UNIT-2 TWO DIMENSIONAL RANDOM VARIABLES

In Previous Chapter, Outcomes of an experiment are the Values assumed by Single Random Variable.

But there are Situations with Outcomes of Several R.V

For Eg: We might measure the Pressure P & Volume V of a gas released from a controller Chemical experiment giving rise to 2D sample space consisting of outcomes of (P, V) so there 2D RV P + V are involved simultaneously.

Types of 2D Random Variable:-

Discrete Random Variable:-

If the possible values of (X, Y) are finite (or) countably infinite, then (X, Y) is called a 2.D Discrete R.V.

Continuous Random Variable

If the values of (X, Y) is an uncountably Infinite set (ie) the range space is a region in the XY plane, then (X, Y) is called 2.D continuous R.V

Joint Prob Mass function	Joint Prob density function
$P(X,Y) = P[X = x_i, Y = y_j]$	$P\left[x - \frac{dx}{2} \le x \le x + \frac{dx}{2}, y - \frac{dy}{2} \le y \le y + \frac{dy}{2}\right] = f(x,y)\,dx\,dy$
(i) $P(x_i, y_j) \ge 0$ $\forall$ $i = 1,2,3 \ldots$, $j = 1,2,3 \ldots$	(i) $f(x,y) \ge 0$ $\forall (x,y) \in R$
(ii) $\sum_i \sum_j P(x_i, y_j) = 1$	(ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$
Marginal pmf	**Marginal Pdf**
Marginal PMf of X: $f(x) = \sum_j P_{ij} = P_{i*}$ $= P_{i1} + P_{i2} + P_{i3} + \ldots + P_{ij}$	**Marginal Pdf of X:** $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$
Marginal PMf of Y: $f(y) = \sum_i P_{ij} = P_{*j}$ $= P_{1j} + P_{2j} + P_{3j} + \ldots + P_{ij}$	**Marginal Pdf of Y:** $f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

Cumulative distribution function :-

Discrete : cdf $F(x,y) = P[X \le x, Y \le y]$

Continuous : cdf $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dx\, dy$

Properties

Discrete R.V	Continuous R.V
(i) $0 \le P(x_i, y_j) \le 1$	1) $0 \le F(x,y) \le 1$
(ii) $\sum_i \sum_j P(x_i, y_j) = 1$	2) $F(-\infty, \infty) = 1$
(iii) $P(x_i) = \sum_j P(x_i, y_j)$	3) $F(-\infty, y) = 0$
(iv) $P(y_j) = \sum_i P(x_i, y_j)$	4) $F(x, -\infty) = 0$
	(5) $F(x,y) = \frac{\partial^2}{\partial x \partial y}[F(x,y)]$

Conditional PMF	Conditional PDF
(i) $P(x/y) = \frac{P_{ij}}{P_{*j}}$	(i) $f(x/y) = \frac{f(x,y)}{f(y)}$
(ii) $P(y/x) = \frac{P_{ij}}{P_{i*}}$	(ii) $f(y/x) = \frac{f(x,y)}{f(x)}$
(iii) $\sum P_{y/x}(y) = 1$	(iii) $\int f_{y/x}(y)\, dy = 1$
(iv) $\sum P_{x/y}(x) = 1$	(iv) $\int f_{x/y}(x)\, dx = 1$

Independent 2.D R.V :-

In discrete R.V $P_{ij} = P_{i*} \times P_{*j}$

In continuous R.V $f(x,y) = f(x) \cdot f(y)$

To calculate the Marginal distribution when the R.V "X" takes horizontal values and "y" takes Vertical Values.

$Y \downarrow$ \ $X \rightarrow$	x_1	x_2	x_3	$P_Y(y) = f(y)$
y_1	$P(1,1)$ P_{11}	$P(1,2)$ P_{12}	$P(1,3)$ P_{13}	$P_{11}+P_{12}+P_{13} = P(Y=y_1)$ Sum of 1st Row
y_2	$P(2,1)$ P_{21}	$P(2,2)$ P_{22}	$P(2,3)$ P_{23}	$P_{21}+P_{22}+P_{23} = P(Y=y_2)$ Sum of 2nd Row
y_3	$P(3,1)$ P_{31}	$P(3,2)$ P_{32}	$P(3,3)$ P_{33}	$P_{31}+P_{32}+P_{33} = P(Y=y_3)$ Sum of 3rd Row.
$P_X(x) = f(x)$	$P_{11}+P_{21}+P_{31} = P(X=x_1)$ Sum of 1st column	$P_{12}+P_{22}+P_{32} = P(X=x_2)$ Sum of 2nd column	$P_{13}+P_{23}+P_{33} = P(X=x_3)$ Sum of 3rd column	

PROBLEMS:- Discrete Random Variable

(1) The Joint PMF of (X,Y) is given by $P(x,y) = K(2x+3y)$ $x = 0,1,2$ & $y = 1,2,3$ (i) find all Marginal distributions of x and y. (2) conditional Prob distributions (3) Prob distribution of x and y $(x+y)$

(4) $P(X+Y > 3)$

X \ Y	1	2	3
0	$x=0\ y=1$ 3K	$x=0\ y=2$ 6K	$x=0\ y=3$ 9K
1	$x=1\ y=1$ 5K	$x=1\ y=2$ 8K	$x=1\ y=3$ 11K
2	$x=2\ y=1$ 7K	$x=2\ y=2$ 10K	$x=2\ y=3$ 13K.

$$\sum_{j=1}^{3} \sum_{i=0}^{2} P(x_i, y_j) = 1.$$

$$\left.\begin{array}{l} 3K+6K+9K+5K+8K+11K \\ +7K+10K+13K \end{array}\right\} = 1$$

$$72K = 1 \Rightarrow \boxed{K = 1/72}$$

$X \downarrow$ \ $Y \rightarrow$	1	2	3	$P_X(x) = f(x) = P_{i*}$
0	$3/72$ $\boxed{P_{01}}$	$6/72$ $\boxed{P_{02}}$	$9/72$ $\boxed{P_{03}}$	$P(X=0) = 18/72$
1	$5/72$ $\boxed{P_{11}}$	$8/72$ $\boxed{P_{12}}$	$11/72$ $\boxed{P_{13}}$	$P(X=1) = 24/72$
2	$7/72$ $\boxed{P_{21}}$	$10/72$ $\boxed{P_{22}}$	$13/72$ $\boxed{P_{23}}$	$P(X=2) = 30/72$
$P_Y(y) = f(y) = P_{*j}$	$P(Y=1) = 15/72$	$P(Y=2) = 24/72$	$P(Y=3) = 33/72$	1

(i) Marginal distribution of X :-

$P(X=0) = 18/72$ $\quad P(X=1) = \frac{24}{72}$ $\quad P(X=2) = \frac{30}{72}$

Marginal distribution of Y :-

$P(Y=1) = 15/72,$ $\quad P(Y=2) = \frac{24}{72}$ $\quad P(Y=3) = \frac{33}{72}$

(ii) Conditional distribution of X, given Y $P[X=x_i / Y=y_j]$

$X = 0, 1, 2 \qquad Y = 1, 2, 3$

First fix $Y=1$

$$P[X=0/Y=1] = \frac{P[X=0, Y=1]}{P[Y=1]} = \frac{P(0,1)}{P(Y=1)} = \frac{9/72}{15/72} = \frac{1}{5}$$

$$P[X=1/Y=1] = \frac{P[X=1, Y=1]}{P[Y=1]} = \frac{P[1,1]}{P(Y=1)} = \frac{5/72}{15/72} = 1/3$$

$$P[X=2/Y=1] = \frac{P[X=2, Y=1]}{P[Y=1]} = \frac{P[2,1]}{P(Y=1)} = \frac{7/72}{15/72} = 7/15$$

$Y=2$

$$P(X=0, Y=2) = \frac{P(0,2)}{P(Y=2)} = \frac{6/72}{24/72} = \frac{1}{4}$$

$$P(X=1, Y=2) = \frac{P(1,2)}{P(Y=2)} = \frac{8/72}{24/72} = \frac{1}{3}$$

$$P(X=2, Y=2) = \frac{P(2,2)}{P(Y=2)} = \frac{10/72}{24/72} = 5/12$$

$Y=3$

$$P(X=0, Y=3) = \frac{P(0,3)}{P(Y=3)} = \frac{9/72}{33/72} = \frac{9}{33}$$

$$P(X=1, Y=3) = \frac{P(1,3)}{P(Y=3)} = \frac{11/72}{33/72} = \frac{1}{3}$$

$$P(X=2, Y=3) = \frac{P(2,3)}{P(Y=3)} = \frac{13/72}{33/72} = \frac{13}{33}$$

Conditional distribution of Y, given X : $P[Y=y_j / X=x_i]$

$x = 0, 1, 2 \qquad y = 1, 2, 3$

fix $x=0$

$$P[Y=1, X=0] = \frac{P(0,1)}{P(X=0)} = \frac{3/72}{18/72} = \frac{1}{6}$$

$$P[Y=2, X=0] = \frac{P(2,0)}{P(X=0)} = \frac{6/72}{18/72} = \frac{1}{3}$$

$$P[Y=3, X=0] = \frac{P(3,0)}{P(X=0)} = \frac{9/72}{18/72} = 1/2$$

fix $x=1$

$$P[Y=1, X=1] = \frac{P(1,1)}{P(X=1)} = \frac{5/72}{24/72} = 5/24$$

$$P[Y=2, X=1] = \frac{P(2,1)}{P(X=1)} = \frac{8/72}{24/72} = 1/3$$

$$P[Y=3, X=1] = \frac{P(3,1)}{P(X=1)} = \frac{11/72}{24/72} = \frac{11}{24}$$

fix $x=2$.

