Course: HPS/MAT390

The Handling of Imaginary Numbers by Cardano and Bombelli

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Gerolamo Cardano (1501 - 1576) and Rafeal Bombelli (1526 - 1573) were some of the first mathematicians who encountered and handled imaginary numbers. Although their conceptions and intuitions of imaginary numbers were different from the current ones, their discoveries and early works inspired further investigation of imaginary numbers. However, one may ask to what extent did Cardano and Bombelli's work contribute to shaping the concept of imaginary numbers? Were their reactions to imaginary numbers mathematically appropriate? Their mathematical investigation of imaginary numbers of unquestionably importance, but they did little to uncover the identity of imaginary numbers.

Imaginary numbers, also known as complex numbers, are nowadays known as numbers with both a real part and an imaginary part. Most students first encounter imaginary numbers in high school in solutions to quadratic equations involving square roots of negative numbers. However, most historians do not believe that it was the quadratic equations that lead to the discovery of imaginary numbers, instead it was Cardano's solution of the cubic equation in his 1545 *Ars Magna*. The term "imaginary numbers" were not used by mathematicians like Cardano, who described these as "fictitious" and "sophisticated" terms. On the other hand, Bombelli referred to imaginary roots of equations as *quantità silvestri*, which roughly translates into "wild quantities" (Curcio, 2017, p.127). Although this paper will refer to this concept as simply imaginary numbers, it is important to take note of the first steps that Cardano and Bombelli took, and how they inspired a new mathematical concept.

The discovery of imaginary numbers is largely linked to Cardano's solution of the cubic equation. The solution to the cubic equation is a major part of *Ars Magna*, and the significance of this solution is that it empowered the theory of equations. The theory of equations involves solving equations with unknown variables (often denoted by 'x'), and mathematicians who pursued the theory of equations were known as the 'cossist algebraists' (Fraser, 2020). Ars Magna presented the solution of the cubic as well as the quartic (fourth degree) equation, which were previously unavailable to mathematicians. Although Cardano was the main figure, it shouldn't go unnoticed that Lodovico Ferrari (1522 - 1565) – one of Cardano's pupils – discovered a formula to solve the fourth-degree equation (Curcio, 2017, p.126). Furthermore, it's known that Cardano's publication of the solution of the cubic equation was against his promise to Niccolò Fontana (1499 - 1557), also known as Tartaglia (Curcio, 2017, p.126). It is known that Tartaglia inspired Cardano's solution of the cubic equation, although Cardano still deserves the credit for the rigorous geometric proof of the solution to the cubic equation. Hence, the method that Cardano presented in *Ars Magna* is also known as the Cardan-Tartaglia method (Bagni, 2009).

The Cardan-Tartaglia method of reducing cubic equations into quadratic equations played a key role in Cardano's success of finding the solution. Cardano's strategy targets depressed cubic equations (cubic equations without the square term) in the following form:

$$x^3 + mx = n$$

It is important to note that Cardano had not yet adopted a full symbolic presentation for mathematics at the time of writing *Ars Magna*. Instead of using abstract coefficients like m and n, Cardano based his solution on specific examples, such as:

$$x^3 + 6x = 20$$

William Dunham presents a detailed interpretation of Cardano's solution in his book *Journey Through Genius*. According to Dunham (1990), Cardano's solution has a geometric intuition. First, Cardano imagines a cube with side u within a larger cube with side t (see appendix 1). Since the large cube with side t has the same volume as all the other (non-overlapping) solids contained within it, we arrive at

$$t^3 = u^3 + u(t-u)^3 + 2tu(t-u) + u^2(t-u) + u(t-u)^2$$

Upon rearranging, we get

$$\begin{aligned} &u(t-u)^3 + 2tu(t-u) + u^2(t-u) + u(t-u)^2 = t^3 - u^3 \\ &\therefore (t-u)^3 + (t-u)[2tu + u^2 + u(t-u)] = t^3 - u^3 \\ &\therefore (t-u)^3 + 3tu(t-u) = t^3 - u^3 \end{aligned}$$

Cardano noticed that the above equation had the same form as the depressed cubic equation mentioned above, that is if we let $x = t - u$, $m = 3tu$, and $n = t^3 - u^3$. Now all that is left to do is solve for $t - u$. We start by rearranging the conditions:

$$m = 3tu, \qquad \therefore u = \frac{m}{3t}$$

Then we substitute into the other condition and rearrange to get:

$$n = t^3 - \left(\frac{m}{3t}\right)^3 = t^3 - \frac{m^3}{27t^3}$$

$$\therefore t^3 - \frac{m^3}{27t^3} - n = 0$$

$$\therefore t^3 \left[t^3 - \frac{m^3}{27t^3} - n\right] = 0$$

$$\therefore (t^3)^2 - n(t^3) - \frac{m^3}{27} = 0$$

The result we obtain is a quadratic equation. Thus, the problem is now reduced from a cubic problem to a quadratic problem, where the solution was already available to Cardano at the time. Solving this yield:

$$t^3 = \frac{1}{2}\left(n \pm \sqrt{n^2 + \frac{4m^3}{27}}\right) = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

