Unit - IV Queueing Theory

Queueing Theory objective:

The objective is to have good economic balance between weighting time cost and service cost.

It is used to find optimum solution for minimizing the waiting time and service cost.

Arrival → [Queue] → [Service] → Departure →

→ System ←

Queue discipline:

1. FIFO - First In First Out
2. FILO - First In Last Out
3. Service in Random order
4. Priority

Customer Behavior:

+ Balking - Customer refused to enter queueing system
+ Jockeying - when there are parallel queues, the customer jump from one queue to another.
+ Reneging - Customer who leave the queue without receiving service due to in-patience (to much waiting)

KENDALL'S Notation for representating queing models:

A queue is generally represented by $(a/b/c):(d/e)$

where a - no. of arrival per unit time $= \lambda$

b - no. of service per unit time $= \mu$

c - no. of service

d - capacity of the system

e - queue discipline.

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Type I :

$(M \mid M \mid 1) : (\infty \mid FIFO)$

Arrival, Service/μ, No. of Servers, Queue capacity, Queue Discipline

(λ)

Formula :

1. Average No. of Customers in the System :

$$L_s = \frac{\lambda}{\mu - \lambda}$$

2. Average No. of customers in the queue :

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

3. Average waiting time of the customers in the System, $w_s = \frac{1}{\mu - \lambda}$ (or) $\frac{L_s}{\lambda}$

4. Average waiting time of the customer is the queue :

$$w_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. P(The system is Idle) $= P_0 = 1 - \left(\frac{\lambda}{\mu}\right)$

6. P(System is busy) $= P_1 = \frac{\lambda}{\mu}$

7. Waiting time of system exceed 't' $\} = P[w_s > t] = e^{-(\mu - \lambda)t}$

8. Prob of no. of customer's in the system exceed 'k' $\} = P[N > k] = \left(\frac{\lambda}{\mu}\right)^k$

9. Prob that waiting time of the Customers in queue exceeds 't' $= P[w_q > t] = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}$

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Type I :

$(M \mid M \mid 1) : (\infty \mid FIFO)$

(λ) Arrival, Service (μ), No. of Servers, Queue capacity, Queue Discipline

Formula :

1. Average No. of Customers in the System :

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2. Average No. of customers in the queue :

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4. Average waiting time of the customer is the queue :

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

5. P(The system is Idle) $= P_0 = 1 - \left(\frac{\lambda}{\mu}\right)$

6. P(System is busy) $= P_1 = \frac{\lambda}{\mu}$

7. Waiting time of system exceed 't' $\Big\} = P[W_s > t] = e^{-(\mu - \lambda)t}$

8. Prob of no. of customer's in of the system exceed 'k' $\Big\} = P[N > k] = \left(\frac{\lambda}{\mu}\right)^{k+1}$

9. Prob that waiting time of the customer in queue exceeds 't' $= P[W_q > t]$

$$= \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}$$

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10. Length of non-empty queue
or
Avg length of queue that formed from time to time $\Big\} = \frac{\mu}{\mu - \lambda}$

11. Traffic Intensity
(or)
Utilization factor $\Big\} = \rho = \frac{\lambda}{\mu}$

12. Prob that there are 'n' customers in the system $= P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

<u>Note</u>:

1. If the arrival rate is "a" <u>per unit</u>
then $\lambda = a$

2. If the arrival time is average 'a' Unit
$\lambda = \frac{1}{a}$ per unit

3. If the inter arrival time is a unit
$\lambda = \frac{1}{a}$

4. If the service rate is "b" <u>per</u> unit.
then $\mu = b$

5. If the mean service time, is "b" unit then
$\mu = \frac{1}{b}$ per unit

6. If $\lambda = a$ per minute $= a \times 60$ per hour

7. $\lambda = a$ per hour $= \frac{a}{60}$ per minute.

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Customers arrive at a one-man barber shop according to a poisson process with a mean inter arrival time of 12 min. Customers spend an average of 10 min in the barber's chair.

a) What is the expected number of customers in the barber shop and in the queue?

b) Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.

c) How much time can a customer expect to spend in the barber's shop?

d) Management will provide another chair and hire another barber, when a customer's waiting time in the shop exceeds 1.25 h. How much must the average rate of arrivals increase to warrant a second barber?

e) What is the average time customers spend in queue?

f) What is the probability that the waiting time in system is greater than 30 min?

g) Calculate the percentage of customers who have to wait prior to getting into the barber's chair.

h) What is the probability that more than 3 customers are in the system.

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Inter arrival time = 12 mins

$\therefore \lambda = \frac{1}{12}$ per minute

Mean service time = 10 mins

$\therefore \mu = \frac{1}{10}$ per minutes

$$\frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{10}} = 0.833$$

$$\mu - \lambda = \frac{1}{10} - \frac{1}{12} = \frac{1}{60}$$

i) a. $$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{60}} = 5$$

b. $$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\left(\frac{1}{12}\right)^2}{\frac{1}{10}\left(\frac{1}{60}\right)} = 4.166$$

ii) P(~~no.~~ NO customer in the system) $= P_0 = 1 - \frac{\lambda}{\mu}$

$= 1 - 0.833$

$= 0.167$

Percentage of time an arrival need not wait $\} = 0.167 \times 100$

$= 16.7\%$

iii) $$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{60}} = 60$$

iv) Given, $W_s > 1.25$ hrs

$W_s > 75$ mins

Let λ_n be the new arrival rate

$$\frac{1}{\mu - \lambda_n} > 75$$

$$1 > 75(\mu - \lambda_n)$$

$$1 > 75\mu - 75\lambda_n$$

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$$75\lambda_2 > 75\mu - 1$$

$$\lambda_2 > \frac{75}{75}\mu - \frac{1}{75}$$

$$\lambda_2 > \mu - \frac{1}{75}$$

$$\lambda_2 > \frac{1}{10} - \frac{1}{75}$$

$$\lambda_2 > 0.086 = \frac{13}{50}$$

Difference $= \lambda_2 - \lambda$

$$= 0.086 - \frac{1}{12}$$

$$= 0.0026 = \frac{1}{375} \quad \frac{1}{300}$$

If the management fix the second barber then the average increased by $\frac{1}{300}$

e) $$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\frac{1}{12}}{\frac{1}{10}\left(\frac{1}{60}\right)} = 50 \text{ mins}$$