$$P[Y=1, X=2] = \frac{P[1,2]}{P(X=2)} = \frac{7/72}{30/72} = 7/30$$

$$P[Y=2, X=2] = \frac{P[2,2]}{P(X=2)} = \frac{10/72}{30/72} = 1/3$$

$$P[Y=3, X=2] = \frac{P[3,2]}{P(X=2)} = \frac{13/72}{30/72} = 13/30.$$

(5) Probability Distribution of $X+Y$

$(X+Y)$	P
1 $P(0,1)$	$3/72$
2 $P(0,2) + P(1,1)$	$\frac{6}{72} + \frac{5}{72} = \frac{11}{72}$
3 $P(0,3) + P(1,2) + P(2,1)$	$\frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$
4 $P(1,3) + P(2,2)$	$\frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5 $P(2,3)$	$13/72$
Total	1

(6) $P[X+Y>3]$

$$P[X+Y>3] = P(X+Y=4) + P(X+Y=5)$$

$$= \frac{21}{72} + \frac{13}{72}$$

$$= \frac{34}{72}$$

(2) The Joint distribution of X and Y is given by

$$f(x,y) = \frac{x+y}{21}, \quad x = 1,2,3 \ \& \ y = 1,2$$

(i) find Marginal Distribution of x & y

(ii) find $E[XY]$

Given $f(x,y) = \frac{x+y}{21}$ $\quad x = 1,2,3$, $y = 1,2$

$y \downarrow$ \ $X \rightarrow$	1	2	3	$P_Y(y) = f(y)$
1	2/21 P_{11}	3/21 P_{12}	4/21 P_{13}	9/21 $P(Y=1)$
2	3/21 P_{21}	4/21 P_{22}	5/21 P_{23}	12/21 $P(Y=2)$
$P_X(x) = f(x)$	5/21 $P(X=1)$	7/21 $P(X=2)$	9/21 $P(X=3)$	1

1) Marginal Distribution of X :-

$P(X=1) = 5/21 \quad P(X=2) = 7/21 \quad P(X=3) = 9/21$

Marginal Distribution of Y :-

$P(Y=1) = 9/21 \quad P(Y=2) = 12/21$

2) $$E[XY] = \sum_{j=1}^{2} \sum_{i=1}^{3} x_i y_j \, P(x_i) P(y_j)$$

$$= x_1 y_1 P(x_1) P(y_1) + x_1 y_2 P(x_1) P(y_2) + x_2 y_1 P(x_2) P(y_1) + x_2 y_2 P(x_2) P(y_2) + x_3 y_1 P(x_3) P(y_1) + x_3 y_2 P(x_3) P(y_2)$$

$$= (1)(1) \frac{5}{21} \times \frac{9}{21} + (1)(2) \frac{5}{21} \times \frac{12}{21} + (2)(1) \frac{7}{21} \times \frac{9}{21} + (2)(2) \frac{7}{21} \times \frac{12}{21} + (3)(1) \frac{9}{21} \times \frac{9}{21} + (3)(2) \frac{9}{21} \times \frac{12}{21}$$

$$= \frac{1}{441} \left[45 + 120 + 126 + 336 + 243 + 648 \right]$$

$$= \frac{1518}{441} = \frac{506}{147}$$

(3) The 2D R.V (X, Y) has the Joint density function $f(x,y) = \frac{x+2y}{27}$

$x = 0, 1, 2$ & $y = 0, 1, 2$

(i) Find conditional distribution of Y given $X = x$

(ii) Find conditional distribution of X given $Y = 1$

$Y \downarrow$ / $X \rightarrow$	0	1	2	$P_Y(y) = f(y)$
0	0 P_{00}	$1/27$ P_{10}	$2/27$ P_{20}	$P(Y=0)$ $3/27$
1	$2/27$ P_{01}	$3/27$ P_{11}	$4/27$ P_{21}	$P(Y=1)$ $9/27$
2	$4/27$ P_{02}	$5/27$ P_{12}	$6/27$ P_{22}	$P(Y=2)$ $15/27$
$P_X(x) = f(x)$	$6/27$ $P(X=0)$	$9/27$ $P(X=1)$	$12/27$ $P(X=2)$	1

Conditional Distribution of Y given $X = x$

Fix $x=0$

$$P(y=0, x=0) = \frac{P(0,0)}{P(X=0)} = 0$$

$$P(y=1, x=0) = \frac{P(0,1)}{P(X=0)} = \frac{2/27}{6/27} = 1/3$$

$$P(y=2, x=0) = \frac{P(0,2)}{P(X=0)} = \frac{4/27}{6/27} = \frac{4}{6} = 2/3$$

Fix $x=1$

$$P(y=0, x=1) = \frac{P(1,0)}{P(X=1)} = \frac{1/27}{9/27} = 1/9$$

$$P(y=1, x=1) = \frac{P(1,1)}{P(X=1)} = \frac{3/27}{9/27} = 1/3$$

$$P(y=2, x=1) = \frac{P(1,2)}{P(X=1)} = \frac{5/27}{9/27} = 5/9$$

Fix $x=2$

$$P(y=0, x=2) = \frac{P(2,0)}{P(X=2)} = \frac{2/27}{12/27} = 1/6$$

$$P(y=1, x=2) = \frac{P(2,1)}{P(X=2)} = \frac{4/27}{12/27} = 1/3$$

$$P(y=2, x=2) = \frac{P(2,2)}{P(X=2)} = \frac{6/27}{12/27} = 1/2$$

(2) Conditional distribution of X, given $Y = 1$

Fix $Y=1$

$$P[X=0, Y=1] = \frac{P[0,1]}{P(Y=1)} = \frac{2/27}{9/27} = 2/9$$

$$P[X=1, Y=1] = \frac{P[1,1]}{P(Y=1)} = \frac{3/27}{9/27} = 3/9 = 1/3$$

$$P[X=2, Y=1] = \frac{P[2,1]}{P(Y=1)} = \frac{4/27}{9/27} = 4/9.$$

Problems on continuous R.V.

Continuous Random Variable

1) The Joint Pdf of R-V (x,y) is given by $f(x,y) = Kxy\,e^{-(x^2+y^2)}$ $x>0,\ y>0$ (i) Find the Value of K (ii) P.T X & Y are Independent.

To find K

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \quad \text{but } x>0 \ \& \ y>0 \text{ is given}$$

$$\int_{0}^{\infty}\int_{0}^{\infty} Kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

$$K\left[\int_0^\infty x\,e^{-x^2}dx\right]\left[\int_0^\infty y\,e^{-y^2}dy\right] = 1$$

Let $t = x^2$ $\qquad v = y^2$

$dt = 2x\,dx$ $\qquad dv = 2y\,dy$

$\Rightarrow x\,dx = \frac{dt}{2}$ $\qquad \Rightarrow y\,dy = \frac{dv}{2}$

$x=0 \Rightarrow t=0$ $\qquad y=0 \Rightarrow v=0$

$x=\infty \Rightarrow t=\infty$ $\qquad y=\infty \Rightarrow v=\infty$

$$K\left(\int_0^\infty e^{-t}\cdot\frac{dt}{2}\right)\left(\int_0^\infty e^{-v}\cdot\frac{dv}{2}\right) = 1$$

$$\frac{K}{4}\left(\int_0^\infty e^{-t}dt\right)\left(\int_0^\infty e^{-v}dv\right) = 1$$

$$K/4\left[\left[\frac{e^{-t}}{-1}\right]_0^\infty \left(\frac{e^{-v}}{-1}\right)_0^\infty\right] = 1$$

$$\frac{K}{4}\left[\frac{e^{-\infty}-e^{0}}{-1}\right]\left[\frac{e^{-\infty}-e^{0}}{-1}\right] = 1$$

$$K/4 = 1 \Rightarrow \boxed{K=4}$$

$\therefore$ $f(x,y) = 4xy\,e^{-(x^2+y^2)}$

Marginal density of "x"

$$f_X(x) = \int_0^\infty f(x,y)\,dy \quad \text{as } y>0$$

$$= 4\int_0^\infty xy\,e^{-(x^2+y^2)}\,dy$$

$$= 4xe^{-x^2}\underbrace{\int_0^\infty y\,e^{-y^2}\,dy}$$

$$\left\{\begin{array}{l}\int_0^\infty y\,e^{-y^2}\,dy = 1/2 \\ \int_0^\infty x\,e^{-x^2}\,dx = 1/2\end{array}\right\}$$

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$$= 4xe^{-x^2}(1/2)$$

$$f_X(x) = 2xe^{-x^2}, \quad x>0$$

Marginal density of "y"

$$f_Y(y) = \int_0^\infty f(x,y)\,dx \quad \text{as } x>0$$

$$= 4\int_0^\infty xy\,e^{-(x^2+y^2)}\,dx$$

$$= 4ye^{-y^2}\underbrace{\int_0^\infty x\,e^{-x^2}\,dx}$$

$$= 4ye^{-y^2}(1/2)$$

$$f_Y(y) = 2ye^{-y^2}$$

Independent

$$f(x,y) = f(x)\cdot f(y)$$

$$4xy\,e^{-(x^2+y^2)} = 2y\,e^{-y^2}\cdot 2x\,e^{-x^2}$$

$$= 4xy\,e^{-x^2}\cdot e^{-y^2}$$

$$= 4xy\,e^{-(x^2+y^2)}$$

$\therefore$ X and Y are Independent.

