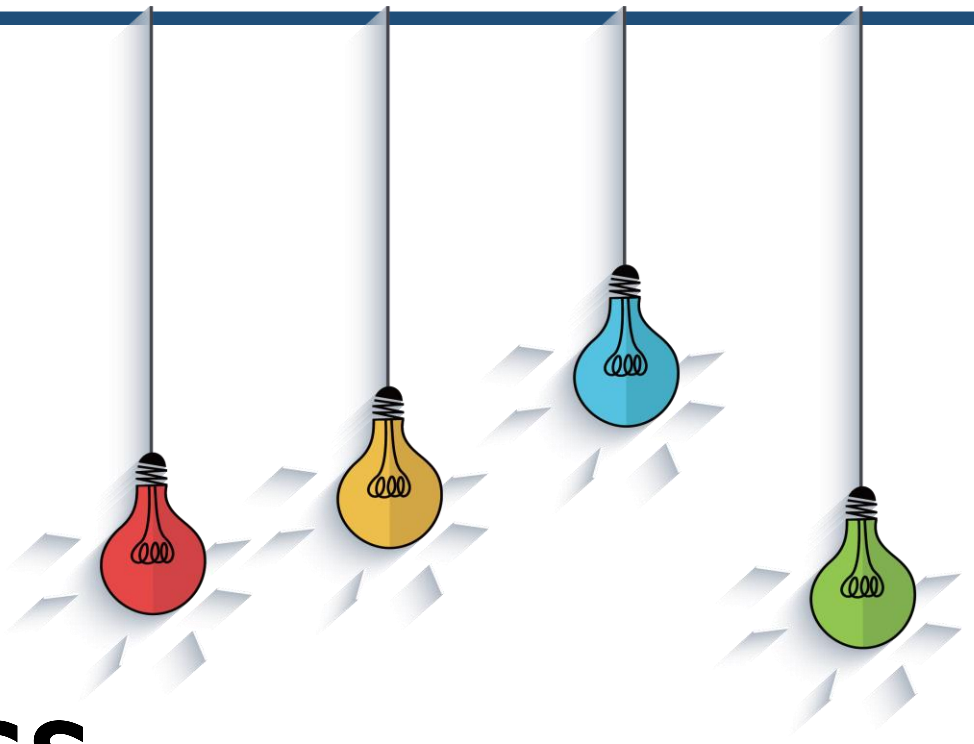

IF120 Discrete Mathematics

04 Functions, Sequences, Series

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REVIEW

- Direct Proofs and Indirect Proofs
- Other Proofs Methods
- Proofs Strategy
- Mathematical Induction

OUTLINE

- Functions
- Sequences
- Strings

Functions

- Let X and Y be sets.
- A *function* f from X to Y is a subset of the Cartesian product $X \times Y$ having the property that for each $x \in X$, there is exactly one $y \in Y$ with $(x, y) \in f$.
- We sometimes denote a function f from X to Y as $f : X \to Y$.
- The set X is called the **domain of f** and the set Y is called the **codomain of f**.
- The set

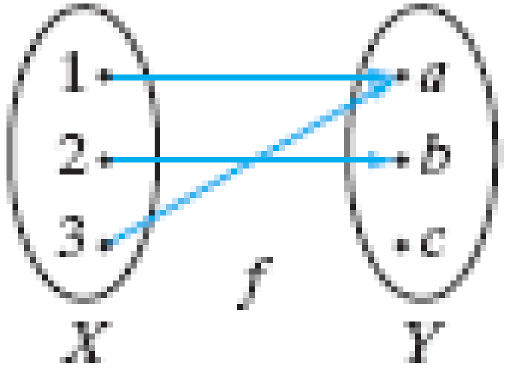
$$\{y \mid (x, y) \in f\}$$

(which is a subset of the codomain Y) is called the **range of f**.

