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IF120
Discrete Mathematics

11 Graph 2

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REVIEW

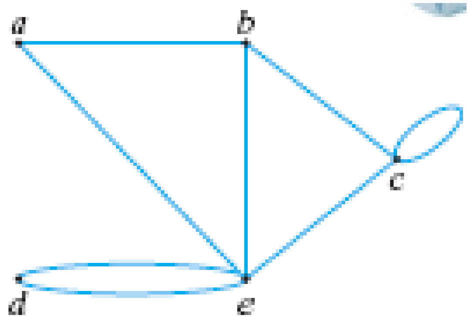
- Graph Terminologies
- Path and Cycle
- Euler Cycle and Hamiltonian Cycle

OUTLINE

- Graph Representation
- Isomorphism
- Planar Graphs
- Shortest Path Problem

Graph Representation

- There are some other methods to represent a graph besides by drawing it, such as by using **adjacency matrix** and **incidence matrix**.
- Consider the graph of Figure below. To obtain the **adjacency** matrix of this graph, we first select an **ordering** of the vertices, say *a*, *b*, *c*, *d*, e. Next, we **label** the rows and columns of a matrix with the ordered vertices. The entry in this matrix in row *i* , column *j* , $i \neq j$, is the **number** of **edges incident** on *i* and *j*. If $i = j$, the entry is **twice** the number of loops incident on *i* .



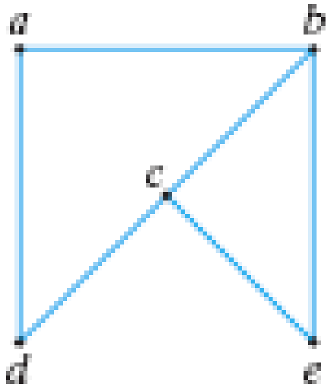
- The adjacency matrix for this graph is

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix} \end{array}.$$

- Notice that we can obtain the **degree** of a **vertex** *v* in a graph *G* by **summing** row *v* or column *v* in *G*'s adjacency matrix.

Adjacency Matrix

- The adjacency matrix of the simple graph of Figure below is



$$A = \begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

- **Theorem 11.1**: *If A is the adjacency matrix of a simple graph, the ijth entry of A^n is equal to the number of paths of length n from vertex i to vertex j , n = 1, 2, ...*

Adjacency Matrix

- After previous Example, we showed that if *A* is the matrix of the graph, then

$$A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \end{array}.$$

By multiplying,

$$A^4 = A^2A^2 = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 & 1 \\ 2 & 1 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 9 & 3 & 11 & 1 & 6 \\ 3 & 15 & 7 & 11 & 8 \\ 11 & 7 & 15 & 3 & 8 \\ 1 & 11 & 3 & 9 & 6 \\ 6 & 8 & 8 & 6 & 8 \end{pmatrix} \end{array}.$$

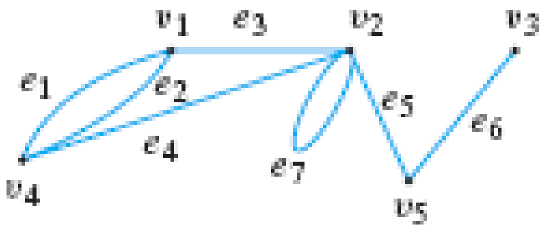
The entry from row *d*, column e is 6, which means that there are six paths of length 4 from *d* to e. By inspection, we find them to be

$$(d, a, d, c, e), \quad (d, c, d, c, e), \quad (d, a, b, c, e),$$
$$(d, c, e, c, e), \quad (d, c, e, b, e), \quad (d, c, b, c, e).$$

Incidence Matrix

- Another useful matrix representation of a graph is known as the **incidence matrix.**

❑ To obtain the incidence matrix of the graph in Figure below, we label the rows with the vertices and the columns with the edges (in some arbitrary order). The entry for row *v* and column e is 1 if e is incident on *v* and 0 otherwise. Thus the incidence matrix for the graph is

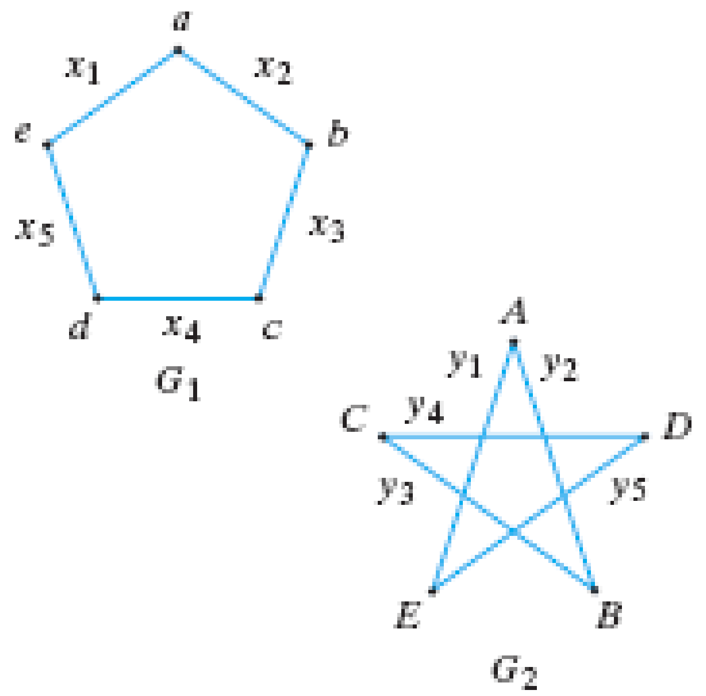


$$
\begin{array}{c|ccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
v_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
v_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
v_4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}.
$$

- A column such as e_7 is understood to represent a loop.
- Notice that in a graph **without loops** each **column** has **two** 1's and that the sum of a **row** gives the **degree** of the vertex identified with that row.

