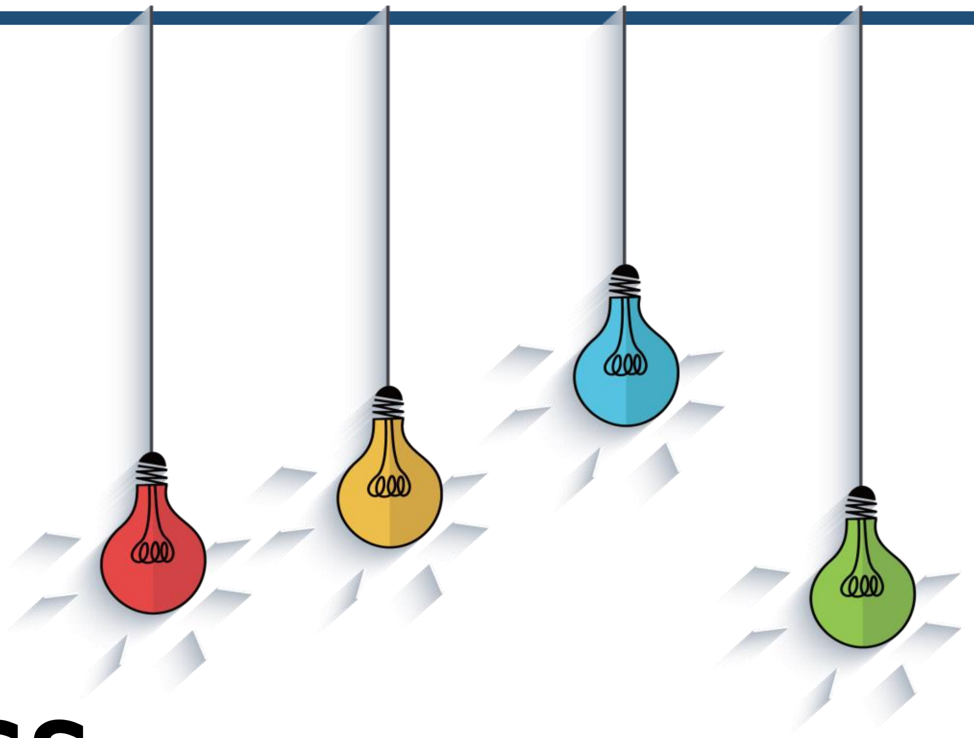

IF120 Discrete Mathematics

07 Counting Methods

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REVIEW

- Number Theory
- Binary, Octal, and Hexadecimal System Numbers
- Euclid Algorithm

OUTLINE

- Counting Methods Principle
- Permutation and Combination

Counting Methods

- The menu for Kay's Quick Lunch is shown here.
- As you can see, it features two appetizers, three main courses, and four beverages.
- How many different dinners consist of one main course and one beverage?

- If we list all possible dinners consisting of one main course and one beverage,

  HT, HM, HC, HR, CT, CM, CC, CR, FT, FM, FC, FR,
- we see that there are 12 different dinners.
- Notice that there are three main courses and four beverages and $12 = 3 \cdot 4$.
- How many possible dinners consisting of one appetizer, one main course, and one beverage?

APPETIZERS	
Nachos	2.15
Salad	1.90
MAIN COURSES	
Hamburger	3.25
Cheeseburger	3.65
Fish Filet	3.15
BEVERAGES	
Tea	.70
Milk	.85
Cola	.75
Root Beer	.75

Multiplication Principle

- *If an activity can be constructed in t successive steps and step 1 can be done in* n_1 *ways, step 2 can then be done in* n_2 *ways,* $\ldots$ *, and step t can then be done in* n_t *ways, then the number of different possible activities is* $n_1 \cdot n_2 \cdots n_t$.

❑ How many dinners are available from Kay's Quick Lunch consisting of one main course and an *optional* beverage?

We may construct a dinner consisting of one main course and an optional beverage by a two-step process. The first step is "select the main course" and the second step is "select an optional beverage." There are $n_1 = 3$ ways to select the main course (hamburger, cheeseburger, fish filet) and $n_2 = 5$ ways to select the optional beverage (tea, milk, cola, root beer, none).

