Laplacian Ω-Closure and a Conditional Framework toward the Hodge Conjecture

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Abstract

This note refines and consolidates a numerical and conceptual program around an "Ω-closure" operator. In its simplest form, the operator is a self-adjoint endomorphism $C\colon V \to V$ on a finite-dimensional real inner product space V, fixing a distinguished "algebraic" subspace $A \subset V$ pointwise and uniformly contracting its orthogonal complement $T = A^{\perp}$. Earlier versions of this project (under the name *HodgeClean*) constructed and stress–tested a variety of such Ω-closures using finite-dimensional toy models and multi-stage numerical pipelines.

In this WRAPPED version we isolate a particularly clean mechanism: if a self-adjoint "Laplacian" L has kernel A and a positive spectral gap on T, then its heat-flow $C_t = e^{-tL}$ is an explicit Ω-closure for each $t > 0$. We formulate a conditional Hodge–theoretic statement assuming the existence of such Laplacians on actual Hodge structures, and we report new numerical experiments that mimic a K3–like H^2 with Picard rank 20 in dimension $n = 1000$, together with variable-rank experiments (rank 1, 10, 20) at a common heat time.

The work is exploratory and does *not* claim a proof of any new Hodge case. Its value is to present a concrete spectral mechanism, a finite dimensional model with robust numerical support, and a clearly stated conditional theorem whose geometric hypotheses can be tested in specific settings.

1 From Ω-closure to a Laplacian picture

Let (V, Q) be a finite-dimensional real inner product space with an orthogonal decomposition

$$V = A \oplus T,$$

where A is intended to model the span of algebraic classes and T its transcendental complement. Denote by P_A and P_T the orthogonal projectors onto A and T.

Definition 1.1. An Ω-*closure operator* on $(V, Q, A \oplus T)$ is a Q-selfadjoint linear map $C\colon V \to V$ such that

(i) $C|_A = \mathrm{id}_A$;

(ii) $C(T) \subseteq T$ and $C|_T$ is a strict contraction for the Q-norm;

(iii) the spectrum of C is contained in $\{1\} \cup [0, \gamma]$ for some $0 \leq \gamma < 1$, with 1 having multiplicity $\dim A$.

In many of the earlier HodgeClean stages the operator C was built directly and then numerically tested for these properties. The key observation of the present note is that such a C arises naturally from a Laplacian-type operator.

1.1 Abstract Laplacian model

Assume we are given a Q-selfadjoint operator $L\colon V \to V$ such that

(L1) $\mathrm{Ker}(L) = A$;

(L2) L preserves the decomposition $V = A \oplus T$ and $L|_A = 0$;

(L3) the restriction $L|_T$ is positive definite, with spectrum contained in $[\lambda_{\min}, \lambda_{\max}]$ for some $0 < \lambda_{\min} \leq \lambda_{\max}$.

The heat-flow semigroup is defined by functional calculus:

$$C_t \;=\; e^{-tL}, \qquad t > 0.$$

Lemma 1.2 (Heat-flow as Ω-closure)**.** *Under* (L1)–(L3)*, each $C_t = e^{-tL}$ is Q-selfadjoint, fixes A pointwise, preserves T, and has spectrum*

$$\{1\} \cup \{e^{-t\lambda} : \lambda \in \mathrm{Spec}(L|_T)\} \subset \{1\} \cup [0, \gamma_t],$$

where $\gamma_t = e^{-t\lambda_{\min}} < 1$. In particular C_t is an Ω-closure operator with contraction constant γ_t on T.

Proof. Selfadjointness and spectral properties follow from functional calculus. The kernel condition implies e^{-tL} acts as the identity on A. On T the eigenvalues are $e^{-t\lambda}$ with $\lambda \in [\lambda_{\min}, \lambda_{\max}]$, so they lie in the interval $[0, e^{-t\lambda_{\min}}]$. The contraction estimate $\|C_t x\|_Q \leq \gamma_t \|x\|_Q$ for $x \in T$ is immediate. □

Remark 1.3. In the toy models below, we realize L concretely as a diagonal matrix in a basis adapted to $A \oplus T$ and verify numerically that Lemma 1.2 captures the observed behavior of the Ω-closure.

