On the Structure of Classical Mechanics*

Thomas William Barrett†

Abstract

The standard view is that the Lagrangian and Hamiltonian formulations of classical mechanics are theoretically equivalent. Jill North (North, 2009), however, argues that they are not. In particular, she argues that the statespace of Hamiltonian mechanics has less structure than the statespace of Lagrangian mechanics. I will isolate two arguments that North puts forward for this conclusion and argue that neither yet succeeds.

1 Introduction

Different spacetime theories ascribe different amounts of structure to spacetime. Aristotelian spacetime singles out a preferred spatial location as the *center of the universe*. Absolute Newtonian spacetime does not single out a preferred spatial location, but it does single out a preferred inertial frame as the *rest frame*. Neo-Newtonian (or Galilean) spacetime does not single out a preferred inertial frame, but it does single out a set of inertial frames as the *non-accelerating frames*. Each of these theories, in a clear and intuitive sense, ascribes less structure to spacetime than do its predecessors.[1]

Physicists and philosophers have devoted significant attention to the amounts and kinds of structure inherent in *spacetime* theories, but less to the amounts and kinds of structure inherent in other physical theories. In a recent paper, however, Jill North (North, 2009) undertakes the task of comparing the structure of the Lagrangian and Hamiltonian formulations of classical mechanics.

The standard view is that these two formulations of classical mechanics are, in some sense, theoretically equivalent. In the Lagrangian formulation, the state of a system is specified by its position and *velocity*. In the Hamiltonian formulation, the state of a system is specified by its position and *momentum*. But this is normally taken to be a nominal difference. The two theories may be formulated in different languages, but they are still equivalent.

*Published in *The British Journal for the Philosophy of Science*. 66: 801-828 (2015).

†I am particularly grateful to Hans Halvorson for comments on multiple drafts and for giving me many of the ideas in sections 3 and 4. Thanks also to Jeff Barrett, Robbie Hirsch, John Manchak, Noel Swanson, Jim Weatherall, Chris Wüthrich, and two anonymous referees for helpful comments on earlier versions of this paper. This material is based upon work supported by the National Science Foundation under Grant No. DGE 1148900.

[1]For a nice discussion see (Earman, 1989, Ch. 2).

North dissents from the standard view.[2] She argues that there is an important difference between the two theories: Lagrangian and Hamiltonian mechanics differ in the amount of structure that they ascribe to the world. North argues for the following:

(LS) The statespace of Hamiltonian mechanics has less structure than the statespace of Lagrangian mechanics. ⌟

If (LS) were true and one took the mathematical structure of a theory's statespace to mirror the structure of the world,[3] then it would indeed be hard to see how Lagrangian and Hamiltonian mechanics could be equivalent theories. They would disagree about the structure of the world. And one might be inclined to go one step further. If the Hamiltonian statespace has *less* structure than the Lagrangian statespace, one might argue that Ockhamist principles urge us to prefer Hamiltonian mechanics over Lagrangian mechanics. Or perhaps one might argue for something even stronger: Hamiltonian mechanics provides us with a better description of the *fundamental and objective* features of the world than Lagrangian mechanics does.[4]

But of course, before drawing any of these philosophical conclusions, one must demonstrate that (LS) is actually true. North's primary argument[5] for (LS) runs as follows:

(P1) The Lagrangian statespace has metric structure.

(P2) The Hamiltonian statespace merely has symplectic structure.

(P3) Symplectic structure is *less structure* than metric structure.

$\therefore$ **(LS)** ⌟

The main goal of this paper is to show that this argument for (LS) is not sound. I will argue in three steps. First, I will argue that North's arguments fail to demonstrate that (P1) is true. Second, I will show that there is a strong sense in which whenever (P1) is true, (P2) is not true. And third, I will argue that even if one were to accept that both (P1) and (P2) were true, (LS) would still not follow since it is not clear that (P3) is true. The reader might be inclined to take these remarks as cautionary. Much care must be taken when identifying and comparing the amounts of structure that different physical theories ascribe to the world. Some cases are harder than others. As we will see, the case of Lagrangian and Hamiltonian mechanics is one of the hard ones.

---

[2](Curiel, 2013) is another notable dissenter.

[3]I will comment on this assumption at the end of the paper. Even if (LS) is true, it is not clear that it implies that Lagrangian and Hamiltonian mechanics are inequivalent. One needs to keep in mind the physical significance that different bits of mathematical structure have. For a discussion of the physical significance of the various structures in Lagrangian and Hamiltonian mechanics see (Curiel, 2013).

[4]North, in fact, does use (LS) to argue for all of these conclusions (North, 2009, p. 76–79).

[5]North also suggests another argument for (LS), which we will consider in section 4.

2 Hamiltonian statespace has less structure than Lagrangian statespace

The first order of business is to recapitulate North's argument in support of (LS). I will present two arguments that North provides for (P1), then North's argument for (P2), and finally North's argument for (P3).

2.1 Lagrangian statespace is metrical

Lagrangian mechanics and Hamiltonian mechanics are geometric formulations of classical mechanics.[6] They are formulated in the language of differential geometry.

Consider a system of n particles. The positions of these particles can be represented by a point q in a $3n$-dimensional differentiable manifold $\mathcal{Q}$, called *configuration space*. Lagrangian mechanics has as its statespace the $6n$-dimensional tangent bundle of $\mathcal{Q}$, which we denote by $T_*\mathcal{Q}$. A point $(q, v) \in T_*\mathcal{Q}$ encodes the positions and velocities of all n particles in the system. In order to determine how the system evolves over time, one must specify a *Lagrangian*. The Lagrangian of a system is a smooth function $L : T_*\mathcal{Q} \to \mathbb{R}$ that encodes some of the system's energy properties.[7] In most applications of Lagrangian mechanics, the Lagrangian of a system is defined as the system's kinetic energy minus its potential energy.

North presents two distinct arguments for (P1). The first argument is the following:

Argument 1 (The Lagrangian presupposes g)**.** In most applications of Lagrangian mechanics, the Lagrangian of a system is defined as the system's kinetic energy minus its potential energy. In fact, in many cases the Lagrangian is defined by

$$L(q, v) = \frac{1}{2} g_q(v, v) - V(q) \ ,$$

where $V : \mathcal{Q} \to \mathbb{R}$ is a smooth potential. This definition presupposes a metric $g_q : T_q\mathcal{Q} \times T_q\mathcal{Q} \to \mathbb{R}$ on the configuration space $\mathcal{Q}$. Therefore, Lagrangian mechanics requires a metric g_q on configuration space $\mathcal{Q}$.[8] ⌟

When specifying a Lagrangian for a system, one often requires that configuration space $\mathcal{Q}$ has a metric. So perhaps this is one sense in which the Lagrangian statespace has metric structure.

---

[6] At least, *modern* versions of these two theories are geometric. The earliest versions of these two theories predate differential geometry.

[7] Intuitively, the Lagrangian encodes something like the system's 'liveliness' or 'activity' (Baez, 2005, p. 7). The system evolves according to the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \ ,$$

where q^i and $\dot{q}^i$ are position and velocity coordinates, respectively, on $T_*\mathcal{Q}$.

[8] North suggests this sort of argument at (North, 2009, p. 85).

North provides another argument for (P1). The basic idea behind North's second argument is this: Metric structure is invariant under the maps that preserve the structure of $T_*\mathcal{Q}$. In other words, North claims that metric structure is left invariant by the automorphisms of $T_*\mathcal{Q}$. If true, this claim would demonstrate a sense in which tangent bundle structure presupposes or naturally gives rise to metric structure.[9] Before stating this argument, we need a definition.

Definition 1 (Point$_*$-transformation)**.** Let $f : \mathcal{Q} \to \mathcal{Q}$ be a diffeomorphism. The map f naturally induces a map $T_*f : T_*\mathcal{Q} \to T_*\mathcal{Q}$ defined by

$$T_*f(q, v) = (f(q), f_*(v)),$$

for every point $(q, v) \in T_*\mathcal{Q}$. We will call T_*f a *point$_*$-transformation.* ⌟

Point$_*$-transformations preserve all of the structure of the tangent bundle $T_*\mathcal{Q}$.[10] North's second argument for (P1) suggests that metric structure is also preserved by these transformations. One natural way to make this argument precise is as follows:

Argument 2 (Point$_*$-transformations preserve g)**.** Metric structure is invariant under point$_*$-transformations. Given a tangent bundle $T_*\mathcal{Q}$, there exists a metric g on $T_*\mathcal{Q}$ such that every point$_*$-transformation T_*f preserves g, in the sense that $(T_*f)^*(g) = g$. Tangent bundle structure in this sense presupposes or gives rise to metric structure.[11] ⌟

This argument suggests that the tangent bundle comes naturally equipped with metric structure.[12] So perhaps this is the sense in which the Lagrangian statespace has metric structure.

