Verifying the existence of ML estimates for GLMs

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Motivation: why should we use generalized linear models?

- Practitioners often prefer least squares when seemingly better alternatives exist. Example:
    - Linear probability model instead of logit/probit
    - Log transformations instead of Poisson
- This comes with several disadvantages:
    - Inconsistent estimates under heteroskedasticity due to Jensen's inequality; bias can be quite severe (Manning & Mullahy 2001, Santos Silva & Tenreyro 2006, etc.)
    - Linear models might lead to a wrong support: predicted probabilities outside [0-1], $\log(0)$, etc.

Digression: genesis of this paper

- We wanted to run pseudo-ML poisson regressions with fixed effects:
    - Paulo: $\log(1 + wages)$
    - Tom: $\log(1 + trade)$
    - Sergio: $\log(1 + credit)$
- Should have been feasible:
    - No incidental parameters problem (Wooldridge 1999, Fernandez-Val and Weidner 2016), Weidner and Zylkin 2019)
    - Works with non-count variables (Gourieroux et al 1984)
    - Practical estimator through IRLS and alternating projections (Guimarães 2014, Correia 2017, Zylkin et al 2018)
- However, there was another obstacle we did not anticipate:
    - Our implementation often failed to converge, or converged to incorrect solutions.
    - Problem was aggravated when working with many levels of fixed effects (our intended goal)

How can maximum likelihood estimates *not* exist?

Consider a Poisson regression on a simple dataset without constant:

- Log-likelihood: $\mathcal{L}(\beta) = \sum [y_i(x_i\beta) - \exp(x_i\beta) - \log(y_i!)]$
- FOC: $\sum x_i[y_i - \exp(x_i\beta)] = 0$

y	x
0	1
0	1
0	0
1	0
2	0
3	0

- In this example, the FOC becomes $\exp(\beta) = 0$, maximized only at infinity!
    - Note that at infinity the first two observations are fit perfectly, with $\mathcal{L}_i = 0$
- More generally, non-existence can arise from any **linear combination of regressors** including fixed effects.

Existing literature

- Non-existence conditions have been independently (re)discovered multiple times:
    - Log-linear frequency table models (Haberman 1973, 1974)
    - Binary choice (Silvapulle 1981, Albert and Anderson 1984)
    - GLM sufficient–but–not–necessary conditions (Wedderburn 1976, Santos Silva and Tenreyro 2010)
    - GLM (Verbeek 1989, Geyer 1990, Geyer 2008, Clarkson and Jenrich 1991; all three unaware of each other).
- Most researchers still unaware of problem outside of binary choice models; no textbook mentions as of 2019.
    - Software implementations either fail to converge or inconspicuously converge to wrong results.

Our contribution

1. Derive existence conditions for a broader class of models than in existing work
    - Including Gamma PML, Inverse Gaussian PML
2. Clarify how to correct for non-existence of *some* parameters.
    - Finite components of β can be consistently estimated; inference is possible
3. Introduce a novel and easy-to-implement algorithm that detects and corrects for non-existence
    - Particularly useful with high-dimensional fixed effects and partialled-out covariates.
    - Can be implemented with run–of–the-mill tools.

Proposition 1: non-existence conditions (1/3)

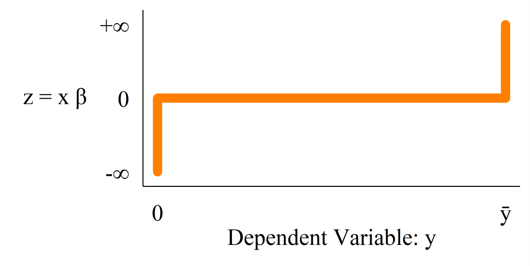
Consider the class of GLMs defined by exponential log-likelihood functions:

$$\mathcal{L} = \sum_i \mathcal{L}_i = \sum_i \left[a(\phi)\, y_i\, \theta_i - a(\phi)\, b(\theta_i) + c(y_i, \phi)\right]$$

- a, b, and c are known functions; ϕ is a scale parameter
- $\theta_i = \theta(x_i\beta)$ is the canonical link function; where $\theta' > 0$
- $y_i \geq 0$ is an outcome variable. Potentially $y \leq \bar{y}$ as in logit/probit but for simplicity we'll ignore this through this talk.
- Its conditional mean is $\mu_i = E[y_i|x_i] = b'(\theta_i)$
- Assume for simplicity that regressors X have full column rank.
- Assume that $\mathcal{L}_i$ has a finite upper bound.