By the Multiplication Principle, there are 3 ·5 = 15 dinners. As confirmation, we list the 15 dinners (N = no beverage):

HT, HM, HC, HR, HN, CT, CM, CC, CR, CN, FT, FM, FC, FR, FN.

- How many strings of length 4 can be formed using the letters *ABCDE* if repetitions are not allowed?

  A string of length 4 can be constructed in four successive steps: Choose the first letter; choose the second letter; choose the third letter; and choose the fourth letter. The first letter can be selected in five ways. Once the first letter has been selected, the second letter can be selected in four ways. Once the second letter has been selected, the third letter can be selected in three ways. Once the third letter has been selected, the fourth letter can be selected in two ways. By the Multiplication Principle, there are

  $$5 \cdot 4 \cdot 3 \cdot 2 = 120$$

  Strings.

- How many strings of part (a) begin with the letter *B*?

  The strings that begin with the letter *B* can be constructed in four successive steps: Choose the first letter; choose the second letter; choose the third letter; and choose the fourth letter. The first letter (*B*) can be chosen in one way, the second letter in four ways, the third letter in three ways, and the fourth letter in two ways. Thus, by the Multiplication Principle, there are

  $$1 \cdot 4 \cdot 3 \cdot 2 = 24$$

  strings that start with the letter *B*.

Multiplication Principle

- We next give a proof using the Multiplication Principle that a set with *n* elements has 2^n subsets.

- Use the Multiplication Principle to show that a set $\{x_1, \ldots, x_n\}$ containing *n* elements has 2^n subsets.

  A subset can be constructed in *n* successive steps: Pick or do not pick x_1; pick or do not pick x_2; . . . ; pick or do not pick x_n. Each step can be done in two ways. Thus the number of possible subsets is

$$\underbrace{2 \cdot 2 \cdots 2}_{n \text{ factors}} = 2^n.$$

Addition Principle

- *Suppose that $X_1, \ldots, X_t$ are sets and that the ith set X_i has n_i elements. If $\{X_1, \ldots, X_t\}$ is a pairwise disjoint family (i.e., if $i \neq j, X_i \cap X_j = \emptyset$), the number of possible elements that can be selected from X_1 or X_2 or ... or X_t is*

$$n_1 + n_2 + \cdots + n_t.$$

- *(Equivalently, the union $X_1 \cup X_2 \cup \cdots \cup X_t$ contains $n_1 + n_2 + \cdots + n_t$ elements.)*

❑How many eight-bit strings begin either 101 or 111?

An eight-bit string that begins 101 can be constructed in five successive steps: Select the fourth bit; select the fifth bit; . . . ; select the eighth bit. Since each of the five bits can be selected in two ways, by the Multiplication Principle, there are

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$$

eight-bit strings that begin 101. The same argument can be used to show that there are 32 eight-bit strings that begin 111. Since there are 32 eight-bit strings that begin 101 and 32 eight-bit strings that begin 111, there are 32 + 32 = 64 eight-bit strings that begin either 101 or 111.

Tips

- If we are counting objects that are constructed in **successive steps**, we use the **Multiplication** Principle.
- If we have **disjoint** sets of objects and we want to know the **total** number of objects, we use the **Addition** Principle.
- It is important to recognize when to apply each principle. This skill comes from practice and careful thinking about each problem.

Multiplication & Addition Principle

- In how many ways can we select two books from different subjects among five distinct computer science books, three distinct mathematics books, and two distinct art books?

  Using the Multiplication Principle, we find that we can select two books, one from computer science and one from mathematics, in $5 \cdot 3 = 15$ways. Similarly, we can select two books, one from computer science and one from art, in $5 \cdot 2 = 10$ ways, and we can select two books, one from mathematics and one from art, in $3 \cdot 2 = 6$ ways. Since these sets of selections are pairwise disjoint, we may use the Addition Principle to conclude that there are

$$15 + 10 + 6 = 31$$

  ways of selecting two books from different subjects among the computer science, mathematics, and art books.

- A six-person committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer.

a) In how many ways can this be done?

