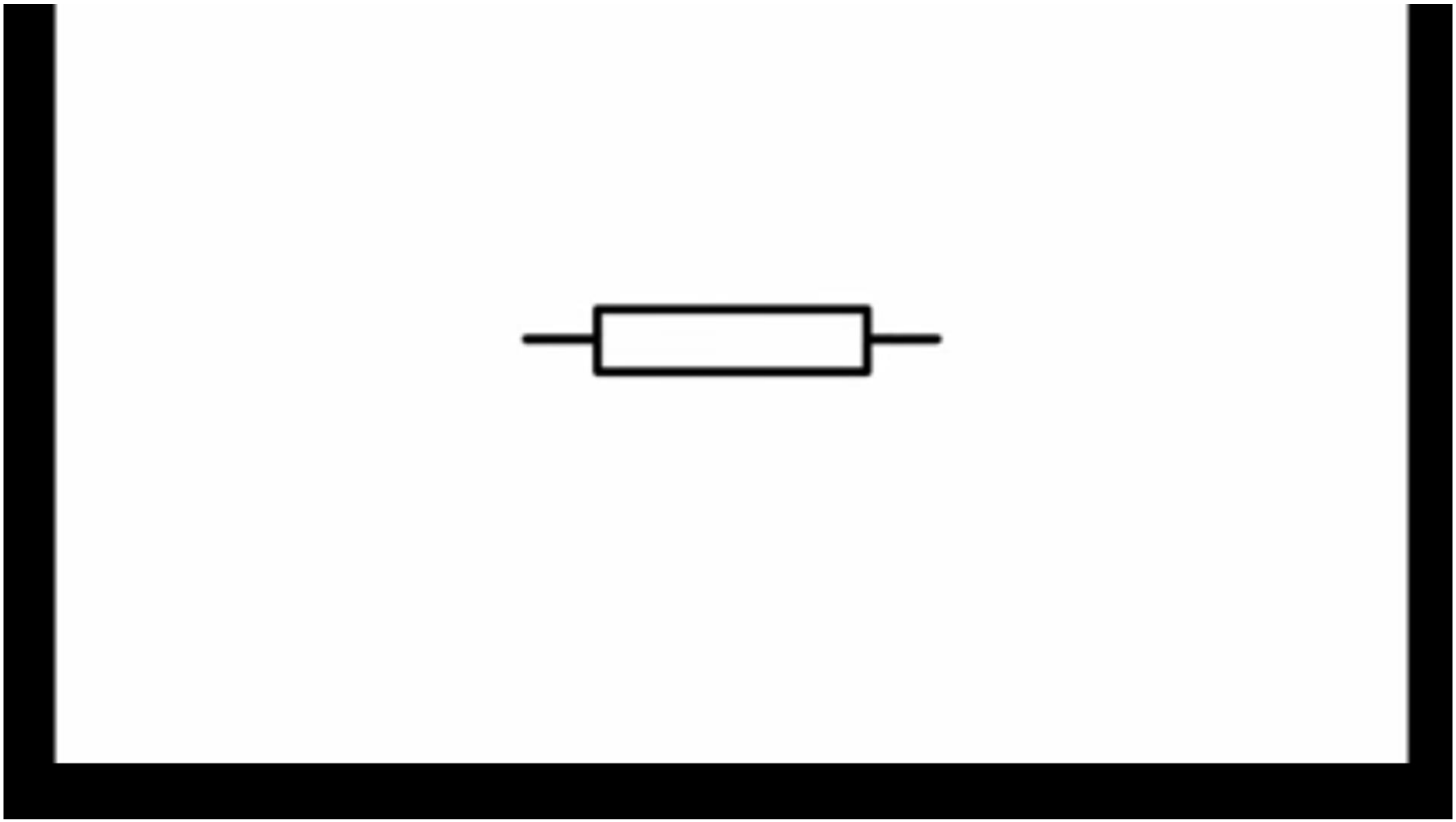


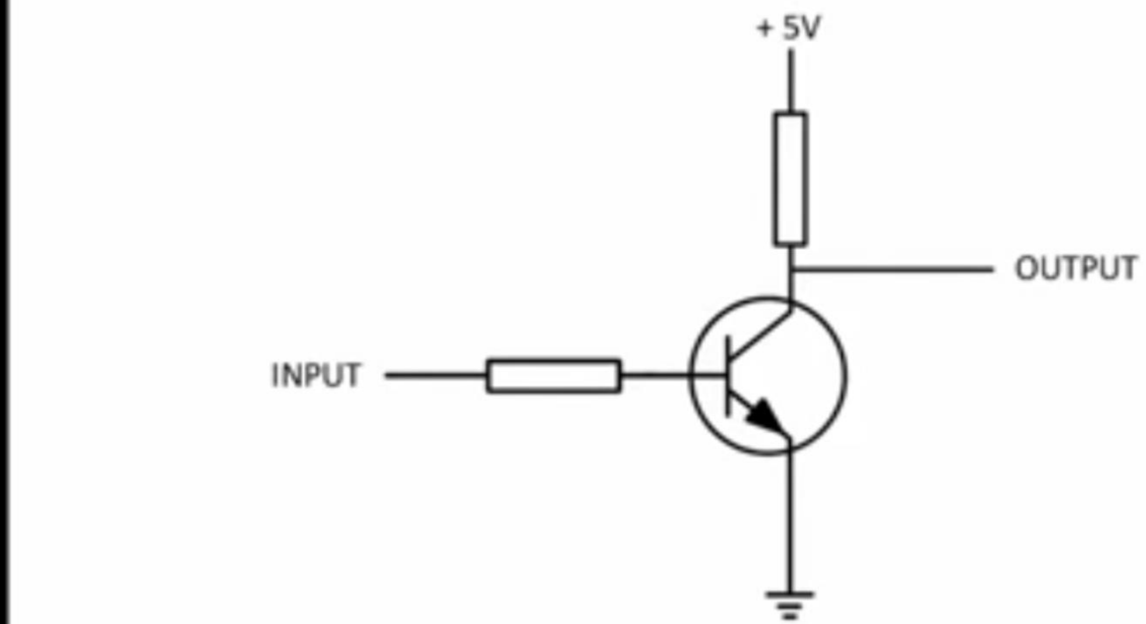

EZ80


PH
2N2222A

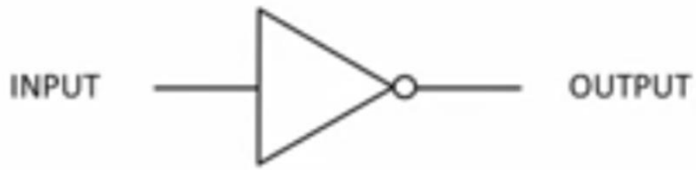







+ 5V
OUTPUT
INPUT

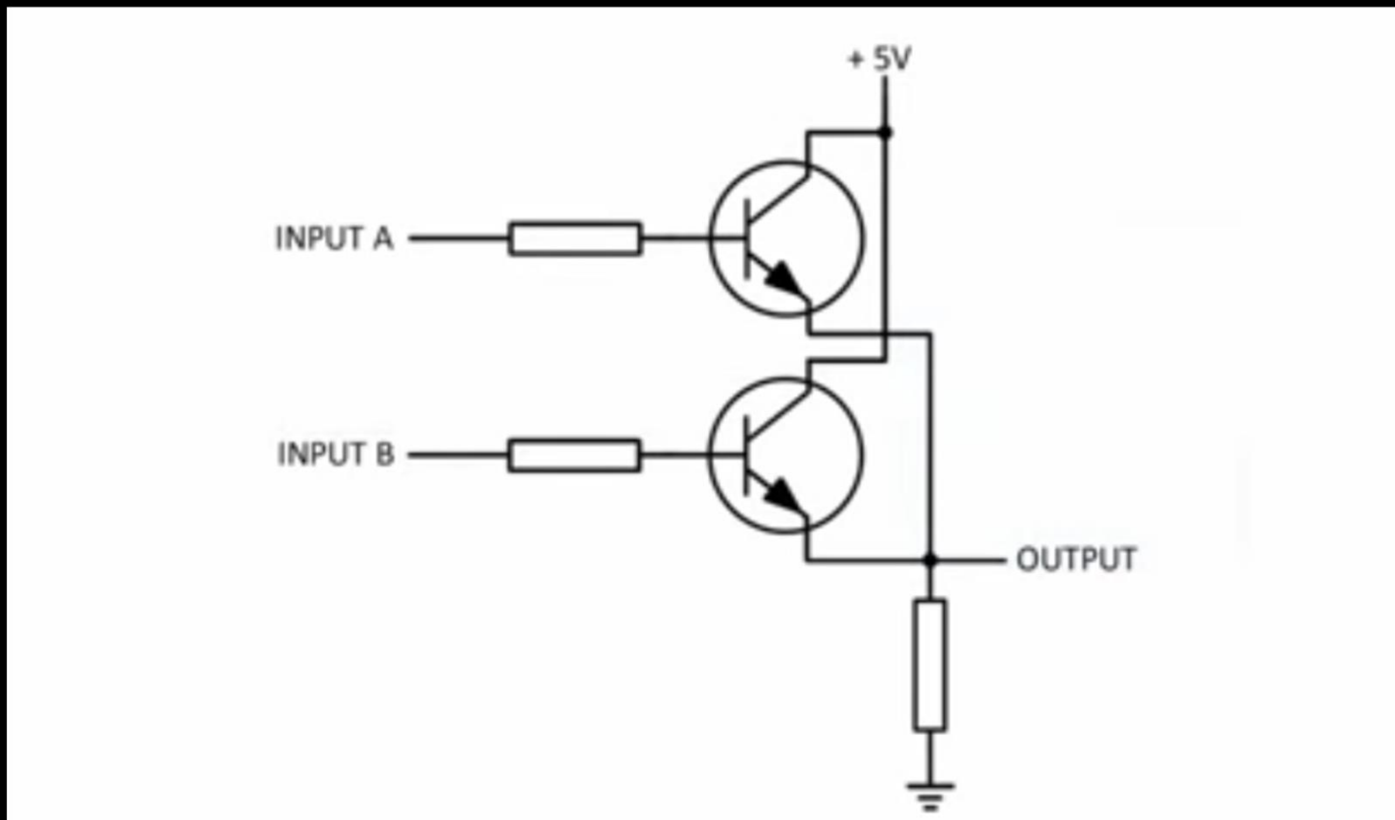
NOT



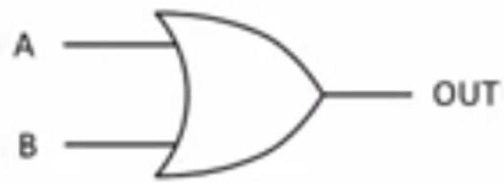
INPUT	OUTPUT
0	1
1	0

AND

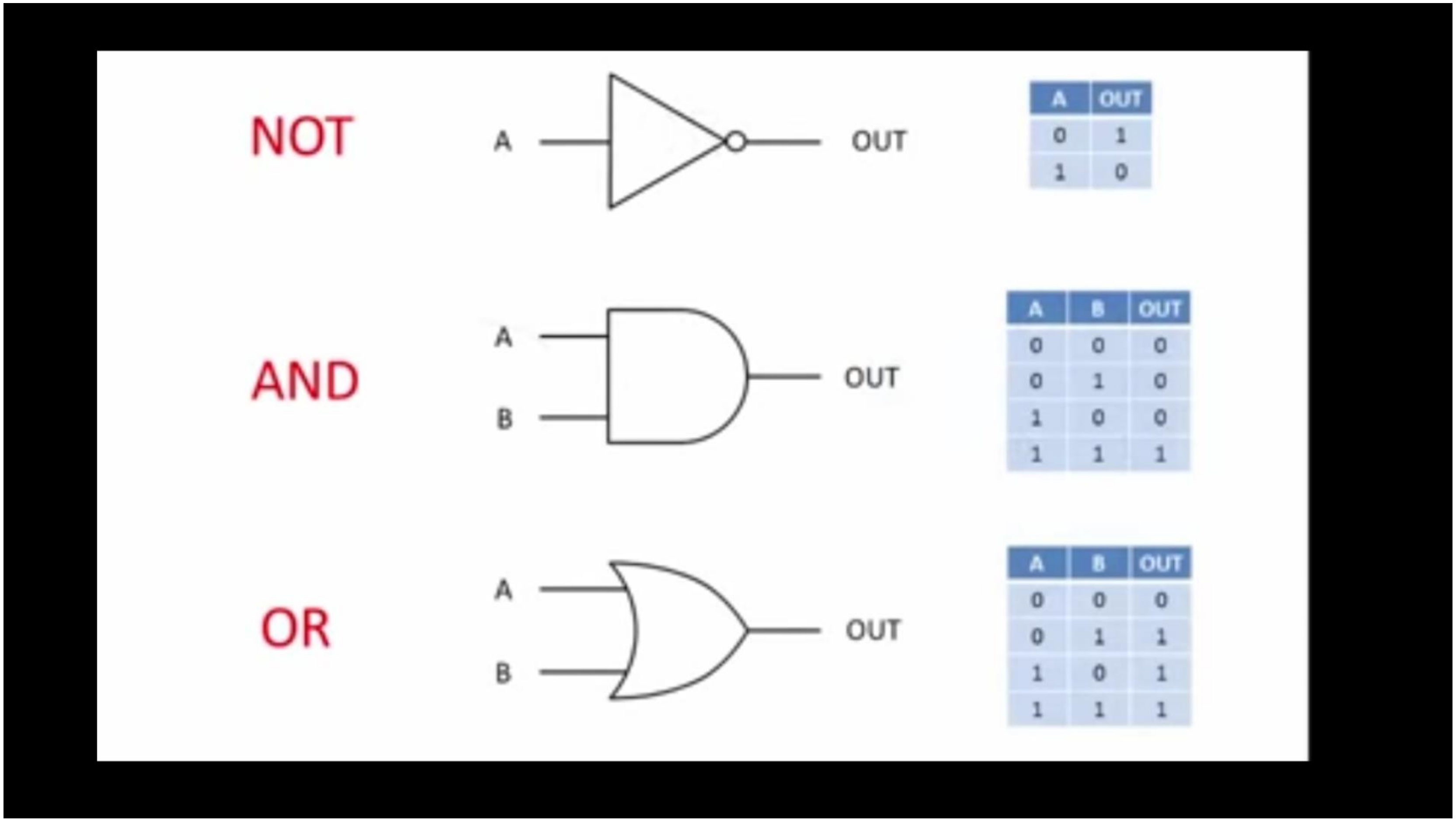
A	B	OUT
0	0	0
0	1	0
1	0	0
1	1	1


+ 5V
INPUT A
INPUT B
OUTPUT

OR



A	B	OUT
0	0	0
0	1	1
1	0	1
1	1	1

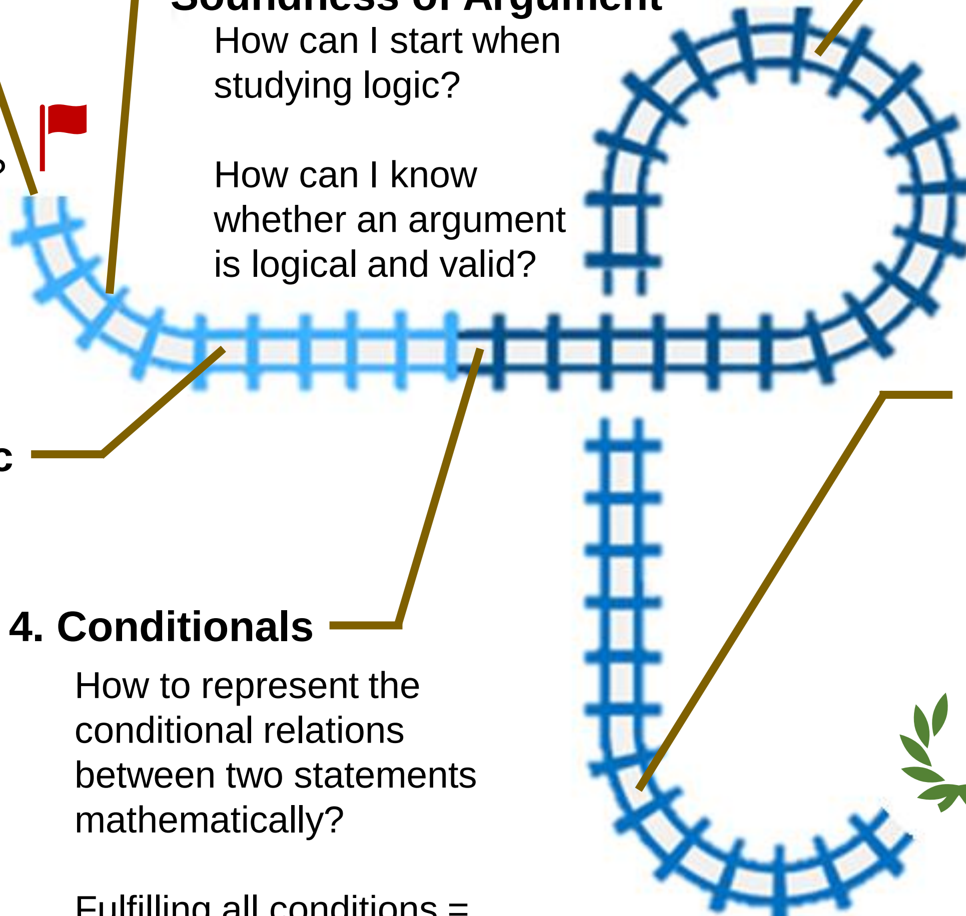


A	OUT
0	1
1	0

A	B	OUT
0	0	0
0	1	0
1	0	0
1	1	1

A	B	OUT
0	0	0
0	1	1
1	0	1
1	1	1

Journey of Logic