ML solution for β will **not** exist iff there is a non-zero vector γ such that:

$$x_i\gamma = z_i \begin{cases} \leq 0 & \text{if } y_i = 0 \\ = 0 & \text{if } 0 < y_i < \bar{y} \\ \geq 0 & \text{if } y_i = \bar{y} \end{cases}$$



Proposition 1: non-existence conditions (3/3)

- Linear combination z is a “certificate of non-existence”: hard to obtain, but can be used to verify non-existence
    - If we add z to the regressor set, its associated FOC will not have a solution.
- Observations where $z_i \neq 0$ will have a perfect fit.
- If $\mathcal{L}_i$ is unbounded above, conditions are slightly more complex; see proposition 2 of the paper.

Addressing non-existence

- As in perfect collinearity, first look for specification problems:
    - In a Poisson wage regression, did we add "unemployment benefits" as covariate?
    - In a Poisson trade regression, did we add an "is embargoed?" indicator?
- If no specification problems, it's due to sampling error.
- Solution: allow estimates to take values in the extended reals: $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$
    - Example: $\hat{\beta} = \lim_{a \to \infty} a + 3$
    - We are mostly interested in the non-infinite component

Proposition 3: Addressing non-existence

- Given a $\mathcal{L}_i$ bounded above, ML solution in the extended reals will always exist.
- Given vector z identifying all instances of non-existence, if we first drop perfectly predicted observations (and resulting perfect collinear variables) ML solution **in the reals** will always exist.
    - It will consistently estimate the non-infinite components of β, allowing for inference on them (proposition 3d)
    - We can recover infinite components by regressing z against x.

Obtaining z: Existing Alternatives

1. Drop boundary observations with $\mathcal{L}_i$ close to 0 (Clarkson and Jenrich 1991)
    - Slow under non-existence; often fails as "close to 0" is data specific.
2. Solve a modified simplex algorithm (Clarkson and Jenrich 1991)
    - Cannot handle fixed effects or other partialled-out covariates
3. Analytically solve computational geometry problem (Geyer 2008), or use eigenvalues of Fischer information matrix (Eck and Geyer 2018).
    - Extremely slow and complex (Geyer 2008); might not converge (Eck and Geyer 2018); cannot handle fixed effects (both).

None works well with fixed effects!

Obtaining z: Iterative Rectifier (our algorithm)

1. Define a working dependent variable $z_i = \mathbb{1}_{y_i=0}$
2. Given an arbitrarily large integer K, set weights $w_i = \begin{cases} 1 & \text{if } y_i = 0 \\ K & \text{if } y_i > 0 \end{cases}$
3. (Weighted least squares) Regress z on X with weights w; potentially allowing for fixed effects
4. Stop if all $\hat{z}_i \geq 0$
5. Else, update $z_i = max(\hat{z}_i, 0)$ and repeat from step 3

- Steps 2-3 are the “weighting method” of solving least squares with equality constraints (Stewart 1997); step 5 is a “rectifier” that enforces a positive dependent variable
- Proofs in proposition 4 and appendix
- Stata implementation in `ppmhdfe` package
- Convergence usually achieved in a few iterations, but choosing weights too large could lead to numerical instability.

Other existing approaches

- Naïve approach: drop the regressors causing non-existence and proceed as usual
  - Leads to non-sensical results (Zorn 2005, Gelman 2008)
- Penalize estimates beyond plausible values (Firth regression, Bayesian aproach)
  - "For Poisson regression and other models with the logarithmic link, we would not often expect effects larger than 5 on the logarithmic scale" (Gelman 2008)
  - Not a ML estimator
  - Many datasets (e.g. in trade) can have plausible effects way beyond 5.
- Solutions specific to binary choice discussed in Konis (2007)

Comparison of solutions

Method	Advantages	Concerns
1. Drop regressors	-	Nonsensical
2. Drop $\mu_i < \varepsilon$ observations	Simple	Fails often: ε is data dependent
3. Bayesian: penalize $\mu_i < \varepsilon$	It's Bayesian	It's Bayesian. ε is data dependent
4. Modified simplex	Fast for small k	Slow for large k Can't handle FEs
5. Directions of recession	Exact answer "at infinity"	Complex, very slow (?) Can't handle FEs
6. Iterative rectifier	Simple works well with large k and FEs	Numerical accuracy (?)

Conclusion

Non-existence of estimates:

- Affects a broad class of GLMs beyond just binary choice models
- Poorly understood (no textbook mentions); not addressed in statistical packages
- Leads practitioners to stay with least squares despite limitations

This paper:

- Presents non-existence conditions for a broad class of GLMs
- Discusses how to address non-existence: drop perfectly predicted observations, then proceed as normal
- Introduces an algorithm for detecting and addressing non-existence that is conceptually simple, easy-to-implement, and allows for fixed effects