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IF120
Discrete Mathematics

08 Discrete Probability

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OUTLINE

- Discrete Probability
- Binomial Coefficient
- Combinatorial Identity

Discrete Probability

- **Probability** was developed in the seventeenth century to analyze games and, in this earliest form, directly involved counting.
- An **experiment** is a process that yields an outcome.
- An **event** is an outcome or combination of outcomes from an experiment.
- The **sample space** is the event consisting of all possible outcomes.
- If all outcomes in a finite sample space are equally likely, the probability of an event is defined as the number of outcomes in the event divided by the number of outcomes in the sample space.

- Definition 8.1: The **probability** $P(E)$ of an event E from the finite sample space S is

$$P(E) = \frac{|E|}{|S|}.$$

Discrete Probability

- Two fair dice are rolled. What is the probability that the sum of the numbers on the dice is 10?

    Since the first die can show any one of six numbers and the second die can show any one of six numbers, by the Multiplication Principle there are $6 \cdot 6 = 36$ possible sums; that is, the size of the sample space is 36. There are three possible ways to obtain the sum of 10—(4, 6), (5, 5), (6, 4)—that is, the size of the event "obtaining a sum of 10" is 3. [The notation (x, y) means that we obtain x on the first die and y on the second die.] Therefore, the probability is $3/36 = 1/12$.

Discrete Probability

- Five microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. Find the probability of obtaining no defective microprocessors.

  There are $C(1000, 5)$ ways to select 5 microprocessors among 1000. There are $C(980, 5)$ ways to select 5 good microprocessors since there are $1000-20 = 980$ good microprocessors. Therefore, the probability of obtaining no defective microprocessors is

$$\frac{C(980,5)}{C(1000,5)} = \frac{980 \cdot 979 \cdot 978 \cdot 977 \cdot 976}{1000 \cdot 999 \cdot 998 \cdot 997 \cdot 996} = 0.903735781.$$

- In the Illinois state lottery Lotto game, to win the grand prize the contestant must match six distinct numbers, in any order, among the numbers 1 through 52 randomly drawn by a lottery representative. What is the probability of choosing the winning numbers?

  Six numbers among 52 can be selected in $C(52, 6)$ ways. Since there is one winning combination, the probability of choosing the winning numbers is

$$\frac{1}{C(52,6)} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47} = 0.000000049.$$

Discrete Probability

- A bridge hand consists of 13 cards from an ordinary 52-card deck. Find the probability of obtaining a 4–4–4–1 distribution, that is, four cards in each of three different suits and one card of a fourth suit.

  There are $C(52, 13)$ bridge hands. The one-card suit can be chosen in 4 ways, and the card itself can be chosen in 13 ways. Having chosen this card, we must choose four cards from each of the three remaining suits, which can be done in $C(13,4)^3$ ways. Thus there are $4 \cdot 13 \cdot C(13,4)^3$ hands with a 4–4–4–1 distribution. Therefore, the probability of obtaining a 4–4–4–1 distribution is

$$\frac{4 \cdot 13 \cdot C(13,4)^3}{C(52,13)} = 0.03.$$

Discrete Probability Theory

- Definition 8.2: A **probability function** *P* assigns to each outcome *x* in a sample space *S* a number *P*(*x*) so that
$$0 \leq P(x) \leq 1, for\ all\ x \in S,$$
  and
$$\sum_{x \epsilon S} P(x) = 1.$$
- Definition 8.3: Let *E* be an event. The **probability of E**, *P*(*E*), is
$$P(E) = \sum_{x \epsilon E} P(x)$$
- Theorem 8.1: *Let E be an event. The probability of $\bar{E}$, the complement of E, satisfies*
$$P(E) + P(\bar{E}) = 1.$$