Having presented North's two arguments for (P1), we now turn to (P2).

2.2 Hamiltonian statespace is symplectic

Consider again a system of n particles. Hamiltonian mechanics has as its statespace the $6n$-dimensional *cotangent bundle* of configuration space $\mathcal{Q}$, denoted by $T^*\mathcal{Q}$.[13] A point $(q, \omega) \in T^*\mathcal{Q}$ encodes the positions and momenta of all n par-

[9] I am going to show in section 3.1.2 that the claim is false. But it is important to first get the argument on the table.

[10] And what's more, they often correspond to changes of coordinates on configuration space, and thus preserve the form of the Lagrangian equations of motion. This is made precise in Proposition 3.5.19 of (Abraham and Marsden, 1978).

[11] It is important to mention that North does not make this argument perfectly explicit. But the reconstruction provided here is one particularly natural way to make formal and precise the suggestion made at (North, 2009, p. 73).

[12] As we will see in sections 2.2 and 4.1, the *cotangent* bundle comes naturally equipped with symplectic structure in exactly this sense: Given a cotangent bundle $T^*\mathcal{Q}$, there is a symplectic form on $T^*\mathcal{Q}$ which is preserved by all of the automorphisms of $T^*\mathcal{Q}$. North's suggestion seems to be that a similar relationship holds between the tangent bundle and metric structure as holds between the cotangent bundle and symplectic structure.

[13] As we will see later, North provides another argument which rejects the assumption that Hamiltonian mechanics must be formulated on a cotangent bundle. At places, however, she seems to refer to this more restricted formulation of Hamiltonian mechanics, so I find it worthy of discussion. See (North, 2009, p. 70–71, p. 71–72, p. 83).

ticles in the system. As in the Lagrangian approach, we need to specify some energy conditions in order to say how the system evolves over time. We do this by specifying a *Hamiltonian*. A Hamiltonian is a smooth function $H : T^*\mathcal{Q} \to \mathbb{R}$ that encodes the total energy of the system.[14]

North's argument for (P2) is simple: The cotangent bundle $T^*\mathcal{Q}$ has a natural symplectic structure.[15] This symplectic structure is constructed by first noticing the following fact.

Fact 1. There is a canonical covector field $\theta_{(q,\omega)}$ on $T^*\mathcal{Q}$, called the *Liouville* (or Poincaré, or tautological) *1-form.* ⌟

The construction of the Liouville 1-form is straightforward. The cotangent bundle $T^*\mathcal{Q}$ comes equipped with a smooth projection $\pi : T^*\mathcal{Q} \to \mathcal{Q}$ defined by $\pi(q, \omega) = q$. Given a point $(q, \omega) \in T^*\mathcal{Q}$, the projection π induces the pullback $\pi^* : T^*_q\mathcal{Q} \to T^*_{(q,\omega)}T^*\mathcal{Q}$ from the cotangent spaces of $\mathcal{Q}$ to the cotangent spaces of $T^*\mathcal{Q}$. To each point $(q, \omega) \in T^*\mathcal{Q}$, we want to assign a covector, an element of $T^*_{(q,\omega)}T^*\mathcal{Q}$. So given such a point $(q, \omega) \in T^*\mathcal{Q}$ we define

$$\theta_{(q,\omega)} = \pi^*(\omega) \ .$$

The covector field $\theta_{(q,\omega)}$ on $T^*\mathcal{Q}$ is the Liouville 1-form.

Of course, we do not yet have a symplectic form on $T^*\mathcal{Q}$; we only have a covector field. But it is not hard to get a symplectic form out of $\theta_{(q,\omega)}$. We simply take its exterior derivative. We define

$$\Omega = d\theta_{(q,\omega)} \ .$$

And as one can easily verify, Ω is a symplectic form on $T^*\mathcal{Q}$. Ω is the canonical symplectic form on the cotangent bundle.

It is in this sense, therefore, that the Hamiltonian statespace has symplectic structure. This structure essentially 'comes for free' since there is a natural symplectic form Ω associated with the cotangent bundle of any manifold.

2.3 Metric > symplectic

We have recounted North's arguments for (P1) and (P2). Now all that remains in this recapitulation of North's argument is to argue for (P3), that symplectic structure is, in some sense, *less structure* than metric structure.

It is clear what North's argument is for (P3). A symplectic form Ω on a $2n$-dimensional manifold determines a volume form $\Omega^n = \Omega \wedge \ldots \wedge \Omega$ on the

[14] The dynamics of Hamiltonian mechanics are given by Hamilton's equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}^i = -\frac{\partial H}{\partial q_i} \ .$$

See (Baez, 2005, Ch. 4).

[15] And, perhaps more importantly, this symplectic structure is left invariant by all the maps which preserve the form of Hamiltonian equations of motion. This fact is made precise in Theorem 3.3.19 of (Abraham and Marsden, 1978).

manifold, the n-fold wedge product of Ω with itself. So the statespace of Hamiltonian mechanics has mere volume structure, while the statespace of Lagrangian mechanics has metric structure. North reminds us that while distance determines volume, volume does not determine distance:

> [...] there is a clear sense in which a space with a metric structure has more structure than one with just a volume element. Metric structure comes with, or determines, or presupposes, a volume structure, but not the other way around. [...] Intuitively, knowing the distances between the points in a space will give you the volumes of the regions, but the volumes will not determine the distances. Metric structure adds a further level of structure. (North, 2009, p. 74–75)

Since the statespace of Lagrangian mechanics has full-blown metric structure, while the statespace of Hamiltonian mechanics only has volume structure, North concludes that the Hamiltonian statespace has *less structure* than the Lagrangian statespace does. This completes North's argument for (LS).

3 Hamiltonian statespace *does not* have less structure than Lagrangian statespace

The goal of this section is to show that North's argument for (LS), as presented in the previous section, is not sound. I will first argue that the Lagrangian statespace *does not* always have metric structure. Then I will show that whenever the Lagrangian statespace has metric structure, the Hamiltonian statespace has *more* structure than mere symplectic structure. It has symplectic structure and a sort of metric structure. Finally, I will conclude by arguing that there is a compelling sense in which metric structure simply *is not* more structure than symplectic structure.

3.1 Lagrangian statespace has less than metric structure

The Lagrangian statespace does not always have metric structure. The tangent bundle does not come naturally equipped with metric structure, and in general, specifying a Lagrangian for a system does not presuppose a metric on configuration space. But before arguing for this, let me assuage another potential worry that one might have with North's argument for (LS).

3.1.1 A potential worry

One might have noticed an asymmetry in the above arguments. Argument 1 for (P1) concludes that specifying a Lagrangian requires a metric *on the configuration space* $\mathcal{Q}$.[16] North's argument for (P2) concludes that there is a natural

[16] Argument 2 for (P1), on the other hand, does not exhibit this asymmetry since it concludes that the *tangent bundle* $T_*\mathcal{Q}$ has metric structure.

symplectic form on the *cotangent bundle* $T^*\mathcal{Q}$. We are therefore comparing metric structure on $\mathcal{Q}$ to symplectic structure on $T^*\mathcal{Q}$. This might strike one as problematic. But this worry can be assuaged by recognizing the following fact.

Fact 2. A metric g_q on $\mathcal{Q}$ can be 'pulled back' onto $T_*\mathcal{Q}$ using the smooth projection π. ⌟

Suppose that, as North argues, Lagrangian mechanics requires a metric g_q on $\mathcal{Q}$. The projection $\pi : T_*\mathcal{Q} \to \mathcal{Q}$ induces a linear pushforward from the tangent spaces of $T_*\mathcal{Q}$ to the tangent spaces of $\mathcal{Q}$ denoted by $\pi_* : T_{(q,v)}T_*\mathcal{Q} \to T_q\mathcal{Q}$, for each $(q, v) \in T_*\mathcal{Q}$. The metric g_q on $\mathcal{Q}$ and the map π_* naturally determine a metric $g'_{(q,v)}$ on $T_*\mathcal{Q}$ defined by

$$g'_{(q,v)}(x, y) = g_q(\pi_*(x), \pi_*(y)) \ ,$$

for each $(q, v) \in T_*\mathcal{Q}$ and $x, y \in T_{(q,v)}T_*\mathcal{Q}$.

This provides at least a semblance of symmetry between the two arguments. If North's arguments for (P1) and (P2) succeed, then the Lagrangian statespace $T_*\mathcal{Q}$ has metric structure and the Hamiltonian statespace $T^*\mathcal{Q}$ has symplectic structure. This asymmetry between Argument 1 for (P1) and the argument for (P2) therefore does not impinge on North's conclusion.

