L19

LCCDE Linear Constant-Coefficient Differential.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where usually $N > M$ and is the order of the system.

LTI!

weighted sum of 0 to Nth derivatives of output

weighted sum of 0 to Mth derivatives of input

Frequency Response

Take FT of both side :

$$FT\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = FT\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

Linearity :

$$\sum_{k=0}^{N} a_k \, FT\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \, FT\left\{\frac{d^k x(t)}{dt^k}\right\}$$

Table 4 :

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

- This means $H(j\omega)$ is in the form of the ratio of a numerator polynomial and a denominator polynomial in $j\omega$. Such a frequency response is said to be ***rational***, or of ***rational form***.

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} = \frac{N(j\omega)}{D(j\omega)}}$$

Numerator Polynomial

Denominator Polynomial

Find Residue:

2) For each partial fraction term, evaluating the numerator constant, called the ***residue***.

residues

$$H(j\omega)=\frac{N(j\omega)}{D(j\omega)}=\frac{N(j\omega)}{(j\omega-\alpha_1)(j\omega-\alpha_2)(j\omega-\alpha_3)}=\frac{c_1}{(j\omega-\alpha_1)}+\frac{c_2}{(j\omega-\alpha_2)}+\frac{c_3}{(j\omega-\alpha_3)}$$

The following trick, illustrated for the case where the denominator polynomial has three distinct roots, will enable you to find the residue quickly:

$$\boxed{c_1=H(j\omega)\,(j\omega-\alpha_1)|_{j\omega=\alpha_1};\; c_2=H(j\omega)\,(j\omega-\alpha_2)|_{j\omega=\alpha_2};\; c_3=H(j\omega)\,(j\omega-\alpha_3)|_{j\omega=\alpha_3}}$$

Notice!! has root

nple:

$$H(j\omega)=\frac{N(j\omega)}{(j\omega-\alpha_1)(j\omega-\alpha_2)^2}=\frac{c_1}{(j\omega-\alpha_1)}+\frac{c_2}{(j\omega-\alpha_2)^2}+\frac{c_3}{(j\omega-\alpha_2)}$$

Freq Response → Impulse Response

1. By Differentiation Equation

$$\underbrace{}_{a_2=1}\frac{d^2y(t)}{dt^2}+\overset{a_1=4}{4}\frac{dy(t)}{dt}+\overset{a_0=3}{3}y(t)=\overset{b_1=1}{\frac{dx(t)}{dt}}+\overset{b_0=2}{2}x(t)$$

2. By Frequency Response

$$H(j\omega)=\frac{\overset{b_1=1}{(j\omega)}+\overset{b_0=2}{2}}{\underset{a_2=1}{(j\omega)^2}+\underset{a_1=4}{4(j\omega)}+\underset{a_0=3}{3}}\qquad H(j\omega)=\frac{1/2}{(j\omega+1)}+\frac{1/2}{(j\omega+3)}$$

3. By Impulse Response

$a=1$ $\quad a=3$

$$h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

Rational FT in Three Forms

1. The Polynomial Form

$$H(j\omega)=\frac{N(j\omega)}{D(j\omega)}=\frac{\sum_{k=0}^{M}b_k\,(j\omega)^k}{\sum_{k=0}^{N}a_k\,(j\omega)^k}=\frac{b_{N-1}(j\omega)^{N-1}+\cdots+b_1(j\omega)+b_0}{a_N(j\omega)^N+a_{N-1}(j\omega)^{N-1}+\cdots+a_1(j\omega)+a_0}$$

where $M=N-1$.

This form allows us to write the differential equation. Usually, we assume the order of $D(j\omega)$ is higher than that of $N(j\omega)$ so that $H(j\omega)\to 0$ as $\omega\to\infty$ (systems are usually bandlimited)

2. The Factored Form

Where both $N(j\omega)$ and $D(j\omega)$ are factorized into a product:

$$H(j\omega) = \frac{b_{N-1}\prod_{i=1}^{N-1}(j\omega - \beta_i)}{\prod_{k=1}^{N}(j\omega - \alpha_k)}$$

Roots: value of $j\omega$ s.t. the polynomial evaluates to 0

This form allows us to plot the magnitude and phase response as function of ω.

$$|H(j\omega)| = \frac{|b_{N-1}|\prod_{i=1}^{N-1}|j\omega - \beta_i|}{\prod_{k=1}^{N}|j\omega - \alpha_k|}$$

