Ex. 1.12.1: Conditional probability: suppose that if $\theta = 1$, then y has a normal distribution with mean 1 and standard deviation σ, and if $\theta = 2$, then y has a normal distribution with mean 2 and standard deviation σ. Also, suppose $\Pr(\theta = 1) = 0.5$ and $\Pr(\theta = 2) = 0.5$.

a For $\theta = 2$, write the formula for the marginal probability density for y and sketch it.

b What is $\Pr(\theta = 1|y = 1)$, again supposing $\sigma = 2$?

c Describe how the posterior density of θ changes in shape as σ is increased and as it is decreased.

Answer:

The formula for the marginal probability is that of $\mathcal{N}(\theta, \sigma)$:

$$p(y|\theta = 2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\left(\frac{x-2}{\sigma}\right)^2\right).$$

As for the sketch, I'm too lazy to do it. It's a bell shape, centered at 2 and σ being half the width at about 0.61 height. There's an xkcd-looking MatPlotLib plot in Figure 1.

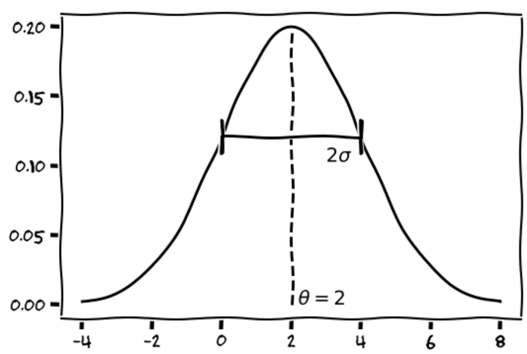

Figure 1: Your typical bell thingy with $\sigma = 2$ and mean 2

Now then,

$$\begin{aligned}
\Pr(\theta = 1|y = 1) &= \frac{\Pr(\theta = 1)\Pr(y = 1|\theta = 1)}{\Pr(\theta = 1)\Pr(y = 1|\theta = 1) + \Pr(\theta = 2)\Pr(y = 2|\theta = 2)} \\
&= \frac{0.5 \cdot \frac{1}{\sqrt{8\pi}}}{0.5 \cdot \frac{1}{\sqrt{8\pi}} + 0.5 \cdot \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{1}{8}\right)} \\
&\approx 0.53.
\end{aligned}$$

The posterior for θ gets more homogeneous as σ increases, and instead gets closer to $(1, 0)$ otherwise, since σ directly defines the overlap between both likelihood functions as functions of y.