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CE 121 – LINEAR ALGEBRA

Pertemuan 12 :
Diagonalisasi

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Capaian Pembelajaran Mingguan Mata Kuliah (Sub-CPMK)

1. Mahasiswa menghitung diagonalisasi matrik

Sub-Pokok Bahasan

- Diagonalisasi

Diagonalisasi

Teorema 1

Jika λ_1, λ_2,…, λ_n adalah nilai-nilai eigen yang berbeda dari matriks A yang berukuran $n \times n$ dan $X_1, X_2, \ldots, X_n$ adalah vektor-vektor eigen yang bersesuaian, maka vektor-vektor eigen $X_1, X_2, \ldots, X_n$ adalah bebas linear.

Diagonalisasi

- Suatu matriks A berorde $n \times n$ disebut **matriks diagonal** jika $a_{ij} = 0$ untuk $i \neq j$.
- Contoh matriks diagonal

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Diagonalisasi

Definisi: Suatu matriks A berorde $n \times n$ disebut dapat didiagonalkan jika terdapat matriks taksingular X dan suatu matriks diagonal D sehingga

$$\boxed{X^{-1}AX = D}$$

Matriks X dikatakan **mendiagonalkan** A

Diagonalisasi

Teorema 2

Suatu matriks A berorde $n \times n$ dapat didiagonalkan jika dan hanya jika A mempunyai n vektor eigen yang bebas linear.

Diagonalisasi

1. Jika A dapat didiagonalkan, maka

a. vektor-vektor kolom dari matriks pendiagonal $\boldsymbol{X}$ adalah vektor-vektor eigen dari $\boldsymbol{A}$

b. elemen-elemen diagonal $\boldsymbol{D}$ adalah nilai-nilai eigen yang bersesuaian

Diagonalisasi

2. Matriks pendiagonal ***X*** tidak tunggal.

Misalkan diberikan matriks $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$

Nilai eigen dan vektor eigen yang bersesuaian :

$$\lambda_1 = 3 \text{ dengan vektor eigen } X_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1 \text{ dengan vektor eigen } X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Berdasarkan teorema 1, matriks A dapat didiagonalkan dengan matriks pendiagonal :

$$X = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Diagonalisasi

$$X^{-1}AX = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

- Perhatikan posisi nilai eigen pada matriks D dan vektor eigen pada matriks X.
- Jika posisi vektor eigen ditukar, maka akan diperoleh matriks pendiagonal yang lain.
- Selain itu, vektor eigen suatu matriks tidak tunggal, sehingga matriks pendiagonal pun tidak tunggal.

Diagonalisasi

$A_{n\times n}$ mempunyai *n* nilai eigen yang berbeda

Teorema 1

Vektor-vektor eigen *A* bebas linier

Teorema 2

A dapat didiagonalkan

Tetapi, jika nilai eigen *A* ada yang sama maka:

- *A* dapat didiagonalkan, atau *A* tidak dapat didiagonalkan

Diagonalisasi

Misalkan $A = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$. Vektor eigen yang bersesuaian dengan

$$\lambda_1 = 0 \rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 1 \rightarrow X_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \text{ dan } X_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Matriks $X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}$

Untuk mengetahui bahwa matriks A dapat didiagonalkan,

maka perlu di uji vektor-vektor eigen merupakan bebas linier.

Diagonalisasi

Uji Bebas Linier

$$X_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}; X_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}; X_3 = \begin{pmatrix}0\\-2\\1\end{pmatrix}$$

$$k_1X_1 + k_2X_2 + k_3X_3 = 0$$

$$k_1\begin{pmatrix}1\\1\\1\end{pmatrix} + k_2\begin{pmatrix}1\\2\\0\end{pmatrix} + k_3\begin{pmatrix}0\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 0\\1 & 2 & -2\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{array}{cc}\begin{pmatrix}1 & 1 & 0\\1 & 2 & -2\\1 & 0 & 1\end{pmatrix} & \begin{array}{l} \\ b_2 - b_1\\ b_3 - b_1\end{array}\end{array} \rightarrow \begin{array}{cc}\begin{pmatrix}1 & 1 & 0\\0 & 1 & -2\\0 & -1 & 1\end{pmatrix} & \begin{array}{l}b_1 - b_2\\ \\ b_3 + b_2\end{array}\end{array}$$

$$\rightarrow \begin{array}{cc}\begin{pmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & -1\end{pmatrix} & \begin{array}{l} \\ \\ -b_3\end{array}\end{array} \rightarrow \begin{array}{cc}\begin{pmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix} & \begin{array}{l}b_1 - 2b_3\\ b_2 + 2b_3\\ \end{array}\end{array}$$

$$\rightarrow \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Karena matriks A dapat diubah menjadi matriks Identitas, maka:

$k_1 = k_2 = k_3 = 0$ ⟶ $\boldsymbol{X_1, X_2,}$ **dan** $\boldsymbol{X_3}$ **Bebas Linier**

Teorema 2 ⟶ **Matriks A dapat didiagonalkan**

Diagonalisasi

Matrik invers X

$$\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 1 & 2 & -2 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 & 1\end{array}\right)\begin{array}{l} \\ b_2-b_1\\ b_3-b_1\end{array} \rightarrow \left(\begin{array}{ccc|ccc}1 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & -2 & -1 & 1 & 0\\ 0 & -1 & 1 & -1 & 0 & 1\end{array}\right)\begin{array}{l}b_1-b_2\\ \\ b_3+b_2\end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 2 & 2 & -1 & 0\\ 0 & 1 & -2 & -1 & 1 & 0\\ 0 & 0 & -1 & -2 & 1 & 1\end{array}\right)\begin{array}{l} \\ \\ -b_3\end{array} \rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 2 & 2 & -1 & 0\\ 0 & 1 & -2 & -1 & 1 & 0\\ 0 & 0 & 1 & 2 & -1 & -1\end{array}\right)\begin{array}{l}b_1-2b_3\\ b_2+2b_3\\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & 1 & 2\\ 0 & 1 & 0 & 3 & -1 & -2\\ 0 & 0 & 1 & 2 & -1 & -1\end{array}\right) \qquad X^{-1} = \begin{pmatrix}-2 & 1 & 2\\ 3 & -1 & -2\\ 2 & -1 & -1\end{pmatrix}$$

Diagonalisasi

$$D = X^{-1}\mathrm{AX}$$

$$\mathrm{D} = \begin{pmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonalisasi

4. A dapat didiagonalkan $\rightarrow$ $\mathrm{A} = XDX^{-1}$

Akibatnya

$$A^2 = XDX^{-1}XDX^{-1} = XD^2X^{-1}$$

Secara umum

$$A^k = XD^kX^{-1}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \rightarrow D^k = \begin{pmatrix} {\lambda_1}^k & 0 & \dots & 0 \\ 0 & {\lambda_2}^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {\lambda_n}^k \end{pmatrix}$$

Contoh

Diberikan matriks

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

a) Periksa apakah matriks A dapat didiagonalkan? Jika ya, tentukan matriks X dan D pada diagonalisasi $A = XDX^{-1}$.

b) Tentukan matriks A^3.

Contoh

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

➢ **<u>Nilai Eigen</u>**

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(3-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 3 \qquad \lambda_3 = 3$$

➢ **<u>Vektor Eigen</u>**

- $\lambda_1 = 2$

$$\begin{pmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2 & 0 & -2 \\ 0 & 3-2 & 0 \\ 0 & 0 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Baris 1 : $-2x_3 = 0 \longrightarrow x_3 = 0$

Baris 2 : $x_2 = 0$

Baris 3 : $x_3 = 0$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$X_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Contoh

- **Vektor Eigen**

- $\lambda_2 = \lambda_3 = 3$

$$\begin{pmatrix} 2-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-3 & 0 & -2 \\ 0 & 3-3 & 0 \\ 0 & 0 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Baris 1 : $-x_1 - 2x_3 = 0$

$\longrightarrow \; x_1 = -2x_3$

$$X_{2,3} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad X_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

- **Matriks Pendiagonal $\boldsymbol{X}$**

$$\lambda_1 = 2 \longrightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3 \longrightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 3 \longrightarrow X_3 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{X} = (X_1 \quad X_2 \quad X_3)$$

$$\boldsymbol{X} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Contoh

- **Uji Bebas Linier**

Karena terdapat nilai eigen yang sama, maka vektor eigen belum tentu bebas linier, sehingga perlu diuji bebas linier.

$$k_1X_1 + k_2X_2 + k_3X_3 = 0$$

$$k_1\begin{pmatrix}1\\0\\0\end{pmatrix} + k_2\begin{pmatrix}0\\1\\0\end{pmatrix} + k_3\begin{pmatrix}-2\\0\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & -2\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} b_1 + 2b_3 \quad \rightarrow \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Karena matriks X dapat diubah menjadi matriks Identitas, maka:

$k_1 = k_2 = k_3 = 0$ ⟶ $\boldsymbol{X_1, X_2}$**, dan** $\boldsymbol{X_3}$ **Bebas Linier**

Teorema 2 ⟶ **Matriks A dapat didiagonalkan**

- **Invers Matriks** $\boldsymbol{X}$

$$(X|I) \rightarrow (I|X^{-1})$$

$$\left(\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right) b_1 + 2b_3 \quad \rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 2\\0 & 1 & 0 & 0 & 1 & 0\\0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$$

$$X^{-1} = \begin{pmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Contoh

- **Matriks Diagonal D**

$$D = X^{-1}AX$$

$$D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -6 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \longrightarrow D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