(2) If the Joint Pdf of 2D RV (x, y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3} & 0<x<1,\ 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

Find

(1) $P[x > 1/2]$ (2) $P[y > 1]$ (3) $P[y < x]$ (4) $P[y < 1/2 / x < 1/2]$

(5) $P[x+y \geq 1]$ (6) conditional density function & (7) check whether conditional density functions are valid.

Marginal density function of "x"

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \int_0^2 x^2 dy + \int_0^2 \left(\frac{x}{3}\right) y\,dy$$

$$= x^2 (y)_0^2 + \left(\frac{x}{3}\right)\left(y^2/2\right)_0^2$$

$$= x^2 (2-0) + \left(\frac{x}{3}\right)\left(\frac{4}{2} - 0\right)$$

$$= 2x^2 + (x/3)2 = 2\left[x^2 + x/3\right] \text{ <OR> } \frac{2}{3}\left[3x^2 + x\right]$$

$$f_x(x) = \frac{2}{3}\left[3x^2 + x\right]$$

Marginal density function of "y"

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \int_0^1 x^2 dx + \frac{y}{3}\int_0^1 x\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \frac{y}{3}\left(\frac{x^2}{2}\right)_0^1 = \left(\frac{1-0}{3}\right) + \frac{y}{3}\left(\frac{1-0}{2}\right)$$

$$= \frac{1}{3} + \frac{y}{6} = \frac{2+y}{6} \qquad f_y(y) = \frac{2+y}{6}.$$

i) $P[x > 1/2]$

$$P[x > 1/2] = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} \frac{2}{3}(3x^2 + x)\,dx = \frac{2}{3}\left[\cancel{3}\left(\frac{x^3}{\cancel{3}}\right) + \frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \frac{2}{3}\left[(1 + 1/2) - \left((1/2)^3 + \frac{(1/2)^2}{2}\right)\right]$$

$$= \frac{2}{3}\left((3/2) - (1/4)\right) = \frac{2}{3}\left[\frac{6-1}{4}\right] = \frac{2}{3} \times \frac{5}{4} = 5/6$$

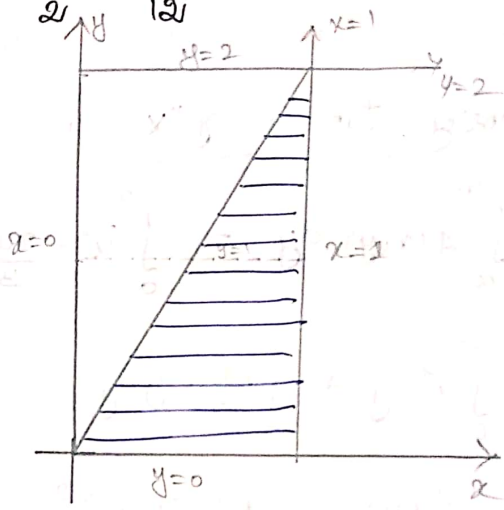
(ii) $P[y>1]$

$$P[y>1] = \int_1^2 \frac{2+y}{6}\,dy = \frac{1}{6}\left[2y+\frac{y^2}{2}\right]_1^2$$

$$= \frac{1}{6}\left[(4+4/2)-(2+1/2)\right]$$

$$= \frac{1}{6}\left[6-5/2\right] = \frac{1}{6}\times\frac{7}{2} = \frac{7}{12}$$

(3) $P[y<x]$



$x=0$ to $x=1$
$y=0$ to $y=x$

$$P[y<x] = \int_0^1\int_0^x f(x,y)\,dy\,dx$$

$$= \int_0^1\int_0^x \left(x^2+\frac{xy}{3}\right)dy\,dx$$

$$= \int_0^1 \left[x^2y+\frac{x}{3}\left(\frac{y^2}{2}\right)\right]_0^x dx$$

$$= \int_0^1 \left(x^3+\frac{x^3}{6}\right)dx$$

$$= \left[\frac{x^4}{4}+\frac{x^4}{24}\right]_0^1 = \left(\frac{1}{4}+\frac{1}{24}\right)-0 = 7/24$$

(4) $P[y<1/2 \,/\, x<1/2]$

$$P[y<1/2 \,/\, x<1/2] = \frac{P[x<1/2,\ y<1/2]}{P[x<1/2]}$$

$$P[x<1/2,\ y<1/2] = \int_0^{1/2}\int_0^{1/2}\left(x^2+\frac{xy}{3}\right)dy\,dx$$

$$= \int_0^{1/2}\left[\frac{x^3}{3}+\frac{y}{3}\left(\frac{x^2}{2}\right)\right]_0^{1/2} dy$$

$$= \int_0^{1/2}\left[\frac{1}{24}+\frac{y}{3}\left(\frac{1}{8}\right)\right]dy$$

$$= \frac{1}{24}\int_0^{1/2}(1+y)\,dy = 1/24\left(y+y^2/2\right)_0^{1/2}$$

$$= 1/24\left(1/2+\frac{1}{8}\right) = \frac{1}{24}(5/8) = \frac{5}{192}$$

$P[X < 1/2] = 1 - P[X > 1/2]$

$= 1 - 5/6 = 1/6.$

$P[Y < 1/2 / X < 1/2] = \dfrac{5/192}{1/6} = 5/32$

(5) Conditional density function of X, given Y

$$f(X/Y) = \frac{f(x,y)}{f(y)} = \frac{x^2 + xy/3}{\frac{2+y}{6}} = \frac{3x^2+xy}{\cancel{3}} \times \frac{\cancel{6}\,2}{2+y}$$

$$f(X/Y) = \frac{6x^2+2xy}{2+y}$$

Marginal density function of Y, given X

$$f(Y/X) = \frac{f(x,y)}{f(x)} = \frac{x^2 + xy/3}{2x^2 + \frac{2x}{3}} = \frac{\frac{3x^2+xy}{\cancel{3}}}{\frac{6x^2+2x}{\cancel{3}}}$$

$$f(Y/X) = \frac{3x^2+xy}{6x^2+2x}$$

(6) Check whether conditional density function are valid

(i) $f(X/Y) \geq 0$ and $f(Y/X) \geq 0$.

(i) $$f(Y/X) = \int_{-\infty}^{\infty} f(Y/X)\,dy = \int_0^2 \frac{3x^2+xy}{6x^2+2x}\,dy$$

$$= \frac{1}{6x^2+2x}\int_0^2 (3x^2+xy)\,dy = \frac{1}{6x^2+2x}\left[3x^2y + x\frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{6x^2+2x}(6x^2+2x) = 1 \geq 0$$

Hence $f(Y/X) \geq 0$

$$f(X/Y) = \int_{-\infty}^{\infty} f(X/Y)\,dx = \int_0^1 \frac{6x^2+2xy}{2+y}\,dx = \frac{1}{2+y}\int_0^1 (6x^2+2xy)\,dx$$

$$= \frac{1}{2+y}\left(\cancel{6}\left(\frac{x^3}{\cancel{3}}\right) + \cancel{2}y\left(\frac{x^2}{\cancel{2}}\right)\right)_0^1 = \frac{1}{2+y}\left(2x^3+yx^2\right)_0^1$$

$$= \frac{1}{2+y}(2+y) = 1 \geq 0$$

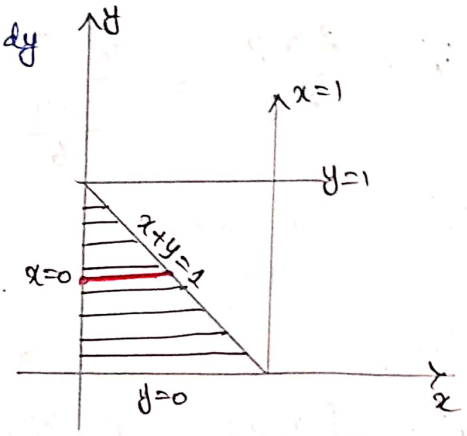
Hence $f(X/Y) \geq 0$

(7) $P[x+y \geq 1]$

$$P[x+y \geq 1] = 1 - P[x+y < 1]$$

If $x+y=1$ Points are $(1,0)$ & $(0,1)$

$$P[x+y<1] = \int_0^1 \int_0^{1-y} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$



$x=0$ & $x=1-y$

$y=0$ to $y=1$

$$= \int_0^1 \left[\frac{x^3}{3} + \frac{y}{3}\left(\frac{x^2}{2}\right)\right]_0^{1-y} dy$$

$$= \int_0^1 \left(\frac{(1-y)^3}{3} + \frac{y}{3}\,\frac{(1-y)^2}{2}\right) dy$$

$$= \frac{1}{3} \int_0^1 \left[(1-y)^3 + \frac{y}{2}(1+y^2-2y)\right] dy$$

$$= \frac{1}{3} \int_0^1 (1-y)^3 + \left(\frac{y}{2} + \frac{y^3}{2} - y^2\right) dy$$

$$= \frac{1}{3} \left[\frac{(1-y)^4}{-4} + \frac{y^2}{4} + \frac{y^4}{8} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{3} \left[\left(\cancel{\frac{(1-1)^4}{-4}}_{0} + \frac{1}{4} + \frac{1}{8} - \frac{1}{3}\right) - \left(\frac{1}{-4}\right)\right]$$

$$= \frac{1}{3}\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{8} - \frac{1}{3}\right) = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{8} - \frac{1}{3}\right)$$

$$= \frac{1}{3}\left(\frac{24+6-16}{48}\right) = \frac{1}{3}\left(\frac{\cancel{14}\,7}{\cancel{48}\,24}\right)$$

$$= \frac{1}{3}(7/24) = \frac{7}{72}$$

$$P[x+y \geq 1] = 1 - P[x+y<1]$$

$$= 1 - 7/72$$

$$= \frac{72-7}{72} = \frac{65}{72}$$

3) Given Joint pdf of X and Y is $f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$

