PROGRAM STUDI TEKNIK KOMPUTER
FAKULTAS TEKNIK DAN INFORMATIKA
UNIVERSITAS MULTIMEDIA NUSANTARA
SEMESTER GANJIL TAHUN AJARAN 2024/2025

UMN
UNIVERSITAS
MULTIMEDIA
NUSANTARA

CE 121 – LINEAR ALGEBRA

Pertemuan 8 :
Kombinasi Linier

Firstka Helianta MS, S.Si., M.Si

Capaian Pembelajaran Mingguan Mata Kuliah (Sub-CPMK)

1. Mahasiswa dapat mencari himpunan span dan melakukan uji independensi linier untuk matrik.

Sub-Pokok Bahasan

- Himpunan Span
- Kombinasi Linear

Kombinasi Linear

Kombinasi Linier (k.l): w k.l. $S = \{v_1, v_2, v_3, \ldots, v_r\}$ jika

$w = k_1v_1 + k_2v_2 + k_3v_3 \ldots + k_rv_r$ $k_1, k_2, k_3, \ldots, k_r$ terdefinisi /ada nilainya

Independensi Linier:

$$S = \{v_1, v_2, v_3, \ldots, v_r\}$$

disebut himpunan bebas linier / tidak-bergantung linier (***linearly independent***) jika solusi Sistem Persamaan Linier Homogen

$$k_1v_1 + k_2v_2 + k_3v_3 \ldots + k_rv_r = 0$$

adalah solusi trivial $k_1, k_2, k_3, \ldots, k_r = 0$

Kombinasi Linear

Dependensi Linier:

$$S = \{v_1, v_2, v_3, \ldots, v_r\}$$

disebut himpunan tidak-bebas linier / bergantung linier (*linearly dependent*) jika solusi Sistem Persamaan Linier Homogen

$$k_1 v_1 + k_2 v_2 + k_3 v_3 \ldots + k_r v_r = 0$$

adalah solusi non-trivial $k_1, k_2, k_3, \ldots, k_r = 0$

dan ada $k_j \neq 0$ (j = 1 .. r)

Kombinasi Linear

Diketahui : himpunan $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r\}$

Ditanyakan: apakah S *linearly independent* atau *linearly dependent*?

Jawab:

1. Bentuk SPL Homogen $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 \ldots + k_r\mathbf{v}_r = \mathbf{0}$
2. Tentukan solusinya
3. Jika solusinya trivial $k_1, k_2, k_3, \ldots, k_r = 0$

   maka S ***linearly independent***
4. Jika solusinya non-trivial maka S ***linearly dependent***

Kombinasi Linear

Contoh(1):

Tentukan apakah u = (1,-2,3), v = (5,6,-1), w = (3,2,1) saling bebas linier

$$k_1 u + k_2 v + k_3 w = 0$$

$$k_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + k_2 \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix} + k_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Didapatkan SPL :

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Kombinasi Linear

Dengan melakukan OBE, didapatkan matrik sebagai berikut :

$$\left[\begin{array}{ccc|c} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

sehingga didapatkan penyelesaian :

$k_2 + 0.5\, k_3 = 0 \;\rightarrow\; k_2 = -0{,}5k_3$

$k_1 + 0.5\, k_3 = 0 \;\rightarrow\; k_1 = -0{,}5k_3$

Jadi, andaikan k3 = t, maka k2 = -0.5 t dan k1 = -0.5 t ⟶

Punya banyak penyelesaian ⟶ tidak bebas linier

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} b_2 + 2b_1 \\ b_3 - 3b_1 \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{array}\right] b_3 + b_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] b_2 \times \left(\frac{1}{16}\right) \quad \left[\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] b_1 - 5b_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{2}u - \frac{1}{2}v + w = 0$$

$k_1 - k_2 = 0 \quad \rightarrow k_1 = k_2$

$k_2 + 0{,}5k_3 = 0 \quad \rightarrow k_3 = -2k_2$

Kombinasi Linear

Tentukan apakah vektor-vektor berikut bebas linier atau terpaut linier?

