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IF120 Discrete Mathematics

10 Graph 1

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REVIEW

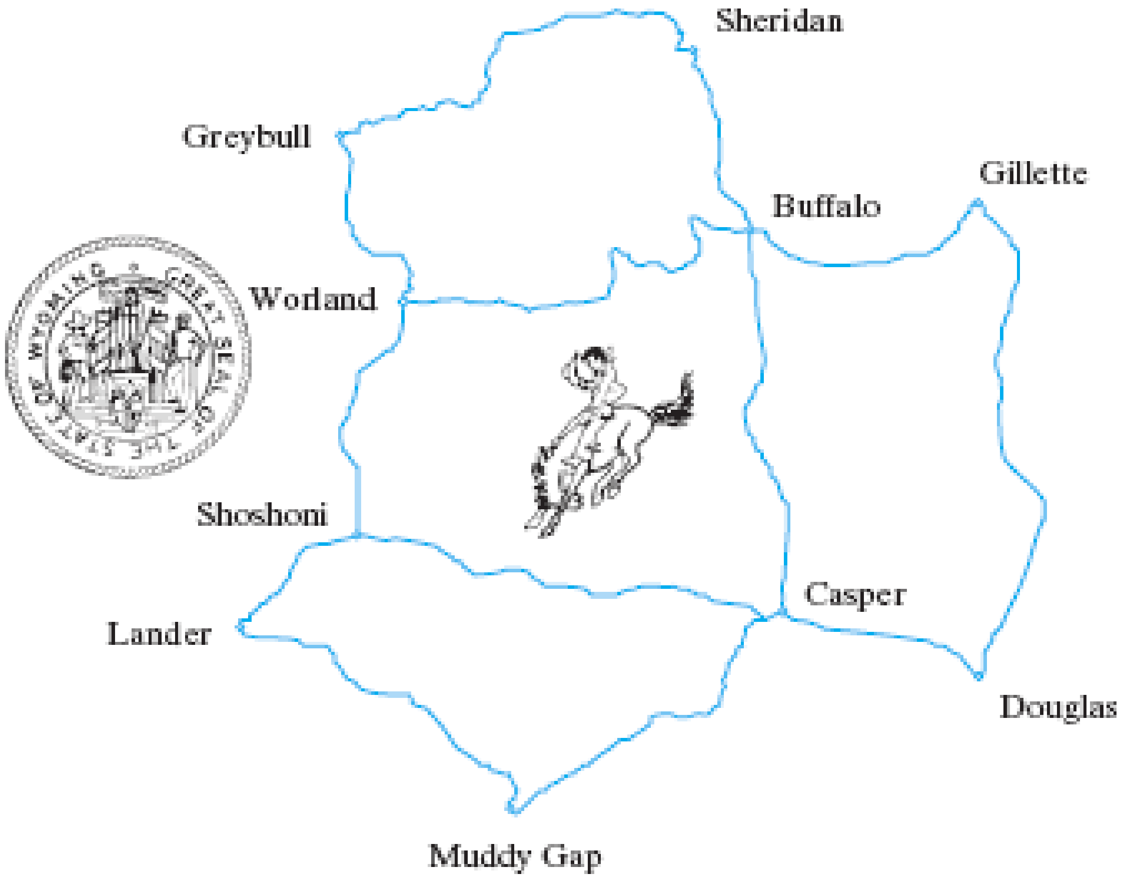
- Recursive Algorithm
- Recurrence Relations
- Solving Recurrence Relations

OUTLINE

- Graph Terminologies
- Path and Cycle
- Euler Cycle and Hamiltonian Cycle

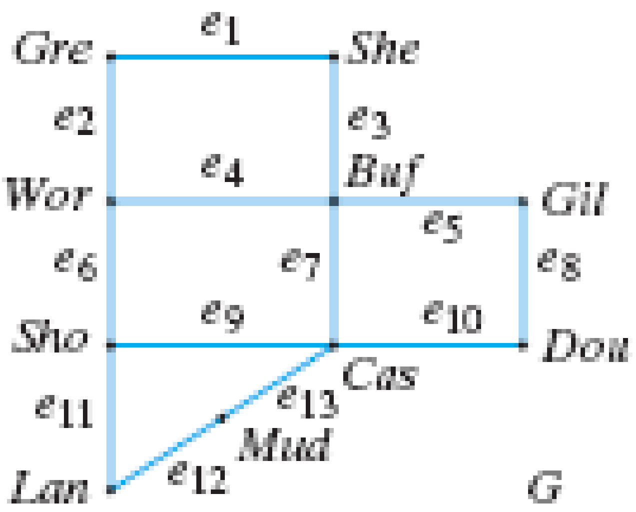
Introduction

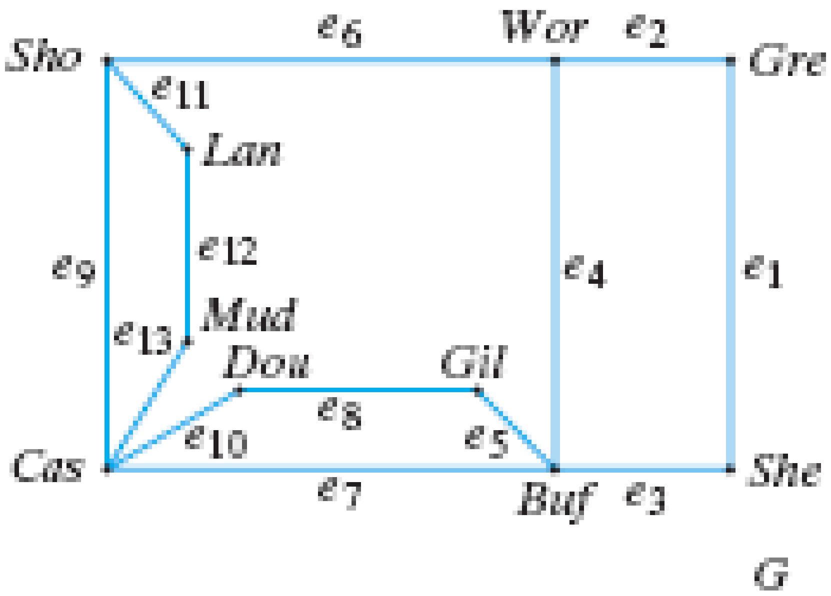
- See the figure on the right. The figure shows the highway system in Wyoming that a particular person is responsible for inspecting. Specifically, this road inspector must travel all of these roads and file reports on road conditions, visibility of lines on the roads, status of traffic signs, and so on. Since the road inspector lives in **Greybull**, the most economical way to inspect all of the roads would be to start in Greybull, travel each of the roads exactly once, and return to Greybull. Is this possible?