Find Marginal & conditional pdf of X & Y also P.T X & Y are Independent

Marginal Density fn of X :-

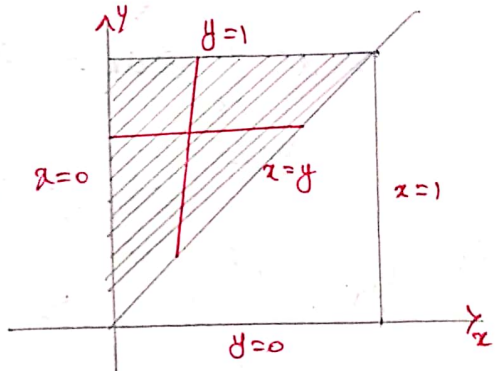
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{1} 8xy\,dy = \overset{4}{\cancel{8}}x\left(\frac{y^2}{\cancel{2}}\right)_x^1 = 4x\left[1-x^2\right]$$

$0<x<1$

Marginal density fn of Y :-

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 8xy\,dx = \overset{4}{\cancel{8}}y\left(\frac{x^2}{\cancel{2}}\right)_0^y = 4y^3 \quad 0<y<1$$

Conditional Pdf of X and Y



$$f(Y/X) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)}$$

$$= \frac{2y}{1-x^2} \quad 0\le x<y<1$$

$$f(X/Y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2} \quad 0<x<y<1$$

To Prove X & Y are Independent

$$f(x,y) = f(x)\cdot f(y)$$

$$8xy = 4x(1-x^2)\cdot 4y^3$$

$$8xy \neq 16xy^3(1-x^2)$$

$\therefore$ X & Y are not Independent.

(4) Find "K" if Joint pdf of bivariate R.V. (X,Y) is given by $f(x,y) = \begin{cases} K(1-x)(1-y) & 0\le x<4,\ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$

W.K.T $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{x=0}^{4}\int_{y=1}^{5} K(1-x)(1-y)\,dy\,dx = 1$

$$1 = K\int_{x=0}^{4}(1-x)\int_1^5(1-y)\,dy\,dx = K\int_0^4(1-x)\left[y-\frac{y^2}{2}\right]_1^5 dx$$

Solving we get $\boxed{K = 1/32}$

(5) Given $f(x,y) = \begin{cases} cx(x-y) & 0<x<2,\ -x<y<x \\ 0 & \text{otherwise} \end{cases}$

find (i) c (ii) $f(x)$ (iii) $f(y/x)$

WKT

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{x=0}^{2}\int_{y=-x}^{x} c(x^2-xy)\,dy\,dx = 1$$

$$1 = \int_0^2 c\left[\left(x^2y - x\left(y^2/2\right)\right)\right]_{-x}^{x} dx = \int_0^2 c\left[\left(x^3-\frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right]dx$$

$$= \int_0^2 c\left(x^3 - \cancel{\frac{x^3}{2}} + x^3 + \cancel{\frac{x^3}{2}}\right)dx$$

$$= 2c\int_0^2 x^3\,dx = \cancel{2}c\left(\frac{x^4}{\cancel{4}_{2}}\right)_0^2 = \frac{2^4 \cdot c}{2} = c2^3 = 8c$$

$$8c = 1 \Rightarrow c = 1/8$$

(ii) $$f(x) = \int_{-x}^{x} f(x,y)\,dy = \int_{-x}^{x} cx(x-y)\,dy = cx\left[xy - x\left(y^2/2\right)\right]_{-x}^{x}$$

$$= c\left[\left(x^3 - \cancel{\frac{x^3}{2}}\right) - \left(-x^3 - \cancel{\frac{x^3}{2}}\right)\right]$$

$$= \frac{1}{\cancel{8}_4}\left(\cancel{2}x^3\right) = \frac{x^3}{4}$$

(iii) $$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}(x^2-xy)}{x^3/4} = \frac{1}{\cancel{8}_2}\times\cancel{4}\left(\frac{x-y}{x^2}\right)$$

$$= \frac{x-y}{2x^2}$$

(6) The Joint Pdf of 2-D R.V (x, y) is given by

$$f(x,y) = xy^2 + \frac{x^2}{8} \qquad 0 \le x \le 2, \quad 0 \le y \le 1$$

find (i) $P(X>1 / Y<1/2)$ (ii) $P(Y<1/2 / X>1)$ (iii) $P(X<Y)$

(iv) $P(X+Y \le 1)$

Given $f(x,y) = xy^2 + \frac{x^2}{8} \qquad 0 \le x \le 2, \quad 0 \le y \le 1$

(i) $P(X>1 / Y<1/2) = \dfrac{P(X>1, Y<1/2)}{P(Y<1/2)}$

$$P(X>1, Y<1/2) = \int_{x=1}^{2} \int_{y=0}^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\, dx$$

$$= \int_1^2 \left[x\left(\frac{y^3}{3}\right) + \frac{x^2}{8}(y)\right]_0^{1/2} dx = \int_1^2 \left[\frac{x}{3}(1/2)^3 + \frac{x^2}{8}(1/2)\right] dx$$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx = \left(\frac{x^2}{48} + \frac{x^3}{48}\right)_1^2 = \frac{1}{48}\left[(2^2+2^3) - (1^2+1^3)\right]$$

$$= \frac{1}{48}\left((4+8)-(1+1)\right) = \frac{10}{48} = 5/24$$

$P(X>1, Y<1/2) = 5/24$

$$P(Y<1/2) = \int_{y=0}^{1/2} \int_{x=0}^{2} (xy^2 + x^2/8)\, dx\, dy = \int_0^{1/2} \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^2$$

$$= \int_0^{1/2} \left[\left(\frac{\cancel{4}^{2} y^2}{\cancel{2}} + \frac{\cancel{8}^{1}}{\cancel{24}_{3}}\right) - (0)\right] dy = \int_0^{1/2} (2y^2 + 1/3)\, dy$$

$$= \left[2(y^3/3) + y/3\right]_0^{1/2} = \frac{2}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{3}(1/2) = \frac{1}{12} + \frac{1}{6} = \frac{3}{12}$$

$P(Y<1/2) = 1/4$

$$P(X>1) = \int_{y=0}^{1} \int_{x=1}^{2} (xy^2 + x^2/8)\, dx\, dy = \int_{x=1}^{2} \left(x(y^3/3) + \frac{x^2}{8}(y)\right)_0^1 dx$$

$$= \int_1^2 \left(\frac{x}{3} + \frac{x^2}{8}\right) dx = \left(\frac{x^2}{6} + \frac{x^3}{24}\right)_1^2 = \left(\frac{4}{6} + \frac{8}{24}\right) - \left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= 1 - \frac{5}{24} = 19/24$$

(i) $P(x>1 / y<1/2) = \dfrac{P[x>1,\ y<1/2]}{P(y<1/2)}$

$$= \frac{5/24}{1/4} = \frac{5}{24} \times 4 = 5/6$$

(ii) $P(y<1/2 / x>1) = \dfrac{P(y<1/2,\ x>1)}{P(x>1)}$

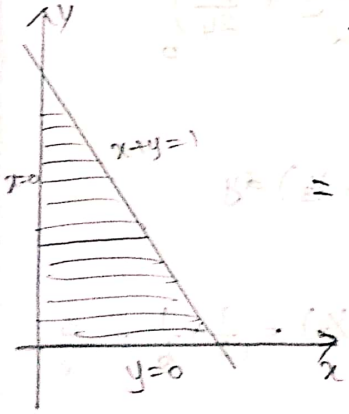
$$= \frac{5/24}{19/24} = \frac{5}{24} \times \frac{24}{19} = 5/19$$

(iii) $P(x<y) = \displaystyle\int_{y=0}^{1}\int_{x=0}^{y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_{y=0}^{1} \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^y dy = \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left(\frac{y^5}{10} + \frac{y^4}{96}\right)_0^1 = \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{106}{960} = \frac{53}{480}$$

iv $P(x+y \le 1) = \displaystyle\int_0^1 \int_0^{1-y} \left(xy^2 + x^2/8\right) dx\, dy$



$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{x^3}{24}\right)_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2y^2}{2} + \frac{(1-y)^3}{24}\right] dy$$

$$\int_0^1 \left(\frac{(1+y^2-2y)y^2}{2} + \frac{(1-y)^3}{24}\right) dy$$

$$= \int_0^1 \left(\frac{1}{2}\left(y^2+y^4-2y^3\right) + \frac{(1-y)^3}{24}\right) dy$$

$$= \left[\frac{1}{2}\left(y^3/3 + \frac{y^5}{5} - \frac{2y^4}{4}\right) + \frac{(1-y)^4}{+96}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{2}{4}\right) + 0\Big) - \left(0 + \frac{1}{96}\right)\Big] = \frac{13}{480}$$