1. $U = (1,1,2);\ V = (1,0,1);\ W = (2,1,3)$

2. $A = (1,-2,1);\ B = (2,2,1);\ C = (-1,1,-1)$

3. $K = (3,4,5);\ L = (2,9,2);\ M = (4,18,4)$

Kombinasi Linear

1. $U = (1,1,2);\ V = (1,0,1);\ W = (2,1,3)$

$$k_1\begin{pmatrix}1\\1\\2\end{pmatrix} + k_2\begin{pmatrix}1\\0\\1\end{pmatrix} + k_3\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$k_1 + k_2 + 2k_3 = 0$$
$$k_1 + k_3 = 0$$
$$2k_1 + k_2 + 3k_3 = 0$$

$$\begin{bmatrix}1 & 1 & 2\\1 & 0 & 1\\2 & 1 & 3\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\1 & 0 & 1 & 0\\2 & 1 & 3 & 0\end{array}\right] \begin{array}{c} \\ b_2 - b_1\\ b_3 - 2b_1\end{array} \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\0 & -1 & -1 & 0\\0 & -1 & -1 & 0\end{array}\right] \begin{array}{c} \\ \\ b_3 - b_2\end{array}$$

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\0 & -1 & -1 & 0\\0 & 0 & 0 & 0\end{array}\right] \begin{array}{c} \\ -b_2\\ \\ \end{array} \quad \left[\begin{array}{ccc|c}1 & 1 & 2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right] \begin{array}{c}b_1 - b_2\\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$k_1 + k_3 = 0 \rightarrow k_1 = -k_3$

$k_2 + k_3 = 0 \rightarrow k_2 = -k_3$

Punya Banyak Solusi $\longrightarrow$ **Tidak Bebas Linear**

Kombinasi Linear

2. $A = (1,-2,1);\ B = (2,2,1);\ C = (-1,1,-1)$

$$k_1\begin{pmatrix}1\\-2\\1\end{pmatrix} + k_2\begin{pmatrix}2\\2\\1\end{pmatrix} + k_3\begin{pmatrix}-1\\1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$k_1 + 2k_2 - k_3 = 0$$
$$-2k_1 + 2k_2 + k_3 = 0$$
$$k_1 + k_2 - k_3 = 0$$

$$\begin{bmatrix}1 & 2 & -1\\-2 & 2 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -1\\-2 & 2 & 1\\1 & 1 & -1\end{bmatrix}\begin{array}{l}\\b_2 + 2b_1\\b_3 - b_1\end{array} \quad \begin{bmatrix}1 & 2 & -1\\0 & 6 & -1\\0 & -1 & 0\end{bmatrix}\begin{array}{l}\\\frac{1}{6}b_2\\\\\end{array}$$

$$\begin{bmatrix}1 & 2 & -1\\0 & 1 & -1/6\\0 & -1 & 0\end{bmatrix}\begin{array}{l}\\\\b_3 + b_2\end{array} \quad \begin{bmatrix}1 & 2 & -1\\0 & 1 & -1/6\\0 & 0 & -1/6\end{bmatrix}\begin{array}{l}b_1 - 6b_3\\b_2 - b_3\\\\\end{array}$$

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & -1/6\end{bmatrix}\begin{array}{l}b_1 - 2b_2\\\\-6b_3\end{array} \quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \quad \begin{array}{l}k_1 = 0\\k_2 = 0\\k_3 = 0\end{array}$$

Punya Satu Solusi $\longrightarrow$ **Bebas Linear**

Himpunan Span

Misalkan $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ adalah vektor-vektor dalam suatu ruang vektor V. Himpunan semua kombinasi linear dari $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ disebut **rentang (span)** dari vektor-vektor $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ dan dituliskan sebagai

$$\text{Rentang}\,(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)\,.$$

Himpunan Span

Ilustrasi

Misalkan $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ dan $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, maka

$$x = 2v_1 + v_2$$

$$\text{Rentang}(\mathbf{v}_1, \mathbf{v}_2) = \left\{ \mathbf{x} \in \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

$\alpha = 2$ $\beta = 1$

Himpunan Span

Definisi:

Himpunan $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ disebut **himpunan perentang** untuk V jika dan hanya jika setiap vektor di V dapat dituliskan sebagai kombinasi linear dari $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Jika $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ adalah himpunan perentang untuk V, maka dapat dikatakan bahwa $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **merentang** V.

Himpunan Span

1. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ dengan $\mathbf{e}_1 = (1,0,0)^T$, $\mathbf{e}_2 = (0,1,0)^T$ $\mathbf{e}_3 = (0,0,1)^T$ merupakan himpunan perentang untuk R^3 karena sembarang vektor $(a,b,c)^T$ di R^3 dapat dituliskan sebagai

$$(a,b,c)^T = a\,(1,0,0)^T + b\,(0,1,0)^T + c\,(0,0,1)^T.$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Himpunan Span

2. $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,2,5)^T\}$ juga merupakan himpunan perentang untuk R^3 karena sembarang vektor $(a,b,c)^T$ di R^3 dapat dituliskan sebagai

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

Himpunan Span

4. Himpunan $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \end{pmatrix} \right\}$ tidak merentang R^2 (*periksa!*).

