UNIT - I - Random Variables

Probability basic concepts and Axioms - conditional probability, Multiplication Theorem - Discrete and continuous random variables - Probability mass function, cdf. Continuous random variables - Pdf and cdf applications - Expectation and variance - Problems on Expectations and Variance - Moment Generating function - Problems on MGF - Functions of random variables - Problems on Functions of random Variables - Tchebycheff's inequality - Introduction to theoretical distribution - Formula and application of Tchebycheff's inequality - Applications of random variables in Engineering.

Random Experiment :

An experiment whose output is uncertain even though all the ~~co~~ outcomes are known.

Ex : 1. Birth of a baby 2. Tossing a fair coin 3. Throwing a fair ~~coin~~ die.

Sample space :

The set of all possible outcomes in a random experiment is called as sample space. It is denoted by S.

Ex :

1. For a birth of baby, $S = \{M, F\}$
2. For Tossing a fair coin, $S = \{H, T\}$
3. For throwing a fair die, $S = \{1, 2, 3, 4, 5, 6\}$.

Event : A subset of a sample space S is called as event. It is denoted by A.

Ex : Find the possible outcomes which is at least two heads, when we tossing three coins simultaneously.

$S = \{HHH, HTH, HHT, THH, TTH, THT, HTT, TTT\}$

$E = \{HHH, HTH, HHT, THH\}$

Mutually Exclusive Events :

Two events A and B are said to be mutually exclusive events if they do not occur simultaneously. If A and B are mutually exclusive, then $A \cap B = \phi$.

Ex: $S = \{HH, HT, TH, \cancel{HH}\ TT\}$

$A = \{HH\}$ $B = \{HT\}$, $A \cap B = \{H\}$

$A = \{HH\}$ $B = \{TT\}$; $A \cap B = \phi$

Independent Events :

Two events A and B are said to be independent if occurence of A does not affect the occurence of B.

If A and B are independent events then $P(A \cap B) = P(A)\, P(B)$.

$S_1 = \{H, T\}$ $S_2 = \{H, T\}$ getting H in S_1 H in S_2 } are independent

Probability :

Probability of an event A is defined by $P(A) = \dfrac{n(A)}{n(S)}$

ie) $P(A) = \dfrac{\text{number of cases favourable to A}}{\text{Total number of cases}}$.

Results :

i) $P(\phi) = 0$

ii) $P(\bar{A}) = 1 - P(A)$, for any event A.

iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events A and B

Conditional Probability :

If the probability of the event A provided the event B has already occured is called the conditional probability and is defined as,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.$$

Similarly, $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$; $P(A) \neq 0$

Note : If A and B are independent events, then

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\, \cancel{P(B)}}{\cancel{P(B)}}$$

$$\therefore P(A/B) = P(A)$$

Similarly $P(B/A) = P(B)$.

Random Variable :

The output of the random experiment is assigned by a real number. That number is called a random variable. It is denoted by X.

Random Variable
- Discrete Random Variable
- Continuous Random Variable

Discrete Random Variable :

A random variable X is said to discrete, if it is finite or countable.

Probability Mass Function (PDF)

Let X be a discrete random variable. Then $P(X = x_i) = p(x_i) = p$ said to be probability mass function,

i) $0 \leq p(x_i) \leq 1$

ii) $\sum p(x_i) = 1$

$X = x_i$	x_1	x_2	...	x_r	...
$P(X = x_i)$	p_1	p_2	...	p_r	...

Cumulative Distribution Function (CDF)

Let X be a RV. Then CDF of X is

i) $F(x) = P(X \le x) = \sum_{X \le x} p(x)$

$p(x_i) = P(X = x_i) = F(x_i) - F(x_{i-1})$

where F is the distribution function of random variable X.

Mathematical Expectation:

$$E(X) = M_X = \sum_i x_i \, p(x_i)$$

$$E(X^2) = \sum_i x_i^2 \, p(x_i)$$

$$Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2.$$

Example : 1 A discrete random variable has the following probability distribution,

$X = x$	-2	-1	0	1	2	3
$p(X = x)$	0.1	k	0.2	$2k$	0.3	$3k$

i) Find value of k

ii) Find $P(X < 2)$

iii) $P(-2 < X < 2)$, iv) Find distribution fn of X.

Solution :

i) To find k :

$$\sum p(X = x_i) = 1$$

$$\therefore \; 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$6k + 0.6 = 1$$

$$6k = 1 - 0.6$$

ie) $6k = 0.4$

$$k = 0.4/6$$

$$\boxed{k = 0.0667}$$

ii) $P(X < 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$

$= 0.1 + k + 0.2 + 2k$

$= 0.3 + 3k$

$= 0.3 + 3(0.0667)$

$= 0.5$

$$\boxed{P(X < 2) = 0.5}$$

iii) $P(-2 < X < 2)$

$= P(X = -1) + P(X = 0) + P(X = 1)$

$= k + 0.2 + 2k$

$= 0.2 + 3k$

$= 0.2 + 3(0.0667)$

$= 0.4$

$$\boxed{P(-2 < X < 2) = 0.4}$$

iv) Distribution Function of x

$F(-2) = P(x=-2) = 0.1$

$F(-1) = P(x=-2) + P(x=-1) = 0.1 + k = 0.1667$

$F(0) = P(X=-2) + P(X=-1) + P(X=0)$

$= 0.1 + k + 0.2$

$= 0.3 + k$

$F(0) = 0.3667$

$F(1) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + k + 0.2 + 2k$

$= 0.3 + (0.0667)3$

$F(1) = 0.5$

$F(2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2)$

$= 0.5 + 0.3$

$F(2) = 0.8$

$F(3) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.8 + 3(0.0667)$

$F(3) = 1$

x	−2	−1	0	1	2	3
$p(x)$	0.1	0.0667	0.2	0.1334	0.3	0.2
$F(x)$	0.1	0.1667	0.3667	0.5	0.8	1

Example : 2 A discrete random varia[ble] has the following probability distributi[on]

x	0	1	2	3	4	5	6	7
$p(x)$	a	3a	5a	7a	9a	11a	13a	15a

i) Find the value of a ii) $P(X<3)$, iii) P(0... iv) $P(X \geq 3)$, v) Find CDF of X vi) Find mean vii) Var(X).

Solution : i) To find a

$$\sum_{i=0}^{8} P(X=x_i) = 1$$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a$

$81a = 1$

$\boxed{a = 1/81}$

x	0	1	2	3	4	5	6	7
$p(x)$	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81

ii) $P(X<3)$

$P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= 1/81 + 3/81 + 5/81 = 9/81$

$\boxed{P(X<3) = 1/9}$

iii) $P(0<X<3)$

$P(0<X<3) = P(X=1) + P(X=2)$

$= 3/81 + 5/81 = 8/81$

(iv) $P(X \geq 3)$

$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$= 7/81 + 9/81 + \frac{11}{81} + \frac{13}{81} + \frac{15}{81} + 17/81$

$= \frac{72}{81} = 8/9$

$\boxed{P(X \geq 3) = 8/9}$

v) Distribution Function of X :

x	0	1	2	3	4	5	6	7	8
$p(x)$	1/81	3/81	5/81	7/81	9/81	11/81	13/81	15/81	17/81
$F(x)$	1/81	4/81	9/81	16/81	25/81	36/81	49/81	64/81	1

vi) $E(x)$

$E(x) = \sum x\, p(x)$

$= 0 \cdot \frac{1}{81} + 1 \cdot 3/81 + 5/81 \cdot 2 + 3 \cdot 7/81 + 4 \cdot 9/81 + 5 \cdot 11/81 + 6 \cdot 13/81 + 7 \cdot 15/81 + 8 \cdot 17/81$