- The set

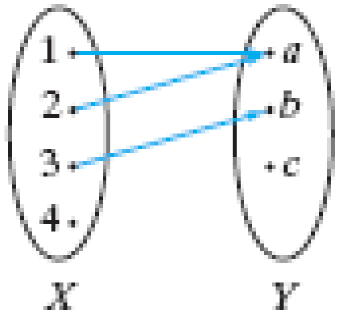
$$f = \{(1, a), (2, b), (3, a)\}$$

is a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$.

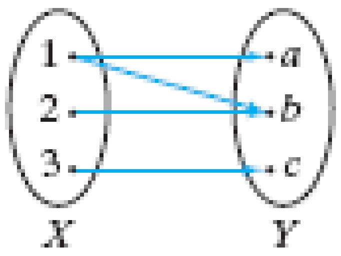


arrow diagram

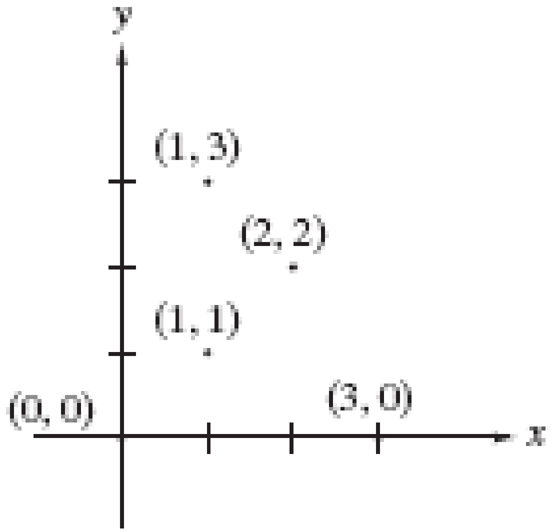
Functions



not a function



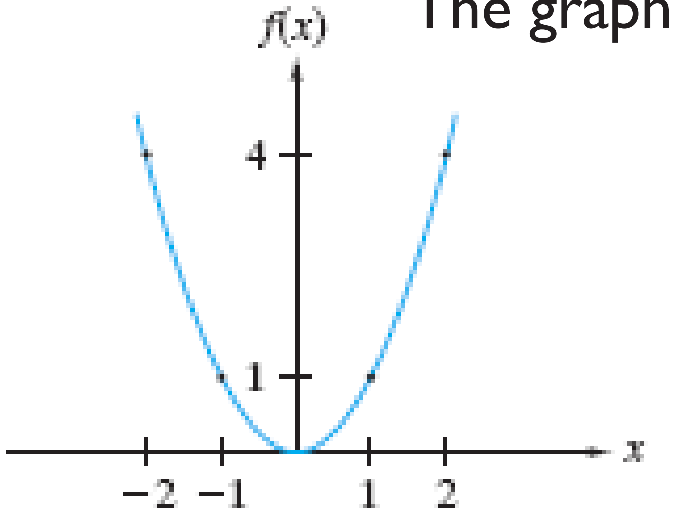
not a function



not a function

- Another way to visualize a function is to draw its graph.
- The **graph of a function** f whose domain and codomain are subsets of the real numbers is obtained by plotting points in the plane that correspond to the elements in f.
- The domain is contained in the horizontal axis and the codomain is contained in the vertical axis.

The graph of $f(x) = x^2$



Modulus Operator

- *Definition*: If x is an integer and y is a positive integer, we define x **mod** y to be the remainder when x is divided by y.

❑We have

  6 mod 2 = 0, 5 mod 1 = 0, 8 mod 12 = 8, 199673 mod 2 = 1.