We use the Multiplication Principle. The officers can be selected in three successive steps: Select the chairperson; select the secretary; select the treasurer. The chairperson can be selected in six ways. Once the chairperson has been selected, the secretary can be selected in five ways. After selection of the chairperson and secretary, the treasurer can be selected in four ways. Therefore, the total number of possibilities is

$$6 \cdot 5 \cdot 4 = 120.$$

b) In how many ways can this be done if either Alice or Ben must be chairperson?

Arguing as in part (a), if Alice is chairperson, we have $5 \cdot 4 = 20$ ways to select the remaining officers. Similarly, if Ben is chairperson, there are 20 ways to select the remaining officers. Since these cases are disjoint, by the Addition Principle, there are

$$20 + 20 = 40$$

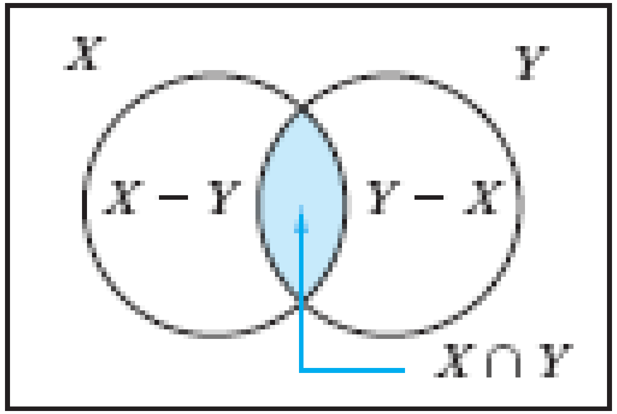
possibilities.

Inclusion-Exclusion Principle

- Suppose that we want to count the number of eight-bit strings that start 10 or end 011 or both. Let X denote the set of eight-bit strings that start 10 and Y denote the set of eight-bit strings that end 011. The goal then is to compute $|X \cup Y|$.
- We *cannot* use the Addition Principle and add $|X|$ and $|Y|$ to compute $|X \cup Y|$ because the Addition Principle requires X and Y to be disjoint. Here X and Y are not disjoint; for example, $10111011 \in X \cap Y$.
- The **Inclusion-Exclusion Principle** generalizes the Addition Principle by giving a formula to compute the number of elements in a union without requiring the sets to be pairwise disjoint.
- **Theorem 7.1: Inclusion-Exclusion Principle for Two Sets**

  If X and Y are finite sets, then

  $|X \cup Y| = |X| + |Y| - |X \cap Y|$.



- A committee composed of Alice, Ben, Connie, Dolph, Egbert, and Francisco is to select a chairperson, secretary, and treasurer. How many selections are there in which either Alice or Dolph or both are officers?

  Let X denote the set of selections in which Alice is an officer and let Y denote the set of selections in which Dolph is an officer. We must compute $|X \cup Y|$. Since X and Y are *not* disjoint (both Alice and Dolph could be officers), we cannot use the Addition Principle. Instead we use the Inclusion-Exclusion Principle.

  We first count the number of selections in which Alice is an officer. Alice can be assigned an office in three ways, the highest remaining office can be filled in five ways, and the last office can be filled in four ways. Thus the number of selections in which Alice is an officer is $3 \cdot 5 \cdot 4 = 60$, that is, $|X| = 60$. Similarly, the number of selections in which Dolph is an officer is 60, that is, $|Y| = 60$.

  Now $X \cap Y$ is the set of selections in which both Alice and Dolph are officers. Alice can be assigned an office in three ways, Dolph can be assigned an office in two ways, and the last office can be filled in four ways. Thus the number of selections in which both Alice and Dolph are officers is $3 \cdot 2 \cdot 4 = 24$, that is, $|X \cap Y| = 24$. The Inclusion-Exclusion Principle tells us that

  $$|X \cup Y| = |X| + |Y| - |X \cap Y| = 60 + 60 - 24 = 96.$$

  Thus there are 96 selections in which either Alice or Dolph or both are officers.

Permutations

- Definition: A **permutation** of *n* distinct elements $x_1, \dots, x_n$ is an ordering of the *n* elements $x_1, \dots, x_n$.