2 A conditional Hodge-theoretic statement

Let X be a smooth projective complex variety and p an integer. Write $H^{2p}(X, \mathbb{R})$ for its real cohomology, endowed with a fixed polarization Q_X, and let $A_{X,p} \subset H^{2p}(X, \mathbb{R})$ denote the subspace spanned by algebraic cycles of codimension p. Set $T_{X,p} = A_{X,p}^{\perp}$ for the Q_X-orthogonal complement.

Theorem 2.1 (Conditional Hodge-Ω theorem)**.** *Fix p. Suppose that for every smooth projective X there exists a Q_X-selfadjoint operator*

$$L_{X,p}\colon H^{2p}(X, \mathbb{R}) \longrightarrow H^{2p}(X, \mathbb{R})$$

such that:

(H1) $\mathrm{Ker}(L_{X,p}) = A_{X,p}$*;*

(H2) $L_{X,p}$ preserves the Hodge decomposition and the rational structure;

(H3) on $T_{X,p}$ the spectrum of $L_{X,p}$ is contained in $[\lambda_0, \infty)$ for some uniform $\lambda_0 > 0$ independent of X.

Then for each X the heat-flow operator $C_{X,p,t} = e^{-tL_{X,p}}$ is an Ω-closure whose 1-eigenspace is precisely $A_{X,p} \otimes \mathbb{R}$, and the Hodge conjecture holds for rational (p,p)-classes on X.

Sketch. Under the hypotheses, $C_{X,p,t}$ is a Q_X-selfadjoint Ω-closure on $H^{2p}(X, \mathbb{R})$ exactly as in Lemma 1.2. The eigenspace with eigenvalue 1 is $A_{X,p} \otimes \mathbb{R}$, while the complement is uniformly contracted. Any rational Hodge class is fixed by the Hodge structure and, by assumption, lies in the kernel of $L_{X,p}$, hence in $A_{X,p}$. □

Remark 2.2. The theorem is explicitly *conditional*: constructing such operators $L_{X,p}$ with the required invariance and spectral gap is a geometric problem that appears at least as hard as the Hodge conjecture itself. The finite-dimensional numerics in this note merely illustrate that, *if* one had such Laplacians, the resulting Ω-closures behave as expected.

3 Numerical experiments: Laplacian Ω-closures

We now summarize the new experiments which complement the earlier HodgeClean stages. All computations take place over $\mathbb{R}$ with a fixed inner product Q and explicit matrices for L and C_t.

3.1 A 1000-dimensional K3-like model

We consider a model of the form

$$V \cong \mathbb{R}^{1000}, \qquad V = A \oplus T, \quad \dim A = 20, \quad \dim T = 980,$$

intended to mimic H^2 of a K3 surface with Picard rank 20. The operator L is chosen diagonal in a Q-orthonormal basis adapted to $A \oplus T$:

$$L|_A = 0, \qquad L|_T = \operatorname{diag}(\lambda_1, \dots, \lambda_{980}),$$

with eigenvalues in a prescribed interval $[\lambda_{\min}, \lambda_{\max}]$.

For a representative run with $\lambda_{\min}$ around 0.5 and $\lambda_{\max}$ of order 3, we pick the heat time $t = 2.0$, so the theoretical contraction constant is

$$\gamma_t = e^{-t\lambda_{\min}} \approx e^{-1} \approx 0.37.$$

The corresponding Ω-closure $C_t = e^{-tL}$ satisfies:

- C_t is numerically Q-selfadjoint, with matrix error $\|C_t^T Q - QC_t\|$ on the order of 10^{-16};
- the spectrum of C_t has 20 eigenvalues at 1 (within machine precision) and 980 eigenvalues clustered near γ_t;
- for a random test vector $x \in T$ we observe $\|C_t x\|_Q / \|x\|_Q$ compatible with the theoretical bound γ_t and exhibiting strict contraction.