Introduction

- The problem can be modeled as a **graph.**
- In fact, since graphs are drawn with dots and lines, they look like road maps.
- In Figures below, we have drawn a graph G that models the map of the figure before (together with its alternative).
- The dots are called **vertices** and the lines that connect the vertices are called **edges.**



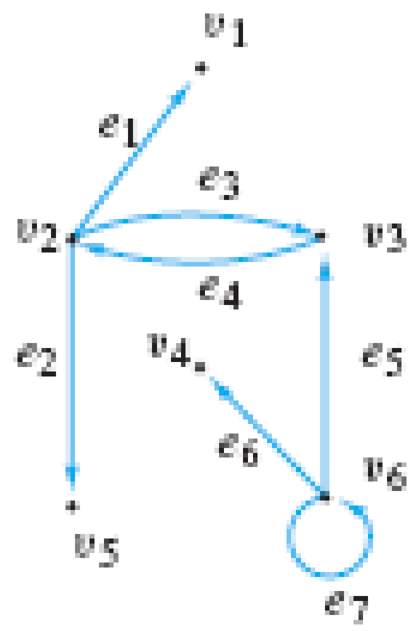


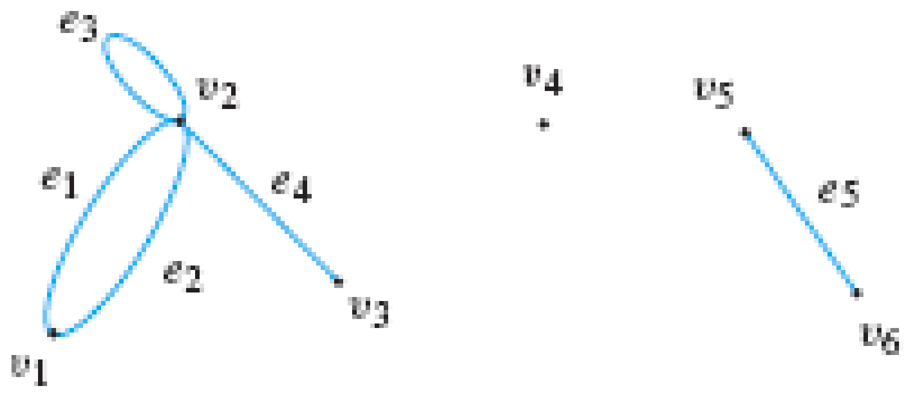
Graph Terminologies

- A **graph** (or *undirected graph*) *G* consists of a set *V* of **vertices** (or *nodes*) and a set *E* of **edges** (or *arcs*) such that each edge $e \in E$ is associated with an unordered pair of vertices. If there is a unique edge e associated with the vertices *v* and *w*, we write $e = (v, w)$ or $e = (w, v)$. In this context, (v, w) denotes an edge between *v* and *w* in an undirected graph and *not* an ordered pair.
- A **directed graph** (or *digraph*) *G* consists of a set *V* of *vertices* (or *nodes*) and a set *E* of *edges* (or *arcs*) such that each edge $e \in E$ is associated with an ordered pair of vertices. If there is a unique edge e associated with the ordered pair (v, w) of vertices, we write $e = (v, w)$, which denotes an edge from *v* to *w*.
- An edge e in a graph (undirected or directed) that is associated with the pair of vertices *v* and *w* is said to be **incident** *on v* and *w*, and *v* and *w* are said to be **incident** *on* e and to be **adjacent vertices**.
- If *G* is a graph (undirected or directed) with vertices *V* and edges *E*, we write $G = (V, E)$.
- Unless specified otherwise, the sets *E* and *V* are assumed to be finite and *V* is assumed to be nonempty.

Graph Terminologies

❑A directed graph is shown on the right figure. The directed edges are indicated by arrows. Edge e_1 is associated with the ordered pair (v_2, v_1) of vertices, and edge e_7 is associated with the ordered pair (v_6, v_6) of vertices. Edge e_1 is denoted (v_2, v_1), and edge e_7 is denoted (v_6, v_6).



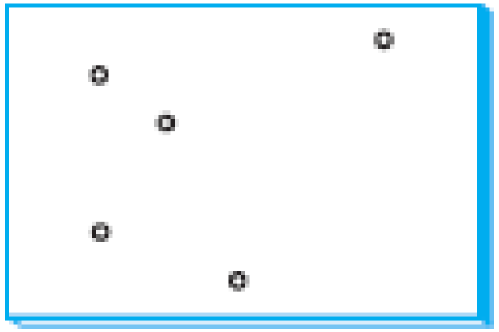


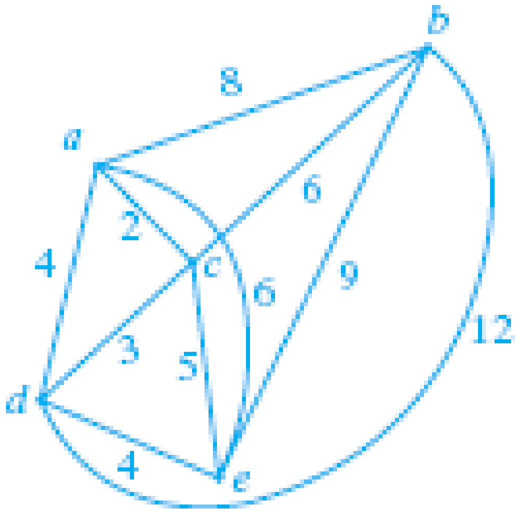
- In Figure on the left, edges e1 and e2 are both associated with the vertex pair {*v*1, *v*2}. Such edges are called **parallel edges.**
- An edge incident on a single vertex is called a **loop.**
- A vertex, such as vertex *v*4 in Figure 8.1.5, that is not incident on any edge is called an **isolated vertex.**
- A graph with neither loops nor parallel edges is called a **simple graph.**

Graph Terminologies

- Frequently in manufacturing, it is necessary to bore many holes in sheets of metal (see left Figure below). Components can then be bolted to these sheets of metal. The holes can be drilled using a drill press under the control of a computer. To save time and money, the drill press should be moved as quickly as possible. We model the situation as a graph.