We can control the magnitude and phase response of a system by controlling the locations of the roots.

$$\sphericalangle H(j\omega) = \sphericalangle b_{N-1} + \sum_{i=1}^{N-1}\sphericalangle(j\omega - \beta_i) - \sum_{k=1}^{N}\sphericalangle(j\omega - \alpha_k)$$

This form also says that we can implement an Nth-order system as a cascade (product) of 1^{st}-order systems

3. The Partial Fraction Form

$$H(j\omega) = \sum_{k=1}^{N}\frac{c_k}{j\omega - \alpha_k}$$

Residues

Roots

This form allows us to find the inverse transform as sum of one-sided exponentials.

From Table 4.2 , the FT of a one-sided exponential is:

$$e^{-at}u(t);\ \ Re\{a\} > 0 \overset{FT}{\longleftrightarrow} \frac{1}{(j\omega + a)}$$

absolute integrable iff $Re\{a\} > 0$

By linearity (and assuming $h(t)$ is causal; more later), the impulse response is a sum of one-sided exponentials:

$$H(j\omega) = \sum_{k=1}^{N}\frac{c_k}{j\omega - \alpha_k} \Rightarrow h(t) = \sum_{k=1}^{N} c_k\, e^{\alpha_k t}u(t)$$

This form also says that we can implement an Nth-order system as a sum of 1^{st}-order systems

$h(t) = \sum_{k=1}^{N} C_k e^{\alpha_k t} u(t)$ is completely specified by roots α_k and residue C_k.

α_k is the exponential constant and tells everything about characteristics of a system.

- For a real system, $h(t)$ is real. Hence, for any complex root α_k, there must be another root $\alpha_{k'} = \alpha_k^*$ with residue $c_{k'} = c_k^*$, so that:

$$c_k e^{\alpha_k t}u(t) + c_k^* e^{\alpha_k^* t}u(t) = 2\mathrm{Re}\{c_k e^{\alpha_k t}u(t)\} = 2|c_k|\,\{e^{\mathrm{Re}\{\alpha_k\}t}\cos(\mathrm{Im}\{\alpha_k\}t + \measuredangle c_k)\}$$

The k-th term and k'-th term form a conjugate pair

Real part of a causal complex exponential multiplied by a complex number c_k

= A damped oscillation scaled by $|c_k|$ and shifted in phase by $\measuredangle c_k$

1. This means complex roots must occur in conjugate pairs, and if there is any complex root, the impulse response is ***oscillatory***.

Complex Root → Oscillatory

2. The roots α_k's are the exponential constants of the right-sided exponentials. Hence, if any root has a ***non-negative real part***, the corresponding exponential grows with time and the system is ***unstable***.

Re(Root) > 0 → Unstable

2nd-Order Systems

- For a 2nd order LCCDE, we can normalize the leading coefficient a_2 to 1 for convenience:

$$y^{(2)}(t) + a_1 y^{(1)}(t) + a_0 y(t) = b_1 x^{(1)}(t) + b_0 x(t)$$

Leading coefficient, a_2, normalized to 1

- For a real system, all coefficients a_0, a_1, b_0, b_1 are real.
- Frequency response is rational:

$$H(j\omega) = \frac{N(j\omega)}{D(j\omega)} = \frac{b_1(j\omega)+b_0}{(j\omega)^2 + a_1(j\omega) + a_0}$$

We can find roots of $D(j\omega)$ and express $H(j\omega)$ in partial fraction form:

Residues

$$H(j\omega) = \frac{b_1(j\omega)+b_0}{(j\omega-\alpha_1)(j\omega-\alpha_2)} = \frac{c_1}{(j\omega-\alpha_1)} + \frac{c_2}{(j\omega-\alpha_2)}$$

Roots

Which gives causal impulse response as: $h(t) = c_1 e^{\alpha_1 t}u(t) + c_2 e^{\alpha_2 t}u(t)$

Roots of $D(j\omega)$: α_1, α_2 are exponential constants in the impulse response and completely determines characteristics of response.