$= \frac{3}{81} + \frac{10}{81} + \frac{21}{81} + \frac{36}{81} + \frac{55}{81} + \frac{78}{81} + \frac{105}{81} + \frac{136}{81} = \frac{444}{81}$

$\boxed{E(x) = \frac{444}{81}}$

vii) $Var(x)$

$Var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \sum x^2 p(x)$

$= 0^2 \cdot \frac{1}{81} + 1^2 \cdot 3/81 + 2^2 \cdot 5/81 + 3^2 \cdot 7/81 + 4^2 \cdot 9/81 + 5^2 \cdot 11/81 + 6^2 \cdot 13/81 + 7^2 \cdot 15/81 + 8^2 \cdot 17/81$

$= 0 + \frac{3}{81} + \frac{20}{81} + \frac{144}{81} + \frac{275}{81} + \frac{468}{81} + \frac{735}{81} + \frac{1088}{81}$

$E(x^2) = \frac{2733}{81}$

$Var(x) = \frac{2733}{81} - \left(\frac{444}{81}\right)^2$

$= \frac{2733}{81} \cdot \frac{81}{81} - \frac{(444)^2}{81^2}$

$= \frac{221373 - 197136}{6561} = \frac{24237}{6561}$

$\boxed{Var(x) = \frac{2693}{729}}$

Example : 3 A discrete random variable x has the probability function shown below.

x	0	1	2	3	4	5	6	7
$p(x)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2+a$

i) Find a ii) $P(X<6)$ (iii) $P(X \geq 6)$,

iv) $P(0<X<4)$, v) $P(X<6)/P(X \geq 4)$

vi) the smallest value of λ such that $P(X \leq \lambda) > 1/2$

Solution :

i) To find a :

WKT $\sum_i p(x_i) = 1$

$0 + a + 2a + 2a + 3a + a^2 + 2a^2 + 7a^2 + a = 1$

$10a^2 + 9a - 1 = 0$

$a = \frac{-9 \pm \sqrt{81 - 4 \cdot 10 \cdot (-1)}}{2 \cdot 10}$

$= \frac{-9 \pm \sqrt{81 + 40}}{20}$

$= \frac{-9 \pm \sqrt{121}}{20} = \frac{-9 \pm 11}{20}$

$= \frac{-9+11}{20}, \frac{-9-11}{20}$

$= \frac{2}{20}, \frac{-20}{20}$

$= 0.1, -1$

$\boxed{a = 0.1}$ $\because 0 \le p(x) \le 1.$

x	0	1	2	3	4	5	6	7
$p(x)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

ii) To find $p(X < 6)$:

$P(X < 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 0 + 0.1 + 0.2 + 0.2 + 0.3 + 0.01 \cancel{+ 0.02}$

$= 0.81$

$\boxed{P(X < 6) = 0.81}$

iii) $P(X \geq 6)$

$P(X \geq 6) = P(X=6) + P(X=7)$

$= 0.02 + 0.17$

$\boxed{P(X \geq 6) = 0.19}$

(or) $P(X \geq 6) = 1 - P(X < 6)$

$= 1 - 0.81$

$\boxed{P(X \geq 6) = 0.19}$

iv) $P(0 < x < 4)$

$P(0 < x < 4) = P(X=1) + P(X=2) + P(X=3)$

$= 0.1 + 0.2 + 0.2$

$\boxed{P(0 < x < 4) = 0.5}$

v) $P(X < 6 / X \geq 4)$

$P(X < 6 / X \geq 4) = \frac{P(X < 6) \cap P(X \geq 4)}{P(X \geq 4)}$

$= \frac{P(4 \le x \le 5)}{P(X \geq 4)} = \frac{P(X=4) + P(X=5)}{P(X=4) + P(X=5) + P(X=6) + P(X=7)}$

$= \frac{0.3 + 0.01}{0.3 + 0.01 + 0.02 + 0.17} = \frac{0.31}{0.5} = \boxed{0.62}$

ii) The smallest value of λ such that

$p(X \leq \lambda) > 1/2$ $\quad$ $F(x) = P(X \leq x)$

$F(0) = P(X \leq 0)$

$p(X \leq \lambda) > 1/2$ is $F(\lambda) > 1/2$

$F(0) = 0 < 1/2$

$F(1) = 0.1 < 1/2$

$F(2) = 0.3 < 1/2$

$F(3) = 0.5 \leq 1/2$

$F(4) = 0.8 > 1/2$ $\quad$ $P(X \leq 4)$ $0.8 > 1/2$

$P(X \leq 5) = 0.89 > 1/2$

$\boxed{\lambda = 4}$

Ex : 4 If $P(X=x) = \begin{cases} \frac{x}{15} & \text{for } x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$

Find i) $P(X=1 \text{ or } X=2)$ ii) $P(1/2 < X < 5/2 \,/\, X > 1)$

Solution :

x	$\cancel{0}$	1	2	3	4	5
$p(x)$		$1/15$	$2/15$	$3/15$	$4/15$	$5/15$

i) $P(X=1 \text{ or } X=2) = P(X=1) + P(X=2)$

$= \frac{1}{15} + \frac{2}{15} = 3/15 = 1/5$

$\boxed{P(X=1 \text{ or } X=2) = 1/5}$

ii) $P(1/2 < X < 5/2 \,/\, X > 1)$

$P(1/2 < X < 5/2 \,/\, X > 1) = \frac{P(1/2 < X < 5/2) \cap P(X > 1)}{P(X > 1)}$

$= \frac{P(X=2)}{P(X=2) + P(X=3) + P(X=4) + P(X=5)}$

$= \frac{2/15}{2/15 + 3/15 + 4/15 + 5/15} = \frac{2/\cancel{15}}{14/\cancel{15}} = \frac{1}{7}$

$P\left(\frac{1}{2} < X < 5/2 \,/\, X > 1\right) = 1/7$

Ex : 5 Let the random variable takes the values 1, 2, 3 and 4 such that

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

Find PMF and CDF.

Solution :

Since

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

Let $2P(X=1) = k$ $\quad 3P(X=2) = k$ $\quad P(X=3) = k$

$5P(X=4) = k$

PMF of X

x	1	2	3	4
$p(x)$	$k/2$	$k/3$	k	$k/5$

To find k :

$\sum_i p(x_i) = 1$

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\frac{15k + 10k + 30k + 6k}{30} = 1$

$61k = 30$

$\Rightarrow \boxed{k = 30/61}$

$= (1/4)\left[(1-1/4)^{-1}\right]$

$= 1/4\left[4/3\right] = 1/3$

$\boxed{P(x \text{ is even}) = 1/3}$

v) P (x is divisible by 3)

P(x is divisible by 3)

$= P(X=3) + P(X=6) + P(X=9) + \dots$

$= (1/2)^3 + (1/2)^6 + (1/2)^9 + \dots$

$= (1/2)^3\left[1 + (1/2)^3 + (1/2)^6 + \dots\right]$

$= \frac{1}{8}\left[1 + 1/8 + (1/8)^2 + (1/8)^3 + \dots\right]$

$= \frac{1}{8}\left[(1-1/8)^{-1}\right] = \frac{1}{8}\left[8/7\right] = 1/7$

$\boxed{P(x \text{ is divisible by } 3) = 1/7}$

vi) $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + \dots$

$= (1/2)^5 + (1/2)^6 + (1/2)^7 + \dots$

$= \frac{1}{32}\left[1 + 1/2 + (1/2)^2 + \dots\right]$

$= \frac{1}{32}\left[(1-1/2)^{-1}\right] = \frac{1}{32} \times 2 = \frac{1}{16}$

$\boxed{P(X \geq 5) = 1/16}$

vii) CDF of X :

$F(x) = P(X \leq x)$

$= P(X=1) + P(X=2) + \dots + P(X=x)$

$= (1/2) + (1/2)^2 + \dots + (1/2)^x$

$= \frac{1}{2}\left[1 + 1/2 + \dots + (1/2)^{x-1}\right]$

$= \frac{1}{2}\left[\frac{1-(1/2)^x}{1-1/2}\right] = \cancel{1/2}\left[\frac{1-(1/2)^x}{\cancel{1/2}}\right]$

$F(x) = 1 - (1/2)^x \; ; x = 1, 2, 3, \dots$

Ex : 7 If X has the probability distribution,

x	-1	0	1	2
$p(x)$	0.3	0.1	0.4	0.2

then find $E(x)$, $E(x^2)$, $Var(x)$, $E(2x+1)$, $Var(2x+1)$.