  The vertices of the graph correspond to the holes (see right Figure below). Every pair of vertices is connected by an edge. We write on each edge the time to move the drill press between the corresponding holes.

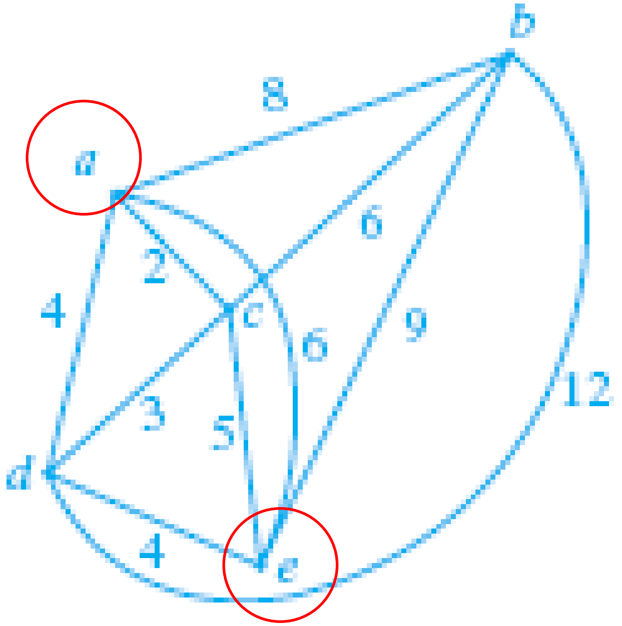




- A graph with numbers on the edges is called a **weighted graph.**
- If edge e is labeled *k*, we say that the **weight of edge** e is *k*.
- In a **weighted** graph, the **length of a path** is the **sum of the weights** of the edges in the path.

Graph Terminologies

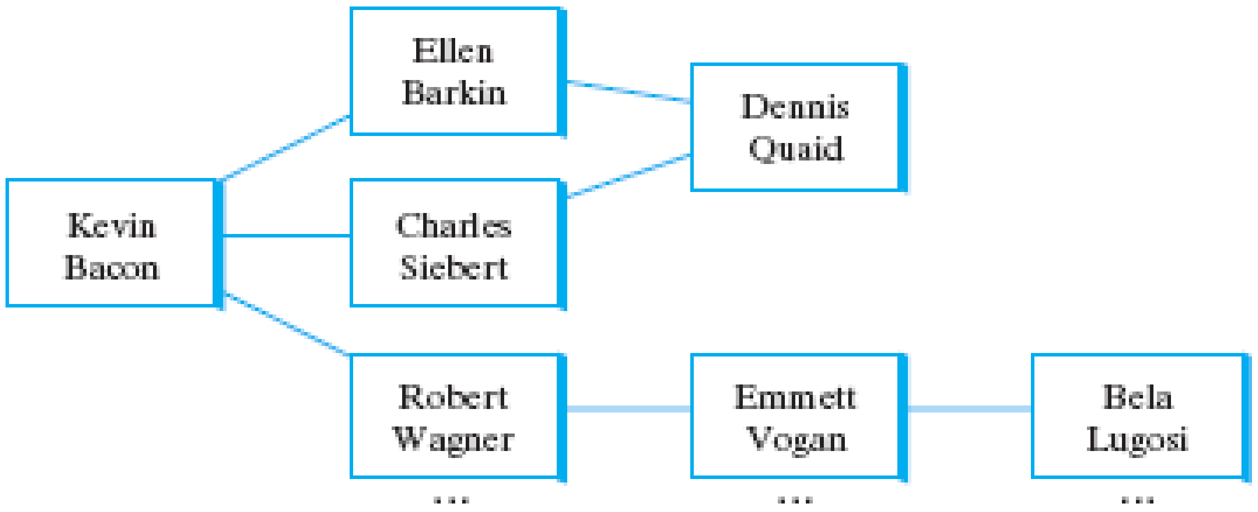
Suppose that in this problem the path is required to begin at vertex *a* and end at vertex e. We can find the minimum-length path by listing all possible paths from *a* to e that pass through every vertex exactly one time and choose the shortest one (see Table below). We see that the path that visits the vertices *a*, *b*, *c*, *d*, e, in this order, has minimum length. Of course, a different pair of starting and ending vertices might produce an even shorter path.



Path	*Length*
a, b, c, d, e	21
a, b, d, c, e	28
a, c, b, d, e	24
a, c, d, b, e	26
a, d, b, c, e	27
a, d, c, b, e	22

- Actor Kevin Bacon has appeared in numerous films including *Diner* and *Apollo 13*. Actors who have appeared in a film with Bacon are said to have *Bacon number one*. For example, Ellen Barkin has Bacon number one because she appeared with Bacon in *Diner*. Actors who did not appear in a film with Bacon but who appeared in a film with an actor whose Bacon number is one are said to have *Bacon number two*. Higher Bacon numbers are defined similarly. For example, Bela Lugosi has Bacon number three. Lugosi was in *Black Friday* with Emmett Vogan, Vogan was in *With a Song in My Heart* with Robert Wagner, and Wagner was in *Wild Things* with Bacon. We next develop a graph model for Bacon numbers.

We let vertices denote actors, and we place one edge between two distinct actors if they appeared in at least one film together.



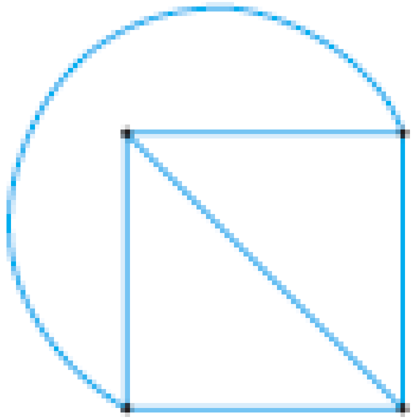
In an **unweighted graph**, the **length of a path** is the **number of edges** in the path. Thus an actor's Bacon number is the length of a shortest path from the vertex corresponding to that actor to the vertex corresponding to Bacon.

It is interesting that *most* actors, even actors who died many years ago, have Bacon numbers of three or less.