3.1.2 General Lagrangians

What does impinge on North's conclusion, however, is that the Lagrangian statespace does not always have metric structure. In general, (P1) is not true.

We will begin by showing that Argument 2 for (P1) does not work. Recall that Argument 2 claimed that point$_*$-transformations leave metric structure invariant, and so the tangent bundle comes naturally equipped with metric structure. The following simple example shows that this cannot be the case. We exhibit a tangent bundle $T_*\mathcal{Q}$ and a point$_*$-transformation $T_*f : T_*\mathcal{Q} \to T_*\mathcal{Q}$ which 'breaks' *any* metric g on $T_*\mathcal{Q}$.

Example 1 (Point$_*$-transformations *do not* preserve g)**.** Let $\mathcal{Q} = \mathbb{R}$, so that $T_*\mathcal{Q} \cong \mathbb{R}^2$. Consider the diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$. One can easily verify that $T_*f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$T_*f(q, v) = (2q, 2v).$$

Now let $g_{(q,v)}$ be an arbitrary metric on $\mathbb{R}^2$. It is easy to see that T_*f does not preserve this metric structure. At the point $p = (0, 0) \in \mathbb{R}^2$, the pushforward $(T_*f)_* : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$ is given by $(T_*f)_*(w) = 2w$, for every $w \in T_p\mathbb{R}^2$. And so for any non-zero $w \in T_p\mathbb{R}^2$ we have

$$g_p(w, w) \neq 4g_p(w, w) = g_p\big((T_*f)_*(w), (T_*f)_*(w)\big).$$

Therefore $(T_*f)^*(g_p) \neq g_p$, and so the point$_*$-transformation T_*f does not preserve the metric structure $g_{(q,v)}$. ⌟

This means that Argument 2 does not work. Point$_*$-transformations on $T_*\mathcal{Q}$ simply do not preserve metric structure. Contrary to the claim of Argument 2, metric structure is *not* invariant under point$_*$-transformations. So we cannot rely on Argument 2 to demonstrate (P1).

If we want to demonstrate (P1), it seems as if we will have to rely on Argument 1. Recall that Argument 1 appeals to the fact that in many applications of Lagrangian mechanics, one presupposes a metric on $\mathcal{Q}$ when specifying a Lagrangian for the system. But as North herself points out (North, 2009, p. 73), this argument also has problems. In general, a Lagrangian does not make reference to a metric on $\mathcal{Q}$. A Lagrangian is merely an arbitrary smooth scalar function $L : T_*\mathcal{Q} \to \mathbb{R}$. It need not be of the form

$$L(q, v) = \frac{1}{2} g_q(v, v) - V(q) \ ,$$

with g_q a metric on $\mathcal{Q}$ and $V : \mathcal{Q} \to \mathbb{R}$ a smooth potential. In general, specifying a Lagrangian L for a system simply does not require a metric g_q on configuration space.

3.2 Hamiltonian statespace (often) has more than symplectic structure

One might respond to these concerns about (P1) by stipulating that the scope of Lagrangian mechanics be restricted to only include systems with Lagrangians that *do* presuppose a metric on $\mathcal{Q}$. One might argue, for example, that systems with Lagrangians that do not presuppose a metric are not *physically reasonable* systems. After all, many common Lagrangian systems do have Lagrangians of the form $L(q, v) = \frac{1}{2} g_q(v, v) - V(q)$. One might argue that this is a sense in which (P1) is true: Most physically reasonable Lagrangian systems have Lagrangians that presuppose a metric on configuration space.

In this section, however, I will argue that even if one restricts the scope of Lagrangian mechanics to only include systems that presuppose a metric on $\mathcal{Q}$, North's argument for (LS) still does not go through. There is a strong sense in which whenever the Lagrangian statespace has metric structure, the Hamiltonian statespace has the same. I will first argue that whenever there is a metric on the Lagrangian statespace, the metric naturally induces a dual structure on the Hamiltonian statespace. Then I will show that many simple Hamiltonians presuppose this dual structure in exactly the same way that many simple Lagrangians presuppose a metric.

3.2.1 A dual structure on the Hamiltonian statespace

Suppose that, as North has argued, in Lagrangian mechanics the configuration space $\mathcal{Q}$ has a metric g_q. It is then crucial to take note of the following fact when comparing the structure of the Lagrangian and Hamiltonian statespaces.

Fact 3. Given a point $q \in \mathcal{Q}$, the metric g_q induces a vector space isomorphism $F_g^q : T_q\mathcal{Q} \to T_q^*\mathcal{Q}$, defined for each $v \in T_q\mathcal{Q}$ by

$$F_g^q(v) = g_q(v, \_\_) \ ,$$

where the covector $g_q(v, \_\_) \in T_q^*\mathcal{Q}$ acts on arbitrary vectors $w \in T_q\mathcal{Q}$ by $w \mapsto g_q(v, w)$. The map $F_g : T_*\mathcal{Q} \to T^*\mathcal{Q}$ defined by $F_g : (q, v) \mapsto (q, F_g^q(v))$ is a diffeomorphism, usually called the *musical isomorphism.* ⌟

The isomorphism F_g lets us essentially 'move' the metric structure g_q from the Lagrangian statespace onto the Hamiltonian statespace. The metric g_q allows one to determine lengths of and angles between vectors — which are supposed to encode *velocities* — at points of $\mathcal{Q}$. And it turns out that g_q naturally gives rise to a corresponding structure g_q^* which allows one to determine lengths of and angles between *covectors* — which are supposed to encode *momenta* — at points of $\mathcal{Q}$.

The metric g_q is a smooth assignment of inner products to the tangent spaces of $\mathcal{Q}$; g_q^* is a smooth assignment of inner products to the *cotangent* spaces of $\mathcal{Q}$. It is easy to define how g_q^* acts on pairs of covectors using the fact that F_g^q is an isomorphism for each $q \in \mathcal{Q}$ and thus invertible and linear:

$$g_q^*(\alpha, \beta) = g_q((F_g^q)^{-1}(\alpha), (F_g^q)^{-1}(\beta)) \ , \tag{1}$$

for every $\alpha, \beta \in T_q^*\mathcal{Q}$.[17] Whenever there is a metric g_q on the Lagrangian statespace, therefore, a dual structure g_q^* naturally arises on the Hamiltonian statespace. If (P1) is true and the Lagrangian statespace has metric structure, then the Hamiltonian statespace has more structure than mere symplectic structure.

3.2.2 Simple Hamiltonians

There is, in fact, a second and more interesting way to see this. Notice that North's argument for (LS) does not compare the structures of Lagrangian and Hamiltonian mechanics on level footing. Argument 1 demonstrates that Lagrangian mechanics has metric structure by considering both the statespace $T_*\mathcal{Q}$ *and* the Lagrangian $L : T_*\mathcal{Q} \to \mathbb{R}$ defined on that statespace. The argument that Hamiltonian mechanics has symplectic structure, however, only relies on the Hamiltonian statespace $T^*\mathcal{Q}$. It does not take into account the Hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$ defined on the statespace.

This might strike one as strange. Lagrangians and Hamiltonians serve similar purposes in Lagrangian and Hamiltonian mechanics.[18] They encode certain energy conditions and, in conjunction with the dynamical laws, dictate how systems will evolve over time. If to determine the structure of the Lagrangian

[17] The metrical structure g_q^* is not just an arbitrary assignment of inner products to the cotangent spaces of $\mathcal{Q}$. Given a metric g_q, g_q^* arises in a perfectly natural way. This is the sense in which I say g_q determines or gives rise to g_q^*.

[18] Although they may have different physical significance. See (Curiel, 2013) for a discussion.

statespace, one considers the Lagrangian L, then to determine the structure of the Hamiltonian statespace one should consider the Hamiltonian H. When this asymmetry is remedied, one can see that insofar as the Lagrangian statespace is equipped with the structure g_q, the Hamiltonian statespace is equipped with the structure g_q^*.

In order to see the precise way in which the Hamiltonian statespace has the structure g_q^*, it will be convenient to first introduce some terminology. A simple mechanical system, defined as follows, is exactly the kind of Lagrangian system which is considered above in Argument 1 for (P1).

Definition 2 (Simple Mechanical System)**.** A *simple mechanical system* is a quadruple $(\mathcal{Q}, g_q, V, L)$, where

- $\mathcal{Q}$ is a differentiable manifold, the configuration space for the system.
- $g_q : T_q\mathcal{Q} \times T_q\mathcal{Q} \to \mathbb{R}$ is a Riemannian metric on $\mathcal{Q}$.
- $V : \mathcal{Q} \to \mathbb{R}$ is a smooth potential on $\mathcal{Q}$.
- $L : T_*\mathcal{Q} \to \mathbb{R}$ is the Lagrangian of the system, and is defined by $L(q, v) = \frac{1}{2}g_q(v, v) - V(q)$. ⌟

We want to see what simple mechanical systems look like from the Hamiltonian perspective. Given a Lagrangian system, there is a way to translate it into a Hamiltonian system. The first step in this translation is to notice the following fact.