Solution :

$E(x) = \sum x\, p(x)$

$= -1(0.3) + 0(0.1) + 1(0.4) + 2(0.2)$

$= -0.3 + 0.4 + 0.4$

$\boxed{E(x) = 0.5}$

$E(x^2) = \sum_i x_i^2\, p(x_i)$

$= (-1)^2(0.3) + 0^2(0.1) + 1^2(0.4) + 2^2(0.2)$

$= 0.3 + 0.4 + 0.8$

$\boxed{E(x^2) = 1.5}$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 1.5 - (0.5)^2$$

$$= 1.5 - 0.25$$

$$\boxed{Var(x) = 1.25}$$

$$E(2x+1) = 2E(x) + 1$$

$$= 2(0.5) + 1$$

$$\boxed{E(2x+1) = 2}$$

$$Var(2x+1) = 4\,Var(x)$$

$$= 4(1.25)$$

$$= 5$$

$$\boxed{Var(2x+1) = 5}$$

Ex : 8 A fair coin is tossed three times. Let x be the number if tails appearing. Find the probability distribution of x. And also calculate E(x).

Solution :

$S = \{HHH, HTH, THH, HHT, TTH, THT, HTT, TTT\}$

Let X be the number of tails appearing.

$x(HHH) = 0$; $X(THH) = 1$, $x(HHT) = 1$,

$x(TTH) = 2$; $x(THT) = 2$; $x(HTT) = 2$;

$x(TTT) = 3$; $x(HTH) = 1$

$X = \{0, 1, 2, 3\}$

$P(X=0) = 1/8$; $P(X=1) = 3/8$

$P(X=2) = 3/8$; $P(X=3) = 1/8$

x	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

To find E(x) :

$$E(x) = \sum x\, p(x)$$

$$= 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8$$

$$= 0 + \frac{3}{8} + \frac{3 \cdot 2}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = 3/2$$

$$\boxed{E(x) = 1.5}$$

Ex : 9 A man draws 3 balls fr[om] an urn containing 5 white and 7 [black] balls. He gets Rs. 10 for each white balls and Rs. 5 each black ball. Fi[nd] his expectation.

Solution :

Totall No of balls = 5 White + 7 black =

Three balls are drawn at random. P[ossible] cases are

i) 3 white balls drawn

ii) 3 black balls drawn

iii) 2 white and 1 black balls drawn

[iv) 1] white and 2 black balls drawn

$P[\text{3 white balls drawn}] = \frac{5C_3}{12C_3} = 1/22$

$P[\text{3 black balls drawn}] = \frac{7C_3}{12C_3} = \frac{\frac{7\times6\times5}{3\times\cancel{2}\times1}}{\frac{12\times11\times10}{3\times2\times1}} = 7/44$

$P[\text{2 white and 1 black balls drawn}] = \frac{5C_2 \times 7C_1}{12C_3}$

$= \frac{\frac{5\times\cancel{4}^{2}}{\cancel{2}\times\cancel{1}}\times 7}{\frac{\cancel{12}^{4}\times11\times10}{\cancel{3}\times2\times\cancel{1}}}$

$= \frac{5\times\cancel{2}\times7}{\cancel{4}_2\times11\times\cancel{10}_2} = 7/22$

$P[\text{1 white and 2 black balls drawn}] = \frac{5C_1 \times 7C_2}{12C_3}$

$= \frac{5\times\frac{7\times\cancel{6}^{3}}{\cancel{2}\times1}}{\frac{\cancel{12}^{2}\times11\times10}{3\times\cancel{2}\times1}} = \frac{5\times7\times3}{2\times11\times\cancel{10}_2} = \frac{21}{44}$

Cost of 3 white balls = 30

Cost of 3 black balls = 15

Cost of 2 white and 1 black balls = 25

Cost of 1 white and 2 black balls = 20

Let x denote cost

S	3W	3B	2W 1B	1W 2B
x	30	15	25	20
$p(x)$	1/22	7/44	7/22	21/44

$E(x) = \sum x\, p(x)$

$= 30\times 1/22 + 15\times 7/44 \times 25\times\frac{7}{22} + 20\times\frac{21}{44}$

$= \frac{30}{22} + \frac{105}{44} + \frac{175}{22} + \frac{420}{44}$

$= \frac{60}{44} + \frac{105}{44} + \frac{350}{44} + \frac{420}{44}$

$= \frac{935}{44} = 21.25$

$\boxed{E(x) = \text{Rs } 21.25}$

<u>Ex: 10</u> Let x be a discrete random variable with distribution function given by

$$F(x) = \begin{cases} 0 & ; x < 1 \\ \frac{1}{3} & ; 1 \le x < 4 \\ \frac{1}{2} & ; 4 \le x < 6 \\ 5/6 & ; 6 \le x < 10 \\ 1 & ; x \ge 10. \end{cases}$$

Find i) PMF of x ii) $p(2 < x < 6)$ iii) mean of x iv) var (x).

<u>Solution:</u>

$P(X=1) = 1/3 - 0 = 1/3$

$P(X=4) = \frac{1}{2} - 1/3 = 1/6$

$P(X=6) = 5/6 - 1/2 = 1/3$

$P(X=10) = 1 - 5/6 = 1/6$

i) Probability distribution of x

x	1	4	6	10
$p(x)$	1/3	1/6	1/3	1/6

ii) $P(2<x<6) = P(x=4) = 1/6$

iii) $E(x) = \sum x\, p(x)$

$= 1 \cdot 1/3 + 4 \cdot 1/6 + 6 \cdot 1/3 + 10 \cdot 1/6$

$= 1/3 + 2/3 + 6/3 + 5/3$

$= 14/3$

$\boxed{E(x) = 14/3}$

iv) Var(x)

$E(x^2) = \sum x^2 p(x)$

$= 1^2 \cdot 1/3 + 4^2 \cdot 1/6 + 6^2 \cdot 1/3 + 10^2 \cdot 1/6$

$= 1/3 + 16/6 + 36/3 + 100/6$

$= 1/3 + 8/3 + 36/3 + 50/3$

$= 95/3$

$Var(x) = E(x^2) - [E(x)]^2$

$= \frac{95}{3} - \frac{196}{9} = \frac{285-196}{9} = \frac{89}{9}$

Moments :

Moments about the origin (Raw moments)

The rth moment about origin is

$\mu_r' = E[x^r]$.

If X is a discrete random variable, then the rth moment about origin is given by

$$\mu_r' = \sum_x x^r p(x)$$

Note : $\mu_1' = E(x)$; $\mu_2' = E(x^2)$

Variance $= \mu_2' - (\mu_1')^2$

Moments about the mean [Central M

The rth moment about mean is

$\mu_r = E[(X-\mu)^r]$, where μ is the mean of X.

$\mu_1 = E[(X-\mu)^1] = E(x) - \mu$

$\mu_2 = E[(X-\mu)^2] = E[X^2] + (E[x])^2$

$\mu_2 = E[x^2] - (E[x])^2 = $ Variance

Ex : 1 The first 3 moments about t origin are 5, 26, 78. Find the first three moments about the value 3.

