Kendal's notation for representing queuing models :-

David Kendal introduced a special notation to describe a queuing system as

$$(a/b/c) : (d/e)$$

Where

a : arrival (or inter arrival) probability distribution

b : service time probability distribution

c : number of servers

d : capacity of the system

e : queue discipline (or service discipline)

Symbols a and b denotes one of the following

M : Markov (Poisson) arrival and departure distribution or exponential distribution.

G : General service time distribution or general departure distribution.

Symbols for e denotes the following.

FCFS : First Come First Served (or)

FIFO : First In First Out

SIRO : Service In Random Order.

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The notations usual used in queuing theory:

n : number of customer in the system.

λ : mean arrival rate

μ : mean service rate

ρ : traffic intensity

L_q : average number of customer in the queue.

L_s : average number of customer in the system

W_q : mean waiting time in the queue.

W_s : mean waiting time in the system.

$P_n(t)$: transient state probability of exactly n customers in the system.

P_n : steady state probability of n customers in the system.

Little's formula:

① $L_s = \lambda W_s$

② $W_q = W_s - \frac{1}{\mu}$

③ $L_q = \lambda W_q$

④ $L_s = L_q + \frac{\lambda}{\mu}$

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Model (1):- Single server model with infinite system capacity

i.e. $(M/M/1):(\infty/FIFO)$

(I) Traffic intensity

$$\rho = \frac{\lambda}{\mu}$$

(II) $$W_s = \frac{1}{\mu-\lambda}$$

(III) $$W_q = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$$

(IV) $$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(V) $$L_s = \lambda W_s = \frac{\lambda}{\mu-\lambda}$$

(VI) Probability that the system is empty

$$P_0 = 1 - \frac{\lambda}{\mu}$$

(VII) Probability that n customer in the system

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

(VIII) $$P(N \geq k) = \left(\frac{\lambda}{\mu}\right)^k$$

(IX) $$P(W_s > t) = e^{-(\mu-\lambda)t}$$

(X) The probability density function of waiting time in the system

$$f(w) = (\mu-\lambda)\, e^{-(\mu-\lambda)w}, \quad w \geq 0$$

(XI) The probability density function of waiting time in the queue

$$g(w) = \frac{\lambda}{\mu}(\mu-\lambda)\, e^{-(\mu-\lambda)w}, \quad w \geq 0$$

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Model (2):- Single server model with finite system capacity

i.e. $(M/M/1) : (K/FIFO)$

(I) $$\rho = \frac{\lambda}{\mu}$$

(II) $$P_n = \begin{cases} \dfrac{\left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} & , \rho \neq 1 \text{ i.e. } \lambda \neq \mu \\ \dfrac{1}{K+1} & , \rho = 1 \text{ i.e. } \lambda = \mu \end{cases}$$

(III) $$P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \rho \neq 1 \text{ i.e } \lambda \neq \mu \\ \dfrac{1}{K+1} & \text{if } \rho = 1 \text{ i.e } \lambda = \mu \end{cases}$$

(IV) Effective arrival rate

$$\lambda_{eff} = \lambda' = \mu(1-P_0)$$

(V) $$E(N) = L_s = \begin{cases} \dfrac{\lambda}{\mu-\lambda} - \dfrac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{if } \lambda \neq \mu \\ \dfrac{K}{2} & \text{if } \lambda = \mu \end{cases}$$

(VI) $$E(N_q) = L_q = E(N) - \frac{\lambda'}{\mu}$$

(VII) $$E(W_s) = \frac{E(N)}{\lambda'}$$

(VIII) $$E(W_q) = \frac{E(N_q)}{\lambda'}$$

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Ex-(1):- Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. b/w one arrival and the next. The length of phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average no. of customers waiting in the system.

(b) What is the prob. that a person arriving at the booth will have to wait in the queue.

(c) What is the prob. that it will take him more than 10 min. altogether to wait for the phone and complete his call?

(d) Estimate the fraction of the day when the phone will be in use.