Fact 4. Given a point $q \in \mathcal{Q}$, a smooth function $L : T_*\mathcal{Q} \to \mathbb{R}$ gives rise to a map $\mathcal{L}_L^q : T_q\mathcal{Q} \to T_q^*\mathcal{Q}$ defined by:

$$\mathcal{L}_L^q(v) \cdot w = \frac{d}{dt} L(q, v + tw)\bigg|_{t=0} ,$$

for each $v, w \in T_q\mathcal{Q}$, where $\cdot$ denotes the application of a covector to a vector. The map $\mathcal{L}_L : T_*\mathcal{Q} \to T^*\mathcal{Q}$ defined by $(q, v) \mapsto (q, \mathcal{L}_L^q(v))$ is usually called the *Legendre transformation* associated with L.[19] ⌟

Using the Legendre transformation, one can turn a Lagrangian $L : T_*\mathcal{Q} \to \mathbb{R}$ into a Hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$. Recall that the Hamiltonian H for a system is supposed to encode the system's total energy. There is also a way to define the total energy of a system in Lagrangian mechanics.

Definition 3 (Total Energy)**.** Given a system with Lagrangian $L : T_*\mathcal{Q} \to \mathbb{R}$, the *total energy* $E_L : T_*\mathcal{Q} \to \mathbb{R}$ of the system is defined by

$$E_L(q, v) = \mathcal{L}_L^q(v) \cdot v - L(q, v) ,$$

for each $(q, v) \in T_*\mathcal{Q}$.[20] ⌟

[19] For a nice geometrical discussion of the Legendre transformation see (Mac Lane, 1970). We will restrict our attention here to 'nice' Lagrangians whose Legendre transformations $\mathcal{L}_L$ are actually diffeomorphisms between the tangent and cotangent bundles. Such Lagrangians are typically called *hyperregular* (Abraham and Marsden, 1978, §3.6).

[20] See (Abraham and Marsden, 1978, §3.5).

The Hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$ associated with a given Lagrangian $L : T_*\mathcal{Q} \to \mathbb{R}$ is obtained by 'translating' the total energy E_L from the Lagrangian framework into the Hamiltonian framework:

$$H = E_L \circ \mathcal{L}_L^{-1} .$$

We now have enough to show that the Hamiltonian H associated with a simple mechanical system presupposes g_q^* in exactly the same way that the Lagrangian L associated with a simple mechanical system presupposes g_q. This fact can be seen in two steps. We first notice that for simple mechanical systems, the Legendre transformation associated with L is equal to the musical isomorphism associated with g_q.

Lemma 1. *Let (Q, g_q, V, L) be a simple mechanical system. Then $\mathcal{L}_L = F_g$.* ⌟

Proof. The proof is a straightforward calculation and is thus omitted. □

We then notice that the form of the Hamiltonian for a simple mechanical system is perfectly dual to the form of the system's Lagrangian.

Proposition 1. *Let (Q, g_q, V, L) be a simple mechanical system. The Hamiltonian $H : T^*\mathcal{Q} \to \mathbb{R}$ associated with this system is given by*

$$H(q, \omega) = \frac{1}{2} g_q^*(\omega, \omega) + V(q) ,$$

where $g_q^* : T_q^*\mathcal{Q} \times T_q^*\mathcal{Q} \to \mathbb{R}$ is defined as in equation (1). ⌟

Proof. Notice that for a simple mechanical system, the energy $E_L : T_*\mathcal{Q} \to \mathbb{R}$ is given by

$$E_L(q, v) = \mathcal{L}_L^q(v) \cdot v - L(q, v) = F_g^q(v) \cdot v - L(q, v) = \frac{1}{2} g_q(v, v) + V(q) ,$$

where the first equality follows from Definition 3, the second equality follows from Lemma 1, and the third from Definition 2 and Fact 3. The Hamiltonian H associated with the system is therefore given by

$$\begin{aligned} H(q, \omega) &= E_L \circ \mathcal{L}_L^{-1}(q, \omega) \\ &= E_L \circ F_g^{-1}(q, \omega) \\ &= \frac{1}{2} g_q((F_g^q)^{-1}(\omega), (F_g^q)^{-1}(\omega)) + V(q) \\ &= \frac{1}{2} g_q^*(\omega, \omega) + V(q), \end{aligned}$$

where the last equality follows from the definition of g_q^* in equation (1). □

In light of this discussion, it is hard to see any sense in which the Lagrangian statespace could have metric structure without the Hamiltonian statespace having the same. If the Lagrangian statespace comes with a metric g_q on configuration space $\mathcal{Q}$, then we have seen that g_q naturally induces a corresponding structure g_q^* on the Hamiltonian statespace. And furthermore, Proposition 1 shows that simple Hamiltonians make reference to g_q^* in *precisely the same way* that simple Lagrangians make reference to g_q.

Argument 1 for (P1) fails because it demonstrates too much. Insofar as Argument 1 demonstrates that Lagrangian mechanics has metric structure, it demonstrates the same of Hamiltonian mechanics. If (P1) is true, then North's argument for (LS) does not go through.[21] The argument is self-undermining.

3.3 Comparing Lagrangian and Hamiltonian structures

Even if there were a sense in which the Lagrangian statespace had metric structure without the Hamiltonian statespace having the same, North's argument for (LS) would encounter another obstacle. It is not clear that symplectic structure is, in fact, less structure than metric structure. According to one plausible criterion for comparing amounts of mathematical structure, (P3) is not true.[22]

3.3.1 Counting mathematical structure

We begin the discussion of (P3) with a brief digression. In order to be able to say that symplectic structure is less structure than metric structure, we need to have a principled way of comparing amounts of mathematical structure.

Given a mathematical object, North suggests, there is a particularly natural way to assess the amount of structure that this object has (North, 2009, p. 87). One simply looks to the automorphisms, or symmetries, of the object. Automorphisms are invertible structure preserving maps from a mathematical object to itself. The idea is this: If a mathematical object has *many* automorphisms, then the object has little structure that the automorphisms need to preserve. On the other hand, if it has *few* automorphisms, then the object has much structure that the automorphisms need to preserve. The amount of structure that a mathematical object has is inversely proportional to, in some sense, the *size* of the object's automorphism group.

We therefore count the structure of mathematical objects, like the statespaces of Lagrangian and Hamiltonian mechanics, with something like the following principle in mind:

(SYM) A mathematical object X has more structure than a mathematical object Y if $\mathrm{Aut}(X)$ is 'smaller than' $\mathrm{Aut}(Y)$.[23] ⌟

Of course, SYM is not useful until we spell out precisely what we mean by 'smaller than.' As a first attempt, North suggests that we use the dimension of

[21] If (P1) is true and North's argument for (P2) is sound, then Hamiltonian statespace has metric structure *and* symplectic structure. And this will clearly not serve the same purpose in North's argument as (P2) did.

[22] In this section I expand upon a remark made in (Swanson and Halvorson, 2012) and (Halvorson, 2011).

[23] Where $\mathrm{Aut}(X)$ is the automorphism group of X. This is the same starting criterion used in (Swanson and Halvorson, 2012).

the automorphism groups as the relevant measure (North, 2009, p. 87). Consider the following criterion for comparing structure:

(DIM) A mathematical object X has more structure than a mathematical object Y if $\dim(\mathrm{Aut}(X)) < \dim(\mathrm{Aut}(Y))$. ⌟

Indeed, when we apply DIM to one particular case, it gives the intuitive answer. According to DIM, an n-dimensional vector space V has less structure than an n-dimensional inner product space (V, g). This is good. An inner product space intuitively has more structure than a mere vector space; it has inner product structure *in addition to* vector space structure.

But DIM suffers from two shortcomings. First, it is not completely general. There are many mathematical objects whose automorphism group does not in any sense have a dimension. Groups do not in general have a dimension.[24] And second, DIM makes some puzzling verdicts. Consider the vector space $\mathbb{R}$ and the vector space $\mathbb{R}^2$. As one can easily verify, according to DIM, $\mathbb{R}^2$ has less structure than $\mathbb{R}$.[25] This is a strange verdict. DIM, therefore, is unappealing as a general criterion for comparing amounts of mathematical structure.

The following example of vector space structure and inner product space structure, however, suggests another more promising way to make SYM precise.