Upon taking only the positive square root term, and further rearranging, we get:

$$t = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

Recall u and t are related by the following equations:

$$n = t^3 - u^3, \qquad \therefore u = \sqrt[3]{t^3 - n}$$

Substituting the former into the above we get

$$u = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}} - n} = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

To conclude the derivation:

$$x = t - u = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

This final form is known as Cardano's formula. Again, Dunham reminds us that these derivations were not presented in *Ars Magna*, since Cardano used specific numbers in his example. However, most historians agree with this modern symbolic interpretation since the results Cardano obtained was in the same form as the derivations above, namely

$$x = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$$

Where originally $m = 6, n = 20$ and we get the above solution by simply substituting m and n into the derivation above. In short, the key to Cardano's solution to depressed cubic equations was reducing the cubic equation to a quadratic with a known solution, this type of solution is also known as solution by radicals or algebraic solution (Dunham, 1990, p. 146).

Although Cardano's solution was considered a great success, yet it still faced many challenges. The first was that it did not explicitly solve cubic equations in the general form, and another was that it did not handle negative numbers for the m and n coefficients well. For example, set $m = -15, n = 4$ we obtain $x^3 - 15x = 4$, which is an irreducible cubic equation. Using Cardano's method on this equation yields

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

This solution is troublesome, since it does not make sense to find the square root of a negative number, or so at the time of *Ars Magna*'s publication. Upon examination of the solution, one may easily dismiss the equation as unsolvable. However, one can show that $x = 4$, and $x = 2 \pm \sqrt{3}$ are three real solutions to the irreducible cubic equation above. More importantly, this marks the discovery of what we now call "imaginary numbers". Cardano noticed this issue, but was quick to disregard the enterprise of imaginary numbers "as subtle as it is useless" without giving much thought (Dunham, 1990, p. 150).

Unlike Cardano, Bombelli saw potential in the seemingly preposterous "imaginary numbers", and published some ideas in his 1572 *Algebra*. Bombelli's work aimed to bypass the issues of the square roots of negative numbers while soling for real solutions of cubic equations, and save Cardano's technique. However, his work raised more questions than it di answered. His first step was to cube the term $2 + \sqrt{-1}$.

$$
\begin{aligned}
\left(2+\sqrt{-1}\right)^{3} &= 8+12\sqrt{-1}-6-\sqrt{-1} \\
&= 2+11\sqrt{-1} \\
&= 2+\sqrt{-121}
\end{aligned}
$$

With this result, we may inversely conclude that

$$\sqrt[3]{2+\sqrt{-121}}=2+\sqrt{-1}$$

Similarly

$$\sqrt[3]{-2+\sqrt{-121}}=-2+\sqrt{-1}$$

Recall Cardano's method to the irreducible cubic $x^3-15x=4$ yields

$$x=\sqrt[3]{2+\sqrt{-121}}-\sqrt[3]{-2+\sqrt{-121}}$$

If we substitute Bombelli's answer, we arrive at one of the real solutions to the irreducible cubic equation above. The derivations are as follows.

$$
\begin{aligned}
x &= \sqrt[3]{2+\sqrt{-121}}-\sqrt[3]{-2+\sqrt{-121}} \\
&= 2+\sqrt{-1}-\left(-2+\sqrt{-1}\right)=4
\end{aligned}
$$

Since the positive and negative $\sqrt{-1}$ cancels each other, we end up with a perfectly real solution (Dunham, 1990, p. 150). One of the many questions that Bombelli's solution raised was how does one know that we can simplify $\sqrt[3]{2+\sqrt{-121}}$ to $2+\sqrt{1}$. His solution seems to be a one-off treatment rather than a general method we could apply to all similar situations.

