Mathematical Overview of the Stokes Equation

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1 Introduction

2 Two-Dimensional Diffusion Problem

2.1 Weak Formulation and Broken Sobolev Spaces

It may help to first motivate the problem by solving the second-order diffusion equation. Namely, let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. The boundary $\partial\Omega$ is partitioned into two disjoint sets, Γ_N and Γ_D, and $\mathbf{n}$ denotes the unit normal vector to $\partial\Omega$. Then

$$-\nabla \cdot (K\nabla u) + \alpha u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = g_D \quad \text{on } \Gamma_D \tag{2.2}$$

$$K\nabla u \cdot \mathbf{n} = g_N \quad \text{on } \Gamma_N \tag{2.3}$$

The weak formulation of this problem is as follows: given $w \in H_0^1(\Omega)$, we seek u such that for all $v \in H_0^1(\Omega)$, we have

$$\int_\Omega (K\nabla w \cdot \nabla v + \alpha wv) = \int_\Omega fv - \int_\Omega (K\nabla u_D \cdot \nabla v + \alpha u_D v) \tag{2.4}$$

In the above relation, the first term is obtained by applying Green's Theorem to the Laplacian. Next, we partition the domain into a conforming mesh (the overlap between any two elements is at most a vertex or edge) $\mathcal{M}_h$. On this partition, we extend the definition of a Sobolev space by introducing the broken Sobolev space for any $s \in \mathbb{R}$:

$$H^s(\mathcal{M}_h) = \left\{ v \in L^2(\Omega) \;:\; \forall E \in \mathcal{M}_h,\ v|_E \in H^s(\Omega) \right\} \tag{2.5}$$

with the broken Sobolev norm:

$$||v||_{H^s(\mathcal{M}_h)} = \left(\sum_{E \in \mathcal{M}_h} ||v||^2_{H^s(E)} \right)^{1/2} \tag{2.6}$$

Note that the broken Sobolev norm simply says that the norm over the entire partition is the square root of the sums of the squares of the Sobolev norms piecewise across each element of the partition. In the case when $s = 1$, we have:

$$||v||_{H^1(\mathcal{M}_h)} = ||\nabla v||_{H^0(\mathcal{M}_h)} = \left(\sum_{E \in \mathcal{M}_h} ||\nabla v||^2_{L^2(E)} \right)^{1/2} \tag{2.7}$$

2.2 Discrete Formulation

We can define the bilinear form $a_\epsilon \ : \ H^s(\mathcal{M}_h) \times H^s(\mathcal{M}_h) \to \mathbb{R}$:

$$\begin{aligned} a_\epsilon(v,w) = & \sum_{E\in\mathcal{M}_h} \int_E (K\nabla v \cdot \nabla w + \alpha v w) \\ & - \sum_{e\in\Gamma_h^i\cup\Gamma_D} \int_e \left(\{K\nabla v\cdot \mathbf{n}_e\}[w] - \epsilon \{K\nabla w\cdot\mathbf{n}_e\}[v]\right) \\ & + \sum_{e\in\Gamma_h^i\cup\Gamma_D} \frac{\sigma_e}{|e|}\int_e [v][w] \end{aligned} \tag{2.8}$$

We also define the linear form $\ell \ : \ H^s(\mathcal{M}_h) \to \mathbb{R}$:

$$\ell(v) = \sum_{E\in\mathcal{M}_h} \int_E fv + \sum_{e\in\Gamma_D}\int_e \left(K\nabla v\cdot\mathbf{n}_e + \frac{\sigma_e}{|e|}v\right) g_D + \sum_{e\in\Gamma_N}\int_e v g_N \tag{2.9}$$

Putting (2.8) and (2.9) together, we have the following: find $u \in H^s(\mathcal{M}_h), \ s > 3/2$ such that for all $v \in H^s(\mathcal{M}_h)$, we have:

$$a_\epsilon(u,v) = \ell(v) \tag{2.10}$$

2.3 Implementation

Let u_h be the discontinuous Galerkin solution to (2.10). We have

$$u_h = \sum_{E\in\mathcal{M}_h}\sum_i c_i^E \phi_i^E$$

where $\phi_i^E \in H^s(\mathcal{M}_h)$ denotes the i^{th} basis function on the mesh element E and c_i^E, its coefficient. Essentially, the discontinuous Galerkin solution on each element is a linear combination of the basis functions on that mesh element.

The basis functions are predetermined by the user. For example, on a triangular mesh, we might choose Lagrange basis functions while on a quadrilateral mesh, we might choose Legendre basis functions. See **??** for an in-depth study of the other types of bases.

On each mesh element, the basis functions are thus known, and the goal is to computer c_i^E. The discrete problem in (2.10) then becomes a linear system:

$$\mathbf{A}\cdot\mathbf{c} = \mathbf{b}$$

where $\mathbf{A}$ is the global element matrix (to be assembled locally, or element-wise) corresponding to $a_\epsilon(u_h, v_h)$, $\mathbf{c}$ is the vector of basis function coefficients, and $\mathbf{b}$ is the global right-hand side (again assembled element-wise) corresponding to $\ell(v_h)$.

3 Model Problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. For an incompressible viscous flow, the Stokes equations are:

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{3.1}$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{3.2}$$
$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{3.3}$$

where

4 Variational Formulation

5 Simulations

6 Results

7 Conclusions