Isomorphisms of Graphs

- The following instructions are given to two persons who cannot see each other's paper: "Draw and label five vertices *a*, *b*, *c*, *d*, and e. Connect *a* and *b*, *b* and *c*, *c* and *d*, *d* and e, and *a* and e." The graphs produced are shown in Figures below. Surely these figures define the same graph even though they appear dissimilar. Such graphs are said to be **isomorphic.**



- **<u>Definition 11.1</u>**: Graphs G_1 and G_2 are **isomorphic** if there is a **one-to-one**, **onto** function f from the vertices of G_1 to the vertices of G_2 and a **one-to-one, onto** function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f\ (v)$ and $f\ (w)$ in G_2. The pair of functions f and g is called an **isomorphism** of G_1 onto G_2.

Isomorphism

- **Theorem 11.2**: *Graphs G_1 and G_2 are isomorphic if and only if for some ordering of their vertices, their adjacency matrices are equal.*
- **Corollary 11.1**: *Let G_1 and G_2 be simple graphs. The following are equivalent:*

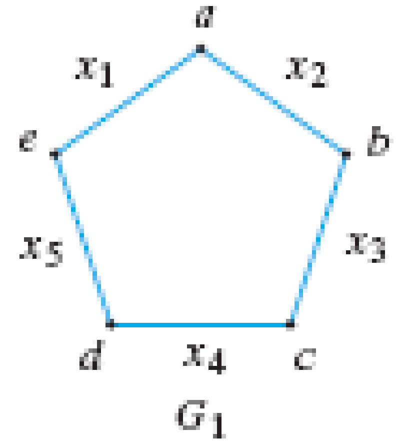
a) *G_1 and G_2 are isomorphic.*

b) *There is a one-to-one, onto function f from the vertex set of G_1 to the vertex set of G_2 satisfying the following: Vertices v and w are adjacent in G_1 if and only if the vertices $f(v)$ and $f(w)$ are adjacent in G_2.*

Isomorphism

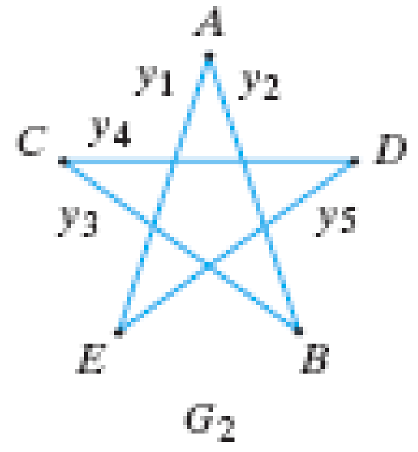
- The adjacency matrix of graph G_1 relative to the vertex ordering *a*, *b*, *c*, *d*, e,

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array},$$



is equal to the adjacency matrix of graph G_2 relative to the vertex ordering *A*, *B*, *C*, *D*, *E*,

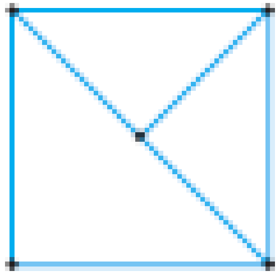
$$\begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$



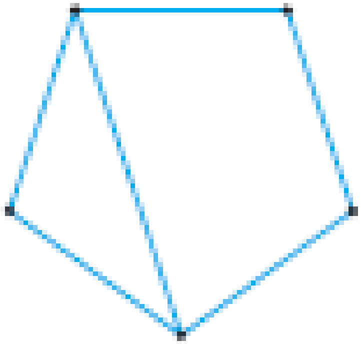
We see again that G_1 and G_2 are isomorphic.

- The following is one way to show that two simple graphs G_1 and G_2 are **not isomorphic**. Find a property of G_1 that G_2 does *not* have but that G_2 *would* have if G_1 and G_2 were isomorphic. Such a property is called an **invariant.**
- By Definition 11.1, if graphs G_1 and G_2 are isomorphic, there are one-to-one, onto functions from the edges (respectively, vertices) of G_1 to the edges (respectively, vertices) of G_2. Thus, if G_1 and G_2 are isomorphic, then G_1 and G_2 have the **same number** of **edges** and the **same number** of **vertices**. Therefore, if e and n are nonnegative integers, the properties "**has e edges**" and "**has n vertices**" are **invariants**.

❑ The graphs G_1 and G_2 in Figures below are not isomorphic, since G_1 has seven edges and G_2 has six edges and "has seven edges" is an invariant.

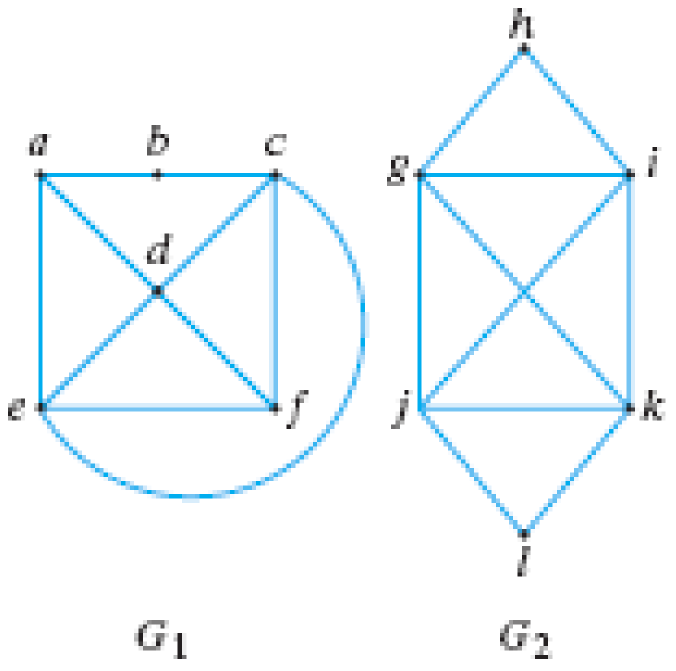


G_1

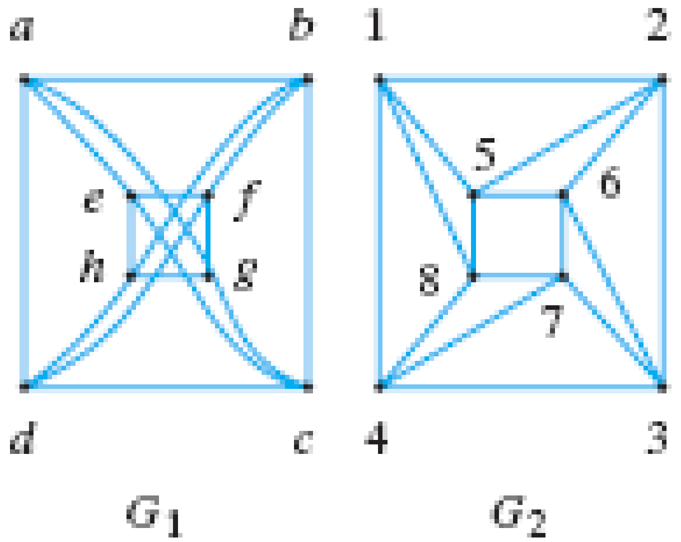


G_2

- Other invariants are:
  - if k is a positive integer, "has a **vertex of degree** k"
  - "has a **simple cycle of length** k."