❑Suppose that we have cells in a computer memory indexed from 0 to 10. We wish to store and retrieve arbitrary nonnegative integers in these cells. One approach is to use a **hash function.** A hash function takes a data item to be stored or retrieved and computes the first choice for a location for the item. For example, for our problem, to store or retrieve the number n, we might take as the first choice for a location, n mod 11. Our hash function becomes

$$h(n) = n \bmod 11$$

collision resolution policy

132			102	15	5	257		558		32
0	1	2	3	4	5	6	7	8	9	10

Floor and Ceiling

- *Definition*: The **floor** of *x*, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to *x*. The **ceiling** of *x*, denoted $\lceil x \rceil$, is the least integer greater than or equal to *x*.

- $\lfloor 8.3 \rfloor = 8, \lfloor -1.3 \rfloor = -2, \lfloor 5 \rfloor = 5, \lceil 0.5 \rceil = 1, \lceil -3.3 \rceil = -3, \lceil -9 \rceil = -9$

- In 2007, the U.S. first-class postage rate for retail flat mail up to 13 ounces was 80 cents for the first ounce or fraction thereof and 17 cents for each additional ounce or fraction thereof. The postage *P*(*w*) as a function of weight *w* is given by the equation

$$P(w) = 80 + 17\lceil w - 1 \rceil, \qquad 13 \geq w > 0$$

The expression $\lceil w - 1 \rceil$ counts the number of additional ounces beyond 1, with a fraction counting as one additional ounce. As examples,

$$P(3.7) = 80 + 17\lceil 3.7 - 1 \rceil = 80 + 17\lceil 2.7 \rceil = 80 + 17 \cdot 3 = 131,$$
$$P(2) = 80 + 17\lceil 2 - 1 \rceil = 80 + 17\lceil 1 \rceil = 80 + 17 \cdot 1 = 97.$$

Injective Functions

- A function f from X to Y is said to be **one-to-one** (or **injective**) if for each $y \in Y$, there is at most one $x \in X$ with $f(x) = y$.
- Equivalent to: For all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
- In symbols,

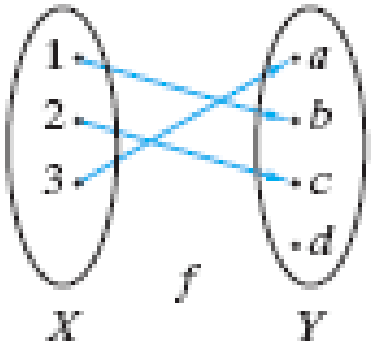
$$\forall x_1 \forall x_2 ((f(x_1) = f(x_2)) \rightarrow (x_1 = x_2))$$

- This form of the definition can often be used to prove that a function is one-to-one.

❑ The function

$f = \{(1, b), (3, a), (2, c)\}$

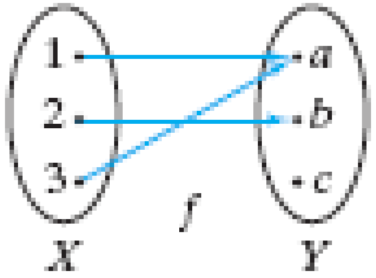
from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is one-to-one.



❑ The function

$f = \{(1, a), (2, b), (3, a)\}$

is not one-to-one since $f(1) = a = f(3)$.



Injective Functions

- Prove that the function

$$f(n) = 2n + 1$$

from the set of positive integers to the set of positive integers is one-to-one.

We must show that for all positive integers n_1 and n_2, if $f(n_1) = f(n_2)$, then $n_1 = n_2$.

So, suppose that $f(n_1) = f(n_2)$. Using the definition of f, this latter equation translates as

$$2n_1 + 1 = 2n_2 + 1$$

Subtracting 1 from both sides of the equation and then dividing both sides of the equation by 2 yields

$$n_1 = n_2$$

Therefore, f is one-to-one.

Injective Functions

- A function is **not** one-to-one if there exist x_1 and x_2 such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

❑ Prove that the function

$$f\ (n)\ =\ 2^n\ -\ n^2$$

from the set of positive integers to the set of integers is *not* one-to-one.