f) $$P[W_s > t] = P[W_s > 30]$$

$$= e^{-(\mu-\lambda)30}$$

$$= e^{-\frac{1}{60} \times 30}$$

$$= e^{-\frac{1}{2}} = 0.606$$

g) $$P(\text{a customer has to wait}) = P(\text{System is busy})$$

$$= \frac{\lambda}{\mu}$$

$$= 0.833$$

h) $$P(W_q > 3) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

$$= (0.833)^{3+1}$$

$$= (0.833)^4 = 0.482$$

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2. Telephone booth Problem

Arrival time = 12 mins

$\lambda = \frac{1}{12}$ per mins

Service time = 4 mins

$\mu = \frac{1}{4}$ per min

$\boxed{S = 1}$ $\boxed{K = \infty}$

$\frac{\lambda}{\mu} = \frac{1}{3} \Rightarrow \mu - \lambda = \frac{1}{6}$

i) $L_s = \frac{\lambda}{(\mu - \lambda)} = \frac{1/12}{1/6} = \frac{1}{2}$

ii) P(system in busy) $= P_0 = \left(\frac{\lambda}{\mu}\right) = \frac{1}{3}$

iii) P(waiting time in the system more than 10 mins) $= e^{-(\mu-\lambda)t}$

$= e^{-\frac{1}{6} \times 10}$

$= e^{-5/3} = 0.188$

d) Fraction of the day $= P_0 = \frac{\lambda}{\mu}$

$= \frac{1}{3}$

e) If waiting time in the queue ≥ 3

$W_q \geq 3$

Let λ_r be new arrival

$\frac{\lambda_r}{\mu(\mu - \lambda)} \geq 3$

$\lambda_r \geq 3\mu(\mu - \lambda_r)$

$\lambda_r \geq 3\mu^2 - 3\mu\lambda_r$

$\lambda_r \geq 3\left(\frac{1}{4}\right)^2 - 3\left(\frac{1}{4}\right)\lambda_r$

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$$\lambda_s + \frac{3}{4}\lambda_s \geq \frac{3}{16}$$

$$\frac{7}{4}\lambda_s \geq \frac{3}{16}$$

$$\lambda_s \geq \frac{3}{16} \times \frac{4}{7}$$

$$\lambda_s \geq \frac{3}{28}$$

Increase in Arrival rate

$$= \lambda_s - \lambda$$

$$= \frac{3}{28} - \frac{1}{12} = \frac{1}{42}$$

v i) Average length of queue from time to time $\Big\} = \frac{\mu}{\mu - \lambda} = \frac{\frac{1}{4}}{\frac{1}{6}} = \frac{3}{2}$

3. Cinema tickets

Given arrival rate = 6 per min

$$\lambda = 6 \text{ per min}$$

Service rate = 7.5 seconds

$$\mu = \frac{1}{7.5} \text{ per seconds}$$

$$\mu = \frac{1}{7.5} \times 60 = 8 \text{ per min}$$

$$S = 1, \; k = \infty$$

i) Time taken to purchase the ticket $\Big\} = W_s = \frac{1}{\mu - \lambda}$

$$= \frac{1}{2} \text{ mins}$$

or

0.5 min

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i) Time taken to ~~purchase the ticket~~ reach the seat } = W_s + Time to reach the seat after getting the ticket

$= 0.5 + 1.5$

$= 2$

Yes,

ii) He came 2 min earlier before movie start

ii) P(Seated for start of picture) =

P (Time taken to purchase the ticket) ≤ 0.5

$\Rightarrow P(W_s) \leq 0.5$

$= 1 - P(W_s > 0.5)$

$= 1 - e^{-(-\mu - \lambda)t}$

$= 1 - e^{-2(0.5)}$

$= 1 - e^{-1}$

P(Seated for start of the picture) = 0.632

iii) $P(W_s \leq t) = 99\%$

$1 - P(W_s > t) = 0.99$

$1 - e^{-(\mu - \lambda)t} = 0.99$

$1 - e^{-2t} = 0.99$

$1 - 0.99 = e^{-2t}$

$0.01 = e^{-2t}$

$-2t = \ln(0.01)$

$-2t = -4.605$

$t = \frac{4.605}{2} = 2.30$

Total time $= 2.3 + 1.5 = 3.8$ mins

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2. Customer arrives at a one man barber shop according Poisson process with a average ~~time~~ inter arrival time of 6 min. Customer spend an average of 8 min. $\lambda = 6, \quad \mu = 8$

To find :

1. $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{8} \Rightarrow 1 - \frac{3}{4} = \frac{1}{4}$

2. $L_s = \frac{\lambda}{\mu - \lambda} = \frac{6}{8-6} = \frac{6}{2} = 3$

3. $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(6)^2}{8(8-6)} = \frac{36}{8(2)} = \frac{36}{16}$

4. $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$

5. $W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{6}{8(8-6)} = \frac{6}{8(2)} = \frac{6}{16}$

Type - II

$(M/M/S) : (\infty/FIFO)$

$\lambda \quad \mu$

In this model

No. of server $= S \geq 2$

Capacity $= \infty$

$\rho = \frac{\lambda}{\mu}$

Formula :

1. Prob that 'n' customers in the system $\Big\} = P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 ; \; 0 \leq n < S$

$\frac{1}{S!\, S^{n-S}}\left(\frac{\lambda}{\mu}\right)^n P_0 ; \; n \geq S$

2. Prob that No customers in the system $\Big\} = P_0 = \left[\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!\left(1 - \frac{\lambda}{\mu S}\right)}\right]^{-1}$

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3. Average no. of customer in the queue

$$= L_q = \frac{P_0 \left(\frac{\lambda}{\mu}\right)^{s+1}}{s \cdot s! \left(1 - \frac{\lambda}{\mu s}\right)^2}$$

4. Average No. of customer in the system $= L_s = L_q + \frac{\lambda}{\mu}$

5. Average time a customer has a spend in the system

$$= W_s = \frac{L_s}{\lambda}$$

6. Average time a customer spend in the queue

$$= W_q = \frac{L_q}{\lambda}$$

7. Prob that an arrival has to wait

$$= P[\text{system is busy}] = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s! \left(1 - \frac{\lambda}{\mu s}\right)}$$

Note:

If all the server are busy, then the system is busy.

If any one of the server is busy, then the system need not to be busy.