On the other hand, Bombelli did introduce a basic rule for computing imaginary roots in the same book. Originally, these rules which are presented in *Algebra* were written in Italian (see appendix 2). The following figure includes Bombelli's rules in both words (left side) and modern symbolic notation (right side):

Più via più di meno, fa più di meno	$+(+i) = +i$
Meno via più di meno, fa meno di meno	$-(+i) = -i$
Più via meno di meno, fa meno di meno	$+(-i) = -i$
Meno via meno di meno, fa più di meno	$-(-i) = +i$
Più di meno via più di meno, fa meno	$(+i)(+i) = -1$
Più di meno via men di meno, fa più	$(+i)(-i) = +1$
Meno di meno via più di meno, fa più	$(-i)(+i) = +1$
Meno di meno via men di meno, fa meno	$(-i)(-i) = -1$

Figure: Bombelli’s rules for calculating imaginary numbers (González-Velasco, 2011, p. 164)

This was a bold step that Bombelli took, but nevertheless an important one. Although he was rightly credited for the creation of these rules, Bombelli did not try to formally define or outline the nature of the quantity we now know as i, the imaginary unit (such that $i^2 = -1$). Instead he proposed few properties of the imaginary unit, arguably for the purpose of computing answers to equations such as the cubic equation. Moreover, he even gave a definition of the concept now known as the conjugate of a complex number, where González-Velasco (2011) quoted from Bombelli:

> “Notice that when we say the Residue of a Binomial [what Cardano called Apotome], what is called più di meno in the Binomial, will be called meno di meno in the Residue.” (González-Velasco, 2011, p.164)

The main idea of this definition is that the product of any pair of complex numbers and its conjugate would produce a real number, and real numbers are nice for calculations. Recall in modern terms, a complex number is a number with a real and imaginary part in the following form:

$$z = a + bi$$

Where z is a complex number, a, b are magnitudes such that $a, b \in \mathbb{R}$ and i is the imaginary unit. An example of an arbitrary pair of a complex number and its conjugate is

$$x = a + bi$$

$$y = a - bi$$

If we multiply the two, we get

$$(a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R}$$

Bombelli had a similar idea, although at the time he did not have the same symbolic notations yet. This concept of complex conjugates is useful in computations, since their product are real numbers and much easier to work with.

Bombelli's work of defining complex conjugation and computation rules suggest that he likely believed imaginary numbers are mere tools one can use in computation to solve more sophisticated problems. This is also evidenced by the fact that Bombelli did not formally define imaginary numbers. From this, it follows that Bombelli may have believed that imaginary numbers would cancel out each other in most situations, such as with solving for the cubic equation using Cardano's formula. As it turned out, it is not always the case that imaginary roots cancel each other. Thus, Bombelli's exploration on imaginary numbers is limited in this sense, although undoubtedly significant.

To summarize, Cardano's and Bombelli's work on the concept now known as imaginary number was an important first step for mathematicians. Although Cardano was first to discover problems involving the square roots of negative numbers, he made little contribution to investigate its consequence. On the other hand, Bombelli was one of the first mathematicians who properly investigated this concept. Although he made significant contributions to outlining some computation rules and defining the conjugate of a complex number, he seemd to focus more on bypassing the issue when computing with imaginary numbers, rather than exploring the true identity of imaginary numbers themselves. Nevertheless, they inspired later mathematicians to investigate this topic further, and in turn they shaped our current understanding of imaginary numbers.

References

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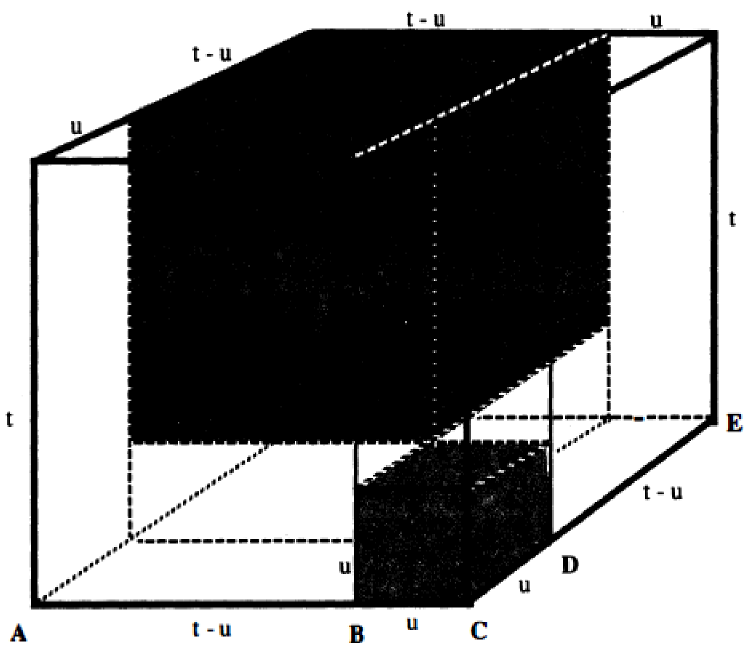
Appendices

Appendix 1: Cardano's imagination of a cube with side t (Dunham, 1990, p. 143)



Appendix 2: Rules of calculations of imaginary numbers (Bombelli, 1572/1966, p. 169)

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.