We must find positive integers n_1 and $n_2, n_1 \neq n_2$, such that

$$f\ (n_1)\ =\ f\ (n_2)$$

By trial-and-error, we find that

$$f\ (2)\ =\ f\ (4)$$

Therefore, *f* is not one-to-one.

Surjective Functions

- If f is a function from X to Y and the range of f is Y, f is said to be **onto** Y (or an **onto function** or a **surjective function**).
- Equivalent to: For all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. In symbols,

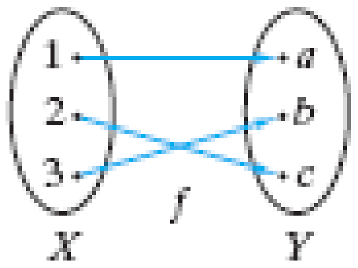
$$\forall y \in Y\, \exists x \in X(f(x) = y)$$

- This form of the definition can often be used to prove that a function is onto.

❑ The function

$f = \{(1, a), (2, c), (3, b)\}$

from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$ is one-to-one and onto Y.



❑ The function

$f = \{(1, b), (3, a), (2, c)\}$

from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$ is *not* onto Y.

Surjective Functions

- Prove that the function

$$f\ (x)\ = \frac{1}{x^2}$$

from the set X of nonzero real numbers to the set Y of positive real numbers is onto Y .

We must show that for every $y \in Y$, there exists $x \in X$ such that $f\,(x) = y$.

Substituting the formula for $f\,(x)$, this last equation becomes

$$\frac{1}{x^2} = y$$

Solving for x, we find

$$x\ = \pm\frac{1}{\sqrt{y}}$$

Notice that $1/\sqrt{y}$ is defined because y is a positive real number. If we take x to be the positive square root

$$x\ = \frac{1}{\sqrt{y}}$$

then $x \in X$. (We could just as well have taken $x = -\ 1/\sqrt{y}$.) Thus, for every $y \in Y$, there exists x, namely $x = 1/\sqrt{y}$ such that

$$f(x) = f\left(\frac{1}{\sqrt{y}}\right) = \frac{1}{(1/\sqrt{y})^2} = y$$

Therefore, f is onto Y.

Surjective Functions

- A function f from X to Y is **not** onto Y if there exists $y \in Y$ such that for all $x \in X$, $f(x) \neq y$.

❑ Prove that the function

$$f(n) = 2n - 1$$

from the set X of positive integers to the set Y of positive integers is *not* onto Y.

We must find an element $m \in Y$ such that for all $n \in X$, $f(n) \neq m$. Since $f(n)$

is an odd integer for all n, we may choose for y any positive, even integer, for example,

$y = 2$. Then $y \in Y$ and

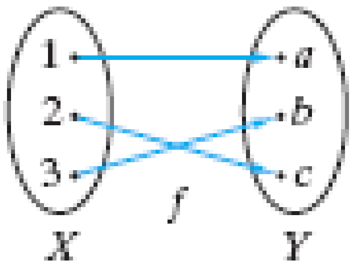
$$f(n) \neq y$$

for all $n \in X$.

Thus f is not onto Y.

Bijection and Inverse Functions

- A function that is both one-to-one and onto is called a **bijection**.



- Suppose that *f* is a one-to-one, onto function from *X* to *Y* . It can be shown that

$$\{(y, x) \mid (x, y) \in f\}$$

is a one-to-one, onto function from *Y* to *X*.

This new function, denoted f^{-1}, is called *f* **inverse.**

- For the function

$$f = \{(1, a), (2, c), (3, b)\}$$

we have

$$f^{-1} = \{(a, 1), (c, 2), (b, 3)\}$$

Inverse Functions

- The function

$$f(x) = 2^x$$

is a one-to-one function from the set **R** of all real numbers onto the set $\mathbf{R}^+$ of all positive real numbers. We will derive a formula for $f^{-1}(y)$.