Example 2 (Vector Space vs. Inner Product Space)**.** Let V be a vector space and g an inner product on V. As mentioned above, there is an intuitive sense in which (V, g) has more structure than V. It has all of the vector space structure of V along with additional inner product structure. And note that every automorphism of (V, g) is also an automorphism of V, but there are some automorphisms of V which are not automorphisms of (V, g).[26] ⌟

The automorphism group of (V, g) is properly contained in the automorphism group of V. This suggests the following revision of SYM:[27]

(SYM*) A mathematical object X has more structure than a mathematical object Y if $\mathrm{Aut}(X) \subsetneq \mathrm{Aut}(Y)$. ⌟

SYM* makes the intuitive verdict for many *easy* cases of structural comparison. In addition to Example 2, the following are some of the cases that SYM* answers correctly:

- A set X has less structure than a group $(X, \cdot)$.

[24] When the automorphism groups are Lie groups, of course, such talk does make sense. This is merely to say that, as it currently stands, DIM is not a completely general criterion for comparing mathematical structures. One might try to instead compare group orders, but as one can easily verify with the case of V vs. (V, g), this criterion will not always give the intuitive verdict.

[25] The automorphism group of $\mathbb{R}^2$ has dimension 4; the automorphism group of $\mathbb{R}$ has dimension 1.

[26] This is easy to see. Automorphisms of V are just bijective linear maps $f : V \to V$, while automorphisms of (V, g) are bijective linear maps $f : V \to V$ such that $g(x, y) = g(f(x), f(y))$ for each $x, y \in V$.

[27] This revision of SYM is the one that is settled upon in (Swanson and Halvorson, 2012).

- A set X has less structure than a topological space (X, τ).
- A differentiable manifold $(\mathcal{M}, U_\alpha, \phi_\alpha)$ has less structure than a Riemannian manifold $(\mathcal{M}, U_\alpha, \phi_\alpha, g_p)$.

And furthermore, SYM* makes the intuitive verdict when applied to the mathematical models of the classical spacetime theories mentioned at the beginning of this paper. According to SYM*, Aristotelian spacetime has more structure than absolute Newtonian spacetime, which in turn has more structure than Neo-Newtonian (or Galilean) spacetime.[28] So SYM* adequately captures our intuition concerning which mathematical object has more structure, at least when applied to certain easy cases. Unfortunately, the case of metric structure and symplectic structure is not one of these easy cases of structural comparison. According to SYM*, these two types of structure are incomparable.[29]

3.3.2 Symplectic and metric structure are incomparable

Recall North's argument that metric structure is more structure than symplectic structure. The argument points out the relationships that the two structures bear to volume structure. Symplectic structure on a manifold gives rise to volume structure on the manifold; a symplectic form naturally determines a volume form on the manifold. But volume structure, as North points out, is intuitively less structure than metric structure. A metric gives rise to a notion of volume, but not vice versa. North concludes that symplectic structure is less structure than metric structure.

The problem with this argument is readily apparent: *Symplectic structure is not the same thing as volume structure.* In fact, symplectic structure is intuitively *more* structure than volume structure. A symplectic form determines a notion of volume on a manifold *and* an orientation on the manifold. If metric structure and symplectic structure are both more structure than volume structure, then one cannot infer from this alone that symplectic structure is less structure than metric structure. Metric structure and symplectic structure are, in fact, incomparable according to SYM*. Metric structure is not more structure than symplectic structure, and symplectic structure is not more structure than metric structure.

This can be seen by considering the following two examples. For simplicity, we restrict attention to the case of inner products and symplectic forms on *vector spaces* rather than metrics and symplectic forms on manifolds. But all of what follows generalizes to the case of manifolds. In what follows, one is invited to think of the vector space V as the tangent space $T_p\mathcal{M}$ to a manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$.

Example 3 (Symplectic $\not\geq$ Metric)**.** Consider a 2-dimensional real vector space V. Let $g : V \times V \to \mathbb{R}$ be an inner product on V, and $\Omega : V \times V \to \mathbb{R}$ be a

[28] The reader can verify this by inspecting (Earman, 1989, Ch. 2).

[29] I borrow this turn of phrase from (Halvorson, 2011).

symplectic form on V. Let $\{e_1, e_2\}$ be a arbitrary basis for V. Define $f_1 : V \to V$ by

$$f_1(e_1) = \frac{1}{2} e_1$$
$$f_1(e_2) = 2e_2 \ ,$$

and extending linearly. One can easily verify that f_1 preserves the symplectic form Ω:

$$\Omega(x, y) = \Omega(f_1(x), f_1(y)) \ ,$$

for every $x, y \in V$. But f_1 *does not* preserve the inner product g. This can be seen simply by noting that $g(e_2, e_2) \neq 4g(e_2, e_2) = g(f_1(e_2), f_1(e_2))$. ⌟

Example 4 (Metric $\not\succ$ Symplectic)**.** Let V, g, Ω, and $\{e_1, e_2\}$ be as in Example 3. Define $f_2 : V \to V$ by

$$f_2(e_1) = e_2$$
$$f_2(e_2) = e_1 \ ,$$

and extending linearly. One can easily verify that $g(x, y) = g(f_2(x), f_2(y))$ for all $x, y \in V$. But f_2 does not preserve the symplectic form Ω, as $\Omega(e_1, e_2) \neq -\Omega(e_1, e_2) = \Omega(f_2(e_1), f_2(e_2))$.[30] ⌟

To summarize, Example 3 presents an automorphism f_1 of the structure (V, Ω) that is not an automorphism of (V, g). The map f_1 preserves the symplectic form, but 'breaks' the inner product. So we cannot apply SYM* to conclude that a symplectic vector space has more structure than an inner product space. Example 4, on the other hand, presents an automorphism f_2 of the structure (V, g) that is not an automorphism of (V, Ω); f_2 preserves the inner product while 'breaking' the symplectic form. So we also cannot apply SYM* to conclude that an inner product space has more structure than a symplectic vector space.

According to the criterion SYM*, therefore, neither is a metric more structure than a symplectic form nor is a symplectic form more structure than a metric. The two structures are in this sense incomparable. This discussion demonstrates that comparing metric structure and symplectic structure is not one of the easy cases of structural comparison. It is not like comparing the structure of a vector space to that of an inner product space, nor is it like any of the other easy cases enumerated above. Something more must be said in support of premise (P3). According to SYM*, (P3) is not true.[31]

[30] This example appears in (Swanson and Halvorson, 2012) and more explicitly in (Halvorson, 2011).

[31] If one *still* wants to argue that metric structure is more structure than symplectic structure, then one would first need to provide a general criterion of structural comparison which makes this verdict.

4 An alternative argument for (LS)

Given these considerations, it seems that we are left with no compelling argument for (LS). But North suggests an alternative argument, which might seem more promising.[32] It runs as follows:

(P1*) Lagrangian mechanics must be formulated on the statespace $T_*\mathcal{Q}$, the tangent bundle of configuration space $\mathcal{Q}$.

(P2*) Hamiltonian mechanics can be formulated on the statespace $(\mathcal{M}, \Omega)$, an arbitrary symplectic manifold.[33]

(P3*) Symplectic manifold structure is *less structure* than tangent bundle structure.

$\therefore$ **(LS)** ⌟

In this section I would like to focus on (P3*). I will first present North's argument for (P3*). Then I will argue that — just as in the metric and symplectic structure case — comparing symplectic manifold structure and tangent bundle structure is *not* one of the easy cases of structural comparison. While this alternative argument for (LS) might appear more promising than the original, more would need to be said in support of (P3*) in order to make this argument compelling.

4.1 The argument for (P3*)

One might begin with the intuition that (P3*) is true. Arbitrary symplectic manifolds do not have vector bundle structure like tangent bundles do.[34] North's argument for (P3*) is an attempt to make this intuition precise. A careful reconstruction of the argument will require a bit of work. We first need to define what the automorphisms of the *cotangent* bundle $T^*\mathcal{Q}$ are.

Definition 4 (Point*-transformation)**.** Let $f : \mathcal{Q} \to \mathcal{Q}$ be a diffeomorphism. Define the map $T^*f : T^*\mathcal{Q} \to T^*\mathcal{Q}$ by

$$T^*f(q, \omega) = (f^{-1}(q), f^*(\omega)),$$

[32] This argument is suggested in Appendix B (North, 2009, p. 84–88), but also at places in the body of the paper (North, 2009, p. 70, p. 77).

[33] $(\mathcal{M}, \Omega)$ is an arbitrary symplectic manifold. This general formulation of Hamiltonian mechanics works like as follows. Given the symplectic manifold $(\mathcal{M}, \Omega)$ and a Hamiltonian $H : \mathcal{M} \to \mathbb{R}$, the condition $\Omega(X_H, \_\_) = dH \cdot (\_\_)$ suffices to define a *Hamiltonian vector field* X_H on $\mathcal{M}$. The integral curves of X_H are the set of allowable trajectories through $\mathcal{M}$. See (Abraham and Marsden, 1978, §3.3) for exposition.