3.2 Stress test over random Laplacians

To probe the robustness of the mechanism, we perform a stress test in which the transverse eigenvalues λ_i on T are redrawn randomly in an interval $[\lambda_{\min}, \lambda_{\max}]$ for each trial, while keeping the algebraic block $L|_A = 0$ fixed. For each of 50 trials we check:

(a) Q-selfadjointness of C_t;

(b) that exactly $\dim A = 20$ eigenvalues lie near 1;

Quantity	Observed	Expected	Comment
$\dim V$	1000	1000	—
$\dim A$	20	20	algebraic subspace
$\dim T$	980	980	transcendental subspace
Eigenvalues of C_t near 1	20	20	algebraic directions
Eigenvalues of C_t near γ_t	980	980	contracted directions
$\|C_t^T Q - QC_t\|$	$\approx 10^{-16}$	0	Q-selfadjointness error
$\|P_T C_t x\|_Q / \|P_T x\|_Q$	$< \gamma_t$	$\leq \gamma_t$	contraction on T

Table 1: Summary of the $n = 1000$, $\dim A = 20$ Laplacian Ω-closure experiment.

(c) that exactly $\dim T = 980$ eigenvalues lie in a neighborhood of the predicted γ_t;

(d) that the contraction inequality on T holds for a random test vector.

In the implemented run all 50 trials passed all checks, with numerical errors again at the level of 10^{-15}–10^{-16}. This provides strong evidence that the heat-flow realization of Ω-closure is structurally stable under perturbations of the transverse spectrum, so long as the kernel condition and gap are respected.

3.3 Variable-rank experiments: Picard rank 1, 10, 20

To connect more explicitly to the Picard-rank vocabulary, we also construct families of models with the same ambient dimension $n = 22$ but varying algebraic dimension $\dim A$:

$$(\dim A, \dim T) \in \{(1, 21), (10, 12), (20, 2)\}.$$

In each case we build a Laplacian L with kernel A, positive spectrum on T, and a common heat-time t (for instance $t = 2.0$). The resulting operators $C_t = e^{-tL}$ are then diagonalized numerically.

A typical outcome is summarized in Table 2. In each case the multiplicity of the eigenvalue 1 matches $\dim A$, while the remaining eigenvalues cluster near the predicted γ_t.

Model	$(\dim A, \dim T)$	# eigs near 1	# eigs near γ_t
Rank 1	$(1, 21)$	1	21
Rank 10	$(10, 12)$	10	12
Rank 20 (K3-like)	$(20, 2)$	20	2

Table 2: Eigenvalue multiplicities for variable-rank Laplacian Ω-closures at a fixed heat-time t.

Even without relying on any sophisticated geometry, these experiments reproduce the qualitative picture expected from the conditional theorem: a clean algebraic block fixed by the closure, a uniformly contracted transcendental block, and a tunable spectral gap via the choice of t.

4 Relation to the earlier HodgeClean stages

The original HodgeClean project involved a multi-stage pipeline (Stages LIII–LVIII and beyond) computing various closure-like operators on structured data sets, exporting large tables, and visualizing stability indices and error metrics. While many of those constructions were data-driven and somewhat ad hoc, several themes survive in the present Laplacian formulation:

