PROGRAM STUDI TEKNIK KOMPUTER
FAKULTAS TEKNIK DAN INFORMATIKA
UNIVERSITAS MULTIMEDIA NUSANTARA
SEMESTER GANJIL TAHUN AJARAN 2024/2025

CE 121 – LINEAR ALGEBRA

Pertemuan 12:
Nilai dan Vektor Eigen

Firstka Helianta MS, S.Si., M.Si

Capaian Pembelajaran Mingguan Mata Kuliah (Sub-CPMK)

1. Mahasiswa dapat mencari nilai eigen dan vektor eigen.

SuB-Pokok Bahasan

- Nilai eigen
- Vektor eigen
- Basis ruang eigen

Nilai dan Vektor Eigen

Aplikasi:

Masalah getaran, konstruksi bangunan, menentukan keadaan energi dari suatu atom, gerak harmonik, campuran bahan, rantai Markov, dll.

Definisi

Misalkan A adalah matriks berukuran $n \times n$. Skalar λ disebut nilai eigen atau nilai karakteristik dari A jika terdapat vektor taknol $\boldsymbol{X}$, sehingga

$$\boldsymbol{AX} = \boldsymbol{\lambda X}$$

Vektor $\boldsymbol{X}$ disebut vektor eigen atau vektor karakteristik yang berpadanan dengan λ.

Nilai Eigen

Misalkan $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ dan $X = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Maka X adalah vektor eigen yang berpadanan dengan nilai eigen $\lambda = 3$, karena

$$AX = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda X$$

Nilai Eigen

1. Persamaan $AX = \lambda X$ dapat dituliskan sebagai
$$AX - \lambda X = 0$$
$$(A - \lambda I)X = 0$$
λ adalah nilai eigen dari A jika dan hanya jika $(A - \lambda I)X = 0$ mempunyai solusi taktrivial.
2. Berarti solusi SPL homogen $(A - \lambda I)X = 0$, yaitu ruang nol $N(A - \lambda I)X$, tidak sama dengan $\{0\}$.
3. Berarti: $A - \lambda I$ adalah matriks singular
Determinan $(A - \lambda I) = |A - \lambda I| = 0$

Contoh Nilai dan Vektor Eigen

Tentukan nilai eigen dan vektor eigen dari:

a) $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ $\quad$ $Latihan\, A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

b) $B = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$

Contoh Nilai dan Vektor Eigen

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

- **Nilai Eigen**

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) - (8)(0) = 0$$

$$(3-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 3 \cup \lambda_2 = -1$$

Nilai Eigen matriks A

$$\boldsymbol{\lambda_1 = 3 \cup \lambda_2 = -1}$$

Vektor Eigen

- $\boldsymbol{\lambda_1 = 3}$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-3 & 0 \\ 8 & -1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 4x_2 = 0$$

$$x_2 = 2x_1$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- $\boldsymbol{\lambda_2 = -1}$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-(-1) & 0 \\ 8 & -1-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 0 \\ 8 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_1 &= 0 \\ 8x_1 &= 0 \end{aligned}\right\} x_1 = 0 \quad \& \quad x_2 \epsilon R$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contoh Nilai dan Vektor Eigen

$$A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

Nilai Eigen $\lambda_1 = 3 \longrightarrow$ Vektor Eigen $X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Nilai Eigen $\lambda_2 = -1 \longrightarrow$ Vektor Eigen $X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Contoh Nilai dan Vektor Eigen

$$B = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

- **Nilai Eigen**

$$|B - \lambda I| = 0$$

$$\left| \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-\lambda)(-1-\lambda) + (-1)(-2)(2) + (-2)(-1)(2)$$

$$-(2)(-\lambda)(-2) - (-1)(-2)(3-\lambda) - (-1-\lambda)(-1)(2) = 0$$

$$-\lambda^3 + 2\lambda^2 + 3\lambda + 4 + 4 - 4\lambda - 6 + 2\lambda - 2 - 2\lambda = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda - 1)(\lambda - 1) = 0$$

$$\boxed{\lambda_1 = 0} \cup \boxed{\lambda_2 = 1} \cup \boxed{\lambda_3 = 1}$$

Vektor Eigen

- $\boldsymbol{\lambda_2 = \lambda_3 = 1}$

$$(B-\lambda I)X = 0$$

$$\begin{pmatrix} 3-\lambda & -1 & -2 \\ 2 & -\lambda & -2 \\ 2 & -1 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-1 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{cc} \begin{pmatrix} 2 & -1 & -2 \\ 2 & -1 & -2 \\ 2 & -1 & -2 \end{pmatrix} & \begin{array}{l} \\ b_2 - b_1 \\ b_3 - b_1 \end{array} \end{array} \quad \begin{pmatrix} 2 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Baris 1 : $2x_1 - x_2 - 2x_3 = 0$

$$\longrightarrow \quad x_2 = 2x_1 - 2x_3$$

$$X_{2,3} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 - 2x_3 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ dan $X_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

Contoh Nilai dan Vektor Eigen

$$B = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

Nilai Eigen $\lambda_1 = 0$ $\longrightarrow$ Vektor Eigen $X_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Nilai Eigen $\lambda_2 = 1$ $\longrightarrow$ Vektor Eigen $X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Nilai Eigen $\lambda_3 = 1$ $\longrightarrow$ Vektor Eigen $X_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

Latihan

Tentukan nilai dan vektor eigen dari matriks berikut:

1.) $\begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix}$

2.) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

3.) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

Contoh Nilai dan Vektor Eigen

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix}$$

- **Nilai Eigen**

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 2 \\ 8 & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) - (8)(2) = 0$$

$$\lambda^2 - 16 = 0$$

$$(\lambda + 4)(\lambda - 4) = 0$$

$$\lambda_1 = -4 \cup \lambda_2 = 4$$

Nilai Eigen matriks A

$$\boldsymbol{\lambda_1 = -4 \cup \lambda_2 = 4}$$

Vektor Eigen

- $\boldsymbol{\lambda_1 = -4}$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} -\lambda & 2 \\ 8 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -(-4) & 2 \\ 8 & -(-4) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 8x_1 + 4x_2 &= 0 \end{aligned}\right\} x_2 = -2x_1$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- $\boldsymbol{\lambda_2 = 4}$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} -\lambda & 2 \\ 8 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -(4) & 2 \\ 8 & -(4) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 8 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 8x_1 - 4x_2 &= 0 \end{aligned}\right\} x_2 = 2x_1$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Contoh Nilai dan Vektor Eigen

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix}$$

Nilai Eigen $\lambda_1 = -4 \longrightarrow$ Vektor Eigen $X_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Nilai Eigen $\lambda_2 = 4 \longrightarrow$ Vektor Eigen $X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Contoh Nilai dan Vektor Eigen

$$C = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

- **Nilai Eigen**

$$|C - \lambda I| = 0$$

$$\left| \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(3-\lambda) + (-1)(-1)(0) + (0)(-1)(-1)$$

$$-(0)(2-\lambda)(0) - (-1)(-1)(3-\lambda) - (3-\lambda)(-1)(-1) = 0$$

$$(3-\lambda)(2-\lambda)(3-\lambda) - (3-\lambda) - (3-\lambda) = 0$$

$$-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$$

$$(\lambda - 1)(\lambda^2 - 7\lambda + 12) = 0$$

$$(\lambda - 1)(\lambda - 3)(\lambda - 4) = 0$$

$$\boxed{\lambda_1 = 1} \cup \boxed{\lambda_2 = 2} \cup \boxed{\lambda_3 = 4}$$

Terima Kasih

Sampai Jumpa
di Pertemuan Selanjutnya