Solution : Given that $\mu_1' = 5$; $\mu_2' = 2$ $\mu_3' = 78$.

$\mu_r = E[(x-\mu)^r]$; $\mu = 5$

$\cancel{\mu_1} = \cancel{E[(x-5)]} = \cancel{E(x) - E[5]}$

$\cancel{= 5-5}$

$\cancel{\boxed{\mu_1 = 0}}$

$= E\left[\cancel{(x-3+5-5)}\right]$

$= \cancel{E\left[(x-3+2)\right]}$

$= \cancel{E(x-3)}$

$\mu_r = E\left[(x-\mu^r)\right]$

$\mu_1 = E\left[(x-3)\right] = E(x) - 3E(1)$

$= 5 - 3$

$\mu_1 = 2$

$\therefore \boxed{E\left[(x-3)\right] = 2}$

$E\left[(x-3)^2\right] = E\left[(x^2 - 6x + 9)\right]$

$= E(x^2) - 6E(x) + 9E(1)$

$= 26 - 6(5) + 9$

$\boxed{\mu_2 = E\left[(x-3)^2\right] = 5}$

$E\left[(x-3)^3\right] = E\left[x^3 - 3 \cdot x^2 \cdot 3 + 3 \cdot x \cdot 9 - 27\right]$

$= E\left[x^3\right] - 9E\left[x^2\right] + 27E\left[x\right] - 27E\left[1\right]$

$= 78 - 9(26) + 27(5) - 27$

$= -48$

$\boxed{\mu_3 = E\left[(x-3)^2\right] = -48}$

Moment Generating Function (MGF).

Let X be a discrete random variable, then MGF is given by $\left(M_x(t) = E[e^{tx}]\right)$

$$M_x(t) = \sum_x e^{tx} p(x)$$

Note :

$$M_x(t) = \mu_r' = \left[\frac{d^r}{dt^r}\left[M_x(t)\right]\right]_{t=0}.$$

Ex : 1 Let X be a number occur when a die is thrown. Find MGF and hence find mean and variance of X.

Solution :

Let X be a number occur when a die is thrown.

x	1	2	3	4	5	6
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

To find MGF :

$M_x(t) = E(e^{tx}) = \begin{cases} \sum e^{tx} p(x) & \text{if } x \text{ is d} \\ \int e^{tx} f(x)dx & \text{if } x \text{ is c} \end{cases}$

$$M_x(t) = \sum_x e^{tx} p(x)$$

$$= e^t \cdot 1/6 + e^{2t} \cdot 1/6 + e^{3t} \cdot 1/6 + e^{4t} \cdot 1/6 + e^{5t} \cdot 1/6 + e^{6t} \cdot 1/6$$

$$M_x(t) = \frac{1}{6}\left\{e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}\right\}$$

To find Mean :

$$M_x(0) = \mu_1' = \left[\frac{d}{dt}\left[M_x(t)\right]\right]_{t=0}$$

$$= \left[\frac{1}{6}\left\{ e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t} \right\}\right]_{t=0}$$

$$= \frac{1}{\cancel{6}_2}\left(\frac{\cancel{6}^3 \times 7}{\cancel{2}}\right) = 7/2$$

$$\boxed{E(x) = 7/2}$$

To find Variance :

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[\frac{1}{6}\left\{ e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t} \right\}\right]_{t=0}$$

$$= \frac{1}{6}\left\{ 1+4+9+16+25+36 \right\} = \frac{91}{6}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{91}{6} - \frac{49}{4}$$

$$= \frac{364 - 294}{24} = \frac{70}{24}$$

$$\boxed{Var(x) = 70/24}$$

Ex:2 Let x be a Random Variable with PMF $p(x) = (1/2)^x$; $x = 1,2,3.$

Find MGF and hence find mean and variance of x.

Solution :

To find MGF :

$$M_x(t) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} (1/2)^x$$

$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left(\frac{e^t}{2}\right)^1 + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2}\left\{ 1 + e^t/2 + (e^t/2)^2 + \dots \right.$$

$$= \frac{e^t}{2}\left[\left(1 - \frac{e^t}{2}\right)^{-1}\right]$$

$$= e^t/2 \left[\left(\frac{2 - e^t}{2}\right)^{-1}\right]$$

$$= \frac{e^t}{\cancel{2}} \cdot \frac{\cancel{2}}{2 - e^t} = \frac{e^t}{2 - e^t}$$

$$\boxed{M_x(t) = \frac{e^t}{2 - e^t}}$$

To find Mean :

$$E(x) = \left(\frac{d}{dt}\left[M_x(t)\right]\right)_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0} = \left(\frac{\cancel{e^t}}{2 - \cancel{e^t}}\right)$$

$$= \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0} =$$

$$= 2/1 = 2 \quad \therefore \boxed{E(x) = 2}$$

To find Variance :

$$E(X^2) = \left[\frac{d^2}{dt^2}\left[M_x(t)\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t(0-e^t).2(2-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \frac{4+2}{1} = 6$$

$E(X^2) = 6$

$Var(X) = E(X^2) - (E(X))^2$

$= 6 - 4 = 2$

$\boxed{Var(X) = 2}$

Continuous Random Variable :

A random variable X is said to be continuous if it can be taken all possible values in an interval.

Probability Density Function :

The probability density function of a random variable X denoted by f(x) has the following properties,

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Note :

i) $P(X=a) = 0$

ii) $P(X \leq a) = \int_{-\infty}^{a} f(x)\,dx = P(X < a)$

iii) $P(X \geq a) = \int_{a}^{\infty} f(x)\,dx = P(X > a)$

Cumulative distribution Function :

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx.$$

Ex : 1 A continuous random variable X has pdf, $f(x) = kx^2 e^{-x}; x \geq 0$. Find the value of k.

Solution :

Given that X is a continuous random variable in $x \geq 0$.

WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} k e^{-x}. x^2\,dx = 1$$

$$k\int_{0}^{\infty} x^2 e^{-x}\,dx = 1$$

$u = x^2$ $\quad v = e^{-x}$

$u' = 2x$ $\quad v_1 = -e^{-x}$

$u'' = 2$ $\quad v_2 = e^{-x}$

$\quad v_3 = -e^{-x}$

$$k\left[-x^2e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k\left[0-(-2)\right] = 1$$

$$2k = 1 \Rightarrow \boxed{k = 1/2}$$

Ex : 2 The pdf of a continuous random variable X is given by $f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

Solution :

WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx + 0 = 1$$

$$a\left(x^2/2\right)_0^1 + a\left(x\right)_1^2 + \left(3ax - ax^2/2\right)_2^3 = 1$$

$$a\left(1/2 - 0\right) + a(2-1) + \left(3a \cdot 3 - \frac{a \cdot 9}{2} - 6a + 2a\right)$$

$$a/2 + a + \left(5a - \frac{9a}{2}\right) = 1$$

$$\frac{a}{2} + a + \frac{a}{2} = 1$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

Ex : 3 If the pdf of X is $f(x)$ ... then find $f(x) = \begin{cases} k & 0 < x < 1 \\ 2k & 1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

Find i) k, ii) $P(1/2 < x < 3/2)$

Solution :

WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$0 + \int_0^1 k\,dx + \int_1^2 2k\,dx + 0 = 1$$

$$k(x)_0^1 + 2k(x)_1^2 = 1$$

$$k(1-0) + 2k(2-1) = 1$$

$$3k = 1 \Rightarrow \boxed{k = 1/3}$$

Ex : 4 A RV X has the PDF $f(x) = \ldots$

Find i) k, ii) $P(X < 1/2)$ iii) $P(1/4 < x < 1/2)$

iv) $P(X > 3/4 \,/\, x > 1/2)$.