Function of Random Variable:-

If X is a R.V with pdf $f(x)$ and $y = g(x)$ is a function of R.V "X" then pdf of y is

$$f(y) = f(x) \cdot \left|\frac{dx}{dy}\right|$$

1) Let X be a R.V with pdf $f(x) = 2x \quad 0<x<1$ find pdf of

(i) $y = 3x+6$ (ii) $y = 8x^3$ (iii) $y = \tan^{-1}(x)$

) Pdf of $y = 3x+6$, $f(x) = 2x \quad 0<x<1$

$$f(y) = f(x) \cdot \left|\frac{dx}{dy}\right| \qquad \frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3} \Rightarrow \left|\frac{dx}{dy}\right| = \left|\frac{1}{3}\right| = 1/3$$

$$y = 3x+6 \Rightarrow y-6 = 3x \Rightarrow \frac{y-6}{3} = x$$

$$f(y) = 2(x) \cdot \frac{1}{3} = \frac{2}{3}\left(\frac{y-6}{3}\right) = \frac{2}{9}(y-6)$$

$$0 < x < 1$$

$$0 < \frac{y-6}{3} < 1 \Rightarrow 0 < y-6 < 3 \Rightarrow 6 < y < 9.$$

$$f(y) = \frac{2}{9}(y-6) \quad 6<y<9$$

(ii) $y = 8x^3$, $f(x) = 2x \quad 0<x<1$

$$\frac{dy}{dx} = 24x^2 \Rightarrow \frac{dx}{dy} = \frac{1}{24x^2} \Rightarrow \left|\frac{dx}{dy}\right| = \left|\frac{1}{24x^2}\right| = \frac{1}{24x^2}$$

let $y = 8x^3 \Rightarrow x^3 = y/8 \Rightarrow x = \sqrt[1/3]{y/8} = y^{1/3}/2$

$$f(y) = 2(x) \cdot \left|\frac{dx}{dy}\right| = \cancel{2}\left(\frac{y^{1/3}}{2}\right) \cdot \frac{1}{\underset{6}{\cancel{24y}}\left(\frac{y^{1/3}}{2}\right)^{\cancel{2}}} = \frac{1}{6y^{1/3}}$$

$$0 < x < 1$$

$$0 < x^3 < 1.$$

$$0 < 8x^3 < 8 \Rightarrow 0 < y < 8.$$

(iii) $y = \tan^{-1}(x)$ $f(x) = 2x \quad 0<x<1$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \Rightarrow \frac{dx}{dy} = 1+x^2 \Rightarrow \left|\frac{dx}{dy}\right| = |1+x^2| = 1+x^2$$

$y = \tan^{-1}(x)$
$\Rightarrow x = \tan y$

$$f(y) = 2x(1+x^2)$$
$$= 2\tan y(1+\tan^2 y)$$
$$= 2\tan y \sec^2 y$$

$$0 < x < 1$$
$$0 < \tan^{-1}(y) < 1$$
$$\tan^{-1}(0) < y < \tan^{-1}(1)$$
$$0 < y < \pi/4$$

Transformation of 2-D Random Variables :-

Transformation of Random Variable

Hints :-

1) Find the Joint density function of (X, Y)

2) Consider new R.V U and V, try to express the relation in the form $x = g_1(u, v)$, $y = g_2(u, v)$

3) Find the Jacobian of (x, y) w.r.to (u, v)

(ie) $\dfrac{\partial(x, y)}{\partial(u, v)} = J$

4) Formula to find pdf of (u, v)

$$f(u, v) = f(x, y) \cdot |J|$$

5) Find $f_u(u)$ + $f_v(v)$ using method of finding Marginal densities

6) Change the domain values $G_n(x, y) \rightarrow (u, v)$

PROBLEMS :-

1) If the Joint Pdf of (X, Y) is given by $f_{XY}(x, y) = x + y \quad 0 \le x \le 1, \; 0 \le y \le 1$ find Pdf of $U = XY$

Given $f_{XY}(x, y) = x + y$ & $U = XY$

let $V = y$ (or) [$V = x$ can be taken]

If $y = v$, $x = u/v$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \quad \frac{\partial x}{\partial v} = \frac{-u}{v^2} \quad \frac{\partial y}{\partial u} = 0 \quad \frac{\partial y}{\partial v} = 1$$

$$|J| = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0 = \frac{1}{v}$$

Joint Pdf (U, V) is given by

$$f(U, V) = f(x, y)\,|J| = (x + y)\frac{1}{v} = \left(\frac{u}{v} + v\right)\frac{1}{v}$$

$$= \frac{u + v^2}{v^2}$$

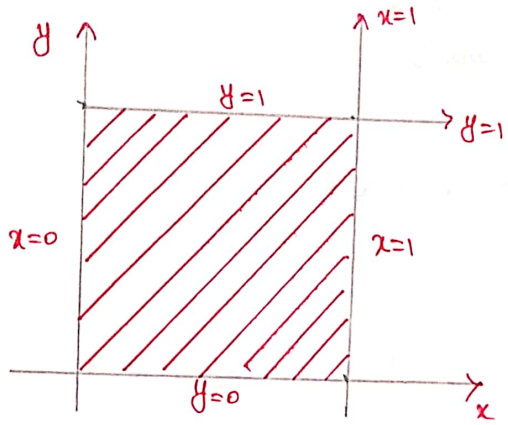
Range of v

$$0 \le y \le 1 \Rightarrow 0 \le v \le 1 \quad (\because v = y)$$

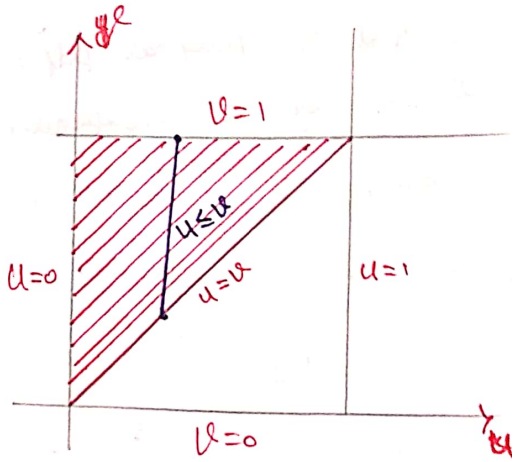
Range of u :-

$$0 \le x \le 1 \Rightarrow 0 \le u/v \le 1 \quad 0 \le u \le v$$

Change of domain :-



$0 \le x \le 1 \quad 0 \le y \le 1$



(u, v) domain

$(0 \le v \le 1) \quad (0 \le u \le v)$

$0 \le u \le v \le 1$

Pdf of (U, V)

$$f(u, v) = \frac{u + v^2}{v^2} \qquad \begin{matrix} 0 \le u \le v \\ 0 \le v \le 1 \end{matrix}$$

To find pdf of U :-

$$f_U(u) = \int_{-\infty}^{\infty} f(u, v)\, dv = \int_u^1 \frac{u + v^2}{v^2}\, dv = \int_u^1 \left(\frac{u}{v^2} + 1\right) dv$$

$$= \left[v + \frac{u(v^{-1})}{-1}\right]_u^1 = \left(v - \frac{u}{v}\right)_u^1 = (1 - u) - (u - 1)$$

$$= 2(1 - u) \quad 0 < u < 1$$

2) The waiting times of X & Y of two customers entering a bank at different times are assumed to be Independent R.V with Pdf's $f_X(x) = \begin{cases} e^{-x} & x>0 \\ 0 & \text{otherwise} \end{cases}$ $f_Y(y) = \begin{cases} e^{-y} & y>0 \\ 0 & \text{otherwise} \end{cases}$

Find the Joint pdf of the sum of their waiting times $U = X+Y$ and the fraction of this time that the first customer spreads waiting i.e) $V = \frac{X}{X+Y}$

(i) Find the Marginal pdf's of U and V

2) S.T they are Independent

Given $f_X(x) = e^{-x}$ $x>0$ $f_Y(y) = e^{-y}$ $y>0$

Since X and Y are Independent

$$f_{XY}(x,y) = f(x)\cdot f(y) = e^{-x}\cdot e^{-y} = e^{-(x+y)}, \quad x>0 \; y>0$$

Given $U = X+Y$ $V = \frac{X}{X+Y}$

Sub $\boxed{x = uv}$ $V = \frac{x}{U}$ $\frac{\partial x}{\partial u} = v$ $\frac{\partial x}{\partial v} = u$

$u = uv + y$ $\boxed{x = uv}$ $\frac{\partial y}{\partial u} = 1-v$ $\frac{\partial y}{\partial v} = -u$

$u - uv = y$

$y = u(1-v)$

<u>To find $|J|$</u>

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$= |-uv - (u(1-v))| = |-\cancel{uv} - u + \cancel{uv}|$$

$$= |-u| = u$$

<u>To find $f(u,v)$</u>

$$f(u,v) = f(x,y)\cdot |J|$$

$$= e^{-(x+y)}\cdot |u| = u\cdot e^{-u}$$

given $u = x+y$

To find Range of v

$x>0,\ y>0 \Rightarrow u>0 \qquad y = u - uv > 0 \Rightarrow \cancel{u} > \cancel{u}v$

