Complejos

Z^2
$Z = a + bi$
$w = c + di$
$z \cdot w = (a + bi)(c + di)$
$= a(c, di) + bi(c + di)$
$= ac + iad + ibc + bdi^2$
$= ac - bd + i(ad + bc)$

Consideramos el conjunto

$$R^2 = R \times R = \{(a,b) : a, beR\}$$

Ejemplos:

$$(1,2) + (3,4) = (1+3, 2+4) = (4,6)$$
$$\left(\frac{2}{5}, \frac{1}{3}\right) + \left(\frac{3}{4}, \frac{1}{2}\right) = \left(\frac{8+15}{20}, \frac{2-3}{6}\right)$$

25 de Julio

Resumen Numeros complejos: Conjunto de los reales que poseen suma y Producto

$$C := (\mathbb{R}^2, +, \cdot)\ 00$$
$$(a,b) = (a,0) + (b,0)$$
$$= (a,0)\cdot(1,0) + (b,0)\cdot(0,1)$$
$$= a\,(1,0) + b\,(0,1)$$
$$= a + bi$$

Ejercicio:

$$\frac{1}{1-i} + \frac{1}{1+i} = \frac{2}{2} = 1 \qquad \frac{1\cdot(1+i)}{1-i} + \frac{1\cdot(1-i)}{1+i} = \frac{1+i+1-i}{2} = \frac{2}{2}$$

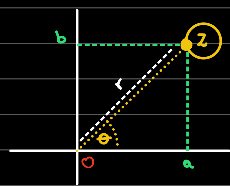
Forma Polar de un Complejo

Dado $Z \in \mathbb{C}$, cuya Forma Binomial es: $z = a + bi$

Consideramos: r : distancia de Z a O

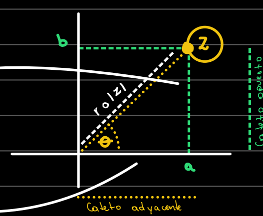
θ : angulo que forma la recta (argumento)

OZ : con el eje x



Relación entre Forma Binomica y Polar

- Si: r y θ son conocidos:
  - $\sin\theta = \frac{cat.op}{hip} = \frac{b}{r} \Rightarrow b = r\cdot\sin\cdot\theta$
  - $\cos\theta = \frac{cat.ad}{hip} = \frac{a}{r} \Rightarrow\ = r\cdot\cos\cdot\theta$



Asi:
$$z = r\cdot\cos\cdot\theta + i\cdot r\cdot\sin\cdot\theta$$
$$= r\,(\cos\cdot\theta + i\cdot\sin\cdot\theta)$$

Notación: Cualquiera de los dos se conoce como forma Polar del numero Complejo

$1)\, r\cdot cis\,(\theta)$

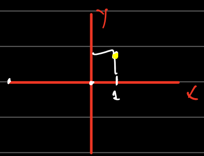
$2)\, r\cdot e^{i\theta}$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0$$
$$e^{i\pi} + 1 = 0$$

Modulo de un complejo

dado
$$z = a + bi$$
$$r = \sqrt{a^2 + b^2} = |z|$$

Ejercicios: $a + bi$



a) $z = 4 = 4e^{i\cdot 0}$

b) $z = -8 = 8e^{i\cdot\pi}$

c) $z = i \longrightarrow 1\cdot e^{i\cdot\frac{\pi}{2}}$

d) $z = (2i)^3 = -8i = 8e^{-i\cdot\frac{\pi}{2}}$

e) $z = 1 + \sqrt{3i} = 2\left(\frac{1}{2} + \frac{\sqrt{3i}}{2}\right) \longrightarrow |z| = \sqrt{1^2+(\sqrt{3})^2} = 2 \longrightarrow 2e^{i\cdot\frac{\pi}{3}}$

f) $z = -1 + i = \sqrt{1^2+1^2} = \sqrt{2} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\, e^{i\cdot\frac{\pi}{4}} = \sqrt{2}\, e^{i\cdot\left(\frac{3}{4}\pi\right)}$

$\tan\theta = \left(\frac{b}{a}\right)$

$\theta = \text{Arctan}\,\frac{b}{a} \longrightarrow \frac{\sqrt{3}}{1} \quad \sqrt{3}$

Propiedades 1.

