L14 DTFT

Synthesis Equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Analysis Equation

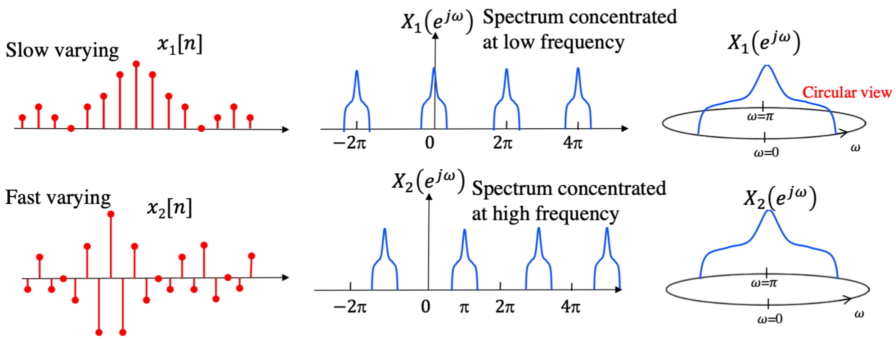
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$X(e^{j\omega})$: DTFT of $h[n]$

↳ 2π-periodic : $e^{j(\omega+2\pi)} = e^{j\omega}\underbrace{e^{j2\pi}}_{1} = e^{j\omega}$

$\omega = m2\pi \rightarrow$ low freq

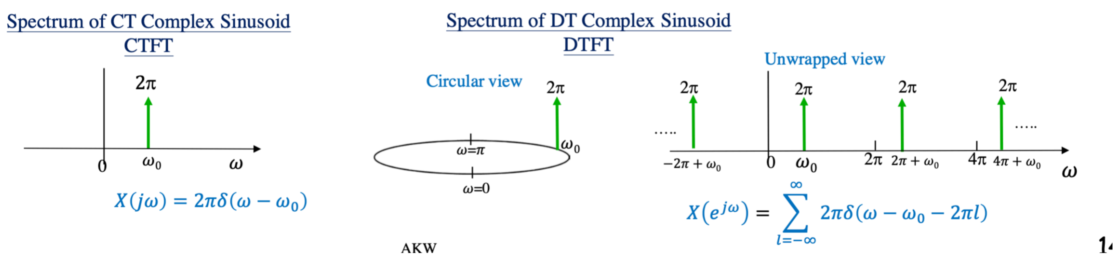
$\omega = (2m+1)2\pi \rightarrow$ high freq



DTFT → Periodic

Complex sinusoid $e^{j\omega_0 n}$ $\longleftrightarrow$ $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi l)$

↑ 2π-periodic



(FS expansion)

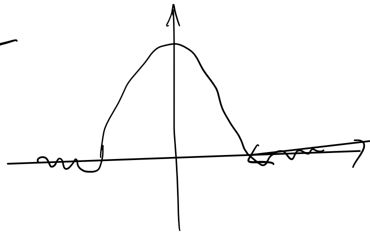
Recall: If $x[n]$ periodic: $x[n] = \sum_{\langle N \rangle} a_k e^{jk\omega_0 n}$ $\omega_0 = \frac{2\pi}{N}$

$$X(e^{j\omega}) = \sum_{\langle N \rangle} a_k \sum_{l=-\infty}^{\infty} 2\pi \delta(\omega - k\omega_0 - 2\pi l)$$

$$= \sum_{-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Properties of DTFT: Table 5

L13



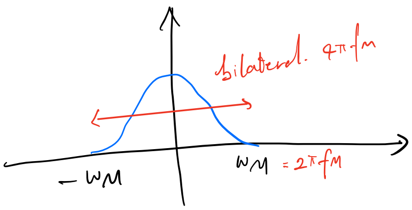
bandwidth: width of frequency range spectrum occupies

bandlimited: bandwidth is finite

maximum frequency above which spectrum is 0/negligible

maximum freq: unilateral bandwidth f_M

bilateral bandwidth: total width of spectrum including negative freq. $2 \times f_M$



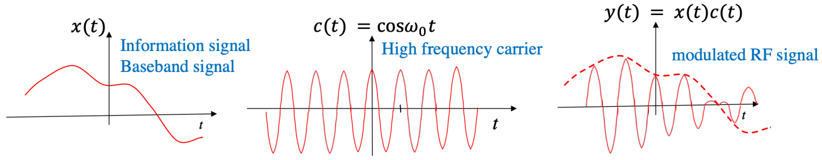
bilateral = 2x maximum frequency / unilateral

Hard to transmit at low freq: $\lambda = \frac{c}{f}$ $f \to 0$ $\lambda \to \infty$

So multiply baseband by radio carrier to shift frequency

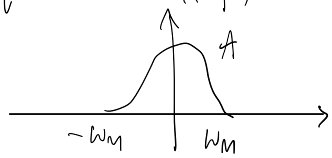
Modulation

Example 4.21: In ***Amplitude Modulation*** (AM), we ***multiply*** the information signal $x(t)$ by a higher frequency sinusoid ***carrier*** to produce a ***modulated signal*** before transmission. The modulated signal is also called a Radio Frequency (RF) signal.

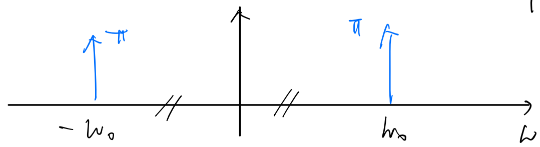


- Multiplication of two signals is also called ***mixing***.

Freq. Domain

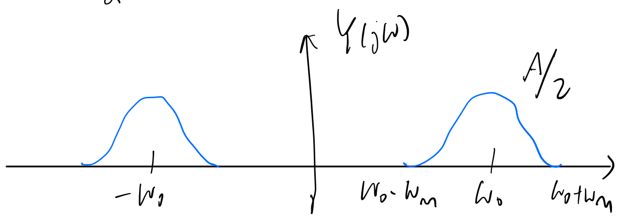


$C(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ FT of $\cos\omega_0 t$



$y(t) = x(t)c(t) = \frac{1}{2\pi} X(j\omega) * C(j\omega)$ ↙ $X(j\omega) * \frac{1}{2}\delta(\omega+\omega_0)$

$= \frac{1}{2} X(j(\omega-\omega_0)) + \frac{1}{2} X(j(\omega+\omega_0))$



Modulated Signal

Shifted copies of $FT\{x(t)\}$

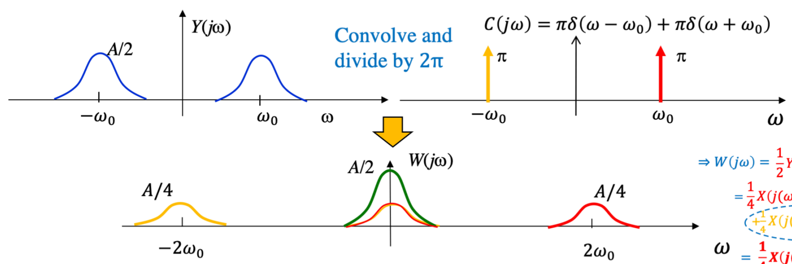
Now at higher freq. range and good for transmission.

Demodulation

Recover $x(t)$ from $y(t) = x(t)c(t)$: demodulation

- Multiply $y(t)$ by $c(t)$ again.

