Mathematical Overview of the Stokes Equation

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1 Introduction

2 Two-Dimensional Diffusion Problem

2.1 Weak Formulation and Broken Sobolev Spaces

It may help to first motivate the problem by solving the second-order diffusion equation. Namely, let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. The boundary $\partial\Omega$ is partitioned into two disjoint sets, Γ_N and Γ_D, and $\mathbf{n}$ denotes the unit normal vector to $\partial\Omega$. Then

$$-\nabla \cdot (K\nabla u) + \alpha u = f \quad \text{in } \Omega \tag{2.1}$$

$$u = g_D \quad \text{on } \Gamma_D \tag{2.2}$$

$$K\nabla u \cdot \mathbf{n} = g_N \quad \text{on } \Gamma_N \tag{2.3}$$

The weak formulation of this problem is as follows: given $w \in H_0^1(\Omega)$, we seek u such that for all $v \in H_0^1(\Omega)$, we have

$$\int_\Omega (K\nabla w \cdot \nabla v + \alpha w v) = \int_\Omega f v - \int_\Omega (K\nabla u_D \cdot \nabla v + \alpha u_D v) \tag{2.4}$$

In the above relation, the first term is obtained by applying Green's Theorem to the Laplacian. Next, we partition the domain into a conforming mesh (the overlap between any two elements is at most a vertex or edge) $\mathcal{M}_h$. On this partition, we extend the definition of a Sobolev space by introducing the broken Sobolev space for any $s \in \mathbb{R}$:

$$H^s(\mathcal{M}_h) = \left\{ v \in L^2(\Omega) \ : \ \forall E \in \mathcal{M}_h,\ v|_E \in H^s(\Omega) \right\} \tag{2.5}$$

with the broken Sobolev norm:

$$||v||_{H^s(\mathcal{M}_h)} = \left(\sum_{E \in \mathcal{M}_h} ||v||^2_{H^s(E)} \right)^{1/2} \tag{2.6}$$

Note that the broken Sobolev norm simply says that the norm over the entire partition is the square root of the sums of the squares of the Sobolev norms piecewise across each element of the partition. In the case when $s = 1$, we have:

$$||v||_{H^1(\mathcal{M}_h)} = ||\nabla v||_{H^0(\mathcal{M}_h)} = \left(\sum_{E \in \mathcal{M}_h} ||\nabla v||^2_{L^2(E)} \right)^{1/2} \tag{2.7}$$

2.2 Discrete Formulation

We can define the bilinear form $a_\epsilon : H^s(\mathcal{M}_h) \times H^s(\mathcal{M}_h) \to \mathbb{R}$:

$$
\begin{aligned}
a_\epsilon(v, w) = & \sum_{E \in \mathcal{M}_h} \int_E (K \nabla v \cdot \nabla w + \alpha v w) \\
& - \sum_{e \in \Gamma_h^i \cup \Gamma_D} \int_e (\{K \nabla v \cdot \mathbf{n}_e\} [w] - \epsilon \{K \nabla w \cdot \mathbf{n}_e\} [v]) \\
& + \sum_{e \in \Gamma_h^i \cup \Gamma_D} \frac{\sigma_e}{|e|} \int_e [v][w]
\end{aligned} \tag{2.8}
$$

We also define the linear form $\ell : H^s(\mathcal{M}_h) \to \mathbb{R}$:

$$
\ell(v) = \sum_{E \in \mathcal{M}_h} \int_E f v + \sum_{e \in \Gamma_D} \int_e \left(K \nabla v \cdot \mathbf{n}_e + \frac{\sigma_e}{|e|} v \right) g_D + \sum_{e \in \Gamma_N} \int_e v g_N \tag{2.9}
$$

Putting (2.8) and (2.9) together, we have the following: find $u \in H^s(\mathcal{M}_h),\ s > 3/2$ such that for all $v \in H^s(\mathcal{M}_h)$, we have:

$$
a_\epsilon(u, v) = \ell(v)
$$

2.3 Implementation

3 Model Problem

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. For an incompressible viscous flow, the Stokes equations are:

$$
-\mu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{3.1}
$$

$$
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{3.2}
$$

$$
\mathbf{u} = 0 \quad \text{on } \partial\Omega \tag{3.3}
$$

where

4 Variational Formulation

5 Simulations

6 Results

7 Conclusions