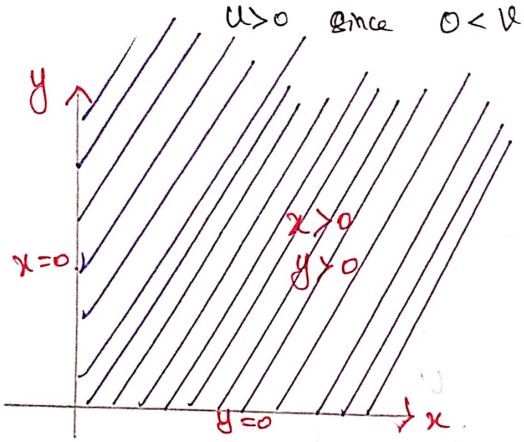
$\Rightarrow v < 1$

$\therefore x>0,\ y>0 \Rightarrow 0 < v < 1$

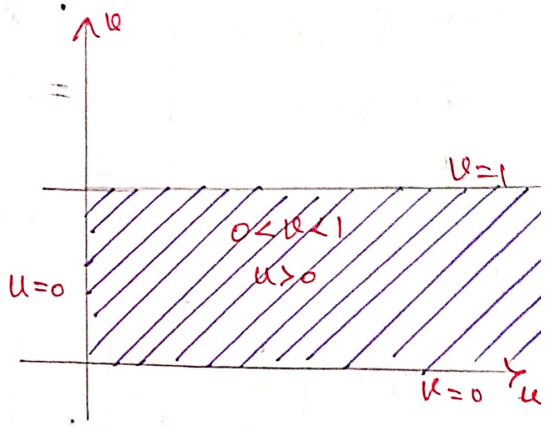
Range of u:-

$x = uv \qquad uv > 0$ as $x>0$

$u>0$ since $0<v<1$



xy domain



(u,v) domain

Pdf of (u,v)

$$f(u,v) = ue^{-u} \qquad 0<v<1,\ u>0$$

Marginal density function of U:-

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv = \int_0^1 ue^{-u}\,dv = ue^{-u}(v)_0^1 = ue^{-u}$$

$$f(u) = ue^{-u} \ ;\ u>0$$

Marginal density function of v:-

$$f(v) = \int_{-\infty}^{\infty} f(u,v)\,du = \int_0^{\infty} ue^{-u}\,du = \left[u(-e^{-u}) - (1)(e^{-u})\right]_0^{\infty}$$

$$= (e^{-\infty} - e^{-\infty}) - (0e^0 - e^0) = 1$$

$$f(v) = 1$$

To Prove u & v are Independent

$$f(u,v) = f(u)\cdot f(v)$$

$$ue^{-u} = ue^{-u}\cdot 1 = ue^{-u}$$

Hence u and v are Independent

3) Let (X,Y) be 2.D R.V having the Joint density function

$$f(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)}, & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of $U = \sqrt{X^2+Y^2}$

$$f(x,y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Given $U = \sqrt{X^2+Y^2}$

Let $V = X$

$\frac{\partial x}{\partial u} = 0 \quad \frac{\partial x}{\partial v} = 1$

$\Rightarrow u^2 = x^2+y^2$

$u^2 = v^2+y^2$

$y^2 = u^2-v^2$

$y = \sqrt{u^2-v^2}$

$\frac{\partial y}{\partial u} = \frac{\cancel{2}u}{\cancel{2}\sqrt{u^2-v^2}} = \frac{u}{\sqrt{u^2-v^2}}$

$\frac{\partial y}{\partial v} = \frac{-v}{\sqrt{u^2-v^2}}$

To Find $|J|$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\cancel{2}u}{\sqrt{u^2-v^2}} \\ 1 & \frac{-\cancel{2}v}{\sqrt{u^2-v^2}} \end{vmatrix}$$

$$= \left| 0 - \frac{u}{\sqrt{u^2-v^2}} \right| = \frac{u}{\sqrt{u^2-v^2}}$$

To find pdf of $f(u,v)$

$$f(u,v) = f(x,y)\cdot |J|$$

$$= 4xy\, e^{-(x^2+y^2)} \frac{u}{\sqrt{u^2-v^2}}$$

$$= 4v\cancel{\sqrt{u^2-v^2}}\, e^{-u^2} \cdot \frac{u}{\cancel{\sqrt{u^2-v^2}}}$$

$$= 4uv\, e^{-u^2}$$

Range of u & v

$y = \sqrt{u^2-v^2}$

$y \geq 0$

$u^2-v^2 \geq 0$

$u \geq v \Rightarrow u>0$

$u > v > 0 \Rightarrow \boxed{0 \leq v \leq u}$

$x = v$

$x>0$

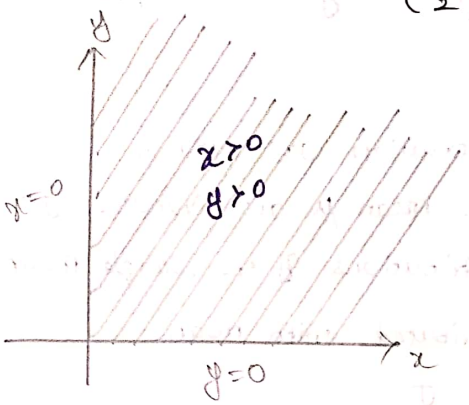
$\Rightarrow v>0$

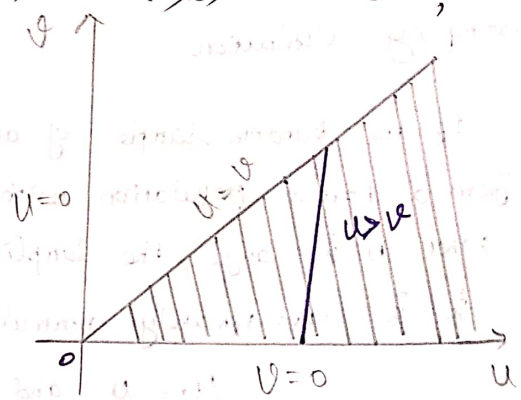
$$f(u,v) = \begin{cases} 4uv\, e^{-u^2} & 0<v<u \\ 0 & \text{otherwise} \end{cases}$$

To find Pdf of U

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv = \int_0^u 4uve^{-u^2}\,dv = 4ue^{-u^2}\int_0^u v\,dv$$

$$= 4ue^{-u^2}\left(\frac{v^2}{2}\right)_0^u = \cancel{4}^{2}ue^{-u^2}\left(\frac{u^2}{\cancel{2}}\right) = 2u^3e^{-u^2}, \quad u>0$$





UNIT - II

Co-Variance & Correlation

Covariance

Covariance and Correlation

If X and Y are Random Variables, then the co-variance between them is defined as

$$cov(X,Y) = E(XY) - E(X)\cdot E(Y)$$

Note:

1) $cov(aX, bY) = ab\ cov(X,Y)$
2) $cov(X+a, Y+b) = cov(X,Y)$
3) $cov(aX+b, cY+d) = ac\ cov(X,Y)$
4) $var(X_1 + X_2) = var(X_1) + var(X_2) + 2\ cov(X_1, X_2)$
5) $var(X_1 - X_2) = var(X_1) + var(X_2) - 2\ cov(X_1, X_2)$
6) If X_1 and X_2 are Independent then

$$var(X_1 \pm X_2) = var(X_1) \pm var(X_2) \quad [\because cov(X_1, X_2) = 0]$$

(7) $\sigma_x = \sqrt{\frac{1}{n}(\Sigma x^2) - \left(\frac{\Sigma x}{n}\right)^2}$ $\quad \sigma_y = \sqrt{\frac{1}{n}(\Sigma y^2) - \left(\frac{\Sigma y}{n}\right)^2}$

(8) $\left.\begin{matrix} r(X,Y) \\ \text{Correlation} \end{matrix}\right\} = \dfrac{Cov(X,Y)}{\sigma_x \cdot \sigma_y}$

(9) Correlation co-efficient (r) is Independent of Change of origin & scale

(ie) $r(X,Y) = r(U,V)$ where $U = \frac{X-a}{h}$, $V = \frac{Y-b}{k}$

Where a and b are some arbitrary constants usually the mid value of the given data X and Y respectively.

Problem:-

Let X and Y be discrete R.V with Prob fn $f(x,y) = \frac{x+y}{21}$

$x = 1,2,3$, $y = 1,2$ find (i) Mean and Variance of X and Y

(ii) $Cov(X,Y)$ (iii) correlation of X and Y.

$y \downarrow$ \ $x \rightarrow$	1	2	3	Total $P(Y=y)$
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
$P(X=x)$	5/21	7/21	9/21	1

$$E(X) = \Sigma x P(x) = 1 \times (5/21) + 2(7/21) + 3(9/21)$$

$$= \frac{5}{21} + \frac{14}{21} + \frac{27}{21} = \frac{46}{21}$$

$$E(Y) = \Sigma y P(x) = 1 \times \frac{9}{21} + 2 \times \frac{12}{21} = \frac{9+24}{21} = \frac{33}{21}$$

$$E(X^2) = 1^2 \times (5/21) + 2^2 (7/21) + 3^2 (9/21)$$

$$= \frac{5 + 28 + 81}{21} = \frac{114}{21}$$

$$E(Y^2) = 1^2 \left(\frac{9}{21}\right) + 2^2 \left(\frac{12}{21}\right) = \frac{9+48}{21} = \frac{57}{21}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{114}{21} - \left(\frac{46}{21}\right)^2 = \frac{278}{441}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \left(\frac{57}{21}\right) - \left(\frac{33}{21}\right)^2 = \frac{108}{441}$$

$$E(XY) = 1 \cdot 1 \cdot \left(\frac{2}{21}\right) + 1 \cdot 2 \cdot \left(\frac{3}{21}\right) + 1 \cdot 3 \cdot \left(\frac{4}{21}\right) +$$
$$2 \cdot 1 (3/21) + 2 \cdot 2 (4/21) + 2 \cdot 3 \cdot (5/21)$$

$$= \frac{2}{21} + \frac{6}{21} + \frac{12}{21} + \frac{6}{21} + \frac{16}{21} + \frac{30}{21} = \frac{72}{21}$$

$$Cov(XY) = E(XY) - E(X)\cdot E(Y)$$

$$= \frac{72}{21} - \frac{46}{21} \times \frac{33}{21} = \frac{1512 - 1518}{441} = \frac{-6}{441} = -0.0136$$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-0.0136}{\frac{\sqrt{276}}{21} \cdot \frac{\sqrt{108}}{21}} = \frac{-6}{(16.67)(10.39)} = -0.0346$$