(e) When the waiting time is atleast 3 min. the second booth will be install. By how much the flow of arrivals should increase in order to justify a second booth?

(f) What is the average length of the queue that forms from time to time?

Sol:- Here $\lambda = \frac{1}{12}$, $\mu = \frac{1}{4}$ $\therefore$ $\rho = \frac{\lambda}{\mu} = \frac{1}{3}$

(a) $E(N_s) = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{4} - \frac{1}{12}} = 0.5$

(b) $P(W>0) = 1 - P_0 = \frac{\lambda}{\mu} = \frac{1}{3}$

(c) $P(W>10) = e^{-(\mu-\lambda)\times 10}$

$= e^{-(\frac{1}{4}-\frac{1}{12})\times 10} = e^{-\frac{5}{3}} = 0.1889$

(d) P(the phone will be in use) $= 1 - P_0 = 1 - \frac{2}{3}$

$= 1/3$

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(e) The second bath will be installed if

$$E(W_q) > 3$$

i.e $\frac{\lambda}{\mu(\mu-\lambda)} > 3$ i.e. $\frac{\lambda_R}{\frac{1}{4}(\frac{1}{4}-\lambda_R)} > 3$

$$\Rightarrow \lambda_R > \frac{3}{28}$$

$\therefore$ The arrival rate should increase by

$$\frac{3}{28} - \frac{1}{12} = \frac{1}{42} \text{ per minute.}$$

(f) $E(N_q /$ the queue is always available$)$

$$= E(N_q / N_q > 0)$$

$$= E(N_q / N > 1)$$

$$= \frac{E(N_q)}{P(N>1} = \frac{E(N_q)}{1-P_0-P_1}$$

$$= \frac{\lambda^2}{\mu(\mu-\lambda)} \cdot \frac{1}{1-(1+\frac{\lambda}{\mu})(1-\frac{\lambda}{\mu})}$$

$$= \frac{\mu}{\mu-\lambda} = \frac{\frac{1}{4}}{\frac{1}{4}-\frac{1}{12}}$$

$$= 1.5 \text{ person.}$$

Answer

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Ex-(2):- Customers arrive at a one man barber shop acc^n to a Poisson process with a mean interval time of 12 min. Customers spend an average of 10 min in the barber's chair.

(a) What is the expected no. of customers in the barber shop and in the queue?

(b) Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.

(c) How much time can a customer expect to spend in the barber's shop?

(d) If the waiting time exceeds 1.25 hr, management provide another chair, find increase arrival rate.

(e) What is the average time customers spend in the queue?

(f) What is the prob. that the waiting time in the system is greater than 30 min?

(g) Calculate the percentage of customers who have to wait prior to getting into the barber's chair.

(h) What is the probabability that more that 3 customers are in the system?

Sol:- we have

$\frac{1}{\lambda} = 12 \Rightarrow \lambda = \frac{1}{12}$ per minute

$\frac{1}{\mu} = 10 \Rightarrow \mu = \frac{1}{10}$ per minute

(a) $E(N_s) = \frac{\lambda}{\mu - \lambda} = 5$

$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = 4.17$

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(b) P(a customer straight goes to the barber's chair)

i.e. $P_0 = 1 - \frac{\lambda}{\mu} = \frac{1}{6}$

(c) $E(W) = \frac{1}{\mu - \lambda} = 60$ min.

(d) $E(W) > 75$ i.e. $\frac{1}{\mu - \lambda_r} > 75$

$\Rightarrow \lambda_r > \frac{13}{150}$

Increase arrival rate

$= \frac{13}{150} - \frac{1}{12} = \frac{1}{300}$ per minute

(e) $E(W_q) = \frac{\lambda}{\mu(\mu - \lambda)} = 50$ min.