- The roots α_1, α_2 are given by the quadratic formula:

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_0}}{2}$$

the "radical"

$$\overset{1}{A}x^2 + \overset{a_1}{B}x + \overset{a_0}{C} = 0$$

$$\Rightarrow \text{Roots} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

- We note that α_1, α_2 are complex if $a_1{}^2 < 4a_0$, or $|a_1| < 2\sqrt{a_0}$

Radical is imaginary if the term inside the square root is negative

- We note also that: 1. $\alpha_1 \alpha_2 = a_0$;

  and 2. $\alpha_1 + \alpha_2 = -a_1$

  Because $(j\omega - \alpha_1)(j\omega - \alpha_2) = (j\omega)^2 \underbrace{-(\alpha_1 + \alpha_2)}_{a_1} j\omega + \underbrace{\alpha_1 \alpha_2}_{a_0}$

Characterization

1: Oscillatory Response : Oscillatory if roots of $D(j\omega)$ complex

- If α_1, α_2 are real,

  then $h(t) = c_1 e^{\alpha_1 t} u(t) + c_2 e^{\alpha_2 t} u(t)$ is a sum of two real one-sided exponentials.

- For a real second order system, if α_1, α_2 are complex, they must be a conjugate pair and so are c_1, c_2 , and

  the impulse response is a damped oscillation:

Real part of complex exponential is a damped oscillation

$$h(t) = c_1 e^{\alpha_1 t} u(t) + c_1^* e^{\alpha_1^* t} u(t) \;= 2Re\{c_1 e^{\alpha_1 t} u(t)\} \;= 2|c_1| e^{\sigma t} \cos(\omega t + \measuredangle c_1)\, u(t)$$

where $\sigma = \mathrm{Re}\{\alpha_1\}$ and $\omega = \mathrm{Im}\{\alpha_1\}$

2. Stability

$$h(t) = c_1 e^{\alpha_1 t} u(t) + c_2 e^{\alpha_2 t} u(t)$$

For $h(t)$ to be stable, both α_1, α_2 must have negative real parts. This means *with a_2 normalized to 1, all coefficients in the denominator must be positive* $(a_0, a_1 > 0)$.

This is because, with leading coefficient a_2 normalized to 1:

- Recall $\alpha_1\alpha_2 = a_0$. If $\alpha_1\alpha_2 = a_0 < 0$, it implies α_1, α_2 must be real with one being positive and one negative. α_1, α_2 cannot be complex because if they are, they must be a conjugate pair and their product cannot be <0.
- If $a_0 = 0$, at least one root is equal to 0, meaning $h(t)$ contains a unit step. not stable
- Recall $\alpha_1 + \alpha_2 = -a_1$. If $a_1 \leq 0$, it means $\alpha_1 + \alpha_2 = -a_1 \geq 0$. Then:

  If α_1, α_2 are complex, then $\alpha_1 = \alpha_2{}^*$, implying that Re{α_1}=Re{α_2}= $\frac{\alpha_1 + \alpha_2}{2} \geq 0$

  If α_1, α_2 are real, at least one of them is non-negative

Characterization by Natural Frequency and Zeta Parameter

We often express a 2nd order system using two alternative parameters :

i. The *natural frequency* $\omega_n = \sqrt{a_0}$, which provides a scaling in frequency.

ii. A *zeta parameter* $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$ which conveniently shows if the roots are complex.

$a_1 = 2\zeta\omega_n$

Re-express $D(j\omega)$ in terms of ω_n and ζ as:

$$D(j\omega) = (j\omega)^2 + a_1(j\omega) + a_0 = (j\omega)^2 + \underbrace{2\zeta\omega_n}_{a_1}(j\omega) + \underbrace{\omega_n{}^2}_{a_0}$$

Then:

$$\alpha_1, \alpha_2 = \frac{-a_1 \pm \sqrt{a_1{}^2 - 4a_0}}{2} = \frac{\overbrace{2\zeta\omega_n}^{a_1} \pm \sqrt{(\overbrace{2\zeta\omega_n}^{a_1})^2 - \overbrace{4\omega_n{}^2}^{a_0}}}{2} = \omega_n\,(-\zeta \pm \sqrt{\zeta^2 - 1})$$

scaling

$|\zeta| \geq 1 \Rightarrow \alpha_1, \alpha_2$ real

$|\zeta| < 1 \Rightarrow \alpha_1, \alpha_2$ complex

While ω_n provides a scaling in frequency, ζ makes explicit whether $a_1{}^2 - 4a_0 \geq 0$ and it also fully characterizes the system. (Recall $\omega_n = \sqrt{a_0}$, $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$, $\alpha_1, \alpha_2 = \omega_n\,(-\zeta \pm \sqrt{\zeta^2 - 1})$)

3a. If $\zeta \leq 0$, system is unstable (since it means $a_1 \leq 0$)

3b. If $|\zeta| < 1$, roots have imaginary part and system is *oscillatory* (if $|\zeta| < 1$ means $a_1{}^2 - 4a_0 < 0$)

3c. If $\zeta \gg 1$, one root is nearly $-2\zeta\,\omega_n$ but the other root is only slightly less than 0. Therefore the impulse response decays very slowly. The system is *over-damped*.

