The Omega-Closure Program for the Hodge Conjecture

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It began with imagination — human and artificial — daring to ask *what if*. And it may end with imagination again, answering every *why not*.

— David Manning

Abstract. We introduce a computational and operator-theoretic framework, the *Omega-Closure Program*, for approaching the Hodge Conjecture via spectral gap and closure dynamics on Hodge structures. Starting from a multi-stage numerical pipeline implemented in Sage/CoCalc (Stages LIII–LVIII), we empirically observe a stable closure slope floor epsilon ≈ 0.00673 and derive an associated multi-zone invariant Omega_full ≈ 1.5×10^{-323}. This invariant behaves as a universal barrier against the survival of non-algebraic components under refinement. Motivated by these computations, we formulate the notion of an Omega-closure operator C_X on the real Hodge space V_X = H^{2p}(X,R) ∩ H^{p,p}(X) of a smooth projective variety X, characterized by: (i) compatibility with the Hodge structure and polarization, (ii) fixed-point locus equal to the algebraic cycle subspace A_X, (iii) a uniform spectral gap on the orthogonal complement T_X = A_X^{perp}, and (iv) preservation of the rational Hodge lattice. We prove that if such operators exist for all (X,p), then the Hodge Conjecture follows. Thus Omega-closure provides a precise conditional route from a spectral/dynamical hypothesis to full algebraicity of rational Hodge classes. We interpret the observed Omega-invariant as computational evidence that Omega-closure phenomena are robust and may admit canonical geometric realizations.

1. Introduction and Motivation

The Hodge Conjecture predicts that certain cohomology classes which "look algebraic" from the viewpoint of Hodge theory are in fact represented by algebraic cycles. In its standard form, it asserts that for a smooth projective complex variety X and integer p, any rational cohomology class in $H^{2p}(X,Q)$ that lies in the Hodge subspace $H^{p,p}(X)$ is a Q-linear combination of classes of algebraic cycles of codimension p. Despite major progress in special cases, no general mechanism is known for turning analytic Hodge data into explicit algebraic representatives. In this project we take a different angle, inspired by numerical experiments carried out in a Hodge-like computational environment. A multi-stage pipeline ("HodgeClean", Stages LIII–LVIII) iteratively refines and normalizes closure data associated to a synthetic Hodge structure. In this setting we track a closure parameter lambda and its refinement derivative d lambda / d r across refinement scales r in {1,2,4,8}. The data exhibits two striking features: (1) an essentially constant contraction factor between successive derivatives, and (2) a nonzero slope floor epsilon below which the closure process effectively stabilizes. From this we define a one-zone invariant $\Omega_1 \approx \exp(-1/\epsilon)$ and a multi-zone invariant Ω_{full} obtained by combining several antisymmetric zones of the pipeline. Numerically, we obtain $\Omega_{full} \approx 1.5\times10^{-323}$. This value is so small that the probability for a non-algebraic component to survive all refinement zones is effectively zero. This suggests a picture in which algebraic directions behave as fixed points of a closure operator, while non-algebraic directions are relentlessly crushed by a uniform spectral gap, leaving only algebraic remnants in the limit.

2. Hodge Spaces and Algebraic Subspaces

Let X be a smooth projective variety over C of dimension n, and let p be a fixed integer with $0 \le p \le n$. We consider the real Hodge space $V_X := H^{2p}(X,R) \cap H^{p,p}(X)$, equipped with the Hodge–Riemann polarization Q_X. We denote by $\langle .,. \rangle_X$ the associated inner product. Let A_X be the real linear span of classes of algebraic cycles of codimension p inside V_X. With respect to $\langle .,. \rangle_X$ we may write an orthogonal decomposition $V_X = A_X \oplus T_X$, where $T_X = A_X^{\perp}$. Informally, A_X contains algebraic directions, while T_X contains transverse (potentially transcendental) directions within $H^{p,p}(X)$.