Solution :

To find k :

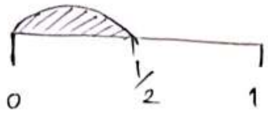
WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 kx\,dx = 1$$

$$\Rightarrow k\left(\frac{x^2}{2}\right)_0^1 = 1$$

$$\Rightarrow k/2 = 1 \Rightarrow \boxed{k = 2}$$

ii) $P(X < 1/2)$.



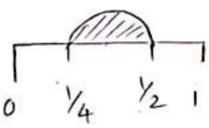
$$P(X < 1/2) = \int_0^{1/2} f(x)\,dx$$

$$= \int_0^{1/2} 2x\,dx$$

$$= 2\left(x^2/2\right)_0^{1/2} = 2\left(\frac{(1/2)^2}{2} - 0\right) = 1/4$$

$$\boxed{P(X < 1/2) = \frac{1}{4}}$$

iii) $P(1/4 < X < 1/2)$



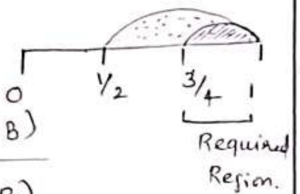
$$P(1/4 < X < 1/2) = \int_{1/4}^{1/2} f(x)\,dx$$

$$= \int_{1/4}^{1/2} 2x\,dx = \cancel{2}\left(x^2/\cancel{2}\right)_{1/4}^{1/2}$$

$$= 1/4 - 1/16 = 3/16$$

$$\boxed{P(1/4 < X < 1/2) = 3/16}$$

iv) $P(X > 3/4 \,/\, X > 1/2)$



WKT $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$P(X > 3/4 \,/\, X > 1/2) = \frac{P(X > 3/4)}{P(X > 1/2)}$$

$$P(X > 3/4) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx$$

$$= \cancel{2}\left(x^2/\cancel{2}\right)_{3/4}^{1} = 1 - \frac{9}{16} = 7/16$$

$$\boxed{P(X > 3/4) = 7/16}$$

$$P(X > 1/2) = 1 - P(X \le 1/2) = 1 - 1/4 = 3/4$$

$$\therefore P(X > 3/4 \,/\, P(X > 1/2)) = \frac{7/16}{3/4} = \frac{7}{\cancel{16}_4} \times \frac{4}{3}$$

$$\boxed{P(X > 3/4 \,/\, P(X > 1/2)) = 7/12}$$

Ex : 5 The diameter of an electric bulb is a continuous random variable X with pdf $f(x) = kx(1-x)$, $0 \le x \le 1$. Find i) k ii) $F(x)$ iii) $P\left(X \le 1/2 \,\middle|\, \frac{1}{3} < x < 2/3\right)$

Solution :

i) To find k :

WKT, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

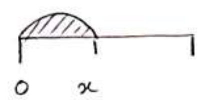
$$\int_0^1 kx(1-x)\,dx = 1$$

$$k\left[\frac{x^2}{2} - x^3/3\right]_0^1 = 1$$

$$\Rightarrow k\left[\left(\frac{1}{2} - \frac{1}{3}\right) - 0\right] = 1$$

$$\Rightarrow k/6 = 1 \Rightarrow \boxed{k = 6}$$

To find $F(x)$:



$$F(x) = P(X \le x) = \int_0^x f(x)\,dx$$

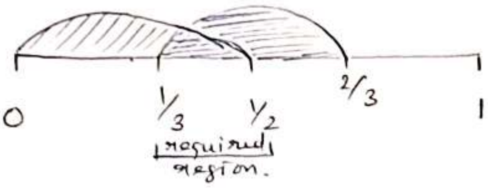
$$= \int_0^x 6x(1-x)\,dx$$

$$= 6\left(x^2/2 - x^3/3\right)_0^x = \cancel{6}\left(\frac{3x^2 - 2x^3}{\cancel{6}}\right)$$

$$F(x) = \begin{cases} 0 ; & x < 0 \\ 3x^2 - 2x^3 & 0 \le x < 1 \\ 1 & x \geq 1 \end{cases}$$

iii) To find $P(X \le 1/2 \,/\, 1/3 < X < 2/3)$

WKT $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$



$$P(X \le 1/2 \,/\, 1/3 < X < 2/3) = \frac{P(1/3 < X < 1/2)}{P(1/3 < X < 2/3)}$$

$$P(1/3 < X < 1/2) = \int_{1/3}^{1/2} f(x)\,dx = \int_{1/3}^{1/2} 6x(1-x)\,dx$$

$$6\left(\frac{x^2}{2} - x^3/3\right)_{1/3}^{1/2} = 6\left(3x^2/2 - 2x^3/3\right)_{1/3}^{1/2}$$

$$= 6\left(\frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\right)^3\right) \quad \frac{6}{6}\left(\left(3/4 - 2/8\right) - \left(3/9 - 2/27\right)\right)$$

$$- \left(\frac{1}{2}\left(\frac{1}{3}\right)^2 - \frac{1}{3}\left(\frac{1}{3}\right)^3\right) \quad \left[3/4 - 1/4 - 1/3 + 2/27\right)$$

$$= 6\left[\frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{27}\right] \left(\frac{1}{2} - 1/3 + 2/27\right) = \frac{1}{6} + \frac{2}{27}$$

$$= 6\left[\frac{3-1}{24} - \right. \quad = \frac{27+12}{6 \times 27} = \frac{39}{162} = \frac{13}{54}$$

$$\therefore \boxed{P(1/3 < X < 1/2) = \frac{13}{54}}$$

$$P(1/3 < X < 2/3) = \int_{1/3}^{2/3} 6x(1-x)\,dx$$

$$= \frac{6}{6}\left(3x^2 - 2x^3\right)_{1/3}^{2/3}$$

$$= \left[3(4/9) - 2(8/27) - 3(1/9) + 2(1/27)\right]$$

$$= \frac{4}{3} - \frac{16}{27} - \frac{1}{3} + \frac{2}{27} = \frac{36 - 16 - 9 + \ldots}{27}$$

$$P(1/3 < X < 2/3) = 13/27$$

$$\frac{P(1/3 < X < 1/2)}{P(1/3 < X < 2/3)} = \frac{13}{54} \times \frac{27}{13} = \frac{1}{2}$$

$$\boxed{P(X \le 1/2 \,/\, 1/3 < X < 2/3) = 1/2}$$

Ex: 6 If $f(x) = \begin{cases} x e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$

i) S.T $f(x)$ is a PDF of CR X. (ii) Find

Solution:

i) To show that $f(x)$ is a PDF:

$$\text{WKT} \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_0^{\infty} x e^{-x^2/2}\,dx$$

put $\frac{x^2}{2} = t \Rightarrow \frac{2x\,dx}{2} = dt \Rightarrow x\,dx = dt$

When $x = 0$, $t = 0$; When $x = \infty$, $t = \infty$

$$= \int_0^{\infty} e^{-t}\,dt = \left[-e^{-t}\right]_0^{\infty} = -\left[e^{-\infty} - \ldots\right]$$

$$= 1$$

$\therefore$ $f(x)$ is a PDF.

(ii) To Find CDF ($F(x)$):

$$F(x) = P(X \le x) = \int_0^x f(x)\,dx$$

$$= \int_0^x x\, e^{-x^2/2}\, dx \qquad x^2/2 = t \quad x\,dx = dt \quad x^2 = 2t \quad x = \sqrt{2t}$$

$$= \int_0^{\sqrt{2t}} e^{-t}\, dt = \left(-e^{-t}\right)_0^t = -e^{-t} + 1$$

$$= 1 - e^{-x^2/2}$$

$$\boxed{F(x) = \begin{cases} 1 - e^{-x^2/2} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}}$$

Ex: 7 The CDF of a continuous RV of x

is $F(x) = \begin{cases} 0 & ; x \leq 0 \\ x^2 & ; 0 < x \leq 1/2 \\ 1 - \frac{3(3-x)^2}{25} & ; 1/2 \leq x < 3 \\ 1 & ; x \geq 3 \end{cases}$

Find i) PDF of x ii) $P(|x| \leq 1)$

iii) $P(1/3 \leq x \leq 4)$.