- Let $w(t) = y(t)c(t)$. Then $W(j\omega) = \frac{1}{2\pi} Y(j\omega) * C(j\omega)$:

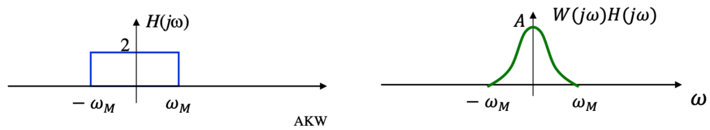


$\Rightarrow W(j\omega) = \frac{1}{2} Y(j(\omega-\omega_0)) + \frac{1}{2} Y(j(\omega+\omega_0))$

$= \frac{1}{4} X(j(\omega-\omega_0-\omega_0)) + \frac{1}{4} X(j(\omega+\omega_0-\omega_0)) + \frac{1}{4} X(j(\omega-\omega_0+\omega_0)) + \frac{1}{4} X(j(\omega+\omega_0+\omega_0))$

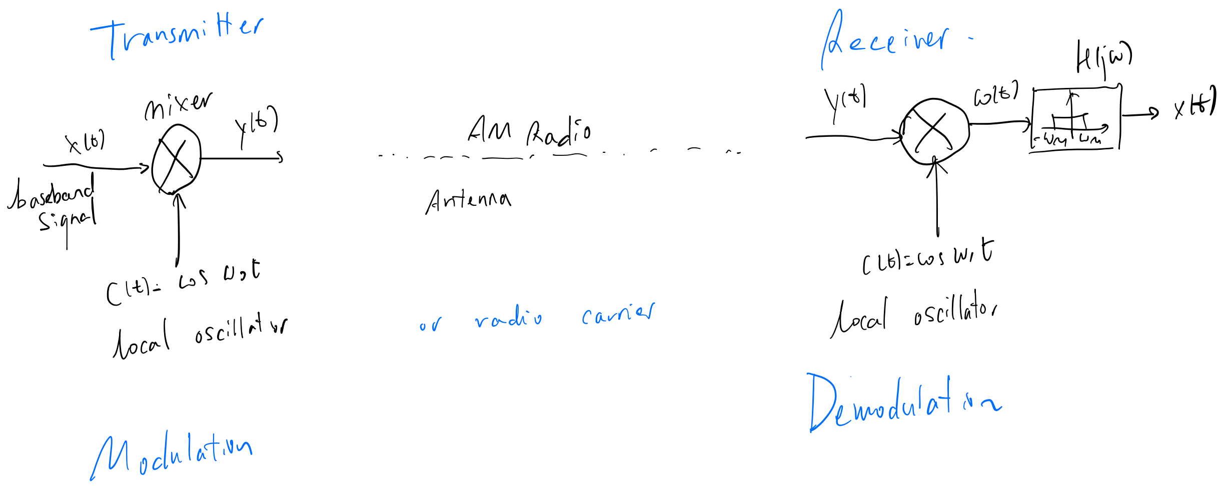
$= \frac{1}{4} X(j(\omega-2\omega_0)) + \frac{1}{2} X(j\omega) + \frac{1}{4} X(j(\omega+2\omega_0))$

Then we apply an ILP with a scaling factor of 2 to $w(t)$, we recover $x(t)$.



↳ Ideal Low Pass Filter.

AM Modu/Demodn System



- Going back to the mathematics of demodulation, we observe that:

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ ← $\cos(A)\cos(B) = \frac{1}{2}(\cos(A+B) + \cos(A-B))$ and let A=B

$$w(t) = y(t)c(t) = x(t)\cos^2\omega_0 t \;\; = \frac{1}{2}x(t)(1 + \cos 2\omega_0 t)$$

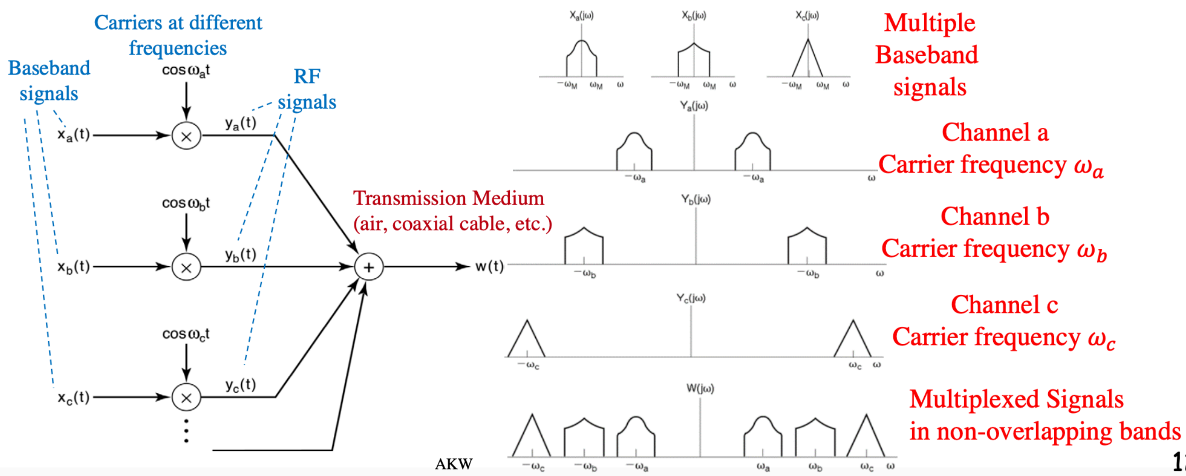
$$= \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_0 t$$

So $w(t)$ has one part that is the original baseband signal,

and another that is shifted to high frequency by $2\omega_0$ and is removed by low-pass filtering

II. Use of Radio Spectrum

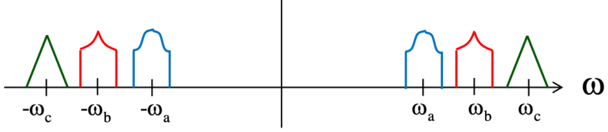
- Another important reason for modulation is for *Frequency Division Multiplexing* (FDM) which allows multiple signals to be transmitted over the same medium at the same time.
- In FDM, we modulate different baseband signals using different carrier frequencies. The resulting RF signals can be transmitted together at the same time because they remain separate in the frequency domain..



FDM Demodulation

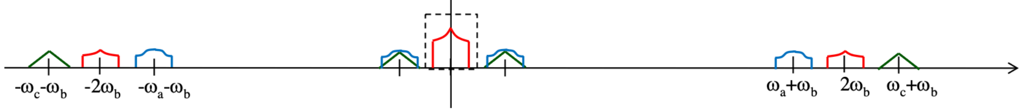
- To demodulate a specific channel within the multiplexed signal, one way is to mix the multiplexed signal with the corresponding carrier frequency and then LPF.