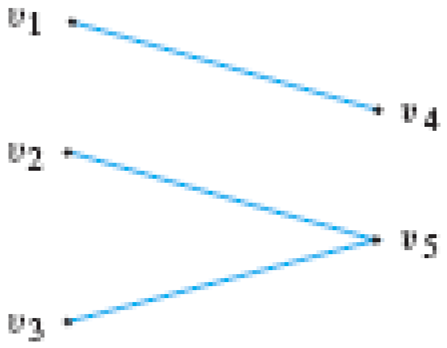
Complete and Bipartite Graph

- Definition 10.1: The **complete graph on n vertices**, denoted K_n, is the simple graph with *n* vertices in which there is an edge between every pair of distinct vertices.

❑ The complete graph on four vertices, K_4, is shown in Figure on the right.

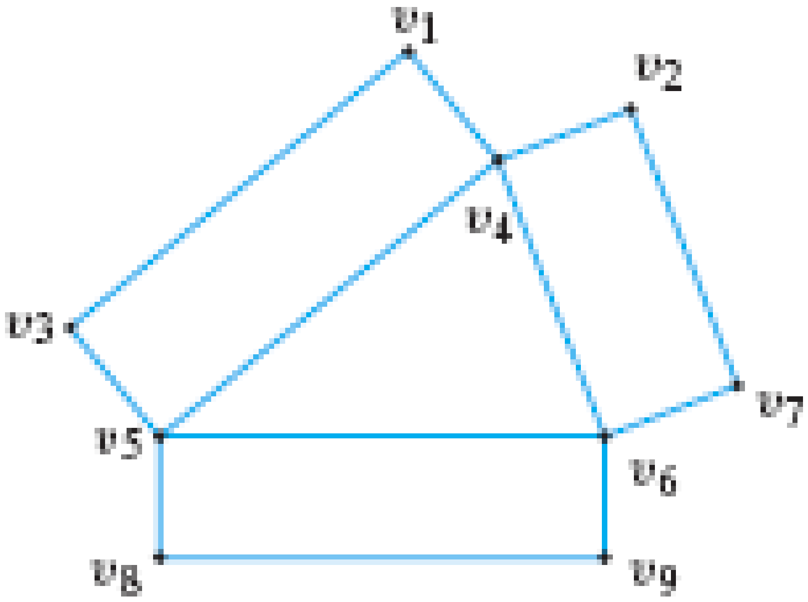


- Definition 10.2: A graph $G = (V, E)$ is **bipartite** if there exist subsets V_1 and V_2 (either possibly empty) of *V* such that $V_1 \cap V_2 = \emptyset$, $V_1 \cup V_2 = V$, and each edge in *E* is incident on one vertex in V_1 and one vertex in V_2.



❑ The graph in the left Figure is bipartite since if we let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5\}$, each edge is incident on one vertex in V_1 and one vertex in V_2.

- The graph in Figure below is *not* bipartite. It is often easiest to prove that a graph is not bipartite by arguing by contradiction.



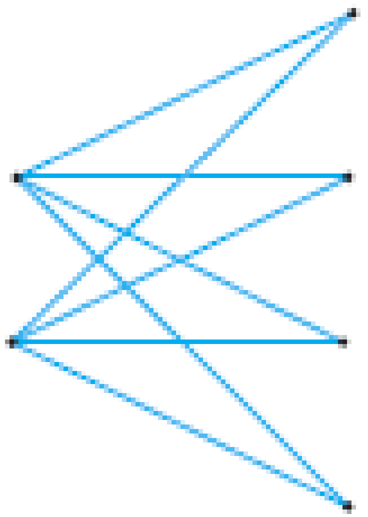
Suppose that the graph is bipartite. Then the vertex set can be partitioned into two subsets V_1 and V_2 such that each edge is incident on one vertex in V_1 and one vertex in V_2. Now consider the vertices v_4, v_5, and v_6. Since v_4 and v_5 are adjacent, one is in V_1 and the other in V_2. We may assume that v_4 is in V_1 and that v_5 is in V_2. Since v_5 and v_6 are adjacent and v_5 is in V_2, v_6 is in V_1. Since v_4 and v_6 are adjacent and v_4 is in V_1, v_6 is in V_2. But now v_6 is in both V_1 and V_2, which is a contradiction since V_1 and V_2 are disjoint. Therefore, the graph is not bipartite.

- The complete graph K_1 on one vertex is bipartite. We may let V_1 be the set containing the one vertex and V_2 be the empty set. Then each edge (namely none!) is incident on one vertex in V_1 and one vertex in V_2.

Complete Bipartite Graph

- Definition 10.3: The **complete bipartite graph on m and n vertices**, denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into sets V_1 with *m* vertices and V_2 with *n* vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.

❑ The complete bipartite graph on two and four vertices, $K_{2,4}$, is shown in Figure below.



Path and Cycle

- Definition 10.4: Let v_0 and v_n be vertices in a graph. A **path** from v_0 to v_n of length *n* is an alternating sequence of *n*+1 vertices and *n* edges beginning with vertex v_0 and ending with vertex v_n,

$$(v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n)$$

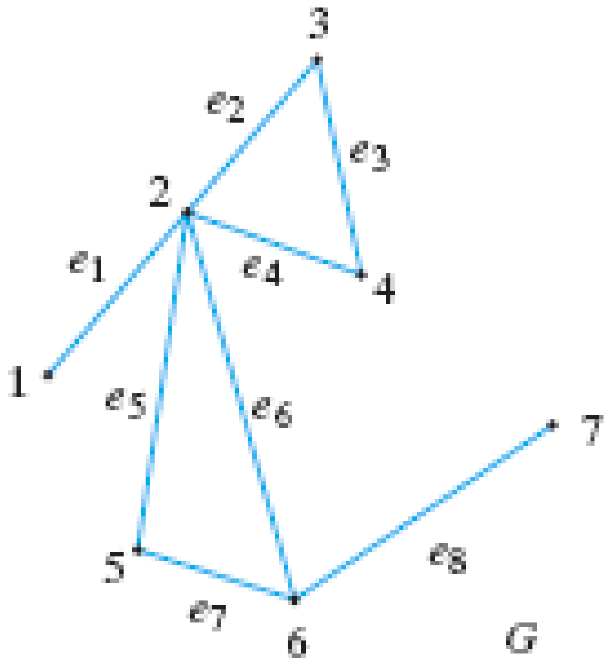
in which edge e_i is incident on vertices v_{i-1} and v_i for $i = 1,2, \dots, n$.

- In the graph of Figure on the right,

$$(1, e_1, 2, e_2, 3, e_3, 4, e_4, 2)$$

is a path of length 4 from vertex 1 to vertex 2.