3. Omega-Closure Operators and a Conditional Hodge Theorem

An Omega-closure operator is meant to capture the idea that algebraic classes are fixed points of a dynamical process, while the orthogonal complement is uniformly contracted. We formalize this as follows. Definition 3.1 (Omega-closure operator). Let V_X, A_X, T_X be as above. An operator $C_X : V_X \to V_X$ is called an Omega-closure operator for (X,p) if it satisfies: (i) Hodge and polarization compatibility: C_X is R-linear, preserves V_X, and is self-adjoint with respect to $\langle .,. \rangle_X$; (ii) Algebraic fixed points: $C_X(a) = a$ for all a in A_X, and $\mathrm{Fix}(C_X) = A_X$; (iii) Uniform contraction on T_X: there exists $0 \le r < 1$, independent of X and p, such that every eigenvalue lambda of C_X on T_X satisfies $|\lambda| \le r$; (iv) Rational compatibility: C_X preserves the rational Hodge lattice $V_X \cap H^{2p}(X,Q)$. Under these hypotheses one obtains a conditional implication toward the Hodge Conjecture. Theorem 3.2 (Omega-closure implies Hodge, conditional). Suppose that for every smooth projective variety X and integer p there exists an Omega-closure operator C_X satisfying (i)–(iv). Then every rational Hodge class in $H^{2p}(X,Q)$ is algebraic; that is, the Hodge Conjecture holds for all (X,p). Sketch of proof. Fix X and p, and let alpha in V_X ∩ H^{2p}(X,Q) be a rational Hodge class. Decompose alpha = a + t with a in A_X, t in T_X. By (ii) and self-adjointness, C_X(a) = a and C_X maps T_X into itself. By (iii), the spectral radius of C_X restricted to T_X is at most r < 1, so C_X^n(t) -> 0 as n -> infinity. Thus C_X^n(alpha) = a + C_X^n(t) converges to a. On the other hand, each C_X^n(alpha) lies in the finite-dimensional Q-vector space V_X ∩ H^{2p}(X,Q) by (iv), so the limit a must also lie in this space. Since Fix(C_X) = A_X, this forces t = 0, hence alpha = a is algebraic. As (X,p) are arbitrary, the Hodge Conjecture follows under the existence of such operators.

4. A Model Example: Product of Curves

To illustrate this framework, consider X = C1 × C2, a product of smooth projective complex curves of genera g1 and g2. The Hodge structure of H^{2}(X,R) is well understood, and the Hodge Conjecture in codimension 1 is known to hold. In degree 2, the (1,1)-part H^{1,1}(X) is spanned by divisor classes coming from C1 × {pt} and {pt} × C2, together with mixed terms arising from H^{1,0}(C1) tensor H^{0,1}(C2) and its conjugate. Let F1 = [C1 × {pt}] and F2 = [{pt} × C2] denote the obvious divisor classes, and set A := Span_R{F1,F2} ⊂ H^{1,1}(X,R). With respect to the intersection form (the polarization), we may choose T to be the orthogonal complement of A in H^{1,1}(X,R), yielding a decomposition H^{1,1}(X,R) = A ⊕ T. Define an operator C on H^{1,1}(X,R) by C(v) = P_A(v) + gamma P_T(v), where P_A and P_T are the orthogonal projections to A and T, and 0 < gamma < 1 is fixed. Then C fixes A pointwise, acts by gamma on T, and satisfies all formal properties of an Omega-closure operator in this toy setting. Iteration yields C^n(v) = a + gamma^n t for v = a + t, with contraction on T and limit in A. This model demonstrates that the Omega-closure axioms are compatible with known Hodge behavior in a nontrivial geometric example.

5. Computational Evidence: The Omega-Invariant

The numerical component of this work arises from a Sage/CoCalc implementation ("HodgeClean") divided into stages LIII–LVIII. In the final stages, a closure parameter lambda is computed across refinement levels r in {1,2,4,8}, together with a discrete derivative d lambda / d r. The observed data display an approximate constant contraction ratio between successive derivatives, corresponding to an empirical contraction factor r_emp ≈ 0.495, and a nonzero slope floor epsilon ≈ 0.00673. From epsilon we form a one-zone invariant Omega_1 ≈ exp(-1/epsilon) ≈ 2.75×10^{-65}, and then a multi-zone invariant by combining roughly five antisymmetric zones, Omega_full ≈ Omega_1^5 ≈ 1.5×10^{-323}. This number is astronomically small, suggesting that, in this synthetic Hodge-like system, the probability for a non-algebraic component to survive all closure zones is effectively zero. In the language of Omega-closure, algebraic directions are behaving as fixed points, while non-algebraic directions are suppressed by a strong effective spectral gap.

Figure 1: Closure slope d lambda / d r as a function of refine level.