Spectrum of Multiplexed Signal $w(t)$: $W(j\omega)$



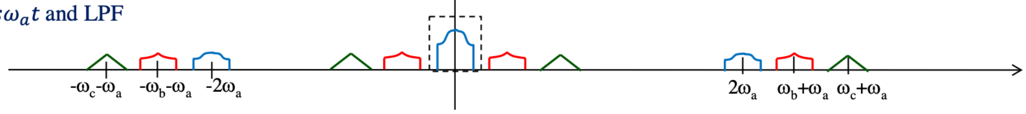
To demodulate channel b, mix with $cos\omega_b t$ and LPF

Spectrum of $w(t)cos\omega_b t$



To demodulate channel a, mix with $cos\omega_a t$ and LPF

Spectrum of $w(t)cos\omega_a t$



AM Modulation/Dem.

$$w(t) = x(t)\, C_{Tx}(t)\, C_{Rx}(t) = x(t)(\cos w_0 t)\cos(w_0 t + \theta) = \frac{1}{2}x(t)\cos\theta + \cdots$$

Have to make phase offset between radio carrier small!

If offset $= 90°$, output from LP $= 0$ since $\cos 90° = 0$

Transmit: $\cos w_0 t$

Receive: $\cos(w_0 t + \theta)$

I/Q Channels

Two Information Channels for the Same Carrier

- On the other hand, the previous slide implies that we can use two carriers at the same frequency but 90° out of phase (cosine and sine waves) to transmit two information signals at the same time.
- By convention, the carrier lagging in phase by 90° is called the I (In-Phase) channel, and the carrier leading in phase by 90° is called the Q (Quadrature Phase) channel.

Now I simply use ω instead of ω_0 for convenience

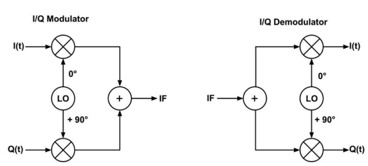
I/Q Transmitter

$$x_I(t)\cos(\omega t) + x_Q(t)\cos\left(\omega t + \frac{\pi}{2}\right)$$

or $x_I(t)\cos(\omega t) - x_Q(t)\sin(\omega t)$

$x_I(t), x_Q(t)$, are two information signals.

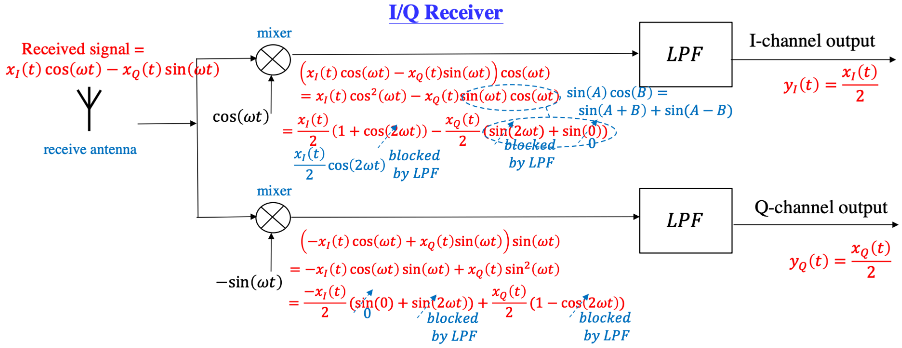
We can think of them as one complex-valued information signal, with $x_I(t)$ being the real part of the signal and $x_Q(t)$ being the imaginary part.



I/Q Channel Receiver

- At the receiver, we mix the received signal separately with the in-phase carrier and quadrature carrier to recover the I-channel and Q-channel information signals. For each of the I/Q receiver, the signal with carrier that is 90º out of phase will not produce any output!

I/Q Receiver



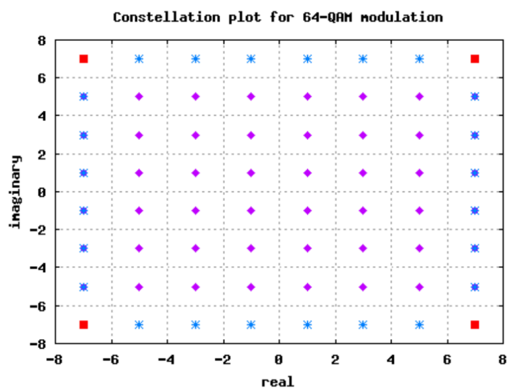
QAM

$x(t) = \sigma_i = \{\alpha_i + j\beta_i\}.$

16-QAM : $2^4 = 16$ 4 bits

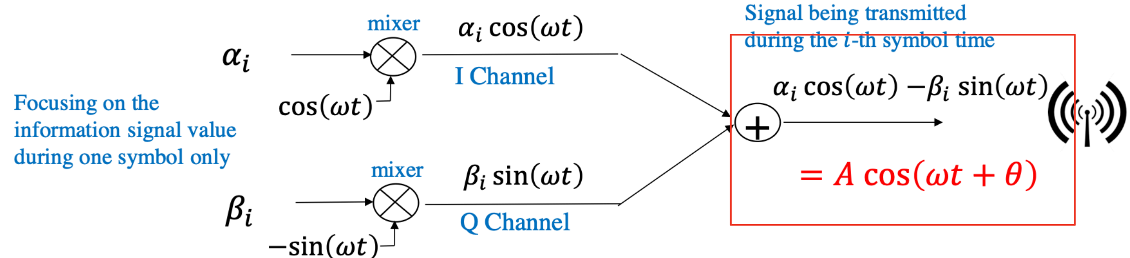
64-QAM : $2^6 = 64$ 6 bits

Signal constellation



I/Q Channels - Two Waves = One Wave

- So what does a complex σ_i mean?
- During each symbol time, to transmit σ_i, we modulate the in−phase carrier by $\alpha_i = Re\{\sigma_i\}$ and the quadrature carrier by $\beta_i = Im\{\sigma_i\}$. Then we transmit the sum:



$\alpha_i = Re\{\sigma_i\} = A\cos(\theta)$

- Signal transmitted for the i-th symbol is $\alpha_i \cos(\omega t) - \beta_i \sin(\omega t)$.
  We can re-express it as $A\cos(\theta)\cos(\omega t) - A\sin(\theta)\sin(\omega t)$ where A and θ are the magnitude of phase of the symbol σ_i.
- Using the trigonometric identity $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we see that the transmit signal is simply $A\cos(\omega t + \theta)$! The transmitted signal is simply one wave with an amplitude and phase as specified by σ_i!

Complex Representation of Modulated Signal

- The transmitted signal for the i-th symbol is $\alpha_i \cos(\omega t) - \beta_i \sin(\omega t) = A\cos(\omega t + \theta)$.
- A more concise representation of the transmitted signal is $\sigma_i e^{j\omega t}$.