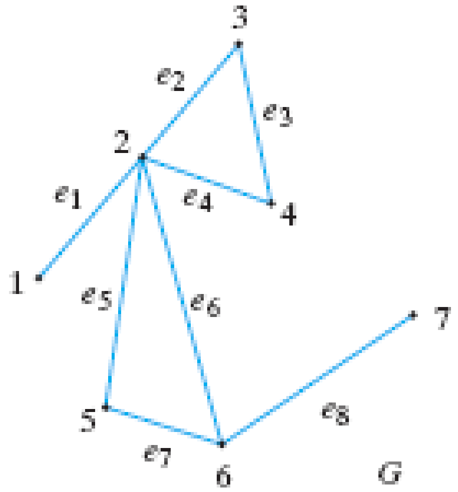
- In the graph of the same figure, the path (6) consisting solely of vertex 6 is a path of length 0 from vertex 6 to vertex 6.

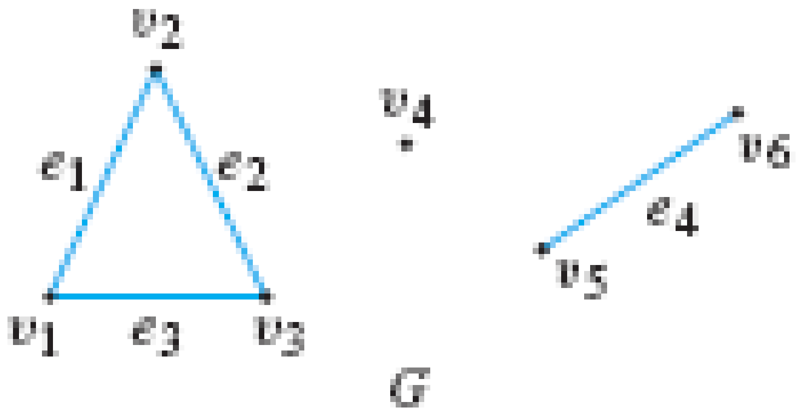


Path and Cycle

- Definition 10.5: A graph G is **connected** if given any vertices v and w in G, there is a path from v to w.

❑The graph G of Figure below (left) is connected since, given any vertices v and w in G, there is a path from v to w.





❑The graph G of Figure above (right) is not connected since, for example, there is no path from vertex v_2 to vertex v_5.

Subgraph

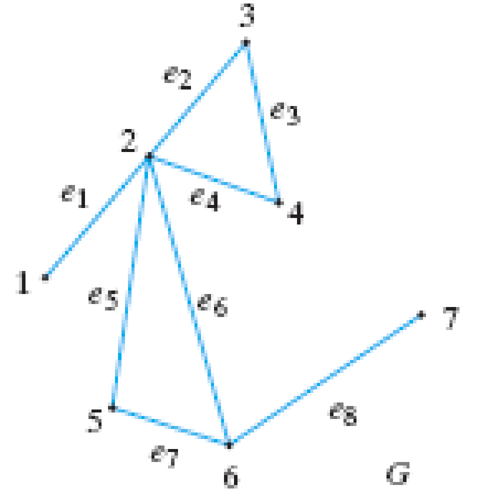
- Definition 10.6: Let $G = (V, E)$ be a graph. We call (V', E') a **subgraph** of G if

a) $V' \subseteq V$ and $E' \subseteq E$.

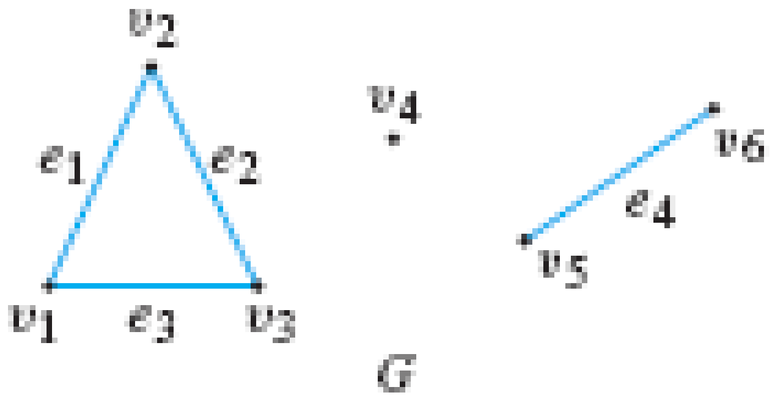
b) For every edge $e' \in E'$, if e' is incident on v' and w', then $v', w' \in V'$.

- The graph $G' = (V', E')$ of Figure below (left) is a subgraph of the graph $G = (V, E)$ of Figure below (right) since $V' \subseteq V$ and $E' \subseteq E$.

- Definition 10.7: Let G be a graph and let v be a vertex in G. The subgraph G' of G consisting of all edges and vertices in G that are contained in some path beginning at v is called the **component** of G containing v.



- The graph G of Figure on the right has one component, namely itself. Indeed, a graph is connected if and only if it has exactly one component.



- Let G be the graph of Figure on the left. The component of G containing v_3 is the subgraph

$$G_1 = (V_1, E_1), V_1 = \{v_1, v_2, v_3\}, E_1 = \{e_1, e_2, e_3\}.$$

The component of G containing v_4 is the subgraph

$$G_2 = (V_2, E_2), V_2 = \{v_4\}, E_2 = \emptyset.$$

The component of G containing v_5 is the subgraph

$$G_3 = (V_3, E_3), V_3 = \{v_5, v_6\}, E_3 = \{e_4\}.$$

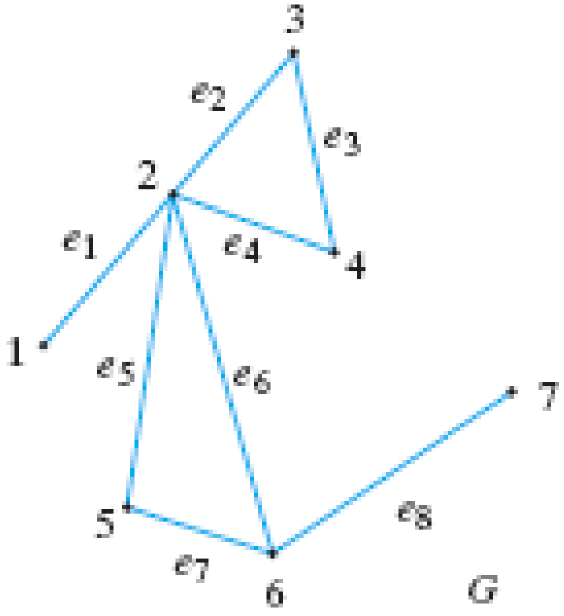
Simple Path and Cycle

- Definition 10.8: Let v and w be vertices in a graph G.