Prob of a server busy $= \frac{\lambda}{\mu s}$

Prob of ~~the~~ a server idle $= 1 - \frac{\lambda}{\mu s}$

8. Mean waiting time in the queue for those who actually wait $= E\left[\frac{W_q}{W_s}\right] = \frac{1}{\mu s - \lambda}$

9. Average No of customers (in non-empty queue) who has to actually wait

$$= L_w = \frac{\frac{\lambda}{\mu s}}{\left(1 - \frac{\lambda}{\mu s}\right)}$$

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Note :

1. In this model every formula depends on P_0. So find the P_0 value first.

2. $\frac{\lambda}{\mu s} < 1$

1. Typist Problem.

Given, $s = 3$

Arrival rate = 15 per hour

λ = 15 per hour

Service rate = 6 per hour

μ = 6 per hour

$$\frac{\lambda}{\mu} = \frac{15}{6} = 2.5 \quad \Big| \quad \frac{\lambda}{\mu s} = \frac{5}{6} = 0.833$$

$$1 - \frac{\lambda}{\mu s} = \frac{1}{6} = 0.167$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1 - \frac{\lambda}{\mu s}\right)} \right]^{-1}$$

$$= \sum_{n=0}^{2} \frac{(2.5)^n}{n!} + \frac{(2.5)^3}{3!\,(0.167)} \Big]^{-1}$$

$$= \left[1 + \frac{2.5}{1} + \frac{(2.5)^2}{2} + 15.59 \right]^{-1}$$

$$P_0 = [22.218]^{-1}$$

$$P_0 = \frac{1}{22.218}$$

$$= 0.045$$

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i) P(all the typist are busy) $= \dfrac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$

$$= \frac{(2.5)^3 (0.045)}{3!\,(0.167)}$$

$$= 0.701$$

ii) P(Average no of letter waiting to be typed) = The average no of letters in the queue.

$$= L_q = \frac{\left(\frac{\lambda}{\mu}\right)^{s+1} P_0}{s\, s!\left(1-\frac{\lambda}{\mu s}\right)^2}$$

$$= \frac{(2.5)^4 (0.045)}{3(3!)(0.167)^2}$$

$$= 3.501$$

iii) The avg time a letter has to spend for waiting and for being typed $= W_s = \dfrac{L_s}{\lambda}$

$$L_s = L_q + \frac{\lambda}{\mu}$$

$$= 3.501 + 2.5$$

$$= 6.001$$

$$W_s = \frac{L_s}{\lambda} = \frac{6.001}{15} = 0.40006$$

iv) P(one letter is the system) $= P_1$ $[\because 1 < 3]$

$$P_1 = \frac{1}{1!}\left(\frac{\lambda}{\mu}\right)^1 P_0$$

$$= (2.5)(0.045) = 0.1125$$

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Formula :

$$P(T>t) = e^{-\mu t}\left[1 + \frac{\left(\frac{\lambda}{\mu}\right)^{c}\left(1-e^{-\mu t\left(c-1-\lambda/\mu\right)}\right)}{c!\left(1-\frac{\lambda}{\mu s}\right)\left(c-1-\lambda/\mu\right)}\right]p_0$$

v) P(that will take longer than 20 mins waiting to be typed) $= P(T>20) = P\left(T>\frac{1}{3}\right)$

$$= e^{-\mu t}\left[1 + \frac{\left(\frac{\lambda}{\mu}\right)^{s}\left(1-e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right)}{s!\left(1-\frac{\lambda}{\mu s}\right)\left(s-1-\frac{\lambda}{\mu}\right)}\right)p_0$$

$$= e^{-6\times\frac{1}{3}}\left[1 + \frac{(2.5)^3\left(1-e^{-6\times\frac{1}{3}(3-1-2.5)}\right)}{3!\left(\cancel{1-2}\ 0.167\right)(3-1-2.5)}\right]\times 0.045$$

$$= 0.135\left[1 + \frac{(-1.208)}{-0.501}\right]$$

$$= 0.4605$$

2. Supermarket :

Given : $s = 2$,

Arrival rate $= \lambda = 10$ per hour

Service rate $= 4$ mins

$\mu = \frac{1}{4}$ per min $= \frac{1}{4}\times 60 = 15$ per hour

$$\frac{\lambda}{\mu} = \frac{10}{15} = \frac{2}{3} = 0.667$$

$$\frac{\lambda}{\mu s} = \frac{10}{15\times 2} = 0.333$$

$$1-\frac{\lambda}{\mu s} = 1-0.333 = 0.667$$

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$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!} \frac{\left(\frac{\lambda}{\mu}\right)^s}{\left(1 - \frac{\lambda}{\mu s}\right)} \right]^{-1}$$

$$P_0 = \left[\sum_{n=0}^{1} \frac{1}{n!} (0.667)^n + \frac{1}{2!} \frac{(0.667)^2}{0.667} \right]^{-1}$$

$$P_0 = [1.99]^{-1}$$

$$P_0 = 0.5 = \frac{1}{2}$$

i) $\left.\begin{array}{l} P(\text{arrival have to wait}) \\ P(\text{system is busy}) \end{array}\right\} = \dfrac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1 - \frac{\lambda}{\mu s}\right)}$

$$= \frac{\left(\frac{2}{3}\right)^2 \left(\frac{1}{2}\right)}{2! \left(\frac{2}{3}\right)}$$

$$= \frac{1}{6}$$

ii) $\left.\begin{array}{l} \text{Expected idle time of} \\ \text{each give} \end{array}\right\} = 1 - \dfrac{\lambda}{\mu s}$

$$= \frac{2}{3}$$

$\left.\begin{array}{l} \%\text{ of Expected idle} \\ \text{time of each girle} \end{array}\right\} = \dfrac{2}{3} \times 100$

$$= 66.7\%$$

iii) Average queue length $= L_q = \dfrac{P_0 \left(\frac{\lambda}{\mu}\right)^{s+1}}{s \cdot s! \left(1 - \frac{\lambda}{\mu s}\right)^2}$

$$= \frac{\frac{1}{2}\left(\frac{2}{3}\right)^3}{2 \times 2! \left(\frac{2}{3}\right)^2}$$

$$= \frac{1}{12}$$

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iii) No. of unit in the system $= L_s = L_q + \frac{\lambda}{\mu}$

$= \frac{1}{12} + \frac{2}{3} = \frac{3}{4}$

iv) Customer has to wait what is expected length of is waiting time $\Big\} = E[w_q / w_s]$

$= \frac{1}{\mu s - \lambda}$

$= \frac{1}{(15)(2) - 10}$

$= \frac{1}{20}$

v) What is expected No. of idle girls in any these $\Big\} =$

3. Petrol Bunk :

Given :

Arrival rate = 30 per hours

$\lambda = 30$ per hours

Service rate = 6 mins

$\mu = \frac{1}{6}$ per min

$\mu = \frac{1}{6} \times 60$

$\mu = 10$ per hour.