$Ae^{j\theta}$. $Ae^{j(\theta+\omega t)}$

$\downarrow$ Re

$A\cos(\omega t+\theta)$

We can think of the transmitted RF signal $A\cos(\omega t + \theta)$ as the real part of $\sigma_i e^{j\omega t}$,
we can also think of $\sigma_i e^{j\omega t}$ as twice the positive frequency part of $A\cos(\omega t + \theta)$,

$$\text{since} \quad A\cos(\omega t + \theta) = \frac{1}{2}(\sigma_i e^{j\omega t} + \sigma_i e^{-j\omega t})$$

- Over all symbols, the transmitted RF signal can be presented mathematically by:

$$\text{transmitted RF signal} = \underbrace{\sum_i \sigma_i r(t - iT)}_{\substack{x(t) \\ \text{The information signal} \\ \text{that convey bits}}} \underbrace{e^{j\omega t}}_{\substack{\text{The complex sinusoid} \\ \text{representing the} \\ \text{I- and Q- carriers}}}$$

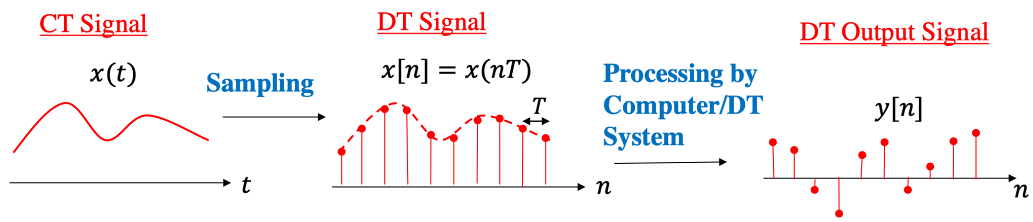
L16

Nyquist's Sampling Theorem

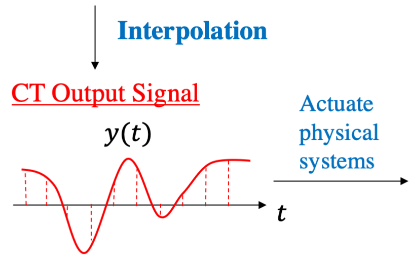
Today CT signals $\xrightarrow{\text{sampling}}$ DT digital signal

$\xleftarrow{\text{interpolation}}$

Sampling, Processing, and Interpolation



Intuitively, the DT signal $x[n] = x(nT)$ can represent the CT signal $x(t)$ very well if the sampling interval T is "small enough".



T : sampling interval

$\frac{1}{T}$: Sampling rate

$x[n] = x(nT)$

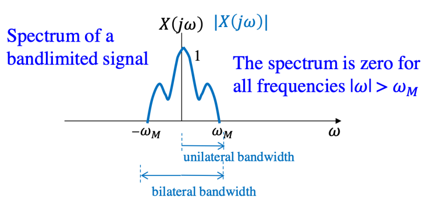
Ideally T should be small and $\frac{1}{T}$ high.
But costly for small Ts.
Determine a good sampling rate:

Nyquist's Sampling Theorem

If CT smooth enough + T small enough:
CT signal uniquely recoverable from samples DT

- Specifically, $x(t)$ with FT $X(j\omega)$ is band-limited with ***unilateral bandwidth*** ω_M if :

$$\boxed{X(j\omega) = 0 \quad \forall\, \omega \text{ s.t. } |\omega| > \omega_M}$$



spectrum is conjugate symmetric for real signal so bilateral $= 2\omega_M$

Stated as follow:

- Let $x(t)$ be a ***band-limited*** signal so that $X(j\omega) = 0$ for $|\omega| > \omega_M$.

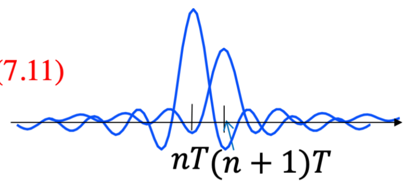
  Then $x(t)$ *can be uniquely reconstructed* by its samples $x(nT), n = -\infty, \ldots \infty$, if T is *small enough* such that $\boxed{\omega_s = \frac{2\pi}{T} > 2\omega_M}$ by:

  bilateral bandwidth

  ILPF

  Value of CT signal at any time t — Sampled values of $x(t)$ at instances nT. — A shifted sinc function centered at nT.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{T\sin(\omega_c(t-nT))}{\pi(t-nT)} \quad \ldots(7.11)$$



  for any ω_c such that $\omega_M < \omega_c < \omega_s - \omega_M$

- $\omega_s = \frac{2\pi}{T}$ is the *sampling frequency* in angular frequency.

  $f_s = \frac{1}{T}$ is the *sampling frequency* in ordinary frequency

Nyquist's Rate

- Nyquist theorem states that the sampling frequency needs to be greater than two times the signal's maximum frequency (=bilateral bandwidth):

$$f_S > 2f_M \text{ or } \omega_s > 2\omega_M$$

- This minimum sampling rate required is called the ***Nyquist rate*** or ***Nyquist frequency***.

The Sampling Theorem – Final Explanation

- From table 4.2, the impulse response of an ILPF with cut-off frequency ω_c is

$$h_{lp}(t) = \frac{\sin \omega_c t}{\pi t}$$

- Therefore:

  Low-pass filtering of $x_p(t)$ yields $x(t)$ if no aliasing

$$x(t) = Tx_p(t) * h_{lp}(t) = T\Bigg(\underbrace{\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)}_{x_p(t)}\Bigg) * h_{lp}(t)$$

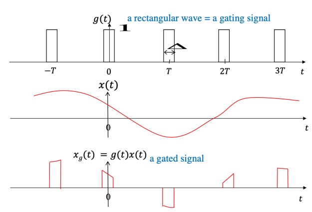
$$= T\sum_{n=-\infty}^{\infty} x(nT)\, h_{lp}(t-nT) = T\sum_{n=-\infty}^{\infty} x(nT)\frac{\sin \omega_c (t-nT)}{\pi(t-nT)} \quad \text{which is eqn. (7.11).}$$

  $\delta(t-nT) * h_{lp}(t) = h_{lp}(t-nT)$

- Eqn. (7.11) represents the convolution of a sampled impulse train with the ILPF.

- You can also view eq. (7.11) as an *interpolation formula* that interpolates $x(t)$ from all of its sampled values.

- Sampling by gating instead of impulse train



Aliasing

Recall: $e^{j(\Omega + m2\pi)n} \equiv e^{j\Omega n}$

then means:

while $e^{j(\omega+m\omega_s)t} \neq e^{j\omega t}$

$$e^{j(\omega+m\omega_s)nT} \equiv e^{j\omega nT}$$

This is because $e^{j(\omega+m\omega_s)nT} = e^{j(\omega nT+\cancel{mn2\pi})} = e^{j\omega nT}$

since $m\omega_s nT = m\frac{2\pi}{T}nT = mn2\pi$

Complex sinusoids $e^{j\omega n}$ at ordinary freq. f and $f + m f_s$ produce same set of samples

Sampling of a real sinusoid

- In many physical problems, we are concerned with the real sinusoid (e.g., our ears hear real sinusoids).
- A real sinusoid is a conjugate pair of complex sinusoids, and therefore we can say even more.

Consider the cosine:

Like the complex sinusoid, if sampled at ω_s, increasing frequency by $m\omega_s$ does not change any sample value:

$$\cos((m\omega_s + \omega)nT) = \cos\left(\left(m\frac{2\pi}{T} + \omega\right)nT\right) \overset{\cancel{+mn2\pi}}{=} \cos(\omega nT)$$

$\cos(-\theta) = \cos(\theta)$

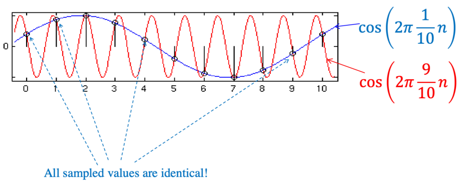
But cosine is even; meaning: $\cos(\omega nT) = \cos(-\omega nT) = \cos\big((m\omega_s - \omega)nT\big) = \cos\big((m\omega_s + \omega)nT\big)$

Therefore $\Rightarrow \cos(m\omega_s \pm \omega)nT = \cos(\omega nT)$

This means if we sample at 1Hz, real sinusoids at 1.1 Hz, 1.9 Hz, 2.1 Hz, 2.9 Hz, -0.1 Hz, -0.9Hz …, etc., will all become the same as 0.1 Hz!