$\zeta \gg 1$ implies $\sqrt{\zeta^2 - 1} = \zeta - \varepsilon$ where $\varepsilon = 0^+$ <---- *Means just slightly larger than 0*

Hence, $\alpha_1, \alpha_2 = \omega_n\,(-\zeta \pm \sqrt{\zeta^2 - 1}) = \omega_n\,(-2\zeta + \varepsilon),\ \omega_n\,(-\varepsilon)$ Close to 0; decay very slowly

3d. If $\zeta \cong 1$, both roots are near $-\omega_n$, and system impulse response decays at the fastest rate possible. System is *critically damped* – desirable in a suspension system. close to $\frac{1}{(a+j\omega)^2}$

3e. If $\zeta \to 0^+$, system is *under-damped*. $|H(j\omega)|$ may become very large around ω_n as we will show in a few slides.

L20 LT and RoC

Laplace Transform and RoC

- For a signal $x(t)$, its LT is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt; \qquad x(t) \overset{L}{\leftrightarrow} X(s)$$

$$X(s) = L\{x(t)\}; \quad \text{and} \quad x(t) = L^{-1}\{X(s)\}$$

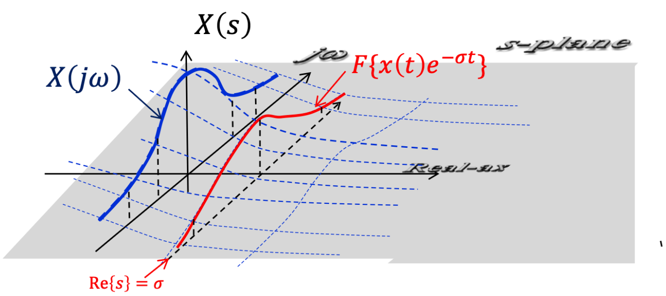
- In *Fourier analysis*, we limit our attention to the case of complex sinusoids where $s = j\omega$, and focus on the integral:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt; \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$s = \sigma + j\omega$$

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt = FT\{x(t)e^{-\sigma t}\}$$

which means that the LT of a signal $x(t)$ evaluated at $s = \sigma + j\omega$ can be viewed as the FT of $x(t)e^{-\sigma t}$.



We refer to the values of s for which $X(s)$ converges as the Region of Convergence (ROC) of the Laplace transform.

- If $X(s)$ exists for $s = \sigma + j\omega$, it means that the FT of $x(t)e^{-\sigma t}$ converges, which also means $x(t)e^{-\sigma t}$ is *absolute integrable*:

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$

- Note that the Laplace transforms in examples 9.1 and 9.2 have the same algebraic form. The only difference is in the ROC.

- **Hence, in specifying a Laplace Transform, we must also specify its ROC**.

LT in Rational Forms and Poles and Zeros

Rational Form of LT:

$$X(s) = \frac{N(s)}{D(s)}$$

$D(s)$: Denominator Polynomial

Roots s for $D(s) = 0$ are the poles of $X(s)$

for those s, $X(s)$ is infinite.

$N(s)$: Numerator Polynomial

Root s for $N(s) = 0$ are the zeros of $X(s)$

for those s, $X(s) = 0$

- Locations of poles and zeros completely specify the algebraic form of the LT except for a scaling factor.

- We will see that LT in rational form naturally arises in the study of differential equations. Recall that in the study of rational frequency response, the locations of the roots of $D(j\omega)$ tell us a lot about the system.

Properties of ROC

Property 1 **The ROC of an LT consists of strips (vertical regions) parallel to the $j\omega$-axis on the s-plane.**

- The ROC is the set of values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges, or $x(t)e^{-\sigma t}$ is absolute integrable:

$$\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}| dt < \infty$$

- The condition depends on σ only and hence the ROC is determined by σ only. That is, the ROC must be made up of vertical strips on the s-plane.