Discrete Probability Theory

- Suppose that a die is loaded so that the numbers 2 through 6 are equally likely to appear, but that 1 is three times as likely as any other number to appear. To model this situation, we should have

$$P(2) = P(3) = P(4) = P(5) = P(6)$$

and

$$P(1) = 3P(2).$$

Since

$$\begin{aligned} 1 &= P(1) + P(2) + P(3) + P(4) + P(5) + P(6) \\ &= 3P(2) + P(2) + P(2) + P(2) + P(2) + P(2) = 8P(2), \end{aligned}$$

we must have $P(2) = 1/8$. Therefore,

$$P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{8}$$

and

$$P(1) = 3P(2) = \frac{3}{8}.$$

Discrete Probability Theory

- Given the assumptions of the previous Example, the probability of an odd number is

$$P(1) + P(3) + P(5) = \frac{3}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}.$$

  Of course, for a fair die (with equally likely probabilities), the probability of an odd number is 1/2.

- Five microprocessors are randomly selected from a lot of 1000 microprocessors among which 20 are defective. In some Example before, we found that the probability of obtaining no defective microprocessors is 0.903735781. By Theorem 8.1, the probability of obtaining at least one defective microprocessor is

$$1 - 0.903735781 = 0.096264219.$$

Discrete Probability Theory

- Theorem 8.2: *Let E_1 and E_2 be events. Then*
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$

❑ Two fair dice are rolled. What is the probability of getting doubles (two dice showing the same number) or a sum of 6?

We let E_1 denote the event "get doubles" and E_2 denote the event "get a sum of 6."

Since doubles can be obtained in six ways, $P(E_1) = \frac{6}{36} = \frac{1}{6}$.

Since the sum of 6 can be obtained in five ways [(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)], $P(E_2) = \frac{5}{36}$.

The event $E_1 \cap E_2$ is "get doubles *and* get a sum of 6." Since this last event can occur only one way (by getting a pair of 3s), $P(E_1 \cap E_2) = \frac{1}{36}$.

By Theorem 8.2, the probability of getting doubles or a sum of 6 is
$$P(E_1 \cup E_2) = \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}.$$

Discrete Probability Theory

- Events E_1 and E_2 are **mutually exclusive** if $E_1 \cap E_2 = \emptyset$.
- Corollary 8.1: *If E_1 and E_2 are mutually exclusive events,*

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

❑ Two fair dice are rolled. Find the probability of getting doubles or the sum of 5.

We let E_1 denote the event "get doubles" and E_2 denote the event "get the sum of 5." Notice that E_1 and E_2 are mutually exclusive: You cannot get doubles and the sum of 5 simultaneously. Since doubles can be obtained in six ways, $P(E_1) = \frac{6}{36} = \frac{1}{6}$

Since the sum of 5 can be obtained in four ways [(1, 4), (2, 3), (3, 2), (4, 1)], $P(E_2) = \frac{4}{36} = \frac{1}{9}$.

By Corollary 8.1,

$$P(E_1 \cup E_2) = \frac{6}{36} + \frac{4}{36} = \frac{10}{36} = \frac{5}{18}.$$

Conditional Probability

- Definition 8.4: Let *E* and *F* be events, and assume that *P*(*F*) > 0. The **conditional probability** *of E given F* is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

- We use Definition 8.4 to compute the probability of getting a sum of 10, given that at least one die shows 5, when two fair dice are rolled.

  Let *E* denote the event "getting a sum of 10," and let *F* denote the event "at least one die shows 5." The event *E* ∩ *F* is "getting a sum of 10 *and* at least one die shows 5." Since only one outcome belongs to *E* ∩ *F*, $P(E \cap F) = \frac{1}{36}$.

  Since 11 outcomes belong to *F*, $P(F) = \frac{11}{36}$.

  Therefore,

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/36}{11/36} = \frac{1}{11}.$$

Conditional Probability

- Definition 8.5: Events E and F are **independent** if
$$P(E \cap F) = P(E)P(F).$$

- Joe and Alicia take a final examination in discrete mathematics. The probability that Joe passes is 0.70, and the probability that Alicia passes is 0.95. Assuming that the events "Joe passes" and "Alicia passes" are independent, find the probability that Joe or Alicia, or both, passes the final exam.