❑ There are six permutations of three elements. If the elements are denoted *A*, *B*, *C*, the six permutations are

ABC, ACB, BAC, BCA, CAB, CBA.

- **Theorem 7.2:** *There are n! permutations of n elements.*

❑ There are

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$$

permutations of 10 elements.

Permutations

- How many permutations of the letters *ABCDEF* contain the substring *DEF*?

  To guarantee the presence of the pattern *DEF* in the substring, these three letters must be kept together in this order. The remaining letters, *A*, *B*, and *C*, can be placed arbitrarily. We can think of constructing permutations of the letters *ABCDEF* that contain the pattern *DEF* by permuting four tokens—one labeled *DEF* and the others labeled *A*, *B*, and *C*. By Theorem 7.2, there are 4! permutations of four objects. Thus the number of permutations of the letters *ABCDEF* that contain the substring *DEF* is $4! = 24$.

- How many permutations of the letters *ABCDEF* contain the letters *DEF* together in any order?

  We can solve the problem by a two-step procedure: Select an ordering of the letters *DEF*; construct a permutation of *ABCDEF* containing the given ordering of the letters *DEF*. By Theorem 7.2, the first step can be done in $3! = 6$ ways and, according to the above Example, the second step can be done in 24 ways. By the Multiplication Principle, the number of permutations of the letters *ABCDEF* containing the letters *DEF* together in any order is $6 \cdot 24 = 144$.

Permutations

- Definition: An **r-permutation** of *n* (distinct) elements $x_1, \ldots, x_n$ is an ordering of an *r*-element subset of $\{x_1, \ldots, x_n\}$. The number of *r*-permutations of a set of *n* distinct elements is denoted $P(n, r)$.

❑ Examples of 2-permutations of *a*, *b*, *c* are

ab, *ba*, *ca*.

- **Theorem 7.3:** *The number of r-permutations of a set of n distinct objects is*

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1), \qquad r \le n.$$

❑ According to Theorem 7.3, the number of 2-permutations of $X = \{a, b, c\}$ is

$$P(3, 2) = 3 \cdot 2 = 6.$$

These six 2-permutations are

ab, *ac*, *ba*, *bc*, *ca*, *cb*.

Permutations

- In how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

  We need to count the number of orderings of four persons selected from a group of 10, since an ordering picks (uniquely) a chairperson (first pick), a vice-chairperson (second pick), a secretary (third pick), and a treasurer (fourth pick). By Theorem 7.3, the solution is

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

  Since

$$P(n,r) = \frac{n(n-1)\ldots(n-r+1)(n-r)\ldots 2.1}{(n-r)\ldots 2.1} = \frac{n!}{(n-r)!}$$

  We may rewrite the solution as

$$P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

Permutations

- In how many ways can seven distinct Martians and five distinct Jovians wait in line if no two Jovians stand together?

  We can line up the Martians and Jovians by a two-step process: Line up the Martians; line up the Jovians. The Martians can line up in $7! = 5040$ ways. Once we have lined up the Martians (e.g., in positions $M1$–$M7$), since no two Jovians can stand together, the Jovians have eight possible positions in which to stand (indicated by blanks):

$$- M1 - M2 - M3 - M4 - M5 - M6 - M7 - .$$

  Thus the Jovians can stand in $P(8, 5) = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$ ways. By the Multiplication Principle, the number of ways seven distinct Martians and five distinct Jovians can wait in line if no two Jovians stand together is

$$5040 \cdot 6720 = 33{,}868{,}800.$$

Combinations

- Definition: *Given a set* $X = \{x_1, \dots, x_n\}$ *containing n (distinct) elements,*

a. *An* **r-combination** *of X is an unordered selection of r-elements of X (i.e., an r-element subset of X).*

b. *The number of r-combinations of a set of n distinct elements is denoted C(n, r) or* $\binom{n}{r}$.

- <u>**Theorem 7.4:**</u> *The number of r-combinations of a set of n distinct objects is*

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}, \qquad r \le n.$$

Combinations

- A group of five students, Mary, Boris, Rosa, Ahmad, and Nguyen, has decided to talk with the Mathematics Department chairperson about having the Mathematics Department offer more courses in discrete mathematics. The chairperson has said that she will speak with three of the students. In how many ways can these five students choose three of their group to talk with the chairperson?