  A **simple path** from v to w is a path from v to w with no repeated vertices.

  A **cycle** (or **circuit**) is a path of nonzero length from v to v with no repeated edges.

  A **simple cycle** is a cycle from v to v in which, except for the beginning and ending vertices that are both equal to v, there are no repeated vertices.



❑ For the graph on the left, we have the following information.

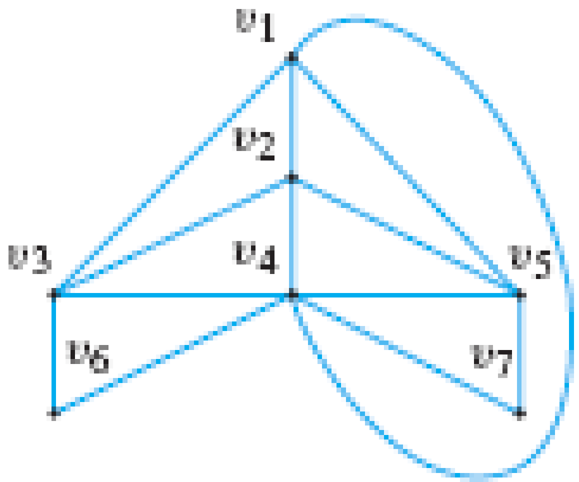
Path	*Simple Path?*	*Cycle?*	*Simple Cycle?*
(6, 5, 2, 4, 3, 2, 1)	No	No	No
(6, 5, 2, 4)	Yes	No	No
(2, 6, 5, 2, 4, 3, 2)	No	Yes	No
(5, 6, 2, 5)	No	Yes	Yes
(7)	Yes	No	No

Euler Cycle

- The solution to the existence of Euler cycles is nicely stated by introducing the degree of a vertex. The **degree of a vertex** *v*, $\delta(v)$, is the number of edges incident on *v*.
- Theorem 10.1: *If a graph G has an* **Euler cycle**, *then G is* **connected** *and every vertex has* **even degree**.
- Theorem 10.2: *If G is a connected graph and every vertex has even degree, then G has an Euler cycle.*

❑Let *G* be the graph of Figure on the right. Use Theorem 10.2 to verify that *G* has an Euler cycle. Find an Euler cycle for *G*.

We observe that *G* is connected and that
$\delta(v_1) = \delta(v_2) = \delta(v_3) = \delta(v_5) = 4, \delta(v_4) = 6,$
$\delta(v_6) = \delta(v_7) = 2.$
Since the degree of every vertex is even, by Theorem 10.2, *G* has an Euler cycle. By inspection, we find the Euler cycle
$(v_6, v_4, v_7, v_5, v_1, v_3, v_4, v_1, v_2, v_5, v_4, v_2, v_3, v_6).$



Euler Cycle

- Theorem 10.3: *If G is a graph with m edges and vertices* $\{v_1, v_2, \dots, v_n\}$, *then*

$$\sum_{i=1}^{n} \delta(v_i) = 2m.$$

  In particular, the sum of the degrees of all the vertices in a graph is even.

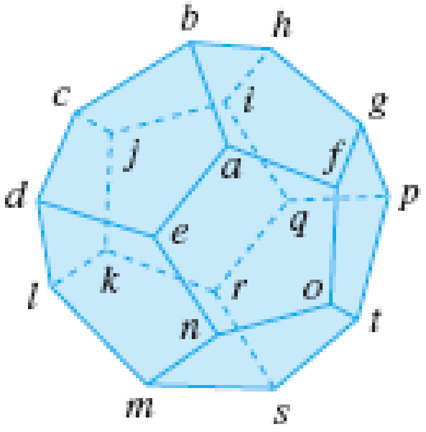
- Corollary 10.1: *In any graph, the number of vertices of odd degree is even.*

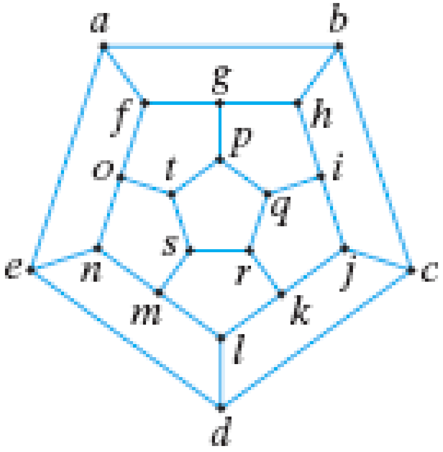
- Theorem 10.4: *A graph has a path with no repeated edges from v to w* $(v \neq w)$ *containing all the edges and vertices if and only if it is connected and v and w are the only vertices having odd degree.*

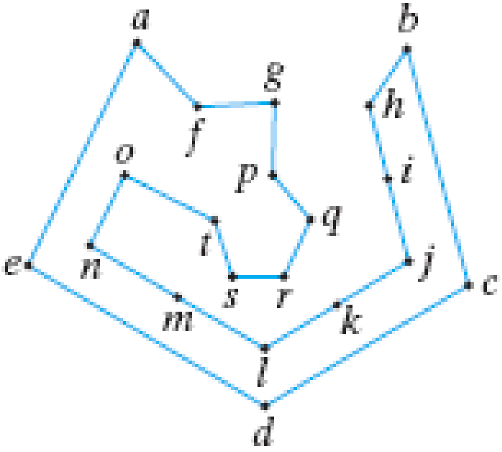
- Theorem 10.5: *If a graph G contains a cycle from v to v, G contains a simple cycle from v to v.*

Hamiltonian Cycle

- Sir William Rowan Hamilton marketed a puzzle in the mid-1800s in the form of a dodecahedron.
- Each corner bore the name of a city and the problem was to start at any city, travel along the edges, visit each city exactly one time, and return to the initial city.
- We can solve Hamilton's puzzle if we can find a cycle in the graph that contains each vertex exactly once (except for the starting and ending vertex that appears twice).