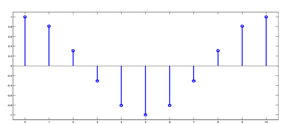
Sampling of a real sinusoid

The diagram below illustrates that two sinusoids, one at frequency 0.1 Hz and the other at frequency 0.9 Hz, give identical samples when sampled at T=1 (f_s = 1):



Aliasing for real sinusoid

If I just show you the sampled DT sequence from the previous slide, what frequency will you perceive?



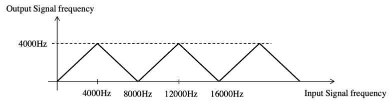
- We perceive 0.1 Hz.
- But CT sinusoids at 0.9 Hz, 1.1 Hz, 1.9 Hz, 2.1 Hz, etc., will all produce the DT sequence above and be perceived as 0.1 Hz!
- This is ***aliasing***, the *disguising of high frequency signals as low frequency*.

Aliasing: Disguising of high frequency signals as low frequency
伪装

Demo – Sampling for Digital Telephony

- We mentioned that the modern digital telephone network samples voice signal at 8,000 Hz. Input frequencies above 4,000 Hz will alias into lower frequencies. For example, 5000 Hz and 11000 Hz will appear as 3000 Hz, 7000 Hz and 9000 Hz will appear as 1000 Hz, etc.
- We can plot the perceived output frequency against the input CT signal frequency. As we sample at 8 KHz, the highest perceived frequency possible is 4 KHz.

In the telephone network, we sample at 8 KHz. That means input signals at 1 KHz, 7 KHz, 9 KHz, 15 KHz, 17 KHz, 23 KHz, etc., will all produce identical samples and



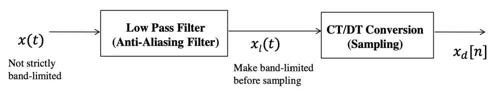
Sample at 8kHz
highest possible : 4kHz

$\frac{1}{2} f_s$

Anti-Aliasing Filter

In many CT to DT conversion systems, we want to keep the sampling rate low, but the input CT signal may not be strictly band-limited to ½ of the sampling rate.

- A ***anti-aliasing filter*** (**AAF**), which is simply a low-pass filter, is often used to remove the high frequency components in the CT signal before sampling.



- With anti-aliasing filter, the error is *only* in the high frequency components lost.
- Without anti-aliasing filter, *the high frequency components are not only lost, they turn into low frequency signals that should not be there*!

高频信号会伪装成低频,
造成干扰

If sample at 8kHz,
filter to remove components
above 4kHz

Aliasing in Image: Moire Effect

L17

Recall : $x_p(t) = x_c(t)\,p(t)$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\omega - k\omega_s)\right)$$

Poisson Sum of $x_c(t)$ scaled $\frac{1}{T}$

$$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) \quad \text{or} \quad X_p(j\omega) = X_d(e^{j\omega T})$$

Same except for scaling in frequency

$X_c(t)$: CT signal

$X_p(t)$: Sampled Impulse train $= x_c(t)\,p(t)$

$X_d[n]$: Sampled sequence $= X_c(nT)$

- Again $X_p(j\omega)$ is a scaled Poisson sum of $X_c(j\omega)$: $X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\omega - k\frac{2\pi}{T}\right)\right)$ $\omega_s = \frac{2\pi}{T}$
- If $x_c(t)$ is bandlimited to $\omega_M < \frac{\pi}{T}$ so that there is no aliasing, then:

Eq.7.21 of preceding slide

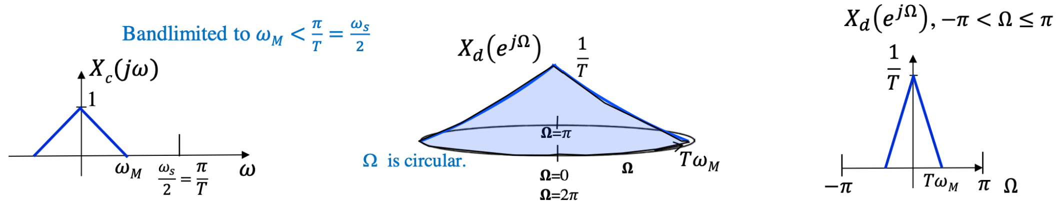
$\omega_M < \frac{1}{2}\omega_s$

$\omega_s > 2\omega_M$

$$X_c(j\omega) = \begin{cases} TX_p(j\omega) = TX_d(e^{j\omega T}) & -\frac{\pi}{T} \le \omega \le \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$

$-\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}$

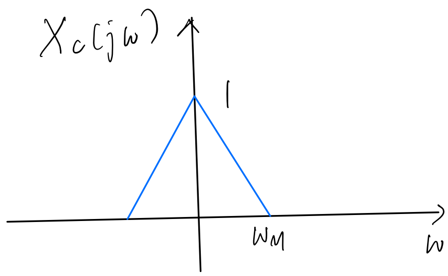
- Ω is circular. We can regard the DTFT $X_d(e^{j\Omega})$ as a frequency-scaled and circular version of $X_c(j\omega)$; i.e., $\boxed{X_d(e^{j\Omega}) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right) \quad \text{for } -\pi < \Omega \le \pi}$, as shown below:



$\Omega = \omega T$

Relationship among $X_c(j\omega)$, $X_p(j\omega)$, $X_d(e^{j\Omega})$

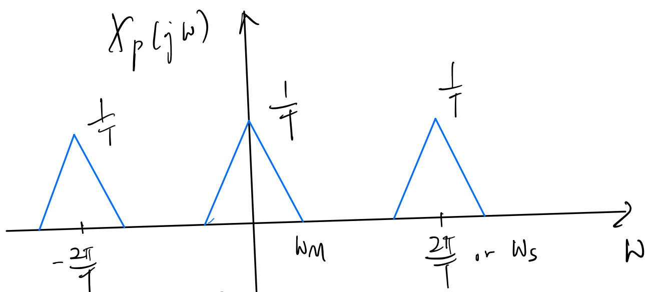
CTFT of bandlimited signal $x_c(t)$ (CT)



CTFT of sampled impulse train

$x_p(t) = x_c(t)\,p(t)$

$\boxed{\text{Scaled } \frac{1}{T} \text{ and repeat every } \omega_s = \frac{2\pi}{T}}$