(2) The Joint Prob Mass fn of X & Y is given below

$y \backslash x$	-1	1	$P(Y=y)$
0	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{4}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{4}{8}$
$P(X=x)$	$\frac{3}{8}$	$\frac{5}{8}$	1

find correlation co-eff of (X,Y)

$$E(X) = \Sigma x \cdot P(x) = (-1)(3/8) + 1(5/8) = 2/8 = 1/4$$

$$E(Y) = \Sigma y\, P(y) = 0(4/8) + 1(4/8) = 4/8 = 1/2$$

$$E(X^2) = \Sigma x^2 P(x) = (-1)^2(3/8) + 1^2(5/8) = 8/8 = 1$$

$$E(Y^2) = \Sigma y^2 P(y) = 0^2(4/8) + 1^2(4/8) = 4/8$$

$$Var(X) = E(X^2) - (E(X))^2 = 1 - (1/4)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{4}{8} - (4/8)^2 = \frac{32-16}{64} = \frac{16}{64} = \frac{1}{4}$$

$$E(XY) = (-1)\underbrace{\cancel{(0)(1/8)}}_{0} + (1)\underbrace{\cancel{(0)(3/8)}}_{0} + 1(-1)(2/8) + (1)(1)(2/8)$$

$$E(XY) = -2/8 + 2/8 = 0.$$

$$r(X,Y) = \frac{E(XY) - E(X)\cdot E(Y)}{\sigma_X \cdot \sigma_Y}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$0 - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$$

$$Cov(XY) = -\frac{1}{8}$$

$$r(X,Y) = \frac{-(1/8)}{\left(\frac{\sqrt{15}}{4}\right)\left(1/2\right)} = \frac{-1/8}{\sqrt{15}/8} = \frac{-1}{\sqrt{15}} = -0.258.$$

(3) Two R.V X and Y have the Joint density

$$f(x,y) = \begin{cases} 2-x-y & 0<x<1, \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

S.T $\text{cov}(X,Y) = \frac{-1}{144}$ also find Co-eff of Correlation

Soln

$$f(x) = \int_{y=0}^{1} f(x,y)\,dy = \int_0^1 (2-x-y)\,dy = \left[(2-x)y - y^2/2\right]_0^1$$

$$= (2-x)(1) - 1/2 = \frac{2(2-x)-1}{2} = \frac{4-2x-1}{2} = \frac{3-x2}{2}$$

$$\boxed{f(x) = \frac{3-2x}{2} \qquad 0<x<1}$$

$$f(y) = \int_{x=0}^{1} (2-x-y)\,dx = \left[(2-y)x - x^2/2\right]_0^1 = \frac{3-2y}{2} \qquad 0<y<1$$

Now

$$E(X) = \int_0^1 x\cdot f(x)\,dx = \int_0^1 x\cdot\left(\frac{3-2x}{2}\right)dx = \frac{1}{2}\int_0^1 (3x-2x^2)\,dx$$

$$= \frac{1}{2}\left[3\left(\frac{x^2}{2}\right) - 2\left(\frac{x^3}{3}\right)\right]_0^1 = \frac{1}{2}\left[\left(\frac{3}{2}-\frac{2}{3}\right)\right]$$

$$= \frac{1}{2}\left(\frac{9-4}{6}\right) = \frac{1}{2}\times\frac{5}{6} = \frac{5}{12}$$

Illy

$$E(Y) = \int_0^1 y\cdot f(y)\,dy = \int_0^1 y\cdot\left(\frac{3-2y}{2}\right)dy = 5/12$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3-2x}{2}\right)dx = \frac{1}{2}\int_0^1 (3x^2-2x^3)\,dx$$

$$= \frac{1}{2}\left[\cancel{3}\left(\frac{x^3}{\cancel{3}}\right) - \cancel{2}\left(\frac{x^4}{\cancel{4}_2}\right)\right]_0^1 = \frac{1}{2}\left(1-1/2\right) = 1/4$$

$$E(Y^2) = \int_0^1 y^2 . f(y)\, dy = \int_0^1 y^2 \left(\frac{3-2y}{2}\right) dy = 1/4$$

$$E(XY) = \int_0^1 \int_0^1 xy\, f(x,y)\, dx\, dy = \int_0^1 \int_0^1 xy\,(2-x-y)\, dy\, dx$$

$$= \int_0^1 \int_0^1 (2xy - x^2y - y^2x)\, dy\, dx = \int_0^1 \left[2x(y^2/2) - \frac{x^2y^2}{2} - x\frac{y^3}{3}\right]_0^1 dx$$

$$= \int_0^1 \left[x(1-0) - \frac{x^2}{2}(1-0) - x(1/3 - 0)\right] dx = \int_0^1 \left(x - \frac{x^2}{2} - x/3\right) dx$$

$$= \int_0^1 \left(\frac{2x}{3} - \frac{x^2}{2}\right) dx = \left(\frac{2}{3}\left(\frac{x^2}{2}\right) - \frac{x^3}{6}\right)_0^1 = \left(\frac{1}{3} - \frac{1}{6}\right) - (0)$$

$$E(X,Y) = \left(\frac{2-1}{6}\right) = \frac{1}{6}$$

$$Cov(X,Y) = E(X,Y) - E(X)\,E(Y)$$

$$= \frac{1}{6} - \frac{5}{12} \times \frac{5}{12} = \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144}$$

$$= -\frac{1}{144}$$

$$r(x,y) = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-1/144}{\left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)} = \frac{-1}{144} \times 4 = \frac{-1}{36}$$

$$r(x,y) = -0.02777$$

(3 (A) Suppose that the 2-D RV's (X, Y) has the Joint Pdf

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain correlation co-eff between X and Y, Check whether X and Y are Independent

$$f(x) = \int_0^1 (x+y)\,dy = \left(xy + y^2/2\right)_0^1 = x + 1/2$$

$$f(x) = x + 1/2 \qquad 0<x<1$$

$$f(y) = \int_0^1 (x+y)\,dx = \left(x^2/2 + xy\right)_0^1 = 1/2 + y$$

$$f(y) = y + 1/2 \qquad 0<y<1$$

$$E(X) = \int_0^1 x \cdot f(x)\,dx = \int_0^1 x(x+1/2)\,dx = \int_0^1 (x^2 + x/2)\,dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1$$

$$= \left(\frac{1}{3} + \frac{1}{4}\right) - 0 = \frac{4+3}{12} = 7/12$$

$$E(Y) = \int_0^1 y \cdot f(y)\,dy = \int_0^1 y(y+1/2)\,dy = \int_0^1 (y^2 + y/2)\,dy = 7/12$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2(x+1/2)\,dx = \int_0^1 \left(x^3 + \frac{x^2}{2}\right)dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{6}\right)_0^1 = \left(\frac{1}{4} + \frac{1}{6}\right) - (0) = \frac{6+4}{24} = \frac{10}{24} = \frac{5}{12}$$

$$E(Y^2) = \int_0^1 (y^3 + y^2/2)\,dy = 5/12$$

$$E(XY) = \int_0^1 \int_0^1 xy(x+y)\,dy\,dx = \int_0^1 \int_0^1 (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_0^1 dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right)dx$$

$$= \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1 = \left(\frac{1}{6} + \frac{1}{6}\right) - 0 = \frac{2}{6} = 1/3$$

$$\text{Cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$= \frac{1}{3} - \frac{7}{12} \times \frac{7}{12} = \frac{1}{3} - \frac{49}{144} = \frac{48-49}{144} = \frac{-1}{144}$$

$$\text{cov}(XY) = -1/144 \neq 0$$

$\therefore$ X and Y are Not Independent

Now $\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{5}{12} - (7/12)^2 = \frac{5}{12} - \frac{49}{144}$

$$\sigma_X^2 = \frac{60-49}{144} = \frac{11}{144} \Rightarrow \sigma_X = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12}$$

IIIly $\text{Var}(Y) = E(Y^2) - (E(Y))^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$

$$\sigma_Y = \frac{\sqrt{11}}{12}$$

$$r(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{-1/144}{\frac{\sqrt{11}}{12} \times \frac{\sqrt{11}}{12}} = \frac{-1}{\cancel{144}} \times \frac{\cancel{144}}{11} = \frac{-1}{11}$$

$= 0.0909.$ (+ve)

Regression line x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$= -0.134\left(\frac{0.83}{0.9}\right)$$

$$b_{xy} = -0.124$$

$$\bar{x} = E(x) = 0.9; \quad \bar{y} = E(y) = 1$$

$$x - 0.9 = -0.124(y-1)$$

$$x - 0.9 = -0.124y + 0.124$$

$$\boxed{x = 1.024 - 0.124y}$$

Regression line y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} \Rightarrow b_{yx} = -0.145$$

$$y - 1 = -0.145(x - 0.9)$$

$$\boxed{y = 1.1305 - 0.145x}$$

Ex: 2 The JPDF of a 2DRV is given by $f(x,y) = \frac{1}{3}(x+y)$; $0 \le x \le 1$, $0 \le y \le 2$. Find the eqn. of two lines of regression.