Solution:

i) $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0 & ; x \leq 0 \\ 2x & ; 0 < x \leq 1/2 \\ \frac{-6(3-x)}{25} & ; 1/2 \leq x < 3 \\ 0 & ; x \geq 3 \end{cases}$$

ii) $P(|x| \leq 1) = P(-1 < x < 1)$

$= P(x \leq 1) = F(1)$

$= 1 - \frac{3(4)}{25} = 13/25$

$$\boxed{P(|x| \leq 1) = \frac{13}{25}}$$

iii) $P(1/3 \leq x \leq 4) = F(4) - F(1/3)$

$= 1 - (1/3)^2 = 8/9$

$$\boxed{P(1/3 \leq x \leq 4) = 8/9}$$

Ex: 8 Experience has shown that while walking in a certain park, the time X (in minutes), between two people smoking has a density function of the form

$f(x) = \begin{cases} \lambda e^{-x} x & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$. i) Calculate λ ii) Find the distribution function of X. iii) What is the probability that Murugan, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? iv) at least 7 minutes?

Solution:

i) To find λ :

WKT $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\int_0^{\infty} \lambda x\, e^{-x}\, dx = 1 \qquad \int_0^{\infty} x^n e^{-x}\, dx = n!$

$\lambda (1!) = 1 \Rightarrow \boxed{\lambda = 1}$

ii) To find $F(x)$:

$F(x) = P(X \leq x) = \int_0^x f(x)\, dx$

The CDF of X is given by

x	1	2	3	4
$p(x)$	$15/61$	$10/61$	$30/61$	$6/61$
$F(x)$	$15/61$	$25/61$	$55/61$	1

Ex: 6 The probability function of an infinite distribution is given by $P(X=j) = (1/2)^j$; $j = 1, 2, 3, \ldots, \infty$. Verify that the total probability is 1 and also find mean, variance, P(X is even), P(X is divisible by 3), $P(X \geq 5)$, CDF of X.

Solution: i) To verify the total probability is 1.

x	1	2	3	4	5	6	...
$p(x)$	$1/2$	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$(1/2)^5$	$(1/2)^6$	...

$$\sum_i p(x_i) = 1$$

$$\Rightarrow 1/2 + (1/2)^2 + (1/2)^3 + (1/2)^4 + \ldots \neq \cancel{1}$$

$$\Rightarrow \frac{1}{2}\left[1 + 1/2 + (1/2)^2 + (1/2)^3 + \ldots\right] \neq \cancel{1}$$

$$= \frac{1}{2}\left[(1-1/2)^{-1}\right] = \frac{1}{2}\left[(1/2)^{-1}\right] = 1/\cancel{2} \times \cancel{2} = 1$$

$$\therefore \sum_i p(x_i) = 1$$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$(1+x)(1-x)^{-3} = 1 + 4x + 9x^2 + 16x^3 + \ldots$

(ii) To find mean

$$\text{mean} = E(X) = \sum_i x\, p(x)$$

$$= 1 \cdot 1/2 + 2(1/2)^2 + 3(1/2)^3 + \ldots$$

$$= \frac{1}{2}\left[1 + 2(1/2) + 3(1/2)^2 + \ldots\right]$$

$$= \frac{1}{2}\left[(1-1/2)^{-2}\right] = \frac{1}{2} \cdot 4 = 2$$

$$\boxed{\text{Mean} = 2}$$

(iii) To find variance

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = 1^2\, 1/2 + 2^2 (1/2)^2 + 3^2 (1/2)^3 + \ldots$$

$$= \frac{1}{2}\left[1 + 4(1/2) + 9(1/2)^2 + \ldots\right]$$

$$= \frac{1}{2}\left[(1+1/2)(1-1/2)^{-3}\right]$$

$$= 1/\cancel{2}\left[3/\cancel{2} \cdot \cancel{8}\right] = 6$$

$$\boxed{E(X^2) = 6}$$

$$Var(X) = 6 - 4$$

$$\boxed{Var(X) = 2}$$

iv) P(X is even)

$$P(X \text{ is even}) = P(X=2) + P(X=4) + P(X=6) + \ldots$$

$$= (1/2)^2 + (1/2)^4 + (1/2)^6 + \ldots$$

$$= (1/2)^2\left[1 + (1/2)^2 + (1/2)^4 + \ldots\right]$$

$$= (1/2)^2\left[1 + 1/4 + (1/4)^2 + \ldots\right]$$

$$= \int_0^x x\, e^{-x} dx \qquad u = x \quad v = e^{-x}$$
$$u' = 1 \quad v_1 = -e^{-x} \quad v_2 = e^{-x}$$

$$= \left[x\left(\frac{e^{-x}}{-1}\right) - 1 \cdot e^{-x} \right]_0^x$$

$$F = xe^{-x} - e^{-x} + 1$$

$$F(x) = \begin{cases} 0 & ; x \le 0 \\ 1 - xe^{-x} - e^{-x} & ; x > 0 \end{cases}$$

iii) $P(2 < X < 5) = F(5) - F(2)$

$$= \cancel{1} - 5e^{-5} - e^{-5} - \cancel{1} + 2e^{-2} + e^{-2}$$

$$\boxed{P(2 < X < 5) = 3e^{-2} - 6e^{-5}}$$

iv) $P(\text{at least } 7 \text{ min}) = P(X \geq 7) =$

$$1 - P(X < 7) = 1 - F(7)$$

$$= 1 - 1 + 7e^{-7} + e^{-7} = 8e^{-7}$$

$$\boxed{P(\text{at least } 7 \text{ min}) = 8e^{-7}}$$

Ex: 9 Let X be a Continuous RV with PDF $f(x) = kx(2-x)$; $0 < x < 2$. Find i) k ii) mean iii) Variance iv) Distribution function of X.

Solution :

i) To find k :

WKT $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^2 k\, x(2-x)\, dx = 1$$

$$k\left(\frac{2x^2}{2} - \frac{x^3}{3}\right)_0^2 = 1$$

$$k\left(4 - 8/3 - 0\right) = 1$$

$$k\left(4/3\right) = 1 \Rightarrow \boxed{k = 3/4}$$

ii) To find Mean :

WKT, $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$E(X) = \frac{3}{4}\int_0^2 x^2(2-x)\, dx$$

$$= 3/4 \left(2x^3/3 - x^4/4\right)_0^2$$

$$= \frac{3}{4}\left(16/3 - \cancel{16}^{4}/\cancel{4}_{1} - 0\right)$$

$$= 3/4\left(4/3\right) = 1$$

$$\boxed{E(X) = 1}$$

iii) To find Variance :

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \frac{3}{4}\int_0^2 x^3(2-x)\, dx$$

$$= 3/4 \left(2x^4/4_{2} - x^5/5\right)_0^2$$

$$= \frac{3}{4}\left(8 - \frac{32}{5} - 0\right) = 3/4\left(8/5\right)$$

$$\boxed{E(X^2) = 6/5}$$

$$Var(X) = E(X^2) - E(X) = 6/5 - 1 = 1/5$$

iv) To find CDF of x :

$$F(x) = P(X \le x) = \int_0^x f(x)dx$$

$$= \int_0^x \frac{3}{4} x(2-x)\,dx$$

$$= \frac{3}{4}\left(\cancel{2}x^2/\cancel{2} - x^3/3\right)_0^x$$

$$= \cancel{3}/4\left(\frac{3x^2 - x^3}{\cancel{3}}\right) = \frac{3x^2-x^3}{4}$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{3x^2-x^3}{4} & 0 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

Ex : 10 Let x be a continuous RV of. with PDF $f(x) = kx^2e^{-x}$; $x \ge 0$. Find i) k ii) Mean iii) Variance. iv) r^{th} moment and hence find Mean, Variance.