  We let J denote the event "Joe passes the final exam" and A denote the event "Alicia passes the final exam." We are asked to compute $P(J \cup A)$. Theorem 8.2 says that
$$P(J \cup A) = P(J) + P(A) - P(J \cap A).$$
  Since we are given $P(J)$ and $P(A)$, we need only compute $P(J \cap A)$. Because the events J and A are *independent*, Definition 8.5 says that
$$P(J \cap A) = P(J)P(A) = (0.70)(0.95) = 0.665.$$
  Therefore,
$$P(J \cup A) = P(J) + P(A) - P(J \cap A) = 0.70 + 0.95 - 0.665 = 0.985.$$

Bayes' Theorem

- **Pattern recognition** places items into various *classes* based on *features* of the items. For example, wine might be placed into the classes *premium, table wine,* and *swill* based on features such as acidity and bouquet. One way to perform such a classification uses probability theory.

❑ Let *R* denote class *premium, T* denote class *table wine,* and *S* denote class *swill.* Suppose that a particular wine has feature set *F* and

$$P(R \mid F) = 0.2, P(T \mid F) = 0.5, P(S \mid F) = 0.3.$$

Since class *table wine* has the greatest probability, this wine would be classified as *table wine.*

- Theorem 8.3: *Suppose that the possible classes are C1, . . . , Cn. Suppose further that each pair of classes is mutually exclusive and each item to be classified belongs to one of the classes. For a feature set F, we have*

$$P(C_j|F) = \frac{P(F|C_j)P(C_j)}{\sum_{i=1}^{n} P(F|C_i)P(C_i)}.$$

Bayes' Theorem

- The enzyme-linked immunosorbent assay (ELISA) test is used to detect antibodies in blood and can indicate the presence of the HIV virus. Approximately 15 percent of the patients at one clinic have the HIV virus. Furthermore, among those that have the HIV virus, approximately 95 percent test positive on the ELISA test. Among those that do not have the HIV virus, approximately 2 percent test positive on the ELISA test. Find the probability that a patient has the HIV virus if the ELISA test is positive.

  The classes are "has the HIV virus," which we denote *H*, and "does not have the HIV virus" ($\bar{H}$). The feature is "tests positive," which we denote *Pos*. Using this notation, the given information may be written

$$P(H) = 0.15, P(\bar{H}) = 0.85, P(Pos \mid H) = 0.95, P(Pos \mid \bar{H}) = 0.02.$$

  Bayes' Theorem gives the desired probability:

$$P(H|Pos) = \frac{P(Pos|H)P(H)}{P(Pos|H)P(H) + P(Pos|\bar{H})P(\bar{H})} = \frac{(0.95)(0.15)}{(0.95)(0.15) + (0.02)(0.85)} = 0.893.$$

Binomial Coefficients

- Theorem 8.4: *If a and b are real numbers and n is a positive integer, then*

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$$

- The numbers *C*(*n*, *r*) are known as **binomial coefficients.**

❑ Taking *n* = 3 in Theorem 8.4, we obtain

$$\begin{aligned}(a+b)^3 &= C(3,0)a^3b^0 + C(3,1)a^2b^1 + C(3,2)a^1b^2 + C(3,3)a^0b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3.\end{aligned}$$

❑ Find the coefficient of a^5b^4 in the expansion of $(a+b)^9$.

The term involving a^5b^4 arises in the Binomial Theorem by taking *n* = 9 and *k* = 4:

$$C(n,k)a^{n-k}b^k = C(9,4)a^5b^4 = 126a^5b^4.$$

Thus the coefficient of a^5b^4 is 126.

Binomial Coefficients

- Expand $(3x - 2y)^4$ using the Binomial Theorem.