(f) $P(W > t) = e^{-(\mu - \lambda)t}$

$\therefore P(W > 30) = e^{-\left(\frac{1}{10} - \frac{1}{12}\right) \times 30} = 0.6065$

(g) P(a customer has to wait) $= P(W > 0)$

$= 1 - P_0 = \frac{\lambda}{\mu} = \frac{5}{6}$

$\therefore$ Percentage of customers who have to wait

$= \frac{5}{6} \times 100 = 83.33$

(h) $P(N > 3) = 1 - (P_0 + P_1 + P_2 + P_3)$

$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{5}{6}\right)^4$

$= 0.4823$

Answer

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Ex-(3):- At what average rate must a clerk in a supermarket work in order to ensure a probability of 0.90 that the customer will not wait longer than 12 min? It is assume that there is only one counter at which customer arrive in a poisson fashion at an average rate of 15 per hour and that the length of the service by the clerk has an exponential distribution

Sol:- Here

$$\lambda = 15 \text{ per hours} \quad , \quad \mu = \mu_R \text{ / hours}$$

According to question

$$P\left(W_q \le \frac{1}{5}\right) = 0.90$$

$$\therefore \; P\left(W_q > \frac{1}{5}\right) = 1 - 0.90 = 0.10$$

$$\int_{\frac{1}{5}}^{\infty} g(w)\, dw = 0.10$$

$$\int_{0.2}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)\, e^{-(\mu-\lambda)w}\, dw = 0.10$$

$$\left[\frac{\lambda}{\mu} \cancel{(\mu-\lambda)} \frac{e^{-(\mu-\lambda)w}}{-\cancel{(\mu-\lambda)}}\right]_{0.2}^{\infty} = 0.10$$

$$\frac{15}{\mu_R} e^{-(15-\mu_R)\times 0.2} = \frac{0.10}{100}$$

$$150\, e^{-0.2(15-\mu_R)} = \mu_R$$

taking log

$$\log 150 + 0.2(15-\mu_R) = \log \mu_R$$

Solving $\mu_R = 24$ approximately

Answer

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Ex-(15):- A duplicating machine maintained for office use is operated by an office assistant who earns Rs 5 per hour. The time to complete each job varies according to an exponential distribution with mean 6 min. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If a 8-hr day is used as base, determine

(a) The percentage idle time of the machine

(b) The average time a job is in the system and

(c) the average earning per day of the assistant.

Sol:- Here

$$\lambda = 5/\text{hour}, \quad \mu = \frac{60}{6} = 10/\text{hour}.$$

(a) $P(\text{the machine is idle}) = P_0$

$$= 1 - \frac{\lambda}{\mu} = 1 - \frac{5}{10} = \frac{1}{2} = 50 \text{ percent}.$$

(b)

$$E(W) = \frac{1}{\mu - \lambda}$$

$$= \frac{1}{10-5} = \frac{1}{5} \text{ hr or } 12 \text{ minutes}.$$

(c) $E(\text{earning per day})$

$= E(\text{Number of jobs done/day}) \times \text{earning per job}.$

$= E(\text{Number of jobs done/day}) \times E(\text{time in hour/job}) \times \text{earning per hour}$

$$= (8 \times 5) \times \frac{1}{5} \times 5 = \text{Rs } 40$$

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Ex-(13):- In a single server queuing system with poisson input and exponential service rate, if the mean arrival rate is 3 calling units per hours, the expected service time is 0.25 hrs and the maximum possible number of calling unit in the system is 2, find P_n $(n \geq 0)$, average number of calling units in the system and in the queue. and average waiting time in the system ~~and~~ in the queue.