$\frac{\lambda}{\mu} = \frac{30}{10} = 3$

$\frac{\lambda}{\mu s} = \frac{3}{4}$

$1 - \frac{\lambda}{\mu s} = 1 - \frac{3}{4} = \frac{1}{4}$

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$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\left(1-\frac{\lambda}{\mu s}\right)} \right]^{-1}$$

$$= \left[\sum_{n=0}^{3} \frac{1}{n!} (3)^n + \frac{(3)^4}{4!\left(\frac{1}{4}\right)} \right]^{-1}$$

$$= \left[1 + \frac{3}{1!} + \frac{(3)^2}{2!} + \frac{(3)^3}{3!} + \frac{3^4}{4!\left(\frac{1}{4}\right)} \right]^{-1}$$

$$= \left[1 + 3 + \frac{9}{2} + \frac{27}{6} + \frac{81}{24 \times \frac{1}{4}} \right]^{-1}$$

$$= \left[\frac{159}{6} \right]^{-1}$$

$$= \frac{2}{53} = 0.037.$$

i) P(Arrival has to wait) = P(System is busy)

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!\left(1-\frac{\lambda}{\mu s}\right)}$$

$$= \frac{(3)^4 \times 0.037}{4! \times \left(\frac{1}{4}\right)}$$

$$= \frac{81 \times \frac{2}{53}}{24 \times \frac{1}{4}} \Rightarrow \frac{81 \times \frac{2}{53}}{6}$$

$$\Rightarrow \frac{27}{53} = 0.509$$

ii) $L_q = 1.528$

$$W_s = \frac{L_s}{\lambda} = \frac{4.528}{30} = 0.1509$$

$L_s = 4.528$

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iii) a Pump be Idle on Average $\Big\} = 1 - \frac{\lambda}{\mu s}$

$= \frac{1}{4}$

Percentage a Pump be Idle $= \frac{1}{4} \times 100$

$= 25\%$

Type - III $(M|M|1) : (k|FIFO)$

Formula :

1. $P\{$ that there 'n' customers in the system $\} = P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

2. $P\{$System is empty$\} = P_0 = \begin{cases} \dfrac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} & ; \lambda \neq \mu \\ \dfrac{1}{k+1} & ; \lambda = \mu \end{cases}$

3. $L_s = \begin{cases} \dfrac{\lambda}{\mu-\lambda} - \left[\dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right] & ; \lambda \neq \mu \\ \dfrac{k}{2} & ; \lambda = \mu \end{cases}$

Note :1. $\boxed{\rho = \frac{\lambda}{\mu}}$

2. $L_s = \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(k+1)\rho^{k+1}}{1-(\rho)^{k+1}} & ; \lambda \neq \mu \\ \dfrac{k}{2} & ; \lambda = \mu \end{cases}$

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4. Effective Arrival rate $\lambda = \lambda' = \mu(1 - P_0)$

5. Avg no. of customers in the queue, $L_q = L_s - \frac{\lambda'}{\mu}$

6. Avg waiting time of customers in system, $W_s = \frac{L_s}{\lambda'}$

7. Avg waiting time of customers in queue, $W_q = \frac{L_q}{\lambda'}$

8. P(customer turned away) $= \left(\frac{\lambda}{\mu}\right)^k P_0$

1. Patient

Given, Arrival rate $= \lambda = 30$ per hour

service rate $= \mu = 20$ per hour

$(\lambda \neq \mu)$

$k = 14 + 1 = 15$

In waiting Room — Getting Service

$\frac{\lambda}{\mu} = \frac{30}{20} = \frac{3}{2} = 1.5$

ii) $P_0 = \frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1 - 1.5}{1 - (1.5)^{16}} = 0.00076$

i) Effective Arrival rate $= \lambda' = \mu(1 - P_0)$

$= 20(1 - 0.00076)$

$= 19.98$

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ii) P (Patient does not have to wait) = P_0 = 0.00076

iii) Expected waiting time of Patient discharge fee from the clinic $= W_S = \frac{L_S}{\lambda'}$

$$L_S = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{30}{20-30} - \left[\frac{(16)\ (1.5)^{16}}{1-(1.5)^{16}}\right]$$

$$= 13.024$$

$$W_S = \frac{13.024}{19.98}$$

$W_S = 0.651$ hours

3. Barber Shop :

Given, Arrival rate, $\lambda = 5$ per hour

Service rate, $\mu = 4$ per hour

Capacity, $k = 5$ [4 + 1]

$P_0 = +0.00888\ 2$

$\lambda' = 3.64472$

a) P (Prob Idle time) = P_0

$= 0.00888$

Percentage of Idle time $= 0.0888 \times 100$

$= 8.88$

b) P(Customer turned away) $= \left(\frac{\lambda}{\mu}\right)^k P_0$

$= (1.25)^5 \cdot (0.0888)$

$= 0.278$

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iii) Expected No. of Customer waiting for hair cut } $= L_q$

$= L_s - \frac{\lambda'}{\mu}$

$$\cancel{L_s = \frac{\rho}{1-\rho} - \frac{(k+1)\rho^{k+1}}{1-(\rho)^{k+1}}}$$

$$L_s = \frac{\lambda}{\mu-\lambda} - \left[\frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1+\left(\frac{\lambda}{\mu}\right)^{k+1}}\right] = 3.131$$

$$= L_s - \frac{\lambda'}{\mu}$$

$$= 3.131 - \frac{3.645}{4} \Rightarrow 2.2110$$

iv) Average a customer spending in Barber shop } $= W_s = \frac{L_s}{\lambda'}$

$= 0.858$

4. ~~Petrol Pump~~ :

4. Railway Yard :

Given,

Arrival rate $= \lambda = 30$ per day

Service rate $= 36$ mins , $k = 9$

$\mu = \frac{1}{36}$ per mins

$= \frac{1}{36} \times 60 \times 24$

$\mu = 40$ per day

$$\frac{\lambda}{\mu} = \frac{30}{40} = \frac{3}{4} = 0.75$$

i) $$P_0 = \frac{1-\left(\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1-0.75}{1-(0.75)^{10}} = \frac{0.25}{0.9436}$$