Property 2 **The ROC of a rational LT does not contain any pole.**

- The poles of a rational function is the zeros of its denominator.
- The value of $X(s)$ is infinite at a pole, which means that the LT does not converge at a pole.

Property 3 **If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire complex plane.**

- If a signal has *finite duration* from T_1 and T_2 (it has *finite support*), we can write

$$X(s) = \int_{T_1}^{T_2} x(t)e^{-\sigma t}e^{-j\omega t} dt \qquad e^{-st} = e^{-\sigma t}e^{-j\omega t}$$

- But, for $t \in [T_1, T_2]$, $|x(t)e^{-\sigma t}| \le |x(t)| \max(e^{-\sigma T_1}, e^{-\sigma T_2})$

where $$\max(e^{-\sigma T_1}, e^{-\sigma T_2}) = \begin{cases} e^{-\sigma T_1} & \sigma > 0 \\ e^{-\sigma T_2} & \sigma < 0 \end{cases} \qquad \begin{matrix} e^{-\sigma t} \text{ decaying, hence larger at } T_1 \\ e^{-\sigma t} \text{ growing, hence larger at } T_2 \end{matrix}$$

- Hence, if $\int_{-\infty}^{\infty} |x(t)| dt < B$ ($x(t)$ absolute integrable), then

$$\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}| dt < B \times \max(e^{-\sigma T_1}, e^{-\sigma T_2})\ ;$$

i.e., $x(t)e^{-\sigma t}$ is absolute integrable for any σ, and the ROC is the entire s-plane.

Property 4 **Suppose $x(t)$ is right-sided. If $Re(s) = \sigma_0$ is in the ROC, then all values of s for which $Re(s) = \sigma_1 > \sigma_0$ will also be in the ROC; i.e., the ROC must be a *right-half plane*.** must extend to infinity to the right

If x(t) right-sided, then ROC must be right-half plane.

Definition: A signal is right-sided if there exist a T_1 for which $x(t) = 0$, $\forall\, t < T_1$. A right-sided signal is a signal with initial rest.

- All causal signals are right-sided but right-sided signals are not necessarily causal

Initial rest
$x(t) = 0\ \forall t < T_1$

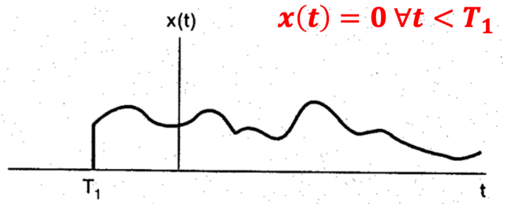
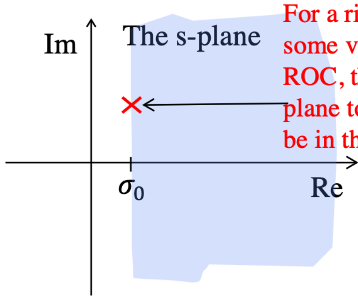
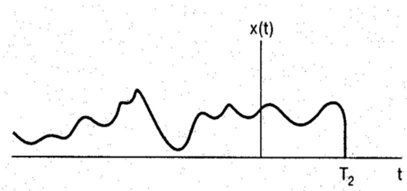


Figure 9.6
A Right-Sided Signal
*But not causal



For a right-sided signal, if some value of s is in the ROC, the entire right-half plane to the right of s must be in the ROC

Property 5 **Suppose $x(t)$ is left-sided. If $Re(s) = \sigma_0$ is in the ROC, then all values of s for which $Re(s) = \sigma_1 < \sigma_0$ will also be in the ROC; i.e., the ROC must be a *left-half plane*.**



For a left-sided signal, if some value of s is in the ROC, the entire left-half plane to the leftof s must be in the ROC

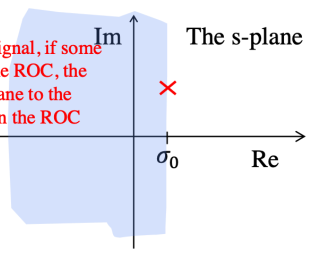


Figure 9.8

- A left-sided signal is a signal with final rest (i.e., there is some T_2 for which $x(t) = 0, \forall\, t > T_2$).
- All anti-causal signals are left-sided but left-sided signals are not necessarily anti-causal.
- We follow same proof as previous slide but with $\sigma_1 < \sigma_0$ and $t < T_2$

