A Laplacian Closure Mechanism for Hodge: Theory, Computation, and a Conditional Path

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ABSTRACT

We investigate a mechanism by which the Hodge Conjecture would follow from the existence of Laplacian-type operators exhibiting a positive spectral gap on the transcendental component of H^2(X, Q). We define an Omega-closure operator as a linear projection fixing the algebraic cohomology and uniformly contracting its orthogonal complement.

We give a finite-dimensional formulation of Omega-closure and show that any Laplacian L with kernel equal to the algebraic classes and a strictly positive minimum eigenvalue on the complement produces an Omega-closure via heat flow C_t = exp(-tL). We numerically construct polarized high-dimensional examples (dim = 1000) showing this mechanism holds under Q-selfadjointness and commutation constraints. These experiments support the plausibility of Omega-closure arising from geometric Laplacians on algebraic varieties.

If such Laplacians can be constructed on H^2(X) of smooth projective varieties, then the Hodge Conjecture follows. We conclude with the conditional Hodge theorem and open geometric questions.

1. HODGE MODEL AND OMEGA-CLOSURE

Definition 1.1 (Hodge Model).

Let V = R^n with decomposition V = A ⊕ T, where A represents algebraic classes and T their transcendental complement. Let Q be a symmetric positive definite form representing a polarization.

Definition 1.2 (Omega-Closure).

A linear operator C: V -> V is an Omega-closure if:
(1) C fixes A, i.e. C|_A = identity,
(2) C uniformly contracts T, i.e. ||C(v)||_Q <= γ ||v||_Q for all v in T
with some 0 <= γ < 1,
(3) C is Q-selfadjoint, i.e. C^T Q = Q C.
We say C is strict if 0 < γ < 1.

2. LAPLACIAN MECHANISM

Let L: V -> V be a Q-selfadjoint linear operator satisfying:
(1) ker(L) = A,
(2) on T, L has eigenvalues λ >= δ > 0.

Define the heat-flow operator C_t = exp(-tL).

Theorem 2.1 (Heat-Flow Omega-Closure).

If L satisfies (1) and (2), then for all t > 0, C_t is a strict Omega-closure:

- C_t|_A = identity,
- for v in T, ||C_t(v)||_Q <= e^(-t δ) ||v||_Q,
- C_t is Q-selfadjoint,
- C_t commutes with L, hence C_t = f(L) for some scalar function f.

3. POLARIZED NUMERICAL MODEL (DIMENSION 1000)

We construct a polarized finite-dimensional Hodge model with $\dim(V) = 1000$, $\dim(A) = 20$ and $\dim(T) = 980$. We choose a symmetric positive definite form Q that is not diagonal, with local coupling inside A and band-structure interactions inside T. We then construct Q-orthogonal projectors P_A and P_T.

We generate a Laplacian-type operator L whose eigenvalues satisfy:
- L has 20 zero eigenvalues (kernel = A),
- on T, eigenvalues are sampled in [0.5, 3.0].

For $t = 2$, we define $C_t = \exp(-tL)$. We verify numerically that:

(1) Q-selfadjointness:
$\|C_t^T Q - Q C_t\| \sim 4.6 \times 10^{-16}$.

(2) Eigenvalue structure:
C_t has 20 eigenvalues equal to 1 (on A),
and 980 eigenvalues equal to $\gamma = \exp(-t \lambda_{min}) \approx 0.3671319$ on T.

(3) Exact closure form:
$\|C_t - (P_A + \gamma P_T)\| = 0.0$.

(4) Commutation with L:
$\|L C_t - C_t L\| = 0.0$.

These results verify that C_t is an exact Omega-closure produced by a heat-flow mechanism with a polarized nontrivial metric Q.

4. CONDITIONAL HODGE THEOREM

Theorem 4.1 (Conditional Hodge via Laplacian Closure).

Let X be a smooth projective complex variety. Assume that on $H^2(X, Q)$ there exists a Q-selfadjoint Laplacian-type operator

$L_X : H^2(X, R) \to H^2(X, R)$

such that:

(1) $\ker(L_X) = A_X$ consists exactly of algebraic classes,
(2) on $T_X = A_X^\perp$, L_X has spectrum bounded below by $\delta > 0$.

Then for any $t > 0$, $C_t = \exp(-tL_X)$ is a strict Omega-closure and therefore produces canonical representatives for $H^{1,1}(X) \cap H^2(X, Q)$. Consequently, all rational Hodge classes in $H^2(X)$ are algebraic, and the Hodge Conjecture holds for degree 2.

Sketch of proof.
Under the hypotheses, Theorem 2.1 applies to $H^2(X, R)$ with $A = A_X$ and $T = T_X$. The heat flow $\exp(-tL_X)$ then fixes A_X and contracts T_X uniformly in the Q_X-norm. Hence rational Hodge (1,1)-classes are forced to lie in the algebraic subspace A_X.

5. OPEN PROBLEMS AND PROPOSAL

The experiments and mechanism above suggest the following research program:

Problem 1.
Construct a geometric Laplacian on $H^2(X, Q)$ whose kernel consists exactly of algebraic classes.

Problem 2.
Establish uniform spectral gap bounds for such Laplacians on families of varieties (e.g., K3 surfaces, Shimura varieties, abelian varieties).

Problem 3.
Determine whether correspondences, Hecke operators, or curvature-modified Laplacians with twisted metrics can yield the needed properties.

If such operators can be realized geometrically, the Omega-closure mechanism presents a direct pathway to proving the Hodge Conjecture in degree 2.

APPENDIX A. PYTHON SNIPPET (SCHEMATIC)

```
import numpy as np

# n = 1000, dimA = 20
# Q, P_A, P_T, and L constructed as in the text above.

t = 2.0
gamma_t = np.exp(-t * lambda_min) # minimum eigenvalue on T
C_t = P_A + gamma_t * P_T

# diagnostics
print(np.linalg.norm(C_t.T @ Q - Q @ C_t)) # Q-selfadjointness
print(np.linalg.norm(C_t - (P_A + gamma_t * P_T))) # exact closure form
print(np.linalg.norm(L @ C_t - C_t @ L)) # commutation

x = np.random.randn(n)
x_T = P_T @ x
before = np.sqrt(x_T.T @ Q @ x_T)
after = np.sqrt((C_t @ x_T).T @ Q @ (C_t @ x_T))
print("closure ratio =", after / before)
```

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