Suppose that (y, x) is in f^{-1}; that is,

$$f^{-1}(y) = x \qquad (4.1)$$

Then (x, y) ∈ f. Thus,

$$y = 2^x$$

By the definition of logarithm,

$$\log_2 y = x \qquad (4.2)$$

Combining (4.1) and (4.2), we have

$$f^{-1}(y) = x = \log_2 y$$

That is, for each $y \in \mathbf{R}^+$, $f^{-1}(y)$ is the logarithm to the base 2 of y. We can summarize the situation by saying that the inverse of the exponential function is the logarithm function.

Composition

- Let g be a function from X to Y and let f be a function from Y to Z. The **composition of f with g**, denoted $f \circ g$, is the function

$$(f \circ g)(x) = f(g(x))$$

  from X to Z.

❑Given

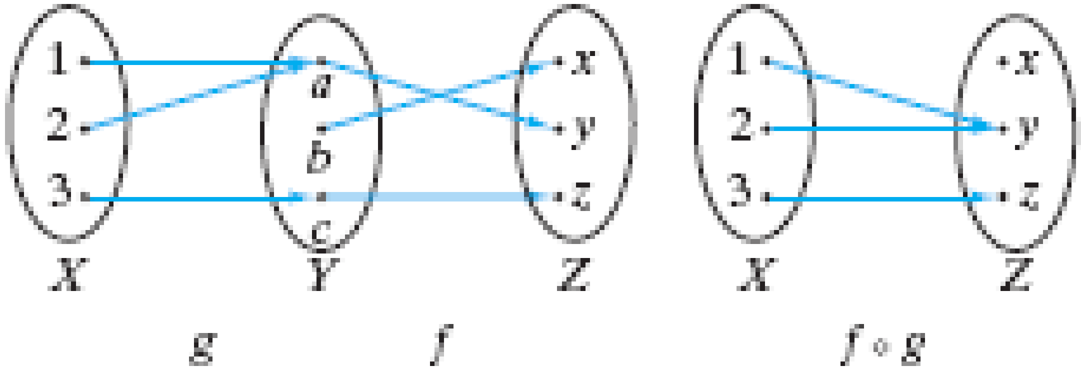
$$g = \{(1,a),(2,a),(3,c)\}$$

a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c\}$, and

$$f = \{(a,y),(b,x),(c,z)\}$$

a function from Y to $Z = \{x, y, z\}$, the composition function from X to Z is the function

$$f \circ g = \{(1,y),(2,y),(3,z)\}$$



Binary and Unary Operator

- A function from $X \times X$ to X is called a **binary operator** on X.

❑Let $X = \{1, 2, \ldots\}$. If we define

$$f\,(x, y) = x + y,$$

where $x, y \in X$, then f is a binary operator on X.

- A function from X to X is called a **unary operator** on X.

❑Let U be a universal set. If we define

$$f\,(X) = \bar{X},$$

where $X \in P(U)$, then f is a unary operator on $P(U)$.

Sequences

- A **sequence** named, say, *s* is denoted *s* or $\{s_n\}$. Here *s* or $\{s_n\}$ denotes the entire sequence

$$s_1, s_2, s_3, \ldots.$$

- We use the notation s_n to denote the single element of the sequence *s* at index *n*.

❑Consider the sequence *s*

$$2, 4, 6, \ldots, 2n, \ldots.$$

The first element of the sequence is 2, the second element of the sequence is 4, and so on. The *n*th element of the sequence is 2*n*. If the first index is 1, we have

$$s_1 = 2, s_2 = 4, s_3 = 6, \ldots, s_n = 2n, \ldots.$$

Sequences

- Define a sequence *b* by the rule b_n is the *n*th letter in the word *digital*. If the index of the first term is 1, then b_1 = *d*, b_2 = b_4 = *i* , and b_7 = *l*. This sequence is a finite sequence. It can be denoted $\{b_k\}_{k=1}^{7}$.
- If *x* is the sequence defined by