1. Need for Logic

What is logic?

Why do we study logic?

Where can I find logic in engineering?

2. Validity and Soundness of Argument

How can I start when studying logic?

How can I know whether an argument is logical and valid?

3. Propositional Logic

What can I visualize different conditions (true/false) of propositions?

How can I formulate propositions mathematically?

What can I connect multiple propositions?

4. Conditionals

How to represent the conditional relations between two statements mathematically?

Fulfilling all conditions = success?

5. From Proposition to Predicate

How can I construct a valid argument systematically?

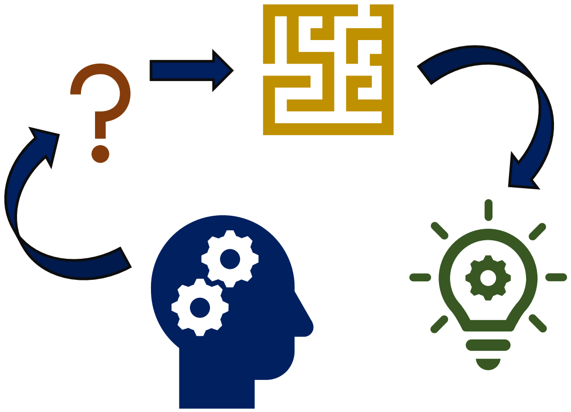
6. Predicate Logic

How to determine whether a statement is true for:
all cases (variables) or
some cases (variables) ?

In the end, you can think logically, analyze problem logically, and deduce a logical conclusion

1. Need for Logic

What is Logic?



Thinking Process?

A	B	A AND B	A OR B	NOT A
False	False	False	False	True
False	True	False	True	True
True	False	False	True	False
True	True	True	True	False

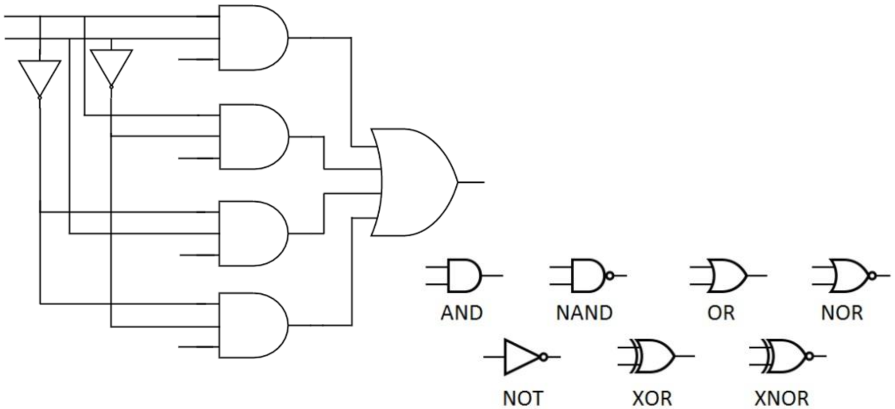
Truth / False?

All animals with wings can fly.
Penguins have wings.

Therefore, penguins can fly.

Valid? Sound?

Validating Arguments?



Circuit?

What is Logic?

- **Logic** ('possessed of reason, intellectual, dialectical, argumentative')
  - Originally, meaning “the word” or “what is spoken”
  - But coming to mean “thought” or “reason”
  - Consists of the systematic study of valid rules of inference, i.e., the relations that lead to the acceptance of one proposition (the conclusion) on the basis of a set of other propositions (premises). More broadly, logic is the analysis and appraisal of arguments

Warm-up Questions of Logic

Question 1

- If you overslept, you'll be late for class
  - You didn't oversleep.
    - Therefore,
      1) You' re late.
      2) You aren't late.
      3) You didn't attend the class.
      4) None of these follow.

Which one is correct?

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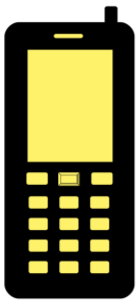
Warm-up Questions of Logic

<u>Question 2</u>

**Statement 1:
Good phone is
not cheap**



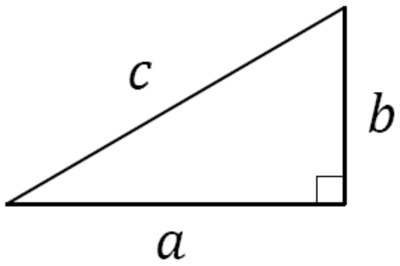
**Statement 2:
Cheap phone is
not good**

- These two statements mean
  1) different things.
  2) the same thing.

Which one is correct?

Warm-up Questions of Logic

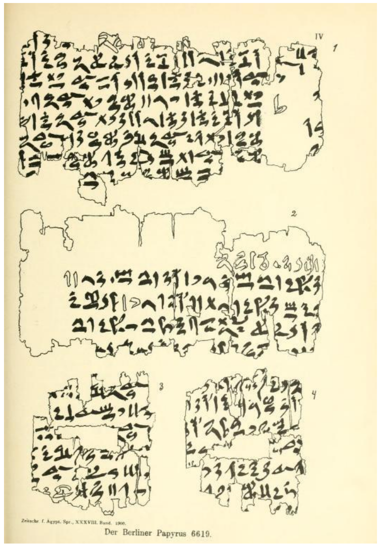
Question 3



Claim:

$a^2 + b^2 = c^2$

How to prove it?



- Found in the Middle Kingdom Egyptian Berlin Papyrus, a solution of the Pythagorean triple is **6:8:10**
- Since $\mathbf{6^2 + 8^2 = 10^2}$ is true, the claim is **true**

Is this a proof?

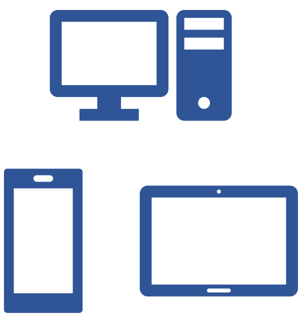
Need for Logic

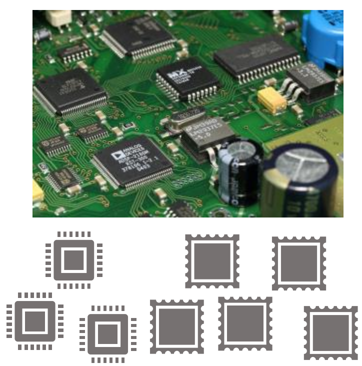
- Logic is not only the foundation of mathematics, but also is important in numerous fields including law, medicine, engineering and science.
- George Boole (Boolean algebra) introduced mathematical methods to logic in 1847

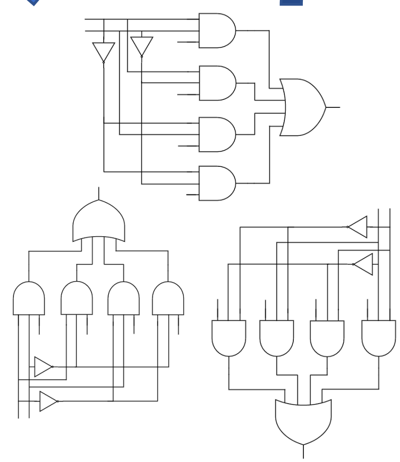
Property	AND	OR
Commutative	AB = BA	A+B = B+A
Associative	(AB)C = A(BC)	(A+B)+C = A+(B+C)
Distributive	A(B+C) = (AB)+(AC)	A+(BC) = (A+B)(A+C)
Identity	A1 = A	A+0 = A
Complement	A(A') = 0	A+(A')= 1
De Morgan's Law	(AB)' = A' OR B'	(A+B)' = A'B'

Need for Logic

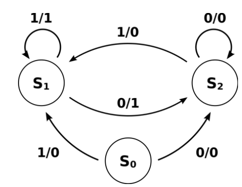
- Logic is essential to electronic/computer/smartphone/engineering



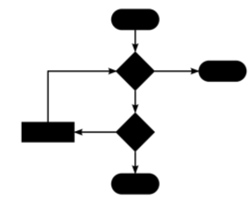




State Diagram



Flow Chart



Truth Table

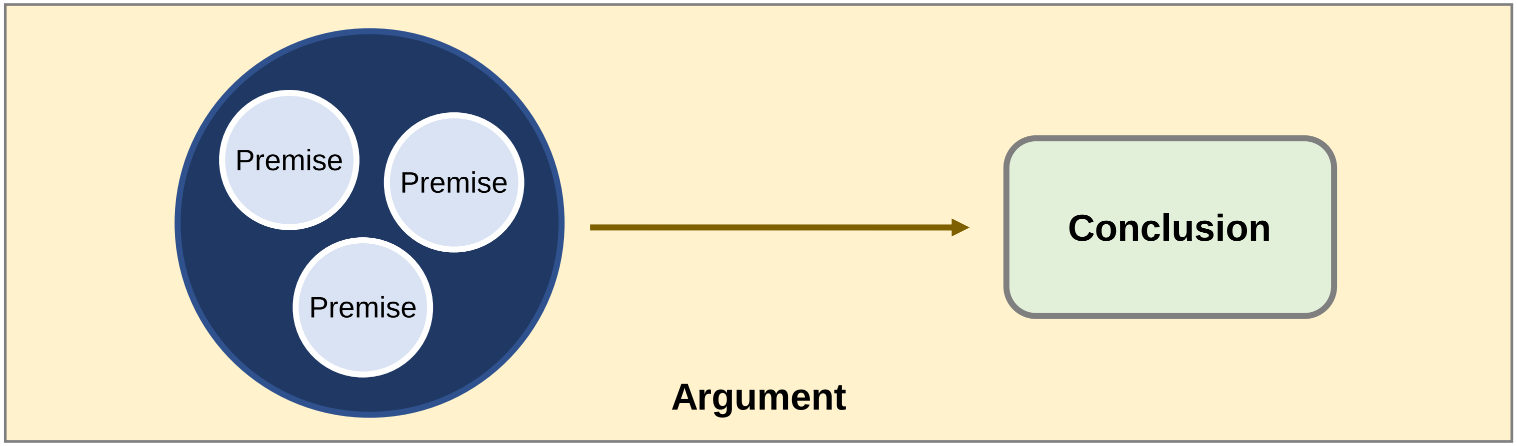
A	B	A AND B	A OR B	NOT A
False	False	False	False	True
False	True	False	True	True
True	False	False	True	False
True	True	True	True	False

Validity and Soundness of Argument

Validity of Argument

- Logic is the study of methods for evaluating whether the **premises** of an argument adequately **support** (or provide evidence for) its conclusion.

- A **valid argument** is one where there is **a specific relation of logical support** between **premises** of the argument and its **conclusion**



Validity of Argument

- An argument is a list of statements
  - The last statement is called a conclusion
  - All preceding statements are called premises (or assumptions/hypotheses)
- Validity and invalidity apply only to arguments, not statements
  - An argument is *valid* if and only if it is <u>not</u> possible that all of its premises are true and its conclusion is false.
  - An argument is *invalid* if and only if it is possible that all of its premises are true and its conclusion is false.

Examples

Argument 1

All smartphones can send emails.

My phone is a smartphone.

So, my phone can send email.

Premises

Conclusion

Argument 2

All smartphones can send emails.

My desktop can send email.

So, my desktop is a smartphone.

Premises

Conclusion

Are these two arguments valid?

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Two Major Types of Arguments

Deductive Argument

- ❖ The conclusion flows from the premises necessarily
- ❖ The claim that the truth of its premises guarantees the truth of its conclusion

Example

Hong Kong is an Asian city.
I am living in Hong Kong.

So, I am living in Asia.

Inductive Argument

- ❖ The conclusion flows from the premises probably
- ❖ The claim that the truth of its premises provides some grounds for its conclusion or makes the conclusion more probable

Example

Most who live in Hong Kong speak Cantonese
Susan lives in Hong Kong.

So, Susan speaks Cantonese.

Deductive Argument

- In this course, we focus on deductive argument.

- Usefulness: It reveals consequences of premises.
  - Mathematics is based on deductive arguments.
    - Theorems are derived from axioms.

- Limitation: it doesn't tell which premises are actually true
  - Science is mainly based on inductive arguments
    - You need to observe the world and collet empirical evidence.

Valid and Sound Arguments

- An argument is valid if its conclusion is a logical consequence of the premises
  1) An argument can have one or more false premises and still be valid.
  2) You can't assume that an argument is valid just because all its premises are true.
  3) Valid arguments never go from true premises to a false conclusion
  4) A valid argument can go from false premises to a true conclusion

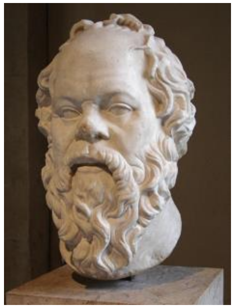
- An argument is sound if it is valid and the premises are true

- Note:

  An invalid or unsound argument may still have a true conclusion

In-class Exercises

<u>Statement 1</u>
All men are mortal.
Socrates was a man.
Therefore, Socrates was mortal.



Socrates (469/470- 399BC) was a Greek philosopher and is considered the father of western philosophy.

<u>Statement 2</u>
All birds with wings can fly.
Penguins have wings.
Therefore, penguins can fly.

<u>Statement 3</u>
All bears are birds.
All birds are mammals.
So, all bears are mammals.

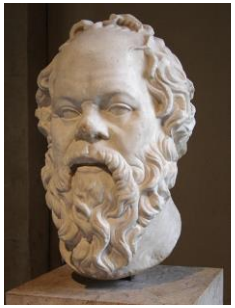
Valid? Sound?

In-class Exercises

Valid

Sound

<u>Statement 1</u>
All men are mortal.
Socrates was a man.
Therefore, Socrates was mortal.



Socrates (469/470- 399BC) was a Greek philosopher and is considered the father of western philosophy.

Invalid

Unsound

<u>Statement 2</u>
All birds with wings can fly.
Penguins have wings.
Therefore, penguins can fly.

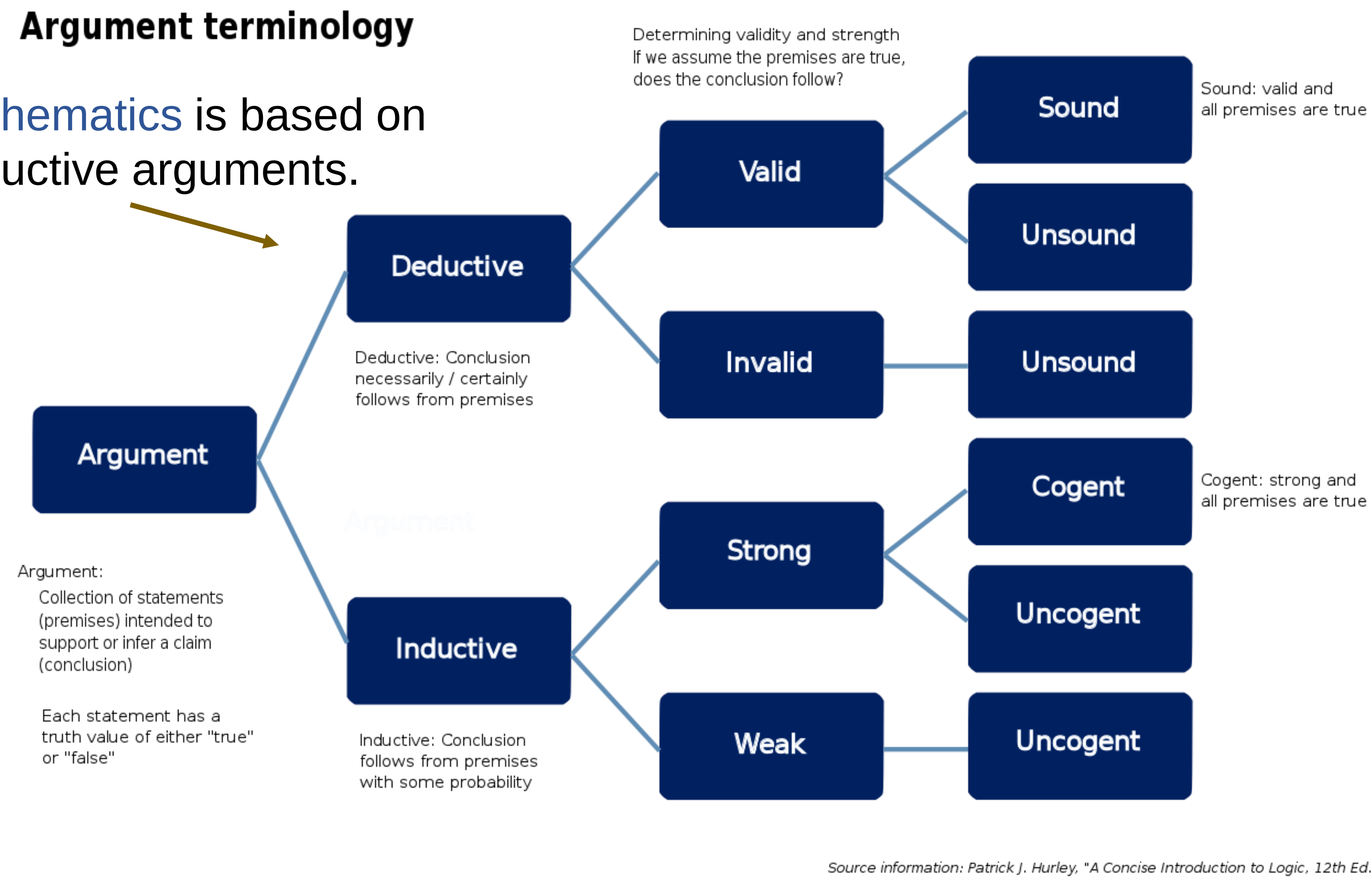
Valid

Unsound

<u>Statement 3</u>
All bears are birds.
All birds are mammals.
So, all bears are mammals.

Argument Terminology

Mathematics is based on deductive arguments.



Propositional Logic

Propositions

- A *proposition* is a statement that is **either True or False**, but not both.
  - Each proposition has a **truth value**, either **T** of **F**

- Are they propositions?
  1) " 1+1 = 11 "
  2) " God exists "
  3) " Can you explain what logic is? "
  4) " Turn on the computer, please "

Logical Connectives

- Propositions can be combined to form compound propositions by using **logical connectives**

  e.g., *p*: Hardworking students will not be distracted, and

  q: Hardworking students will get A

  Compound proposition: (*p* and *q*) or ($p \wedge q$)

- A premise can be either a proposition or compound proposition

- Three fundamental logical connectives:
  - Negation, NOT
  - Conjunction, AND
  - Disjunction, OR

Negation - NOT

- Given a proposition p, the compound proposition $\sim p$ is called the **negation** of p
  - It is read as "not p" or " It is not the case that p "
  - Alternative notation of negation: $\neg p$

- Truth table of negation

p	$\sim p$
T	F
F	T

Interpretation:
- If p is true, then its negation is false
- If p is false, then its negation is true

Conjunction - AND

- Given propositions *p* and *q*, the compound proposition *p* ∧ *q* is called the **conjunction** of *p* and *q*
  - It is read as "*p* and *q*"

- Truth table of conjunction

2 propositions

$2^2 = 4$ rows

Each proposition has 2 conditions (T/F)

p	*q*	*p* ∧ *q*
T	T	T
T	F	F
F	T	F
F	F	F

Interpretation:
A conjunction is **only true** when **both *p* and *q* are true**.

Otherwise, a conjunction of two statements will be **false**

Disjunction - OR

- Given propositions p and q, the compound proposition $p \vee q$ is called the **disjunction** of p and q
  - It is read as "*p* or *q*"

- Truth table of disjunction

2 propositions

$2^2 = 4$ rows

Each proposition has 2 conditions (T/F)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Interpretation:
A disjunction is true in all cases **except** when **both *p* and *q* are false**

Well-Formed Formulas

- A well-formed formula (or simply **formula**) is an expression constructed according to the following rules
    1) Each propositional variable is a formula

       e.g., p, q, r, and s

    2) Formulas combined by logical connectives (with parenthesis if involving two formulas) is a formula

       e.g., $\sim p$, $(p \land q)$, $\sim(p \lor q)$

       Outer parenthesis is often omitted, e.g., $p \land q$

In-class Exercises

- Are they formulas?

  1) $(p \land {\sim}{\sim}q)$

  2) $(p \lor q \land s)$

  3) $(p \lor q) \land {\sim}(p \land q)$

Compound Truth Tables

- The compound truth table for $(p \lor q) \land \sim(p \land q)$

p	q	$p \lor q$	$p \land q$	$\sim(p \land q)$	$(p \lor q) \land \sim(p \land q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Intermediate result

Final result

The formula means "(p or q) and not (p and q)"
It is called **Exclusive-OR (XOR)**, and denoted by $p \oplus q$

Logical Equivalence

- Two formulas *A* and *B* are logically equivalent (or simply equivalent) if they have identical truth values in all cases
  - $A \equiv B$ denotes *A* is equivalent to *B*

Example: $p \lor q \equiv q \lor p$

p	q	$p \lor q$	$q \lor p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$p \lor q$ and $q \lor p$ always have the same truth values.
Therefore, $p \lor q \equiv q \lor p$

De Morgan’s Law

- The negation of a conjunction is the disjunction of the negations:

$$\sim(p \land q) \equiv \sim p \lor \sim q$$

- The negation of a disjunction is the of the conjunction of the negations

$$\sim(p \lor q) \equiv \sim p \land \sim q$$

Augustus De Morgan (1806-1871), a British mathematician and logician.

Tautologies

- A statement that is always true is called a tautology.

- Example: $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

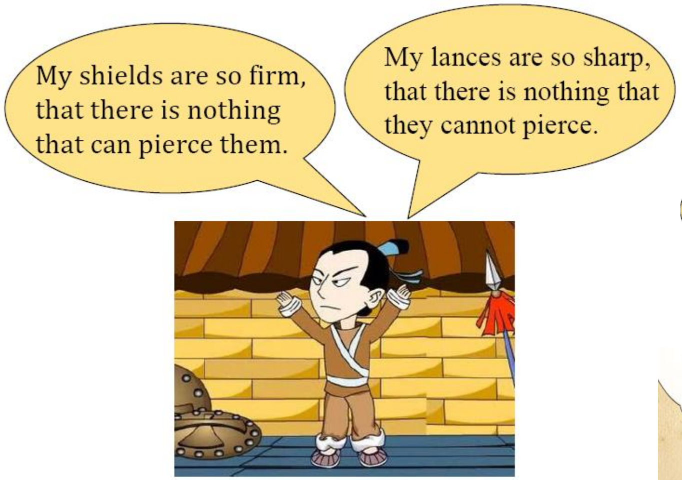
- Tautologies reveal hidden truths. For example, all theorems in mathematics are tautologies.
  - De Morgan's Law, the Pythagorean theorem

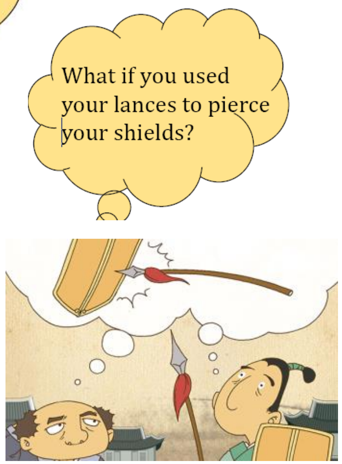
Contradictions

- A statement that is always false is called a contradiction.