[34] North puts this intuition as follows: "[the Hamiltonian statespace] is a kind of structure that need not be easily or naturally splitable into a configuration space plus associated tangent spaces [...]. The Lagrangian statespace structure is therefore a more restrictive, less general structure" (North, 2009, p. 86).

for every point $(q, \omega) \in T^*\mathcal{Q}$, where f^* is the pullback associated with f. We will call T^*f a *point*-transformation*.[35] ⌟

Just as point$_*$-transformations preserve all of the structure of the tangent bundle $T_*\mathcal{Q}$, point*-transformations preserve all of the structure of the cotangent bundle $T^*\mathcal{Q}$. In other words, point*-transformations are the automorphisms of $T^*\mathcal{Q}$. We also need to define what the automorphisms of a symplectic manifold $(\mathcal{M}, \Omega)$ are.

Definition 5 (Symplectic Map)**.** Let $(\mathcal{M}, \Omega)$ be a symplectic manifold. A diffeomorphism $f : \mathcal{M} \to \mathcal{M}$ is called a *symplectic map* if $f^*(\Omega) = \Omega$.[36] ⌟

The intuition behind this definition should be clear. We are simply requiring that an automorphism of the object $(\mathcal{M}, \Omega)$ preserve *both* manifold structure and symplectic structure. Using these definitions and the criterion SYM*, we can see that there is a sense in which cotangent bundle structure is *more* structure than symplectic manifold structure. This can be seen in two steps. The first step is to notice the following fact.

Fact 5. Let $\mathcal{Q}$ be a manifold and Ω the canonical symplectic form on $T^*\mathcal{Q}$. Then for all point*-transformations T^*f it is the case that $(T^*f)^*(\Omega) = \Omega$, where $(T^*f)^*$ is the pullback of T^*f. ⌟

This fact simply says that all point*-transformations $T^*f : T^*\mathcal{Q} \to T^*\mathcal{Q}$ are symplectic maps from the symplectic manifold $(T^*\mathcal{Q}, \Omega)$ to itself.[37] In other words, all automorphisms of the *cotangent bundle* $T^*\mathcal{Q}$ are automorphisms of the *symplectic manifold* $(T^*\mathcal{Q}, \Omega)$. This means that every point*-transformation preserves the natural symplectic structure of $T^*\mathcal{Q}$. The converse, however, does not hold. There are symplectic maps from $T^*\mathcal{Q}$ to itself which are not point*-transformations, and thus do not preserve the cotangent bundle structure of $T^*\mathcal{Q}$. This is illustrated by the following example.[38]

Example 5 (Not all symplectic maps are point*-transformations)**.** Let $\mathcal{Q}$ be arbitrary manifold and let β be a non-zero covector field on $\mathcal{Q}$ such that $d\beta = 0$. Define the diffeomorphism $f : T^*\mathcal{Q} \to T^*\mathcal{Q}$ by

$$f(q, \omega) = (q, w + \beta_q),$$

for each $(q, \omega) \in T^*\mathcal{Q}$, where $\beta_q \in T_q^*\mathcal{Q}$ is the covector that the field β assigns to the point q.

We show that f is a symplectic map but not a point*-transformation. We first show that $f^*(\theta_{f(q,\omega)}) = \theta_{(q,\omega)} + \pi^*(\beta_q)$. Let $v \in T_{(q,\omega)}T^*\mathcal{Q}$. Then we see

[35] Point*-transformations are also sometimes just called *point-transformations*, or also *cotangent lifts*.

[36] Symplectic maps are sometimes called *canonical transformations* or also *symplectomorphisms*.

[37] For proof of Fact 5, the reader is invited to consult (Abraham and Marsden, 1978, Thm. 3.2.12).

[38] See (Abraham and Marsden, 1978, Problem 3.2E).

that

$$\begin{aligned}
f^*(\theta_{f(q,\omega)}) \cdot v = \theta_{f(q,\omega)} \cdot f_*(v) &= \pi^*(\omega + \beta_q) \cdot f_*(v) \\
&= (\omega + \beta_q) \cdot \big((\pi_* \circ f_*)(v)\big) \\
&= (\omega + \beta_q) \cdot \big((\pi \circ f)_*(v)\big) \\
&= (\omega + \beta_q) \cdot \big(\pi_*(v)\big) = \pi^*(\omega + \beta_q) \cdot (v) \\
&= \big(\theta_{(q,\omega)} + \pi^*(\beta_q)\big) \cdot (v),
\end{aligned}$$

where the second equality follows from the definitions of θ and f, the fifth equality follows from the fact that $\pi \circ f = \pi$, and the seventh equality follows from the definition of θ and linearity of π^*. Since v was arbitrary, we see that $f^*(\theta_{f(q,\omega)}) = \theta_{(q,\omega)} + \pi^*(\beta_q)$ for every $(q, \omega) \in T^*\mathcal{Q}$. And since $d\beta = 0$ it must be that

$$f^*(\Omega) = f^*(d\theta) = d(f^*(\theta)) = d(\theta + \pi^*(\beta)) = d\theta + \pi^*(d\beta) = d\theta = \Omega,$$

so f is a symplectic map, as desired.

But f is not a point*-transformation. If it were a point transformation, then it would have to be generated by the identity map $\mathrm{id} : \mathcal{Q} \to \mathcal{Q}$. That is, it would have to be that $f = T^*(\mathrm{id})$. But since β is non-zero one can easily verify that $f \neq T^*(\mathrm{id})$. So f is a symplectic map but not a point*-transformation. $\lrcorner$

This example shows that there are symplectic maps on $T^*\mathcal{Q}$ which are not point*-transformations. In other words, there are elements of the automorphism group of the symplectic manifold $(T^*\mathcal{Q}, \Omega)$, which are not elements of the automorphism group of the *cotangent bundle* $T^*\mathcal{Q}$.

Let's briefly summarize this discussion. Fact 5 tells us that all of the automorphisms of the cotangent bundle $T^*\mathcal{Q}$ are automorphisms of the symplectic manifold $(T^*\mathcal{Q}, \Omega)$. Example 5 shows us, however, that there are automorphisms of the symplectic manifold $(T^*\mathcal{Q}, \Omega)$ which are *not* automorphisms of the cotangent bundle $T^*\mathcal{Q}$. So we can apply the criterion SYM* to conclude that the symplectic manifold $(T^*\mathcal{Q}, \Omega)$ has less structure than the cotangent bundle $T^*\mathcal{Q}$. This gives us a strong sense in which cotangent bundle structure is more structure than symplectic manifold structure. North concludes that vector bundle structure, and in particular, *tangent bundle structure*, is more structure than symplectic manifold structure. This completes the argument for (P3*).[39]

4.2 Symplectic manifold vs. tangent bundle structure

I would like to raise a few concerns with this argument for (P3*). I will argue that comparing tangent bundle structure with symplectic manifold structure is another one of the hard cases of structural comparison.

[39] North puts the argument this way: "[. . .] the Lagrangian equations of motion are invariant under a set of point transformations; the Hamiltonian, under the canonical transformations. Whereas all point transformations are canonical transformations — point transformations form a subgroup of the set of all canonical transformations — point transformations are [. . .] only a special type of canonical transformation" (North, 2009, p. 87–88).

It is important to point out that the above argument for (P3*) fails to respect a crucial distinction. The argument demonstrates that *cotangent* bundle structure is, according to SYM*, more structure than symplectic manifold structure. But this, of course, is not the question at issue. We need to compare *tangent* bundle structure — not cotangent bundle structure — with symplectic manifold structure. And we cannot infer from the fact that cotangent bundle structure is more structure than symplectic manifold structure that *tangent* bundle structure is too. Simply put, this is because tangent bundle structure is different from cotangent bundle structure. As the following example illustrates, unlike the cotangent bundle, the tangent bundle does not come naturally equipped with a symplectic form.[40]

Example 6 ($T_*\mathcal{Q}$ is not symplectic)**.** We demonstrate that the tangent bundle does not come naturally equipped with a symplectic form simply by exhibiting a tangent bundle $T_*\mathcal{Q}$ and an automorphism of $T_*\mathcal{Q}$ — a point$_*$-transformation — which 'breaks' any symplectic form on $T_*\mathcal{Q}$.[41]

Let $\mathcal{Q} = \mathbb{R}$, so that $T_*\mathcal{Q} \cong \mathbb{R}^2$. Let Ω be an arbitrary symplectic form on $\mathbb{R}^2$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$. As we computed in Example 1, at the point $p = (0, 0) \in \mathbb{R}^2$ the pushforward $(T_*f)_* : T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$ is given by

$$(T_*f)_*(w) = 2w,$$

for every vector $w \in T_p\mathbb{R}^2$. We now see that T_*f 'breaks' the symplectic form Ω. This is clear since if $\{w_1, w_2\} \subset T_p\mathbb{R}^2$ is a basis, then

$$\begin{aligned}\Omega_p(w_1, w_2) \neq 4\Omega_p(w_1, w_2) &= \Omega_p(2w_1, 2w_2)\\ &= \Omega_p\big((T_*f)_*w_1, (T_*f)_*w_2\big)\\ &= (T_*f)^*(\Omega_p)(w_1, w_2)\end{aligned}$$

So $\Omega \neq (T_*f)^*\Omega$, and thus $T_*f : T_*\mathbb{R} \to T_*\mathbb{R}$ does not preserve the symplectic form Ω. ⌟

This example demonstrates that the tangent bundle $T_*\mathcal{Q}$ *does not* come naturally equipped with a symplectic form like the cotangent bundle does. The point$_*$-transformation T_*f preserves all of the tangent bundle structure, but it does not preserve symplectic structure.[42] This means that tangent bundle structure does not presuppose or give rise to symplectic structure.

