Ex. 2.1: Is it possible to find a general formula for $p(C|A+B)$, analogous to $(2-48)$, from the product and sum rules? If so, derive it; if not, explain why this cannot be done.

Answer:

Sure. We've derived Bayes theorem without naming it, which is simply product rule and commutativity, so we can say

$$\begin{aligned} p(C|A+B) &= \frac{p(C)\,p(A+B|C)}{p(A+B)} \\ &= \frac{p(C)\ (p(A)+p(B)-p(AB|C))}{p(A)+p(B)-p(AB)}. \end{aligned}$$

Ex. 2.2: Now suppose we have a set of propositions $\{A_1, \cdots, A_n\}$ which on information X are mutually exclusive: $p(A_iA_j|X) = p(A_i|X)\delta_{ij}$. Show that $p(C|(A_1+A_2+\cdots+A_n)X)$ is a weighted average of the separate plausibilities $p(C|A_iX)$:

$$p(C|(A_1+\cdots+A_n)X) = p(C|A_1X + A_2X + \cdots + A_nX) = \frac{\sum_i p(A_i|X)p(C|A_iX)}{\sum_i p(A_i|X)}.$$

Answer: For $i \neq j$,

$$\begin{aligned} p(C|A+B) &= \frac{p(A+B|C)p(C)}{p(A+B)} \\ &= \frac{(p(A)+p(B)-p(AB|C))p(C)}{p(A)+p(B)-p(AB|C)}. \end{aligned}$$

Iterating this over $\sum_i A_i$ we get

$$p\left(\sum_i A_i \middle| X\right) = \sum_i p(A_i|X).$$

Thus, and in a similar manner to the above excercise,

$$\begin{aligned} p\left(C \middle| \sum_i A_i X\right) &= \frac{p\left(\sum_i A_i|CX\right)p(C|X)}{p(\sum_i A_i|X)} \\ &= \frac{\sum_i p(A_i|CX)p(C|X)}{\sum_i p(A_i|X)} \\ &= \frac{\sum_i p(A_i|X)p(C|A_iX)}{\sum_i p(A_i|X)}. \end{aligned}$$

Ex.: Let A_i mutually exclusive ($P(A_iA_j) = P(A_i)\delta_{ij}$). Then, $P(\sum_i A_i) = \sum_i P(A_i)$.

Answer:

If $i \in \{1, 2\}$, then

$$P(\sum_i A_i) = P(A_1 + A_2) \tag{1}$$

$$= P(A_1) + P(A_2) - \cancelto{0}{P(A_1A_2)} \tag{2}$$

$$= \sum_i P(A_i). \tag{3}$$

Then, by induction,

$$P\left(\sum_i^{N+1} A_i\right) = P\left(\sum_i^{N} A_i\right) + P(A_{N+1}) - P\left(\sum_i^{N} A_i A_{N+1}\right) \tag{4}$$

$$= \sum_i^N P(A_i) + P(A_{N+1}) - \sum_i^N \cancelto{0}{(A_i A_{N+1})} \tag{5}$$

$$= \sum_i P(A_i). \tag{6}$$

Ex. 2.3: Limits on Probability Values: As soon as we have the numerical values $a = P(A|C)$ and $b = P(B|C)$, the product and sum rules place some limits on the possible numerical values for their conjunction and disjuncion. Supposing that $a \leq b$, show that the probability of the conjunction cannot exceed that of the least possible proposition: $0 \leq P(AB|C) \leq a$, and the probability of the disjunction cannot be less than that of the most probable proposition: $b \leq P(A+B|C) \leq 1$. Then show that, if $a + b > 1$, there is a stronger inequality for the conjunction; and if $a + b < 1$ there is a stronger one for the disjunction. These necessary general inequalities are helpful in detecting errors in calculations.

Answer:

For the conjunction,

$$P(AB|C) = P(A|C)P(B|AC) \tag{7}$$

$$= aP(B|AC) \qquad \leq a. \tag{8}$$

Since $P(AB|C) = b + a - P(A+B|C)$, if $b + a > 1$ then $P(AB|C) > 1 - P(A+B|C) = P(\overline{A+B}|C)$.

For the disjunction,

$$P(A+B|C) = P(A|C) + P(B|C) - P(AB|C) \tag{9}$$

$$= b + a - P(AB|C) \tag{10}$$

$$\geq b. \tag{11}$$

And also, $b + a < 1$ implies $P(A+B|C) < 1 - P(AB|C) = P(\overline{AB}|C)$.