$= 0.2649$

$$\lambda' = \mu(1-P_0) = 40(1-0.2649)$$

$= 40(0.7351)$

$= 29.404.$

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ii) (Length of queue) = $L_q = L_s - \frac{\lambda'}{\mu}$

$$L_s = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{30}{40 - 30} - \frac{10\left(\frac{30}{40}\right)^{10}}{1 - \left(\frac{30}{40}\right)^{10}}$$

$$= 3 - \frac{10(0.75)^{10}}{1 - (0.75)^{10}}$$

$$= 3 - \frac{0.563}{0.943}$$

$$= 3 - 0.597$$

$$= 2.403$$

$$L_q = 2.403 - \frac{29.404}{40}$$

$$= 2.403 - 0.7351$$

$$= 1.66$$

4. Petrol Pump :

Given

Arrival rate $= \lambda = 10$ per hour

Service rate $= 2$ min

$\mu = \frac{1}{2}$ per min $\qquad k = 5$

$= \frac{1}{2} \times 60$

$\mu = 30$ per hour

$$\frac{\lambda}{\mu} = \frac{10}{30} = \frac{1}{3} = 0.333$$

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$$P_0 = \frac{1 - \left(\frac{\lambda}{\mu}\right)}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} = \frac{1 - 0.333}{1 - (0.333)^6} = 0.667$$

$$\lambda' = \mu(1 - P_0) = 30(1 - 0.667) = 30(0.333)$$
$$= 9.999$$

i) $$L_s = \frac{10}{30-10} - \frac{6(0.333)^6}{1-(0.333)^6}$$

$$= \frac{10}{20} - \frac{0.008}{0.99}$$

$$= 0.5 - 0.0080$$

$$= 0.491$$

ii) $$P(\text{turned away}) = \left(\frac{\lambda}{\mu}\right)^K P_0$$

$$= (0.333)^6 \cdot 0.667$$

$$= 0.002$$

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Unit - 5 Markov Process

Defn:

A Random process in which the future depends only on the present and not on the past values are called markov Process

Markov chain:

A discrete state ~~to~~ markov process is called markov chain.

If for all n,

$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots X_0 = a_0]$

$= P[X_n = a_n | X_{n-1} = a_{n-1}]$

Then the process $\{X_n\}$ is called markov chain.

$a_0, a_1, a_2, \dots a_n$ are states of markov chain.

One - step transition probability:

Let $\{X_n, n \geq 0\}$ be a markov chain, then the conditional probability $P[X_{n+1} = j / X_n = i]$ is called one - step transition probability from the state i to the state j at time t_n to time t_{n+1}.

It is denoted by $P_{ij}(n, n+1)$ or P_{ij}^1

Homogenous markov chain:

A markov chain is homogenous in time if one - step transition probability is independent of step.

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N - step transition Probability :

The conditional probability $P[X_n = a_j / X_0 = i]$ is called N - step transition probability and it is denoted by P_{ij}^{n}

Transition Probability Matrix (or) Probability Vector :

A Vector $P_1, P_2 \ldots, P_0$ is called Probability vector, if i) $P_i \geq 0 \quad \forall i$

ii) $\sum_i P_i = 1$ [i.e Row total = 1]

Regular Matrix :

A stochastic matrix P is said to be regular matrix if all the entries of $P^m > 0$ for some positive integer 'm'

Formula :

Note :

1. Steady state - condition / long run

$\pi = (\pi_1, \pi_2 \ldots \pi_n)$ is said to be steady state

if (i) $\pi P = \pi$

(ii) $\pi_1 + \pi_2 + \ldots + \pi_n = 1$

2. $P[X_n = j / X_m = i] = P_{ij}^{n-m}$

3. $P[X_n = j] = \sum_{i=0}^{k} P[X_n = j / X_0 = i] P[X_0 = i]$

4. $P[A, B, C, D] = P(A/B) P(B/C) P(C/D) P(D)$

5. $P^{(n)} = P^{(n-1)} P$

(i.e) $P^{(1)} = P^{(0)} . P$

$P^{(0)} = P^{(1)} P$

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Imp (Sem) 1. The TPM of a markov chain has three states 1, 2 & 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

and initial distribution is $p^{(0)} = (0.7 \quad 0.2 \quad 0.1)$

Find i) $P[X_2 = 3]$

ii) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

Given

State space = $\{1, 2, 3\}$

Initial Prob dist

$$p^{(0)} = \begin{array}{ccc} 1 & 2 & 3 \\ [0.7 & 0.2 & 0.1] \end{array}$$

$\therefore P[X_0 = 1] = 0.7$

$P[X_0 = 2] = 0.2$

$P[X_0 = 3] = 0.1$

i) $P[X_2 = 3] = \sum_{i=1}^{3} P[X_2 = 3 / X_0 = i] \, P[X_0 = i]$

$= P[X_2 = 3 / X_0 = 1] P[X_0 = 1] +$

$P[X_2 = 3 / X_0 = 2] P[X_0 = 2] +$

$P[X_2 = 3 / X_0 = 3] P[X_0 = 3]$

$= P_{13}^{2-0} \cdot (0.7) + P_{23}^{2-0} (0.2) + P_{33}^{2-0} (0.1)$

$= P_{13}^{2} (0.7) + P_{23}^{2} (0.2) + P_{33}^{2} (0.1)$

To find P_{13}^{2}, P_{23}^{2} & P_{33}^{2}

we find P^2.

$P^2 = P \cdot P$

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$$= \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.43 & 0.31 & 0.26 \\ 2 & 0.24 & 0.42 & 0.34 \\ 3 & 0.36 & 0.35 & 0.29 \end{array}$$

$= (0.26)(0.7) + (0.34)(0.2) + (0.29)(0.1)$

$P[X_2 = 3] = 0.279$

ii) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

$= P[X_3 = 2 / X_2 = 3]\, P(X_2 = 3 / X_1 = 3]\, P(X_1 = 3 / X_0 = 2]\, P(X_0 = 2]$

$= P_{32}^{3-2} \times P_{33}^{2-1} \times P_{23}^{1-0} \times P[X_0 = 2]$

$= P'_{32} \times P'_{33} \times P'_{23} \times 0.2$

$= 0.4 \times 0.3 \times 0.2 \times 0.2$

$= 0.0048$

2. The TPM of a Markov chain with 3 states $(0, 1, 2)$ is

$$P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

and initial state distribution of the chain is $P(X_0 = i)$

Find i) $P[X_2 = 2]$

ii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

Given state space $= \{0, 1, 2\}$

Given initial state prob dist is

$P(X_0 = i) = \frac{1}{3}, \quad i = 0, 1, 2.$

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(i.e) $P(X_0 = 0) = \frac{1}{3}$