- **Matriks A^3** $\quad A^k = XD^kX^{-1}$

$$A^3 = XD^3X^{-1}$$

$$A^3 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 & 0 & 16 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 8 & 0 & -38 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$$

Latihan

Periksa apakah matriks A, B, dan C dapat didiagonalkan? Jika ya, tentukan matriks X dan D pada diagonalisasi $A = XDX^{-1}$

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{6} & \mathbf{-1} \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Latihan A

$$A = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

Nilai Eigen

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - (6)(0) = 0$$

$$(1-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

Vektor Eigen

- $\lambda_1 = 1$

$$\begin{pmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-1 & 0 \\ 6 & -1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Baris 2 : $6x_1 - 2x_2 = 0$

$x_2 = 3x_1$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Vektor Eigen

- $\lambda_2 = -1$

$$\begin{pmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-(-1) & 0 \\ 6 & -1-(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \text{Baris 1 : } 2x_1 = 0 \\ \text{Baris 2 : } 6x_1 = 0 \end{array}\right\} x_1 = 0$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Latihan A

- **<u>Matriks Pendiagonal X</u>**

$$\lambda_1 = 1 \longrightarrow X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\lambda_2 = -1 \longrightarrow X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{X} = (X_1 \quad X_2)$$

$$\boldsymbol{X} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Matriks A dapat didiagonalkan karena memiliki nilai eigen yang berbeda.

- **<u>Invers Matriks X</u>**

$$(X|I) \to (I|X^{-1})$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right) b_2 - 3b_1$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right)$$

$$X^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

- **<u>Matriks Diagonal D</u>**

$$D = X^{-1}AX$$

$$D = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Latihan B

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- **Nilai Eigen**

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-\lambda) - (1)(1) = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda - 1)(\lambda + 1) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

- **Vektor Eigen**

- $\lambda_1 = 1$

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{ll} \text{Baris 1 :} & -x_1 + x_2 = 0 \\ \text{Baris 2 :} & x_1 - x_2 = 0 \end{array}\right\} x_2 = x_1$$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **Vektor Eigen**

- $\lambda_2 = -1$

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -(-1) & 1 \\ 1 & -(-1) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{ll} \text{Baris 1 :} & x_1 + x_2 = 0 \\ \text{Baris 2 :} & x_1 + x_2 = 0 \end{array}\right\} x_2 = -x_1$$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Latihan B

Matriks Pendiagonal $\boldsymbol{X}$

$$\lambda_1 = 1 \longrightarrow X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \longrightarrow X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$$

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Matriks B dapat didiagonalkan karena memiliki nilai eigen yang berbeda.

Invers Matriks $\boldsymbol{X}$

$$(X|I) \rightarrow (I|X^{-1})$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) b_2 - b_1$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) -1/2b_2$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) b_1 - b_2$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right)$$

$$X^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

$$X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Matriks Diagonal $\boldsymbol{D}$

$$D = X^{-1}BX$$

$$D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$D = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Latihan C

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

- **Nilai Eigen**

$$|C - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 2 \qquad \lambda_3 = 2$$

- **Vektor Eigen**
  - $\lambda_1 = 3$

$$\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-3 & 0 & 0 \\ 0 & 2-3 & 0 \\ 0 & 1 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Baris 2 : $-x_2 = 0 \longrightarrow x_2 = 0$

Baris 3 : $x_2 - x_3 = 0 \longrightarrow x_3 = 0$

$$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$$

$$X_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Latihan C

➢ **Vektor Eigen**

- $\lambda_2 = \lambda_3 = 2$

$$\begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 1 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Baris 1 : $x_1 = 0$

Baris 3 : $x_2 = 0$

$$X_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \& \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

➢ **Matriks Pendiagonal $\boldsymbol{X}$**

$$\lambda_1 = 2 \longrightarrow X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3 \longrightarrow X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 3 \longrightarrow X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{X} = (X_1 \quad X_2 \quad X_3)$$

$$\boldsymbol{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Latihan C

- **Uji Bebas Linier**

Karena terdapat nilai eigen yang sama, maka vektor eigen belum tentu bebas linier, sehingga perlu diuji bebas linier.

$$k_1X_1 + k_2X_2 + k_3X_3 = 0$$

$$k_1\begin{pmatrix}1\\0\\0\end{pmatrix} + k_2\begin{pmatrix}0\\0\\1\end{pmatrix} + k_3\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$$

$$\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}k_1\\k_2\\k_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 1\end{pmatrix}$$

Karena matriks X **tidak dapat** diubah menjadi **matriks Identitas**, maka:

$$\boldsymbol{X_1, X_2, \text{dan } X_3 \text{ Tidak Bebas Linier}}$$

Teorema 2 :

Matriks C tidak dapat didiagonalkan

Terima Kasih

Sampai Jumpa
di Pertemuan Selanjutnya