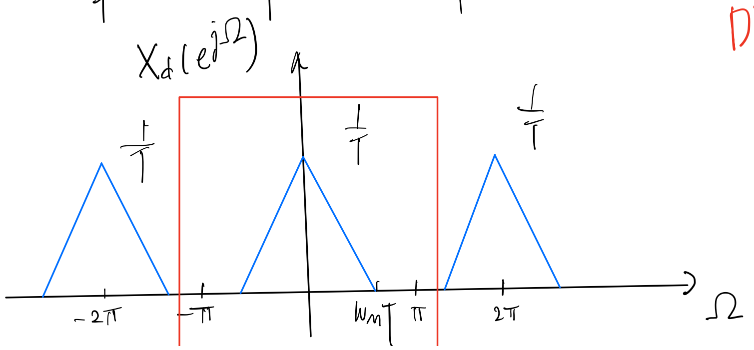


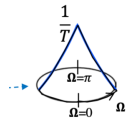
DTFT of sampled DT signal

$x_d[n] = x_c(nT)$

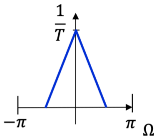
$X_d(e^{j\Omega}) = X_p\left(j\frac{\Omega}{T}\right) \quad \Omega = \omega T$

frequency scaled by T





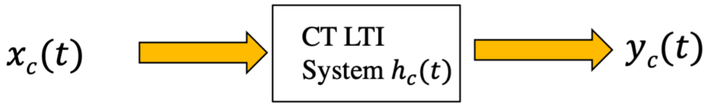
DT frequency is circular



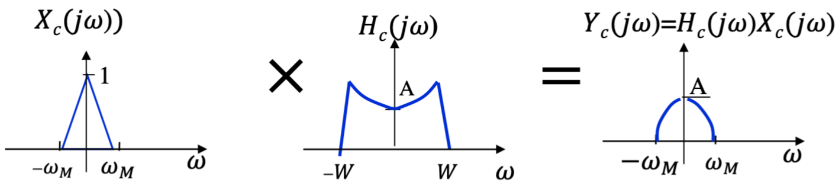
DT frequency is from $-\pi$ to π only

DT processing of CT signals

- Suppose we want to filter a bandlimited CT signal $x_c(t)$ using an LTI system to produce the output $y_c(t)$.



- In frequency domain, $Y_c(j\omega) = H_c(j\omega)X_c(j\omega)$.



- Given the equivalency of a bandlimited CT signal and its sampled DT signal, we can produce the same output by converting $x_c(t)$ to DT and applying a DT filter $H_d(e^{j\Omega})$ that is "equivalent" to $H_c(j\omega)$.

  - To process CT signals in DT, conceptually we need to take the following steps:

    1. Convert CT input to its DT sampled sequence: $x_d[n] = x_c(nT)$
    2. Apply an equivalent DT filter $H_d(e^{j\Omega})$
    3. Convert DT output to CT by interpolation/ILP with scaling

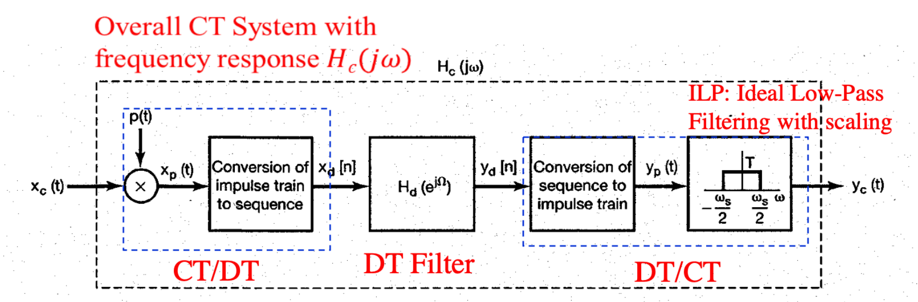


Figure 7.24 Overall system for filtering a continuous-time signal using a discrete-time filter.

Again, we use the symbols Ω and ω to refer to DT and CT frequency to avoid confusion.

Equivalent DT Filter

- The frequency response of the DT filter simply needs to be the same frequency scaled version of the CT filter:

$$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \le \pi}$$

$\Omega = \omega T$

- Then in the frequency range $|\Omega| \le \pi$, the output

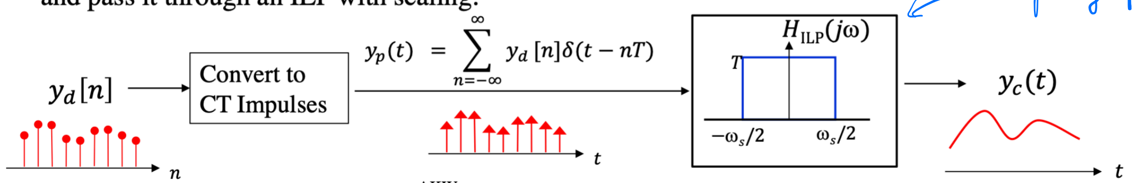
$$Y_d(e^{j\Omega}) = X_d(e^{j\Omega})H_d(e^{j\Omega}) = \frac{1}{T}X_c\left(j\frac{\Omega}{T}\right)H_c\left(j\frac{\Omega}{T}\right) = \frac{1}{T}Y_c\left(j\frac{\Omega}{T}\right) \quad \text{for } |\Omega| \le \pi$$

- This means $y_d[n]$ is the sample sequence of the CT output $y_c(t)$:

$$y_d[n] = y_c(nT)$$

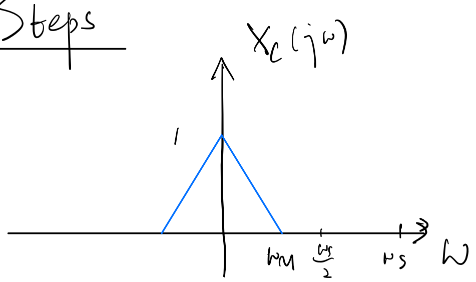
and we can convert $y_d[n]$ to $y_c(t)$ by converting $y_d[n]$ to a sampled impulse train and pass it through an ILP with scaling:

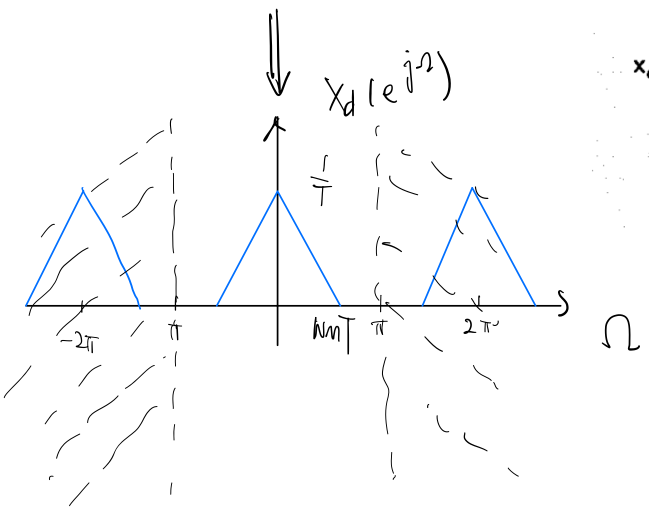
Scaling by T

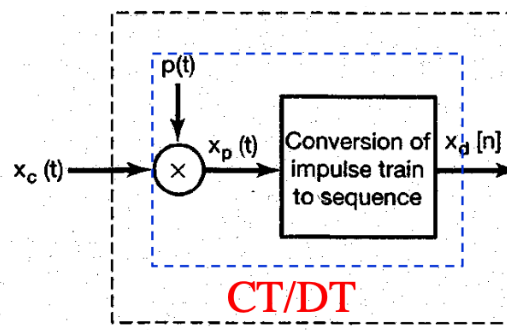


AKW

Steps

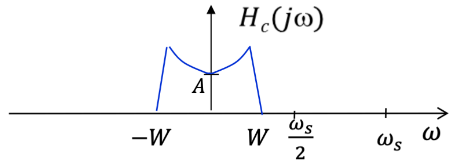


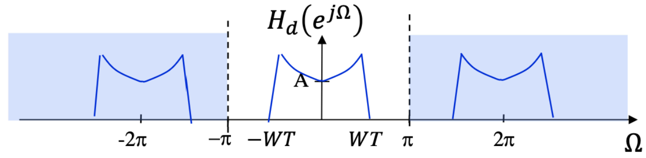


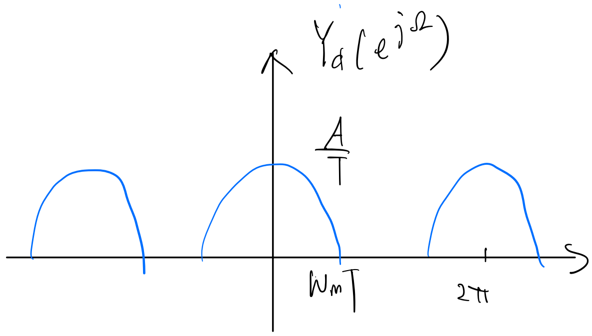


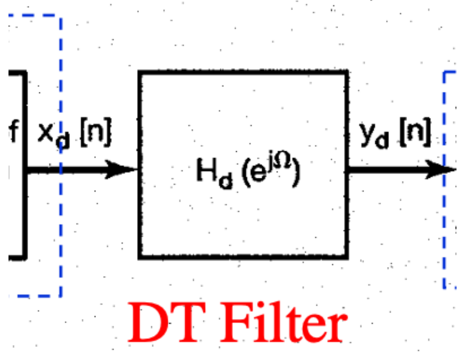
Apply DT filter w/ frequency-scaled frequency response

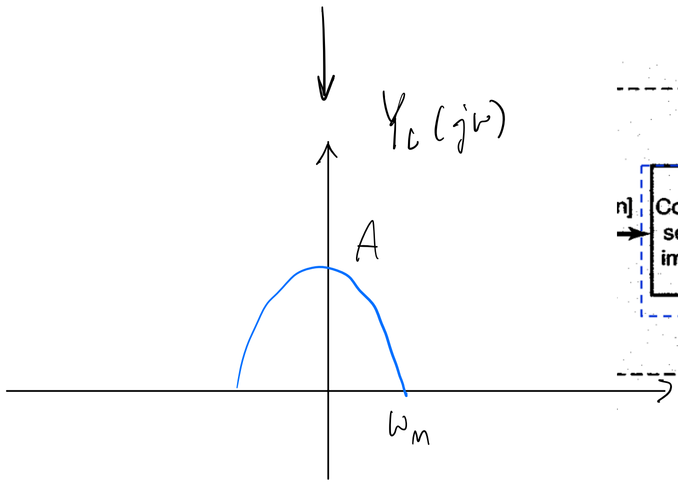
$$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad -\pi \le \Omega < \pi}$$

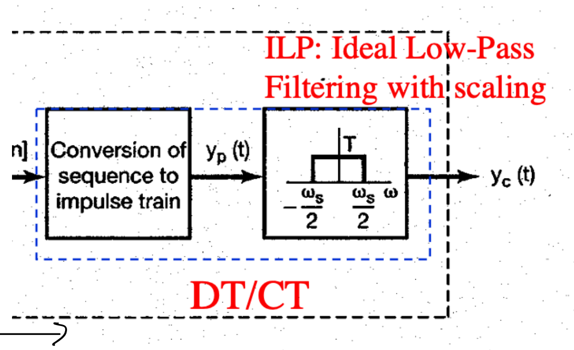












Impulse Response of the Equivalent DT Filter

- The frequency response of the equivalent DT filter is a frequency-scaled version of the CT filter:

  DTFT of equivalent DT filter CTFT of $h_c(t)$

  $$\boxed{H_d(e^{j\Omega}) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \le \pi}$$

- Can we also say that the impulse response $h_d[n]$ of the equivalent DT filter are the sample values (with scaling) of the impulse response $h_c(t)$ of the CT filter?

  The answer is **yes** *if* $H_c(j\omega)$ is also $\frac{\omega_s}{2}$-bandlimited by the same argument we went through for signals.

- All these constant scaling by T and $\frac{1}{T}$ seem very confusing. In real implementation we can simply do one scaling once and for all at the end. Sometimes we do not care about the scaling at all (we need to apply amplification anyway or we simply want to compare results)

Summary - Discrete-time processing of continuous-time signals

$x_c(t)$ is band-limited and we want to filter it with a CT filter $H_c(j\omega)$.

1. Sample $x(t)$ at above Nyquist rate to obtain DT signal $x_d[n] = x_c(nT)$
2. Apply DT filter $H_d(e^{j\Omega})$ such that

$$\boxed{H_d\left(e^{j\Omega}\right) = H_c\left(j\frac{\Omega}{T}\right) \qquad |\Omega| \leq \pi}$$

3. DT output will be sampled value of intend CT output; i.e., $y_d[n] = y_c(nT)$, and we can recover $y_c(t)$ by converting $y_d[n]$ to a sampled impulse train $y_p(t)$, low-pass filtering, and scaling by T.
4. If the impulse response $h_c(t)$ of the CT filter is band-limited, we can also say that the impulse response of the needed DT filter is the sampled value of $h_c(t)$ with scaling by T. That is, $h_d[n] = Th(nT)$.

1. Sample
2. Apply DT filter
3. $y_d[n] \rightarrow y_p(t) \xrightarrow{\text{LP by } T} y_c(t)$

L18

Spectrum analyzer

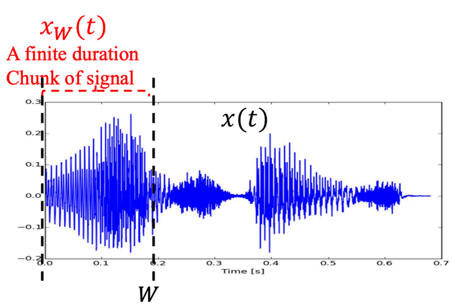
↳ Analyze spectrum of signals

SA Operation (I) – Taking a finite chunk of signal

- Can the SA compute the FT integral?

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{Cannot wait till } t = \infty \text{ to compute this integral!}$$

- The SA has to analyze one "chunk" of the signal at a time. ***That's why the spectrum in the demo changes with time***. Assume each chunk has duration W and let's call it $x_W(t)$.



⇒ ***How should we compute its spectrum?***

Spectrum of Finite Duration Chunk

FT: $X_W(j\omega) = \int_{-\infty}^{\infty} x_W(t)\, e^{-j\omega t}\, dt = \int_0^W x_W(t)\, e^{-j\omega t}\, dt$

FS: $a_k = \frac{1}{W}\int_0^W x_W(t)\, e^{-jk\frac{2\pi}{W}t}\, dt = \frac{1}{W} X_W\left(k\frac{2\pi}{W}\right)$

- We observe:
  1. The FS coefficients are sampled values of the FT (with scaling of $1/W$)
  2. Either the FT or FS fully specifies $x_W(t)$ - we can synthesize $x_W(t)$ from either one. So for a finite duration signal, ***the sampled values of the FT fully specify the FT***! This is analogous to sampling theorem!

Freq. Resolution

FS: fundamental frequency $= \frac{1}{W}$ Hz or $\frac{2\pi}{W}$ rads

↳ also frequency resolution.