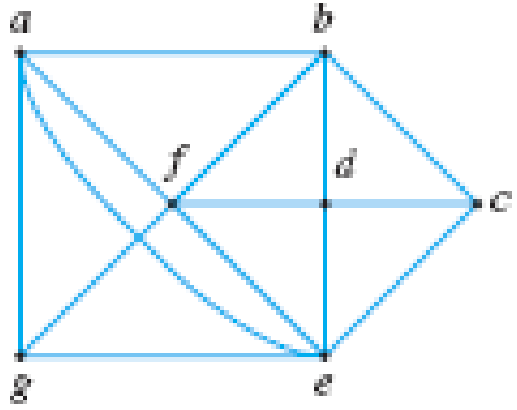




- In honor of Hamilton, we call a cycle in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice, a **Hamiltonian cycle.**

Hamiltonian Cycle

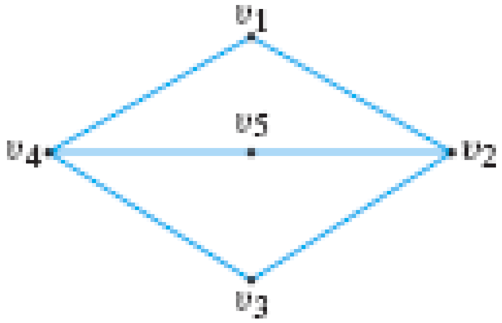
- The cycle (a, b, c, d, e, f, g, a) is a Hamiltonian cycle for the graph of Figure below.

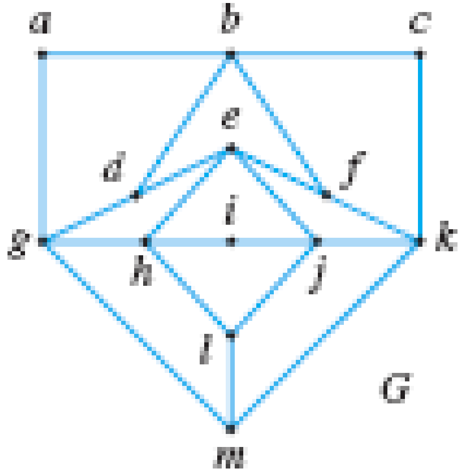


- The problem of finding a Hamiltonian cycle in a graph sounds similar to the problem of finding an Euler cycle in a graph.
- An Euler cycle visits each edge once, whereas a Hamiltonian cycle visits each vertex once; however, the problems are actually quite distinct.
- Moreover, unlike the situation for Euler cycles, no easily verified necessary and sufficient conditions are known for the existence of a Hamiltonian cycle in a graph.

- Show that the graph of Figure below does not contain a Hamiltonian cycle.

  Since there are five vertices, a Hamiltonian cycle must have five edges. Suppose that we could eliminate edges from the graph, leaving just a Hamiltonian cycle. We would have to eliminate one edge incident at v_2 and one edge incident at v_4, since each vertex in a Hamiltonian cycle has degree 2. But this leaves only four edges—not enough for a Hamiltonian cycle of length 5. Therefore, the graph does not contain a Hamiltonian cycle.



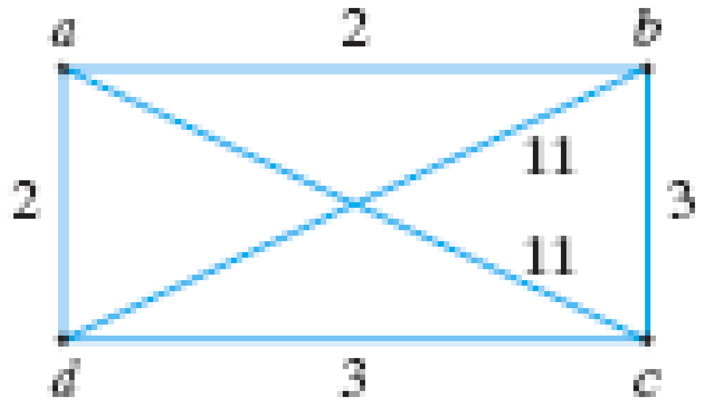


- Show that the graph G of Figure at the bottom left does not contain a Hamiltonian cycle.

  Suppose that G has a Hamiltonian cycle H. The edges (a, b), (a, g), (b, c), and (c, k) must be in H since each vertex in a Hamiltonian cycle has degree 2. Thus edges (b, d) and (b, f) are not in H. Therefore, edges (g, d), (d, e), (e, f), and (f, k) are in H. The edges now known to be in H form a cycle C. Adding an additional edge to C will give some vertex in H degree greater than 2. This contradiction shows that G does not have a Hamiltonian cycle.

Traveling Salesperson Problem

- The **traveling salesperson problem** is related to the problem of finding a Hamiltonian cycle in a graph.
- The problem is: Given a weighted graph *G*, find a minimum-length Hamiltonian cycle in *G*.
- If we think of the vertices in a weighted graph as cities and the edge weights as distances, the traveling salesperson problem is to find a shortest route in which the salesperson can visit each city one time, starting and ending at the same city.

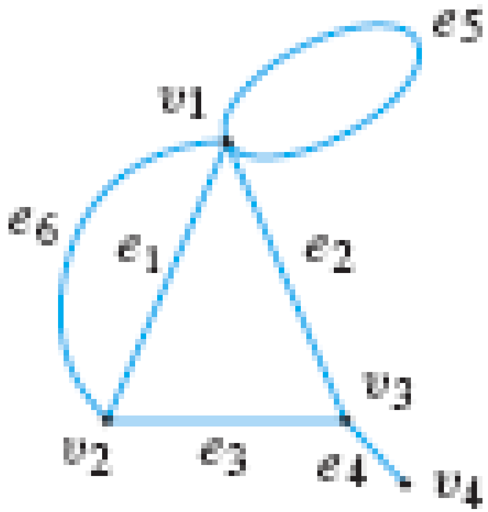


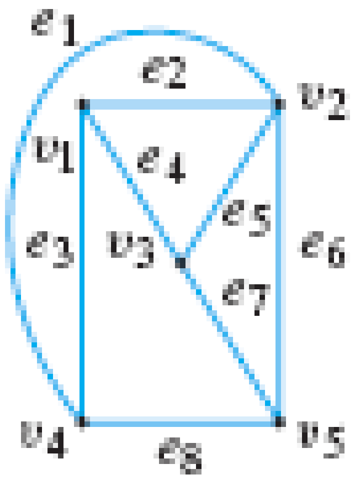
- The cycle $C = (a, b, c, d, a)$ is a Hamiltonian cycle for the graph *G* of the left figure. Replacing any of the edges in *C* by either of the edges labeled 11 would increase the length of *C*; thus *C* is a minimum-length Hamiltonian cycle for *G*. Thus *C* solves the traveling salesperson problem for *G*.