Dados los complejos $z, w \in \mathbb{C}$:

- $\overline{\overline{z}} = z$
- $\overline{z \pm w} = \overline{z} \pm \overline{w}$
- $\overline{z\cdot w} = \overline{z}\cdot\overline{w}$
- $z + \overline{z} = 2Re(z), \qquad z - \overline{z} = 2Im(z)$
- $\overline{z} = z \Leftrightarrow z \in \mathbb{R}$
- $\overline{z} = -z \Leftrightarrow z \in$ $i\cdot\mathbb{R} = \{it : t \in \mathbb{R}\}$

7) $|z\cdot w| = |z|\cdot|w|$

8) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

9) $|z \pm w| \leq |z| + |w|$

10) $\big||z| - |w|\big| \leq |z \pm w|$

Producto de Numeros Complejos

• Sean $z = r_1 e^{i\cdot\theta_1}$ y $w = r_2 e^{i\cdot\theta_2}$

Luego $z\cdot w = r_1\cdot r_2\cdot e^{i(\theta_1+\theta_2)}$

$= r_1\cdot r_2 \text{ cis }(\theta_1+\theta_2)$

$\text{cis}\,\theta_1\cdot\text{cis}\,\theta_2 = \text{cis}(\theta_1+\theta_2)$

$e^{i\cdot\theta_1}\cdot e^{i\cdot\theta_2} = e^{i(\theta_1+\theta_2)}$

Ejemplos:

Sean $z = 4\sqrt{3} + 4i$; $w = -1 + \sqrt{3}i$

En primer lugar se tiene:

$z = 8e^{i\cdot\frac{\pi}{6}}$ $\quad |z| = \sqrt{(4\sqrt{3})^2+(4)^2} = 8 \longrightarrow \theta = \arctan\left(\frac{4}{4\sqrt{3}}\right) = \frac{\pi}{6}$

$w = 2e^{i\cdot\frac{2}{3}\cdot\pi}$ $\quad |w| = \sqrt{(-1)^2+(\sqrt{3})^2} = 2 \longrightarrow \theta = \arctan\left(\frac{\sqrt{3}}{1}\right) + \frac{\pi}{2} = \frac{2\pi}{3}$

$z\cdot w = 16\, e^{i\left(\frac{\pi}{6} + 2\cdot\frac{\pi}{3}\right)}$ $\quad \frac{\pi}{6} + \frac{2\pi}{3} \quad \frac{\pi + 4\pi}{6} \quad \frac{5\pi}{6}$

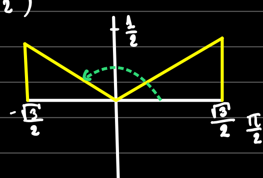
$= 16e^{i\frac{5\pi}{6}}$

$= 16\left[\cos\left(\frac{5\pi}{6}\right) + i\cdot\sin\left(\frac{5\pi}{6}\right)\right]$ $\quad \text{Cis} = (\text{Cos}\cdot\theta + i\cdot\text{Sin}\cdot\theta)$

$= 16\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$

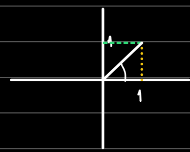
$= -8\sqrt{3} + i8$

$\left(16\cdot -\frac{\sqrt{3}}{2}\right) + \left(16\cdot\frac{1}{2}\right)i$



$Z = 1 + i \cdot \quad \sqrt{1^2 + 1^2} = \sqrt{2}$

$w = -2 + 2i \cdot$



$\frac{1}{1} = 1$

$Z = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$w = 2\sqrt{2}\, e^{i\frac{3}{4}\pi}$ $\quad 2 \cdot \sqrt{2}^{\,2} = 4$