$P(X_0 = 1) = \frac{1}{3}$

$P(X_0 = 2) = \frac{1}{3}$

i) $P[X_2 = 2] = \sum_{i=0}^{2} P[X_2 = 2 / X_0 = i] P[X_0 = i]$

$P[X_2 = 2] = P\left[\frac{X_2 = 2}{X_0 = 0}\right] P[X_0 = 0] + P\left[\frac{X_2 = 2}{X_0 = 1}\right]$

$P[X_0 = 1] + P[X_2 = 2 / X_0 = 2] P[X_0 = 2]$

$= P_{02}^{2-0}\left(\frac{1}{3}\right) + P_{12}^{2-0}\left(\frac{1}{3}\right) + P_{22}^{2-0}\left(\frac{1}{3}\right)$

$= P_{02}^{2}\left(\frac{1}{3}\right) + P_{12}^{2}\left(\frac{1}{3}\right) + P_{22}^{2}\left(\frac{1}{3}\right)$

$= \frac{1}{3}\left[P_{02}^{2} + P_{12}^{2} + P_{22}^{2}\right]$

To compute P_{02}^2, P_{12}^2 & P_{22}^2 find P^2

$$P^2 = P \times P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{6} \\ \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{2}{16} & \frac{9}{16} & \frac{1}{4} \end{bmatrix}$$

$= \frac{1}{3}\left[\frac{1}{16} + \frac{3}{16} + \frac{1}{4}\right]$

$= 0.167 = \frac{1}{6}$

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ii) $P[X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2]$

$= P\left[\frac{X_2 = 1}{X_2 = 2}\right] P\left[\frac{X_2 = 2}{X_1 = 1}\right] P\left[\frac{X_1 = 1}{X_0 = 2}\right] P[X_0 = 2]$

$= P_{21}^{3-2} \times P_{12}^{2-1} \times P_{21}^{1-0} \times \left(\frac{1}{3}\right)$

$= P_{21}^{1} \times P_{12}^{1} \times P_{21}^{1} \times \frac{1}{3}$

$= \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3}$

$= \frac{3}{64}$

3 A Salesman ~~tr~~ territory consist of three cities A, B, C. He never sells in the same city on the successive days. If he sell in city A, then the next day he sells in city B. However if he sells in either B or C then the next day he is twice as likely to sell in city A as in the other city. How often does he sell in each of the city in the steady state.

State space = {A, B, C}

The TPM is given by

$$P = \begin{array}{c|ccc} \text{Today} \downarrow & A & B & C \rightarrow \text{Tomorrow} \\ \hline A & 0 & 1 & 0 \\ B & \frac{2}{3} & 0 & \frac{1}{3} \\ C & \frac{2}{3} & \frac{1}{3} & 0 \end{array}$$

To find steady state

$\pi = (\pi_1 \ \pi_2 \ \pi_3)$

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Conditions are

$$\pi P = \pi \quad - (1)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad - (2)$$

Sub eq (1),

$$(\pi_1 \quad \pi_2 \quad \pi_3) \begin{bmatrix} 0 & 1 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\left(0 + \frac{2}{3}\pi_2 + \frac{2}{3}\pi_3 \quad \pi_1 + 0 + \frac{1}{3}\pi_3 \quad 0 + \frac{\pi_2}{3} + 0\right) = (\pi_1 \quad \pi_2 \quad \pi_3)$$

Equating the three term on both sides

$$\frac{\pi_2}{3} = \pi_3$$

$$\boxed{\pi_2 = 3\pi_3} \rightarrow (3)$$

Equating the 2nd term on b.s

$$\pi_1 + \frac{\pi_3}{3} = \pi_2$$

$$\pi_1 + \frac{\pi_3}{3} = 3\pi_3 \quad \text{using } (3)$$

$$\pi_1 = 3\pi_3 - \frac{\pi_3}{3}$$

$$\pi_1 = \frac{9\pi_3 - \pi_3}{3} = \frac{8\pi_3}{3}$$

Sub this in eq (2)

$$(2) \Rightarrow \pi_1 + \pi_2 + \pi_3 = 1$$

$$\frac{8\pi_3}{3} + 3\pi_3 + \pi_3 = 1$$

$$\frac{8\pi_3}{3} + 3\pi_3 + \pi_3 = 1$$

$$\frac{8\pi_3 + 9\pi_3 + 3\pi_3}{3} = 1$$

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$$\frac{20\pi_3}{3} = 1$$

$$\pi_3 = \frac{3}{20}$$

$$\pi_2 = 3\pi_3 = 3\left(\frac{3}{30}\right) = \frac{9}{20}$$

$$\pi_1 = \frac{8}{3}\pi_3 = \frac{8}{3} \times \frac{3}{20} = \frac{8}{20}$$

Steady state distribution $= \pi = \left(\overset{A}{\frac{8}{20}}, \overset{B}{\frac{9}{20}}, \overset{C}{\frac{3}{20}}\right)$

4. A man either drives a car or catch a train, to go office each day. He never goes, two days in a row by train. But he drives one day, that the next day, he is as likely to drive again, as he used to travel like train. Suppose that on the 1st day of the week, the man tosses the fair die & drive to work, if only if 6 appears

Find i) Prop that he taken train on 3rd day.

ii) The prop that he drives to work in the long run. (steady state)

State space $= \{$ Train, Car $\}$

The TPM is given by

$$P = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} \end{array}$$

P (first day he drives the car) = P(getting 6 in a die) $= \frac{1}{6}$

P (first day he goes by train) $= 1 - \frac{1}{6} = \frac{5}{6}$

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Prob dist of the first day

$$p^{(1)} = \begin{array}{cc} T & C \\ \end{array} \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

Prob dist of the 2nd day :

$$p^{(2)} = p^{(1)} \cdot P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}$$

Prob dist of 3rd day

$$p^{(3)} = p^{(2)} \cdot P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{array}{cc} T & C \\ \end{array} \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

P(Third day he Catches the train) = $\frac{11}{24}$

<u>Steady</u> <u>State</u> :