  In solving this problem, we must *not* take order into account. (For example, it will make no difference whether the chairperson talks to Mary, Ahmad, and Nguyen or to Nguyen, Mary, and Ahmad.) By simply listing the possibilities, we see that there are 10 ways that the five students can choose three of their group to talk to the chairperson:

  $$MBR, MBA, MRA, BRA, MBN, MRN, BRN, MAN, BAN, RAN.$$

  In the terminology of Definition of Combination, the number of ways the five students can choose three of their group to talk with the chairperson is $C(5, 3)$, the number of 3-combinations of five elements. We have found that

  $$C(5, 3) = 10.$$

Combinations

- In how many ways can we select a committee of three from a group of 10 distinct persons?

  Since a committee is an unordered group of people, the answer is

$$C(10,3) = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!} = \frac{10.9.8}{3.2.1} = 240.$$

- In how many ways can we select a committee of two women and three men from a group of five distinct women and six distinct men?

  We find that the two women can be selected in

$$C(5,2) = 10$$

  ways and that the three men can be selected in

$$C(6,3) = 20$$

  ways. The committee can be constructed in two successive steps: Select the women; select the men. By the Multiplication Principle, the total number of committees is

$$10 \cdot 20 = 200.$$

PRACTICE

PRACTICE 1

- *Use the Multiplication Principle to solve these problems.*

1. A man has eight shirts, four pairs of pants, and five pairs of shoes. How many different outfits are possible?
2. The options available on a particular model of a car are five interior colors, six exterior colors, two types of seats, three types of engines, and three types of radios. How many different possibilities are available to the consumer?
3. The Braille system of representing characters was developed early in the nineteenth century by Louis Braille. The characters, used by the blind, consist of raised dots. The positions for the dots are selected from two vertical columns of three dots each. At least one raised dot must be present. How many distinct Braille characters are possible?
4. Two dice are rolled, one blue and one red. How many outcomes are possible?
5. How many different car license plates can be constructed if the licenses contain three letters followed by two digits if repetitions are allowed? if repetitions are not allowed?

PRACTICE 2

- *Use the Addition Principle to solve these problems.*

1. Given that there are 32 eight-bit strings that begin 101 and 16 eight-bit strings that begin 1101, how many eight-bit strings begin either 101 or 1101?
2. Two dice are rolled, one blue and one red. How many outcomes give the sum of 2 or the sum of 12?
3. A committee composed of Morgan, Tyler, Max, and Leslie is to select a president and secretary. How many selections are there in which Tyler is president or not an officer?
4. How many times are the print statements executed?

   for i = 1 to m

       println(i)

   for j = 1 to n

       println(j)

PRACTICE 3

- *Use the Inclusion-Exclusion Principle to solve these problems.*

1. How many eight-bit strings either begin 100 or have the fourth bit 1 or both?
2. How many eight-bit strings either start with a 1 or end with a 1 or both?
3. Two dice are rolled, one blue and one red. How many outcomes have either the blue die 3 or an even sum or both?
4. How many integers from 1 to 10,000, inclusive, are multiples of 5 or 7 or both?

PRACTICE 4

- *Exercises below refer to a club consisting of six distinct men and seven distinct women.*

1. In how many ways can we select a committee of five persons?
2. In how many ways can we select a committee of three men and four women?
3. In how many ways can we select a committee of four persons that has at least one woman?
4. In how many ways can we select a committee of four persons that has at most one man?

NEXT WEEK'S OUTLINE

- Mid-term Exam

REFERENCES

- Johnsonbaugh, R., 2005, *Discrete Mathematics*, New Jersey: Pearson Education, Inc.
- Rosen, Kenneth H., 2005, *Discrete Mathematics and Its Applications*, 6th edition, McGraw-Hill.
- Hansun, S., 2021, *Matematika Diskret Teknik*, Deepublish.
- Lipschutz, Seymour, Lipson, Marc Lars, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, McGraw-Hill.
- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

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