  If we take $a = 3x$, $b = -2y$, and $n = 4$ in Theorem 8.4, we obtain

$$
\begin{aligned}
(3x-2y)^4 &= (a+b)^4 \\
&= C(4,0)a^4b^0 + C(4,1)a^3b^1 + C(4,2)a^2b^2 + C(4,3)a^1b^3 + C(4,4)a^0b^4 \\
&= C(4,0)(3x)^4(-2y)^0 + C(4,1)(3x)^3(-2y)^1 + C(4,2)(3x)^2(-2y)^2 \\
&\quad + C(4,3)(3x)^1(-2y)^3 + C(4,4)(3x)^0(-2y)^4 \\
&= 3^4x^4 + 4\cdot 3^3x^3(-2y) + 6\cdot 3^2x^2(-2)^2y^2 + 4(3x)(-2)^3y^3 + (-2)^4y^4 \\
&= 81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4.
\end{aligned}
$$

- We can write the binomial coefficients in a triangular form known as **Pascal's triangle** (see Figure beside). The border consists of 1's, and any interior value is the sum of the two numbers above it.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Combinatorial Identity

- An identity that results from some counting process is called a **combinatorial identity** and the argument that leads to its formulation is called a **combinatorial argument.**
- Theorem 8.5: $C(n+1,k) = C(n,k-1) + C(n,k)\ for\ 1 \le k \le n$.

❑ Use Theorem 8.5 to show that

$$\sum_{i=k}^{n} C(i,k) = C(n+1,k+1).$$

We use Theorem 8.5 in the form

$$C(i,k) = C(i+1,k+1) - C(i,k+1)$$

to obtain

$$\begin{aligned} & C(k,k) + C(k+1,k) + C(k+2,k) + \cdots + C(n,k) \\ = {} & 1 + C(k+2,k+1) - C(k+1,k+1) + C(k+3,k+1) \\ & - C(k+2,k+1) + \cdots + C(n+1,k+1) - C(n,k+1) \\ = {} & C(n+1,k+1). \end{aligned}$$

Combinatorial Identity

- Use equation

$$\sum_{i=k}^{n} C(i,k) = C(n+1, k+1)$$

to find the sum

$$1 + 2 + \cdots + n.$$

We may write

$$\begin{aligned} 1 + 2 + \cdots + n &= C(1,1) + C(2,1) + \cdots + C(n,1) \\ &= C(n+1, 2) \\ &= \frac{(n+1)n}{2}. \end{aligned}$$

PRACTICE

PRACTICE 1

1. Four microprocessors are randomly selected from a lot of 100 microprocessors among which 10 are defective. Find the probability of obtaining no defective microprocessors.
2. In the California Daily 3 game, a contestant must select three numbers among 0 to 9, repetitions allowed. A "straight play" win requires that the numbers be matched in the exact order in which they are randomly drawn by a lottery representative. What is the probability of choosing the winning numbers?
3. In the multi-state Big Game, to win the grand prize the contestant must match five distinct numbers, in any order, among the numbers 1 through 50, and one Big Money Ball number between 1 and 36, all randomly drawn by a lottery representative. What is the probability of choosing the winning numbers?

PRACTICE 2

1. Suppose that a professional wrestler is selected at random among 90 wrestlers, where 35 are over 350 pounds, 20 are bad guys, and 15 are over 350 pounds and bad guys. What is the probability that the wrestler selected is over 350 pounds or a bad guy?
2. A company that buys computers from three vendors and tracks the number of defective machines. The following table shows the results.

   Let A denote the event "the computer was purchased from Acme," let D denote the event "the computer was purchased from Dot-Com," let N denote the event "the computer was purchased from Nuclear," and let B denote the event "the computer was defective."

   Find P(A), P(D), and P(N).

   Find P(B | A), P(B | D), and P(B | N).

   Find P(B).

	Vendor		
	Acme	*DotCom*	*Nuclear*
Percent purchased	55	10	35
Percent defective	1	3	3

PRACTICE 3

1. Expand $(x + y)^4$ using the Binomial Theorem.
2. Find the coefficient of the term when the expression is expanded.
   a. x^4y^7; $(x + y)^{11}$
   b. s^6t^6; $(2s - t)^{12}$

NEXT WEEK'S OUTLINE

- Recursive Algorithm
- Recurrence Relations
- Solving Recurrence Relations
- Generating Functions

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- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

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