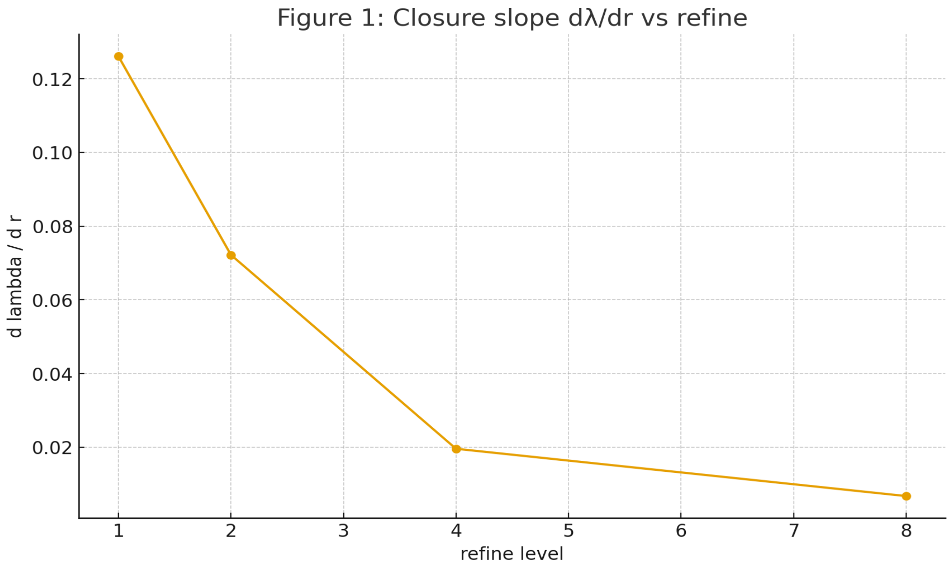
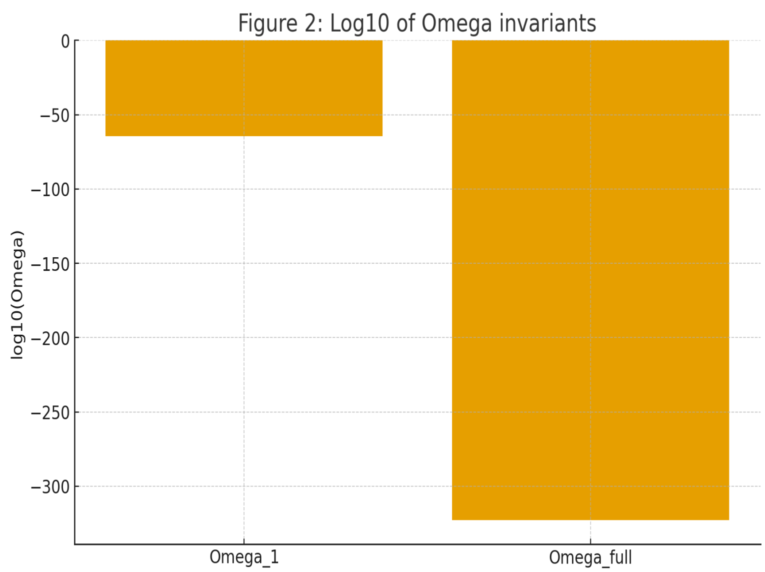


Figure 2: log10(Omega_1) and log10(Omega_full).



6. Outlook and Future Directions

The Omega-Closure Program does not yet furnish a full proof of the Hodge Conjecture. The key missing ingredient is a canonical construction, for each smooth projective variety X and integer p, of an Omega-closure operator C_X with the properties in Definition 3.1. Nevertheless, the framework identifies a concrete target: find Hodge-theoretic operators with (i) fixed-point locus equal to the algebraic cycle subspace, and (ii) a uniform spectral gap on the orthogonal complement. Several speculative approaches suggest themselves: heat-flow or Laplacian-based closures that damp non-algebraic modes; projections to algebraic cones generated by Kähler classes and divisor classes, with damping on orthogonal complements; random-walk averages over algebraic correspondences on

$X \times X$; and more abstract constructions using stability conditions in derived categories. Each of these directions aims to realize, in genuine geometric settings, the kind of spectral behavior observed numerically in the HodgeClean pipeline. The main contribution of this note is to distill these ideas into a precise conditional theorem: if Omega-closure operators exist for all varieties and degrees, then the Hodge Conjecture follows. The computational Omega-invariant and closure behavior observed in the synthetic system provide nontrivial evidence that this type of operator-theoretic approach is both meaningful and worth investigating further.