Soln:

$$f_X(x) = \int_0^2 f(x,y)\,dy = \int_0^2 \frac{1}{3}(x+y)\,dy$$

$$= \frac{1}{3}\left(xy + \frac{y^2}{2}\right)_0^2 = \frac{1}{3}(2x+2),\ 0 \le x \le 1$$

$$f_X(x) = \frac{1}{3}(2x+2),\ 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 f(x,y)\,dx$$

$$= \int_0^1 \frac{1}{3}(x+y)\,dx = \frac{1}{3}\left[\frac{x^2}{2} + yx\right]_0^1$$

$$f_Y(y) = \frac{1}{3}\left[\frac{1}{2} + y\right] = \frac{1}{6}(1+2y),\ 0 \le y \le 2$$

$$E(X) = \int_0^1 x\, f(x)\,dx$$

$$= x\int_0^1 \frac{1}{3}(2x+2)\,dx$$

$$= \frac{1}{3}\left[2\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1$$

$$\boxed{E(X) = \frac{5}{9}}$$

$$E(Y) = \int_0^2 y\, f(y)\,dy$$

$$= \int_0^2 \frac{1}{6}(1+2y)\,y\,dy$$

$$= \frac{1}{6}\left(\frac{y^2}{2} + 2\frac{y^3}{3}\right)_0^2$$

$$\boxed{E(Y) = \frac{11}{9}}$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\,\frac{2}{3}(1+x)\,dx$$

$$= \frac{2}{3}\int_0^1 (x^2 + x^3)\,dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{7}{18}$$

$$E(Y^2) = \int_0^2 y^2 f(y)\,dy = \int_0^2 y^2\,\frac{1}{6}(1+2y)\,dy$$

$$= \frac{1}{6}\int_0^2 y^2(1+2y)\,dy = \frac{1}{6}\left[\frac{y^3}{3} + \frac{2y^4}{4}\right]_0^2$$

$$= \frac{16}{9}$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= \frac{7}{18} - \left(\frac{5}{9}\right)^2 \Rightarrow \sigma_x = \sqrt{13/162}$$

$$\sigma_y^2 = E(y^2) - [E(y)]^2$$

$$= \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81} \Rightarrow \boxed{\sigma_y = \frac{\sqrt{23}}{9}}$$

$$E(xy) = \int_0^2 \int_0^1 xy\, f(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^1 xy \frac{1}{3}(x+y)\, dx\, dy$$

$$= \frac{1}{3} \int_0^2 \int_0^1 (x^2y + xy^2)\, dx\, dy$$

$$= \frac{1}{3} \int_0^2 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2\right)_0^1 dy$$

$$= \frac{1}{3} \int_0^2 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy = \frac{1}{3}\left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^2$$

$$= \frac{1}{3}\left(\frac{4}{6} + \frac{8}{6}\right) = \frac{2}{3}$$

$$Cov(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{2}{3} - \left(\frac{5}{9}\right)\left(\frac{11}{9}\right) = -\frac{1}{81}$$

$$\rho(x,y) = -\sqrt{\frac{2}{299}}$$

<u>Regression line x on y:</u>

$$(x - \bar{x}) = b_{xy}(y - \bar{y})$$

$$x - \frac{5}{9} = -\frac{1}{23}\left(y - \frac{11}{9}\right)$$

$$x = -\frac{1}{23} + \frac{14}{23}$$

<u>Regression line y on x</u>

$$(y - \bar{y}) = b_{yx}(x - \bar{x})$$

$$y - \frac{11}{9} = -\frac{2}{3}\left(x - \frac{5}{9}\right)$$

$$\boxed{y = -\frac{2x}{3} + \frac{17}{13}}$$

Ex: 3: In a bivariate population, $\sigma_x = \sigma_y = \sigma$, and the angle b/w the regression lines is $\tan^{-1}(6)$. obtain the value of Corr. coeff

Soln: Given $\sigma_x = \sigma_y = \sigma$

$\theta = \tan^{-1}(6) \Rightarrow \tan\theta = 6$ —— (1)

WKT $\tan\theta = \frac{(1-r^2)}{r} \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$

$= \frac{(1-r^2)}{r} \frac{\sigma^2}{2\sigma^2} = \frac{1-r^2}{r} \cdot \frac{1}{2}$

$$\boxed{\tan\theta = \frac{1-r^2}{2r}}$$

From (1), $6 = \frac{1-r^2}{2r}$

$1 - r^2 = 12r$

$r^2 + 12r - 1 = 0$

$r = \frac{-12 \pm \sqrt{144 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{-12 \pm \sqrt{148}}{2}$

$= \frac{-12 \pm 2\sqrt{37}}{2} = -6 \pm \sqrt{37}$

$r = -6 + \sqrt{37}$ (or) $r = -6 - \sqrt{37}$

$\boxed{r = 0.0828}$ $\boxed{r = -12.0828}$

Central Limit Theorem:

The central limit theorem states that under general conditions the sums and means of sample of random measurement drawn from a population tend to possess an approximately well shaped sampling distribution.

If $X_1, X_2, \ldots, X_n$ is a sequence of n independent and identically distributed (i.i.d) R.V, each having mean μ and variance σ^2 and if $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$, then the variate $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ has a distn. that approaches the std. normal distn. as $n \to \infty$ provided mgf exists.

Other forms

1) Liapounoff's form of CLT

If $X_1, X_2, \ldots, X_n$ is a sequence of ind. R.V with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots$ and if $S_n = X_1 + X_2$ then under general conditions S_n follows a N.D with mean $\mu = \mu_i$ & $\sigma^2 = \sigma_i^2$, $i = 1, 2, \ldots n$ as $n \to \infty$

If $\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$

$$E(\bar{x}) = \frac{n\mu}{n} = \mu \; ; \; V(\bar{x}) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$\sigma_i = \frac{\sigma}{\sqrt{n}}$$

$\bar{x} \sim N(\mu, \sigma/\sqrt{n})$ as $n \to \infty$

2) Lindeberg - Levy CLT :

If R.V $X_1, X_2, \dots X_n$ have the same pdf with mean μ and variance σ^2 then R.V $S_n = X_1 + X_2 + \dots + X_n$ follows N.D with mean $(n\mu)$ and variance $(n\sigma^2)$

i.e., $S_n \sim N(n\mu, \sigma\sqrt{n})$

Problems

1) Let $X_1, X_2, \dots, X_{100}$ be independent & identically dist. R.V with mean $\mu = 2$ & $\sigma^2 = \frac{1}{4}$.

Find $P(192 < X_1 + X_2 + \dots + X_{100} < 210)$

Soln: Given $\mu = E(X) = 2$

$\sigma^2 = Var(X) = \frac{1}{4}$

$n = 100$

$\bar{X} = X_1 + X_2 + \dots + X_{100}$

$\bar{X} \sim N(n\mu, \sigma\sqrt{n}) \Rightarrow \bar{X} \sim N(2 \times 100, \frac{1}{2} \times 10)$

$\bar{X} \sim N(200, 5)$

$$P(192 < \bar{X} < 210) = P\left(\frac{192-\mu}{\sigma\sqrt{n}} < \frac{\bar{X}-\mu}{\sigma\sqrt{n}} < \frac{210-200}{\sigma\sqrt{n}}\right)$$

$$= P\left(\frac{192-200}{5} < Z < \frac{210-200}{5}\right)$$

$$= P(-1.6 < Z < 2) = P(0 < Z < 1.6) + P(0 < Z < 2)$$

$$= 0.4452 + 0.4772$$

$$= 0.9224$$

2) The life time of a certain brand of an electric bulb may be considered as a R.V with mean 1200 hr and S.D 250 hr. Find the prob. using CLT that average life time of 60 bulbs exceeds 1250 hr.

Soln: Given $\mu = E(x) = 1200$

$\sigma = 250$

$Var(x) = \sigma^2 = (250)^2$

$n = 60$

and $\bar{x} \rightarrow$ Mean life time of 60 bulbs

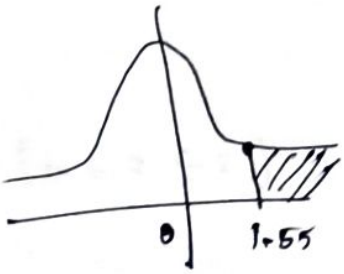
$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$ i.e, $\bar{X} \sim N\left(1200, \frac{250}{\sqrt{60}}\right)$

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 1200}{250/\sqrt{60}}$$

Given $\bar{x} = 1250$

$$P(\bar{x} > 1250) = P\left(\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} > \frac{1250 - 1200}{250/\sqrt{60}}\right)$$

$$= P\left(Z > \frac{50 \times \sqrt{60}}{250}\right)$$



$$= P(Z > 1.55)$$

$$= 0.5 - P(0 < Z < 1.55)$$

$$= 0.5 - 0.4394$$

$$= 0.0606$$

3) A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using CLT, with what prob. can we assert that mean of the sample will not differ from $\mu = 60$ by more than 4

Soln: Given $\mu = E(x) = 60$

$\sigma^2 = Var(x) = 400$

$n = 100$

$\bar{x}$ be sample mean

$\bar{x} \sim$ N.D. $\bar{x} \sim N(\mu, \sigma/\sqrt{n})$

$\bar{x} \sim N(60, 2)$

$$P[|\bar{x} - 60| \leq 4] = P[60 - 4 \leq \bar{x} \leq 60 + 4]$$

$$= P[56 \leq \bar{x} \leq 64]$$

$$= P\left[\frac{56-60}{2} \leq \frac{\bar{x}-60}{2} \leq \frac{64-60}{2}\right]$$

$$= P[-2 \leq z \leq 2]$$

$$= P[-2 \leq z \leq 0] + P[0 \leq z \leq 2]$$

$$= 2P[0 < z < 2]$$

$$= 2(0.4772) = 0.9544$$