Sol:- The given problem is of $(M/M/1):(K/FIFO)$

Here

$$\lambda = 3, \quad \mu = 4 \quad \text{and} \quad K = 2$$

We have

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right] \qquad (\because \lambda \neq \mu)$$

$$= \left(\frac{3}{4}\right)^n \left[\frac{1-\frac{3}{4}}{1-\left(\frac{3}{4}\right)^{2+1}}\right] = (0.4324)(0.75)^n$$

Again

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} = 0.4324$$

Also

$$E(N) = \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}$$

$$= \frac{3}{4-3} - \frac{(2+1)\left(\frac{3}{4}\right)^{2+1}}{1-\left(\frac{3}{4}\right)^{2+1}} = \frac{30}{37} \approx 0.8$$

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Now

$$\lambda' = \mu(1-P_0)$$

$$= 4\left(1-\frac{16}{37}\right) = \frac{84}{37}$$

$$E(N_q) = E(N) - \frac{\lambda'}{\mu}$$

$$= \frac{30}{37} - \frac{1}{4}\left(\frac{84}{37}\right) = \frac{9}{37} = 0.24$$

$$E(W_s) = \frac{E(N)}{\lambda'}$$

$$= \frac{\left(\frac{30}{37}\right)}{\frac{84}{37}}$$

$$= \frac{5}{14} \text{ hrs or } 21.4 \text{ minutes}$$

$$E(W_q) = \frac{E(N_q)}{\lambda'}$$

$$= \frac{\left(9/37\right)}{\left(84/37\right)}$$

$$= \frac{9}{84}$$

$$= \frac{3}{28} \text{ hr}$$

or 6.4 minutes

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Ex-(14):- The local one-person barber shop can accommodate a maximum of 5 peoples at a time (4 waiting and 1 getting hair-cut). Customers arrive according to a Poisson distribution with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (Exponential service time).

(a) What percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is the expected number of customers waiting for a hair-cut?

(d) How much time can a customers expect to spend in the barber shop?

Sol: Here $\lambda = 5$, $\mu = 4$, $k = 5$

(a) P(the barber is idle)

i.e. $$P(N=0) = P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6} = 0.0888$$

$\therefore$ Percentage of time when the barber is idle

$$= 8.88 \approx 9$$

(b) P(a customer is turned away)

$$P(N>5) = \left(\frac{\lambda}{\mu}\right)^5 \left[\frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}\right]$$

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$$= \left(\frac{5}{4}\right)^5 \left[\frac{1-\frac{5}{4}}{1-\left(\frac{5}{4}\right)^6} \right] = \frac{3125}{11529} = 0.2711$$

(c)

$$E(N_2) = E(N) - (1-P_0)$$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(K+1)\left(\frac{\lambda}{\mu}\right)^{K+1}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} - (1-P_0)$$

$$= \left(\frac{5}{4-5}\right) - \frac{6\left(\frac{5}{4}\right)^6}{1-\left(\frac{5}{4}\right)^6} - (1-0.0888)$$

$\approx$ 2.2 customers.

(d)

$$E(W) = \frac{E(N)}{\lambda'}$$

$$= \frac{E(N)}{\mu(1-P_0)}$$

$$= \frac{3.1317}{3.6448}$$

$= 0.8592$ hrs

$= 51.5$ minutes

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Ex-(16):- Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hours. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(a) Find the effective arrival rate at the clinic.

(b) What is the probability that an arriving patient will not wait?

(c) What is the expected waiting time until a patient is discharged from the clinic?

Sol:- Here

$\lambda = 30/\text{hour}$ $\mu = 20/\text{hour}$, $k = 14+1 = 15$

(a) As $\lambda \neq \mu$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1-\frac{30}{20}}{1-\left(\frac{30}{20}\right)^{16}} = 0.00076$$

Effective arrival rate

$$\lambda' = \mu(1-P_0)$$

$$= 20(1-0.00076)$$

$$= 19.98 \text{ per hour}$$

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(b) P(a patient will not wait)

$= P_0 = 0.00076$

(c) $$E(N) = \frac{\lambda}{\mu - \lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{30}{20-30} - \frac{16\left(\frac{30}{20}\right)^{16}}{1 - \left(\frac{30}{20}\right)^{16}}$$

≈ 13 patients nearly

$$E(W) = \frac{E(N)}{\lambda'}$$

$$= \frac{13}{19.98} = 0.65 \text{ hrs}$$

$= 39$ minutes

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