$\pi = (\pi_1, \pi_2)$

Conditions (1) $\pi P = \pi$

(2) $\pi_1 + \pi_2 = 1$

$$① \Rightarrow (\pi_1, \pi_2) \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$\left(\frac{1}{2}\pi_2 \,.\, \pi_1 + \frac{1}{2}\pi_2\right) = (\pi_1, \pi_2)$$

Comparing the 1st term on b.s

$$\frac{1}{2}\pi_2 = \pi_1$$

$$\boxed{\pi_2 = 2\pi_1}$$

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(2) $\Rightarrow \pi_1 + \pi_2 = 1$

$\pi_1 + 2\pi_1 = 1$

$3\pi_1 = 1$

Steady state dist $\pi = \begin{array}{cc} T & C \\ \left(\frac{1}{3} & \frac{2}{3}\right) \end{array}$

ii) P (he drive the cab in long run) $= \frac{1}{3}$

5. A gambler has ₹ 2. He bets ₹ 1 at a time and wins ₹ 1 with the prob $\left(\frac{1}{2}\right)$. He stops playing if he losses ₹ 2 or he wins ₹ 4.

i) Write down the TPM of the associate markov chain.

ii) what is the prop that he lost his money at end of the s th place.

iii) what is the prop the game last more the 7 ~~place~~ game. (continue till the end)

Let X_n represent the amount with the player at the end of the ~~s th~~ n th round

Initially he has Rs. 2, he bets Rs 1 at a time. The game ends. when he lose Rs 2 [2 - 2 = 0]

(or) he wins Rs 4 [2 + 4 = 6]

State space = {0, 1, 2, 3, 4, 5, 6}

P(Player wins) = $\frac{1}{2}$

P(player loss) = $\frac{1}{2}$

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TPM is given by:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

i) Initial prob dist

$$p^{(0)} = \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Prob dist of the 1st round:

$$p^{(1)} = p^{(0)} P$$

$$= \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Prob dist of the 2nd round:

$$p^{(2)} = p^{(1)} P$$

$$= \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{pmatrix} \end{array}$$

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Prob dist of 3rd round.

$= p^{(3)} = p^{(2)} \cdot p$

$$= \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\frac{1}{4} \quad \frac{1}{4} \quad 0 \quad \frac{3}{8} \quad 0 \quad \frac{1}{8} \quad 0 \right)$$

Prob dist of 4th round

$p^{(4)} = p^{(3)} \cdot p$

$$= \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\frac{3}{8} \quad 0 \quad \frac{5}{16} \quad 0 \quad \frac{5}{16} \quad 0 \quad \frac{1}{16} \right)$$

Prob dist of 5th round

$p^{(5)} = p^{(4)} \cdot p$

$$= \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\frac{3}{8} \quad \frac{5}{32} \quad 0 \quad \frac{9}{32} \quad 0 \quad \frac{1}{8} \quad \frac{1}{16} \right)$$

ii) P[he lost his money at end of the 5th play] $= \frac{3}{8}$

$p^{(6)} = p^{(5)} \cdot p$

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\frac{29}{64} \quad 0 \quad \frac{7}{32} \quad 0 \quad \frac{13}{64} \quad 0 \quad \frac{1}{8} \right)$$

$p^{(7)} = p^{(6)} \cdot p$

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$
$$\left(\frac{29}{64} \quad \frac{7}{64} \quad 0 \quad \frac{27}{128} \quad 0 \quad \frac{13}{128} \quad \frac{1}{8} \right)$$

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iii) P (the game last more than 7 th round)

= P (the system is neither at state '0' nor at state 6.)

$= \frac{7}{64} + 0 + \frac{21}{128} + 0 + \frac{13}{128}$

$= \frac{21}{64}$

6. Suppose the prob of a dry day, following a rainy day is 1/3 and that prob of a rainy day following $\frac{1}{2}$. Given may 1 is a dry day. Find the prob that

i) May 3 is also dry day.

ii) May 5 is also a dry day

State space = { dry day, Rainy day }

The TPM is given by

$$P = \begin{array}{c|cc} & \text{dry} & \text{Rainy} \\ \hline \text{dry} & \frac{1}{2} & \frac{1}{2} \\ \text{Rainy} & \frac{1}{3} & \frac{2}{3} \end{array}$$

Given May 1 is dry day.

P [May 1 dry day] = 1

P [May 1 rain day] = 0

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$$\therefore \quad P^{(1)} = \begin{array}{cc} \text{Dry} & \text{Rainy} \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P^{(2)} = P^{(1)} \cdot P$$
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} = \left(\frac{1}{2} + 0 \quad \frac{1}{2} + 0 \right)$$
$$= \left(\frac{1}{2} \quad \frac{1}{2} \right)$$

$$P^{(3)} = P^{(2)} \cdot P = \left(\frac{1}{2} \quad \frac{1}{2} \right) \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$
$$= \left(\frac{1}{4} + \frac{1}{6} \quad \frac{1}{4} + \frac{1}{3} \right)$$
$$= \begin{array}{cc} \text{Dry} & \text{Rainy} \\ \end{array}$$
$$\left(\frac{5}{12} \quad \frac{7}{12} \right)$$

i) $P(\text{May 3 is dry day}) = \frac{5}{12}$

$$P^{(4)} = P^{(3)} \cdot P = \left(\frac{5}{12} \quad \frac{7}{12} \right) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$
$$= \left(\frac{29}{72}, \frac{43}{72} \right)$$

$$P^{(5)} = P^{(4)} \cdot P = \left(\frac{29}{72} \quad \frac{43}{72} \right) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$
$$= \left(\frac{173}{432} \quad \frac{259}{432} \right)$$

ii) $P[\text{May 5 is also dry day}] = \frac{173}{432}$

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7. A fair die is tossed repeatedly. If ~~x~~ x_n denotes maximum number occuring in 'n' tosses, Find the TPM of the markov chain x_n. Also find P^2 & P_{rep} of $P(X_2 = 6)$

State space = {1, 2, 3, 4, 5, 6}

X_n = The Maximum of the numbers occuring in the first n tosses

i) The TPM is given by

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \{1, 2, 3\} = 3 \\ \{2, 3, 4\} = 3 \\ \{5, 3, 3\} = 4 \\ \\ \\ \end{array}$$

ii) $P^2 = P \times P$

$$= \frac{1}{36} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

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$$= \frac{1}{36}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 36 \end{bmatrix}$$

iii) Since initial prob dist is given

$$P^{(0)} = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$
$$\left(\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \right)$$

$$P[X_2 = 6] = \sum_{i=1}^{6} P\left[\frac{X_2 = 6}{X_0 = i}\right] P(X_0 = i)$$

$$= \frac{1}{6} \sum_{i=1}^{6} P_{i6}^{2-0}$$

$$= \frac{1}{6} \times \frac{1}{36}\left[11 + 11 + 11 + 11 + 11 + 36\right] = \frac{91}{216}$$

Classification of Markov chain :

1. Irreducible Markov chain:

A markov chain is said to be irreducible if every state is reachable from every other state.