- Since "has a vertex of degree 3" is an invariant, the graphs G_1 and G_2 of Figure on the right are not isomorphic; G_1 has vertices (a and f) of degree 3, but G_2 does not have a vertex of degree 3. Notice that G_1 and G_2 have the same numbers of edges and vertices.



- Since "has a simple cycle of length 3" is an invariant, the graphs G_1 and G_2 of Figure on the left are not isomorphic; the graph G_2 has a simple cycle of length 3, but all simple cycles in G_1 have length at least 4. Notice that G_1 and G_2 have the same numbers of edges and vertices and that every vertex in G_1 or G_2 has degree 4.

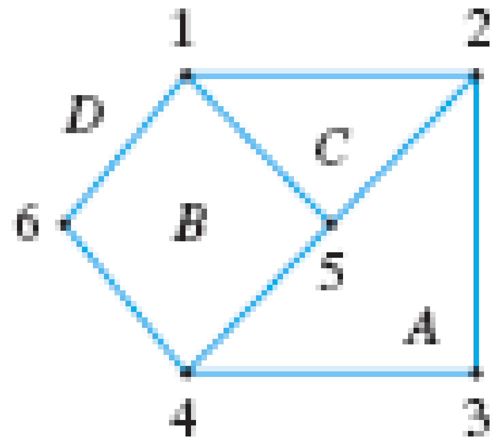
Planar Graphs

- **Definition 11.2**: A graph is **planar** if it can be drawn in the plane without its edges crossing.
- If a connected, planar graph is drawn in the plane, the plane is divided into contiguous regions called **faces.** A face is characterized by the cycle that forms its boundary.

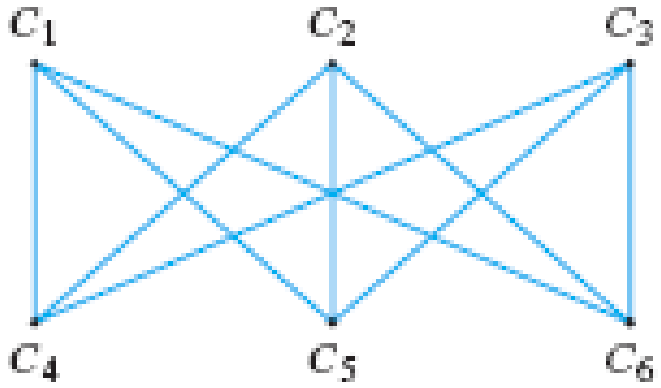
- For example, in the graph of Figure on the right, face *A* is bounded by the cycle $(5, 2, 3, 4, 5)$ and face *C* is bounded by the cycle $(1, 2, 5, 1)$. The outer face *D* is considered to be bounded by the cycle $(1, 2, 3, 4, 6, 1)$. The graph of the Figure has *f* = 4 faces, e = 8 edges, and *v* = 6 vertices. Notice that *f* , e, and *v* satisfy the equation

$$f = e - v + 2.$$

The formula also known as Euler formula for Graph.



Planar Graphs



$K_{3,3}$

- Show that the graph $K_{3,3}$ of Figure on the right is not planar.

  Suppose that $K_{3,3}$ is planar. Since every cycle has at least four edges, each face is bounded by at least four edges. Thus the number of edges that bound faces is at least $4f$. In a planar graph, each edge belongs to at most two bounding cycles. Therefore,

$$2e \geq 4f.$$

  Using previous formula, we find that

$$2e \geq 4(e - v + 2).$$

  For the graph, e = 9 and v = 6, so the above equation becomes

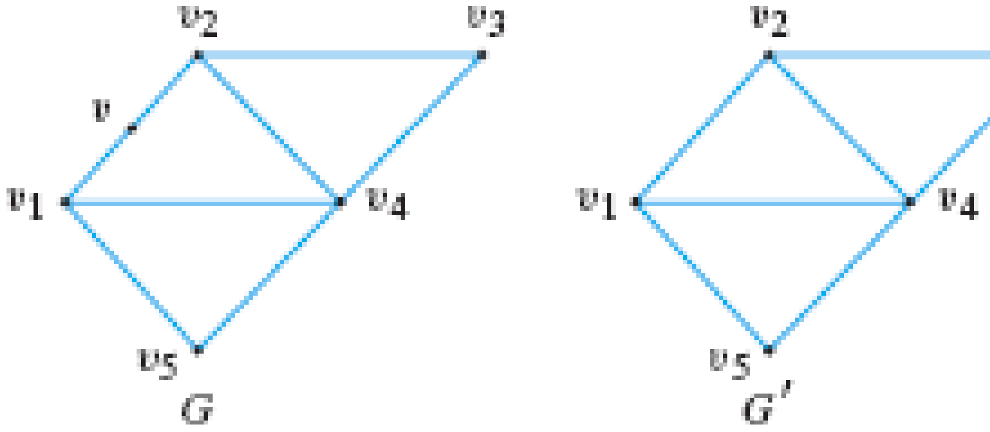
$$18 = 2 \cdot 9 \geq 4(9 - 6 + 2) = 20,$$

  which is a contradiction. Therefore, $K_{3,3}$ is not planar.

Planar Graphs

- **Definition 11.3**: If a graph G has a vertex v of degree 2 and edges (v, v_1) and (v, v_2) with $v_1 \neq v_2$, we say that the edges (v, v_1) and (v, v_2) are in **series.** A **series reduction** consists of deleting the vertex v from the graph G and replacing the edges (v, v_1) and (v, v_2) by the edge (v_1, v_2). The resulting graph G' is said to be **obtained from G by a series reduction**. By convention, G is said to be obtainable from itself by a series reduction.

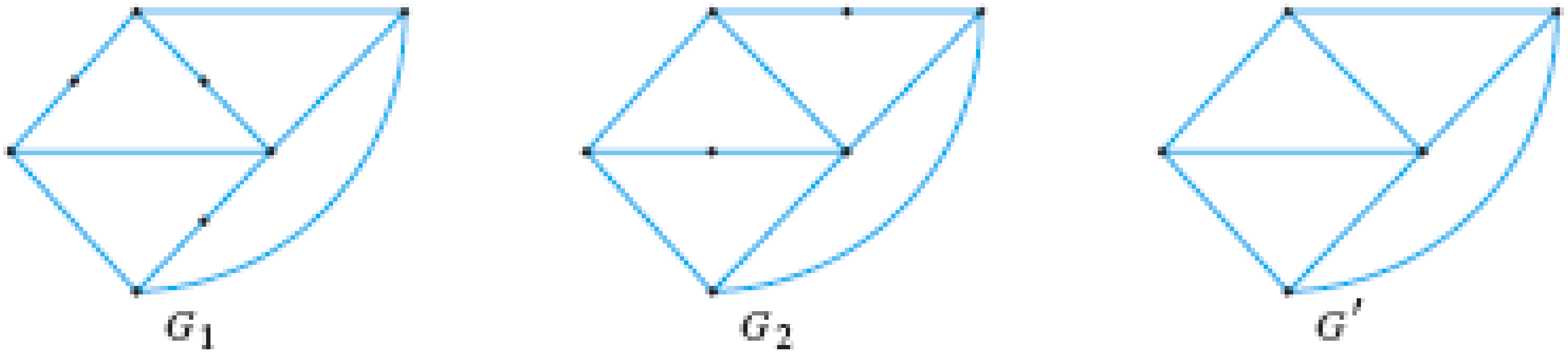
❑ In the graph G of Figure below, the edges (v, v_1) and (v, v_2) are in series. The graph G' is obtained from G by a series reduction.



Homeomorphic

- **Definition 11.4**: Graphs G_1 and G_2 are **homeomorphic** if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.

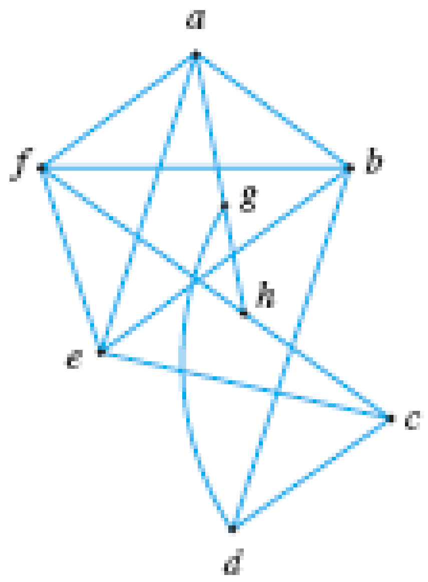
❑The graphs G_1 and G_2 of Figure below are homeomorphic since they can both be reduced to the graph G by a sequence of series reductions.



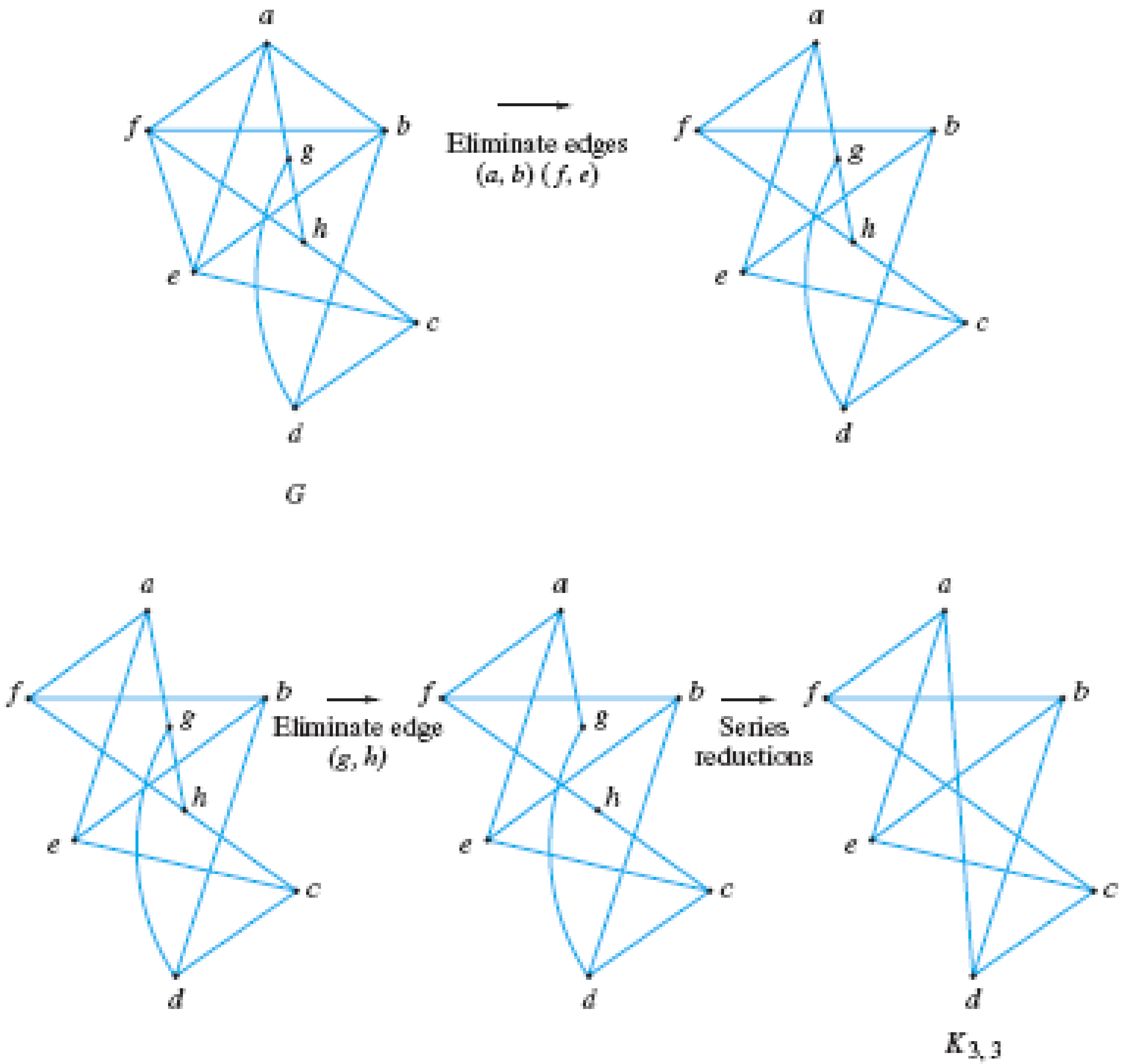
Kuratowski's Theorem

- **Theorem 11.3**: *A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.*

❑Show that the graph G of Figure below is not planar by using Kuratowski's Theorem.



Let us try to find $K_{3,3}$ in the graph G. We first note that the vertices a, b, f , and e each have degree 4. In $K_{3,3}$ each vertex has degree 3, so let us eliminate the edges (a, b) and (f, e) so that all vertices have degree 3. We note that if we eliminate one more edge, we will obtain two vertices of degree 2 and we can then carry out two series reductions. The resulting graph will have nine edges; since $K_{3,3}$ has nine edges, this approach looks promising. Using trial and error, we finally see that if we eliminate edge (g, h) and carry out the series reductions, we obtain an isomorphic copy of $K_{3,3}$. Therefore, the graph G is not planar, since it contains a subgraph homeomorphic to $K_{3,3}$.


Eliminate edges
(a, b) (f, e)
G
Eliminate edge
(g, h)
Series
reductions
$K_{3,3}$

Shortest Path Problem

- Recall that a weighted graph is a graph in which values are assigned to the edges and that the length of a path in a weighted graph is the sum of the weights of the edges in the path. We let $w(i, j)$ denote the weight of edge (i, j). In weighted graphs, we often want to find a **shortest path** (i.e., a path having minimum length) between two given vertices. Algorithm 11.1, due to E.W. Dijkstra (1930-2002), which efficiently solves this problem, is the topic of this section.
- Throughout this section, *G* denotes a connected, weighted graph. We assume that the weights are positive numbers and that we want to find a shortest path from vertex *a* to vertex z. The assumption that *G* is connected can be dropped.

Dijkstra's Shortest-Path Algorithm

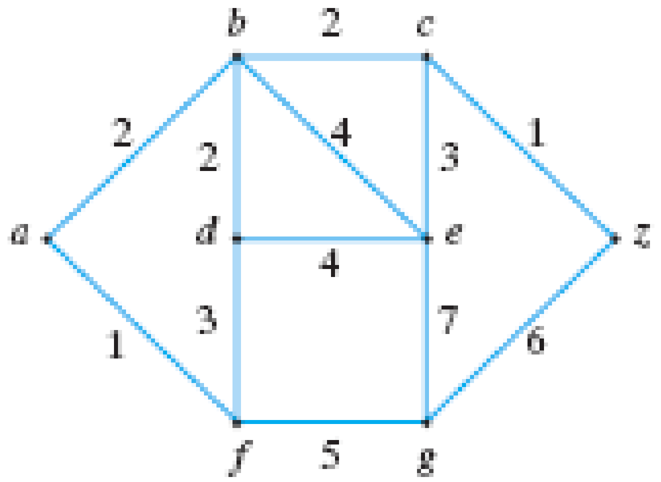
This algorithm finds the length of a shortest path from vertex a to vertex z in a connected, weighted graph. The weight of edge (i, j) is $w(i, j) > 0$ and the label of vertex x is $L(x)$. At termination, $L(z)$ is the length of a shortest path from a to z.

Input: A connected, weighted graph in which all weights are positive; vertices a and z

Output: $L(z)$, the length of a shortest path from a to z

```
dijkstra(w, a, z, L) {
    L(a) = 0
    for all vertices x ≠ a
        L(x) = ∞
    T = set of all vertices
    // T is the set of vertices whose shortest distance from a has
    // not been found
    while (z ∈ T) {
        choose v ∈ T with minimum L(v)
        T = T − {v}
        for each x ∈ T adjacent to v
            L(x) = min{L(x), L(v) + w(v, x)}
    }
}
```

- We show how Algorithm 11.1 finds a shortest path from *a* to *z* in the graph of Figure below. (The vertices in *T* are uncircled and have temporary labels. The circled vertices have permanent labels.)



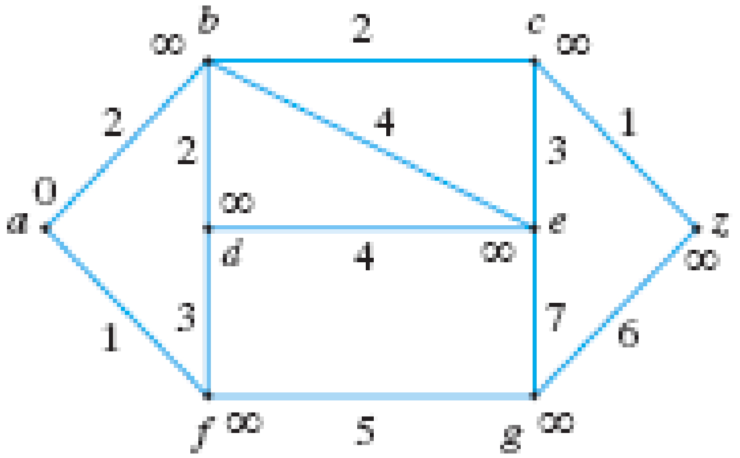
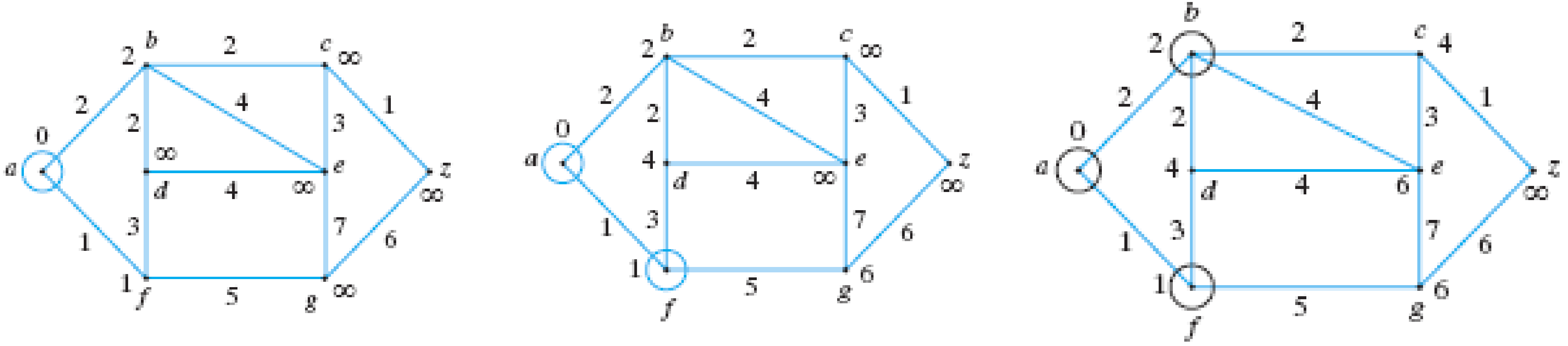


Figure on the left shows the result of executing lines 2–5. At line 8, *z* is not circled. We proceed to line 9, where we select vertex *a*, the uncircled vertex with the smallest label, and circle it (see Figure on the next slide). At lines 11 and 12 we update each of the uncircled vertices, *b* and *f*, adjacent to *a*. We obtain the new labels

$L(b) = \min\{\infty, 0 + 2\} = 2, L(f) = \min\{\infty, 0 + 1\} = 1$

At this point, we return to line 8.

Since z is not circled, we proceed to line 9, where we select vertex f, the uncircled vertex with the smallest label, and circle it (see Figure in the middle). At lines 11 and 12 we update each label of the uncircled vertices, d and g, adjacent to f. We obtain the labels shown in the middle figure.

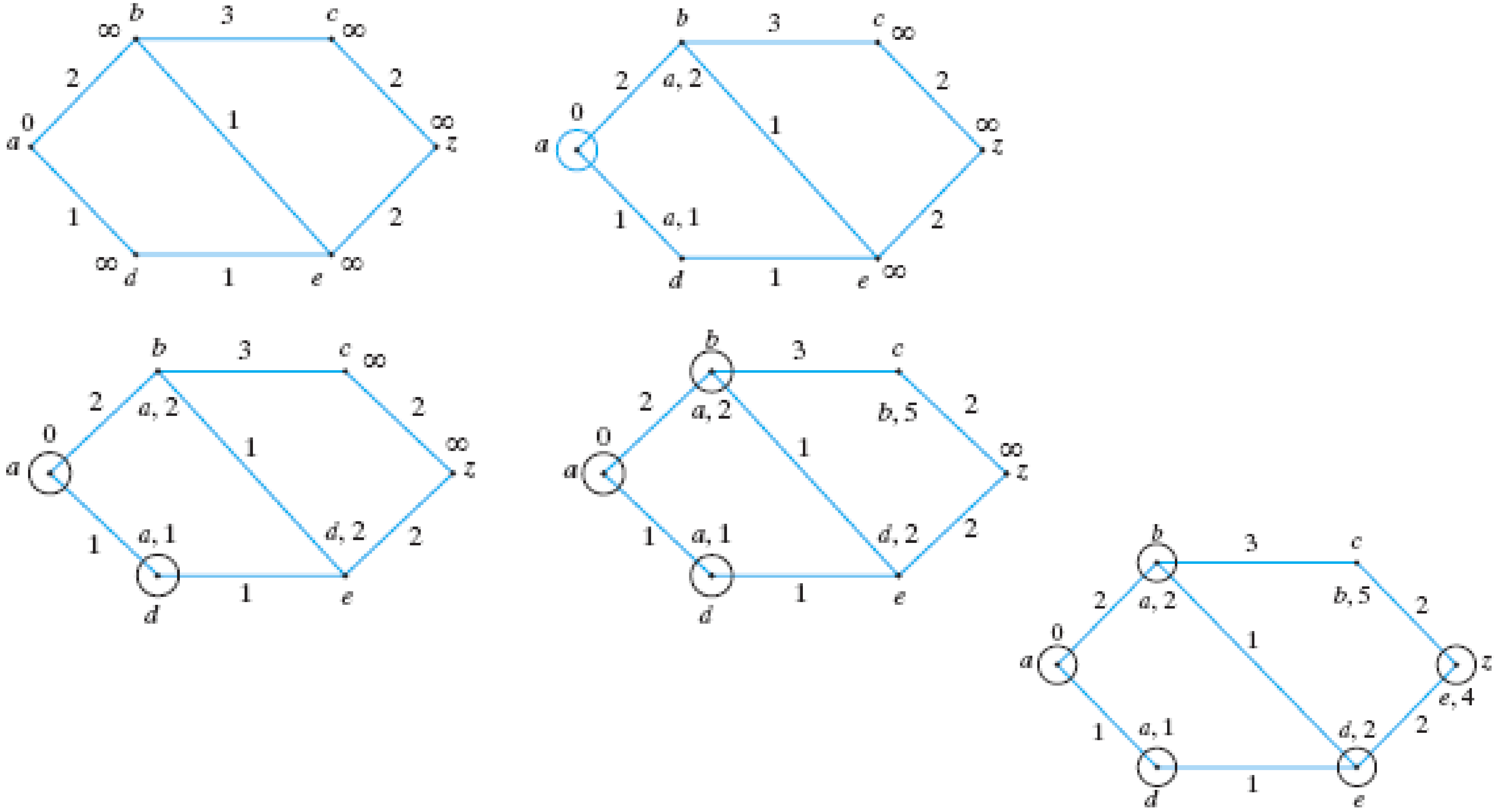


You should verify that the next iteration of the algorithm produces the labeling shown in the last Figure (on the right) and that at the termination of the algorithm, z is labeled 5, indicating that the length of a shortest path from a to z is 5.

A shortest path is given by (a, b, c, z).

Shortest Path Problem

- Find a shortest path from *a* to *z* and its length for the graph of Figure on the next slide.
  - We will apply Algorithm 11.1 with a slight modification. In addition to circling a vertex, we will also label it with the name of the vertex from which it was labeled.
  - The first figure shows the result of executing lines 2–4 of Algorithm 11.1. First, we circle *a* (see the next Figure). Next, we label the vertices *b* and *d* adjacent to *a*. Vertex *b* is labeled "*a*, 2" to indicate its value and the fact that it was labeled from *a*. Similarly, vertex *d* is labeled "*a*, 1."
  - Next, we circle vertex *d* and update the label of the vertex e adjacent to *d* (see the next Figure). Then we circle vertex *b* and update the labels of vertices *c* and e (see the next Figure). Next, we circle vertex e and update the label of vertex *z* (see the last Figure).
  - At this point, we may circle *z*, so the algorithm terminates. The length of a shortest path from *a* to *z* is 4. Starting at *z*, we can retrace the labels to find the shortest path (*a*, *d*, e, *z*).



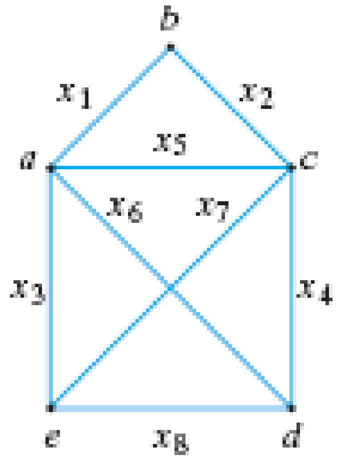
a 0 b ∞ c ∞ z ∞ d ∞ e ∞
3 2 1 2 1 2 1
a 0 b a, 2 c ∞ z ∞ d a, 1 e ∞
3 2 1 2 1 2 1
a 0 b a, 2 c ∞ z ∞ d a, 1 e d, 2
3 2 1 2 1 2 1
a 0 b a, 2 c b, 5 z ∞ d a, 1 e d, 2
3 2 1 2 1 2 1
a 0 b a, 2 c b, 5 z e, 4 d a, 1 e d, 2
3 2 1 2 1 2 1

PRACTICE

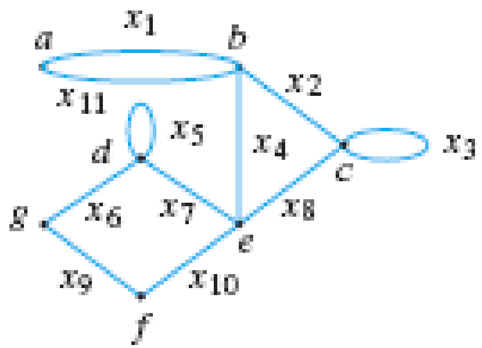
PRACTICE 1

- *In Exercises below, write the adjacency and incidence matrices of each graph.*

1.



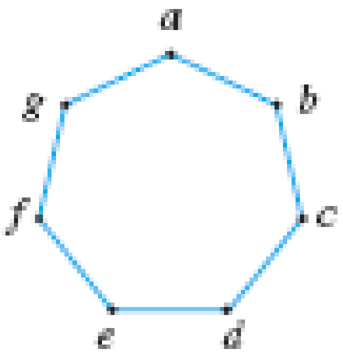
2.



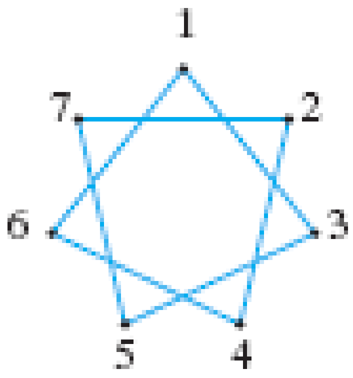
PRACTICE 2

- *In Exercises below, prove that the graphs G_1 and G_2 are isomorphic.*

1.

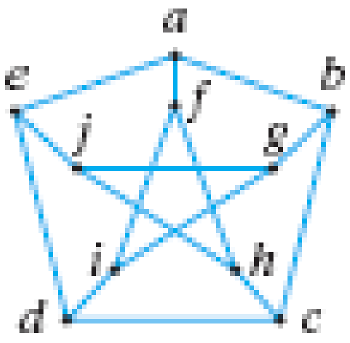


G_1

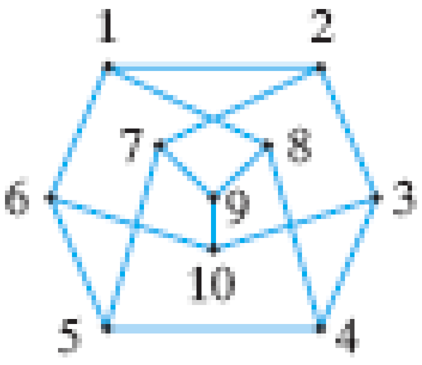


G_2

2.



G_1

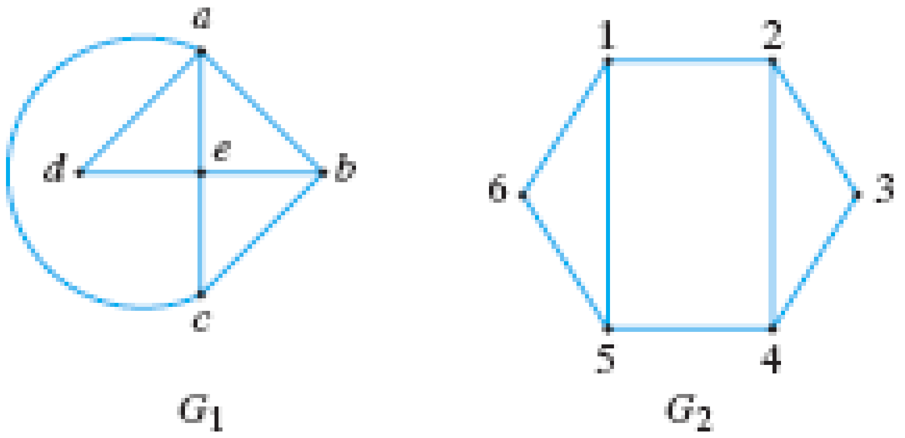


G_2

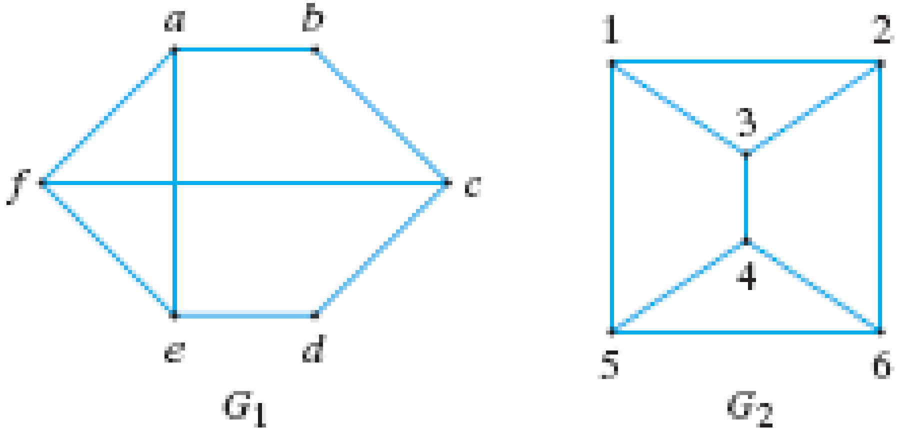
PRACTICE 3

- *In Exercises below, prove that the graphs G_1 and G_2 are not isomorphic.*

1.



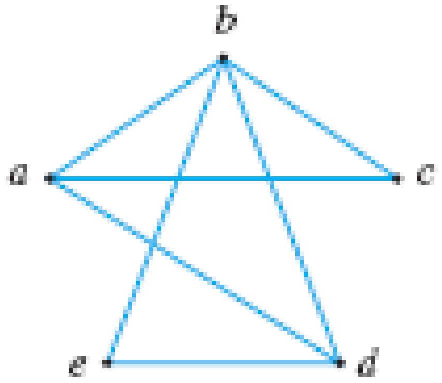
2.



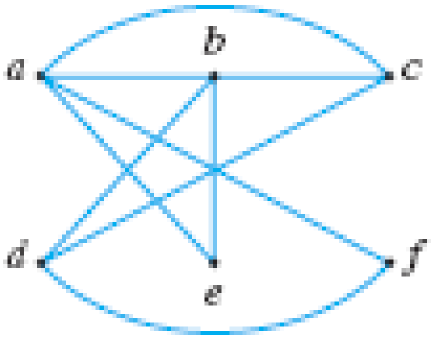
PRACTICE 4

- *In Exercises below, show that each graph is planar by redrawing it so that no edges cross.*

1.



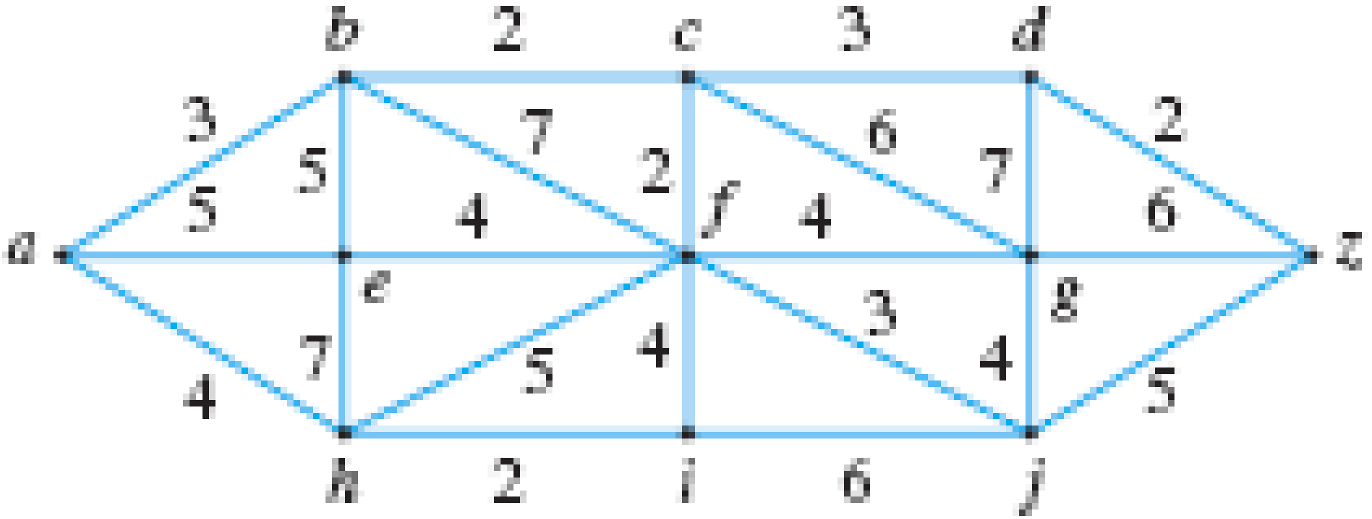
2.



PRACTICE 5

- *In Exercises below, find the length of a shortest path and a shortest path between each pair of vertices in the weighted graph.*

1. a, f
2. b, j



NEXT WEEK'S OUTLINE

- Trees Terminologies
- Spanning Trees
- Binary Trees

REFERENCES

- Johnsonbaugh, R., 2005, *Discrete Mathematics*, New Jersey: Pearson Education, Inc.
- Rosen, Kenneth H., 2005, *Discrete Mathematics and Its Applications*, 6th edition, McGraw-Hill.
- Hansun, S., 2021, *Matematika Diskret Teknik*, Deepublish.
- Lipschutz, Seymour, Lipson, Marc Lars, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, McGraw-Hill.
- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

INFORMATIKA
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Misi

1. Menyelenggarakan pembelajaran dengan teknologi dan kurikulum terbaik serta didukung tenaga pengajar profesional.
2. Melaksanakan kegiatan penelitian di bidang Informatika untuk memajukan ilmu dan teknologi Informatika.
3. Melaksanakan kegiatan pengabdian kepada masyarakat berbasis ilmu dan teknologi Informatika dalam rangka mengamalkan ilmu dan teknologi Informatika.