$Z \cdot w = 2\sqrt{2} \cdot \sqrt{2}\, e^{i\cdot\left(\frac{\pi}{4} + \frac{3}{4}\pi\right)}$

$= 4 \cdot e^{i\cdot\pi} = 4\left(\cancel{\cos\pi}^{-1} + i \cdot \cancel{\sin\pi}^{0}\right)$

$= -4$

Corolario ; $Z = r_1 e^{i\cdot\theta}$ y $w = r_2 e^{i\theta_2}$

1) $\frac{Z}{w} = \frac{r_1}{r_2} \cdot e^{i(\theta_1 - \theta_2)}$

2) $\frac{1}{Z} = \frac{1}{r_1} e^{-i\theta_1}$

3) $\forall n \in \mathbb{Z}$; $Z^n = r_1^n e^{in\cdot\theta_1}$

Ejercicios:

3) $\frac{Z}{w} = \frac{1}{2} \cdot e^{i\left(\frac{\pi}{4} - \frac{3}{4}\pi\right)} = \frac{1}{2} e^{-i\frac{\pi}{2}}$

4) $(1+i)^{20} \qquad 1 - i$

5) $\frac{1}{(1-i)^{20}}$

Para determinar θ en la forma polar de $z = a + bi$

$$\theta := \begin{cases} \operatorname{Arctan}\left(\frac{b}{a}\right) & ; \text{ si } a > 0 \\ \pi/2 & ; \text{ si } a = 0,\ b > 0 \\ \pi - \operatorname{Arctan}\left(\left|\frac{b}{a}\right|\right) & ; \text{ si } a < 0,\ b \geq 0 \\ \pi + \operatorname{arctan}\left(\left|\frac{b}{a}\right|\right) & ; \text{ si } a < 0,\ b < 0 \\ \frac{3\pi}{2} & \end{cases}$$

$\longrightarrow \pi + \operatorname{Arctan}\left(\frac{b}{a}\right)$ si $a < 0$

Calcular:

$$i^{101} = \left(1 \cdot e^{i\frac{\pi}{2}}\right)^{101}$$
$$= e^{i\,101 \cdot \pi/2}$$
$$= e^{i\,50\pi + \pi/2}$$
$$= e^{i\,(25 \cdot 2\pi + \pi/2)}$$
$$= e^{i\frac{\pi}{2}}$$
$$= i$$

Teorema de Moivre

Sea $z \in \mathbb{C} - \{0\}$ y sea $n \in \mathbb{N}$

Luego la ecuación
$$x^n = z$$

tiene exactamente n soluciones distintas.

- mas precisamente, si $z = re^{i\theta}$, las n- soluciones $z_0, z_1, \ldots, z_{n-1}$, estan dados por

$$z_k = r^{\frac{1}{n}} \cdot e^{i\frac{\theta + 2\pi k}{n}} \; ; \; k = k = 0, \ldots, n-1$$

Ejemplos:

1) $z = 4$ Hallamos las 2 raices de 4.

$$z = 4e^{i \cdot 0}$$

$$z_0 = 4^{1/2} \cdot e^{i\frac{0 + 2\pi \cdot 0}{2}} = 2 \cdot e^{i \cdot 0} = 2$$
$$z_1 = 4^{1/2} \cdot e^{i\frac{0 + 2\pi \cdot 0}{2}} = 2 \cdot \underbrace{e^{i\pi}}_{-1} = -2$$

Ejercicio:

$$z_0 = 1^{1/4} \cdot e^{i\frac{0 + 2\pi \cdot 0}{4}} = e^{i \cdot 0} = 1$$
$$z_1 = 1^{1/4} \cdot e^{i\frac{0 + 2\pi \cdot 1}{4}} = e^{i\pi} = -1$$
$$z_2 = 1^{1/4} \cdot e^{i\frac{0 + 2\pi \cdot 2}{4}} = e^{i\frac{4\pi}{4}} =$$
$$z_3 = 1^{1/4} \cdot e^{i\frac{0 + 2\pi \cdot 3}{4}} = e^{i3}$$