Solution : i) To find k :

WKT, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^\infty kx^2e^{-x}dx = 1 \quad \because \int_0^\infty x^n e^{-x}dx = n!$$

$$k\,2! = 1 \Rightarrow \boxed{k = 1/2}$$

ii) To find Mean :

$$\text{Mean} = E(X) = \int_0^\infty kx^3e^{-x}dx$$

$$= 1/2 \int_0^\infty x^3e^{-x}dx = 1/2\,(3!) = 3$$

$$\boxed{E(X) = 3}$$

iii) To find Variance :

$$E(X^2) = \int_0^\infty \frac{1}{2} x^2 f(x)\,dx$$

$$= \frac{1}{2}\int_0^\infty x^4 \cancel{f(x)}\, e^{-x}dx$$

$$= 1/2\; 4! = 12$$

$$\boxed{E(X^2) = 12}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= 12 - 9$$

$$\boxed{Var(X) = 3}$$

iv) WKT $\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx$

$$\mu_r' = E(X^r) = \int_0^\infty x^r \, 1/2\, x^2e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty x^{r+2}e^{-x}dx$$

$$\boxed{\mu_r' = 1/2\,(r+2)!}$$

$$E(X) = \mu_1' = \frac{1}{2}(3!) = 3$$

$$E(X^2) = \mu_2' = \frac{1}{2}(4!) = 12$$

$$Var(X) = E(X^2) - E(X)^2 = 12 - 9 = 3$$

Ex : 11 A continuous RV X has pdf $f(x) = kx(2-x)$, $0 \le x \le 2$. i) Find k, ii) r^{th} moment about origin iii) Find first 3 central moments.

Solution :

i) To find k :

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$k\int_0^2 x(2-x) = 1$

$k\left[\cancel{2}x^2/\cancel{2} - x^3/3\right]_0^2 = 1$

$k\left[4 - 8/3\right] = 1$

$k(4/3) = 1 \Rightarrow \boxed{k = 3/4}$

ii) rth moment about origin :

$\mu_r' = E(X^r)$

$= \int_0^2 x^r f(x)\,dx$

$= \frac{3}{4}\int_0^2 x^r . x(2-x)\,dx$ $\quad \int x^n dx = \frac{x^{n+1}}{n+1}+C$

$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx$

$= \frac{3}{4}\left[2\frac{x^{r+1+1}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$

$= \frac{3}{4}\left[\frac{2.2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right]$

$= \frac{3}{4}\left[2^{r+3}\left(\frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)}\right)\right]$

$= \frac{3}{4}\left[\frac{2^{r+3}}{(r+2)(r+3)}\right]$

$\boxed{\mu_r' = \frac{3.\ 2^{r+1}}{(r+2)(r+3)}}$

iii) $\mu_1' = \frac{\cancel{3}.2^2}{\cancel{3}.\cancel{4}} = 1$

$\boxed{\mu_1' = 1}$

$\mu_1 = E(X-\mu) = E(X) - E(\mu) = 1$

$\boxed{\mu_1 = 0}$

Moment Generation Function :

$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$\mu_r' = M_X(t) = r! \times$ Coefficient of in $M_X(t)$

Ex: 1 If a random variable X has the MGF, $M_X(t) = \frac{3}{3-t}$, find the standard deviation of X.

Solution :

$M_X(t) = \frac{3}{3-t} = \frac{\cancel{3}}{\cancel{3}(1-t/3)} = (1-t/3)^{-1}$

$= 1 + t/3 + (t/3)^2 + \dots$

Coefficient of $t = 1/3$; coeff of t^2

$\mu_1' = 1! \times 1/3 = 1/3$; $\mu_2' = 2! \times 1/9 = 2/9$

$Var(X) = \mu_2' - \mu_1'^2 = 2/9 - 1/9 = 1/9$

$\sigma(X) = \sqrt{1/9} = 1/3$ $\boxed{S.D = 1/3}$

Ex: 2 A RV X have pdf $f(x) = \frac{1}{2} e^{-x/2}, x > 0$. Find the MGF, mean and variance of X.

Solution :

i) To find MGF :

$$M_x(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{tx} \cdot e^{-x/2} dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{(t-1/2)x} dx = \frac{1}{2}\int_0^{\infty} e^{-(\frac{1}{2}-t)x} dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1/2-t)x}}{-(1/2-t)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[0 + \frac{1}{\left(\frac{1-2t}{2}\right)}\right] = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{1-2t}$$

$$\boxed{M_x(t) = \frac{1}{1-2t}}$$

ii) To find Mean :

$$M_x(t) = (1-2t)^{-1} = 1 + 2t + 4t^2 + \dots$$

Coeff of $t = 2$; Coeff. of $t^2 = 4$

Mean $= 1! \times$ coeff of t in $M_x(t)$

$= 1 \times 2$

$$\boxed{Mean = 2}$$

$$E(X^2) = 2! \times 4 = 8$$

iii) Variance

$$Var(X) = E(X^2) - E(X)^2$$

$$= 8 - 4$$

$$\boxed{Var(X) = 4}$$

Ex: 3 Find the MGF of a random variable X whose PDF is defined by

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Also fine mean and variance of X.

Solution :

To find MGF :

$$M_x(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx$$

$$= \int_0^1 e^{tx} \cdot x \cdot dx + \int_1^2 (2-x) e^{tx} \cdot dx$$

$u = x \qquad v = e^{tx}$

$u' = 1 \qquad v_1 = \frac{e^{tx}}{t}$

$\qquad\qquad v_2 = \frac{e^{tx}}{t^2}$

$$= \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[2\frac{e^{tx}}{t} - \frac{x e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} - 0 + \frac{1}{t^2} + \cancel{\frac{2e^{2t}}{t}} - \cancel{\frac{2e^{2t}}{t}} + \frac{e^{2t}}{t^2} - \cancel{\frac{2e^t}{t}} + \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2} - \frac{2e^t}{t^2} + \frac{e^{2t}}{t^2}. \quad \boxed{M_x(t) = \frac{1-2e^t+e^{2t}}{t^2}}$$

To find Mean :

$$M_x(t) = \frac{1}{t^2}\left[1-2\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots\right) + \left(1+\frac{2t}{1!}+\frac{(2t)^2}{2!}+\cdots\right)\right]$$

Coefficient of t in $M_x(t) = \frac{-2}{3!}+\frac{2^3}{3!}$

$$= -\frac{\cancel{2}^{1}}{\cancel{6}_{3}}+\frac{\cancel{8}^{4}}{\cancel{6}_{3}}$$

Coeff of t in $M_x(t) = 1$

Coefficient of t^2 in $M_x(t)$

$$= \frac{-2}{4!}+\frac{2^4}{4!} = \frac{-2}{24}+\frac{16}{24} = \frac{14}{24} = \frac{7}{12}$$

$\mu_1' = 1! \times 1 = 1$

$$\boxed{\mu_1' = \text{Mean} = 1}$$

To find Variance :

$$Var(x) = E(x^2) - (E(x))^2 = \mu_2' - (\mu_1')^2$$

$$\mu_2' = 2! \times 7/12 = 7/6$$

$$= \frac{7}{6} - 1 = 1/6$$

$$\boxed{Var(x) = 1/6}$$

Functions of one Random variable :

Let X be a RV with the associated sample space S_x and a known probability distribution. Let g be a scalar function that maps each $x \in S_x$ into $y = g(x)$.