PRACTICE

PRACTICE 1

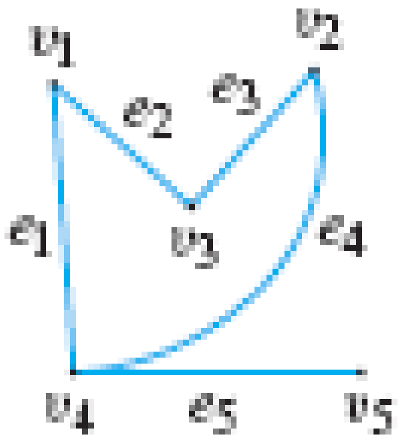
- *For each graph $G = (V, E)$ below, find V, E, all parallel edges, all loops, all isolated vertices, and tell whether G is a simple graph. Also, tell on which vertices edge e_1 is incident.*

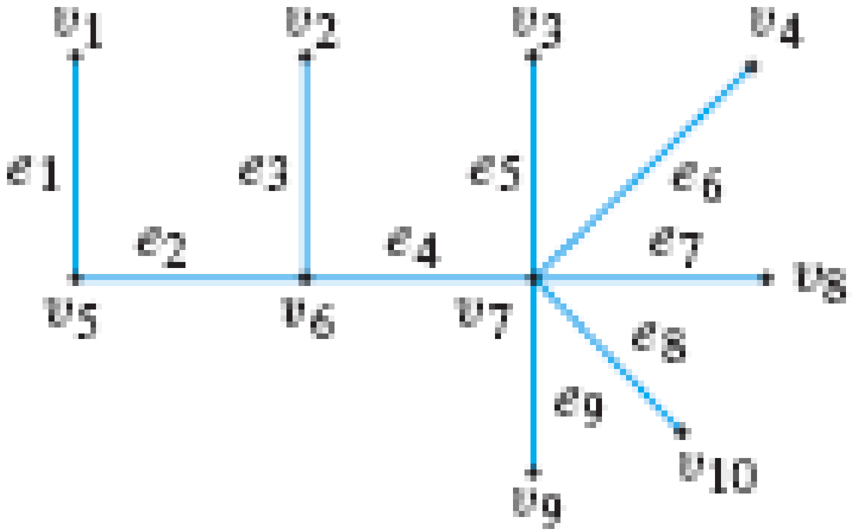




PRACTICE 2

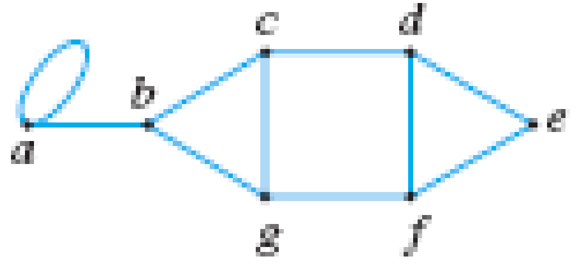
- *State which graphs below are bipartite graphs. If the graph is bipartite, specify the disjoint vertex sets.*



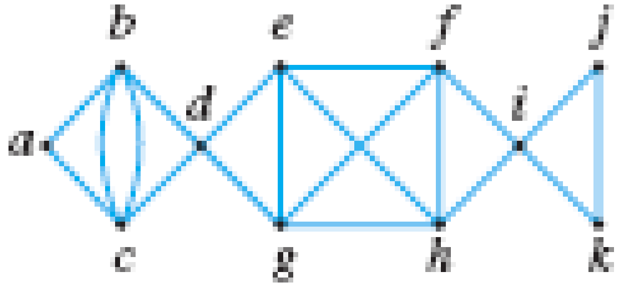


PRACTICE 3

- *Find all the simple cycles and simple paths (from a to e) in the following graph.*

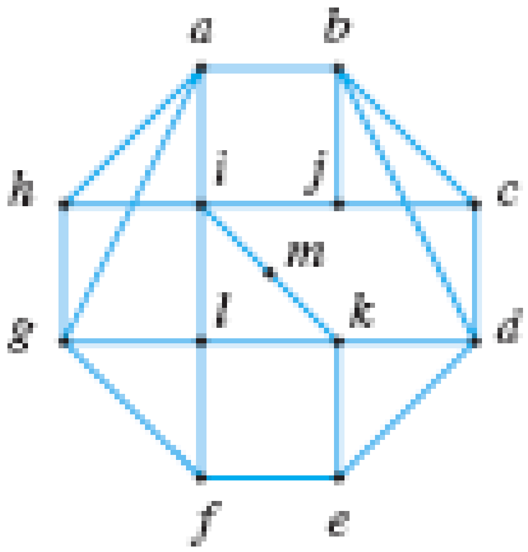


- *Decide whether the graph below has an Euler cycle. If the graph has an Euler cycle, exhibit one.*



PRACTICE 4

- *Determine whether or not the graph below contains a Hamiltonian cycle. If there is a Hamiltonian cycle, exhibit it; otherwise, give an argument that shows there is no Hamiltonian cycle.*



NEXT WEEK'S OUTLINE

- Graph Representation
- Isomorphism
- Planar Graph
- Shortest Path Problem

REFERENCES

- Johnsonbaugh, R., 2005, *Discrete Mathematics*, New Jersey: Pearson Education, Inc.
- Rosen, Kenneth H., 2005, *Discrete Mathematics and Its Applications*, 6^{th} edition, McGraw-Hill.
- Hansun, S., 2021, *Matematika Diskret Teknik*, Deepublish.
- Lipschutz, Seymour, Lipson, Marc Lars, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, McGraw-Hill.
- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

INFORMATIKA
UMN
UNIVERSITAS MULTIMEDIA NUSANTARA

Misi

1. Menyelenggarakan pembelajaran dengan teknologi dan kurikulum terbaik serta didukung tenaga pengajar profesional.
2. Melaksanakan kegiatan penelitian di bidang Informatika untuk memajukan ilmu dan teknologi Informatika.
3. Melaksanakan kegiatan pengabdian kepada masyarakat berbasis ilmu dan teknologi Informatika dalam rangka mengamalkan ilmu dan teknologi Informatika.