$$x_n = \frac{1}{2^n}, \qquad -1 \le n \le 4$$

the elements of *x* are

$$2, 1, 1/2, 1/4, 1/8, 1/16.$$

Sequences

- A sequence *s* is **increasing** if $s_n < s_{n+1}$ for all *n* for which *n* and *n* + 1 are in the domain of the sequence.
- A sequence *s* is **decreasing** if $s_n > s_{n+1}$ for all *n* for which *n* and *n* + 1 are in the domain of the sequence.
- A sequence *s* is **nondecreasing** if $s_n \leq s_{n+1}$ for all *n* for which *n* and *n* + 1 are in the domain of the sequence. (A nondecreasing sequence is like an increasing sequence except that "<" is replaced by "≤.")
- A sequence *s* is **nonincreasing** if $s_n \geq s_{n+1}$ for all *n* for which *n* and *n* + 1 are in the domain of the sequence. (A nonincreasing sequence is like a decreasing sequence except that ">" is replaced by "≥.")

Sequences

- The sequence

$$2, 5, 13, 104, 300$$

  is increasing and nondecreasing.

- The sequence

$$a_i = \frac{1}{i}, \qquad i \geq 1$$

  is decreasing and nonincreasing.

- The sequence

$$100, 90, 90, 74, 74, 74, 30$$

  is nonincreasing, but it is *not* decreasing.

- The sequence

$$100$$

  is increasing, decreasing, nonincreasing, and nondecreasing since there is no value of *i* for which both *i* and *i* + 1 are indexes.

Subsequence

- Let $\{s_n\}$ be a sequence defined for $n = m, m+1, \ldots,$ and let $n_1, n_2, \ldots$ be an increasing sequence whose values are in the set $\{m, m + 1, \ldots\}$.
- We call the sequence $\{s_{n_k}\}$ a **subsequence** of $\{s_n\}$.

❑The sequence

$$b, c \tag{4.3}$$

is a subsequence of the sequence

$$t_1 = a, t_2 = a, t_3 = b, t_4 = c, t_5 = q \tag{4.4}$$

Subsequence (4.3) is obtained from sequence (4.4) by choosing the third and fourth terms. The expression n_k of 'subsequence' definition tells us which terms of (4.4) to choose to obtain subsequence (4.3); thus, $n_1 = 3, n_2 = 4$. The subsequence (4.3) is t_3, t_4 or t_{n_1}, t_{n_2}. Notice that the sequence

$$c, b$$

is *not* a subsequence of sequence (4.4) since the order of terms in the sequence (4.4) is not maintained.

Sigma and Product Notation

- If $\{a_i\}_{i=m}^{n}$ is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \dots a_n$$

- The formalism

$$\sum_{i=m}^{n} a_i$$

is called the **sum** (or **sigma**) **notation** and

$$\prod_{i=m}^{n} a_i$$

is called the **product notation**.

Sigma and Product Notation

- The geometric sum

$$a + ar + ar^2 + \cdots + ar^n$$

can be rewritten compactly using the sum notation as

$$\sum_{i=0}^{n} ar^i$$

- Rewrite the sum

$$\sum_{i=0}^{n} ir^{n-i}$$

replacing the index *i* by *j* , where *i* = *j* – 1.

$$\sum_{i=0}^{n} ir^{n-i} = \sum_{j=1}^{n+1} (j-1)r^{n-j+1}$$

Strings

- A **string** is a finite sequence of characters.
- In programming languages, strings can be used to denote text.
- Within a computer, **bit strings** (strings of 0's and 1's) represent data and instructions to execute.
- *Definition*: A *string over X*, where *X* is a finite set, is a finite sequence of elements from *X*.

❑ Let $X = \{a, b, c\}$. If we let

$$\beta_1 = b, \beta_2 = a, \beta_3 = a, \beta_4 = c,$$

we obtain a string over *X*. This string is written *baac*.