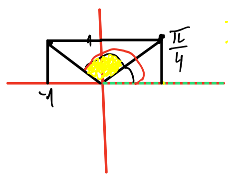
Ejercicios:

4) Hallar las 3 raices cubicas de

a) $z = i$
b) $z = -i$

Para obtener θ en el intervalo [0, 2π), se deben usar las siguientes fórmulas (arctan denota la inversa de la función tangente):

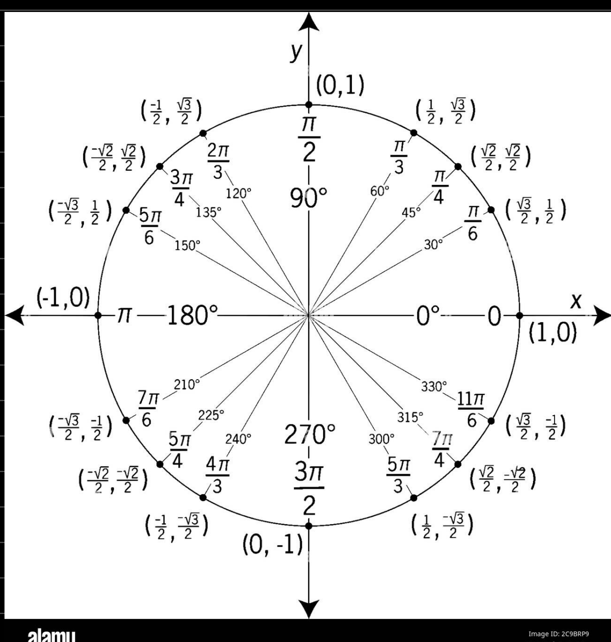
$$\theta = \begin{cases} \arctan(\frac{y}{x}) & \text{si } x > 0 \text{ y } y \geq 0 \\ \frac{\pi}{2} & \text{si } x = 0 \text{ y } y > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{si } x < 0 \\ \frac{3\pi}{2} & \text{si } x = 0 \text{ y } y < 0 \\ \arctan(\frac{y}{x}) + 2\pi & \text{si } x > 0 \text{ y } y < 0 \end{cases}$$



$\frac{\pi}{4} + \frac{\pi}{2} =$ $\frac{\pi + 2\pi}{4} = \frac{3\pi}{4}$

Para obtener θ en el intervalo $(-\pi, \pi]$, se considera que $\arctan\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ es una función creciente en su dominio:

$$\theta = \begin{cases} \arctan(\frac{y}{x}) - \pi & \text{si } x < 0 \text{ y } y < 0 \\ -\frac{\pi}{2} & \text{si } x = 0 \text{ y } y < 0 \\ \arctan(\frac{y}{x}) & \text{si } x > 0 \\ \frac{\pi}{2} & \text{si } x = 0 \text{ y } y > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{si } x < 0 \text{ y } y \geq 0 \end{cases}$$



$\frac{5\pi}{12} = \frac{2\pi}{12} + \frac{3\pi}{12}$

$\frac{\pi}{6} + \frac{\pi}{4}$

$Z = 1 + \sqrt{3}i$ llevarlo a forma polar ✓

$\rightarrow |Z| = |1 + \sqrt{3}i| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$

$\rightarrow r \cdot e^{i\pi} = 2 \cdot e^{i\frac{\pi}{3}}$

$\sin\theta = \frac{\text{Cateto opuesto}}{\text{hipotenusa}}$

$\cos\theta = \frac{\text{Cateto adyacente}}{\text{hipotenusa}}$

$\tan\theta = \frac{\text{Cateto opuesto}}{\text{Cateto adyacente}}$

a) $Z = 4\,\text{cis}\left(\frac{\pi}{2}\right) = 4 \cdot e^{i\frac{\pi}{2}}$

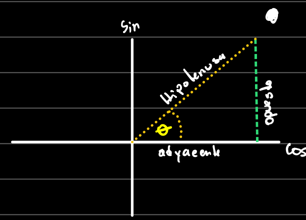
$\rightarrow (\cos(\theta) + i \cdot \sin(\theta))$