Property 6 **Suppose $x(t)$ is two-sided. If $Re(s) = \sigma_0$ is in the ROC, then the ROC consists of a strip in the complex plane that includes the line $Re(s) = \sigma_0$.**

- "Two-sided" means neither right- nor left-sided.
- But we can divide a two-sided signal into a right-sided part and a left-sided part; i.e., let $x(t) = x_L(t) + x_R(t)$ where $x_L(t)$ is left-sided and $x_R(t)$ is right-sided.

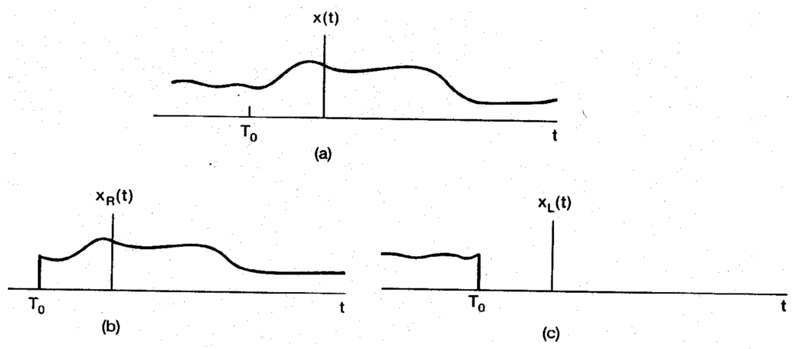


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided sig-

Properties of ROC - Remarks

- ***Remarks***
  - A signal either does not have a Laplace transform, or its ROC must fall into one of the four categories covered by Properties 3-6.
  - **The ROC of a Laplace transform must be either:**

i) the whole *s* plane, or $x(t)$ absolute integrable on finite duration

ii) a right-half plane, or $x(t)$ right-sided

iii) a left-half plane, or $x(t)$ left-sided

iv) a single strip $x(t)$ two-sided

ROC for Rational LT

Property 7 **For a rational Laplace transform, the ROC is bounded by poles or extends to infinity. No poles are contained in the ROC.**

A signal with rational LT is a linear combination of causal and anti-causal exponentials, the ROC of each being a right-half or left-half plane. the overall ROC is the intersection of the individual ROCs.

Property 8 **Suppose the Laplace transform of *x*(*t*) is rational. If *x*(*t*) is right sided, the ROC is the right-half plane to the right of the rightmost pole. If *x*(*t*) is left sided, the ROC is the left-half plane to the left of the leftmost pole.**

No Fourier Transform if $j\omega$-axis not in ROC

L21

I. Inverse Laplace Transform by Partial Fraction (9.3)

- Recall that $X(s=\sigma+j\omega) = FT\{x(t)e^{-\sigma t}\}$.
- That means we can apply inverse FT to $X(s)$ along a fixed σ to obtain:

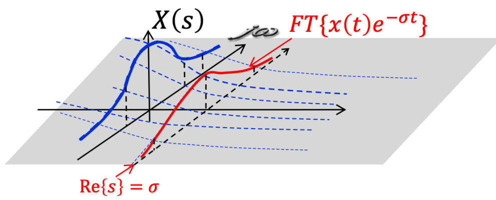
$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\sigma+j\omega)e^{j\omega t}d\omega \quad \text{IFT}$$

Multiplying both sides by $e^{\sigma t}$, we obtain:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega$$

$$= \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st}\, ds$$

$s=\sigma+j\omega \Rightarrow d\omega = \frac{1}{j}ds$ A superposition of e^{st}



the inverse LT integral, which is a *contour integral* on the complex s-plane

- For LTs in rational form, instead of contour integration, we typically use ***partial fraction expansion*** to do the inverse transform.