- Since a string is a sequence, **order** is taken into account. For example, the string *baac* is different from the string *acab*.
- Repetitions in a string can be specified by **superscripts**. For example, the string *bbaaac* may be written b^2a^3c.

Strings

- The string with no elements is called the **null string** and is denoted λ.
- We let X^* denote the set of all strings over X, including the null string, and we let X^+ denote the set of all nonnull strings over X.

❑Let X = {a, b}. Some elements in X^* are

$$\lambda, a, b, abab, b^{20}a^5ba.$$

- The **length** of a string α is the number of elements in α.
- The length of α is denoted $|\alpha|$.

❑If $\alpha = aabab$ and $\beta = a^3b^4a^{32}$, then

$$|\alpha| = 5 \; and \; |\beta| = 39.$$

- If α and β are two strings, the string consisting of α followed by β, written $\alpha\beta$, is called the **concatenation** of α and β.

❑If $\gamma = aab$ and $\theta = cabd$, then

$$\gamma\,\theta = aabcabd, \theta\gamma = cabdaab, \gamma\lambda = \gamma = aab, \lambda\gamma = \gamma = aab.$$

Substrings

- A **substring** of a string α is obtained by selecting some or all consecutive elements of α.
- *Definition:* A string β is a **substring** of the string α if there are strings γ and δ with $\alpha = \gamma\beta\delta$.

❑ The string $\beta = add$ is a substring of the string $\alpha = aaaddad$ since, if we take $\gamma = aa$ and $\delta = ad$, we have $\alpha = \gamma\beta\delta$. Note that if β is a substring of α, γ is the part of α that precedes β (in α), and δ is the part of α that follows β (in α).

PRACTICE

PRACTICE 1

- *Determine whether each set below is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.*

1. $\{(1, a), (2, a), (3, c), (4, b)\}$
2. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$
3. $\{(1, c), (2, d), (3, a), (4, b)\}$

PRACTICE 2

- *Determine whether each function below is one-to-one, onto, or both. Prove your answers. The domain of each function is the set of all integers. The codomain of each function is also the set of all integers.*

1. $f\,(n) = n + 1$
2. $f\,(n) = n^2 - 1$
3. $f\,(n) = \lceil n/2 \rceil$

PRACTICE 3

- Let f and g be functions from the positive integers to the positive integers defined by the equations

1. $f(n) = 2n + 1, g(n) = 3n - 1$
2. $f(n) = n^2, g(n) = 2^n$

   Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

PRACTICE 4

- The sequence t defined by

$$t_n = 2n - 1, n \geq 1.$$

1. Find t_7.
2. Find t_{100}.
3. Find $\sum_{i=3}^{7} t_i$
4. Find $\prod_{i=1}^{3} t_i$
5. Temukan rumus yang merepresentasikan barisan ini sebagai barisan yang indeks bawahnya 0.

PRACTICE 5

- Compute the given quantity using the strings

$$\alpha = baab, \beta = caaba, \gamma = bbab.$$

1. $\alpha\beta$
2. $\beta\alpha$
3. $\alpha\alpha$
4. $\beta\beta$
5. $|\alpha\beta|$
6. $|\beta\alpha|$

NEXT WEEK'S OUTLINE

- Relations
- Matrices of Relations

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- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

INFORMATIKA UMN UNIVERSITAS MULTIMEDIA NUSANTARA

Misi

1. Menyelenggarakan pembelajaran dengan teknologi dan kurikulum terbaik serta didukung tenaga pengajar profesional.
2. Melaksanakan kegiatan penelitian di bidang Informatika untuk memajukan ilmu dan teknologi Informatika.
3. Melaksanakan kegiatan pengabdian kepada masyarakat berbasis ilmu dan teknologi Informatika dalam rangka mengamalkan ilmu dan teknologi Informatika.