- Example: $p \wedge \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	F
F	T	F





A fable from an ancient Chinese book 《韓非子-Hanfeizi》

Equivalence Involving Tautologies and Contradictions

- If **t** is a tautology and **c** is a contradiction, then

$$p \wedge \mathbf{t} \equiv p$$

$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

p	$\mathbf{t}$	$p \wedge \mathbf{t}$
T	T	T
F	T	F

The truth values are the same, so
$p \wedge \mathbf{t} \equiv p$

p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	F	F
F	F	F

The truth values are the same, so
$p \wedge \mathbf{c} \equiv \mathbf{c}$

Theorem of Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Applying Theorem for Proofing

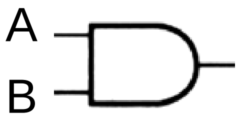
- Prove that $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$$
\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.}
\end{aligned}
$$

Truth table can be used to prove both equivalence and non-equivalence.

Truth Tables in Engineering

Convert **Truth/False** (**T/F**) to **binary number** (**1/0**)

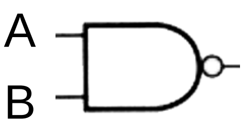


AND Gate

Input		Output
A	B	A ∧ B
1	1	1
1	0	0
0	1	0
0	0	0

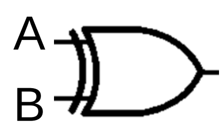
OR Gate

Input		Output
A	B	A ∧ B
1	1	1
1	0	1
0	1	1
0	0	0



NAND Gate

Input		Output
A	B	~(A ∧ B)
1	1	0
1	0	1
0	1	1
0	0	1

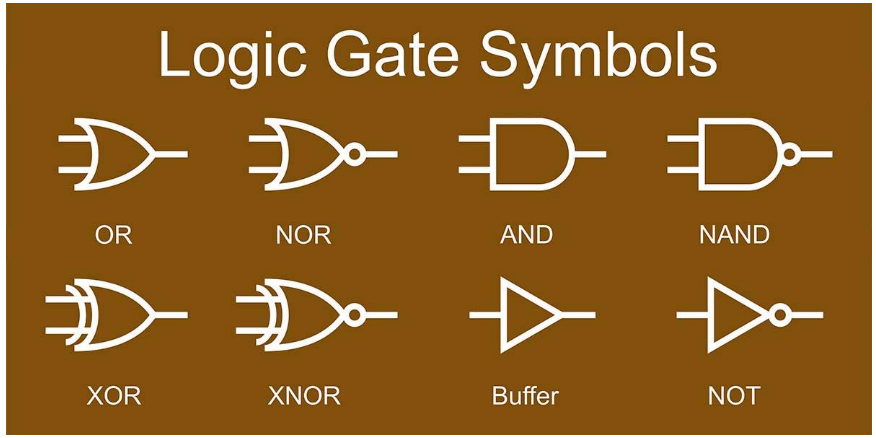


XOR Gate

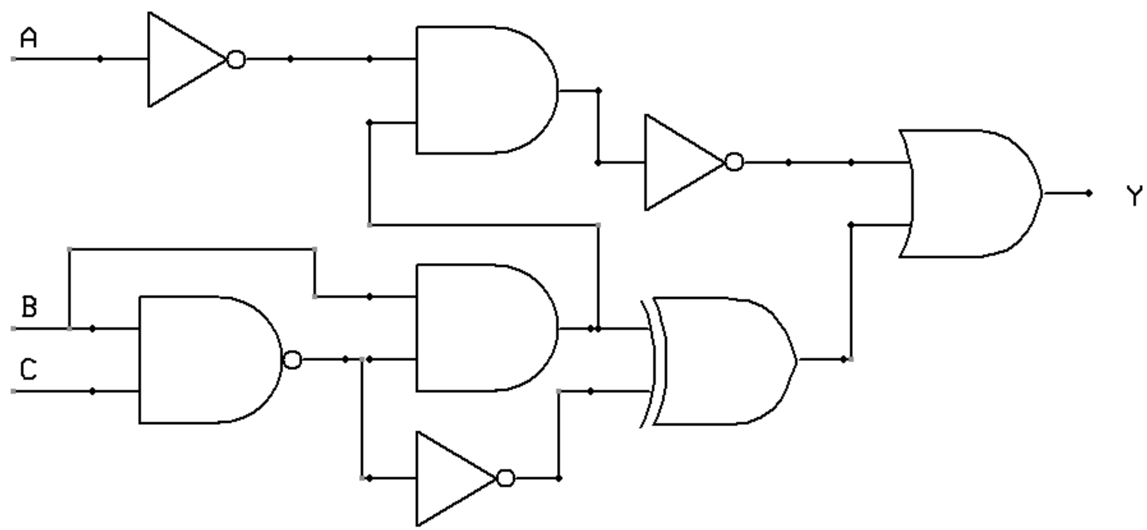
Input		Output
A	B	A ⊕ B
1	1	0
1	0	1
0	1	1
0	0	0

More details will be studied in the section of Boolean Algebra, and you will exploit them during the labs

Logical Connectives in Engineering



A	B	C	Y
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	



Conditionals

Conditional Statement

- A conditional statement, denoted by $p \rightarrow q$, is an if-then statement
  - i.e., If p then q, or equivalently, p implies q
  - p is called the antecedent (hypothesis)
  - q is called the consequent (conclusion)
  - $p \rightarrow q$ is sometimes written as $p \supset q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

If p is **true**,
then $p \rightarrow q$ takes the truth value of q.

If p is **false**,
then $p \rightarrow q$ is regarded to be true.

Counter-Intuitive?

" If the sun is made of gas,
then 1+1 = 11 "

True or False?

- For $p \rightarrow q$ to be true, there is not necessarily any relevance between p and q
  - It depends only on the truth values of p and q

- $p \rightarrow q$ is called material implication
  - Its meaning is not exactly the same as "if … then…" in daily life.