The expression $Y = g(x)$ defines a new RV Y.

Case (i) :

If $g(x)$ is a strictly increasing function of x.

$$f_Y(y) = f_X(x)\cdot\left|\frac{dx}{dy}\right|, \quad x = g^{-1}(y)$$

Case (ii) :

If $g(x)$ is a strictly decreasing function of x.

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}$$

Ex : 1 Given the RV X with density function $f(x) = \begin{cases} 2x & ; 0<x<1 \\ 0 & ; \text{otherwise} \end{cases}$

Find the PDF of i) $Y = 8X^3$ ii) $Y = 3X+6$ iii) $Y = \tan^{-1}X$

Solution : i) Since $y = 8x^3$ is a strictly increasing function in $(0,1)$.

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y^{1/3} = (2x)$

$y^{1/3} = 2x$

$y = 8x^3 \Rightarrow \cancel{\frac{dy}{dx}} =$ $\qquad x = \frac{1}{2}y^{1/3}$

$\frac{dy}{dx} = 24x^2 \Rightarrow \frac{dx}{dy} = \frac{1}{24x^2}$

$$f_Y(y) = 2x\cdot\left|\frac{1}{24x^2}\right| = \frac{1}{12x}$$

$$= \frac{1}{12}\cdot\frac{2}{y^{1/3}} = \frac{2y^{-1/3}}{\cancel{12}} = y$$

$f_Y(y) = y^{-1/3}/6 \quad ; \quad 0<x<1 \Rightarrow$

$0 < \frac{1}{2}y^{1/3} < 1$

$0 < y^{1/3} < 2$

$$\therefore \boxed{f_Y(y) = \frac{y^{-1/3}}{6} ; 0<y<8}$$

ii) Since $Y = 3X+6$ is a strictly increasing function of x in $(0,1)$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y = 3x+6$	$0<x<1$
$\frac{y-6}{3} = x$	$0 < \frac{y-6}{3} < 1$
$dy/dx = 3$	$0<y<9$

$$f_Y(y) = 2x\cdot(1/3)$$

$$= 2\left(\frac{y-6}{3}\right)\times 1/3 = \frac{2(y-6)}{9}$$

$$\boxed{f_Y(y) = \frac{2(y-6)}{9} ; 0<y<9}$$

iii) Since $Y = \tan^{-1}(X)$ is a strictly increasing function of x in $(0,1)$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y = \tan^{-1}(x)$	$0<x<1$
$x = \tan y$	$0<\tan^{-1}y<1$
$\frac{dy}{dx} = \frac{1}{1+x^2}$	$\tan(0) < \tan(\tan^{-1}(y)) < \tan(1)$
$\frac{dx}{dy} = 1+x^2$	$0<y<\pi/4$

$$f_Y(y) = 2x(1+x^2)$$

$$= 2\tan y(1+\tan^2 y)$$

$$\boxed{f_Y(y) = 2\tan y\sec^2 y} ; 0<y<\pi/4$$

Ex : 2 Given the random variable X with density function $f(x) = \begin{cases} 4x & 0<x<2 \\ 0 & \text{otherwise.} \end{cases}$

Find the PDF of $Y = 2X^3$

Soln : Since $y = 2x^3$ is a strictly increasing function of X in $(0,2)$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$y = 2x^3$	$\frac{y^{1/3}}{2^{1/3}} = x$
$\frac{dy}{dx} = 6x^2$	$x = 2^{-1/3}y^{1/3}$
$\frac{dx}{dy} = \frac{1}{6x^2}$	$0<x<2$
	$0 < 2^{-1/3}\cdot y^{1/3} < 2$
	$0 < (y/2)^{1/3} < 2$
	$0 < (y/2) < 8$
	$0<y<16$

$$f_Y(y) = \cancel{4}^{2}x\cdot\frac{1}{\cancel{6}_{3}x^{\cancel{2}}} = \frac{2}{3x}$$

$$= \frac{2}{3}\cdot(2/y)^{1/3} = \frac{2^{4/3}}{3}y^{-1/3}$$

$$\boxed{f_Y(y) = \frac{2^{4/3}}{3}y^{-1/3} ; 0<y<16}$$

Tchebycheff's Inequality :

Statement :

If X is a RV with $E(X) = \mu$ and $Var(X) = \sigma^2$, then $P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$, $c>0$.

Alternative Form :

$0 \le x \le 1$

If $c = k\sigma$,

$$P\left\{ \left| \frac{X-\mu}{k} \right| \geq \sigma \right\} \leq \frac{1}{k^2}$$

$$\therefore P\left\{ \left| \frac{X-\mu}{k} \right| \leq \sigma \right\} \geq 1 - 1/k^2$$

Ex : 1 A RV x has mean $\mu = 12$ and variance $\sigma^2 = 9$ and an unknown probability distribution. Find $P(6 < x < 18)$.

Solution : Since the probability distribution is not known, we cannot find the value of the required probability. But we can find only a lower bound.

$|z| \le a$

$$P\left\{ \left| \frac{X-\mu}{k} \right| \geq 6 \right\} \leq \frac{\sigma^2}{k^2}$$

$-a \le z \le a$

$$P\left\{ \left| \frac{X-\mu}{k} \right| \leq \sigma \right\} \geq 1 - \frac{\cancel{\sigma^2}\,1}{k^2}$$

$$\therefore P\{\cancel{c \le} X - \cancel{\mu} \le c\} \cancel{\geq} 1 - \frac{\sigma^2}{c^2}$$

$$\cancel{P\{\mu - c \le x \le \mu + c\} \geq 1 - \sigma^2/c^2}$$

$$\cancel{P\{12 - c \le x \le 12 + c\} \geq 1 - \frac{9}{c^2}}$$

$$\cancel{\text{Let } c = 6}$$

$$\cancel{P\{6 \le x \le 18\} \geq 1 - \frac{9}{36} = 25/36}$$

$$P\left\{ -\sigma \le \frac{x-\mu}{k} \le \sigma \right\} \geq 1 - \frac{1}{k^2}$$

$$P\{-3k \le x - 12 \le 3k\} \geq 1 - \frac{1}{k^2}$$

$$P(-3k + 12 \le x \le 3k + 12) \geq 1 - 1/k^2$$

$$-3k + 12 = 6$$

$$-3k = -6$$

$$\boxed{k = 2}$$

$$\therefore P(6 < x < 18) \geq 1 - 1/4 = 3/4$$

$$\boxed{P(6 < x < 18) \geq 3/4}$$

Ex : 2 A fair dice is tossed 720 times. Use Tchebycheff's inequality to find a lower bound for the probability of getting 100 to 140 sixes.

Solution : Let x be the number of sixes obtained when a fair dice is tossed 720 times.

$p = P\{\text{getting 6 in a single toss}\}$

$q = 1 - 1/6 = 5/6 \qquad n = 720.$

$\mu = np$

$V = npq$

$\mu = 120 \quad \sigma = 10.$

$$P\{|x - \mu| \le k\sigma\} \geq 1 - 1/k^2$$

$$P\{|x - 120| \le 10k\} \geq 1 - 1/k^2$$

$$P\{-10k \le x - 120 \le 10k\} \geq 1 - 1/k^2$$

$$P\{120 - 10k \le x \le 120 + 10k\} \geq 1 - 1/k^2$$

$$120 - 10k = 100 \Rightarrow 10k = 20 \Rightarrow \boxed{k = 2}$$

$$P(100 \le x \le 140) \geq 3/4$$

$\therefore$ The required lower bound is 0.75.