Transforms in Rational Form

Again, a rational Laplace transform is one that is in the form $N(s)/D(s)$. It can be expressed in three alternate forms as we have seen for rational Fourier transform:

1. <u>The Polynomial Form</u> which will allow us to write the differential equation

or $H(s)$

$$X(s) = \frac{b_{N-1}s^{N-1}+\cdots+b_1 s+b_0}{s^N+a_{N-1}s^{N-1}+\cdots+a_1 s+a_0} \quad \frac{N(s)}{D(s)}$$

We usually assume the order of $N(s)$ < order of $D(s)$ so that $X(s)\to 0$ when $|s|\to\infty$

2. **The Factored Form** which tells us the locations of the poles (α_k) and zeros (β_i):

Form 2a:

zeros

$$X(s) = \frac{b_{N-1} \prod_{i=1}^{N-1} (s - \beta_i)}{\prod_{k=1}^{N} (s - \alpha_k)}$$

a_N assumed normalized to 1

poles

or Form 2b:

$$X(s) = \frac{A \prod_{i=1}^{N-1} \left(1 - \frac{s}{\beta_i}\right)}{\prod_{k=1}^{N} \left(1 - \frac{s}{\alpha_k}\right)}$$

$1 - \frac{s}{\alpha_k} = 0$ when $s = \alpha_k \Rightarrow$ Hence a pole

The advantage of Form 2b is function value at $s = 0$ is explicitly given:

$$X(0) = A \quad \text{by Form 2b}$$

$$X(0) = \frac{b_{N-1} \prod_{i=1}^{N-1} (-\beta_i)}{\prod_{k=1}^{N} (-\alpha_k)} \quad \text{by Form 2a}$$

3. The Partial Fraction Form which with ROC allows you to find the inverse transform

$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s - \alpha_k}$$

residues

Only the poles are explicit in this form

Inverse LT with Partial Fraction

With rational LT in partial fraction form, can write down the inverse transform as sum of causal/anti-causal exponentials.

$$X(s) = \sum_{k=1}^{N} \frac{c_k}{s - \alpha_k} \Rightarrow x(t) = \sum_{k=1}^{N} \chi_k c_k \, e^{\alpha_k t} u(\chi_k t)$$

$u(t)$ or $u(-t)$

where χ_k is a sign variable:

$$\chi_k = \begin{cases} +1 & \text{if ROC is right of } \alpha_k \\ -1 & \text{if ROC is left of } \alpha_k \end{cases}$$

ROC 在 root 右边 : RHP + u(t)   causal

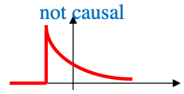
左四 : LHP — $u(-t)$ anti-causal

Causality & Stability

9.7.1 Causality

- The impulse response of a causal LTI system is right-sided. Hence the ROC of the system function for a causal system must be a right-half plane (RHP).
- However, ROC being RHP means impulse response is right-sided but not necessarily causal.
- But if the system function is rational and ROC is a RHP, then the system is causal, because the inverse of a rational LT must be a combination of causal and anti-causal exponentials.

Right-sided but not causal



Rational + RHP ROC ⟶ Causal

If $H(s)$ not rational then RHP RoC don't imply causality

9.7.2 Stability

An LTI system is BIBO stable if and only if the ROC of its system function includes the $j\omega$-axis.

It is because all the following conditions are equivalent:

- ROC of system function includes the $j\omega$-axis,
- The Fourier transform converges, $FT\{h(t)\} = H(j\omega) = H(s)|_{s=j\omega} < \infty\ \forall\omega$
- $h(t)$ is absolute integrable: $\int_{-\infty}^{\infty}|h(t)|dt < \infty$ $h(t)$ absolute integrable $\Rightarrow H(j\omega) < \infty\ \forall\omega$
- Frequency response of the system exists,

Frequency response = $H(j\omega)$. The existence of frequency response means all sinusoidal inputs leads to a bounded output; i.e., $|H(j\omega)| < \infty\ \forall\omega$

Causal System with rational $H(s)$ is stable
iff all poles lie on the left half plane :

ROC is intersection of RHP's for causal, so it is
to the right of rightmost pole.
To include $j\omega$-axis, the pole must be to the left

of $j\omega$-axis.

Properties of LT

If $X(s)$ has a pole at $s = 0$, multiplying $X(s)$ by s cancels this pole and may expand the ROC (if this pole bounds the ROC). Therefore, the new ROC is at least the original ROC.