Example of conditional statement

"If you attend all classes, I will give you an A+ for this course"

p = statement "If you attend all classes"
q = statement "I will give you an A+ for this course"

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If p is **true** (you attended all classes) and I give you an A+
$p \to q$ is true (I **keep** the promise)

If p is **true** and q is **false** (you did not get the A+)
$p \to q$ is false (I **break** the promise)

If p is **false** and there was no obligation to get A+
so truth value of $p \to q$ must be true
(I do not violate the promise)

Conditional defined by NOT and OR

- $\rightarrow$ can be defined by ~ and ∨

$$p \rightarrow q \equiv \sim p \vee q$$

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Variations of Conditionals

- Contrapositive
  - The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

- Converse and Inverse
  - The converse of $p \rightarrow q$ is $q \rightarrow p$
  - The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$
  - Converse is equivalent to Inverse

    $q \rightarrow p \equiv \sim p \rightarrow \sim q$

Example

- A contrapositive example in an English sentence:

  p = statement "**you get 95 marks**"

  q = statement "you get an A+ in this course"

  $p \rightarrow q$

  If **you get 95 marks**, then you get an A+ in this course

  Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

  i.e.,

  If you don't get an A+ in this course, then **you didn't get 95 marks**

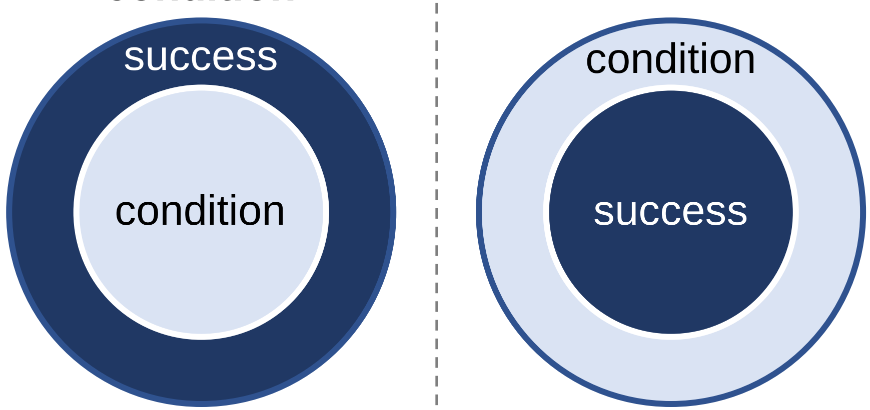
Necessary & Sufficient Conditions

- When $p \rightarrow q$,
  - p is called a sufficient condition for q
  - q is called a necessary condition for p

Condition leads to success, but is not the only condition leading to outcome

Sufficient condition

Necessary condition



Whenever success, the condition is present

"If you are in CityU, you are in Hong Kong."

- Being in CityU is a sufficient condition of being in Hong Kong.
- Being in Hong Kong is a necessary condition of being in CityU.

The Biconditional

- "p only if q" means
  - "If not q then not p", or equivalently,
  - "If p then q"
  - $p \rightarrow q$

- The biconditional is denoted by $p \leftrightarrow q$
  - i.e., **If and only if (*iff*)**
  - $p \leftrightarrow q$ is true only when p and q have **identical** truth values
  - $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Operator Precedence

- It is more convenient if some parenthesis can be omitted

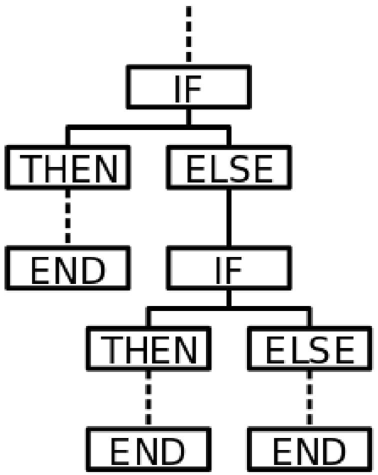
- To avoid ambiguity, the operators have different priorities:

  from high to low: ( ), ~, ∧∨, →, ↔

- Example:

  $p \vee \sim q \rightarrow \sim p$ means $(p \vee (\sim q)) \rightarrow (\sim p)$.

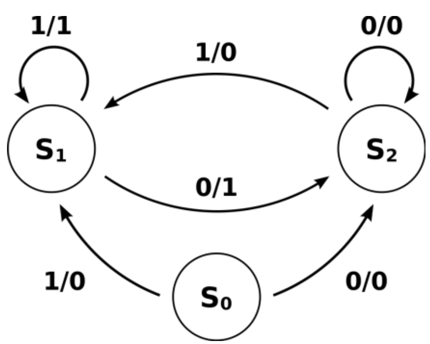
Conditionals in Engineering



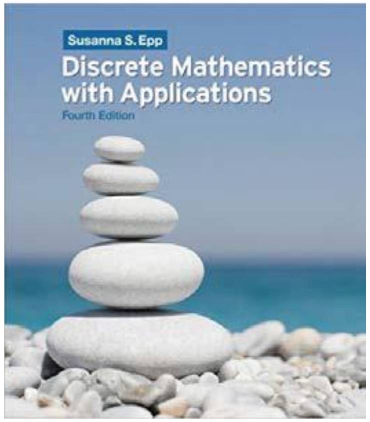
From conditional logic to programming, algorithm development

```
If Number = 1 Then
    Count1 = Count1 + 1
ElseIf Number = 2 Then
    Count2 = Count2 + 1
ElseIf Number = 3 then
    Count3 = Count3 + 1
Else
    CountX = CountX + 1
End If
```

Procedure $LimitedLevelsetCIR(E, X, Y, \hat{Y})$
Input S, D; %'S' for local slope and 'D' for no. of beakpoints
$m := 1;$
$L^1_{1j} = [e_{1j}, 0, 0]$ for $j = 1, 2, ..., A;$
for $(i = 2$ to $i = A)$
for $(j = 1$ to $j = A)$
for $(k = max(1, j - S)$ to $k = j)$
for $(l = 1$ to $l =$ the number of states in $L_{i-1,k})$
if $(i = 2)$
$D_0 := 0;$
else
if $(L^l_{i-1,k}[2] \neq j - k)$
$D_0 := L^l_{i-1,k}[3] + 1;$
else
$D_0 := L^l_{i-1,k}[3];$
end
end
if $(D_0 \leq D)$
$L^m_{ij} = [e_{ij} + L^l_{i-1,k}[1], j - k, D_0];$
$m := m + 1;$
end
end
end
$n := 1;$
for $(brkPoint = 0$ to $brkPoint = D)$
for $(slope = 0$ to $slope = S)$
$V = \{ce \in \mathbf{R} \mid [ce,\ slope,\ brkPoint] \in L^l_{i,j}\};$
if $(V \neq \phi)$
$\hat{L}^n_{ij} = [min(V), slope, brkPoint];$
$n := n + 1;$
end
end
end
replace $L_{i,j}$ with $\hat{L}_{ij}$;
end
end



Recommended Readings



Sections 2.1-2.2, Susanna. S. Epp, Discrete Mathematics with Applications, 4th edition, Brooks Cole, ISBN 978-1111775780, 2011.