$$\begin{pmatrix} a \\ b \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$-\alpha + \beta + 2\gamma = a$$
$$2\alpha - 2\beta - 4\gamma = b$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & a \\ 2 & -2 & -4 & b \end{array}\right] b_2 + 2b_1 \quad \left[\begin{array}{ccc|c} -1 & 1 & 2 & a \\ 0 & 0 & 0 & b + 2a \end{array}\right]$$

$$-\alpha + \beta + 2\gamma = a$$
$$0 + 0 + 0 = b + 2a$$

Himpunan Span

1. Periksa apakah himpunan-himpunan yang diberikan merentang R^2 :

(a) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$ (b) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right\}$

Himpunan Span

$\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}3\\2\end{pmatrix}\right\}$ *Pilih vektor di* $R^2 = \begin{pmatrix}a\\b\end{pmatrix}$

$$\begin{pmatrix}a\\b\end{pmatrix} = \alpha\begin{pmatrix}2\\1\end{pmatrix} + \beta\begin{pmatrix}3\\2\end{pmatrix}$$

$$2\alpha + 3\beta = a$$
$$\alpha + 2\beta = b$$

$$\begin{pmatrix}2 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\left[\begin{array}{cc|c}2 & 3 & a\\1 & 2 & b\end{array}\right] b_1 \leftrightarrow b_2 \quad \left[\begin{array}{cc|c}1 & 2 & b\\2 & 3 & a\end{array}\right] b_2 - 2b_1$$

$$\left[\begin{array}{cc|c}1 & 2 & b\\0 & -1 & a - 2b\end{array}\right] -b_2 \quad \left[\begin{array}{cc|c}1 & 2 & b\\0 & 1 & -a + 2b\end{array}\right] b_1 - 2b_2$$

$$\left[\begin{array}{cc|c}1 & 0 & 2a - 3b\\0 & 1 & -a + 2b\end{array}\right] \quad \begin{aligned}\alpha &= 2a - 3b\\ \beta &= -a + 2b\end{aligned}$$

$\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}2\\3\end{pmatrix}\right\}$ merentang di R^2

pilih $\begin{pmatrix}1\\1\end{pmatrix}$ $\quad \alpha = 2 - 3 = -1$

$$\beta = -1 + 2 = 1$$

$$\begin{pmatrix}1\\1\end{pmatrix} = (-1)\begin{pmatrix}2\\1\end{pmatrix} + (1)\begin{pmatrix}2\\3\end{pmatrix}$$

$$x = -v_1 + v_2$$

Himpunan Span

$\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right\} \textit{Pilih vektor di } R^2 = \begin{pmatrix}a\\b\end{pmatrix}$

$$\begin{pmatrix}a\\b\end{pmatrix} = \alpha\begin{pmatrix}2\\1\end{pmatrix} + \beta\begin{pmatrix}1\\-2\end{pmatrix} + \gamma\begin{pmatrix}2\\4\end{pmatrix}$$

$$2\alpha + \beta + 2\gamma = a$$
$$\alpha - 2\beta + 4\gamma = b$$

$$\begin{pmatrix}2 & 1 & 2\\1 & -2 & 4\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}$$

$$\left[\begin{array}{ccc|c}2 & 1 & 2 & a\\1 & -2 & 4 & b\end{array}\right] b_1 \leftrightarrow b_2 \quad \left[\begin{array}{ccc|c}1 & -2 & 2 & b\\2 & 1 & 2 & a\end{array}\right] b_2 - 2b_1$$

$$\left[\begin{array}{ccc|c}1 & -2 & 2 & b\\0 & 5 & -2 & a-2b\end{array}\right]$$

$$\alpha - 2\beta + 2\gamma = b$$
$$5\beta - 2\gamma = a - 2b$$

$\left\{\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\-2\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}\right\}$ tidak merentang di R^2

Himpunan Span

2. Periksa apakah himpunan-himpunan yang diberikan merentang R^3 :

$$\text{(a)} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{(b)} \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

Himpunan Span

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$$ Merentang di R^3

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = k_1\begin{pmatrix}1\\0\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\\1\end{pmatrix} + k_3\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1 & 0 & 1 & a\\0 & 1 & 0 & b\\0 & 1 & 1 & c\end{array}\right) b_3 - b_2$$

$$\left(\begin{array}{ccc|c}1 & 0 & 1 & a\\0 & 1 & 0 & b\\0 & 0 & 1 & c-b\end{array}\right)^{b_1 - b_3} \left(\begin{array}{ccc|c}1 & 0 & 0 & a+b-c\\0 & 1 & 0 & b\\0 & 0 & 1 & c-b\end{array}\right)$$

$$k_1 = a + b - c = 1 + 2 - 3 = 0$$

$$k_2 = b \qquad k_2 = 2$$

$$k_3 = c - b \qquad k_3 = 3 - 2 = 1$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Terima Kasih

Sampai Jumpa
di Pertemuan Selanjutnya