Recall: ω_n and ζ (zeta) param

$\omega_n = \sqrt{a_0}$ $\qquad\qquad \alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$

$a_1 = 2\zeta\omega_n$

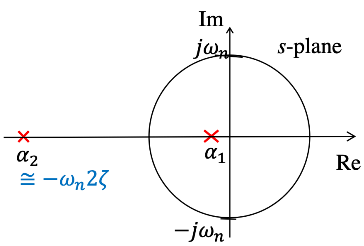
$\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$

ω_n provides a scaling in frequency, and ζ characterizes the system

$\zeta > 1$: Both roots are real and < 0 ; system has no oscillation and is stable for causal systems.

$\zeta >> 1$ Over-damped

↳ decays slowly.



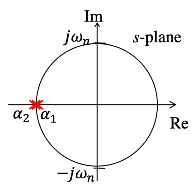
near 0

If $\zeta \gg 1$, then one root is at $\alpha_2 = -\omega_n(2-\varepsilon)\,\zeta$ where $\epsilon = 1 - \sqrt{1-\frac{1}{\zeta^2}}$

but the other root is at $\alpha_1 = -\omega_n\varepsilon\zeta = 0^-$. Means "a little smaller than"

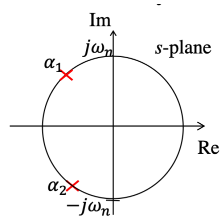
The pole at 0^- means the response decays very slowly in time. The system is overdamped.

$\zeta = 1$ Critically damped $(\alpha_1 = \alpha_2 = -\omega_n)$



$\zeta = 1$: the roots are equal and the impulse response decays at the fastest rate

$0 < \zeta < 1$ Oscillatory



$\alpha_1 = \alpha_2^*$;
$\alpha_1 \ \alpha_2 = a_0 = {\omega_n}^2$
$|\alpha_1| = |\alpha_2| = \omega_n$
so α_1, α_2 are on circle of radius ω_n

$1 > \zeta > 0$: the roots become complex and system is oscillatory

4. $\zeta \to 0^+$ Means "a little greater than"

Under-damped

Impulse response is a damped oscillation that decays very slowly. System may have large response when stimulated at the natural frequency:

$$\cos(\omega_1 t) \to \underbrace{|H(j\omega_1)|}_{\text{Magnitude response}}\cos(\omega_1 t + \underbrace{\sphericalangle H(j\omega_1)}_{\text{Phase response}})$$

For a pole-only 2nd-order system,

$$H(j\omega_1) = \frac{c}{(j\omega_1 - \alpha_1)(j\omega_1 - \alpha_2)}$$

The magnitude response is:

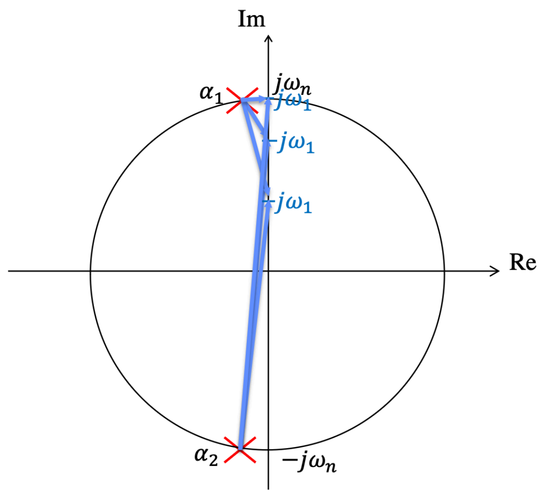
$$|H(j\omega_1)| = \frac{c}{\underbrace{|j\omega_1 - \alpha_1|}_{\text{Distant to } \alpha_1}\underbrace{|j\omega_1 - \alpha_2|}_{\text{Distant to } \alpha_2}}$$

Real part of the root is very small: root is very close to the $j\omega$ axis, near $\pm j\omega_n$

$$\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

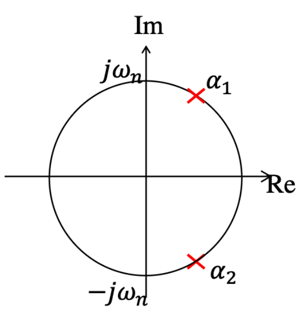
Radical is near $\sqrt{-1} = j$

$\text{Re}\{\alpha_1\} = \text{Re}\{\alpha_2\} = -\zeta\omega_n$ if roots are complex



5. $\zeta \leq 0$

Unstable



$\text{Re}\{\alpha_1\}, \text{Re}\{\alpha_2\} \geq 0$

Impulse response grows or stays flat toward the right in time

Unstable for causal system