The tangent bundle and the cotangent bundle come equipped with different kinds of structures. The mere fact that cotangent bundle structure is more

[40] This means that one cannot use the criterion SYM* to compare tangent bundle structure to cotangent bundle structure. Point$_*$-transformations are not the same as point*-transformations.

[41] Recall how we argued in Example 1. We argue in precisely the same manner here.

[42] This shows a sense in which North's claim quoted in footnote 39 is false. The Lagrangian equations of motion are invariant under point$_*$-transformations. And while point*-transformations *are* always symplectic maps, point$_*$-transformations simply are not. Under normal circumstances it does not make sense to call them "symplectic" since there is usually no symplectic structure on $T_*\mathcal{Q}$ for them to preserve. Unlike $T^*\mathcal{Q}$, the tangent bundle $T_*\mathcal{Q}$ is not naturally a symplectic manifold.

structure than symplectic manifold structure does not allow one to conclude that tangent bundle structure is more structure than symplectic manifold structure too. On the face of it the situation is this: A tangent bundle comes with some structure that a symplectic manifold lacks, linear vector bundle structure. But a symplectic manifold *also* comes with some structure that a tangent bundle lacks, a symplectic form. So something more needs to be said in support of (P3*).

4.3 Trying to patch up the argument for (P3*)

One might respond to these concerns with North's argument for (P3*) by appealing to the fact that there is a crucial similarity between the tangent bundle and the cotangent bundle. Although the two objects do not have *exactly the same* structure, the automorphism group of the tangent bundle $T_*\mathcal{Q}$ is isomorphic to the automorphism group of the cotangent bundle $T^*\mathcal{Q}$. One might take this to be a precise sense in which tangent bundle structure and cotangent bundle structure, although they are different structures, are the same *amount of structure*. We isolate this in the following fact.

Fact 6. The group $\mathrm{Aut}(T_*\mathcal{Q})$ of all point$_*$-transformations is isomorphic to the group $\mathrm{Aut}(T^*\mathcal{Q})$ of all point*-transformations. ⌟

The proof of this fact is as follows.

Proof of Fact 6. We prove that the function $A : \mathrm{Aut}(T_*\mathcal{Q}) \to \mathrm{Aut}(T^*\mathcal{Q})$ defined by

$$A : T_*f \mapsto T^*(f^{-1})$$

is a group isomorphism. We see trivially that A is bijective. Furthermore, we see that

$$\begin{aligned} A(T_*f \circ T_*g) &= A\big(T_*(f \circ g)\big) = T^*((f \circ g)^{-1}) \\ &= T^*(g^{-1} \circ f^{-1}) = T^*(f^{-1}) \circ T^*(g^{-1}) \\ &= A(T_*f) \circ A(T_*g), \end{aligned}$$

where the fourth equality follows from the easily verifiable fact that $T^*(g \circ f) = T^*f \circ T^*g$ for every pair of diffeomorphisms $f, g : \mathcal{Q} \to \mathcal{Q}$. Thus $A : \mathrm{Aut}(T_*\mathcal{Q}) \to \mathrm{Aut}(T^*\mathcal{Q})$ is a group isomorphism. □

One might feel that Fact 6 provides a sense in which the tangent bundle and the cotangent bundle have the *same amount of structure*. And intuitively, since we have already demonstrated a sense in which cotangent bundle structure is more structure than symplectic manifold structure, one might be inclined to conclude from this that tangent bundle structure is *also* more structure than symplectic manifold structure.

Unfortunately, this intuition is hard to make precise. One cannot apply the criterion SYM* to conclude that tangent bundle structure is more structure than symplectic manifold structure.[43] Example 6 shows us this; it is not the case that every point$_*$-transformation on an arbitrary tangent bundle is a symplectic map.

But SYM* is a very restrictive criterion. Since it does not judge tangent bundle structure to be more structure than symplectic manifold structure, one might desire a more general and flexible way to compare amounts of mathematical structure. In light of Fact 6, one might appeal to the following criterion of structural comparison:

(SYM**) A mathematical object X has more structure than a mathematical object Y if $\mathrm{Aut}(X)$ is isomorphic to a proper subgroup of $\mathrm{Aut}(Y)$. $\lrcorner$

SYM** is another way to formalize the imprecise, yet intuitive, criterion SYM. If $\mathrm{Aut}(X)$ is isomorphic to a proper subgroup of $\mathrm{Aut}(Y)$, then that captures a sense in which $\mathrm{Aut}(X)$ is 'smaller than' $\mathrm{Aut}(Y)$. Notice that SYM** captures all of the intuitive cases of structural comparison that SYM* captures, but it also makes verdicts in cases where SYM* abstains. And it turns out that according to SYM**, tangent bundle structure *is* more structure than symplectic manifold structure.

In fact, we have already seen this. Fact 6 shows us that $\mathrm{Aut}(T_*\mathcal{Q})$ is isomorphic to $\mathrm{Aut}(T^*\mathcal{Q})$. And as we saw in section 4.1, by Fact 5 and Example 5, the automorphism group of the cotangent bundle $\mathrm{Aut}(T^*\mathcal{Q})$ is a proper subgroup of the automorphism group of the symplectic manifold $\mathrm{Aut}(T^*\mathcal{Q}, \Omega)$. So indeed, $\mathrm{Aut}(T_*\mathcal{Q})$ is isomorphic to a proper subgroup of the automorphism group of the symplectic manifold $(T^*\mathcal{Q}, \Omega)$. And thus according to SYM**, tangent bundle structure is more structure than arbitrary symplectic manifold structure. This argument provides a sense in which (P3*) might be true.

Unfortunately, it is not at all clear that SYM** is a reliable criterion for comparing amounts of mathematical structure. The following two examples demonstrate that there are some problems with SYM**. Example 7 shows us that SYM** makes some unintuitive verdicts. Example 8 then goes one step further and shows that SYM** makes some nonsensical verdicts: There are mathematical objects which, according to SYM** have more structure than themselves.

Example 7 (Group vs. Vector Space)**.** Consider the three element group $(\mathbb{Z}_3, +)$ and the vector space $\mathbb{R}$. According to SYM**, $(\mathbb{Z}_3, +)$ has more structure than $\mathbb{R}$. To see this, notice that the automorphism group of $(\mathbb{Z}_3, +)$ has as elements just two functions:[44]

$$\mathrm{Aut}(\mathbb{Z}_3, +) = \{f_1 : x \mapsto x, f_2 : x \mapsto -x\},$$

where both $f_1, f_2 : \mathbb{Z}_3 \to \mathbb{Z}_3$. It is easy to see that $\mathrm{Aut}(\mathbb{Z}_3, +)$ is isomorphic to the subgroup $G \subsetneq \mathrm{Aut}(\mathbb{R})$, defined by

$$G = \{g_1 : x \mapsto x, g_2 : x \mapsto -x\},$$

[43] One can also easily see that the criterion DIM is not applicable here.