(i.e) $P_{ij}^{(n)} > 0$ for some n

2. Non - Null Persistent :

A finite and irreducible markov chain is called non - null persistent markov chain.

A periodic :

The period d_i for a return state i, if the GCD for all integers 'm' such that $P_{ij}^{(m)} > 0$

(ie) $d_i = GCD\{m / P_{ij}^{(m)} > 0\}$

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State i is said to be aperiodic if $\boxed{d_i = 1}$

Ergodic :

A non-null persistant and aperiodic markov chain is called ergodic

Imp. 8. 3 boys A, B, C are throwing a ball to each other. A always throws ball to B & B always throws ball to C. C is just likely to throw the ball to B as to A. Show that the process is markovian and classify the state of the markov chain.

State space = { A, B, C }

The TPM is given by

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

x_{n+1} x_n

Since x_{n+1} depends only on x_n

therefore it is markovian (not depends on $x_{n-1}, x_{n-2} \ldots$)

$\therefore$ It is Markovian.

$$P^2 = P \times P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(Do matrix multiplication till no 0 in matrix)

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$$P^3 = P^2 \times P = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$P^4 = P^3 \times P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$P^5 = P^4 \times P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{pmatrix}$$

$P_{11}^{(3)} > 0$ $\quad P_{12}^{(1)} > 0$ $\quad P_{13}^{(2)} > 0$

$P_{21}^{(2)} > 0$ $\quad P_{22}^{(2)} > 0$ $\quad P_{23}^{(1)} > 0$

$P_{31}^{(1)} > 0$ $\quad P_{32}^{(1)} > 0$ $\quad P_{33}^{(2)} > 0$

since $P_{ij}^{(n)} > 0 \quad \forall n$

Given Markov chain is irreducible

Since there are only 3 state [A, B, C]

Markov chain is finite.

Since irreducible and finite therefore markov chain is non-null persistant.

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To find period of states

State A

$P_{11}^{(3)} > 0$, $P_{11}^{(5)} > 0$

Period = GCD $\{3, 5\}$

State A is aperiodic

State B

$P_{22}^{(2)} > 0$, $P_{22}^{(3)} > 0$, $P_{22}^{(4)} > 0$, $P_{22}^{(5)} > 0$

Period = GCD $\{2, 3, 4, 5\} = 1$

State B is aperiodic

State C

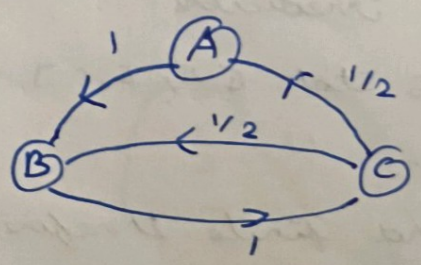
$P_{33}^{(2)} > 0$, $P_{33}^{(3)} > 0$, $P_{33}^{(4)} > 0$, $P_{33}^{(5)} > 0$

Period = GCD $\{2, 3, 4, 5\} = 1$

State C is aperiodic

Since the markov chain is non-null persistent and all its states are aperiodic, therefore it is ergodic.

Transition Diagram :



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State A

A -> B -> C -> A

A -> B -> C -> B -> C -> A

GCD = { 3, 5, ... }

State B

B -> C -> B

B -> C -> A -> B

GCD = { 2, 3, ... }

9. Classify the nature of the markov chain.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$ having state space { 0, 1, 2 }

State space = { 0, 1, 2 }

TPM $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 1 & 0 \end{array}$

$$P^2 = P \times P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^3 = P^2 \times P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = P$$

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$P^4 = P^3 \times P$

$$= \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = P^2$$

$P_{11}^{(2)} > 0$ $\quad P_{12}^{(1)} > 0$ $\quad P_{13}^{(2)} > 0$

$P_{21}^{(1)} > 0$ $\quad P_{22}^{(2)} > 0$ $\quad P_{23}^{(1)} > 0$

$P_{31}^{(2)} > 0$ $\quad P_{32}^{(1)} > 0$ $\quad P_{33}^{(2)} > 0$

since $P_{ij}^{(n)} > 0$

It is irreducible

Also give state space $= \{0, 1, 2\}$ are finite

$\therefore$ It is non - null Persistant.

<u>Period of states</u>:

<u>State 0</u>:

$P_{11}^{(2)} > 0$, $P_{11}^{(4)} > 0, \ldots$

Period $= GCD(2, 4) = 2$

$\therefore$ It has period '2'

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State 1

$P_{22}^{(2)} > 0$, $P_{22}^{(4)} > 0 \ldots$

Period = GCD (2, 4) = 2

State 2 :

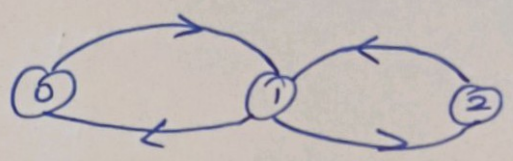
$P_{33}^{(2)} > 0$, $P_{33}^{(4)} > 0$

Period = GCD (2, 4) = 2

∴ All the states have period '2'
(i.e periodic)

∴ Markov chain is not Ergodic

(or)



Since all the states are reachable from other state.

∴ It is irreducible.

The given states are 0, 1, 2 therefore it is finite

Since it is ~~finite~~, ~~it~~ irreducible and finite, it is not null persistent.

Period of state :

State 0

0 → 1 → 0 (2 steps)

0 → 1 → 2 → 1 → 0 (4 steps)

Period = GCD{2, 4} = 2

It is Periodic.

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State 1

1 → 0 → 1 (2 steps)

1 → 2 → 1 → 0 → 1 (4 steps)

Period = GCD (2, 4) = 2

State 2

2 → 1 → 2

2 → 1 → 0 → 1 → 2

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ & \frac{1}{2} & 0 & \frac{1}{2} \\ & 0 & 1 & 0 \end{array}$$

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