Higher W → Can resolve lower frequency $\frac{1}{W}$

longer time

SA Operation (II) – Sampling and Numerical Integration

- Instead of doing integration, the SA, as a digital device, samples the integrand at interval Δ, and performs a numerical integration (i.e., summation) which is again a staircase approximation of the integral:

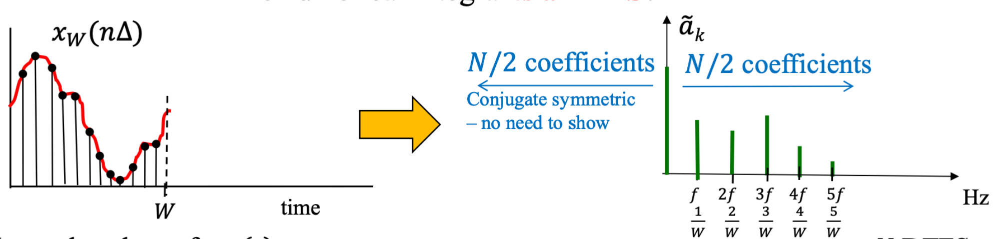
FS Analysis Integral: $$a_k = \frac{1}{W}\int_0^W x_W(t)\, e^{-jk\frac{2\pi}{W}t}\, dt$$

Number of samples $N = \frac{W}{\Delta} \Rightarrow \frac{\Delta}{W} = \frac{1}{N}$

Numerical Integral/ Staircase approximation: $$\tilde{a}_k = \frac{1}{W}\sum_{n=0}^{N-1} x_W(n\Delta) e^{-jk\frac{2\pi}{W}n\Delta}\,\Delta = \frac{1}{N}\sum_{n=0}^{N-1} x_W(n\Delta) e^{-jk\frac{2\pi}{N}n}$$

This is a DTFS analysis sum!

The numerical integral ***is a DTFS***!



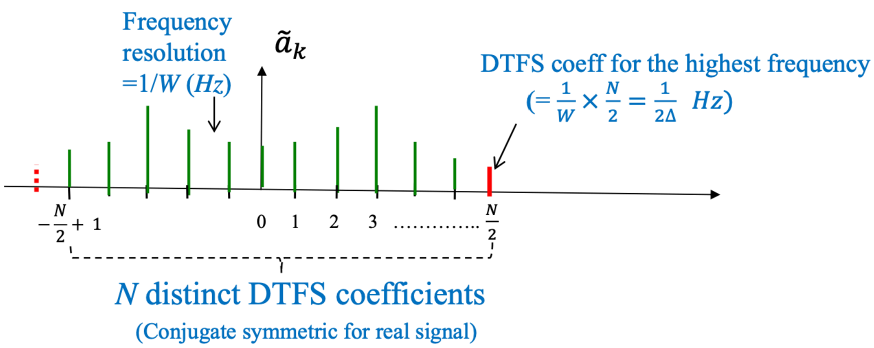
N sample values of $x_W(t)$ $\quad$ N DTFS coefficients

Choice of the Sampling Interval for Numerical Integration

- The numerical integral is the **DTFS of the sampled sequence**: $x_W[n] = x_W(n\Delta)$

$$\tilde{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x_W(n\Delta)e^{-jk\frac{2\pi}{N}n} \quad ; \quad x_W[n] = x_W(n\Delta)$$

- For a DTFS, the FS is periodic; i.e., $\tilde{a}_{k+N}=\tilde{a}_k$. Or viewed alternatively, there are only $N = W/\Delta$ distinct $\tilde{a}_k$, with $k = N/2$ (assume N even) representing the highest frequency harmonic at frequency of $1/2\Delta$.



Summary

In SA:

To receive fine frequency resolution → Long Chunk of Signal

↳ Higher W

freq. resolution $= \frac{1}{W}$ Hz

duration of chunk

To handle high frequency → High Sampling rate

Highest frequency $= \frac{1}{2\Delta} = \frac{1}{2} f_s$ Hz

sampling rate

DFT & FFT

Modem based on DFT instead of DTFS

- ***DFT is the same as DTFS***, except for the absence of the scaling factor. The difference is only a matter of conventional preference.

DTFS $\quad a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$

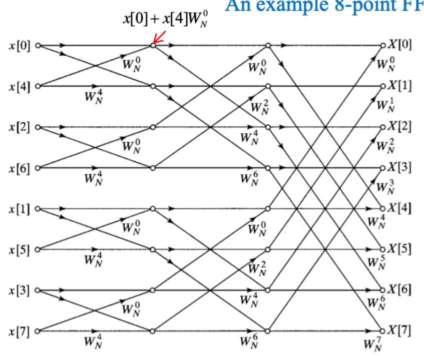
DFT $\quad X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{jk\frac{2\pi}{N}n} \qquad X[k] = a_k N$

DFT and DTFS are the same except for where you put the 1/N!

FFT is a faster algorithm for Calculating DTFS & DFT

- For a DT sequence of duration N, FFT takes order of $O(Nlog_2N)$ multiplications to compute the entire set of $X[k]$ (or DTFS coefficients a_k).
- FFT is based on $\log_2 N$ stages of computation of intermediate values as illustrated by the computation network below:

An example 8-point FFT; N=8



$\boxed{W_N^k = e^{-j2\pi k/N}}$

Each butterfly handle 2 multiplication

$2^3 = 8 \quad$ 3 stages

Each stage $\frac{8}{2} = 4$ butterflies.

So 12 butterflies

$12 \times 2 = 24$ Addition.

Digital Comm. & ODFM

Digital Comm: Send sequence of 0 and 1 bits.

- Recall we introduce a simple data stream signal $x(t) = \sum_i \sigma_i r(t - iT)$ in Lecture 2, Section 4, where:

  T is the symbol duration,

  $r(t - iT) = \begin{cases} 1 & iT \le t < (i+1)T \\ 0 & otherwise \end{cases}$ is a window covering the i-th symbol,

  and σ_i is some signal amplitude value conveying 0 and 1 bits.

- In radio and optical communications, the data stream signal $x(t)$ is not transmitted as is in baseband. Instead, it is used to modulate, or vary, a sinusoidal electro-magnetic (EM) wave carrier. We mentioned previously that we can vary the *amplitude*, *phase*, or *frequency* of the EM carrier to convey bits.
- In this section, we will show that we can easily represent amplitude and phase modulation by allowing σ_i to be complex.

I-Q Channel

Recall: I/Q transmitter.

$$x_I(t)\cos(\omega t) + x_Q(t)\cos\left(\omega t + \frac{\pi}{2}\right)$$

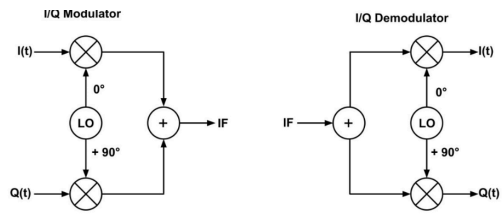
$$- x_Q(t)\sin(\omega t)$$

lags by 90°
I-channel

leads by 90°
Q-channel

$x_I(t), x_Q(t)$, are two information signals.

We can think of them as one complex-valued information signal, with $x_I(t)$ being the real part of the signal and $x_Q(t)$ being the imaginary part.



Recall : **I/Q Channel Receiver**

- At the receiver, we mix the received signal separately with the in-phase carrier and quadrature carrier to recover the I-channel and Q-channel information signals: