IF120 Discrete Mathematics

05 Relations

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REVIEW

- Functions
- Sequences and Strings
- Series

OUTLINE

- Relations
- Equivalence Relations
- Matrices of Relations

Relations

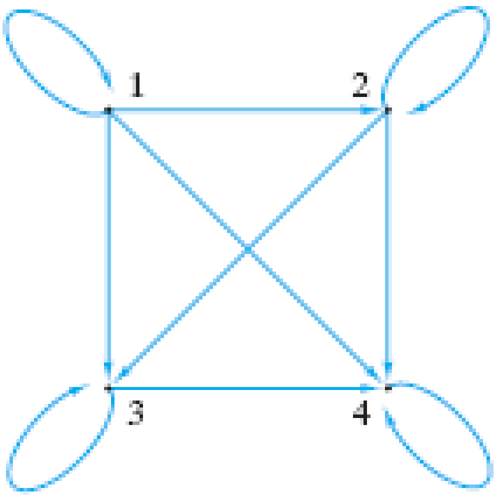
- A *(binary)* **relation** *R from a set X to a set Y* is a subset of the Cartesian product $X \times Y$.
- If $(x, y) \in R$, we write $x\ R\ y$ and say that *x **is related to** y*.
- If $X = Y$, we call R a *(binary) relation on X*.
- A function (remember W04) is a special type of relation.
- A function *f* from *X* to *Y* is a relation from *X* to *Y* having the properties:

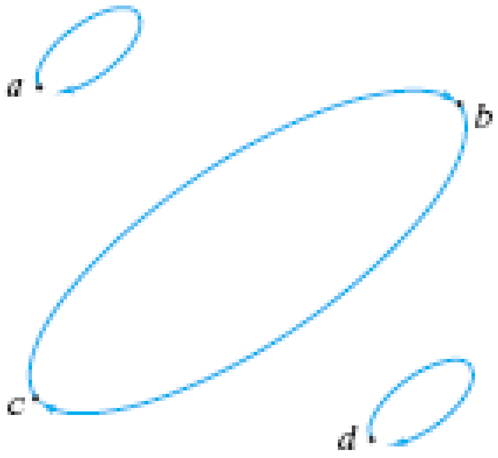
a. The domain of *f* is equal to *X*.

b. For each $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in f$.

Relations

- Let R be the relation on X = {1, 2, 3, 4} defined by $(x, y) \in R$ if $x \leq y, x, y \in X$. Then R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}.





- The relation R on X = {a, b, c, d} given by the digraph of Figure above is R = {(a, a), (b, c), (c, b), (d, d)}.

Relations' Properties: Reflexive

- A relation R on a set X is **reflexive** if $(x, x) \in R$ for every $x \in X$.

❑The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$, is reflexive because for each element $x \in X$, $(x, x) \in R$; specifically, (1, 1), (2, 2), (3, 3), and (4, 4) are each in R. The digraph of a reflexive relation has a loop at every vertex. Notice that the digraph of this relation (see Figure 1 on the slide before) has a loop at every vertex.

❑The relation

$$R = \{(a, a), (b, c), (c, b), (d, d)\}$$

on $X = \{a, b, c, d\}$ is not reflexive. For example, $b \in X$, but $(b, b) \notin R$. That this relation is not reflexive can also be seen by looking at its digraph (see Figure 2 on the slide before); vertex b does not have a loop.

Relations' Properties: Symmetric

- A relation *R* on a set *X* is **symmetric** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.
- In symbols, a relation *R* is symmetric if
$$\forall x \forall y[(x, y) \in R] \rightarrow [(y, x) \in R].$$
- Thus *R* is not symmetric if
$$\neg[\forall x \forall y[(x, y) \in R] \rightarrow [(y, x) \in R]].$$
  or, equivalently,
$$\exists x \exists y[[(x, y) \in R] \wedge [(y, x) \notin R]].$$
- In words, a relation *R* is **not symmetric** if there exist *x* and *y* such that (*x*, *y*) is in *R* and (*y*, *x*) is not in *R*.

- The relation *R* on *X* = {1, 2, 3, 4} defined by $(x, y) \in R$ if $x \leq y, x, y \in X$, is not symmetric. For example, $(2, 3) \in R$, but $(3, 2) \notin R$. The digraph of this relation has a directed edge from 2 to 3, but there is no directed edge from 3 to 2.

Relations' Properties: Antisymmetric

- A relation R on a set X is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

❑The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x \leq y, x, y \in X$, is antisymmetric because for all x, y, if $(x, y) \in R$ (i.e., $x \leq y$) and $(y, x) \in R$ (i.e., $y \leq x$), then $x = y$.

- In symbols, a relation R is antisymmetric if

$$\forall x \forall y [(x, y) \in R \wedge (y, x) \in R] \rightarrow [x = y].$$

- Thus R is **not antisymmetric** if

$$\neg[\forall x \forall y [(x, y) \in R \wedge (y, x) \in R] \rightarrow [x = y]].$$

is equivalent to

$$\exists x \exists y [(x, y) \in R \wedge (y, x) \in R \wedge (x \neq y)].$$

❑The relation

$$R = \{(a, a), (b, c), (c, b), (d, d)\}$$

on $X = \{a, b, c, d\}$ is not antisymmetric because both (b, c) and (c, b) are in R.

Relations' Properties: Transitive

- A relation R on a set X is **transitive** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.
- In symbols, a relation R is transitive if
$$\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R] \rightarrow [(x, z) \in R].$$
- Thus R is **not transitive** if
$$\neg[\forall x \forall y \forall z[(x, y) \in R \wedge (y, z) \in R] \rightarrow [(x, z) \in R]].$$
  or, equivalently,
$$\exists x \exists y \exists z[(x, y) \in R \wedge (y, z) \in R \wedge (x, z) \notin R].$$

❑The relation
$$R = \{(a, a), (b, c), (c, b), (d, d)\}$$
on $X = \{a, b, c, d\}$ is not transitive. For example, (b, c) and (c, b) are in R, but (b, b) is not in R. Notice that in the digraph of this relation there are directed edges from b to c and from c to b, but there is no directed edge from b to b.

Partial Orders

- A relation *R* on a set *X* is a **partial order** if *R* is reflexive, antisymmetric, and transitive.

- Since the relation *R* defined on the positive integers by

$$(x, y) \in R$$

  if *x* divides *y* is reflexive, antisymmetric, and transitive, *R* is a partial order.

- Suppose that *R* is a partial order on a set *X*.
- If $x, y \in X$ and either $x \preccurlyeq y$ or y $\preccurlyeq x$, we say that *x* and *y* are **comparable.**
- If $x, y \in X$ and $x \npreceq y$ and $y \npreceq x$, we say that *x* and *y* are **incomparable.**
- If every pair of elements in *X* is comparable, we call *R* a **total order.**

Inverse

- Let R be a relation from X to Y .
- The **inverse** of R, denoted R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

❑ If we define a relation R from X = {2, 3, 4} to Y = {3, 4, 5, 6, 7} by

$$(x, y) \in R \; if \; x \; divides \; y,$$

we obtain

$$R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}.$$

The inverse of this relation is

$$R^{-1} = \{(4, 2), (6, 2), (3, 3), (6, 3), (4, 4)\}.$$

In words, we might describe this relation as "is divisible by."

Composition

- Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z.
- The **composition** of R_1 and R_2, denoted $R_2 \circ R_1$, is the relation from X to Z defined by
$$R_2 \circ R_1 = \{(x,z) | (x,y) \in R_1 \; and \; (y,z) \in R_2 \; for \; some \; y \in Y\}.$$

❑ The composition of the relations
$$R_1 = \{(1,2),(1,6),(2,4),(3,4),(3,6),(3,8)\}$$
and
$$R_2 = \{(2,u),(4,s),(4,t),(6,t),(8,u)\}$$
is
$$R_2 \circ R_1 = \{(1,u),(1,t),(2,s),(2,t),(3,s),(3,t),(3,u)\}.$$
For example, $(1,u) \in R_2 \circ R_1$ because $(1,2) \in R_1$ and $(2,u) \in R_2$.

Equivalence Relations

- A relation that is reflexive, symmetric, and transitive on a set X is called an **equivalence relation** on X.

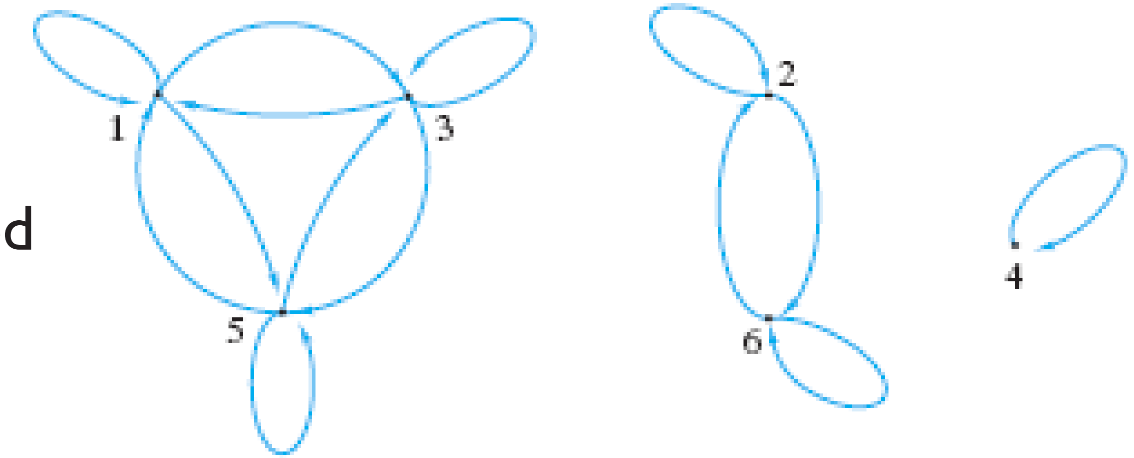
❑ Consider the partition

$$S = \{\{1,3,5\},\{2,6\},\{4\}\}$$

of $X = \{1, 2, 3, 4, 5, 6\}$. The complete relation is

$$R = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,2),(2,6),(6,2),(6,6),(4,4)\}.$$

We see that R is reflexive (there is a loop at every vertex), symmetric (for every directed edge from v to w, there is also a directed edge from w to v), and transitive (if there is a directed edge from x to y and a directed edge from y to z, there is a directed edge from x to z).



Equivalence Relations

- Consider the relation

$R = \{(1,1),(1,3),(1,5),(2,2),(2,4),(3,1),(3,3),(3,5),(4,2),(4,4),(5,1),(5,3),(5,5)\}$ on $\{1, 2, 3, 4, 5\}$. The relation is reflexive because $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \in R$. The relation is symmetric because whenever (x, y) is in R, (y, x) is also in R. Finally, the relation is transitive because whenever (x, y) and (y, z) are in R, (x, z) is also in R. Since R is reflexive, symmetric, and transitive, R is an equivalence relation on $\{1, 2, 3, 4, 5\}$.

- The relation R on $X = \{1, 2, 3, 4\}$ defined by $(x, y) \in R \; if \; x \leq y, x, y \in X$, is not an equivalence relation because R is not symmetric. [For example, $(2,3) \in R$, but $(3,2) \notin R$.] The relation R is reflexive and transitive.

- The relation

$$R = \{(a,a),(b,c),(c,b),(d,d)\}$$

on $X = \{a, b, c, d\}$ is not an equivalence relation because R is neither reflexive nor transitive. [It is not reflexive because, for example, $(b,b) \notin R$. It is not transitive because, for example, (b, c) and (c, b) are in R, but (b, b) is not in R.]

Matrices of Relations

- A matrix is a convenient way to represent a relation R from X to Y .
- Such a representation can be used by a computer to analyze a relation.
- We label the rows with the elements of X (in some arbitrary order), and we label the columns with the elements of Y (again, in some arbitrary order).
- We then ,set the entry in row x and column y to 1 if $x\ R\ y$ and to 0 otherwise.
- This matrix is called the **matrix of the relation** R (relative to the orderings of X and Y).

❑The matrix of the relation

$$R = \{(1, b), (1, d), (2, c), (3, c), (3, b), (4, a)\}$$

from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$ relative to the orderings 1, 2, 3, 4 and a, b, c, d is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Matrices of Relations

- The matrix of the relation R from {2, 3, 4} to {5, 6, 7, 8}, relative to the orderings 2, 3, 4 and 5, 6, 7, 8, defined by

$$x\ R\ y\ if\ \ x\ divides\ y$$

is

$$\begin{array}{c c} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

- The matrix of the relation

$$R\ =\ \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$$

on $\{a, b, c, d\}$, relative to the ordering a, b, c, d, is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

Matrices of Relations

- Theorem 5.1: *Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Choose orderings of X, Y , and Z. Let A_1 be the matrix of R_1 and let A_2 be the matrix of R_2 with respect to the orderings selected. The matrix of the relation $R_2 \circ R_1$ with respect to the orderings selected is obtained by replacing each nonzero term in the matrix product $A_1 A_2$ by* 1.
- Theorem 5.1 provides a quick test for determining whether a relation is transitive. If *A* is the matrix of *R* (relative to some ordering), we compute A^2. We then compare *A* and A^2. The relation *R* is transitive if and only if whenever entry *i*, *j* in A^2 is nonzero, entry *i*, *j* in *A* is also nonzero. The reason is that entry *i*, *j* in A^2 is nonzero if and only if there are elements (*i*, *k*) and (*k*, *j*) in *R*. Now *R* is transitive if and only if whenever (*i*, *k*) and (*k*, *j*) are in *R*, then (*i*, *j*) is in *R*. But (*i*, *j*) is in *R* if and only if entry *i*, *j* in *A* is nonzero. Therefore, *R* is **transitive** if and only if whenever entry *i*, *j* in A^2 is nonzero, entry *i*, *j* in *A* is also nonzero.

Matrices of Relations

- The matrix of the relation

$$R = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$$

  on $\{a, b, c, d\}$, relative to the ordering a, b, c, d, is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  Its square is

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

  We see that whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero. Therefore, R is transitive.

Matrices of Relations

- The matrix of the relation

$$R = \{(a,a),(b,b),(c,c),(d,d),(a,c),(c,b)\}$$

on $\{a, b, c, d\}$, relative to the ordering a, b, c, d, is

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its square is

$$A^2 = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The entry in row 1, column 2 of A^2 is nonzero, but the corresponding entry in A is zero. Therefore, R is *not* transitive.

PRACTICE

PRACTICE 1

- *In Exercises below, write the relation as a table.*

1. *R* = {(Roger, Music), (Pat, History), (Ben, Math), (Pat, PolySci)}
2. The relation *R* on {1, 2, 3, 4} defined by $(x, y) \in R \ if \ x^2 \geq y$
3. The relation *R* from the set *X* of planets to the set *Y* of integers defined by $(x, y) \in R$ if x is in position y from the sun (nearest the sun being in position 1, second nearest the sun being in position 2, and so on)

PRACTICE 2

- *In Exercises below, determine whether each relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.*

1. $(x, y) \in R \; if \; x = y^2$.
2. $(x, y) \in R \; if \; x > y$.
3. $(x, y) \in R \; if \; x = y$.

PRACTICE 3

- *In Exercises below, determine whether the given relation is an equivalence relation on* {1, 2, 3, 4, 5}.

1. $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1)\}$
2. $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(3,4),(4,3)\}$
3. $\{(1,1),(2,2),(3,3),(4,4)\}$

PRACTICE 4

- *In Exercises below, find the matrix of the relation R on X relative to the ordering given. Then, Use the matrix of the relation to test for transitivity.*

1. $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$; ordering of X: 1, 2, 3, 4, 5
2. R as in Exercise 3; ordering of X: 5, 3, 1, 2, 4
3. $R = \{(x, y) \mid x < y\}$; ordering of X: 1, 2, 3, 4

NEXT WEEK'S OUTLINE

- Number System
- Binary, Octal, and Hexadecimal System Numbers
- Divisors and Prime Numbers
- Euclid Algorithm

REFERENCES

- Johnsonbaugh, R., 2005, *Discrete Mathematics*, New Jersey: Pearson Education, Inc.
- Rosen, Kenneth H., 2005, *Discrete Mathematics and Its Applications*, 6th edition, McGraw-Hill.
- Hansun, S., 2021, *Matematika Diskret Teknik*, Deepublish.
- Lipschutz, Seymour, Lipson, Marc Lars, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, McGraw-Hill.
- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

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