$4 \cdot \left(\cos\left(\frac{\pi}{2}\right) + i \cdot \sin\left(\frac{\pi}{2}\right)\right)$

$0 \qquad 1i$

$0 + 4i$

$= 4i$



b) $2\,\text{cis}\left(\frac{\pi}{3}\right)$

$2 \cdot \left(\cos\left(\frac{\pi}{3}\right) + i \cdot \sin\left(\frac{\pi}{3}\right)\right)$

$\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$

$1 + \sqrt{3}i$

c) $Z = 6\,\text{cis}\left(\frac{2\pi}{3}\right)$

$6 \cdot \left(\cos\left(\frac{2}{3}\pi\right) + i \cdot \sin\left(\frac{2\pi}{3}\right)\right)$

$6\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$-3 + 3\sqrt{3}i$

d) $3\,\text{cis}\left(\frac{4\pi}{3}\right)$

$\rightarrow 3($

$\rightarrow \left(\cos\left(\frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{4\pi}{3}\right)\right)$

$3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ $\qquad \frac{3}{1} \cdot \frac{\sqrt{3}}{2} =$

$-\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

e) $7\,\text{cis}\left(\frac{\pi}{6}\right)$

$7\left(\cos\left(\frac{\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6}\right)\right)$

$7\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$\frac{7\sqrt{3}}{2} + \frac{7i}{2}$

o) $Z = \frac{1}{3}\,\text{cis}\left(\frac{7\pi}{4}\right)$

$\frac{1}{3}\left(\cos\left(\frac{7\pi}{4}\right) + i \cdot \sin\left(\frac{7\pi}{4}\right)\right)$

$\frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$\frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{6}i$

p) $Z = \frac{5}{3}\,\text{cis}\left(\frac{9\pi}{4}\right)$

$\frac{5}{3}\left(\cos\left(\frac{9\pi}{4}\right) + i \cdot \sin\left(\frac{9\pi}{4}\right)\right)$

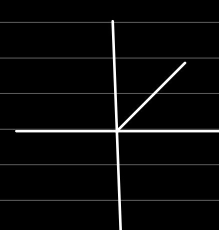
$\frac{5}{3}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$\underbrace{\frac{5\sqrt{2}}{6} + \frac{5\sqrt{2}}{6}i}$

$\sqrt{\left(\frac{5\sqrt{2}}{6}\right)^2 + \left(\frac{5\sqrt{2}}{6}\right)^2}$

$|z| = \frac{5}{3}$

$\theta = \arctan = 1 = \frac{\pi}{4}$



K) $5\,\text{cis}\left(\frac{5\pi}{12}\right)$

$5\left(\cos\left(\frac{5\pi}{12}\right) + i \cdot \sin\left(\frac{5\pi}{12}\right)\right)$

$5\frac{\left(\sqrt{6} - \sqrt{2}\right)}{4} + 5\frac{\left(\sqrt{6} + \sqrt{2}\right)}{4}i$

e) $-3+2i \longrightarrow \sqrt{13} \cdot e^{i}$

l) $3-4i \longrightarrow 5 \cdot e^{i}$

Producto de un Complejo

$z = 4\sqrt{3} + 4i \quad ; \quad w = -1 + \sqrt{3}i$

$|z| = \sqrt{(4\sqrt{3})^2 + (4)^2} = 8 \quad , \theta = \frac{\pi}{6}$ 1

$|w| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \quad \theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{2\pi}{6} + \frac{3\pi}{6} = \frac{5\pi}{6}$ 2

$r_1 \cdot r_2 \, e^{i(\theta_1 + \theta_2)}$

$8 \cdot 2 \, e^{i\left(\frac{\pi}{6} + \frac{5\pi}{6}\right)}$

$16 \, e^{i\left(\frac{6\pi}{6}\right)}$

$16 \, e^{i\pi}$ - 1

$\cos(\pi) + i \cdot \sin(\pi)$

$16(-1+0)$

-16