[44] One can see that this must be the case since group isomorphisms must map group generators to generators. The elements 1 and 2 are the generators of $(\mathbb{Z}_3, +)$.

where both $g_1, g_2 : \mathbb{R} \to \mathbb{R}$. The group G is clearly a proper subgroup of $\mathrm{Aut}(\mathbb{R})$. So according to SYM**, the group $(\mathbb{Z}_3, +)$ has more structure than the vector space $\mathbb{R}$. ⌟

Example 8 ($X < X$)**.** Let X be a countably infinite set. In this case $\mathrm{Aut}(X) = \{\text{bijections } f : X \to X\}$. Let $x \in X$ and consider the subgroup of $\mathrm{Aut}(X)$ defined by

$$G = \{\text{bijections } f : X \to X \text{ s.t. } f(x) = x\}.$$

Since X is countably infinite, we know that there is a bijection $h : X \to X \setminus \{x\}$. One can verify that the function $F : \mathrm{Aut}(X) \to G$ defined by

$$F(f)(y) = \begin{cases} x & \text{if } y = x \\ h \circ f \circ h^{-1}(y) & \text{if } y \neq x \end{cases}$$

is a group isomorphism. Therefore $\mathrm{Aut}(X)$ is isomorphic to a proper subgroup of itself. Applying the criterion SYM** yields that the set X has less structure than itself. ⌟

These two examples give one an uneasy feeling about the criterion SYM**. In the first case, one might have the intuition that the vector space $\mathbb{R}$ has more structure than the finite group $(\mathbb{Z}_3, +)$; vector spaces have an underlying group structure *and* some additional vector space structure.[45] In the second case, SYM** appears to make a verdict which is simply nonsensical. No mathematical object should have more or less structure than itself, but according to SYM**, a countably infinite set X has more structure than itself. Something needs to be said to dispel these worries if one wants to use the criterion SYM** to conclude that tangent bundle structure is more structure than symplectic manifold structure.

One might not be inclined to take this discussion to be a decisive refutation of North's second argument for (LS).[46] But at the very least, it shows that something more needs to be said in support of (P3*). Comparing tangent bundle structure with symplectic manifold structure is not one of the easy cases of structural comparison. It is not, for example, like comparing inner product space structure to vector space structure, as we did in Example 2. North's argument from the previous section merely demonstrates a sense in which *cotangent* bundle structure is more structure than symplectic manifold structure. But this, of course, does not demonstrate (P3*). If one wants to extend North's argument to argue for (P3*) by appealing to the criterion SYM**, then one needs to address the concerns that have been raised here.

[45] Note that SYM* refuses to make a verdict in this case. The functions in the two automorphism groups are different kinds of objects — they have different domains and codomains. This means that neither is $\mathrm{Aut}(\mathbb{Z}_3, +) \subsetneq \mathrm{Aut}(\mathbb{R})$ nor is $\mathrm{Aut}(\mathbb{Z}_3, +) \supsetneq \mathrm{Aut}(\mathbb{R})$, and so SYM* is not applicable.

[46] Indeed, one should not take this discussion to have demonstrated that (P3*) is false. There is certainly an intuitive sense in which the automorphism group a symplectic manifold is *bigger* than the automorphism group of the tangent bundle. This discussion has simply shown that this intuition is hard to formalize. And in order to make this alternative argument for (LS) compelling, one needs to formalize the intuition behind (P3*).

5 Interpreting mathematical structure

Even if one were to provide a compelling argument for (LS), it is not immediately clear what this would tell us about the comparative structures that Lagrangian and Hamiltonian mechanics *ascribe to the world*. (LS) is a comparative claim about the mathematical structure of the statespaces of Lagrangian and Hamiltonian mechanics. And one might worry that this does not imply that Lagrangian and Hamiltonian mechanics are inequivalent theories. There are two particular worries of this type which are worth pointing out.

5.1 Statespace realism

In order to build some intuitions, consider again the case of spacetime theories. There is a clear and intuitive sense in which the mathematical structure of a spacetime theory *mirrors* the structure of spacetime. Recall the classical spacetime theories which were mentioned earlier. In absolute Newtonian spacetime there is a mathematical structure which picks out a preferred rest frame.[47] Neo-Newtonian spacetime does not have this structure. This is a sense in which these two theories tell us different things about the structure of spacetime.

But there is room for a disanalogy between spacetime theories and *statespace* theories. One might take the mathematical structure of a spacetime theory to mirror the structure of the world, but deny that this is the case for statespace theories. One might claim that statespace in classical mechanics merely provides a convenient way to encode dynamical properties — for example, positions, velocities, and momenta — of bodies which are located *in spacetime*. If this is the case, then some of the mathematical structure of statespace might not correspond to any structure whatsoever in the world. Certain mathematical structures on statespace might make the description of what is happening in spacetime easier, but that does not mean that these structures *mirror* or *correspond to* some actual structure in the world.

This is merely to point out the extent to which this discussion connects with concerns about statespace realism. If one thinks, as statespace realists do, that statespace exists as a concrete thing, then the mathematical structure of statespace could mirror the structure of the world in exactly the same way in which the mathematical structure of spacetime mirrors the structure of the world.[48] This is not the place for an evaluation of statespace realism. But it is important to mention that in order to get philosophical mileage out of (LS) — for example, in order to use (LS) to argue that Hamiltonian mechanics ascribes less structure *to the world* than Lagrangian mechanics does — it seems inevitable that some form of statespace realism will be relied upon.[49]

[47] Specifically, this structure is a covariantly constant unit timelike vector field (Earman, 1989, Ch. 2).

[48] North points this out as well (North, 2009, p. 80–81).

[49] One might find the truth or falsity of (LS) to be of independent interest, regardless of what broader philosophical issues turn on (LS). I have strong sympathies here.

5.2 Model Isomorphism and Theoretical Equivalence

There is a second worry. (LS) is a claim about the comparative structure of models of Lagrangian and Hamiltonian mechanics. And much care must be taken when trying to address questions of theoretical equivalence or inequivalence merely by examining the models of the theories in question. The following example illustrates that equivalent physical theories sometimes have non-isomorphic models.

Example 9 (General Relativity)**.** General relativity can be formulated in two ways that are manifestly equivalent: One can define a relativistic spacetime as a pair $(\mathcal{M}, g)$ where g has signature $(1, -1, -1, -1)$ or as a pair $(\mathcal{M}, g')$ where g' has signature $(-1, 1, 1, 1)$. The only difference between these two formulations of general relativity is a sign convention. In the former, one defines timelike vectors v as those vectors that satisfy $g(v, v) > 0$; in the latter, one defines timelike vectors v as those vectors that satisfy $g'(v, v) < 0$. One does not want to say that these two formulations are inequivalent or that they ascribe different structures to spacetime. But the models of these two theories are non-isomorphic.[50] ⌟

When determining the structure that a physical theory ascribes to the world, one cannot always just compare the mathematical structure of the models of the theory. This tactic would lead one to conclude that the two equivalent formulations of general relativity ascribe different, non-isomorphic structures to the world. And that would be an undesirable conclusion. The difference between the two formulations is a sign convention. And that should not in any way change the structure that the two theories ascribe to the world.

The moral of Example 9 extends well beyond the case of Lagrangian and Hamiltonian mechanics. If models of Theory 1 are not isomorphic to models of Theory 2, one cannot conclude from this fact alone that Theory 1 and Theory 2 are inequivalent.[51] So even if (LS) were conclusively established, that would not imply the inequivalence of Lagrangian and Hamiltonian mechanics.

6 Conclusion

I hope to have shown that there are some significant obstacles that must be navigated by anyone interested in comparing the structure of the Hamiltonian and Lagrangian statespaces. The first type of obstacle arises in the task of identifying the precise *kinds of structure* that the statespaces have, and the second type arises in the task of *comparing the amounts* of structure that the statespaces have. Unfortunately, both types of obstacle have proven to be particularly troublesome. I have argued here that neither has yet been successfully navigated.

[50] One can easily verify that if $f : (\mathcal{M}, g) \to (\mathcal{N}, g')$ is an isometry, then g and g' have the same signature.

[51] For a more careful discussion of this issue, see (Halvorson, 2012).

References

Abraham, Ralph, and Jerrold E. Marsden. (1978). *Foundations of Mechanics.* Reading, MA: Benjamin/Cummings Pub. Print.

Baez, John. (2005). "Lectures on Classical Mechanics." Manuscript.

Curiel, Erik. (2013). "Classical Mechanics is Lagrangian; It is Not Hamiltonian." *British Journal for the Philosophy of Science.*

Earman, John. (1989). *World Enough and Space-time: Absolute versus Relational Theories of Space and Time.* Cambridge, MA: MIT. Print.

Halvorson, Hans. (2011). "Natural structures on state space." Manuscript.

Halvorson, Hans. (2012). "What Scientific Theories Could Not Be." *Philosophy of Science* 79, 2: 183–206.

Mac Lane, Saunders. (1970). "Hamiltonian Mechanics and Geometry." *The American Mathematical Monthly* 77.6: 570–586.

North, Jill. (2009). "The 'Structure' of Physics: A Case Study." *Journal of Philosophy* 106: 57–88.

Swanson, Noel, and Hans Halvorson. (2012). "On North's 'The Structure of Physics'". Manuscript.