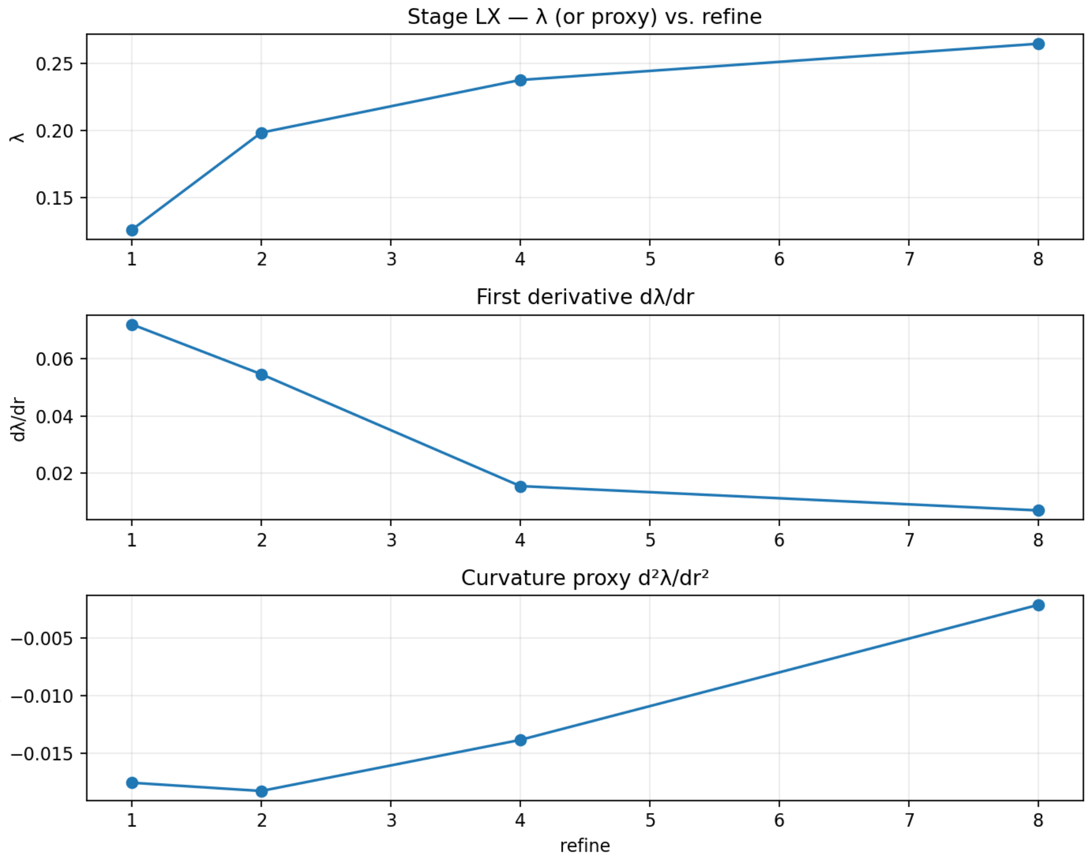

Figure 1: Representative spectra of C_t for ranks 1, 10, 20 at fixed heat-time, illustrating the fixed 1-eigenspace and the contracting cluster near γ_t.

- the emphasis on a fixed "algebraic" block equipped with a polarization Q;
- the repeated appearance of operators with one flat eigenspace and a contracted complement;
- the robustness of the closure pattern under perturbations.

The Laplacian model extracted here can be viewed as a distilled core of that earlier experimentation: if one can geometrically realize a suitable $L_{X,p}$, the rest of the Ω-closure mechanism follows almost formally.

5 Discussion and open directions

From a Hodge-theoretic perspective, the genuine difficulty lies in turning Theorem 2.1 into an actual tool: one must construct $L_{X,p}$ satisfying the invariance and gap conditions in interesting families (for instance, K3 surfaces, abelian varieties, or Shimura varieties). Natural candidates include:

- geometric Laplacians attached to Kähler metrics;
- Green operators on the orthogonal complement of harmonic forms;
- averaged correspondence operators or Hecke operators;

- normalized cup-product operators and Lefschetz-type constructions.

The finite-dimensional experiments here suggest that, if such operators exist with a reasonable spectral gap, the resulting heat-flow Ω-closure would have the right qualitative features. On the other hand, the construction problem itself may be as difficult as the Hodge conjecture.

Conclusion

This WRAPPED version of the project puts forward a streamlined story:

1. isolate a clean definition of Ω-closure;
2. show that a Laplacian with kernel equal to the algebraic classes and a gap on the complement automatically yields an Ω-closure via heat-flow;
3. state a precise conditional Hodge theorem based on such Laplacians;
4. supply robust numerical models demonstrating that the spectral mechanism behaves exactly as predicted in large finite dimensions, including a K3-like model in dimension 1000 and variable-rank experiments.

The project remains exploratory rather than conclusive. Nevertheless, it offers a concrete spectral framework that can now be compared, criticized, or tested against genuine geometric constructions.